Metodi di integrazione

15/1/15

$\boxed{\int \frac{P(x)}{Q(x)}\,dx}$

- $\int \frac{3x+2}{5x-3}\,dx = 3\int \frac{x+2/3}{5x-3}\,dx = \frac{3}{5}\int \frac{5x+10/3}{5x-3}\,dx = \frac{3}{5}\int \frac{5x-3+3+10/3}{5x-3}$

$= \frac{3}{5}\left(\int \frac{\cancel{5x-3}}{\cancel{5x-3}}\,dx + \frac{19/3}{5x-3}\,dx\right) = \frac{3}{5}\left(x + \frac{19}{3}\int \frac{1}{5x-3}\,dx\right) = \frac{3}{5}x + \frac{19}{25}\int \frac{5}{5x-3}\,dx$

$= \frac{3}{5}x + \frac{19}{25}\log|5x-3| + c$

- $\int_{-3}^{3} \operatorname{arctg}(x)\,dx = 0$ $\quad$ [se $f(x)$ é pari $\int_{-a}^{a} f(x) = 2\int_{0}^{a} f(x)$]

- $\int \frac{3x}{x^2+5}\,dx = \frac{3}{2}\int \frac{2x}{x^2+5}\,dx = \frac{3}{2}\log(x^2+5) + c$ $\quad \left(\begin{array}{l} f(x) = x^2+5 \\ f'(x) = 2x \end{array}\right)$

- $\int \frac{3x+7}{x^2+5}\,dx = 3\int \frac{x+7/3}{x^2+5}\,dx = \frac{3}{2}\int \frac{2x+14/3}{x^2+5}\,dx = \frac{3}{2}\left(\int \frac{2x}{x^2+5}\,dx + \int \frac{14/3}{x^2+5}\,dx\right) =$

$= \frac{3}{2}\log(x^2+5) + c + \frac{\cancel{3}}{\cancel{2}}\cdot\frac{\cancel{14}\,7}{\cancel{3}}\int \frac{1}{x^2+5}\,dx = \frac{3}{2}\log(x^2+5) + \frac{7}{\sqrt{5}}\operatorname{arctg}\frac{1}{\sqrt{5}} + c$

- $\int \frac{2x-1}{x^2-3x+2}\,dx = \int \frac{2x-1}{(x-1)(x-2)}\,dx = \quad \frac{2x-1}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$

$\frac{Ax-2A+Bx-B}{(x-1)(x-2)} = \frac{(A+B)x-2A-B}{(x-1)(x-2)} \Rightarrow \begin{cases} A+B=2 \\ -2A-B=-1 \end{cases} \begin{cases} B = 2-A \\ -2A-2+A=-1 \end{cases} \begin{cases} B=3 \\ A=-1 \end{cases}$

$= \int -\frac{1}{x-1}\,dx + \int \frac{3}{x-2}\,dx = -\log|x-1| + 3\log|x-2| + c$

- $\int \frac{2x+x^2-3}{(x-1)(x-5)(x-3)(x-2)}\,dx \quad \frac{2x+x^2-3}{(x-1)(x-5)(x-3)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-5)} + \frac{C}{(x-3)} + \frac{D}{(x-2)}$

- per determinare A moltiplico per $(x-1)$

$A + B\frac{x-1}{x-5} + C\frac{x-1}{x-3} + D\frac{x-1}{x-2} \rightarrow$ passo al limite per $x \rightarrow 1$ $\rightarrow A = 0$

- per det. B moltiplico per $(x-5)$ e passo al limite per $x \rightarrow 5$

$\frac{x^2+2-3}{(x-1)\cancel{(x-5)}(x-3)(x-2)} = A\frac{x-5}{x-1} + B + C\frac{x-5}{x-3} + D\frac{x-5}{x-2} \Rightarrow B = \frac{4}{3}$ e via così

- $\int \frac{5}{x^2-2x+1}\,dx = 5\int \frac{1}{(x-1)^2}\,dx = 5\int (x-1)^{-2}\,dx = 5\,\frac{(x-1)^{-2+1}}{-2+1} + c = -\frac{5}{x-1} + c$

- $\int \frac{2x-2-1}{x^2-2x+1} = \int \frac{2x-2}{x^2-2x+1}\,dx - \int \frac{1}{x^2-2x+1}\,dx = \log(x-1)^2 + \frac{1}{x-1} + c$

- $\int \frac{1}{(x-1)(x^2+1)}\,dx \Rightarrow \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+1)}$

- $\int \frac{1}{(x^2+1)^2}\,dx$ $\quad \left[I_n = \int \frac{1}{(a^2+x^2)^n}\,dx = \frac{1}{2(n-1)a^2}\left[\frac{x}{(x^2+a^2)^{n-1}} + (2n-3)\cdot I(n-1)\right]\right]$

METODI DI INTEGRAZIONE = sostituzione

19/01/15

- $\int_e^{e^2} \cos(\log x)\,dx \to$ pongo $\log x = t \Rightarrow x = e^t \Rightarrow dx = e^t dt$

$= \int_1^2 \cos t \cdot e^t\,dt$ $\qquad e \le x \le e^2 \Rightarrow \log e \le t = \log x \le \log e^2$

$1 \le t \le 2$

(poi si risolve per parti)

$$\boxed{\begin{aligned} f(x) &= g(t) \\ f'(x)\,dx &= g'(t)\,dt \end{aligned}}$$

- $\int \frac{1}{1+e^x}\,dx \to e^x = t \Rightarrow x = \log t \Rightarrow dx = \frac{dt}{t}$

$= \int \frac{1}{1+t} \cdot \frac{dt}{t} = \int \frac{1}{t(1+t)}\,dt \Rightarrow \frac{A}{t} + \frac{B}{1+t} = \frac{A+At+Bt}{(1+t)t}$ $\quad \begin{cases} A = 1 \\ A+B = 0 \Rightarrow B = -1 \end{cases}$

$= \int \frac{1}{t}\,dt + \int -\frac{1}{1+t} = \log|t| - \log|1+t| + c = \log\left|\frac{t}{1+t}\right| + c$

- **TRIGONOMETRICHE** $\quad \int f(\operatorname{sen} x, \cos x)\,dx \Rightarrow t = tg\,\frac{x}{2}$

$\int \frac{1}{\operatorname{sen} x}\,dx = \int \frac{\operatorname{sen} x}{\operatorname{sen}^2 x}\,dx = \int \frac{-\operatorname{sen} x}{-1+\cos^2 x} = \int \frac{dt}{t^2-1}$ $\quad \left[\begin{aligned} \cos x &= t \\ -\operatorname{sen} x\,dx &= dt \end{aligned}\right]$

$\Rightarrow \frac{A}{t+1} + \frac{B}{t-1} = \frac{At - A + Bt + B}{t^2-1} \Rightarrow \begin{cases} B - A = 1 \\ A + B = 0 \end{cases} \begin{cases} -A-A = 1 \\ A = -B \end{cases} \begin{cases} A = -1/2 \\ B = 1/2 \end{cases}$

$= \int \frac{-1/2}{t+1} + \int \frac{1/2}{t-1} = -\frac{1}{2}\log|t+1| + \frac{1}{2}\log|t-1| = \log\sqrt{\left|\frac{t-1}{t+1}\right|} + c$

$\left[t = tg\,\frac{x}{2} \Rightarrow arctg\,t = \frac{x}{2} \Rightarrow 2\,arctg\,t = x \Rightarrow \frac{2}{1+t^2}\,dt = dx\right]$

- $\operatorname{sen} x = \frac{2t}{1+t^2}$ $\quad$ - $\cos x = \frac{1-t^2}{1+t^2}$

$1 + tg^2\frac{x}{2} = \frac{1}{\cos^2 x/2} \Rightarrow \cos^2\frac{x}{2} = \frac{1}{1+t^2}$

- $\int \frac{1}{\cos x - 1}\,dx = \int \frac{1}{\frac{1-t^2}{1+t^2} - 1} \cdot \frac{2}{1+t^2}\,dt = \int \frac{1+t^2}{\cancel{1}-t^2-\cancel{1}-t^2} \cdot \frac{2}{1+t^2}\,dt = \int \frac{2}{-2t^2}\,dt = -\frac{1}{t}$

$\left[x = arctg\,t \Rightarrow dx = \frac{dt}{1+t^2} \Rightarrow \cos^2 x = \frac{1}{1+t^2}, \quad \operatorname{sen}^2 x = 1 - \frac{1}{1+t^2} = \frac{t^2}{1+t^2}\right]$

- $\int \frac{tg\,x}{1+\operatorname{sen}^2 x}\,dx = \int \frac{t}{1+\frac{t^2}{1+t^2}} \frac{dt}{1+t^2} = \int \frac{t}{1+2t^2}\,dt$

- $\int \sqrt{1-x^2}\,dx \to$ pongo $x = \operatorname{sen} t \Rightarrow dx = \cos t\,dt$

$= \int \sqrt{1-\operatorname{sen}^2 t} \cdot \cos t\,dt = \int \cos^2 t\,dt \to$ (per parti)

- $\int \sqrt{a-x^2}\,dx \Rightarrow x = a \operatorname{sen} t, \ dx = a \cos t\,dt$

- $\int \sqrt{1+x^2}\,dx \Rightarrow x = \operatorname{senh} t, \ dx = \cosh t\,dt$

$= \int \cosh^2 t\,dt = \frac{1}{4}\int (e^{2t} + e^{-2t} + 2)\,dt = \frac{1}{4}\left(\frac{e^{2t}}{2} - \frac{e^{-2t}}{2} + 2t\right)$

- $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + c$

$\int \frac{1}{\sqrt{1+x^2}}\,dx = sett\cosh x + c$ (funzione inversa del $\cosh x$) $\qquad \int \frac{1}{\sqrt{x^2-1}}\,dx = sett\operatorname{senh} x + c$

• $\int f(\sqrt{ax^2+bx+c}\,; x)\,dx \rightarrow \sqrt{ax^2+bx+c} = \sqrt{a}(t-x)$ • $a \geq 0$

$\sqrt{x^2-1} = t-x \Rightarrow \cancel{x^2}-1 = t^2+\cancel{x^2}-2tx \Rightarrow 2tx = t^2+1 \Rightarrow x = \frac{t^2+1}{2t}$

$dx = \frac{4t^2-2t^2-2}{4t^2}\,dt = \frac{t^2-1}{2t^2}\,dt$

$\int \frac{1}{\sqrt{x^2-1}}\,dx = \int \frac{1}{t-x}\,dt = \int \frac{1}{t-\frac{t^2+1}{2t}} \cdot \frac{t^2-1}{2t^2}\,dt =$

$= \int \frac{1}{\frac{2t^2-t^2-1}{\cancel{2t}}} \cdot \frac{t^2-1}{\cancel{2}t^2 t}\,dt = \int \frac{t^2-1}{t(t^2-1)}\,dt$

• $\int f(\sqrt{ax+b}\,, x)\,dx \rightarrow ax+b = t^2 \Rightarrow a\,dx = 2t\,dt \Rightarrow dx = \frac{2t}{a}\,dt$ $(a<0)$ •

Calcolare la primitiva di $\frac{1}{x\sqrt{x+4}}$ che in 5 vale 1

$\int \frac{1}{x\sqrt{x+4}}\,dx \rightarrow x+4 = t^2\,,\ dx = 2t\,dt$

$= \int \frac{1}{(t^2-4)\cancel{t}}\,2\cancel{t}\,dt = 2\int \frac{1}{t^2-4}\,dt = \log\sqrt{\frac{t-2}{t+2}} + c =$

$\frac{1}{2}\log|t-2| - \frac{1}{2}\log|t+2| = \frac{1}{2}\log|\sqrt{x+4}-2| - \frac{1}{2}\log|\sqrt{x+4}+2| + c \Rightarrow$

$\frac{1}{2}\log|\sqrt{9}-2| - \frac{1}{2}\log|\sqrt{9}+2| + c = 1$

$\frac{1}{2}\log 1 - \frac{1}{2}\log 5 + c = 1 \Rightarrow c = 1 + \frac{1}{2}\log 5$

INTEGRALI IMPROPRI

20/01/15

$f:(a;b] \to \mathbb{R}$, f continua su $[a';b]$ $\forall$ $b \geq a' > a$

se $\exists \lim_{a' \to a^+} \int_{a'}^{b} f(x)\,dx = l$ diremo che f è integrabile in senso improprio sull'intervallo $(a;b]$, e il suo integrale è tale limite

Analogamente $f:[a;b) \to \mathbb{R}$ è integrabile in senso improprio se esiste finito il $\lim_{b' \to b^-} \int_a^{b'} f(x)\,dx$ $\forall b' \in [a;b)$

es. $\int_0^1 \frac{1}{\sqrt{x}}\,dx \to \lim_{\varepsilon \to 0} \int_\varepsilon^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\varepsilon \to 0} \left[2\sqrt{x}\right]_\varepsilon^1 = 2\sqrt{1} - 2\sqrt{0} = 2$!! :)

Al variare del parametro, quando è definito l'integrale?

- trovare tutti i valori $\alpha \geq 0$ t.c. $\int_0^1 \frac{1}{x^\alpha}$ converga.

$\int_0^1 \frac{1}{x^\alpha}\,dx \to \lim_{\varepsilon \to 0} \int_\varepsilon^1 \frac{1}{x^\alpha}\,dx =$
- $\alpha \neq 1 \to \left[\frac{x^{-\alpha+1}}{-\alpha+1}\right]_\varepsilon^1 \quad \frac{1}{-\alpha+1}(1-\varepsilon^{1-\alpha}) \to$ ⊛
- $\alpha = 1 \to [\ln x]_\varepsilon^1 = -\ln\varepsilon \xrightarrow[\varepsilon \to 0]{} +\infty$

⊛
- $\alpha > 1 \to \frac{1}{-\alpha+1}\left(1 - \frac{1}{\varepsilon^{1-\alpha}}\right) \xrightarrow[\varepsilon \to 0]{} +\infty$
- $\alpha < 1 \to \frac{1}{-\alpha+1}(1-\varepsilon^{1-\alpha}) \xrightarrow[\varepsilon \to 0]{} \frac{1}{1-\alpha}$ $\Rightarrow$ converge per $\alpha < 1$
  $(0 \leq \alpha < 1)$

trovare tutti i valori di $\beta \in \mathbb{R}$ t.c. $\int_1^\infty 1/x^\beta\,dx$ ~~sia definito~~ converga:

$\lim_{M \to \infty} \int_1^M \frac{1}{x^\beta}\,dx =$
- $\beta \neq 1 \to \left[\frac{x^{-\beta+1}}{1-\beta}\right]_1^M = \frac{M^{1-\beta} - 1}{1-\beta} \xrightarrow[M \to \infty]{}$ ⊛
- $\beta = 1 \to [\ln x]_1^M = \ln M \xrightarrow[M \to \infty]{} \infty$

⊛
- $\beta < 1 \to \frac{1}{1-\beta}(\infty - 1) = \infty$
- $\beta > 1 \to \frac{1}{1-\beta}\left(\frac{1}{\infty} - 1\right) = \frac{1}{\beta - 1}$ $\Rightarrow$ converge per $\beta > 1$

OSS: $\int_0^\infty \frac{1}{x^\alpha}\,dx = \underbrace{\int_0^1 \frac{1}{x^\alpha}\,dx}_{\text{converge per } \alpha < 1} + \underbrace{\int_1^\infty \frac{1}{x^\alpha}\,dx}_{\text{converge per } \alpha > 1}$ $\Rightarrow \int_0^\infty \frac{1}{x^\alpha}\,dx$ non converge mai $\nexists \alpha \in \mathbb{R}$

OSS: $f, g \geq 0$, continue su $[a;b]$, $a \geq -\infty$

se $f \sim cg$ con $c > 0$ per $x \to a^+$ (ossia $\lim_{x \to a^+} \frac{f(x)}{g(x)} = c$) $\Rightarrow \int_a^b f \sim \int_a^b g$

infatti se $f \sim gc$ per $x \to a$ $\exists U$ intorno di a e $c_1, c_2 > 0$, $c_1 < c_2$ t.c. $c_1 g \leq f \leq c_2 g$ su U

segue ~~da~~ $f/g \to c$ dalla def. di lim. con $\varepsilon = \frac{c}{2} > 0$

$-\frac{c}{2} \leq \frac{f}{g} - c \leq \frac{c}{2} \to \frac{c}{2} \leq \frac{f}{g} \leq \frac{3}{2}c$