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SOLUZIONI DEGLI ESERCIZI

Esercizio 1

Calcolare
$$\int_\gamma \frac{y^3+\sqrt{1-z}}{z^2}\,ds$$
dove $\gamma(t) := (\cos t, \sin t, t)$ con $-\frac{1}{2} \le t \le \frac{1}{2}$.

Soluzione

Si calcola $ds = \sqrt{2}\,dt$ da cui segue che
$$\begin{aligned}\int_\gamma \frac{y^3+\sqrt{1-z}}{z^2}\,ds &= \sqrt{2}\int_{-1/2}^{1/2}\frac{\sin^3 t}{t^2}\,dt + \sqrt{2}\int_{-1/2}^{1/2}\frac{\sqrt{1-t}}{t^2}\,dt \\ &= \sqrt{2}\int_{-1/2}^{1/2}\frac{\sqrt{1-t}}{t^2}\,dt\end{aligned}$$

Ponendo $\sqrt{1-t} = x$ si ha
$$\begin{aligned}\int_{-1/2}^{1/2}\frac{\sqrt{1-t}}{t^2}\,dt &= \int_{\frac{\sqrt{3}}{\sqrt{2}}}^{\frac{\sqrt{2}}{2}}\frac{2x^2}{(1-x^2)^2}\,dx \\ &= \int_{\frac{\sqrt{3}}{\sqrt{2}}}^{\frac{\sqrt{2}}{2}}\left[\frac{A}{1-x}+\frac{B}{1+x}+\frac{C}{(1-x)^2}+\frac{D}{(1+x)^2}\right]dx\end{aligned}$$

A, B, C, D soddisfano la seguente relazione
$$A(1+x)(1-x)^2 + B(1-x)(1+x)^2 + C(1+x)^2 + D(1-x)^2 = 2x^2$$

Risolvendo il sistema dei coefficienti si ottiene
$$A = -\frac{1}{2} = B = -C = -D$$

quindi l'integrale diventa

$$\begin{aligned}
\int_{\frac{\sqrt{3}}{\sqrt{2}}}^{\frac{\sqrt{2}}{2}} \frac{2x^2}{(1-x^2)^2}\,dx &= \frac{1}{2}\int_{\frac{\sqrt{3}}{\sqrt{2}}}^{\frac{\sqrt{2}}{2}} \left[-\frac{1}{1-x}-\frac{1}{1+x}+\frac{1}{(1-x)^2}+\frac{1}{(1+x)^2}\right]dx \\
&= \frac{1}{2}\left[\ln|1-x|-\ln(1+x)+\frac{1}{1-x}-\frac{1}{1+x}\right]_{\frac{\sqrt{3}}{\sqrt{2}}}^{\frac{\sqrt{2}}{2}}
\end{aligned}$$

e dunque

$$\int_\gamma \frac{y^3+\sqrt{1-z}}{z^2}\,ds = \frac{\sqrt{2}}{2}\left[\ln|1-x|-\ln(1+x)+\frac{1}{1-x}-\frac{1}{1+x}\right]_{\frac{\sqrt{3}}{\sqrt{2}}}^{\frac{\sqrt{2}}{2}}$$

Esercizio 2

Calcolare

$$\int \arccos\sqrt[3]{x}\,dx.$$

Soluzione

Ponendo $\cos t = \sqrt[3]{x}$ si ha $dx = -3\cos^2 t \sin t\,dt$ e l'integrale diventa

$$\begin{aligned}
\int \arccos\sqrt[3]{x}\,dx &= -3\int t\cos^2 t\sin t\,dt \\
&= t\cos^3 t - \int \cos^3 t\,dt \\
&= t\cos^3 t - \int (1-\sin^2 t)\cos t\,dt \\
&= t\cos^3 t - \sin t + \frac{1}{3}\sin^3 t\,dt
\end{aligned}$$

in termini della variabile x si ha

$$\int \arccos\sqrt[3]{x}\,dx = x\arccos\sqrt[3]{x} - \sqrt{1-x^{2/3}} + \frac{1}{3}(1-x^{2/3})^{\frac{3}{2}}$$

Esercizio 3

Sia S la superficie parametrizzata da

$$\begin{cases} x(u,v) = u \\ y(u,v) = \sin v - u\cos v \\ z(u,v) = \cos v + u \sin v \end{cases}$$

con $v \in (0,\pi)$ e $u \in (0,1)$. Sia $f(x,y,z) = xy + yz + zx$, calcolare

$$\int_S f d\sigma.$$

Soluzione

Si ha $d\sigma = \sqrt{1+2u^2}$ dunque l'integrale diventa

$$\begin{aligned} \int_S f d\sigma &= \int_0^1 \sqrt{1+2u^2} \int_0^\pi \sin v \cos v \, dv \, du \\ &+ \int_0^1 u\sqrt{1+2u^2} \int_0^\pi (\sin v + \cos v + 2\sin^2 v - 1) \, dv \, du \\ &+ \int_0^1 u^2\sqrt{1+2u^2} \int_0^\pi (\sin v - \sin v \cos v - \cos v) \, dv \, du \end{aligned}$$

Si osservi che

$$\int_0^\pi \sin v \cos v \, dv = 0$$

e anche

$$\int_0^\pi \cos v \, dv = 0$$

dunque l'integrale si riduce a

$$\begin{aligned} \int_S f d\sigma &= \int_0^1 u\sqrt{1+2u^2} \int_0^\pi (\sin v + 2\sin^2 v - 1) \, dv \, du \\ &+ \int_0^1 u^2\sqrt{1+2u^2} \int_0^\pi \sin v \, dv \, du \end{aligned}$$

Calcolando gli integrali in dv si ha

$$\int_S f d\sigma = (2 + \pi - \pi) \int_0^1 u\sqrt{1+2u^2} \, du + 2\int_0^1 u^2\sqrt{1+2u^2} \, du$$

$$= \quad 2\int_0^1 u\sqrt{1+2u^2}\,du + 2\int_0^1 u^2\sqrt{1+2u^2}\,du$$

Per quanto riguarda il primo integrale si ha subito che

$$\int_0^1 2u\sqrt{1+2u^2}\,du = \frac{1}{3}\Big[\big(1+2u^2\big)^{3/2}\Big]_0^1.$$

Per il secondo integrale basta fare la sostituzione $u\sqrt{2} = \sinh t$ per cui si ottiene

$$\begin{aligned}
\int_0^1 u^2\sqrt{1+2u^2}\,du &= \frac{1}{2\sqrt{2}}\int_0^{\ln(\sqrt{2}+\sqrt{3})} \sinh^2 t\cosh^2 t\,dt \\
&= \frac{1}{8\sqrt{2}}\int_0^{\ln(\sqrt{2}+\sqrt{3})} (1+\cosh 2t)(\cosh 2t - 1)\,dt \\
&= \frac{1}{8\sqrt{2}}\int_0^{\ln(\sqrt{2}+\sqrt{3})} (\cosh^2 2t - 1)\,dt \\
&= \frac{1}{8\sqrt{2}}\int_0^{\ln(\sqrt{2}+\sqrt{3})} \Big[\frac{1}{2}(1+\cosh 4t) - 1\Big]\,dt \\
&= \frac{1}{8\sqrt{2}}\Big[-\frac{1}{2}t + \frac{1}{8}\sinh 4t\Big]_0^{\ln(\sqrt{2}+\sqrt{3})}
\end{aligned}$$

Se si vuole scrivere il risultato in termini di u si ha

$$\int_S f\,d\sigma = \Big[\frac{1}{3}\big(1+2u^2\big)^{3/2} + \frac{1}{16}\Big(u\sqrt{1+2u^2}(1+4u^2) - \frac{1}{\sqrt{2}}\ln\big(u\sqrt{2}+\sqrt{1+2u^2}\big)\Big)\Big]_0^1$$