The restricted, circular, planar three-body problem

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1 The restricted three-body problem

Roughly speaking the *restricted three-body problem* is the problem of describing the bounded motions of a "zero-mass" body subject to the gravitational field generated by an assigned two-body system[1]. To describe mathematically such system, let P_0, P_1, P_2 be three bodies ("point masses") with masses m_0, m_1, m_2 interacting only through the gravitational attraction. If $u^{(i)} \in \mathbb{R}^3$, $i = 1, 2, 3$, denote the position of the bodies in some (inertial) reference frame (and assuming, without loss of generality, that the gravitational constant is one[2]), the Newton equations for this system have the form

$$\begin{aligned} \frac{d^2 u^{(0)}}{dt^2} &= -\frac{m_1(u^{(0)} - u^{(1)})}{|u^{(1)} - u^{(0)}|^3} - \frac{m_2(u^{(0)} - u^{(2)})}{|u^{(2)} - u^{(0)}|^3} \ , \\ \frac{d^2 u^{(1)}}{dt^2} &= -\frac{m_0(u^{(1)} - u^{(0)})}{|u^{(1)} - u^{(0)}|^3} - \frac{m_2(u^{(1)} - u^{(2)})}{|u^{(2)} - u^{(1)}|^3} \ , \\ \frac{d^2 u^{(2)}}{dt^2} &= -\frac{m_0(u^{(2)} - u^{(0)})}{|u^{(2)} - u^{(0)}|^3} - \frac{m_1(u^{(2)} - u^{(1)})}{|u^{(2)} - u^{(1)}|^3} \ . \end{aligned} \tag{1.1}$$

The *restricted three-body problem (with "primary bodies" P_0 and P_1) is*, by definition, *the problem of studying the bounded motions of the system* (1.1) *after having set* $m_2 = 0$,

[1] For general references, see, e.g., V. Szebehely, *Theory of orbits*, Academic Press, New York and London, 1967; A. E. Roy, *Orbital Motion*, Adam Hilger Ltd., Bristol, 1978.

[2] This amounts to re-scale the time.

i.e., of the system

$$\begin{aligned}
\frac{d^2u^{(0)}}{dt^2} &= -\frac{m_1(u^{(0)}-u^{(1)})}{|u^{(1)}-u^{(0)}|^3} , \qquad \frac{d^2u^{(1)}}{dt^2} = -\frac{m_0(u^{(1)}-u^{(0)})}{|u^{(1)}-u^{(0)}|^3} \\
\frac{d^2u^{(2)}}{dt^2} &= -\frac{m_0(u^{(2)}-u^{(0)})}{|u^{(2)}-u^{(0)}|^3} - \frac{m_1(u^{(2)}-u^{(1)})}{|u^{(1)}-u^{(2)}|^3} .
\end{aligned} \tag{1.2}$$

Notice that the equations for the two primaries P_0 and P_1 *decouple* and describe an *unperturbed two-body system*, which can be solved and the solution can be plugged into the equation for $u^{(2)}$, which becomes a second-order, *periodically forced* equation in $\mathbb{R}^3$.

2 Delaunay action-angle variables for the two-body problem

In this section we review the construction of the classical Delaunay[3] action-angle variables for the two-body problem.

The equations of motion of two bodies P_0 and P_1 of masses m_0 and m_1, interacting through gravitation (with gravitational constant equal to one) are given (as in the first line of (1.2)) by

$$\frac{d^2u^{(0)}}{dt^2} = -\frac{m_1(u^{(0)}-u^{(1)})}{|u^{(1)}-u^{(0)}|^3} , \qquad \frac{d^2u^{(1)}}{dt^2} = -\frac{m_0(u^{(1)}-u^{(0)})}{|u^{(1)}-u^{(0)}|^3} , \qquad u^{(i)} \in \mathbb{R}^3 . \tag{2.3}$$

As everybody knows, the total energy, momentum and angular momentum are preserved. We shall therefore fix an *inertial frame* $\{k_1, k_2, k_3\}$, with origin in the center of mass and with k_3-axis parallel to the total angular momentum. In such frame we have

$$u_3^{(0)} \equiv 0 \equiv u_3^{(1)} , \qquad m_0 u^{(0)} + m_1 u^{(1)} = 0 . \tag{2.4}$$

We pass to a *heliocentric frame* by letting,

$$(x, 0) := u^{(1)} - u^{(0)} , \qquad x \in \mathbb{R}^2 . \tag{2.5}$$

In view of (2.3) and (2.4), the equations for x become

$$\ddot{x} = -M\frac{x}{|x|^3} , \qquad M := m_0 + m_1 . \tag{2.6}$$

[3]C. Delaunay, *Théorie du Mouvement de la Lune*, Mémoires de l'Académie des Sciences **1**, Tome XXVIII, Paris, 1860.

This equation is Hamiltonian: let $\mu > 0$ and set

$$H_{\rm Kep}(x, X) := \frac{|X|^2}{2\mu} - \frac{\mu M}{|x|} , \qquad X := \mu \dot{x} , \tag{2.7}$$

then (2.6) is equivalent to the Hamiltonian equation associated to $H_{\rm Kep}$ with respect to the standard symplectic form $dx \wedge dX$, the phase space being $\mathbb{R}^2 \backslash \{0\} \times \mathbb{R}^2$; the (free) parameter μ is traditionally chosen as the "reduced mass" $m_0 m_1 / M$.
The motion in the u-coordinates is recovered (via (2.4) and (2.5)) by the relation

$$u^{(0)} = \Big(\frac{-m_1}{M} x, 0 \Big) , \qquad u^{(1)} = \Big(\frac{m_0}{M} x, 0 \Big) ,$$

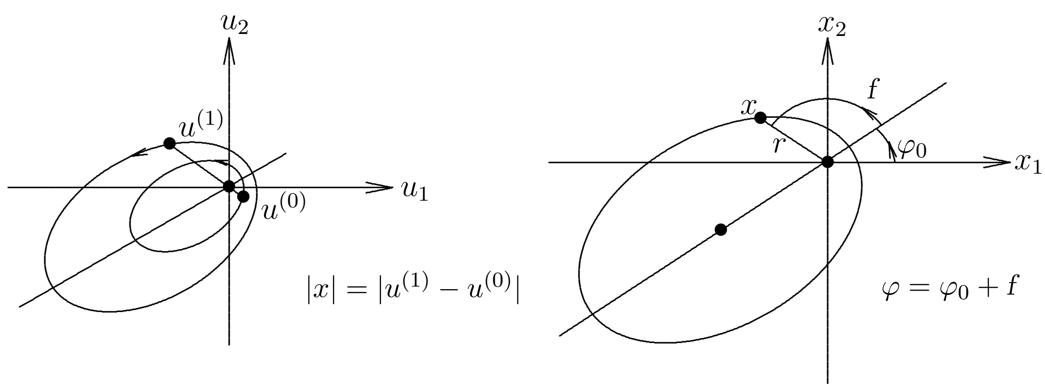

Figure 1: The geometry of the Kepler two–body problem

The dependence of $H_{\rm Kep}$ on x through the absolute value suggests to introduce polar coordinates in the x-plane: $x = r(\cos\varphi, \sin\varphi)$ and, in order to get a symplectic transformation, one is led to the symplectic map $\phi_{\rm pc} : ((r, \varphi), (R, \Phi)) \to (x, X)$ given by

$$\phi_{\rm pc} : \quad \begin{cases} x \ = r(\cos\varphi, \sin\varphi) , \\ X = \Big(R\cos\varphi - \frac{\Phi}{r}\sin\varphi, R\sin\varphi + \frac{\Phi}{r}\cos\varphi \Big) , \\ dx_1 \wedge dX_1 + dx_2 \wedge dX_2 = dr \wedge dR + d\varphi \wedge d\Phi . \end{cases} \tag{2.8}$$

The variables r and φ are commonly called, in celestial mechanics, *the orbital radius* and the *longitude of the planet* P_1.
In the new symplectic variables the Hamiltonian $H_{\rm Kep}$ takes the form

$$H_{\rm pc}(r, \varphi, R, \Phi) := H_{\rm Kep} \circ \phi_{\rm pc}(r, \varphi, R, \Phi) = \frac{1}{2\mu}\Big(R^2 + \frac{\Phi^2}{r^2} \Big) - \frac{\mu M}{r} .$$

The variable φ is *cyclic* (i.e., $\partial H_{\rm pc}/\partial\varphi = 0$ so that $\Phi \equiv$ const), showing that the system with Hamiltonian $H_{\rm pc}$ is actually a *one-degree-of-freedom Hamiltonian system* (in the symplectic variables (r, R)), and is therefore *integrable*. The momentum variable Φ conjugated to φ is an integral of motion and

$$\dot\varphi = \frac{\partial H_{\rm pc}}{\partial \Phi} = \frac{\Phi}{\mu r^2} \qquad \Longrightarrow \qquad \Phi = \mu r^2 \dot\varphi \equiv \text{ const } .$$

Remark 2.1 The total angular momentum, C, in the inertial frame (and referred to the center of mass) is given by[4]

$$C = m_0 u^{(0)} \times \dot u^{(0)} + m_1 u^{(1)} \times \dot u^{(1)} .$$

Taking into account the inertial relation $m_0 u^{(0)} = -m_1 u^{(1)}$ one finds that

$$C = \frac{m_0 m_1}{M} (x, 0) \times (\dot x, 0) = \frac{m_0 m_1}{M\mu} \ (x, 0) \times (X, 0) \ ,$$

and the evaluation of the angular momentum in polar coordinates shows that

$$C = \pm k_3 \ \frac{m_0 m_1}{M} r^2 \dot\varphi = \pm k_3 \frac{m_0 m_1}{M\mu} \Phi \ ;$$

thus *if* μ *is chosen to be the reduced mass* $\frac{m_0 m_1}{M}$, *then* Φ *is exactly the absolute value of the total angular momentum.*

The analysis of the (r, R) motion is standard: introducing the "effective potential"

$$V_{\rm eff}(r) := V_{\rm eff}(r; \Phi) := \frac{\Phi^2}{2\mu r^2} - \frac{\mu M}{r} \ ,$$

one is led to the "effective Hamiltonian" (parameterized by Φ)

$$H_{\rm eff} = \frac{R^2}{2\mu} + V_{\rm eff}(r) \ , \qquad (R = \mu \dot r) \ .$$

[4] "$\times$" denotes, here, the standard "vector product" in $\mathbb{R}^3$.

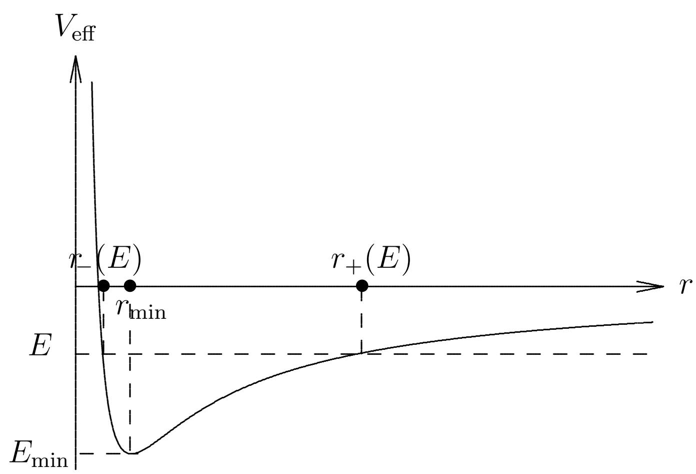

Figure 2: The effective potential of the two–body problem

The motion on the energy level $H_{\mathrm{eff}}^{-1}(E)$ is bounded (and periodic) if and only if

$$E \in [E_{\min}, 0) \ , \qquad E_{\min} := V_{\mathrm{eff}}(r_{\min}) = -\frac{\mu^3 M^2}{2\Phi^2} \ , \quad r_{\min} := \frac{\Phi^2}{\mu^2 M} \ . \tag{2.9}$$

For $E \in (E_{\min}, 0)$ the period $T(E)$ is given by

$$T(E) = 2\int_{r_-(E)}^{r_+(E)} \frac{dr}{\sqrt{\frac{2}{\mu}(E - V_{\mathrm{eff}}(r))}} \ , \tag{2.10}$$

where $r_\pm(E) = r_\pm(E; \Phi)$ are the two positive roots of $E - V_{\mathrm{eff}}(r) = 0$, i.e.,

$$E - V_{\mathrm{eff}}(r) =: \frac{-E}{r^2}(r_+ - r)(r - r_-) \ ,$$

$$r_\pm(E;\Phi) = \frac{\mu M \pm \sqrt{(\mu M)^2 + \frac{2E\Phi^2}{\mu}}}{-2E} \ . \tag{2.11}$$

The integral in (2.10) is readily computed yielding Kepler's second law

$$T(E) = 2\pi \ M \left(\frac{\mu}{-2E}\right)^{3/2} \ .$$

Let us now integrate the motion in the (r, φ) coordinates. The equations of motion in such coordinates are given by

$$\dot{\varphi} = \frac{\Phi}{\mu r^2} \ , \qquad \dot{r}^2 = \frac{2}{\mu}(E - V_{\mathrm{eff}}(r)) \ . \tag{2.12}$$

By symmetry arguments, it is enough to consider the motion for $0 \le t \le T(E)/2$; furthermore, we shall choose the initial time so that $r(0) = r_-$ (i.e., at the initial time the system is at the "*perihelion*"): the corresponding angle will be a certain φ_0 and we shall make the (trivial) change of variables

$$\varphi = \varphi_0 + f \ , \qquad \text{so that} \quad r(0) = r_-(E) \ , \quad f(0) = 0 \ . \tag{2.13}$$

The angle f is commonly called the *true anomaly*; the angle φ_0 (i.e., the angle between the perihelion line, joining the foci of the ellipse and the x_1 axis) is called *the argument of the perihelion* (compare figure at page 3).

Equations (2.12) become

$$\begin{cases} \dot f = \frac{\Phi}{\mu r^2} \ , & f(0) = 0 \\ \dot r^2 = \frac{2}{\mu}(E - V_{\rm eff}(r)) \ , & r(0) = r_-(E) \end{cases}$$

Eliminating time (for $t \in (0, T(E))$, $\dot r > 0$) we find (recall the definitions in (2.9))

$$f = \Phi \int_{r_-(E)}^{r} \frac{d\rho/\rho^2}{\sqrt{2\mu(E - V_{\rm eff}(\rho))}} = \text{Arccos} \ \frac{\frac{r_{\rm min}}{r} - 1}{\sqrt{1 - \frac{E}{E_{\rm min}}}} \ . \tag{2.14}$$

Setting

$$e := \sqrt{1 - \frac{E}{E_{\rm min}}} \ , \qquad p := r_{\rm min} \ , \tag{2.15}$$

we get the classical focal equation

$$r = \frac{p}{1 + e\cos f} = \frac{p}{1 + e\cos(\varphi - \varphi_0)} \ , \tag{2.16}$$

which shows that P_0 and P_1 describe two ellipses of eccentricity $e \in (0, 1)$ with common focus in the center of mass (first Kepler law).

If $a \ge b > 0$ denote the semi-axis of the ellipse, from (2.16) it follows immediately that

$$r_\pm = \frac{p}{1 \mp e} \ , \quad r_+ + r_- = 2a \ , \quad p = a(1 - e^2) \ , \quad r_\pm = a(1 \pm e) \ . \tag{2.17}$$

From the geometry of the ellipse (see Appendix A and in particular (A.45) and the figure at page 18) one knows that

$$r = a(1 - e\cos u) \ , \tag{2.18}$$

where u is the so-called *eccentric anomaly*. Then, from the definition of $E_{\rm min}$, (2.9), the expression for $E - V_{\rm eff}$ in (2.11), the relations (2.15) and (2.17), one finds

$$E_{\rm min} = -\frac{\mu M}{2p} \ , \quad E = -\frac{\mu M}{2a} \ , \quad E - V_{\rm eff} = \frac{\mu M}{2a}\left(\frac{e\sin u}{1 - e\cos u}\right)^2 \ . \tag{2.19}$$

Remark 2.2 The *circular motion* for the two-body problem is obtained for the minimal value of the energy $E = E_{\rm min} = -\frac{\mu^3 M^2}{2\Phi^2}$. In such a case

$$e = 0 \ , \qquad r \equiv p = r_{\rm min} = \frac{\Phi^2}{\mu^2 M} \ ; \tag{2.20}$$

the constant angular velocity and the period are respectively given by

$$\omega_{\rm circ} = \frac{\mu^3 M^2}{\Phi^3} \ , \qquad T_{\rm circ} = 2\pi \frac{\Phi^3}{\mu^3 M^2} \ . \tag{2.21}$$

Eliminating Φ in (2.20) and (2.21) one gets

$$\omega_{\rm circ} = \sqrt{\frac{M}{r^3}} \ , \qquad T_{\rm circ} = 2\pi \sqrt{\frac{r^3}{M}} \ .$$

The motion in the x-variables is given by

$$x(t) = r\Big(\cos(\varphi_0 + \omega_{\rm circ}\ t), \sin(\varphi_0 + \omega_{\rm circ}\ t)\Big) \ . \tag{2.22}$$

We turn to the construction of the *action-angle variables*. For $E \in (E_{\rm min}, 0)$, denote by S_E the curve (energy level) $\{(r,R) : H_{\rm eff}(r,R) = E\}$ (at a fixed value of Φ). The area $A(E)$ encircled by such a curve in the (r,R)-plane is given by

$$A(E) = 2\int_{r_-(E)}^{r_+(E)} \sqrt{2\mu\Big(E - V_{\rm eff}(r)\Big)}dr = 2\pi\ \mu M \ \sqrt{\frac{\mu}{-2E}} - 2\pi\Phi \ .$$

Thus, (by the theorem of Liouville-Arnold) the action variable is given by

$$I(E) = \frac{A(E)}{2\pi} = \mu M \ \sqrt{\frac{\mu}{-2E}} - \Phi \ ,$$

which, inverted, gives the form of the Hamiltonian $H_{\rm eff}$ in the action-angle variables (θ, I) (and parameterized by Φ):

$$h(I) := h(I;\Phi) := -\frac{\mu^3 M^2}{2(I+\Phi)^2} \ .$$

Furthermore (again by Liouville-Arnold), the symplectic transformation between (r,R) (in a neighborhood of a point with $R > 0$) and the action variables, (θ, I), for the Hamiltonian $H_{\rm eff}$ is *generated* by the generating function[5]

$$S_0(I,r;\Phi) := \int_{(r_-(h(I)),0)}^{(r,R_+(r;I))} R dr \ , \qquad R_+(r;I) := \sqrt{2\mu(h(I) - V_{\rm eff}(r))} \ ,$$

[5]Recall that the dependence upon Φ is hidden in r_- and $V_{\rm eff}$.

where the integration is performed over the curve $S_{h(I)}$ oriented *clockwise*: the orientation of $S_{h(I)}$ and the choice of the base point as $(r_-(h(I)), 0)$ is done so that an integration over the closed curve gives $+A(E)$ and so that $\theta = 0$ corresponds to the perihelion position.

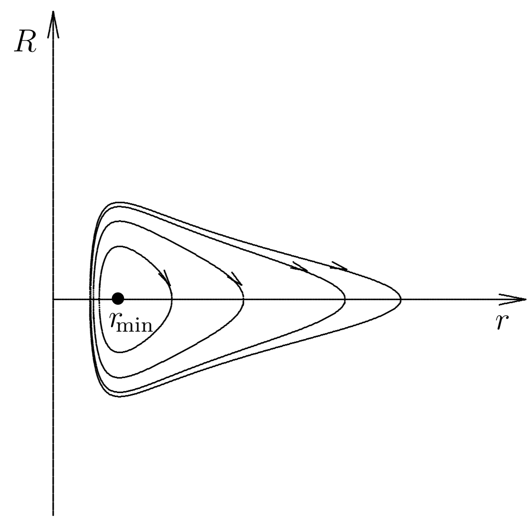

Figure 3: Level curves of the effective Hamiltonian for $E_{\min} < E < 0$

The full symplectic transformation (in the four dimensional phase space of H_{pc})

$$\phi_{\text{aa}} : \quad \begin{cases} (\theta, \psi, I, J) \to (r, \varphi, R, \Phi) \\ d\theta \wedge dI + d\psi \wedge dJ = dr \wedge dR + d\varphi \wedge d\Phi \end{cases}$$

will then be generated by the generating function

$$S_1(I, J, r, \varphi) := S_0(I, r; J) + J\varphi \ , \qquad (J = \Phi) \ .$$

The form of $h(I)$ suggests to introduce one more (linear, symplectic) change of variables given by

$$\phi_{\text{lin}}^{-1} : \quad \begin{cases} \Lambda = I + J \ , & \Gamma = J \ , \\ \lambda = \theta \ , & \gamma = \psi - \theta \ . \end{cases}$$

The variables $(\lambda, \gamma, \Lambda, \Gamma)$ are the celebrated *Delaunay variables for the two-body problem*. If we set

$$\phi_{\text{D}} := \phi_{\text{pc}} \circ \phi_{\text{aa}} \circ \phi_{\text{lin}} \tag{2.23}$$

by the above analysis we get

$$h_{\rm Kep}\circ\phi_{\rm D}(\lambda,\gamma,\Lambda,\Gamma)=h_{\rm Kep}(\Lambda):=-\frac{\mu^3M^2}{2\Lambda^2}\ . \tag{2.24}$$

The symplectic transformation $\phi_{\rm aa}\circ\phi_{\rm lin}$ is generated by $(\Gamma=J=\Phi)$

$$\begin{aligned} S_2(\Lambda,\Gamma,r,\varphi) &:= S_0(\Lambda-\Gamma,r;\Gamma)+\Gamma\varphi \\ &= \int_{r_-(h_{\rm Kep}(\Lambda))}^{r}\sqrt{-\frac{\mu^4M^2}{\Lambda^2}+\frac{2\mu^2M}{\rho}-\frac{\Gamma^2}{\rho^2}}\,d\rho+\Gamma\varphi \\ &= \sqrt{2\mu}\int_{r_-(h_{\rm Kep}(\Lambda))}^{r}\sqrt{h_{\rm Kep}(\Lambda)-V_{\rm eff}(\rho;\Gamma)}\,d\rho\ +\Gamma\varphi\ . \end{aligned}$$

Replacing E by $h_{\rm Kep}(\Lambda)$ and Φ with Γ in the expression for the eccentricity e in (2.15) (recall the definition of $E_{\rm min}$ in (2.9)) one finds

$$e=e(\Lambda,\Gamma)=\sqrt{1-\Big(\frac{\Gamma}{\Lambda}\Big)^2}\ . \tag{2.25}$$

Recalling also the second relation in (2.19) one finds that

$$a=\frac{\Lambda^2}{\mu^2M}\ ,\qquad\qquad \Lambda=\mu\sqrt{Ma}\ . \tag{2.26}$$

Remark 2.3 Recall that

$$\Gamma=\Phi=\frac{\mu M}{m_0m_1}|C|\ ,\qquad C:=\ \text{total angular momentum}\ ,$$

so that

$$\Gamma>0\ .$$

Recall also that

$$E_{\rm min}=-\frac{\mu^3M^2}{2\Gamma^2}\ ,$$

so that $E>E_{\rm min}$ means (by (2.24))

$$\Gamma<\Lambda\ .$$

The momentum space is therefore the *positive cone* $\{0<\Gamma<\Lambda\}$.

The angle λ is computed from the generating function S_2:

$$\begin{aligned}
\lambda &= \frac{\partial S_2}{\partial \Lambda} = \sqrt{\frac{\mu}{2}}\,\frac{\mu^3 M^2}{\Lambda^3}\int_{r_-}^{r}\frac{d\rho}{\sqrt{h_{\rm Kep}(\Lambda)-V_{\rm eff}(\rho;\Gamma)}}\\
&\overset{(2.26)}{=} \sqrt{\frac{\mu M}{2a}}\,\frac{1}{a}\int_{r_-}^{r}\frac{d\rho}{\sqrt{h_{\rm Kep}(\Lambda)-V_{\rm eff}(\rho;\Gamma)}}\\
&\overset{(2.19)}{=} \frac{1}{a}\int_{r_-}^{r}\frac{1-e\cos u}{e\sin u}d\rho\\
&\overset{(2.18)}{=} \int_0^u (1-e\cos u)du\\
&= u-e\sin u\\
&= 2\pi\frac{{\rm Area}(\mathcal{E}(f))}{{\rm Area}(\mathcal{E}(2\pi))}\ ,
\end{aligned} \tag{2.27}$$

where (compare Appendix A), $\mathcal{E}(f)$ is the area (on the ellipse (2.16)) "spanned by the orbital radius":

$$\mathcal{E}(f) := \{x = x(r',f') : 0 \le r' \le r(f)\ , 0 \le f' \le f\}\ ;$$

we have also used the fact that ρ as a function of $u \in [0,\pi]$ is a strictly increasing function and that $\rho(0) = r_-$.

In view of (2.27), λ is called *the mean anomaly*. Analogously, *the angle γ is recognized to be the argument of the perihelion φ_0* introduced above (just before Remark 2.2):

$$\begin{aligned}
\gamma &= \frac{\partial S_2}{\partial \Gamma} = \varphi - \Gamma\int_{r_-}^{r}\frac{1}{\sqrt{2\mu(h_{\rm Kep}(\Lambda)-V_{\rm eff}(\rho))}}\,\frac{d\rho}{\rho^2}\\
&\overset{(2.14)}{=} \varphi - f\\
&\overset{(2.13)}{=} \varphi_0\ .
\end{aligned} \tag{2.28}$$

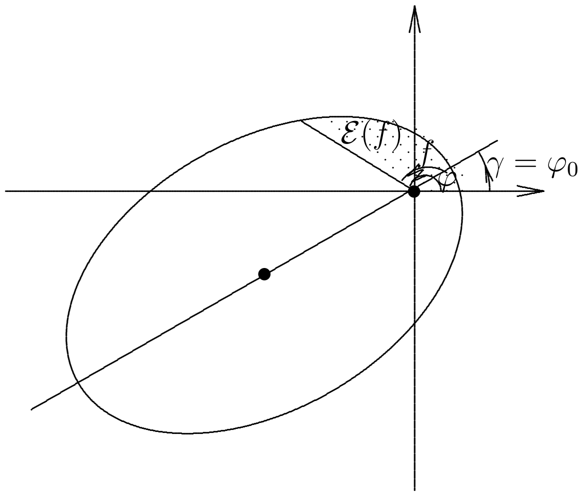

Figure 4: The Delaunay angles

We conclude this classical section by giving *analytical expressions for the eccentric anomaly u, the true anomaly f, the longitude φ and the orbital radius r in terms of the Delaunay variables.*

The (Kepler) equation

$$\lambda = u - e \sin u \ ,$$

(see (2.27)) can be inverted, for $|e|$ small enough, as

$$\begin{aligned} u &= u_0(\lambda, e) := \lambda + e\tilde{u}(\lambda, e) \\ &= \lambda + e\ \sin\lambda + \frac{e^2}{2}\ \sin 2\,\lambda + \frac{e^3}{8}\Big(-\sin\lambda + 3\ \sin 3\,\lambda\Big) + \cdots\ , \end{aligned} \tag{2.29}$$

where $\tilde{u}$ is analytic in $\lambda \in \mathbb{T}$ and $|e|$ small enough; via (2.25), $e = e(\Lambda, \Gamma) = \sqrt{1 - (\Gamma/\Lambda)^2}$, the relation (2.29) yields an analytic expression of the eccentric anomaly as a function of the Delaunay variables λ, Λ, Γ.

From the geometry of the ellipse it follows that (compare (A.45) in Appendix A)

$$\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2}\ ,$$

which can be written, for $|e|$ small enough, as

$$\begin{aligned} f &= \hat{f}_0(u, e) := u + e\hat{f}(u, e) \\ &= u + e\ \sin(u) + \frac{e^2}{4}\ \sin 2\,u + \frac{e^3}{12}\Big(3\ \sin u + \sin 3\,u\Big) + \cdots\ , \end{aligned} \tag{2.30}$$

where $\hat{f}$ is analytic in $u \in \mathbb{T}$ and $|e|$ small enough. Through (2.29) and (2.25), the expression (2.30) yields an analytic expression of the true anomaly f in terms of λ, Λ, Γ:

$$\begin{aligned} f &= f_0(\lambda, e) := \hat{f}_0(u_0(\lambda, e), e) =: \lambda + e\tilde{f}(\lambda, e) \\ &= \lambda + 2\, e\, \sin\lambda + \frac{5}{4} e^2 \sin 2\,\lambda + \frac{e^3}{12}\Big(-3\,\sin\lambda + 13\,\sin 3\,\lambda\Big) + \cdots\ . \end{aligned} \tag{2.31}$$

As above $e = e(\Lambda, \Gamma)$.

The longitude φ by (2.28) is simply $\varphi = \gamma + f$ and can, therefore, be expressed as a function of $\lambda, \gamma, \Lambda, \Gamma$.

From the geometry of the ellipse it follows that (compare (A.45) in Appendix A) r is related to a, e and u by

$$r = a(1 - e\cos u)\ .$$

Thus by (2.26), (2.25) and (2.29) we find

$$\begin{aligned} \frac{r}{a} &= \frac{r_0(\lambda, e)}{a} \\ &:= 1 - e\cos u_0(\lambda, e) \\ &= 1 - e\,\cos\lambda + \frac{e^2}{2}\Big(1 - \cos 2\,\lambda\Big) + \frac{3}{8} e^3\Big(\cos\lambda - \cos 3\,\lambda\Big) + \cdots\ , \end{aligned} \tag{2.32}$$

where $e = e(\Lambda, \Gamma)$ and $a = a(\Lambda) := \Lambda^2/(\mu^2 M)$ (see (2.25) and (2.26)).

3 The restricted, circular, planar three-body problem viewed as nearly-integrable Hamiltonian system

Let us go back to (1.2). Since we shall study the *planar three-body problem*, we *assume that the motion takes place on the plane hosting the Keplerian motion of P_0 and P_1*. This amounts to require

$$u_3^{(i)} \equiv 0 \equiv \dot{u}_3^{(i)}\ , \qquad i = 0, 1, 2\ . \tag{3.33}$$

Observe that, since we are considering the *restricted* problem (i.e. we have set in (1.1) $m_2 = 0$), the "conservation laws" are those of the two-body system $P_0 - P_1$: in particular the total angular momentum is parallel to the u_3-axis (consistently with (3.33)) and the center of mass (and hence the origin of the u-frame) is simply

$$m_0 u^{(0)} + m_1 u^{(1)} = 0\ . \tag{3.34}$$

Next, we pass, as in § 2, to *heliocentric coordinates*:

$$(x^{(1)},0) := u^{(1)} - u^{(0)} \ , \qquad (x^{(2)},0) := u^{(2)} - u^{(0)} \ , \qquad \big(x^{(1)}, x^{(2)} \in \mathbb{R}^2\big) \ ,$$

which transform (1.2) into

$$\ddot{x}^{(1)} := -M_0 \frac{x^{(1)}}{|x^{(1)}|^3} \ , \qquad M_0 := m_0 + m_1 \ , \tag{3.35}$$

$$\ddot{x}^{(2)} := -m_0 \frac{x^{(2)}}{|x^{(2)}|^3} - m_1 \frac{x^{(1)}}{|x^{(1)}|^3} - m_1 \frac{x^{(2)} - x^{(1)}}{|x^{(2)} - x^{(1)}|^3} \ . \tag{3.36}$$

In view of (3.34) the motion in the original u-coordinates is related to the motion in the heliocentric coordinates by

$$u^{(0)} = \Big(\frac{-m_1}{M_0} x^{(1)}, 0\Big) \ , \quad u^{(1)} = \Big(\frac{m_0}{M_0} x^{(1)}, 0\Big) \ , \quad u^{(2)} = \Big(x^{(2)} - \frac{m_1}{M_0} x^{(1)}, 0\Big) \ .$$

The equation in (3.35) describes the decoupled two-body system $P_0 - P_1$, which has been discussed in § 2.

In the restricted, circular, planar three-body problem such motion is assumed to be circular.

It is convenient to *fix the measure units for lengths and masses so that the (fixed) distance between the two primary bodies is one and the sum of their masses is one*:

$$\mathrm{dist}(P_0, P_1) = 1 \ , \qquad M_0 := m_0 + m_1 = 1 \ . \tag{3.37}$$

Recalling Remark 2.2, one sees that *the period of revolution of P_0 and P_1 around their center of mass (the "year") is, in such units,* 2π; the $x^{(1)}$-motion is simply (compare (2.22))

$$\hat{x}^{(1)}_{\mathrm{circ}}(t) = x^{(1)}_{\mathrm{circ}}(t_0 + t) := \Big(\cos(t_0 + t), \sin(t_0 + \ t)\Big) \ .$$

Even though the system of equations (3.35) *and* (3.36) is *not* a Hamiltonian system of equation, (3.35) and (3.36) taken *separately* are Hamiltonian: we have already seen that (3.35) represent just the equations of a two-body system; equations (3.36) represent a $2\frac{1}{2}$-degree-of-freedom Hamiltonian system with Hamiltonian

$$\tilde{H}_1(x^{(2)}, X^{(2)}, t) := \frac{|X^{(2)}|^2}{2\mu} - \mu m_0 \frac{1}{|x^{(2)}|} + \mu m_1 \Big(x^{(2)} \cdot \hat{x}^{(1)}_{\mathrm{circ}}(t)\Big) - \mu m_1 \frac{1}{|x^{(2)} - \hat{x}^{(1)}_{\mathrm{circ}}(t)|} \ ,$$
$$(x^{(2)}, X^{(2)}) \in \mathbb{R}^2 \backslash \{0\} \times \mathbb{R}^2 \ , \ t \in \mathbb{T} \ , \tag{3.38}$$

with respect to the standard symplectic form $dx^{(2)} \wedge dX^{(2)}$; here, $\mu > 0$ is a free parameter. To make the system (3.38) autonomous, we introduce a linear symplectic variable T conjugated to time $\tau = t$:

$$\begin{aligned}
&\tilde{H}_1(x^{(2)}, X^{(2)}, \tau, T) \\
&:= \frac{|X^{(2)}|^2}{2\mu} - \mu m_0 \frac{1}{|x^{(2)}|} + T + \mu m_1 \Big(x^{(2)} \cdot x^{(1)}_{\rm circ}(\tau) \Big) - \mu m_1 \frac{1}{|x^{(2)} - x^{(1)}_{\rm circ}(\tau)|} \ , \\
&(x^{(2)}, X^{(2)}) \in \mathbb{R}^2 \backslash \{0\} \times \mathbb{R}^2 \ , \ (\tau, T) \in \mathbb{T} \times \mathbb{R}. \qquad (3.39)
\end{aligned}$$

Remark 3.1 In the limiting case of a primary body with mass $m_1 = 0$, the Hamiltonian $\tilde{H}_1$ describes a two-body system as in (2.7) with "total mass"

$$M = m_0 \ ,$$

reflecting the fact that the asteroid mass has been set equal to zero.

If the mass m_1 does not vanish but it is small compared to the mass of m_0, the system (3.39) may be viewed as a *nearly-integrable system*. This is more transparent if we use, for the integrable part, the Delaunay variables introduced in § 2 (see in particular (2.23)). Recall that the symplectic transformation $\phi_{\rm D}$, mapping the Delaunay variables to the original Cartesian variables, depends parametrically also on μ and M and that M is now m_0. Next, we choose the free parameter μ so as to make the Keplerian part equal to $-1/(2\Lambda^2)$ (see (3.41) below) and we introduce also a perturbation parameter ε closely related to the mass m_1 of the primary body:

$$\mu := \frac{1}{m_0^{2/3}} \ , \qquad \varepsilon := \frac{m_1}{m_0^{2/3}} = \frac{m_1}{(1 - m_1)^{2/3}} \ . \qquad (3.40)$$

Now, letting

$$\begin{aligned}
&(\lambda, \gamma, \Lambda, \Gamma) = \phi_{\rm D}^{-1}(x^{(2)}, X^{(2)}) \ , \\
&\hat{\phi}_{\rm D}\Big((\lambda, \gamma, \Lambda, \Gamma), (\tau, T)\Big) := \Big(\phi_{\rm D}(\lambda, \gamma, \Lambda, \Gamma), (\tau, T)\Big) \ ,
\end{aligned}$$

we find that[6]

$$\tilde{H}_2 := \tilde{H}_1 \circ \hat{\phi}_{\rm D} = -\frac{1}{2\Lambda^2} + {\rm T} + \varepsilon \left(x^{(2)} \cdot x^{(1)}_{\rm circ}(\tau) - \frac{1}{|x^{(2)} - x^{(1)}_{\rm circ}(\tau)|} \right) \ , \qquad (3.41)$$

where, of course, $x^{(2)}$ is now a function of the new symplectic variables.

[6] Recall (2.24).

Let us now analyze more in detail the perturbing function in (3.41).
Recalling the definition of φ in (2.8), one sees that the angle between the rays $(0, x^{(2)})$ and $(0, x^{(1)}_{\rm circ})$ is $\varphi - \tau$.

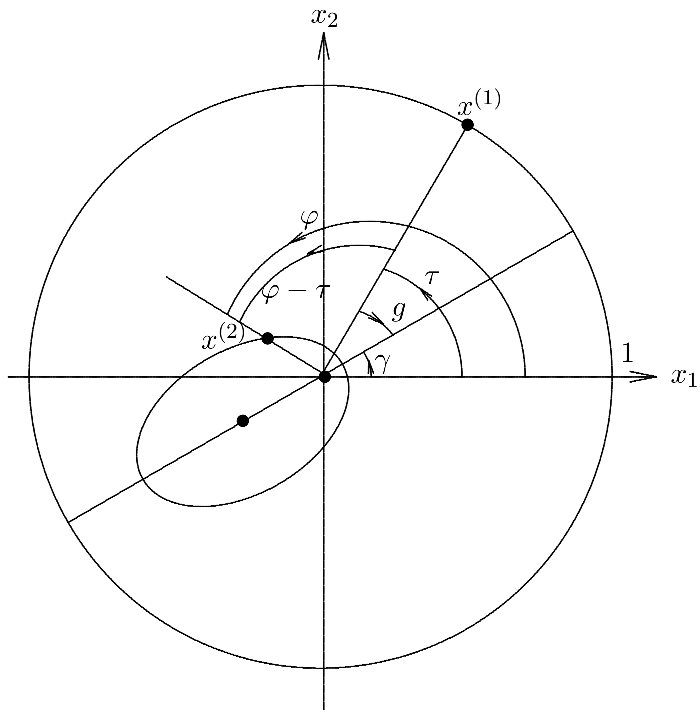

Figure 5: Angle variables for the RCPTBP

Therefore, if we let

$$r_2 := |x^{(2)}| \ ,$$

we get

$$\tilde H_2 = -\frac{1}{2\Lambda^2} + \mathrm{T} + \varepsilon \left(r_2 \cos(\varphi - \tau) - \frac{1}{\sqrt{1 + r_2^2 - 2 r_2 \cos(\varphi - \tau)}} \right) \ .$$

Recall ((2.28)) that $\varphi = \gamma + f$ and ((2.31)) that $f = f_0(\lambda, e) := \lambda + e\tilde f(\lambda, e)$. Thus

$$\varphi - \tau = f + \gamma - \tau = \lambda + \gamma - \tau + e\tilde f(\lambda, e) \ .$$

Such relation suggests to make a new linear symplectic change of variables, by setting

$$\hat\phi_{\rm lin}^{-1}: \quad \begin{cases} L = \Lambda\ , & G = \Gamma\ , \qquad \hat T = T + \Gamma \\ \ell = \lambda\ , & g = \gamma - \tau\ , \qquad \hat\tau = \tau\ . \end{cases}$$

Now, recalling (2.26), (3.40) and (3.37) we see that

$$a = L^2/(\mu^2 M) = m_0^{1/3} L^2\ ,$$

so that, by (2.32), in the new symplectic variables, it is:

$$\begin{aligned} \varphi - \tau &= f + g = f_0(\ell, e) + g = \ell + g + e\tilde f(\ell, e)\ , \\ r_2 &= r_0(\ell, e) = m_0^{1/3} L^2 (1 - e\cos u_0(\ell, e))\ . \end{aligned}$$

where, as above, $e = e(L, G) = \sqrt{1 - (G/L)^2}$.

Notice that the positions (3.40) and (3.37) *define implicitly* m_0 *and hence* $m_0^{1/3}$ *as a (analytic) function of*[7] ε:

$$\begin{aligned} m_0(\varepsilon) \quad &= 1 - \varepsilon + \frac{2}{3}\varepsilon^2 - \frac{1}{3}\varepsilon^3 + \cdots\ , \\ m_0(\varepsilon)^{1/3} &= 1 - \frac{\varepsilon}{3} + \frac{1}{9}\varepsilon^2 - \frac{2}{81}\varepsilon^3 + \cdots\ . \end{aligned}$$

Thus, introducing the functions

$$\begin{aligned} a_\varepsilon &:= a_\varepsilon(L) := m_0(\varepsilon)^{1/3} L^2\ , \\ \rho_\varepsilon &:= \rho_\varepsilon(\ell, L, G) := a_\varepsilon(L)\ \Big(1 - e\cos u_0(\ell, e(L, G))\Big)\ , \\ \sigma &:= \sigma(\ell, L, G) := e(L, G)\ \tilde f(\ell, e(L, G))\ , \end{aligned}$$

we get

$$\tilde H_3 := \tilde H_2 \circ \hat\phi_{\rm lin} = -\frac{1}{2L^2} + \hat T - G + \varepsilon F_\varepsilon(\ell, g, L, G)\ ,$$

where

$$F_\varepsilon := \rho_\varepsilon \cos(\ell + g + \sigma) - \frac{1}{\sqrt{1 + \rho_\varepsilon^2 - 2\rho_\varepsilon\cos(\ell + g + \sigma)}}\ . \tag{3.42}$$

The variable $\hat\tau$ is cyclic (this is the reason for having introduced $\hat\phi_{\rm lin}$) and the linear constant of motion $\hat T$ can be dropped from $\tilde H_3$. The final form of *the Hamiltonian for the restricted, circular, planar, three-body-problem* is:

$$H_{\rm prc}(\ell, g, L, G; \varepsilon) := -\frac{1}{2L^2} - G + \varepsilon F_\varepsilon(\ell, g, L, G)\ ; \tag{3.43}$$

[7] In fact, from (3.40) one can invert the function $m_1 \to \varepsilon(m_1) = m_1/(1 - m_1)^{2/3}$ and check that the inverse function $m_1(\varepsilon) = 1 - m_0(\varepsilon)$ has radius of convergence $(27/4)^{\frac{1}{3}} = 1.889881....$

the phase space is the two-torus $\mathbb{T}^2$ times the positive cone $\{0 < G < L\}$; the symplectic form is the standard two-form $d\ell \wedge dL + dg \wedge dG$.

From the point of view of KAM theory, the integrable part of (3.43)

$$H_0(L, G) := H_{\rm prc}|_{\varepsilon=0} = -\frac{1}{2L^2} - G \ ,$$

is *iso-energetically non-degenerate* since

$$\det \begin{pmatrix} H_0'' & H_0' \\ H_0' & 0 \end{pmatrix} = \det \begin{pmatrix} -\frac{3}{L^4} & 0 & \frac{1}{L^3} \\ 0 & 0 & -1 \\ \frac{1}{L^3} & -1 & 0 \end{pmatrix} = \frac{3}{L^4} > 0 \ .$$

A The ellipse

In this appendix we recall a few classical facts about ellipses.

Cartesian equation. An ellipse is a *set of points in a plane with constant sum of distances from two given points*, called *foci*. The Cartesian equation of an ellipse, with respect to a reference plane $(x_1, x_2) \in \mathbb{R}^2$ with origin chosen as the middle point of the segment joining the two foci, is given by

$$\Big(\frac{x_1}{a}\Big)^2 + \Big(\frac{x_2}{b}\Big)^2 = 1 \ ,$$

where $2a$ is the (constant) sum of distances between x and the foci and

$$\Big(\pm a\sqrt{1 - \Big(\frac{b}{a}\Big)^2}, 0\Big) \tag{A.44}$$

are the coordinates of the foci. The positive numbers a and b are called, respectively, the *major* and the *minor semi-axis* of the ellipse; the number

$$e = \sqrt{1 - \Big(\frac{b}{a}\Big)^2}$$

is called the eccentricity of the ellipse. As it follows from (A.44), the distance c between one focus and the center $x = 0$ of the ellipse is given by

$$c = ea \ .$$

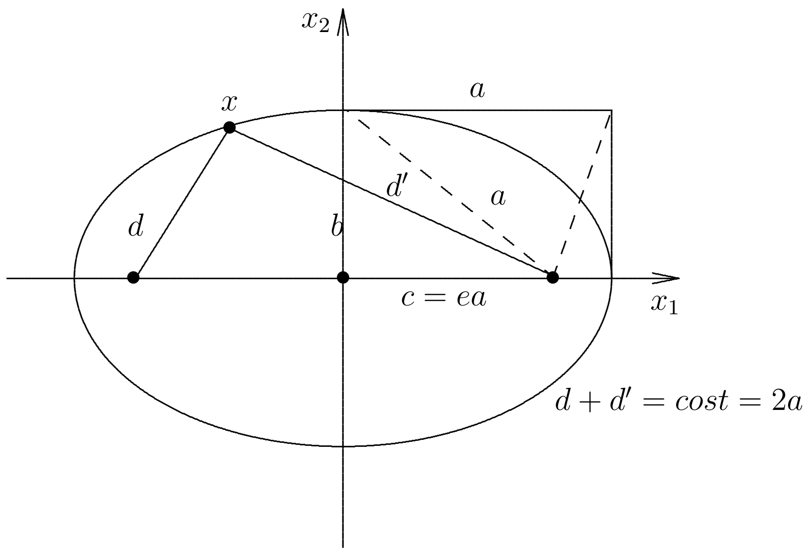

Figure 6: Ellipse of eccentricity 0.78

Focal equation. Introducing polar coordinates (f, r) in the above x-plane taking as pole the focus $O = (c, 0)$, as f the angle between the x_1-axis and the axis joining O with the point x on the ellipse and $r = r(f)$ as the distance $|x - O|$ one finds the following *focal equation*

$$r = r(f) := \frac{p}{1 + e\cos f} \ ,$$

where p is called the *parameter* of the ellipse and is given by

$$p = a(1 - e^2) = \frac{b^2}{a} \ .$$

The angle f is called the *true anomaly*.

Parametric representation. The above ellipse is also described by the following parametric equations

$$x_1 = a\cos u \ , \qquad x_2 = b\sin u \ .$$

The angle u is called the *eccentric anomaly*.

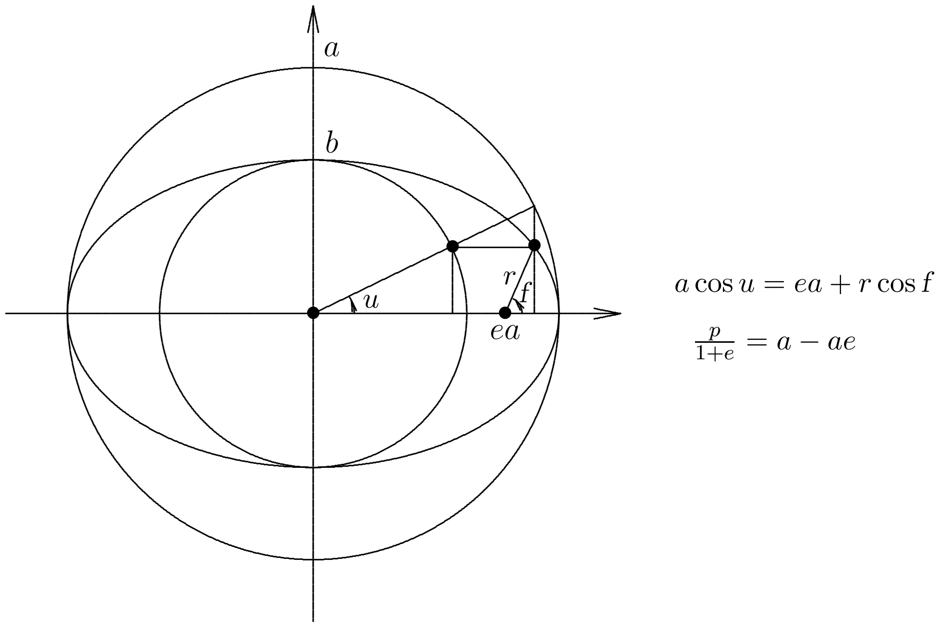

Figure 7: Ellipse parameters

Thus a point x on the ellipse has the double representation:

$$x = (a\cos u, b\sin u) = (ea + r\cos f, r\sin f)\ ,$$

which relates the true and the eccentric anomalies. In particular, one finds:

$$\begin{aligned}
&r\cos f = a(\cos u - e)\ ,\\
&r\sin f = b\sin u = a\sqrt{1-e^2}\sin u\ ,\\
&r = a(1 - e\cos u)\ ,\\
&\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2}\ ,\\
&\mathrm{Area}(\mathcal{E}(f)) = \frac{ab}{2}(u - e\sin u)\ ,
\end{aligned}$$

where

$$\mathcal{E}(f) := \{x = x(r', f') : 0 \le r' \le r(f)\ , 0 \le f' \le f\}\ .$$