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# ITERATION OF ANALYTIC FUNCTIONS

BY CARL LUDWIG SIEGEL

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Let

$$f(z) = \sum_{k=1}^{\infty} a_k z^k \tag{1}$$

be a power series without constant term and denote by $R > 0$ its radius of convergence. The fixed point $z = 0$ of the mapping $z \to f(z)$ is called stable, if there exist two positive finite numbers $r_0 \leqq R$ and $r \leqq R$, such that for all points $z$ of the circle $|z| < r_0$ the set of image points $z_1 = f(z)$, $z_{n+1} = f(z_n)$ $(n = 1, 2, \cdots)$ lies in the circle $|z| < r$.

It is easy to prove the stability in the case $|a_1| < 1$, for then a positive number $r_0 < R$ exists, such that the inequality $|f(z)| \leqq |z|$ holds for $|z| < r_0$, and $r = r_0$ has the required property. Henceforth, the inequality $|a_1| \geqq 1$ is assumed.

If $z = 0$ is stable, then the images $z_n$ $(n = 1, 2, \cdots)$ of the points $z$ of the circle $|z| < r_0$ under the mapping $z \to f(z)$ and its iterations cover a domain $D$ which is connected and contains the point $z = 0$. For all $z$ in $D$, the image point $f(z)$ again lies in $D$. Let

$$z = \varphi(\zeta) = \zeta + \sum_{k=2}^{\infty} c_k \zeta^k \tag{2}$$

be the power series mapping a certain circle $|\zeta| < \rho$ of the $\zeta$ plane conformally onto the universal covering surface of $D$. Then the formula

$$\varphi(\zeta) = z \to f(z) = z_1 = \varphi(\zeta_1)$$

defines a function $\zeta_1 = g(\zeta)$ which is regular in the circle $|\zeta| < \rho$ and satisfies there the inequality $|g(\zeta)| < \rho$; moreover $g(0) = 0$ and $g'(0) = 1$. It follows from Schwarz's lemma that $|a_1| = 1$ and $\zeta_1 = a_1\zeta$. Consequently, the functional equation of Schröder

$$\varphi(a_1\zeta) = f(\varphi(\zeta)) \tag{3}$$

has a convergent solution $\varphi(\zeta) = \zeta + \cdots$.

On the other hand, it is obvious that $z = 0$ is stable, if $|a_1| = 1$ and the functional equation (3) has a convergent solution.

If $a_1$ is an $n^{\text{th}}$ root of unity, then $z = 0$ is stable, if and only if the $(n - 1)^{\text{th}}$ iteration of the mapping $z \to f(z)$ is the identity. This is also easily proved by direct calculation. We assume now that $|a_1| = 1$ and $a_1^n \neq 1$ for $n = 1, 2, \cdots$.

By (1), (2) and (3),

$$\sum_{k=2}^{\infty} c_k(a_1^k - a_1)\zeta^k = \sum_{l=2}^{\infty} a_l\left(\zeta + \sum_{r=2}^{\infty} c_r \zeta^r\right)^l; \tag{4}$$

hence $c_k$ $(k = 2, 3, \cdots)$ is a polynomial in $c_2, \cdots, c_{k-1}$ whose coefficients depend upon $a_1, \cdots, a_k$, and there exists exactly one formal (convergent or divergent) solution $\varphi(\zeta) = \zeta + \cdots$ of (3). The first example of a convergent series $f(z) = a_1 z + \cdots$ with *divergent* Schröder series $\varphi(\zeta)$ has been given by Pfeiffer.[1] Later Cremer[2] has constructed such examples for arbitrary $a_1$ satisfying the condition

$$\liminf_{n\to\infty} |a_1^n - 1|^{1/n} = 0.$$

These $a_1$ are very closely approximated by certain roots of unity, and their linear Lebesgue measure on the unit circle $|a_1| = 1$ is 0.

Until now, however, it was not known if there exists a number $a_1$ of absolute value 1, such that every convergent power series $f(z) = a_1 z + \cdots$ has a *convergent* Schröder series. Julia[3] tried to prove the erroneous hypothesis that the Schröder series is always divergent, if $f(z) - a_1 z$ is a rational function and not identically 0. We shall demonstrate the following

THEOREM: *Let*

$$\log |a_1^n - 1| = O(\log n) \qquad (n \to \infty); \tag{5}$$

*then the Schröder series is convergent.*

Write $a_1 = e^{2\pi i\omega}$; then the condition (5) may be expressed in the form

$$\left|\omega - \frac{m}{n}\right| > \lambda n^{-\mu},$$

for arbitrary integers $m$ and $n$, $n \geqq 1$, where $\lambda$ and $\mu$ denote positive numbers depending only upon $\omega$. It is easily seen that (5) holds for all points of the unit circle $|a_1| = 1$ with the exception of a set of measure 0.

LEMMA 1: *Let $x_p$ $(p = 1, \cdots, r)$ and $y_q$ $(q = 1, \cdots, s)$ be positive integers, $r \geqq 0, s \geqq 2, r + s = t$,*

$$\sum_{p=1}^{r} x_p + \sum_{q=1}^{s} y_q = k, \qquad \sum_{q=1}^{s} y_q > \frac{k}{2}, \qquad y_q \leqq \frac{k}{2}\ (q = 1, \cdots, s);$$

*then*

$$\prod_{p=1}^{r} x_p \prod_{q=1}^{s} y_q^2 \geqq k^3 8^{1-t}. \tag{6}$$

PROOF: Denote the left-hand side of (6) by $L$ and consider first the case $k < 2t - 2$. Then

$$k^{-3}L \geqq k^{-3} > (2t-2)^{-3}. \tag{7}$$

[1] G. A. Pfeiffer, *On the conformal mapping of curvilinear angles. The functional equation $\varphi[f(x)] = a_1\varphi(x)$*, Trans. Amer. Math. Soc. 18, pp. 185–198 (1917).

[2] H. Cremer, *Über die Häufigkeit der Nichtzentren*, Math. Ann. 115, pp. 573–580 (1938).

[3] G. Julia, *Sur quelques problèmes relatifs à l'itération des fractions rationnelles*, C. R. Acad. Sci. Paris 168, pp. 147–149 (1919).

Assume now $k \geqq 2t - 2$ and let

$$\left[\frac{k}{2}\right] = g, \qquad r + \sum_{q=1}^{s} y_q = \eta.$$

Then

$$t \leqq g + 1 \leqq g + 1 + r \leqq \eta \leqq k, \qquad \sum_{p=1}^{r} x_p = k - \eta + r,$$

whence

$$\prod_{p=1}^{r} x_p \geqq k - \eta + 1, \qquad \prod_{q=1}^{s} y_q \geqq \begin{cases} \eta - t + 1, & \text{if } \eta \leqq g - 1 + t \\ (\eta - g - t + 2)g, & \text{if } \eta \geqq g - 1 + t. \end{cases}$$

In the interval $g + 1 \leqq \eta \leqq g - 1 + t$,

$$(k - \eta + 1)(\eta - t + 1)^2 \geqq \min \{(k - g)(g - t + 2)^2, (k - g - t + 2)g^2\};$$

in the interval $g - 1 + t \leqq \eta \leqq k$,

$$(k - \eta + 1)(\eta - g - t + 2)^2 g^2 \geqq (k - g - t + 2)g^2;$$

in the interval $0 \leqq \xi \leqq g$,

$$(k - g)(g - \xi)^2 - (k - g - \xi)g^2 = \{(k - g)\xi - (2k - 3g)g\}\xi \leqq g(2g - k)\xi \leqq 0;$$

consequently

$$L \geqq (k - g)(g - t + 2)^2$$

$$k^{-3} L \geqq \frac{k - g}{k}\left(\frac{g - t + 2}{k}\right)^2 \geqq \tfrac{1}{2}(2t - 2)^{-2} \geqq (2t - 2)^{-3}. \tag{8}$$

Now

$$t - 1 \leqq 2^{t-2} \qquad (t = 2, 3, \cdots),$$

and the lemma follows from (7) and (8).

We use the abbreviation

$$\epsilon_n = |\, a_1^n - 1 \,|^{-1} \qquad (n = 1, 2, \cdots).$$

On account of (5), the inequalities

$$\epsilon_n < (2n)^\nu \qquad (n = 1, 2, \cdots)$$

are fulfilled for a certain constant positive value $\nu$. We define

$$N_1 = 2^{2\nu+1}, \qquad N_2 = 8^\nu N_1 = 2^{5\nu+1}.$$

Lemma 2: *Let $m_l$ $(l = 0, \cdots, r)$ be integral, $r \geqq 0$ and $m_0 > m_1 > \cdots > m_r > 0$; then*

$$\prod_{l=0}^{r} \epsilon_{m_l} < N_1^{r+1} \left\{ m_0 \prod_{l=1}^{r} (m_{l-1} - m_l) \right\}^\nu. \tag{9}$$

PROOF: The assertion is true in the case $r = 0$; assume $r > 0$ and apply induction.

We have the identity

$$a_1^q(a_1^{p-q} - 1) = (a_1^p - 1) - (a_1^q - 1) \qquad (0 < q < p),$$

whence

$$\epsilon_{p-q}^{-1} \leqq \epsilon_p^{-1} + \epsilon_q^{-1}$$

$$\min(\epsilon_p, \epsilon_q) \leqq 2\epsilon_{p-q} < 2^{\nu+1}(p-q)^\nu.$$

This simple remark is the main argument of the whole proof.

Let $\epsilon_{m_l}$ $(l = 0, \cdots, r)$ have its minimum value for $l = h$. Then

$$\epsilon_{m_h} < 2^{\nu+1} \min\{(m_{h-1} - m_h)^\nu, (m_h - m_{h+1})^\nu\}, \tag{10}$$

if we define moreover $m_{-1} = \infty$ and $m_{r+1} = -\infty$. On the other hand, the lemma being true for $r - 1$ instead of $r$, we have

$$\epsilon_{m_h}^{-1} \prod_{l=0}^{r} \epsilon_{m_l} < N_1^r \left\{ \frac{m_0(m_{h-1} - m_{h+1})}{(m_{h-1} - m_h)(m_h - m_{h+1})} \prod_{l=1}^{r} (m_{l-1} - m_l) \right\}^\nu. \tag{11}$$

Since

$$\frac{m_{h-1} - m_{h+1}}{(m_{h-1} - m_h)(m_h - m_{h+1})} = \frac{1}{m_{h-1} - m_h} + \frac{1}{m_h - m_{h+1}} \leqq \frac{2}{\min(m_{h-1} - m_h, m_h - m_{h+1})},$$

the inequality (9) follows from (10) and (11).

Consider now the sequence of positive numbers $\delta_1 = 1, \delta_2, \delta_3, \cdots$ recurrently defined in the following way: For every $k > 1$, let $\mu_k$ denote the maximum of all products $\delta_{l_1} \delta_{l_2} \cdots \delta_{l_r}$ with $l_1 + l_2 + \cdots + l_r = k > l_1 \geqq l_2 \geqq \cdots \geqq l_r \geqq 1$, $2 \leqq r \leqq k$, and put

$$\delta_k = \epsilon_{k-1}\mu_k. \tag{12}$$

LEMMA 3:

$$\delta_k \leqq k^{-2\nu} N_2^{k-1} \qquad (k = 1, 2, \cdots). \tag{13}$$

PROOF: The assertion is true in the case $k = 1$; assume $k > 1$ and apply induction.

The numbers $\alpha_k = k^{-2\nu} N_2^{k-1}$ satisfy the inequalities

$$\frac{\alpha_k \alpha_l}{\alpha_{k+l}} = (k^{-1} + l^{-1})^{2\nu} N_2^{-1} \leqq 2^{2\nu} N_2^{-1} < 1 \qquad (k \geqq 1, l \geqq 1),$$

and consequently

$$\delta_{j_1}\delta_{j_2} \cdots \delta_{j_f} \leqq j^{-2\nu} N_2^{j-1} \qquad (1 \leqq j_1 + \cdots + j_f = j < k; f \geqq 1). \tag{14}$$

By (12), there exists a decomposition

$$\delta_k = \epsilon_{k-1}\delta_{g_1}\delta_{g_2} \cdots \delta_{g_\alpha} \qquad (g_1 + \cdots + g_\alpha = k > g_1 \geqq \cdots \geqq g_\alpha \geqq 1).$$

In the case $g_1 > k/2$, we use this formula with $g_1$ instead of $k$ and find a decomposition

$$\delta_{g_1} = \epsilon_{g_1-1}\delta_{h_1}\delta_{h_2}\cdots\delta_{h_\beta} \qquad (h_1 + \cdots + h_\beta = g_1 > h_1 \geqq \cdots \geqq h_\beta \geqq 1);$$

if also $h_1 > k/2$, we decompose again

$$\delta_{h_1} = \epsilon_{h_1-1}\delta_{i_1}\delta_{i_2}\cdots\delta_{i_\gamma} \qquad (i_1 + \cdots + i_\gamma = h_1 > i_1 \geqq \cdots \geqq i_\gamma \geqq 1),$$

and so on. Writing $k_0 = k$, $k_1 = g_1$, $k_2 = h_1, \cdots$, we obtain in this manner the formula

$$\delta_k = \prod_{p=0}^{r} (\epsilon_{k_p-1}\Delta_p)$$

with $k = k_0 > k_1 > \cdots > k_r > k/2$, where $\Delta_p$ denotes for $p = 0, \cdots, r$ a certain product $\delta_{j_1}\cdots\delta_{j_f}$ and

$$j_1 + \cdots + j_f = \begin{cases} k_p - k_{p+1} & (p = 0, \cdots, r-1) \\ k_r & (p = r), \end{cases}$$

all subscripts $j_1, \cdots, j_f$ being $\leqq k/2$. The number $f$ depends upon $p$; let $f = s$ for $p = r$.

Using (13) for the $s$ single factors of $\Delta_r$ and applying (14) for the estimation of $\Delta_p$ $(p = 0, \cdots, r-1)$, we find the inequality

$$\prod_{p=0}^{r} \Delta_p \leqq N_2^{k-r-s}\left\{\prod_{q=1}^{s} j_q \prod_{p=1}^{r} (k_{p-1} - k_p)\right\}^{-2\nu},$$

where $1 \leqq j_q \leqq k/2$ $(q = 1, \cdots, s)$ and $j_1 + \cdots + j_s = k_r$. By Lemma 2,

$$\prod_{p=0}^{r} \epsilon_{k_p-1} < N_1^{r+1}\left\{k \prod_{p=1}^{r} (k_{p-1} - k_p)\right\}^{\nu},$$

and consequently

$$\delta_k < N_1^{r-1} N_2^{k-t}\left(k^{-1}\prod_{p=1}^{r} x_p \prod_{q=1}^{s} y_q^2\right)^{-\nu}$$

with $t = r + s$, $x_p = k_{p-1} - k_p$, $y_q = j_q$. By Lemma 1,

$$N_2^{1-k} k^{2\nu}\delta_k < N_1^{r+1} N_2^{1-t} 8^{\nu(t-1)} \leqq \left(\frac{8^\nu N_1}{N_2}\right)^{t-1} = 1,$$

and (13) is proved.

Proof of the Theorem: Since the power series (1) has a positive radius of convergence, there exists a positive number $a$, such that $|a_n| \leqq a^{n-1}$ $(n = 2, 3, \cdots)$. The functional equation (3) remains true under the transformation $f(z) \to af(z/a)$, $\varphi(\zeta) \to a\varphi(\zeta/a)$; hence we may assume $|a_n| \leqq 1$ $(n = 2, 3, \cdots)$.

Instead of (4), we consider the functional equation

$$\sum_{k=2}^{\infty} \eta_k\gamma_k\zeta^k = \sum_{l=2}^{\infty}\left(\zeta + \sum_{r=2}^{\infty}\gamma_r\zeta^r\right)^l, \tag{15}$$

where $\eta_2, \eta_3, \cdots$ are positive parameters. Then the coefficients $\gamma_1 = 1, \gamma_2, \gamma_3, \cdots$ are uniquely determined by the formula

$$\gamma_k = \eta_k^{-1} \sum \gamma_{l_1}\gamma_{l_2} \cdots \gamma_{l_r} \qquad (k = 2, 3, \cdots), \tag{16}$$

where $l_1, \cdots, l_r$ run over all positive integral solutions of $l_1 + \cdots + l_r = k$ $(r = 2, \cdots, k)$. Write $\gamma_k = \sigma_k$ in the case $\eta_k = \epsilon_{k-1}^{-1}$ $(k = 2, 3, \cdots)$, and $\gamma_k = \tau_k$ in the case $\eta_k = 1$.

The inequality

$$\sigma_k \leqq \delta_k \tau_k \tag{17}$$

is true for $k = 1$. Applying induction, we infer from (12) and (16) that

$$\sigma_k \leqq \epsilon_{k-1}\mu_k \sum \tau_{l_1}\tau_{l_2} \cdots \tau_{l_r} = \delta_k \tau_k \,;$$

hence (17) holds for all values of $k$.

On the other hand, the power series

$$\psi = \sum_{k=1}^{\infty} \tau_k \zeta^k$$

satisfies the equation

$$\psi - \zeta = (1 - \psi)^{-1}\psi^2,$$

whence

$$4\psi = 1 + \zeta - (1 - 6\zeta + \zeta^2)^{\frac{1}{2}};$$

consequently $\psi$ converges in the circle $|\zeta| < 3 - 2\sqrt{2}$.

By (4), (15) and (17),

$$|c_k| \leqq \delta_k \tau_k \qquad (k = 2, 3, \cdots).$$

It follows now from Lemma 3, that the Schröder series $\varphi(\zeta)$ converges in the circle $|\zeta| < (3 - 2\sqrt{2})2^{-5\nu-1}$.

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