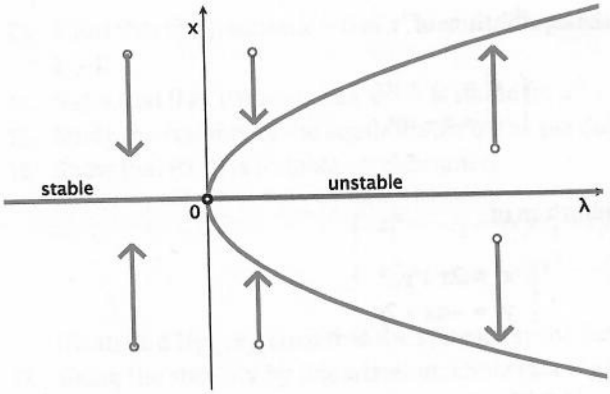

Figure 10.13: Supercritical pitchfork bifurcation for $x' = \lambda x - x^3$.

This pitchfork bifurcation is called *supercritical* to distinguish it from the *subcritical pitchfork bifurcation* that arises dealing with the equation

$$x' = \lambda x + x^3.$$

Notice that in this case the linearized equation at the nontrivial equilibria $x_\lambda = \pm\sqrt{-\lambda}$, $\lambda < 0$, is $y' = (\lambda + 3x_\lambda^2)y = -2\lambda y$. Since $\lambda < 0$, the branches $x = \pm\sqrt{-\lambda}$ are both unstable; see Fig. 10.14.

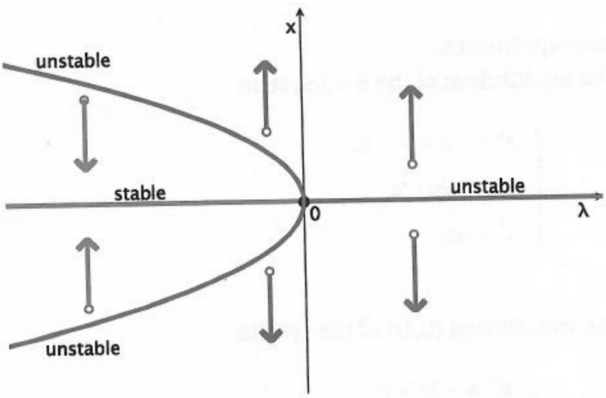

Figure 10.14: Subcritical pitchfork bifurcation for $x' = \lambda x + x^3$.

10.7 Exercises

1. Establish the stability of the equilibrium of

$$\begin{cases} x' = -x + y, \\ y' = -x + 2y. \end{cases}$$

2. Establish the stability of the equilibrium of

$$\begin{cases} x' = -y + ax, \\ y' = x + ay, \end{cases}$$

depending on a.

3. Show that $\forall a \in \mathbb{R}$ the equilibrium of

$$\begin{cases} x' = 2x + y, \\ y' = -ax + 2y, \end{cases}$$

is unstable.

4. * Show that the origin is a saddle for the system

$$\begin{cases} x' = x - y, \\ y' = -y, \end{cases}$$

and find the stable manifolds.

5. Given the 3×3 system

$$\begin{cases} x' = -x + y, \\ y' = -x - y, \\ z' = y - z, \end{cases}$$

show that $(0, 0, 0)$ is a stable equilibrium.

6. Establish the stability of the equilibrium of the 3×3 system

$$\begin{cases} x' = -x + y + z, \\ y' = -2y - z, \\ z' = az, \end{cases}$$

in dependence of $a \neq 0$.

7. Establish the stability of the equilibrium $(0, 0)$ of the system

$$\begin{cases} x'' = -2x + y, \\ y'' = x. \end{cases}$$

8. Show that the origin is a center for the harmonic oscillator $x'' + \omega^2 x = 0$, $\omega \neq 0$.
9. Show that $x = 0$ is a stable solution of the equation $x''' + 3x'' + 3x' + x = 0$.
10. Show that if $a > 0$ the trivial solution $x = 0$ of $x''' - 2x' - ax = 0$ is unstable.
11. Show that $x = 0$ is an unstable solution of $x'''' - 4x'' + x = 0$.
12. Show that $x = 0$ is unstable for any equation of the form $x'''' + 2ax'' - b^2 x = 0$, for any $a, b \neq 0$.

13. Show that the solution $x = 0$ of $x'' + \omega^2 x \pm x^{2k} = 0$, $\omega \neq 0$, $k \in \mathbb{N}$, is stable provided $k \geq 1$.
14. Show that if $F''(0) > -1$ then $x = 0$ is stable for $x'' + x + F'(x) = 0$.
15. Study the stability of the equilibria $k\pi$ of the pendulum equation $x'' + \sin x = 0$.
16. Show that $(0, 0)$ is a stable equilibrium of

$$\begin{cases} x_1'' = -x_1 + 4x_1(x_1^2 - x_2^2), \\ x_2'' = -x_2 - 4x_2(x_1^2 - x_2^2). \end{cases}$$

[Hint: find $U(x_1, x_2)$ such that the system has the form $\overline{x}'' + \nabla U(\overline{x}) = 0$, $\overline{x} = (x_1, x_2)$.]
17. Using the stability by linearization, show that if $g(0) = 0$ then $x = 0$ is unstable for $x'' + g(x') - x = 0$.
18. Using the stability by linearization, show that the trivial solution of $x'' + xx' - x = 0$ is unstable.
19. Using the stability by linearization, show that the trivial solution of $x'' + 2x' + x + x^2 = 0$ is stable
20. Study the stability of the solution $x = 0$ of $x'' - \mu(1 - x^2)x' - x = 0$.