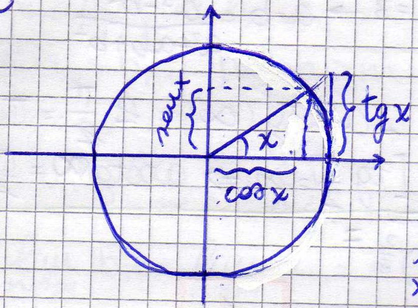
LIMITI NOTEVOLI

11/11/14

① $\lim_{x\to 0} \frac{\operatorname{sen} x}{x} = 1$

[DOBBIAMO DIMOSTRARE CHE: $\lim_{x\to 0^+} \frac{\operatorname{sen} x}{x} = \lim_{x\to 0^-} \frac{\operatorname{sen} x}{x} = 1$]

• $x > 0$



$\operatorname{sen} x \leq x \leq \operatorname{tg} x$

$\cos x \leq \frac{\operatorname{sen} x}{x} \leq 1 \quad \left(x \leq \frac{\operatorname{sen} x}{\cos x}\right)$

$\downarrow$

per il teorema del confronto:

$\lim_{x\to 0^+} \cos x \leq \lim_{x\to 0^+} \frac{\operatorname{sen} x}{\cos x} \leq \lim_{x\to 0^+} 1$

$1 \leq \lim_{x\to 0^+} \frac{\operatorname{sen} x}{x} \leq 1 \Rightarrow \lim_{x\to 0^+} \frac{\operatorname{sen} x}{x} = 1$

• $x < 0 \to \lim_{x\to 0^-} \frac{\operatorname{sen} x}{x} \to$ poniamo $x = -y$

$\lim_{x\to 0^+} \frac{\operatorname{sen}(-y)}{-y} = \lim_{x\to 0^+} \frac{\operatorname{sen} y}{y} = 1 \quad [\operatorname{sen}(-y) = -\operatorname{sen} y]$

DA NON CONFONDERE:

$\lim_{x\to\infty} \frac{\operatorname{sen} x}{x} = 0 \quad \left[-\frac{1}{x} < \frac{\operatorname{sen} x}{x} < \frac{1}{x} \,;\, \frac{1}{x} \text{ per } x\to\infty = 0\right]$

$\lim_{x\to\pi/2} \frac{\operatorname{sen} x}{x} = \frac{2}{\pi}$

CONSEGUENZE: ① $\lim_{x\to 0} \frac{1-\cos x}{x^2} = * = \frac{1}{2}$

$\operatorname{sen}^2 x = 1 - \cos^2 x = (1-\cos x)(1+\cos x)$

$* = \lim_{x\to 0} \frac{1}{x^2} \frac{\operatorname{sen}^2 x}{1+\cos x} = \lim_{x\to 0} \left(\frac{\operatorname{sen} x}{x}\right)^2 \frac{1}{1+\cos x} = (1)^2 \cdot \frac{1}{1+\cos 0} = \frac{1}{2}$

② $\lim_{x\to 0} \frac{\operatorname{tg} x}{x} = \frac{\operatorname{sen} x}{\cos x} \cdot \frac{1}{x} = \lim_{x\to 0} \frac{\operatorname{sen} x}{x} \cdot \frac{1}{\cos x} = 1 \cdot \frac{1}{\cos 0} = 1$

③ $\lim_{x\to 0} \frac{\operatorname{arctg} x}{x} = \begin{cases} x = \tan y \\ y = \arctan x \end{cases} \Rightarrow \lim_{y\to 0} \frac{y}{\operatorname{tg} y} = \lim_{y\to 0} \left(\frac{\operatorname{tg} y}{y}\right)^{-1} = 1^{-1} = 1$

④ $\lim_{x\to 0} \frac{x}{\operatorname{sen} x} = \lim_{x\to 0} \left(\frac{\operatorname{sen} x}{x}\right)^{-1} = 1^{-1} = 1$

altri esempi: ⑤ $\lim_{x\to +\infty} x \operatorname{sen}\left(\frac{1}{x}\right) \Rightarrow$ poniamo $x = \frac{1}{y} \to x\to\infty \Rightarrow y\to 0$

$= \lim_{y\to 0} \frac{1}{y} \operatorname{sen} y = \lim_{y\to 0} \frac{\operatorname{sen} y}{y} = 1$

⑥ $\lim_{x\to 0} \frac{\operatorname{sen} 2x}{x} = \lim_{x\to 0} 2 \frac{\operatorname{sen} 2x}{2x} \Rightarrow$ poniamo $y = 2x$, $x\to 0 \Rightarrow y\to 0$

$\lim_{x\to 0} 2 \frac{\operatorname{sen} y}{y} = 2$

DEFINIAMO: $a_n := \left(1+\frac{1}{n}\right)^n$; $b_n = \left(1+\frac{1}{n}\right)^{n+1}$

$0 < a_n < b_n$ (a_n, b_n sono funzioni con dominio $\mathbb{N}_+$)

b_{n-1} lemma: a_n è strettamente crescente e b_n è strettamente decresc.

[un qualunque b_n è un maggiorante per la successione a_n]

$a_n < b_n < b_m$ per $n > m \Rightarrow \exists \lim_{n\to\infty} a_n = \sup\{a_n | n \in \mathbb{N}_+\} < b_m \ \forall m$

analogamente $\exists \lim_{n\to\infty} b_n = \inf\{b_n | n \in \mathbb{N}_+\} > a_m \ \forall m$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} a_n\left(1+\frac{1}{n}\right) = \lim_{n\to\infty} a_n$$

$\boxed{\text{DEF.}}$ $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = e$: NUMERO DI NEPERO o di EULERO

$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}$ [teorema del confronto] es. per $n=1$ $2 < e < 4$

$\boxed{\text{DEF.}}$ $\ln x := \log_e x$ = logaritmo naturale di x

DIMOSTRIAMO che a_n è strettamente crescente

$$\frac{a_n}{a_{n-1}} = \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{1}{n-1}\right)^{n-1}} = \frac{\left(\frac{n+1}{n}\right)^n}{\left(\frac{n}{n-1}\right)^{n-1}} = \frac{n}{n-1}\left(\frac{n+1}{n}\right)^n\left(\frac{n-1}{n}\right)^n = \frac{n}{n-1}\left(\frac{n^2-1}{n^2}\right)^n =$$

$$= \frac{n}{n-1}\left(1-\frac{1}{n^2}\right)^n > \frac{n}{n-1}\left(1-\frac{1}{n}\right)$$ [PER BERNOULLI: per $n \geq 2$ $(1+h)^n > 1+h\cdot n$ con $h > -1$] $= 1$?

(1) $\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e}$ (2) $\boxed{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e}$

$[x] \leq x < [x]+1$

$$\frac{1}{\underbrace{\left(1+\frac{1}{[x]+1}\right)}_{\to 0}}\cdot\underbrace{\left(1+\frac{1}{[x]+1}\right)^{[x]+1}}_{\to e} \leq \underbrace{\left(1+\frac{1}{x}\right)^x}_{\to e} \leq \underbrace{\left(1+\frac{1}{[x]}\right)^{[x]}}_{\to e}\cdot\underbrace{\left(1+\frac{1}{[x]}\right)}_{\to 1}$$

PER CONFRONTO $\to$ [1^∞: F.I.]

$\begin{cases} \lim_{x\to x_0} f(x) = \ell \\ f(x) \to \ell \text{ per } x \to x_0 \end{cases}$

(3) $\boxed{\lim_{x\to -\infty}\left(1+\frac{1}{x}\right)^x = e}$

$$\left(1+\frac{1}{x}\right)^x \to \text{pongo } \underset{y\to+\infty}{x=-y} \to = \left(1-\frac{1}{y}\right)^{-y} = \frac{1}{\left(1-\frac{1}{y}\right)^y} = \frac{1}{\left(\frac{y-1}{y}\right)^y} = \left(\frac{y}{y-1}\right)^y =$$

$$= \left(\frac{y-1+1}{y-1}\right)^y = \underbrace{\left(1+\frac{1}{y-1}\right)^{y-1}}_{\to e}\underbrace{\left(1+\frac{1}{y-1}\right)}_{\to 1}$$

(4) $\boxed{\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e}$ (segue da (2) e (3) facendo il $\lim_{x\to 0^-}$ e $\lim_{x\to 0^+}$ con il cambio di variab.)

(5) $\boxed{\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1}$

(6) $\boxed{\lim_{x\to 0}\frac{e^x-1}{x} = 1}$ $\to$ poniamo $y = e^x - 1$ $x\to 0 \Leftrightarrow y \to 0$

$$\frac{e^x-1}{x} = \frac{y}{\ln(1+y)} = \frac{1}{\frac{\ln(1+y)}{y}} \to 1$$

(7) $a \in \mathbb{R}$ $\boxed{\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x} = \left(\left(1+\frac{1}{\left(\frac{x}{a}\right)}\right)^{\frac{x}{a}}\right)^a = \boxed{e^a}$

Funzioni Iperboliche

DEF. $\operatorname{senh} x = \operatorname{sh} x = \dfrac{e^x - e^{-x}}{2}$

- $\cosh x = \operatorname{ch} x = \dfrac{e^x + e^{-x}}{2}$
- $\operatorname{tgh} x = \operatorname{sh} x / \operatorname{ch} x$
- $\operatorname{cotgh} x = \operatorname{ch} x / \operatorname{sh} x$

- $\operatorname{sh}(-x) = -\operatorname{sh} x$; $\operatorname{sh} 0 = 0$
- $\operatorname{ch}(-x) = \operatorname{ch} x$; $\operatorname{ch} 0 = 1$
- $\operatorname{ch} x > 1$ per $x > 0$
- $\operatorname{sh} x > 0$ per $x > 0$
- $\lim_{x \to +\infty} \operatorname{sh} x, \operatorname{ch} x = +\infty$

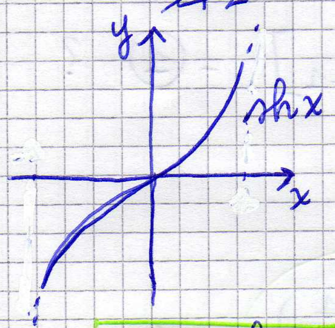
FORMULE

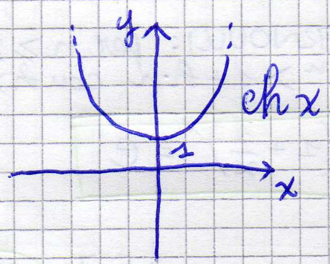
1. $\operatorname{ch}^2 x - \operatorname{sh}^2 x = 1$
2. $\operatorname{ch}(x+y) = \operatorname{ch} x \cdot \operatorname{ch} y + \operatorname{sh} x \cdot \operatorname{sh} y$
3. $\operatorname{sh}(x+y) = \operatorname{sh} x \cdot \operatorname{ch} y + \operatorname{ch} x \cdot \operatorname{sh} y$

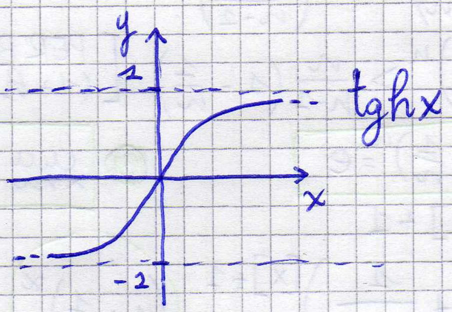
dim 2: $\operatorname{ch}(x+y) = \dfrac{e^x + e^{-x}}{2} \cdot \dfrac{e^y + e^{-y}}{2} + \dfrac{e^x - e^{-x}}{2} \cdot \dfrac{e^y - e^{-y}}{2} =$

$= \dfrac{e^{x+y} + \cancel{e^{x-y}} + \cancel{e^{y-x}} + e^{-(xy)}}{4} + \dfrac{e^{xy} - \cancel{e^{x-y}} - \cancel{e^{y-x}} + e^{-x-y}}{4} =$

$= \dfrac{\cancel{2}(e^{x+y} + e^{-(x+y)})}{\cancel{4}\,2} = \dfrac{e^{x+y} + e^{-(x+y)}}{2} = \operatorname{ch}(x+y)$







(8) $\boxed{\lim_{x \to 0} \dfrac{\operatorname{sh} x}{x} = 1}$

(9) $\boxed{\lim_{x \to 0} \dfrac{\operatorname{ch} x - 1}{x^2} = \dfrac{1}{2}}$

DIM. (8): $\dfrac{\operatorname{sh} x}{x} = \dfrac{1}{2} \dfrac{e^x - e^{-x}}{x} = \dfrac{1}{2} \dfrac{e^x - 1 + 1 - e^{-x}}{x} =$

$= \dfrac{1}{2}\left(\dfrac{e^x - 1}{x} + \dfrac{1 - e^{-x}}{x}\right) \xrightarrow[x \to 0]{} \dfrac{1}{2}(1+1) = 1$

DIM. (9): $\dfrac{\operatorname{ch} x - 1}{x^2} \cdot \dfrac{\operatorname{ch} x + 1}{\operatorname{ch} x + 1} = \dfrac{\operatorname{ch}^2 x - 1}{x^2} \cdot \dfrac{1}{\operatorname{ch} x + 1} = \left(\dfrac{\operatorname{sh} x}{x}\right)^2 \cdot \dfrac{1}{1 + \operatorname{ch} x} \xrightarrow[x \to 0]{} \dfrac{1}{2}$

es. $\lim_{x \to -\infty} x(\operatorname{sen} x - 2) = -\infty$ $\quad -3 < x(\operatorname{sen} x - 2) < -x$