Soluzioni 6-AM4

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1. Si parametrizzi la superficie Γ tramite

$$\varphi(t,u) = (t,u,1-t-u)$$

da cui segue che $d\sigma = \sqrt{3}\,dudt$. Quindi si ha

$$\begin{aligned}
\int_\Gamma yz\sqrt{1-x}\,d\sigma &= \sqrt{3}\int_0^1\int_0^{1-t} u\sqrt{1-t}(1-t-u)dudt \\
&= \sqrt{3}\int_0^1\int_0^{1-t} u\sqrt{1-t}\,dudt - \sqrt{3}\int_0^1\int_0^{1-t} ut\sqrt{1-t}\,dudt \\
&\quad -\sqrt{3}\int_0^1\int_0^{1-t} u^2\sqrt{1-t}\,dudt \\
&= \sqrt{3}\int_0^1 \frac{(1-t)^{\frac{5}{2}}}{2}\,dt - \sqrt{3}\int_0^1\int_0^{1-t} t(1-t)^{\frac{5}{2}}\,dt \\
&\quad -\sqrt{3}\int_0^1 \frac{(1-t)^{\frac{7}{2}}}{2}\,dt \\
&= \left[\frac{\sqrt{3}}{7}(1-t)^{\frac{7}{2}} + 2\frac{\sqrt{3}}{7}t(1-t)^{\frac{7}{2}} + \frac{4\sqrt{3}}{63} - \frac{2\sqrt{3}}{9}(1-t)^{\frac{9}{2}}\right]_0^1
\end{aligned}$$

2. Si calcola $ds = e^t\sqrt{3}\,dt$. L'integrale diventa

$$\begin{aligned}
\int_\gamma x^2\ln z\,ds &= \sqrt{3}\int_0^T te^{3t}\cos^2 t\,dt \\
&= \frac{\sqrt{3}}{2}\int_0^T te^{3t}(1+\cos 2t)\,dt \\
&= \frac{\sqrt{3}}{2}\int_0^T te^{3t}\,dt + \frac{\sqrt{3}}{2}\int_0^T te^{3t}\cos 2t
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sqrt{3}}{2}\left[\frac{t}{3}e^{3t}\right]_0^T - \int_0^T \frac{e^{3t}}{3}\,dt + \frac{\sqrt{3}}{2}\int_0^T te^{3t}\cos 2t\,dt \\
&= \frac{\sqrt{3}}{2}\left[\frac{t}{3}e^{3t} - \frac{e^{3t}}{9}\,dt\right]_0^T + \frac{\sqrt{3}}{2}\int_0^T te^{3t}\cos 2t\,dt
\end{aligned}
$$

Si calcoli ora

$$
\begin{aligned}
\int e^{3t}\cos 2t\,dt &= \frac{e^{3t}}{3}\cos 2t + 2\int \frac{e^{3t}}{3}\sin 2t\,dt \\
&= \frac{e^{3t}}{3}\cos 2t + \frac{2}{3}\int e^{3t}\sin 2t\,dt \\
&= \frac{e^{3t}}{3}\cos 2t + \frac{2}{9}e^{3t}\sin 2t - \frac{4}{9}\int e^{3t}\cos 2t\,dt
\end{aligned}
$$

da cui segue che

$$
\int e^{3t}\cos 2t\,dt = \frac{9}{39}e^{3t}\left(\cos 2t + \frac{2}{3}\sin 2t\right)
$$

Quindi, integrando per parti, si ha che:

$$
\begin{aligned}
&\frac{\sqrt{3}}{2}\left[\frac{t}{3}e^{3t} - \frac{e^{3t}}{9}\,dt\right]_0^T + \frac{\sqrt{3}}{2}\int te^{3t}\cos 2t\,dt = \\
&\qquad = \frac{\sqrt{3}}{2}\left[\frac{t}{3}e^{3t} - \frac{e^{3t}}{9} + \frac{9}{39}te^{3t}\left(\cos 2t + \frac{2}{3}\sin 2t\right)\right]_0^T \\
&\qquad\quad - \frac{\sqrt{3}}{2}\frac{9}{39}\int_0^T e^{3t}\left(\cos 2t + \frac{2}{3}\sin 2t\right)dt \\
&\qquad = \frac{\sqrt{3}}{2}\left[\frac{t}{3}e^{3t} - \frac{e^{3t}}{9} + + \frac{9}{39}te^{3t}\left(\cos 2t + \frac{2}{3}\sin 2t\right)\right. \\
&\qquad\quad \left. - \frac{9^2}{39^2}e^{3t}\left(\cos 2t + \frac{2}{3}\sin 2t\right)\right]_0^T + \frac{2}{3}\int_0^T e^{3t}\sin 2t\,dt
\end{aligned}
$$

Ora basta calcolare

$$
\begin{aligned}
\int_0^T e^{3t}\sin 2t\,dt &= \left[\frac{e^{3t}}{3}\sin 2t\right]_0^T - \frac{2}{3}\int_0^T e^{3t}\cos 2t\,dt \\
&= \left[\frac{e^{3t}}{3}\sin 2t - \frac{2}{9}e^{3t}\cos 2t\right]_0^T - \frac{4}{9}\int_0^T e^{3t}\sin 2t\,dt
\end{aligned}
$$

da cui segue che

$$\int_0^T e^{3t}\sin 2t\,dt = \left[\frac{9}{39}\,e^{3t}\left(\sin 2t - \frac{2}{3}\cos 2t\right)\right]_0^T.$$

Dunque

$$\begin{aligned}\int_\gamma x^2 \ln z\,ds &= \frac{\sqrt{3}}{2}\left[\frac{t}{3}\,e^{3t} - \frac{e^{3t}}{9} + \frac{9}{39}te^{3t}\left(\cos 2t + \frac{2}{3}\sin 2t\right)\right.\\ &\left. -\frac{9^2}{39^2}\,e^{3t}\left(\cos 2t + \frac{2}{3}\sin 2t\right) + \frac{2}{3}\cdot\frac{9}{39}\,e^{3t}\left(\sin 2t - \frac{2}{3}\cos 2t\right)\right]_0^T\end{aligned}$$

3. Ponendo $e^t = x$ si ha, per $x > 0$

$$\begin{aligned}\int e^t \ln\frac{1}{\sqrt{1-e^{-t}}}\,dt &= -\frac{1}{2}\int \ln\frac{x+1}{x}\,dx\\ &= -\frac{1}{2}\int \ln(x+1)\,dx + \frac{1}{2}\int \ln x\,dx\\ &= -\frac{1}{2}x\ln(x+1) + \frac{1}{2}\frac{x}{x+1} + \frac{1}{2}x\ln x - \frac{1}{2}\end{aligned}$$

4. Integrando per parti si ha

$$\begin{aligned}\int \frac{\ln^2 x}{x^2}\,dx &= -\frac{1}{x}\ln^2 x + 2\int \frac{1}{x^2}\ln x\,dx\\ &= -\frac{1}{x}\ln^2 x - \frac{2}{x}\ln x + 2\int \frac{1}{x^2}\,dx\\ &= -\frac{1}{x}\ln^2 x - \frac{2}{x}\ln x - \frac{2}{x}\,dx\end{aligned}$$