52. ON THE PRESERVATION OF CONDITIONALLY PERIODIC MOTIONS UNDER SMALL VARIATIONS OF THE HAMILTON FUNCTION *

We consider in the $2s$-dimensional phase space of a dynamical system with s degrees of freedom a region G, represented as the product of an s-dimensional torus T by a region S in an s-dimensional Euclidean space. The points of the torus will be characterized by circular coordinates $q_1, \dots, q_s$ (the replacement of q_α by $q_\alpha + 2\pi$ does not change the position of the point q), and the coordinates of a point p belonging to S will be denoted by $p_1, \dots, p_s$. Assume that in the region G the equations of motion in the coordinates $(q_1, \dots, q_s, p_1, \dots, p_s)$ have the canonical form

$$\frac{dq_\alpha}{dt} = \frac{\partial}{\partial p_\alpha} H(q,p), \quad \frac{dp_\alpha}{dt} = -\frac{\partial}{\partial q_\alpha} H(q,p). \tag{1}$$

In what follows, the Hamilton function H is assumed to depend on a parameter θ, defined for all $(q,p) \in G$, $\theta \in (-c; +c)$, and to be independent of time. In essence, the consideration below is related to real functions, but imposes rather strong conditions on the smoothness of the function $H(q,p,\theta)$, stronger than the condition of infinite differentiability. For simplicity, in what follows we assume that *the function $H(p,q,\theta)$ is analytic in the variables (q,p,θ) jointly.*

Below the summation over Greek subscripts extends from 1 to s. Ordinary vector notation is used: $(x,y) = \sum_\alpha x_\alpha y_\alpha$ and $|x| = +\sqrt{(x,x)}$. By an integral vector is meant a vector all components of which are integers. A set of points $(q,p) \in G$ with $p = c$ is denoted by T_c. In Theorem 1 it is assumed that S contains the point $p = 0$, that is, $T_0 \subseteq S$.

Theorem 1. *Let*

$$H(q,p,0) = m + \sum_\alpha \lambda_\alpha p_\alpha + \tfrac{1}{2} \sum_{\alpha\beta} \Phi_{\alpha\beta}(q) p_\alpha p_\beta + O(|p|^3), \tag{2}$$

where m and λ_α are constants, and let the inequality

$$|(n,\lambda)| \geq c/|n|^2 \tag{3}$$

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be fulfilled for a certain choice of the constants $c > 0$ and $\eta > 0$ and all integral vectors n. Moreover, let the determinant formed from the average values

$$\phi_{\alpha\beta}(0) = \frac{1}{(2\pi)^s} \int_0^{2\pi} \int_0^{2\pi} \Phi_{\alpha\beta}(q) dq_1 \dots dq_s$$

of the functions

$$\Phi_{\alpha\beta}(q) = \frac{\partial^2 H}{\partial p_\alpha \partial p_\beta}(q, 0, 0)$$

be non-zero:

$$|\phi_{\alpha\beta}(0)| \neq 0. \tag{4}$$

Then there exist analytic functions $F_\alpha(Q, P, \theta)$ and $G_\alpha(Q, P, \theta)$ defined for all sufficiently small θ and all points (Q, P) belonging to a neighbourhood V of the set T_0 that determine a contact transformation

$$q_\alpha = Q_\alpha + \theta F_\alpha(Q, P, \theta), \quad p_\alpha = P_\alpha + \theta G_\alpha(Q, P, \theta)$$

of V into $V' \subseteq G$ reducing H to the form

$$H = M(\theta) + \sum_\alpha \lambda_\alpha P_\alpha + O(P^2) \tag{5}$$

($M(\theta)$ does not depend on Q or P).

The significance of Theorem 1 in mechanics can easily be understood. It shows that, under conditions (2) and (3), an s-parameter family of conditionally periodic motions

$$q_\alpha = \lambda_\alpha t + q_\alpha^{(0)}, \ p_\alpha = 0,$$

existing at $\theta = 0$ cannot disappear under a small variation of the Hamilton function H; namely, the variation results only in a displacement of the s-dimensional torus T_0, along the trajectories of the motions: it is transformed into a torus $P = 0$, which is filled with trajectories of conditionally periodic motions with the same frequencies $\lambda_1, \dots, \lambda_s$.

The transformation

$$(Q, P) = K_\theta(q, p)$$

whose existence is asserted in Theorem 1 can be constructed as the limit of transformations

$$(Q^{(k)}, P^{(k)}) = K_\theta^{(k)}(q, p),$$

where the transformations

$$(Q^{(1)}, P^{(1)}) = L^{(1)}(q, p), \quad (Q^{(k+1)}, P^{(k+1)}) = L_\theta^{(k+1)}(Q^{(k)}, P^{(k)})$$

are found by a "generalized Newton's method" (see [1]). In this paper we confine ourselves to the construction of the transformation $K_\theta^{(1)} = L_\theta^{(1)}$, which makes it possible to understand the role of conditions (3) and (4) of Theorem 1. We determine the transformation $L_\theta^{(1)}$ by means of the formulas

$$\begin{aligned} Q_\alpha^{(1)} &= q_\alpha + \theta Y_\alpha(q), \\ p_\alpha &= P_\alpha^{(1)} + \theta\Big\{\sum_\beta P_\beta^{(1)} \frac{\partial Y_\beta}{\partial q_\alpha} + \xi_\alpha + \frac{\partial}{\partial q_\alpha} X(q)\Big\} \end{aligned} \tag{6}$$

(it can be readily verified that this is a contact transformation) and find the constants ξ_α and ζ and the functions $X(q)$ and $Y_\beta(q)$ by proceeding from the requirement that

$$\begin{aligned} H = m + \sum_\alpha \lambda_\alpha p_\alpha + \tfrac{1}{2} \sum_{\alpha,\beta} \Phi_{\alpha\beta}(q) p_\alpha p_\beta + \\ + \theta\{A(q) + \sum_\alpha B_\alpha(q) p_\alpha\} + O(|p|^3 + \theta|p|^2 + \theta^2) \end{aligned} \tag{7}$$

should have the form

$$H = m + \theta\zeta + \sum \lambda_\alpha P_\alpha^{(1)} + O(|P^{(1)}|^2 + \theta^2). \tag{8}$$

Substituting (6) into (7) we find

$$\begin{aligned} H = m + \sum_\alpha \lambda_\alpha P_\alpha^{(1)} + \theta\Big\{A + \sum_\alpha \lambda_\alpha\Big(\xi_\alpha + \frac{\partial X}{\partial q_\alpha}\Big)\Big\} + \\ + \theta \sum_\alpha P_\alpha^{(1)}\Big\{B_\alpha + \sum_\beta \Phi_{\alpha\beta}(q)\Big(\xi_\beta + \frac{\partial X}{\partial q_\beta}\Big) + \sum_\beta \lambda_\beta \frac{\partial Y_\alpha}{\partial q_\beta}\Big\} + \\ + O(|P^{(1)}|^2 + \theta^2). \end{aligned}$$

Thus, condition (8) reduces to the requirement that the equations

$$A + \sum \lambda_\alpha\Big(\xi_\alpha + \frac{\partial X}{\partial q_\alpha}\Big) = \zeta, \tag{9}$$

$$B_\alpha + \sum_\beta \Phi_{\alpha\beta}\Big(\xi_\beta + \frac{\partial X}{\partial q_\beta}\Big) + \sum_\beta \lambda_\beta \frac{\partial Y_\alpha}{\partial q_\beta} = 0 \tag{10}$$

hold.

We introduce the function

$$Z_\alpha(q) = \sum_\beta \Phi_{\alpha\beta}(q) \frac{\partial}{\partial q_\beta} X(q). \tag{11}$$

Expanding the functions $\Phi_{\alpha\beta}$, A, B_α, X, Y_α, and Z_α into Fourier series of the form

$$X(q) = \sum x(n) e^{i(n,q)}$$

and putting, for the sake of definiteness,

$$x(0) = 0, \quad y_\alpha(0) = 0, \tag{12}$$

we obtain the following equations for the unknown Fourier coefficients $x(n)$, $y_\alpha(n)$, $z_\alpha(n)$ and the constants ξ_α and ζ:

$$a(0) + \sum \lambda_\alpha \xi_\alpha = \zeta, \tag{13}$$

$$a(n) + i(n, \lambda) x(n) = 0 \text{ for } n \neq 0, \tag{14}$$

$$b_\alpha(0) + \sum_\beta \phi_{\alpha\beta}(0) \xi_\beta + z_\alpha(0) = 0, \tag{15}$$

$$b_\alpha(n) + \sum_\beta \phi_{\alpha\beta}(n) \xi_\beta + z_\alpha(n) + i(n, \lambda) y_\alpha(n) = 0 \text{ for } n \neq 0. \tag{16}$$

It is easy to see that the system of equations (11)–(16) possesses a unique solution under conditions (3) and (4). Condition (3) is important when finding $x(n)$ from (14) and $y_\alpha(n)$ from (16). Condition (4) is important for the determination of ξ_β from (15). Since the coefficients of the Fourier series of the analytic functions $\Phi_{\alpha\beta}$, A, and B_α decrease at least as rapidly as ρ^h ($\rho < 1$) with increasing $|n|$, condition (3) implies not only the formal solvability of equations (13)–(16) but also the convergence of the Fourier series of the functions X, Y_α and Z_α and the analyticity of these functions. The construction of further approximations encounters no new difficulties. Only the application of condition (3) in the proof of the convergence of the mappings $K_\theta^{(k)}$ to an analytic limit mapping K_θ is somewhat more intricate.

The condition of absence of "small divisors" (3) should be presumed to be fulfilled "in general" since for any $\eta > s - 1$ there exists $c(\lambda)$ such that

$$|(n, \lambda)| \geq c(\lambda)/|n|^\eta$$

at all points of the s-dimensional space $\lambda = (\lambda_1, \ldots, \lambda_s)$, except at a set of Lebesgue measure zero, for any integers $n_1, n_2, \ldots, n_s$ (see [2]). It is natural to assume that, "in general", condition (4) is also fulfilled. Since

$$\phi_{\alpha\beta}(0) = \frac{\partial \lambda_\beta}{\partial p_\alpha}(0),$$

where

$$\lambda_\beta(p) = \frac{1}{(2\pi)^s} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{dq_\beta}{dt} dq_1 \ldots dq_s,$$

is the "average frequency" corresponding to the coordinate q_β for fixed momenta $p_1, \ldots, p_s$, condition (3) means that the Jacobian of the average frequencies with respect to the momenta is non-zero.

We now proceed to the consideration of the special case when $H(p, q, 0)$ depends only on p, that is, $H(q, p, 0) = W(p)$. In this case, for $\theta = 0$ each torus T_p consists of entire trajectories of conditionally periodic motions with frequencies

$$\lambda_\alpha(p) = \partial W / \partial p_\alpha.$$

If the Jacobian

$$J = \left| \frac{\partial \lambda_\alpha}{\partial p_\beta} \right| = \left| \frac{\partial^2 W}{\partial p_\alpha \partial p_\beta} \right| \tag{17}$$

is non-zero, then Theorem 1 can be applied to almost all tori T_p. There arises the natural conjecture that for small θ the "displaced tori" appearing in accordance with Theorem 1 occupy a major part of the region G. This fact is confirmed by Theorem 2 below. When stating the latter theorem, we assume that the region S is bounded and consider the set M_θ of those points $(q^{(0)}, p^{(0)}) \in G$ for which the solution

$$q_\alpha(t) = f_\alpha(t; q^{(0)}, p^{(0)}, \theta), \quad p_\alpha(t) = g_\alpha(t; q^{(0)}, p^{(0)}, \theta)$$

of the system of equations (1) with initial conditions

$$q_\alpha(0) = q_\alpha^{(0)}, \quad p_\alpha(0) = p_\alpha^{(0)}$$

has trajectories not falling outside the region G for t varying from $-\infty$ to $+\infty$ and is conditionally periodic with periods $\lambda_\alpha = \lambda_\alpha(q^{(0)}, p^{(0)}, \theta)$, that is, has the form

$$f_\alpha(t) = \phi_\alpha(e^{i\lambda_1 t}, \ldots, e^{i\lambda_s t}), \quad g_\alpha(t) = \psi_\alpha(e^{i\lambda_1 t}, \ldots, e^{i\lambda_s t}).$$

Theorem 2. *If $H(q,p,0) = W(p)$ and the determinant* (17) *is non-zero in the region S, then for $\theta \to 0$ the Lebesgue measure of the set M_θ tends to the total measure of the region G.*

It seems that, in a sense, the "general case" is the case when the set M_θ has an everywhere dense complement for all positive θ. Complications of this kind appearing in the theory of analytic dynamical systems were indicated in my paper [3] in connection with a more specific situation.

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References

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