(b) For α with $|\alpha| = 3$ choose a homogeneous harmonic polynomial Q of degree 3 with $D^\alpha Q \neq 0$. With η, t_k and c_k as in part (a), define

$$u(x) = \sum_0^\infty c_k \eta(t_k x) Q(x) = \sum_0^\infty c_k (\eta Q)(t_k x)/t_k^3.$$

Then

$$\Delta u = g(x) = \sum_0^\infty c_k \Delta(\eta Q)(t_k x)/t_k.$$

Show that $g \in C^1$ but that $u \notin C^{2,1}$ in any neighbourhood of the origin. Hence Lemma 4.4 is not valid for $\alpha = 1$.

4.10. Let $u \in C_0^2(B)$ satisfy $\Delta u = f$ in $B = B_R(x_0)$. Show that

$$\text{(a) } |u|_0 \leqslant \frac{R^2}{2n} |f|_0; \qquad \text{(b) } |D_i u|_0 \leqslant R|f|_0, \quad i = 1, \dots, n.$$

Hence in (4.14), $|u|'_{1;B} \leqslant 3R^2 |f|_{0;B}$.

Chapter 5

Banach and Hilbert Spaces

This chapter supplies the functional analytic material required for our study of existence of solutions of linear elliptic equations in Chapters 6 and 8. This material will be familiar to a reader already versed in basic functional analysis but we shall assume some acquaintance with elementary linear algebra and the theory of metric spaces. Unless otherwise indicated, all linear spaces used in this book are assumed to be defined over the real number field. The theory of this chapter, however, carries over almost unchanged if the real numbers are replaced by the complex numbers. Let $\mathscr{V}$ be a linear space over $\mathbb{R}$. A *norm* on $\mathscr{V}$ is a mapping $p: \mathscr{V} \to \mathbb{R}$ (henceforth we write $p(x) = \|x\| = \|x\|_{\mathscr{V}}$, $x \in \mathscr{V}$) satisfying

(i) $\|x\| \geqslant 0$ for all $x \in \mathscr{V}$, $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|\alpha x\| = |\alpha|\ \|x\|$ for all $\alpha \in \mathbb{R}$, $x \in \mathscr{V}$;
(iii) $\|x + y\| \leqslant \|x\| + \|y\|$ for all $x, y \in \mathscr{V}$ (triangle inequality).

A linear space $\mathscr{V}$ equipped with a norm is called a *normed linear space*. A normed linear space $\mathscr{V}$ is a metric space under the metric ρ defined by

$$\rho(x, y) = \|x - y\|, \quad x, y \in \mathscr{V}.$$

Consequently a sequence $\{x_n\} \subset \mathscr{V}$ converges to an element $x \in \mathscr{V}$ if $\|x_n - x\| \to 0$. Also $\{x_n\}$ is a Cauchy sequence if $\|x_n - x_m\| \to 0$ as $m, n \to \infty$. If $\mathscr{V}$ is complete, that is every Cauchy sequence converges, then $\mathscr{V}$ is called a *Banach space*.

Examples. (i) Euclidean space $\mathbb{R}^n$ is a Banach space under the standard norm:

$$\|x\| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}, \quad x = (x_1, \dots, x_n).$$

(ii) For a bounded domain $\Omega \subset \mathbb{R}^n$, the *Hölder spaces* $C^{k,\alpha}(\bar{\Omega})$ are Banach spaces under either of the equivalent norms (4.6) or (4.6)′ introduced in Chapter 4; (see Problems 5.1, 5.2).

(iii) The Sobolev spaces $W^{k,p}(\Omega)$, $W_0^{k,p}(\Omega)$ (see Chapter 7).

Existence theorems in partial differential equations are often reducible to the solvability of equations in appropriate function spaces. For the Schauder theory of linear elliptic equations we will employ two basic existence theorems for operator equations in Banach spaces, namely the Contraction Mapping Principle and the Fredholm alternative.

5.1. The Contraction Mapping Principle

A mapping T from a normed linear space $\mathscr{V}$ into itself is called a *contraction* mapping if there exists a number $\theta < 1$ such that

$$\|Tx - Ty\| \leqslant \theta \|x - y\| \quad \text{for all } x, y \in \mathscr{V}. \tag{5.1}$$

Theorem 5.1. *A contraction mapping T in a Banach space $\mathscr{B}$ has a unique fixed point, that is there exists a unique solution $x \in \mathscr{B}$ of the equation $Tx = x$.*

Proof. (*Method of successive approximations.*) Let $x_0 \in \mathscr{B}$ and define a sequence $\{x_n\} \subset \mathscr{B}$ by $x_n = T^n x_0$, $n = 1, 2, \ldots$ Then if $n \geqslant m$, we have

$$\begin{aligned}
\|x_n - x_m\| &\leqslant \sum_{j=m+1}^{n} \|x_j - x_{j-1}\| \quad \text{by the triangle inequality} \\
&= \sum_{j=m+1}^{n} \|T^{j-1}x_1 - T^{j-1}x_0\| \\
&\leqslant \sum_{j=m+1}^{n} \theta^{j-1}\|x_1 - x_0\| \quad \text{by (5.1)} \\
&\leqslant \frac{\|x_1 - x_0\|\theta^m}{1-\theta} \to 0 \quad \text{as } m \to \infty.
\end{aligned}$$

Consequently $\{x_n\}$ is a Cauchy sequence and, since $\mathscr{B}$ is complete, converges to an element $x \in \mathscr{B}$. Clearly T is also a continuous mapping and hence we have

$$Tx = \lim Tx_n = \lim x_{n+1} = x$$

so that x is a fixed point of T. The uniqueness of x follows immediately from (5.1). □

In the statement of Theorem 5.1, the space $\mathscr{B}$ can obviously be replaced by any closed subset.

5.2. The Method of Continuity

Let $\mathscr{V}_1$ and $\mathscr{V}_2$ be normed linear spaces. A linear mapping $T: \mathscr{V}_1 \to \mathscr{V}_2$ is *bounded* if the quantity

$$\|T\| = \sup_{x \in \mathscr{V}_1, x \neq 0} \frac{\|Tx\|_{\mathscr{V}_2}}{\|x\|_{\mathscr{V}_1}} \tag{5.2}$$

is finite. It is easy to show that a linear mapping T is bounded if and only if it is continuous. The invertibility of a bounded linear mapping may sometimes be deduced from the invertibility of a similar mapping through the following theorem, which is known in applications as the *method of continuity*.

Theorem 5.2. *Let $\mathscr{B}$ be a Banach space, $\mathscr{V}$ a normed linear space and let L_0, L_1 be bounded linear operators from $\mathscr{B}$ into $\mathscr{V}$. For each $t \in [0, 1]$, set*

$$L_t = (1-t)L_0 + tL_1$$

and suppose that there is a constant C such that

$$\|x\|_{\mathscr{B}} \leqslant C\|L_t x\|_{\mathscr{V}} \tag{5.3}$$

for $t \in [0, 1]$. Then L_1 maps $\mathscr{B}$ onto $\mathscr{V}$ if and only if L_0 maps $\mathscr{B}$ onto $\mathscr{V}$.

Proof. Suppose that L_s is onto for some $s \in [0, 1]$. By (5.3), L_s is one-to-one and hence the inverse mapping $L_s^{-1}: \mathscr{V} \to \mathscr{B}$ exists. For $t \in [0, 1]$ and $y \in \mathscr{V}$, the equation $L_t x = y$ is equivalent to the equation

$$\begin{aligned}
L_s(x) &= y + (L_s - L_t)x \\
&= y + (t-s)L_0 x - (t-s)L_1 x
\end{aligned}$$

which in turn, is equivalent to the equation

$$x = L_s^{-1}y + (t-s)L_s^{-1}(L_0 - L_1)x$$

The mapping T from $\mathscr{B}$ into itself given by $Tx = L_s^{-1}y + (t-s)L_s^{-1}(L_0 - L_1)x$ is clearly a contraction mapping if

$$|s-t| < \delta = [C(\|L_0\| + \|L_1\|)]^{-1}$$

and hence the mapping L_t is onto for all $t \in [0, 1]$, satisfying $|s-t| < \delta$. By dividing the interval $[0, 1]$ into subintervals of length less than δ, we see that the mapping L_t is onto for all $t \in [0, 1]$ provided it is onto for any fixed $t \in [0, 1]$, in particular for $t = 0$ or $t = 1$. □

5.3. The Fredholm Alternative

Let $\mathscr{V}_1$ and $\mathscr{V}_2$ be normed linear spaces. A mapping $T: \mathscr{V}_1 \to \mathscr{V}_2$ is called *compact* (or *completely continuous*) if T maps bounded sets in $\mathscr{V}_1$ into relatively compact sets in $\mathscr{V}_2$ or equivalently T maps bounded sequences in $\mathscr{V}_1$ into sequences in $\mathscr{V}_2$ which contain convergent subsequences. It follows that a compact linear mapping is also continuous but the converse is not true in general unless $\mathscr{V}_2$ is finite dimensional.

The Fredholm alternative (or Riesz-Schauder theory) concerns compact linear operators from a space $\mathscr{V}$ into itself and is an extension of the theory of linear mappings in finite dimensional spaces.

Theorem 5.3. *Let T be a compact linear mapping of a normed linear space $\mathscr{V}$ into itself. Then either* (*i*) *the homogeneous equation*

$$x - Tx = 0$$

has a nontrivial solution $x \in \mathscr{V}$ or (ii) *for each $y \in \mathscr{V}$ the equation*

$$x - Tx = y$$

has a uniquely determined solution $x \in \mathscr{V}$. Furthermore, in case (ii), *the operator $(I-T)^{-1}$ whose existence is asserted there is also bounded.*

The proof of Theorem 5.3 depends upon the following simple result of Riesz.

Lemma 5.4. *Let $\mathscr{V}$ be a normed linear space and $\mathscr{M}$ a proper closed subspace of $\mathscr{V}$. Then for any $\theta < 1$, there exists an element $x_\theta \in \mathscr{V}$ satisfying $\|x_\theta\| = 1$ and* $\operatorname{dist}(x_\theta, \mathscr{M}) \geq \theta$.

Proof. Let $x \in \mathscr{V} - \mathscr{M}$. Since $\mathscr{M}$ is closed, we have

$$\operatorname{dist}(x, \mathscr{M}) = \inf_{y \in M} \|x - y\| = d > 0.$$

Consequently there exists an element $y_\theta \in \mathscr{M}$ such that

$$\|x - y_\theta\| \leq \frac{d}{\theta},$$

so that, defining

$$x_\theta = \frac{x - y_\theta}{\|x - y_\theta\|},$$

we have $\|x_\theta\| = 1$ and for any $y \in \mathscr{M}$,

$$\|x_\theta - y\| = \frac{\|x - y_\theta - \|y_\theta - x\|\, y\|}{\|y_\theta - x\|} \geq \frac{d}{\|y_\theta - x\|} \geq \theta.$$

The lemma is thus proved. $\square$

If $\mathscr{V} = \mathbb{R}^n$, it is clear that one can take $\theta = 1$ by choosing x_θ orthogonal to $\mathscr{M}$. This will also be possible in any Hilbert space but in general Lemma 5.4, which asserts the existence of a "nearly orthogonal" element to $\mathscr{M}$, cannot be improved to allow $\theta = 1$.

Proof of Theorem 5.3. It is convenient to split our proof into four stages.

(1) *Let $S = I - T$ where I is the identity mapping and let $\mathscr{N} = S^{-1}(0) = \{x \in \mathscr{V} \mid Sx = 0\}$ be the null space of S. Then there exists a constant K such that*

$$\text{(5.4)} \qquad \operatorname{dist}(x, \mathscr{N}) \leq K\|Sx\| \quad \textit{for all } x \in \mathscr{V}.$$

Proof. Suppose the result is not true. Then there exists a sequence $\{x_n\} \subset \mathscr{V}$ satisfying $\|Sx_n\| = 1$ and $d_n = \operatorname{dist}(x_n, \mathscr{N}) \to \infty$. Choose a sequence $\{y_n\} \subset \mathscr{N}$ such that $d_n \leq \|x_n - y_n\| \leq 2d_n$. Then if

$$z_n = \frac{x_n - y_n}{\|x_n - y_n\|}$$

we have $\|z_n\| = 1$, and $\|Sz_n\| \leq d_n^{-1} \to 0$ so that the sequence $\{Sz_n\}$ converges to 0. But since T is compact, by passing to a subsequence if necessary, we may assume that the sequence $\{Tz_n\}$ converges to an element $y_0 \in \mathscr{V}$. Since $z_n = (S + T)z_n$, we then also have $\{z_n\}$ converging to y_0 and consequently $y_0 \in \mathscr{N}$. However this leads to a contradiction as

$$\begin{aligned} \operatorname{dist}(z_n, \mathscr{N}) &= \inf_{y \in \mathscr{N}} \|z_n - y\| \\ &= \|x_n - y_n\|^{-1} \inf_{y \in \mathscr{N}} \|x_n - y_n - \|x_n - y_n\|\, y\| \\ &= \|x_n - y_n\|^{-1} \operatorname{dist}(x_n, \mathscr{N}) \geq \tfrac{1}{2}. \quad \square \end{aligned}$$

(2) *Let $\mathscr{R} = S(\mathscr{V})$ be the range of S. Then $\mathscr{R}$ is a closed subspace of $\mathscr{V}$.*

Proof. Let $\{x_n\}$ be a sequence in $\mathscr{V}$ whose image $\{Sx_n\}$ converges to an element $y \in \mathscr{V}$. To show that $\mathscr{R}$ is closed we must show that $y = Sx$ for some element $x \in \mathscr{V}$. By our previous result the sequence $\{d_n\}$ where $d_n = \operatorname{dist}(x_n, \mathscr{N})$ is bounded. Choosing $y_n \in \mathscr{N}$ as before and writing $w_n = x_n - y_n$, we consequently have that the sequence $\{w_n\}$ is bounded while the sequence $\{Sw_n\}$ converges to y. Since T is compact, by passing to a subsequence if necessary we may assume that the sequence $\{Tw_n\}$ converges to an element $w_0 \in \mathscr{V}$. Hence the sequence $\{w_n\}$ itself converges to $y + w_0$ and by the continuity of S, we have $S(y + w_0) = y$. Consequently $\mathscr{R}$ is closed. $\square$

(3) *If $\mathscr{N} = \{0\}$, then $\mathscr{R} = \mathscr{V}$. That is, if case* (i) *of Theorem* 5.3 *does not hold, then case* (ii) *is true.*

Proof. By our previous result the sets $\mathscr{R}_j$ defined by $\mathscr{R}_j = S^j(\mathscr{V})$, $j = 1, 2, \ldots$ form a non-increasing sequence of closed subspaces of $\mathscr{V}$. Suppose that no two of these spaces coincide. Then each is a proper subspace of its predecessor. Hence by Lemma 5.4, there exists a sequence $\{y_n\} \subset \mathscr{V}$ such that $y_n \in \mathscr{R}_n$, $\|y_n\| = 1$ and dist $(y_n, \mathscr{R}_{n+1}) \geq \frac{1}{2}$. Thus if $n > m$,

$$\begin{aligned} Ty_m - Ty_n &= y_m + (-y_n - Sy_m + Sy_n) \\ &= y_m - y \quad \text{for some } y \in \mathscr{R}_{m+1}. \end{aligned}$$

Hence $\|Ty_m - Ty_n\| \geq \frac{1}{2}$ contrary to the compactness of T. Consequently there exists an integer k such that $\mathscr{R}_j = \mathscr{R}_k$ for all $j \geq k$. Up to this point we have not used the condition: $\mathscr{N} = \{0\}$. Now let y be an arbitrary element of $\mathscr{V}$. Then $S^k y \in \mathscr{R}_k = \mathscr{R}_{k+1}$ and so $S^k y = S^{k+1} x$ for some $x \in \mathscr{V}$. Therefore $S^k(y - Sx) = 0$ and so $y = Sx$ since $S^{-k}(0) = S^{-1}(0) = 0$. Consequently $\mathscr{R} = \mathscr{R}_j = \mathscr{V}$ for all j. □

(4) *If* $\mathscr{R} = \mathscr{V}$, *then* $\mathscr{N} = \{0\}$. *Consequently either case* (i) *or case* (ii) *holds.*

Proof. This time we define a non-decreasing sequence of closed subspaces $\{\mathscr{N}_j\}$ by setting $\mathscr{N}_j = S^{-j}(0)$. The closure of $\mathscr{N}_j$ follows from the continuity of S. By employing an analogous argument based on Lemma 5.4 to that used in step (3), we obtain that $\mathscr{N}_j = \mathscr{N}_l$ for all $j \geq$ some integer l. Then if $\mathscr{R} = \mathscr{V}$, any element $y \in \mathscr{N}_l$ satisfies $y = S^l x$ for some $x \in \mathscr{V}$. Consequently $S^{2l} x = 0$ so that $x \in \mathscr{N}_{2l} = \mathscr{N}_l$ whence $y = S^l x = 0$. Step (4) is thus proved. □

The boundedness of the operator $S^{-1} = (I - T)^{-1}$ in case (ii) follows from step (1) with $\mathscr{N} = \{0\}$. Note that a slight simplification could be achieved by taking $\mathscr{N} = \{0\}$ at the outset in steps (1) and (2) and that step (4) is independent of the previous steps. Theorem 5.3 is thus completely proved. □

Certain aspects of the spectral behaviour of compact linear operators follow from Theorem 5.3 and Lemma 5.4. A number λ is called an *eigenvalue* of T if there exists a non-zero element x in $\mathscr{V}$ (called an *eigenvector*) satisfying $Tx = \lambda x$. It is clear that eigenvectors belonging to different eigenvalues must be linearly independent. Also the dimension of the null space of the operator $S_\lambda = \lambda I - T$ is called the *multiplicity* of λ. If $\lambda \neq 0$, $\in \mathbb{R}$ is not an eigenvalue of T, it follows from Theorem 5.3 that the *resolvent* operator $R_\lambda = (\lambda I - T)^{-1}$ is a well defined, bounded linear mapping of $\mathscr{V}$ onto itself. From Lemma 5.4 we may deduce the following result.

Theorem 5.5. *A compact linear mapping T of a normed linear space into itself possesses a countable set of eigenvalues having no limit points except possibly $\lambda = 0$. Each non-zero eigenvalue has finite multiplicity.*

Proof. Suppose that there exists a sequence $\{\lambda_n\}$ of not necessarily distinct eigenvalues and a sequence of corresponding linearly independent eigenvectors $\{x_n\}$ satisfying $\lambda_n \to \lambda \neq 0$. Let $\mathscr{M}_n$ be the closed subspace spanned by $\{x_1, \ldots x_n\}$.

By Lemma 5.4, there exists a sequence $\{y_n\}$ such that $y_n \in \mathscr{M}_n$, $\|y_n\| = 1$ and dist $(y_n, \mathscr{M}_{n-1}) \geq \frac{1}{2}$, $(n = 2, 3 \ldots)$. If $n > m$, we have

$$\begin{aligned} \lambda_n^{-1} Ty_n - \lambda_m^{-1} Ty_m &= y_n + (-y_m - \lambda_n^{-1} S_{\lambda_n} y_n + \lambda_m^{-1} S_{\lambda_m} y_m) \\ &= y_n - z \quad \text{where } z \in \mathscr{M}_{n-1}. \end{aligned}$$

For, if $y_n = \sum_{j=1}^{n} \beta_j x_j$ then $y_n - \lambda_n^{-1} Ty_n = \sum_{j=1}^{n} \beta_j (1 - \lambda_n^{-1} \lambda_j) x_j \in \mathscr{M}_{n-1}$ and similarly $S_{\lambda_m} y_m \in \mathscr{M}_m$. Therefore we have

$$\|\lambda_n^{-1} Ty_n - \lambda_m^{-1} Ty_m\| \geq \tfrac{1}{2}$$

which contradicts the compactness of T combined with the hypothesis $\lambda_n \to \lambda \neq 0$. Hence our initial supposition is false and this implies the validity of the theorem. □

5.4. Dual Spaces and Adjoints

For the sake of completeness we mention a few results here that will be proved and applied in this book only in Hilbert spaces. Let $\mathscr{V}$ be a normed linear space. A functional on $\mathscr{V}$ is a mapping from $\mathscr{V}$ into $\mathbb{R}$. The space of all bounded linear functionals on $\mathscr{V}$ is called the *dual space* of $\mathscr{V}$ and is denoted by $\mathscr{V}^*$. It can be shown easily that $\mathscr{V}^*$ is a Banach space under the norm:

$$(5.5) \qquad \|f\|_{\mathscr{V}^*} = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|}$$

Example. The dual space of $\mathbb{R}^n$ is isomorphic to $\mathbb{R}^n$ itself.

The dual space of $\mathscr{V}^*$, denoted $\mathscr{V}^{**}$, is called the second dual of $\mathscr{V}$. Clearly the mapping $J: \mathscr{V} \to \mathscr{V}^{**}$ given by $Jx(f) = f(x)$ for $f \in \mathscr{V}^*$ is a norm preserving, linear, one-to-one mapping of $\mathscr{V}$ into $\mathscr{V}^{**}$. If $J\mathscr{V} = \mathscr{V}^{**}$, then we call $\mathscr{V}$ *reflexive*. Reflexive Banach spaces have certain properties that make them more amenable to applications to differential equations than Banach spaces in general. The Sobolev spaces $W^{k,p}(\Omega)$ introduced in Chapter 7 are reflexive for $p > 1$ but the Hölder spaces $C^{k,\alpha}(\bar{\Omega})$ of Chapter 4 are nonreflexive.

Let T be a bounded linear mapping between two Banach spaces $\mathscr{B}_1$ and $\mathscr{B}_2$. The *adjoint* of T, denoted T^*, is a bounded linear mapping between $\mathscr{B}_2^*$ and $\mathscr{B}_1^*$ defined by

$$(5.6) \qquad (T^* g)(x) = g(Tx) \quad \text{for } g \in \mathscr{B}_2^*,\ x \in \mathscr{B}_1.$$

Letting $\mathcal{N}$, $\mathcal{R}$, $\mathcal{N}^*$, $\mathcal{R}^*$ denote the null spaces and ranges of T, T^* respectively, the following relations hold provided $\mathcal{R}$ is closed,

$$\mathcal{R} = \mathcal{N}^{*\perp} = \{ y \in \mathcal{B}_2 | g(y) = 0 \quad \text{for all } g \in \mathcal{N}^* \},$$

$$\mathcal{R}^* = \mathcal{N}^{\perp} = \{ f \in \mathcal{B}_1^* | f(x) = 0 \quad \text{for all } x \in \mathcal{N} \}.$$

Also the compactness of T implies the compactness of T^*. These two results are proved for example in [YO]. Consequently we see that if case (i) of the Fredholm alternative holds for a Banach space $\mathcal{B}$, then the equation $x - Tx = y$ is solvable for $x \in \mathcal{B}$ if and only if $g(y) = 0$ for all $g \in \mathcal{B}^*$ satisfying $T^*g = g$. This last result will be established directly in Hilbert spaces.

5.5. Hilbert Spaces

We develop here the Hilbert space theory required for our treatment of linear elliptic operators in Chapter 8. A *scalar* (or *inner*) *product* on a linear space $\mathcal{V}$ is a mapping $q: \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ (henceforth we write $q(x, y) = (x, y)$ or $(x, y)_{\mathcal{V}}$, $x, y \in \mathcal{V}$) satisfying

(i) $(x, y) = (y, x)$ for all $x, y \in \mathcal{V}$,

(ii) $(\lambda_1 x_1 + \lambda_2 x_2, y) = \lambda_1(x_1, y) + \lambda_2(x_2, y)$ for all $\lambda_1, \lambda_2 \in \mathbb{R}$, $x_1, x_2, y \in \mathcal{V}$,

(iii) $(x, x) > 0$ for all $x \neq 0, \in \mathcal{V}$.

A linear space $\mathcal{V}$ equipped with an inner product is called an *inner product space* or a *pre-Hilbert space*. Writing $\|x\| = (x, x)^{1/2}$ for $x \in \mathcal{V}$, we have the following inequalities:

Schwarz inequality

(5.7) $$|(x, y)| \leq \|x\| \, \|y\|;$$

Triangle inequality

(5.8) $$\|x + y\| \leq \|x\| + \|y\|;$$

Parallelogram law

(5.9) $$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

In particular an inner product space $\mathcal{V}$ is a normed linear space. A *Hilbert space* is defined to be a complete inner product space.

Examples. (i) Euclidean space $\mathbb{R}^n$ is a Hilbert space under the inner product

$$(x, y) = \sum x_i y_i, \quad x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_n).$$

(ii) The Sobolev spaces $W^{k,2}(\Omega)$; (see Chapter 7).

5.6. The Projection Theorem

Two elements x and y in an inner product space are called *orthogonal* (or *perpendicular*) if $(x, y) = 0$. Given a subset $\mathcal{M}$ of an inner product space we denote by $\mathcal{M}^{\perp}$ the set of elements orthogonal to every element of $\mathcal{M}$. The following theorem asserts the existence of an orthogonal projection of any element in a Hilbert space onto a closed subspace.

Theorem 5.6. *Let $\mathcal{M}$ be a closed subspace of a Hilbert space $\mathcal{H}$. Then for every $x \in \mathcal{H}$ we have $x = y + z$ where $y \in \mathcal{M}$ and $z \in \mathcal{M}^{\perp}$.*

Proof. If $x \in \mathcal{M}$, we set $y = x$, $z = 0$. Hence we may assume $\mathcal{M} \neq \mathcal{H}$ and $x \notin \mathcal{M}$. Define

$$d = \text{dist}(x, \mathcal{M}) = \inf_{y \in \mathcal{M}} \|x - y\| > 0$$

and let $\{y_n\} \subset \mathcal{M}$ be a *minimizing sequence*, that is $\|x - y_n\| \to d$. Using the parallelogram law we obtain

$$4\|x - \tfrac{1}{2}(y_m + y_n)\|^2 + \|y_m - y_n\|^2 = 2(\|x - y_m\|^2 + \|x - y_n\|^2)$$

so that, since $\frac{1}{2}(y_m + y_n) \in \mathcal{M}$, also we have $\|y_m - y_n\| \to 0$ as $m, n \to \infty$; that is the sequence $\{y_n\}$ converges since $\mathcal{H}$ is complete. Also, since $\mathcal{M}$ is closed, $y = \lim y_n \in \mathcal{M}$ and $\|x - y\| = d$.

Now write $x = y + z$ where $z = x - y$. To complete the proof we must show $z \in \mathcal{M}^{\perp}$. For any $y' \in \mathcal{M}$ and $\alpha \in \mathbb{R}$ we have $y + \alpha y' \in \mathcal{M}$ and so

$$d^2 \leq \|x - y - \alpha y'\|^2 = (z - \alpha y', z - \alpha y')$$
$$= \|z\|^2 - 2\alpha(y', z) + \alpha^2 \|y'^2\|.$$

Therefore, since $\|z\| = d$, we obtain for all $\alpha > 0$

$$|(y', z)| \leq \frac{\alpha}{2} \|y'^2\|$$

so that $(y', z) = 0$ for all $y' \in \mathcal{M}$. Hence $z \in \mathcal{M}^{\perp}$. □

The element y is called the *orthogonal projection* of x on $\mathscr{M}$. Theorem 5.6 also shows that any closed proper subspace of $\mathscr{H}$ is orthogonal to some element of $\mathscr{H}$.

5.7. The Riesz Representation Theorem

The Riesz representation theorem provides an extremely useful characterization of the bounded linear functionals on a Hilbert space as inner products.

Theorem 5.7. *For every bounded linear functional F on a Hilbert space $\mathscr{H}$, there is a uniquely determined element $f \in \mathscr{H}$ such that $F(x)=(x, f)$ for all $x \in \mathscr{H}$ and* $\|F\|=\|f\|$.

Proof. Let $\mathscr{N}=\{x \mid F(x)=0\}$ be the null space of F. If $\mathscr{N}=\mathscr{H}$, the result is proved by taking $f=0$. Otherwise, since $\mathscr{N}$ is a closed subspace of $\mathscr{H}$, there exists by Theorem 5.6 an element $z \neq 0, \in \mathscr{H}$ such that $(x, z)=0$ for all $x \in \mathscr{N}$. Hence $F(z) \neq 0$ and moreover for any $x \in \mathscr{H}$,

$$F\left(x-\frac{F(x)}{F(z)} z\right)=F(x)-\frac{F(x)}{F(z)} F(z)=0$$

so that the element $x-\frac{F(x)}{F(z)} z \in \mathscr{N}$. This means that

$$\left(x-\frac{F(x)}{F(z)} z, z\right)=0,$$

that is, that

$$(x, z)=\frac{F(x)}{F(z)}\|z\|^2$$

and hence $F(x)=(f, x)$ where $f=zF(z)/\|z\|^2$. The uniqueness of f is easily proved and is left to the reader. To show that $\|F\|=\|f\|$, we have first, by the Schwarz inequality,

$$\|F\|=\sup_{x \neq 0} \frac{|(x, f)|}{\|x\|} \leqslant \sup_{x \neq 0} \frac{\|x\|\ \|f\|}{\|x\|}=\|f\|;$$

and secondly,

$$\|f\|^2=(f, f)=F(f) \leqslant\|F\|\ \|f\|,$$

so that $\|f\| \leqslant\|F\|$, and hence $\|F\|=\|f\|$. □

Theorem 5.7 shows that the dual space of a Hilbert space may be identified with the space itself and consequently that Hilbert spaces are reflexive.

5.8. The Lax-Milgram Theorem

The Riesz representation theorem suffices for the treatment of linear elliptic equations that are variational, that is, they are the Euler-Lagrange equations of certain multiple integrals. For general divergence structure equations we will require a slight extension of Theorem 5.7 due to Lax and Milgram. A bilinear form **B** on a Hilbert space $\mathscr{H}$ is called *bounded* if there exists a constant K such that

$$(5.10) \qquad |\mathbf{B}(x, y)| \leqslant K\|x\|\ \|y\| \quad \text{for all } x, y \in \mathscr{H}$$

and *coercive* if there exists a number $v>0$ such that

$$(5.11) \qquad \mathbf{B}(x, x) \geqslant v\|x\|^2 \quad \text{for all } x \in \mathscr{H}.$$

A particular example of a bounded, coercive bilinear form is the inner product itself.

Theorem 5.8. *Let **B** be a bounded, coercive bilinear form on a Hilbert space $\mathscr{H}$. Then for every bounded linear functional $F \in \mathscr{H}^*$, there exists a unique element $f \in \mathscr{H}$ such that*

$$\mathbf{B}(x, f)=F(x) \quad \textit{for all } x \in \mathscr{H}.$$

Proof. By virtue of Theorem 5.7, there exists a linear mapping $T: \mathscr{H} \rightarrow \mathscr{H}$ defined by $\mathbf{B}(x, f)=(x, Tf)$ for all $x \in \mathscr{H}$. Furthermore $\|Tf\| \leqslant K\|f\|$ by (5.10) so that T is bounded. By (5.11) we obtain $v\|f\|^2 \leqslant \mathbf{B}(f, f)=(f, Tf) \leqslant\|f\|\ \|Tf\|$, so that

$$v\|f\| \leqslant\|Tf\| \leqslant K\|f\| \quad \text{for all } f \in \mathscr{H}.$$

This estimate implies that T is one-to-one, has closed range (see Problem 5.3) and that T^{-1} is bounded. Suppose that T is not onto $\mathscr{H}$. Then there exists an element $z \neq 0$ satisfying $(z, Tf)=0$ for all $f \in \mathscr{H}$. Choosing $f=z$, we obtain $(z, Tz)=\mathbf{B}(z, z)=0$ implying $z=0$ by (5.11). Consequently T^{-1} is a bounded linear mapping on $\mathscr{H}$. We then have $F(x)=(x, g)=\mathbf{B}(x, T^{-1}g)$ for all $x \in \mathscr{H}$ and some unique $g \in \mathscr{H}$ and the result is proved with $f=T^{-1}g$. □

5.9. The Fredholm Alternative in Hilbert Spaces

Theorems 5.3 and 5.5 are of course applicable to compact operators in Hilbert spaces. Let us derive now for Hilbert spaces our earlier remarks concerning adjoints in Banach spaces. In light of Theorem 5.7, we define the adjoint slightly differently.

If T is a bounded linear operator in a Hilbert space $\mathscr{H}$, its adjoint T^* is also a bounded linear mapping in $\mathscr{H}$ defined by

$$(5.12) \qquad (T^*y, x) = (y, Tx) \quad \text{for all } x, y \in \mathscr{H}.$$

Clearly $\|T^*\| = \|T\|$, where $\|T\| = \sup_{x \neq 0} \|Tx\|/\|x\|$.

Lemma 5.9. *If T is compact, then T^* is also compact.*

Proof. Let $\{x_n\}$ be a sequence in $\mathscr{H}$ satisfying $\|x_n\| \leqslant M$. Then

$$\begin{aligned}\|T^*x_n\|^2 &= (T^*x_n, T^*x_n) = (x_n, TT^*x_n)\\ &\leqslant \|x_n\| \; \|TT^*x_n\|\\ &\leqslant M\|T\| \; \|T^*x_n\|,\end{aligned}$$

so that $\|T^*x_n\| \leqslant M\|T\|$; that is, the sequence $\{T^*x_n\}$ is also bounded. Hence, since T is compact, by passing to a subsequence if necessary, we may assume that the sequence $\{TT^*x_n\}$ converges. But then

$$\begin{aligned}\|T^*(x_n - x_m)\|^2 &= (T^*(x_n - x_m), T^*(x_n - x_m))\\ &= (x_n - x_m, TT^*(x_n - x_m))\\ &\leqslant 2M\|TT^*(x_n - x_m)\| \to 0 \quad \text{as } m, n \to \infty.\end{aligned}$$

Since $\mathscr{H}$ is complete, the sequence $\{T^*x_n\}$ is convergent and hence T^* is compact. □

Lemma 5.10. *The closure of the range of T is the orthogonal complement of the null space of T^*.*

Proof. Let $\mathscr{R}$ = the range of T, $\mathscr{N}^*$ = the null space of T^*. If $y = Tx$, we have $(y, f) = (Tx, f) = (x, T^*f) = 0$ for all $f \in \mathscr{N}^*$ so that $\mathscr{R} \subset \mathscr{N}^{*\perp}$, and since $\mathscr{N}^{*\perp}$ is closed, $\bar{\mathscr{R}} \subset \mathscr{N}^{*\perp}$. Now suppose that $y \notin \bar{\mathscr{R}}$. By the projection theorem, Theorem 5.7, $y = y_1 + y_2$ where $y_1 \in \bar{\mathscr{R}}$, $y_2 \in \bar{\mathscr{R}}^\perp - \{0\}$. Consequently $(y_2, Tx) = (T^*y_2, x) = 0$ for all $x \in \mathscr{H}$, so that $y_2 \in \mathscr{N}^*$. Therefore $(y_2, y) = (y_2, y_1) + \|y_2\|^2 = \|y_2\|^2$ and hence $y \notin \mathscr{N}^{*\perp}$. □

Note that Lemma 5.10 is valid whether or not T is compact. By combining Lemmas 5.9 and 5.10 with Theorems 5.3 and 5.5, we then obtain the following Fredholm alternative for compact operators in Hilbert spaces.

Theorem 5.11. *Let $\mathscr{H}$ be a Hilbert space and T a compact mapping of $\mathscr{H}$ into itself. Then there exists a countable set $\Lambda \subset \mathbb{R}$ having no limit points except possibly $\lambda = 0$, such that if $\lambda \neq 0$, $\lambda \notin \Lambda$ the equations*

$$(5.13) \qquad \lambda x - Tx = y, \; \lambda x - T^*x = y$$

have uniquely determined solutions $x \in \mathscr{H}$ for every $y \in \mathscr{H}$, and the inverse mappings $(\lambda I - T)^{-1}$, $(\lambda I - T^*)^{-1}$ are bounded. If $\lambda \in \Lambda$, the null spaces of the mappings $\lambda I - T$, $\lambda I - T^*$ have positive finite dimension and the equations (5.13) are solvable if and only if y is orthogonal to the null space of $\lambda I - T^*$ in the first case and $\lambda I - T$ in the other.

5.10. Weak Compactness

Let $\mathscr{V}$ be a normed linear space. A sequence $\{x_m\}$ *converges weakly* to an element $x \in \mathscr{V}$ if $f(x_n) \to f(x)$ for all f in the dual space $\mathscr{V}^*$. By the Riesz representation theorem, Theorem 5.7, a sequence $\{x_n\}$ in a Hilbert space $\mathscr{H}$ will converge weakly to $x \in \mathscr{H}$ if $(x_n, y) \to (x, y)$ for all $y \in \mathscr{H}$. The following result is useful in the Hilbert space approach to differential equations.

Theorem 5.12. *A bounded sequence in a Hilbert space contains a weakly convergent subsequence.*

Proof. Let us assume initially that $\mathscr{H}$ is separable and suppose that the sequence $\{x_n\} \subset \mathscr{H}$ satisfies $\|x_n\| \leqslant M$. Let $\{y_m\}$ be a dense subset of $\mathscr{H}$. By the Cantor diagonal process we obtain a subsequence $\{x_{n_k}\}$ of our original sequence satisfying $(x_{n_k}, y_m) \to \alpha_m \in \mathbb{R}$ as $k \to \infty$. The mapping $f: \{y_m\} \to \mathbb{R}$ defined by $f(y_m) = \alpha_m$ may consequently be extended to a bounded linear functional f on $\mathscr{H}$ and hence by the Riesz representation theorem, there exists an element $x \in \mathscr{H}$ satisfying $(x_{n_k}, y) \to f(y) = (x, y)$ as $k \to \infty$, for all $y \in \mathscr{H}$. Hence the subsequence $\{x_{n_k}\}$ converges weakly to x.

To extend the result to an arbitrary Hilbert space $\mathscr{H}$, we let $\mathscr{H}_0$ be the closure of the linear hull of the sequence $\{x_n\}$. Then by our previous argument there exists a subsequence $\{x_{n_k}\} \subset \mathscr{H}_0$ and an element $x \in \mathscr{H}_0$ satisfying $(x_{n_k}, y) \to (x, y)$ for all $y \in \mathscr{H}_0$. But by Theorem 5.5, we have for arbitrary $y \in \mathscr{H}$, $y = y_0 + y_1$, where $y_0 \in \mathscr{H}_0$, $y_1 \in \mathscr{H}_0^\perp$. Hence $(x_{n_k}, y) = (x_{n_k}, y_0) \to (x, y_0) = (x, y)$ for all $y \in \mathscr{H}$ so that $\{x_{n_k}\}$ converges weakly to x, as required. □

The first part of the proof of Theorem 5.12 extends automatically to reflexive Banach spaces with separable dual spaces (see Problem 5.4). The result is true however for arbitrary reflexive Banach spaces (see [YO]).

Notes

The material in this chapter is standard and can be found in texts on functional analysis such as [DS], [EW] and [YO].

Problems

5.1. Prove that the Hölder spaces $C^{k,\alpha}(\bar{\Omega})$, introduced in Chapter 4, are Banach spaces under either of the equivalent norms (4.6) or (4.6)′.

5.2. Prove that the interior Hölder spaces $C_*^{k,\alpha}(\Omega)$ defined by

$$C_*^{k,\alpha}(\Omega)=\{u\in C^{k,\alpha}(\Omega) \mid |u|_{k,\alpha;\Omega}^*<\infty\}$$

are Banach spaces under the interior norms given by (4.17).

5.3. Let $\mathscr{B}$ be a Banach space and T be a bounded linear mapping of $\mathscr{B}$ into itself satisfying

$$\|x\|\leqslant K\|Tx\| \quad \text{for all } x\in\mathscr{B},$$

for some $K\in\mathbb{R}$. Prove that the range of T is closed.

5.4. Prove that a bounded sequence in a separable, reflexive Banach space contains a weakly convergent subsequence.

Chapter 6

Classical Solutions; the Schauder Approach

This chapter develops a theory of second order linear elliptic equations that is essentially an extension of potential theory. It is based on the fundamental observation that equations with Hölder continuous coefficients can be treated locally as a perturbation of constant coefficient equations. From this fact Schauder [SC 4, 5] was able to construct a global theory, an extension of which is presented here. Basic to this approach are apriori estimates of solutions, extending those of potential theory to equations with Hölder continuous coefficients. These estimates provide compactness results that are essential for the existence and regularity theory, and since they apply to classical solutions under relatively weak hypotheses on the coefficients, they play an important part in the subsequent nonlinear theory.

Throughout this chapter we shall denote by $Lu=f$ the equation

$$Lu=a^{ij}(x)D_{ij}u+b^i(x)D_iu+c(x)u=f(x), \quad a^{ij}=a^{ji}, \tag{6.1}$$

where the coefficients and f are defined in an open set $\Omega\subset\mathbb{R}^n$ and, unless otherwise stated, the operator L is *strictly elliptic*; that is,

$$a^{ij}(x)\xi_i\xi_j\geqslant\lambda|\xi|^2, \quad \forall x\in\Omega, \quad \xi\in\mathbb{R}^n, \tag{6.2}$$

for some positive constant λ.

Equations with constant coefficients. Before treating equation (6.1) with variable coefficients, we establish a necessary preliminary result that extends Theorems 4.8 and 4.12 from Poisson's equation to other elliptic equations with constant coefficients. We state these extensions in the following lemma, recalling the interior and partially interior norms defined in (4.17), (4.18) and (4.29). Here and throughout this chapter all Hölder exponents will be assumed to lie in (0, 1) unless otherwise stated.

Lemma 6.1. *In the equation*

$$L_0u=A^{ij}D_{ij}u=f(x), \quad A^{ij}=A^{ji}, \tag{6.3}$$