LIMITI (SUCCESSIONI E SERIE)

Una successione è una funzione $x : \mathbb{N} \to \mathbb{R}$, $n \mapsto x_n$

x si denota $\{x_n\}_{n \in \mathbb{N}}$

$\{x_n\} \neq \{x_n \mid n \in \mathbb{N}\}$

$\{x_n\} \subseteq \mathbb{N} \times \mathbb{R}$, $\{x_n \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$

Esempi

$\{n\}$ (x = identità) $x_n = n$

$x_n = \frac{n^2 - n}{1 - n}$ $\quad x_n = a^n$, con $a \in \mathbb{R}$

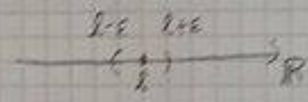
serie $\sigma_n = \sum_{k=1}^{n} a_k$ es. serie geometrica $\sigma_n(x) = \sum_{k=1}^{n} x^k = x + x^2 + x^3 + \dots + x^n$

Def. (i) di limite: data una successione $\{x_n\}$ a valori in $\mathbb{R}$ e $l \in \mathbb{R}$:

si dice che il $\lim_{n \to \infty} x_n = l$ (limite per n che tende a più infinito di x_n è l; "x_n tende ad l")

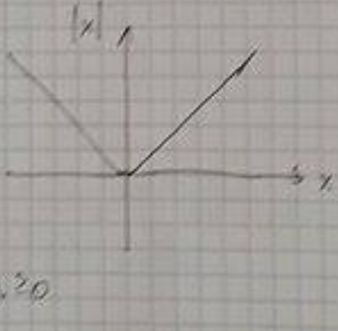
se $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ t.c. $|x_n - l| < \varepsilon$ $\forall n \geq N$.

$\left(|a| := \begin{cases} a & \text{se } a \geq 0 \\ -a & \text{se } a < 0 \end{cases} \right)$



Oss. sul modulo

$|x| \geq 0$, $\forall x \in \mathbb{R}$



$\forall x \in \mathbb{R}$ si ha che:

$-|x| \leq x \leq |x|$

$-|y| \leq y \leq |y|$

$\Downarrow$

$-(|x| + |y|) \leq x + y \leq |x| + |y|$

$\Downarrow$

$|x + y| \leq |x| + |y|$

disuguaglianza triangolare

(se $-a \leq z \leq a$ con $a \geq 0$ $\Leftrightarrow$ $|z| \leq a$

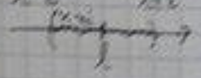
dim. se $z \geq 0$ $\Rightarrow$ $|z| = z \leq a$

se $z < 0$ $\Rightarrow$ $|z| = -z$ $\Rightarrow$ $a \geq -z$

quindi $|z| \leq a$)

$|x - y|$ = distanza tra x e y perché $|x| = |-x|$

$l \in \mathbb{R}$ $\quad \{x \mid |x_n - l| < \varepsilon\} = (l - \varepsilon, l + \varepsilon)$



(ii) $l = +\infty$ ($-\infty$ secondo caso)

$\lim_{n \to +\infty} x_n = +\infty$ se $\forall M \in \mathbb{R}$ $\exists N$: $x_n > M$ $\forall n \geq N$

$\lim_{n\to+\infty} x_n = -\infty$ se $\forall M \in \mathbb{R}\ \exists N: x_n < M \quad \forall n \geq N$

Esempi

(1) $\lim_{n\to+\infty} n = +\infty \quad \forall M \in \mathbb{R}\ \exists N: n > M$ (proprietà archimedea)

$(\exists N \in \mathbb{N}: N > M)$ se $n \geq N > M \Rightarrow n > M \quad \forall n \geq N$.

(2) $c > 0, \ x_n = cn \to +\infty \quad \forall \underset{M>0}{M \in \mathbb{R}}$ per prop. arch. $\exists N: N > \frac{M}{c} \Rightarrow$ se $n \geq N \Rightarrow$

$n > \frac{M}{c} \Rightarrow cn > M$

moltiplico per n e divido per ε

(3) $x_n = \frac{1}{n} \to 0. \quad \forall \varepsilon > 0$ vogliamo che $\left|\underbrace{\frac{1}{n} - 0}_{\frac{1}{n}}\right| < \varepsilon \Rightarrow \frac{1}{n} < \varepsilon \Rightarrow n > \frac{1}{\varepsilon}$, $\exists N;\ \forall n \geq N$

(4) $\left(\frac{(-1)^n}{2n}\right) \to 0$

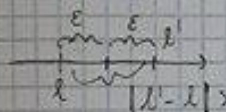
Prop.

Il $\lim x_n = l$, ovvero se esiste, allora è <u>unico</u>.

DIM.

Supponiamo che $x_n \to l \in \mathbb{R}$. Per assurdo, pensiamo che esista $l' \neq l$, $l' \in \mathbb{R}$ t.c. $x_n \to l'$.

$l' > l$



$|l' - l| > 0$

Sia $\varepsilon = \frac{|l' - l|}{2}$, allora $\exists N \in \mathbb{N}: \forall n \geq N \Rightarrow |x_n - l| < \varepsilon$

ed $\exists N' \in \mathbb{N}: \forall n \geq N' \Rightarrow |x_n - l'| < \varepsilon$

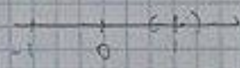
Prendo $\bar{N} = \max\{N, N'\}$ valgono $\forall n \geq \bar{N} \Rightarrow \begin{cases} |x_{\bar{N}} - l| < \varepsilon \\ |x_{\bar{N}} - l'| < \varepsilon \end{cases}$

$2\varepsilon = l' - l = (l' - x_{\bar{N}}) + (x_{\bar{N}} - l) \leq |x_{\bar{N}} - l'| + |x_{\bar{N}} - l| < 2\varepsilon$. contraddizione.

(esempi) non <u>convergenti</u>

(5) $x_n = (-1)^n \quad \{x_n \mid n \in \mathbb{N}\} = \{+1, -1\}$ <u>x_n non converge.</u>

$x_n \not\to +\infty \quad (|x_n| = 1 < 2) \quad x_n \not\to l \in \mathbb{R} \quad \forall l \in \mathbb{R}$, e sia $l \neq 1$



$|l - 1| > 0 \quad (l - \varepsilon, l + \varepsilon) \not\ni 1$ se $\varepsilon = \frac{|l-1|}{2} \quad (-1)^n = 1 \ \forall n$ pari

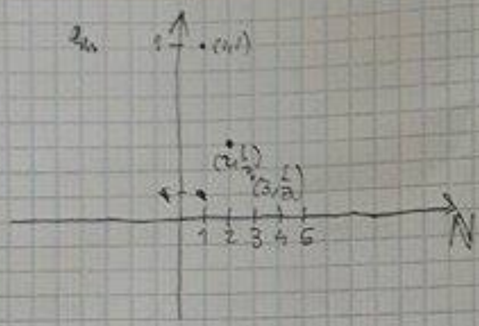
Def.

Sia $\{a_n\}_{n\in\mathbb{N}}$; $l \in \mathbb{R}$. $\lim_{n\to+\infty} a_n = l \overset{\text{def}}{\Leftrightarrow} \underline{\forall \varepsilon > 0}\ \exists N: \forall n \geq N \Rightarrow |a_n - l| < \varepsilon$

$(a_n \cong l \pm \varepsilon)$ da un certo N dipendente da ε.

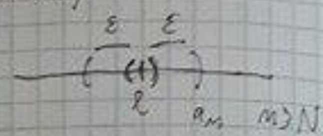
In altri termini, fissato un "errore" arbitrario $\varepsilon > 0$, possiamo trovare un $N = N(\varepsilon)$ t.c.

$\forall n \geq N$ si ha $a_n = l + \underset{\text{errore}}{\varepsilon_n} \quad |\varepsilon_n| < \varepsilon \qquad a_n = 1.4141\ldots a_n \quad a_n = \sqrt{2} + \varepsilon_n \qquad \varepsilon = 0{,}00\ldots$



$\frac{1}{n} \to 0$ (n, a_n)

$a_n \to l$



Es.

$a_n = 2 + \frac{(-1)^n}{\sqrt{n}}$ $a_1 = 2 + (-1) = 1$ $a_2 = 2 + \frac{1}{\sqrt{2}}$ $a_3 = 2 - \frac{1}{\sqrt{3}}$ $a_4 = 2 + \frac{1}{2} = \frac{5}{2}, \dots$

$\lim_{n\to\infty} a_n = 2.$ $|a_n - l| = \left|\cancel{2} + \frac{(-1)^n}{\sqrt{n}} - \cancel{2}\right| = \frac{1}{\sqrt{n}} < \varepsilon$ $|-a| = |a|$

Dato ε, vogliamo risolvere $\frac{1}{\sqrt{n}} < \varepsilon$. Moltiplico per $\sqrt{n}$ $\Leftrightarrow$ $(1 < \sqrt{n}\,\varepsilon)$ e divido per ε $\Leftrightarrow$

$\frac{1}{\varepsilon} < \sqrt{n} \Leftrightarrow \left(\frac{1}{\varepsilon}\right)^2 < n \Leftrightarrow \frac{1}{\varepsilon^2} < n$ $N_\varepsilon := \left[\frac{1}{\varepsilon^2}\right] + 1$

$\frac{1}{\varepsilon^2}$ $\left[\frac{1}{\varepsilon^2}\right]+1$

se $\varepsilon = 10^{-6} = \frac{1}{1\,000\,000}$ $\frac{1}{\varepsilon^2} = 10^{12}$ (mille miliardi)

Def. 2

$a_n \to +\infty$ $\left(\lim_{n\to\infty} a_n = +\infty\right)$ se $\forall M \in \mathbb{R}$ $\exists N \in \mathbb{N}$ t.c. $a_n > M$ $\forall n \geq N$.

$a_n = \sqrt[5]{n^2 - n}$ cresce arbitrariamente.

Teo.
$\forall a \geq 0$, $\forall n \geq 1$
$\exists!\, b \geq 0 : b^n = a$
tale $b := \sqrt[n]{a}$

Teo 2
$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
dim.

esc.

dimostrare che $a_n \to +\infty$ $\sqrt[5]{n^2 - n} = \sqrt[5]{n^2} \cdot \sqrt[5]{\left(1 - \frac{1}{n}\right)}$

Dim. teorema 1 (cenni)

se $a = 0$ tale $b = 0$ banale

Per $n = 1$ $b = a$ v.

Caso $n = 2$ $\Leftrightarrow$ teo. Euclide-Pit.

Per $a > 0$, $n \geq 3$.

$D := \{x > 0 \mid x^n < a\}$ (I) Si fa vedere che D è non vuoto, limitato superiormente. $\Rightarrow$ $\exists\, b = \sup D$

$\Rightarrow b > 0$ (II) $b^n = a$ (si dimostra facendo vedere che $b^n > a$ non è sup D e $b^n < a$ è elemento di D, quindi si hanno due assurdi).

Dimostriamo ora l'unicità di b.

$\forall x, y \in \mathbb{R}$, $\forall n \geq 2$ per $x, y \neq 0$

$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \dots + xy^{n-2} + y^{n-1})$ scende aumenta

Per $n \geq 2 \Rightarrow n = 2$ $x^2 - y^2 = (x-y)(x+y)$ $\to (x-y)\sum_{k=0}^{n-1} x^{n-1-k}y^k$ $y^0 = 1$

Dim. formula * $(x^n - y^n)$

Assumiamo $y \neq 0$, divido per y^n: $\frac{x^n}{y^n} - 1 = \frac{x-y}{y} \cdot \frac{(x^{n-1} + x^{n-2}y + \dots + y^{n-1})}{y^{n-1}}$

$$\left(\frac{x}{y}\right)^n - 1 = \left(\frac{x}{y} - 1\right)\left(\left(\frac{x}{y}\right)^{n-1} + \left(\frac{x}{y}\right)^{n-2} + \dots + 1\right)$$

$* \; t^n - 1 = (t-1)(t^{n-1} + t^{n-2} + \dots + 1)$ $\quad t = \frac{x}{y}$ $\quad$ per $t = 1$ $(x=y)$ V.

$t \neq 1 \Rightarrow \underbrace{1+t+t^2+\dots+t^{n-1}}_{\text{somma geometrica}} = \frac{t^n - 1}{t-1} = \frac{1-t^n}{1-t}$

Dimostriamo per induzione su $n \in \mathbb{N}$, $n \geq 2$.

Per $n = 2 \Rightarrow t^2 - 1 = (t-1)(t+1)$

Assumiamo vera per n e dimostriamola per $n+1$.

$$1+t+\dots+t^n = (1+\dots+t^{n-1}) + t^n \underset{\text{ip. ind.}}{=} \frac{1-t^n}{1-t} + t^n = \frac{1-t^n+t^n(1-t)}{1-t} = \frac{1-\cancel{t^n}+\cancel{t^n}-t^{n+1}}{1-t} =$$

$$\frac{1-t^{n+1}}{1-t}.$$

Se per assurdo $\exists\, b_1 \neq b_2$, $b_1 > 0$, $b_2 > 0$ $\quad b_1^n = a = b_2^n$

Questo implica $0 = b_2^n - b_1^n = (b_2 - b_1)(\underbrace{b_2^{n-1} + b_2^{n-2}b_1 + \dots + b_1^{n-1}}_{n \text{ termini}}) \Rightarrow b_2 - b_1 = 0 \Rightarrow b_2 = b_1$. contr.

Dim. teorema 2 (Dedekind)

Nota: $(xy)^n = x^n y^n$ $\quad \left(\underbrace{\sqrt[n]{a} \cdot \sqrt[n]{b}}_{>0}\right)^n = \left(\sqrt[n]{a}\right)^n \cdot \left(\sqrt[n]{b}\right)^n = a \cdot b$ $\quad$ e per l'unicità (teorema 1)

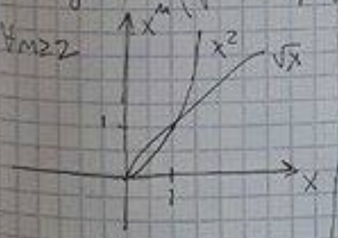
$\Rightarrow \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$

Consideriamo la funzione $f(x) = \sqrt[n]{x} = x^{\frac{1}{n}}$, $x \geq 0$

N.B. $x > y \geq 0 \Rightarrow \sqrt[n]{x} > \sqrt[n]{y}$ $\forall n \geq 1$. $\Rightarrow \frac{\sqrt[n]{x}}{\sqrt[n]{y}} > 1 \Rightarrow \sqrt[n]{\frac{x}{y}}$ $\left(\text{perché } \frac{1}{\sqrt[n]{y}} = \sqrt[n]{\frac{1}{y}}\right)$

$\Rightarrow \frac{x}{y} > 1$ $\quad \left(\sqrt[n]{t} := \sup\{s > 0 : s^n < t\} \Rightarrow t > 1 \Rightarrow \sqrt[n]{t} > 1\right.$

$\forall n \geq 2$



Torniamo ad $a_n = \sqrt[5]{n^2 - n} = \sqrt[5]{n^2 \cdot \left(1 - \frac{1}{n}\right)} = \sqrt[5]{n^2} \cdot \sqrt[5]{1 - \frac{1}{n}} =$

$n^{\frac{2}{5}} \cdot \sqrt[5]{1 - \frac{1}{n}} \geq n^{2/5} \cdot \frac{1}{\sqrt[5]{2}} = c \cdot n^{2/5}$ $\qquad c > 0$, $c = \frac{1}{\sqrt[5]{2}}$

$x^{n/n} = \sqrt[n]{x^n} = \left(\sqrt[n]{x}\right)^n$ $\qquad n \geq 2$

$\forall x \geq 0$ es. $\qquad 1 - \frac{1}{n} \geq \frac{1}{2} \Leftrightarrow 1 - \frac{1}{2} \geq \frac{1}{n} \Rightarrow \frac{1}{2} \geq \frac{1}{n} \Rightarrow n \geq 2$

$\frac{1}{6}\ \frac{1}{6}\ \frac{2}{6} \dots \frac{1}{2}\ \frac{1}{6^n}$

es. $\sqrt[n]{\frac{1}{a}} = \frac{1}{\sqrt[n]{a}}$, $\forall a > 0$, $\forall n \in \mathbb{N}$

$c\,n^{2/5} > M \Rightarrow n^{2/5} > \frac{M}{c} \Leftrightarrow n > \underbrace{\left(\frac{M}{c}\right)^{5/2}}_{N}$

<u>es.</u> $a>0$, $\left(a^{\frac{m}{n}}\right)^{\frac{n}{m}} = a$.

$\left(a^{\frac{n}{m}}\right)^{\frac{m}{n}} = \left(\left(\left(a^{\frac{1}{m}}\right)^{n}\right)^{\frac{1}{n}}\right)^{m} = a$.   <u>es.</u> $\left(a^{\frac{n}{m}}\right)^{\frac{p}{q}} = a^{\frac{np}{mq}}$

<u>Lemma</u> (Bernoulli)

$\alpha > -1$, $n \in \mathbb{N}$, allora $(1+\alpha)^n \geq 1+\alpha n$, $\forall n \geq 0$ $(*)$

($n \geq 2$ vale $(1+\alpha)^n > 1+\alpha n$)

<u>DIM.</u>

$n = 0, 1$ vale l'uguaglianza.

Assumiamo vera $(*)$ per $n \geq 1$, vogliamo dimostrare $(*)_{n+1}$

$$(1+\alpha)^{n+1} = (1+\alpha)^n(1+\alpha) \underset{1+\alpha>0}{\geq} (1+\alpha)(1+\alpha n) = \underbrace{1+\alpha(n+1)+\alpha^2 n}_{1+\alpha+\alpha n+\alpha^2 n} \geq 1+\alpha(n+1).$$

<u>Applicazione</u>

$x > 1 \Rightarrow x^n \to +\infty$ $x = 1+\alpha$ con $\alpha > 0$ $x^n = (1+\alpha)^n \geq 1+\alpha n \overset{?}{>} M$

$1+\alpha n > \alpha n > M \Rightarrow n > \frac{M}{\alpha} = N = \frac{M}{x-1}$

<u>N.B.</u> se $a_n \to +\infty \Rightarrow \frac{1}{a_n} \to 0$. $\frac{1}{a_n} < \varepsilon \Leftrightarrow a_n > \frac{1}{\varepsilon}$ poiché $a_n \to +\infty$ $\exists N$:

$a_n > \frac{1}{\varepsilon}$ $\forall n \geq N$.

se $0<x<1 \Rightarrow y = \frac{1}{x} > 1$ $x^n = \left(\frac{1}{y}\right)^n = \frac{1}{(y^n)} \to 0$ perché $y^n \to +\infty$

$0<x<1 \Rightarrow x^n \to 0 \Rightarrow$ se $x \in \mathbb{R}$, $|x|<1$ $x^n \to 0$ N.B. $\{a_n \to 0 \Leftrightarrow |a_n| \to 0$

<u>TEOREMA</u> Se $0 \leq x < 1 \Rightarrow$

$$\sum_{k=1}^{\infty} x^k := \lim_{n\to\infty} \sum_{k=1}^{n} x^k = \frac{x}{1-x}$$

<u>DIM.</u>

$$\sum_{k=1}^{n} x^k = x+x^2+\ldots+x^n$$

$$1+t+\ldots+t^{n-1} = \frac{1-t^n}{1-t}$$

$$1+t+\ldots+t^n = \frac{1-t^{n+1}}{1-t} - 1 = \frac{\cancel{1}-t^{n+1}-\cancel{1}+t}{1-t} = \frac{t-t^{n+1}}{1-t}$$

da cui segue che $0,\bar{9} = 1$

$$\frac{x}{1-x} \Rightarrow \frac{x-x^{n+1}}{1-x} = \frac{x}{1-x} - \frac{x^{n+1}}{1-x} \to 0$$