Folland Real Analysis (1984).

phenomenon and is much to be prized when it is available. Almost all compactness results in such situations are obtained via two basic theorems, Tychonoff's theorem and the Ascoli–Arzelà theorem, which we present in this section. Our proof of Tychonoff's theorem is based on the following result, which is of independent interest.

(4.42) Alexander's Lemma. Let X be a topological space whose topology is generated by a family $\mathscr{E}$ of sets. If every cover of X by members of $\mathscr{E}$ has a finite subcover, then X is compact. $\mathfrak{A}$ is non empty since X is not compact.

Proof. Suppose X is not compact, and let $\mathfrak{A}$ be the collection of all open covers of X with no finite subcovers. $\mathfrak{A}$ is partially ordered by inclusion. If $\{\mathscr{A}_\beta\}_{\beta\in B}$ is a linearly ordered subcollection of $\mathfrak{A}$ and $U_1, \dots, U_n \in \bigcup_{\beta\in B}\mathscr{A}_\beta$, then $U_1, \dots, U_n \in \mathscr{A}_{\beta_0}$ for some $\beta_0 \in B$, so $\bigcup_1^n U_j \neq X$. It follows that $\bigcup_{\beta\in B}\mathscr{A}_\beta \in \mathfrak{A}$, so by Zorn's lemma $\mathfrak{A}$ has a maximal element $\mathscr{A}$. $\mathscr{A}$ is thus an open cover of X with no finite subcover, and if U is open and $U \notin \mathscr{A}$, $\mathscr{A} \cup \{U\}$ has a finite subcover. Let $\mathscr{B} = \mathscr{A} \cap \mathscr{E}$. We claim that $\mathscr{B}$ covers X; since no finite subfamily of $\mathscr{B}$ covers X, this contradicts our hypothesis.

Suppose to the contrary that there is a point $x \in X \setminus \bigcup\mathscr{B}$. (Here, $\bigcup\mathscr{B}$ is short for $\bigcup_{B\in\mathscr{B}} B$.) Choose $U \in \mathscr{A}$ with $x \in U$. Since $\mathscr{E}$ generates the topology, there exist $V_1, \dots, V_n \in \mathscr{E}$ with $x \in \bigcap_1^n V_j \subset U$. None of the V_j's is in $\mathscr{A}$ since $x \notin \bigcup\mathscr{B}$, so by maximality of $\mathscr{A}$, for each j there is a set W_j which is a finite union of sets in $\mathscr{A}$ such that $V_j \cup W_j = X$. But then

$$U \cup \left(\bigcup_1^n W_j\right) \supset \left(\bigcap_1^n V_j\right) \cup \left(\bigcup_1^n W_j\right) = X,$$

so that $\mathscr{A}$ has a finite subcover, contrary to assumption. □

(4.43) Tychonoff's Theorem. If $\{X_\alpha\}_{\alpha\in A}$ is any family of compact topological spaces, then $X = \prod_{\alpha\in A} X_\alpha$ (with the product topology) is compact.

Proof. The product topology is generated by the sets $\pi_\alpha^{-1}(U)$ where $\pi_\alpha\colon X \to X_\alpha$ is the projection, U is open in X_α, and $\alpha \in A$. By Alexander's lemma it will suffice to prove that every cover $\mathscr{V}$ of X by such sets has a finite subcover. For each $\alpha \in A$ let $\mathscr{V}_\alpha$ be the collection of all open $U \subset X_\alpha$ such that $\pi_\alpha^{-1}(U) \in \mathscr{V}$. There must exist some $\beta \in A$ such that $\mathscr{V}_\beta$ covers X_β; otherwise there would be a point $x \in X$ such that $\pi_\alpha(x) \notin \bigcup\mathscr{V}_\alpha$ for every α, which would mean that $x \notin \bigcup\mathscr{V}$. Since X_β is compact, there exist $U_1, \dots, U_n \in \mathscr{V}_\beta$ such that $\bigcup_1^n U_j = X_\beta$. But then $\pi_\beta^{-1}(U_j) \in \mathscr{V}$ and $\bigcup_1^n \pi_\beta^{-1}(U_j) = \pi_\beta^{-1}(X_\beta) = X$, so we are done. □

We now turn to the Ascoli–Arzelà theorem, which has to do with compactness in spaces of continuous mappings. There are several variants of this result, of which the theorem below contains two of the most useful; see also Exercise 61.