# KAM theory and Celestial Mechanics

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## 1 Introduction

Kolmogorov–Arnold–Moser (KAM) theory deals with the construction of quasi–periodic trajectories in nearly–integrable Hamiltonian systems and it was motivated by classical problems in Celestial Mechanics such as the $n$–body problem. Notwithstanding the formidable bulk of results, ideas and techniques produced by the founders of the modern theory of dynamical systems, most notably by H. Poincaré and G.D. Birkhoff, the fundamental question about the persistence under small perturbations of invariant tori of an integrable Hamiltonian system remained completely open until 1954. In that year A.N Kolmogorov stated what is now usually referred to as the KAM Theorem (in the real–analytic setting) and gave a precise outline of its proof presenting a striking new and powerful method to overcome the so–called small divisor problem (resonances in Hamiltonian dynamics produce, in the perturbation series, divisors which may become arbitrarly small making convergence argument extremely delicate). Subsequently, KAM theory has been extended and applied to a large variety of different problems, including infinite dimensional dynamical systems and partial differential equations with Hamitonian structure. However, establishing the existence of quasi–periodic motions in the $n$–body problem turned out to be a longer story, which only very recently has reached a satisfactory level; the point being that the $n$–body problems present strong degeneracies, which violate the main hypotheses of the KAM Theorem.

In this article we give an account of the ideas and results concerning the construction of quasi–periodic solutions in the planetary $n$ body problem. The synopsis of the article is the following.

In § **2** we give the analytical description of the planetary $(1+n)$–body problem.
In § **3.1** we recall Kolmogorov's 1954 original version of the KAM Theorem, giving an outline of its proof and showing its implications for the simplest many–body case, namely, the restricted, planar, circular three–body problem.
In § **3.2** "Arnold's 1963 theorem" on the existence of a positive measure set of initial data in phase space giving rise to quasi–periodic motions near co–planar and nearly–circular unperturbed Keplerian trajectories is presented. The rest of the section is devoted to the proof of Arnold's Theorem following the hystorical developments: in § **3.2.1** Arnold's proof (1963) for the planar three–body case is presented; in § **3.2.2** the extension to the spatial three–body case due to Laskar and Robutel (1995) is discussed; Herman's proof – in the form given by J. Féjoz in 2004 – of the general spatial $(1+n)$–case is presented in § **3.2.3**.
In § **4** a brief discussion of the construction of lower dimensional elliptic tori bifurcating from the Keplerian unperturbed motions is given (these results have been established in the last two years).
In § **5**, finally, the problem of taking into account real astronomical parameter values is considered and a recent result on an application of (computer–assisted) KAM techniques to the Solar subsystem formed by Sun, Jupiter and the asteroid Victoria is briefly mentioned.

## 2 The planetary $(1+n)$–body problem

The evolution of $(1+n)$ bodies (assimilated to point masses) interacting only through gravitational attraction is governed by Newton's equations. If $u^{(i)} \in \mathbb{R}^3$ denotes the position of the $i^{\text{th}}$ body in a given reference frame and if $m_i$ denotes its mass, then Newton's equations read

$$\frac{d^2 u^{(i)}}{dt^2} = - \sum_{\substack{0\le j\le n\\ j\neq i}} m_j \frac{u^{(i)}-u^{(j)}}{|u^{(i)}-u^{(j)}|^3}\ , \qquad i=0,1,...,n\ ; \tag{2.1}$$

here we have taken the gravitational constant equal to one (which amounts to rescale the time $t$). Eq.'s (2.1) are equivalent to the standard Hamilton's equations corresponding to the Hamiltonian function

$$\mathcal{H}_{\text{New}} := \sum_{i=0}^{n} \frac{|U^{(i)}|^2}{2m_i} - \sum_{0\le i<j\le n} \frac{m_i m_j}{|u^{(i)}-u^{(j)}|}\ , \tag{2.2}$$

where $(U^{(i)}, u^{(i)})$ are standard symplectic variables and the phase space is the "collisionless domain" $\widehat{\mathcal{M}} := \{U^{(i)}, u^{(i)} \in \mathbb{R}^3 : u^{(i)} \neq u^{(j)},\ 0 \le i \neq j \le n\}$; the symplectic form is the standard one: $\sum_i dU^{(i)} \wedge du^{(i)} := \sum_{i,k} dU_k^{(i)} \wedge du_k^{(i)}$; $|\cdot|$ denotes the standard Euclidean norm. Introducing the symplectic coordinate change $(U,u) = \phi_{\text{hel}}(R,r)$,

$$\phi_{\text{hel}} : \left\{ \begin{array}{lll} u^{(0)} = r^{(0)}\ , & u^{(i)} = r^{(0)} + r^{(i)}\ , & (i=1,...,n) \\ U^{(0)} = R^{(0)} - \sum_{i=1}^{n} R^{(i)}\ , & U^{(i)} = R^{(i)}\ , & (i=1,...,n)\ . \end{array} \right. \tag{2.3}$$

one sees that the Hamiltonian $\mathcal{H}_{\text{hel}} := \mathcal{H}_{\text{New}} \circ \phi_{\text{hel}}$ *does not depend upon* $r^{(0)}$; (we recall that a local diffeomorphism is called symplectic if it preserves the symplectic form). This means that $R^{(0)}$ ($\equiv$ total linear momentum) is a global integral of motion. Without loss of generality, one can restrict the attention to the invariant manifold $\mathcal{M}_0 := \{R^{(0)} = 0\}$ ("invariance of Eq. (2.1) by changes of inertial reference frames").

In the "planetary" case one assumes that one of the bodies, say $i = 0$ (the Sun), has mass much larger than that of the other bodies (this accounts for the index "hel", which stands for "heliocentric"). To make transparent the perurbative character of the problem one may introduce the following rescalings. Let

$$m_i = \varepsilon\, \bar{m}_i\ , \qquad X^{(i)} = \frac{R^{(i)}}{\varepsilon\, m_0^{5/3}}\ , \qquad x^{(i)} = \frac{r^{(i)}}{m_0^{2/3}} \qquad (i=1,...,n)\ , \tag{2.4}$$

and rescale time by a factor $\varepsilon m_0^{7/3}$ (which amounts to divide the new Hamiltonian by such a factor); then, the flow of the Hamiltonian $\mathcal{H}_{\text{hel}}$ on $\mathcal{M}_0$ is equivalent to the flow of the Hamiltonian

$$\mathcal{H}_{\text{plt}} := \sum_{i=1}^{n} \left( \frac{|X^{(i)}|^2}{2\mu_i} - \frac{\mu_i M_i}{|x^{(i)}|} \right) + \varepsilon \sum_{1 \le i < j \le n} \left( X^{(i)} \cdot X^{(j)} - \frac{\bar{m}_i \bar{m}_j / m_0^2}{|x^{(i)} - x^{(j)}|} \right), \quad (2.5)$$

on the phase space $\mathcal{M} := \{X^{(i)}, x^{(i)} \in \mathbb{R}^3\colon\ 1 \le i \le n \text{ and } 0 \ne x^{(i)} \ne x^{(j)}\}$ with respect to the standard symplectic form $\sum_{i=1}^n dX^{(i)} \wedge dx^{(i)}$; the mass parameters are defined as

$$M_i := 1 + \varepsilon \frac{\bar{m}_i}{m_0}, \qquad \mu_i := \frac{\bar{m}_i}{m_0 + \varepsilon \bar{m}_i} = \frac{\bar{m}_i}{m_0} \frac{1}{M_i}. \quad (2.6)$$

Let us make a few remarks.

(i) The Hamiltonian $\mathcal{H}_{\text{plt}}^{(0)} := \sum_{i=1}^n \left( \frac{|X^{(i)}|^2}{2\mu_i} - \frac{\mu_i M_i}{|x^{(i)}|} \right)$ is *integrable* and represents the sum of $n$ two–body systems formed by the Sun and the $i^{\text{th}}$ planet (disregarding the interaction with the other planets).

(ii) The transformation $\phi_{\text{hel}}$ in (2.3) preserves the total angular momentum $\widehat{C} := \sum_{i=0}^n U^{(i)} \times u^{(i)}$, which is a (vector–valued) integral for $\mathcal{H}_{\text{New}}$. Thus, also the three components $C_k$, of $C := \sum_{i=1}^n X^{(i)} \times x^{(i)}$, which is proportional to $\widehat{C}$ (and which we shall keep calling "total angular momentum"), are integrals for $\mathcal{H}_{\text{plt}}$. The integrals $C_k$ do not commute: if $\{\cdot,\cdot\}$ denotes the standard Poisson bracket, then $\{C_1, C_2\} = C_3$ (and, cyclically, $\{C_2, C_3\} = C_1$, $\{C_3, C_1\} = C_2$). Nevertheless, one can form two (independent) commuting integrals, such as, for example, $|C|^2$ and $C_3$. This shows that the (spatial) $(1+n)$ body problem has $(3n-2)$ degrees of freedom.

(iii) An important special case is the *planar* $(1+n)$ body problem. In such a case one assumes that all the "single" angular momenta $C^{(i)} := X^{(i)} \times x^{(i)}$ are parallel. In this case the motion takes place on a fixed plane orthogonal to $C$ and (up to a rotation of the reference frame) one can take, as symplectic variables, $X^{(i)}, x^{(i)} \in \mathbb{R}^2$. The Hamiltonian $\mathcal{H}_{\text{pln}}$ governing the dynamics of the *planar* $(1+n)$ body problem is, then, given by the right hand side of (2.5) with $X^{(i)}, x^{(i)} \in \mathbb{R}^2$. Notice that the planar $(1+n)$ body problem has $2n$ degrees of freedom.

(iv) For a deeper understanding of the perturbation theory of the planetary many body problem it is necessary to find "good" sets of symplectic coordinates. The founders of Celestial Mechanics (most notably, Jacobi, Delaunay and Poincaré) have done that for us. In particular, Delaunay introduced an analytic set of symplectic "action–angle" variables. Let us recall the Delaunay variables for the 2–body "reduced Hamiltonian" $\mathcal{H}_{\rm Kep} = \frac{|X|^2}{2\mu} - \frac{\mu M}{|x|}$. Let $\{k_1, k_2, k_3\}$ be a standard orthonormal basis in the $x$–configuration space; let the angular momentum $C = X \times x$ be non parallel to $k_3$ and let the energy $E = \mathcal{H}_{\rm Kep} < 0$. In such a case $x(t)$ describes an ellipse lying in the plane orthogonal to $C$, with focus in the origin and fixed symmetry axes. Let: $a$ be the semi–major axis of the ellipse spanned by $x$; $\imath$ (the inclination) be the angle between $k_3$ and $C$; $G = |C|$; $\Theta = G \cos \imath = C \cdot k_3$; $L = m\sqrt{Ma}$; $\ell$ be the mean anomaly of $x$ (:=$2\pi$ times the normalized area spanned by $x$ measured from the perihelion $P$, which is the point of the ellipse closest to the origin); $\theta$ be the angle between $k_1$ and $N := k_3 \times C$ (:=oriented "node"); $g$ be the argument of the perihelion (:= the angle between $N$ and $(O, P)$). Then (letting $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ )

$$(L, G, \Theta) \in \{L > 0\} \times \{G > \Theta > 0\} , \qquad (\ell, g, \theta) \in \mathbb{T}^3 \tag{2.7}$$

are conjugate symplectic coordinates and if $\phi_{\rm Del}$ is the corresponding symplectic map, then $\mathcal{H}_{\rm Kep} \circ \phi_{\rm Del} = -(\mu^3 M^2)/(2L^2)$.

We note that the Delaunay variables become singular when $C$ is vertical (the node is no more defined) and in the circular limit (the perihelion is not unique). In these cases different variables have to been used.

(v) Let $(X^{(i)}, x^{(i)}) = \phi_{\rm Del}\Big((L_i, G_i, \Theta_i), (\ell_i, g_i, \theta_i)\Big)$. Then $\mathcal{H}_{\rm plt}$ expressed in the Delaunay variables $\{(L_i, G_i, \Theta_i), (\ell_i, g_i, \theta_i) : \ 1 \le i \le n\}$ becomes

$$\mathcal{H}_{\rm Del} = \mathcal{H}^{(0)}_{\rm Del} \ + \ \varepsilon \mathcal{H}^{(1)}_{\rm Del} \ , \qquad \text{with} \qquad \mathcal{H}^{(0)}_{\rm Del} := -\sum_{i=1}^{n} \frac{\mu_i^3 M_i^2}{2L_i^2} \ . \tag{2.8}$$

We remark that the number of action variables on which the integrable Hamiltonian $\mathcal{H}^{(0)}_{\rm Del}$ depends is *striclty less* than the number of degrees of freedom. This "proper degeneracy", as we shall see in next sections, brings in an essential difficulty one has to face in the perturbative approach to the many–body problem. In fact, this feature of the many–body problem is common to several other problems of Celestial Mechanics.

# 3 Maximal KAM tori

## 3.1 Kolmogorov's Theorem and the RPC3BP (1954)

Kolmogorov's invariant tori Theorem deals with the persistence, in nearly–integrable Hamiltonian systems, of Lagrangian (maximal) tori, which, in general, foliate the integrable limit. Kolmogorov stated his theorem in [8] and gave a precise outline of the proof. Let us briefly recall this milestone of the modern theory of Dynamical Systems.

Let $\mathcal{M} := B^d \times \mathbb{T}^d$ ($B^d$ being a $d$–dimensional ball in $\mathbb{R}^d$ centered in the origin) be endowed with the standard symplectic form $dy \wedge dx := \sum dy_i \wedge dx_i$, ($y \in B^d$, $x \in \mathbb{T}^d$). A Hamiltonian function $N$ on $\mathcal{M}$ having a Lagrangian invariant $d$–torus of energy $E$ on which the $N$–flow is conjugated to the linear dense translation $x \to x + \omega t$, $\omega \in \mathbb{R}^d \backslash \mathbb{Q}^d$, can be put in the form

$$N := E + \omega \cdot y + Q(y,x), \quad \text{with} \quad \partial_y^\alpha Q(0,x) = 0,\ \forall\ \alpha \in \mathbb{N}^d,\ |\alpha| \leq 1\ ; \tag{3.9}$$

(as usual, $|\alpha| = \alpha_1 + \cdots + \alpha_d$ and $\partial_y^\alpha = \partial_{y_1}^{\alpha_1} \cdots \partial_{y_d}^{\alpha_d}$); in such a case, the Hamiltonian $N$ is said to be in Kolmogorov normal form. The vector $\omega$ is called the *frequency* vector of the invariant torus $\{y = 0\} \times \mathbb{T}^d$. The Hamiltonian $N$ is said to be *non degenerate* if

$$\det\langle \partial_y^2 Q(0,\cdot)\rangle \neq 0\ , \tag{3.10}$$

where the brackets denote average over $\mathbb{T}^d$ and $\partial_y^2$ the Hessian with respect to the $y$–variablables.

We recall that a vector $\omega \in \mathbb{R}^d$ is said to be *Diophantine* if there exist $\kappa > 0$ and $\tau \geq d - 1$ such that

$$|\omega \cdot k| \geq \frac{\kappa}{|k|^\tau}\ , \qquad \forall\ k \in \mathbb{Z}^d \backslash \{0\}\ ; \tag{3.11}$$

the set $\mathcal{D}^d$ of all Diophantine vectors in $\mathbb{R}^d$ is a set of full Lebesgue measure. We, also, recall that a Hamiltonian trajectory is called *quasi–periodic* with (rationally independent) frequency $\omega \in \mathbb{R}^d$ if it is conjugate to the linear translation $\theta \in \mathbb{T}^d \to \theta + \omega t \in \mathbb{T}^d$.

**Theorem (Kolmogorov, 1954)** *Consider a one–parameter family of real–analytic Hamiltonian functions $H_\varepsilon := N + \varepsilon P$ where $N$ is in Kolmogorov* *normal form (as in* (3.9)*) and* $\varepsilon \in \mathbb{R}$*. Assume that* $\omega$ *is Diophantine and that* $N$ *is non degenerate. Then, there exists* $\varepsilon_0 > 0$ *and for any* $|\varepsilon| \le \varepsilon_0$ *a real–analytic symplectic transformation* $\phi_\varepsilon : \mathcal{M} \to \mathcal{M}$ *putting* $H_\varepsilon$ *in Kolmogorov normal form,* $H_\varepsilon \circ \phi_\varepsilon = N_\varepsilon$*, with* $N_\varepsilon := E_\varepsilon + \omega \cdot y' + Q_\varepsilon(y', x')$*. Furthermore,* $|E_\varepsilon - E|$*,* $\|\phi_\varepsilon - \mathrm{id}\|_{C^2}$*, and* $\|Q_\varepsilon - Q\|_{C^2}$ *are small with* $\varepsilon$*.*

In other words the Lagrangian unperturbed torus $\mathcal{T}_0 := \{y = 0\} \times \mathbb{T}^d$ persists under small perturbation and is smoothly deformed into the $H_\varepsilon$–invariant torus $\mathcal{T}_\varepsilon := \phi_\varepsilon\left(\{y' = 0\} \times \mathbb{T}^d\right)$; the dynamics on such torus, for all $|\varepsilon| \le \varepsilon_0$, consists of dense quasi–periodic trajectories. Note that the $H_\varepsilon$–flow on $\mathcal{T}_\varepsilon$ is analytically conjugated by $\phi_\varepsilon$ to the translation $x' \to x' + \omega t$ with the same frequency vector of $N$, while the energy of $\mathcal{T}_\varepsilon$, namely $E_\varepsilon$, is in general different from the energy $E$ of $\mathcal{T}_0$.

Kolmogorov's proof is based on an iterative (Newton) scheme, which goes as follows. The map $\phi_\varepsilon$ is obtained as $\lim_{k\to\infty} \phi^{(1)} \circ \cdots \circ \phi^{(k)}$, where the $\phi^{(j)}$'s are ($\varepsilon$–dependent) symplectic transformations of $\mathcal{M}$ closer and closer to the identity. It is enough to describe the construction of $\phi^{(1)}$; $\phi^{(2)}$ is then obtained by replacing $H_\varepsilon$ with $H_\varepsilon \circ \phi^{(1)}$, and so on. The map $\phi^{(1)}$ is $\varepsilon$–close to the identity and it is generated by $g(y', x) := y' \cdot x + \varepsilon\Big(b \cdot x + s(x) + y' \cdot a(x)\Big)$ where $s$ and $a$ are (respectively, scalar and vector–valued) real–analytic functions on $\mathbb{T}^d$ with zero average and $b \in \mathbb{R}^d$; we recall that this means that the symplectic map $\phi^{(1)} : (y', x') \to (y, x)$ is implicitely given by the relations $y = \partial_x g$ and $x' = \partial_{y'} g$. It is easy to see that there exists a unique $g$ of the above form such that, for a suitable $\varepsilon_0 > 0$,

$$H_\varepsilon \circ \phi^{(1)} = E_1 + \omega \cdot y' + Q_1(y', x') + \varepsilon^2 P_1 \ , \qquad \forall\ |\varepsilon| \le \varepsilon_0 \ , \tag{3.12}$$

with $\partial_y^\alpha Q_1(0, x') = 0$, for any $\alpha \in \mathbb{N}^d$ and $|\alpha| \le 1$; here $E_1$, $Q_1$ and $P_1$ depend on $\varepsilon$ and, for a suitable $c_1 > 0$ and for $|\varepsilon| \le \varepsilon_0$, $|E - E_1| \le c_1|\varepsilon|$, $\|Q - Q_1\|_{C^2} \le c_1|\varepsilon|$, $\|P_1\|_{C^2} \le c_1$.

Notice that the symplectic transformation $\phi^{(1)}$ is actually the composition of two "elementary" transfomations: $\phi^{(1)} = \phi_1^{(1)} \circ \phi_2^{(1)}$ where $\phi_2^{(1)} : (y', x') \to (\eta, \xi)$ is the symplectic lift of the $\mathbb{T}^d$–diffeomorphism given by $x = \xi + \varepsilon a(\xi)$ (i.e., $\phi_2^{(1)}$ is the symplectic map generated by $y' \cdot \xi + \varepsilon y' \cdot a(\xi)$), while $\phi_1^{(1)}$ : $(\eta, \xi) \to (y, x)$ is the angle–dependent action translation generated by $\eta \cdot x + \varepsilon(b \cdot x + s(x))$; $\phi_2^{(1)}$ acts in the "angle direction" and straightens out the flow up to order $O(\varepsilon^2)$, while $\phi_1^{(1)}$ acts in the "action direction" and is needed to keep the frequency of the torus fixed.

Since $H_\varepsilon \circ \phi_1 =: N_1 + \varepsilon^2 P_1$ is again a perturbation of a non degenerate Kolmogorov normal form (with same frequency vector $\omega$) one can repeat the construction obtaining a new Hamiltonian of the form $N_2 + \varepsilon^4 P_2$. Iterating, after $k$ steps, one gets a Hamiltonian $N_k + \varepsilon^{2^k} P_k$. Carrying out the (straightforward but lengthy) estimates one can check that $\|P_k\|_{C^2} \le c_k \le c^{2^k}$, for a suitable constant $c > 1$ independent of $k$ (the fast growth of the constant $c_k$ is due to the presence of the small divisors appearing in the explicit construction of the symplectic transformations $\phi^{(j)}$). Thus, it is clear that taking $\varepsilon_0$ small enough the iterative procedure converges (super exponentially fast) yielding the thesis of the above theorem.

(vi) While the statement of the invariant tori theorem and the outline of the proof are very clearly explained in [8], Kolmogorov did not fill out the details nor gave any estimates. Some years later (1963), V.I. Arnold [2], published a detailed proof, which, however, did not follow Kolmogorov's idea. In the same years, J.K. Moser published his invariant curve theorem (for area–preserving twist diffeomorphisms of the annulus) in smooth setting. The bulk of techniques and theorems stemmed out from these works is normally referred to as *KAM theory*; for reviews see, e.g, [9] or [1]. A very complete version of the "KAM theorem" both in the real–analytic and in the smooth case (with optimal smoothness assumptions) is given in [15]; the proof of the real–analytic part is based on Kolmogorov's scheme. The KAM theory of Michel Herman, used in his approach to the planetary problem (see § 3.2.3 below), is based on the abstract functional theoretical approach of R. Hamilton (which, in turn, is a development of Nash–Moser implicit function theorem; see [9] for references); it is interesting, however, to note that the heart of Herman's KAM method is based on the above mentioned Kolmogorov's transformation $\phi^{(1)}$ (compare [11]).

(vii) In the nearly–integrable case one considers a one parameter family of Hamiltonians $H_0(I) + \varepsilon H_1(I, x)$ with $(I, x) \in \mathcal{M} := U \times \mathbb{T}^d$ standard symplectic action–angle variables; $U$ being an open subset of $\mathbb{R}^d$. When $\varepsilon = 0$ the phase space $\mathcal{M}$ is foliated by $H_0$–invariant tori $\{I_0\} \times \mathbb{T}^d$, on which the flow is given by $x \to x + \partial_y H_0(I_0)\, t$. If $I_0$ is such that $\omega := \partial_y H_0(I_0)$ is Diophantine and if $\det \partial_y^2 H_0(I_0) \neq 0$, then from Kolmogorov's Theorem it follows that the torus $\{I_0\} \times \mathbb{T}^d$ persists under perturbation. In fact, introduce the symplectic variables $(y, x)$ with

$y = I - I_0$ and let $N(y) := H_0(I_0 + y)$, which by Taylor's formula can be written as $H_0(I_0) + \omega \cdot y + Q(y)$ with $Q(y)$ quadratic in $y$ and $\partial_y^2 Q(0) = \partial_y^2 H_0(I_0)$ invertible. One can then apply Kolmogorov's Theorem with $P_1(y, x) := H_1(I_0 + y, x)$.

Notice that Kolmogorov's nondegeneracy condition $\det \partial_y^2 H_0(I_0) \neq 0$ simply means that the *frequency map*

$$I \in B^d \subset U \to \omega(I) := \partial_y H_0(I) \tag{3.13}$$

is a local diffeomorphism ($B^d$ being a ball around $I_0$).

(viii) The symplectic structure implies that, if $n$ denotes the number of degrees of freedom (i.e., half of the dimension of the phase space), then $d \leq n$; if $d = n$ the quasi–periodic motion is called maximal. Kolmogorov's Theorem gives sufficient conditions in order to get maximal quasi–periodic solutions. In fact, Kolmogorov's nondegeneracy condition is an open condition and the set of Diophantine vectors is a set of full Lebesgue measure. Thus, in general, Kolmogorov's Theorem yields a positive invariant measure set spanned by maximal quasi–periodic trajectories.

As mentioned above, the planetary many–body models are properly degenerate and violate Kolmogorov's non degeneracy conditions and, hence, Kolmogorov's Theorem – clearly motivated by Celestial Mechanics – cannot be applied.

There is, however, an important case to which a slight variation of Kolmogorov's Theorem can be applied (Kolmogorov did not mention this in [8]). The case we are referring to is the simplest nontrivial three–body problem, namely the restricted, planar, circular three body problem (RPC3BP for short). This model, largely investigated by Poincaré, deals with an asteroid of "zero mass" moving on the plane containing the trajectory of two unperturbed major bodies (say, Sun and Jupiter) revolving on a Keplerian circle. The mathematical model for the restricted three body problem is obtained by taking $n = 2$ and setting $m_2 = 0$ in (2.1): the equations for the two major bodies ($i = 0, 1$) decouple from the equation for the asteroid ($i = 2$) and form an integrable two body–system; the problem consists, then, in studying the evolution of the asteroid $u^{(2)}(t)$ in the given gravitational field of the primaries. In the circular, planar case the motion of the two primaries is assumed to be circular and the motion of the asteroid is assumed to take place on the plane containing the motion of the two primaries; in fact (to avoid collisions) one considers either inner or outer (with respect to the circle described by the relative motion of the primaries) asteroid motions. To describe the Hamiltonian $\mathcal{H}_{\rm rcp}$ governing the motion of the RCP3BP problem, introduce planar Delaunay variables $\big((L,G),(\ell,\hat g)\big)$ for the asteroid (better, for the reduced heliocentric Sun–asteroid system). Such variables, which are closely related to the above (spatial) Delaunay variables, have the following physical interpretation: $G$ is proportional to the absolute value of the angular momentum of the asteroid, $L$ is proportional to the square root of the semi–major axis of the istantanueos Sun–asteroid ellipse, $\ell$ is the mean anomaly of the asteroid, while $\hat g$ the argument of the perihelion. Then, in suitably normalized units, the Hamiltonian governing the RPC3BP is given by

$$\mathcal{H}_{\rm rcp}(L,G,\ell,g;\varepsilon):=-\frac{1}{2L^2}-G+\varepsilon\mathcal{H}_1(L,G,\ell,g;\varepsilon)\ , \qquad (3.14)$$

where: $g:=\hat g-\tau$, $\tau\in\mathbb{T}$ being the longitude of Jupiter; the variables $\big((L,G),(\ell,g)\big)$ are symplectic coordinates (w.r.t. the standard symplectic form); the normalizations have been chosen so that the relative motion of the primary bodies is $2\pi$ periodic and their distance is one; the parameter $\varepsilon$ is (essentially) the ratio between the masses of the primaries; the perturbation $\mathcal{H}_1$ is the function $x^{(2)}\cdot x^{(1)}-\frac{1}{|x^{(2)}-x^{(1)}|}$ expressed in the above variables, $x^{(2)}$ being the heliocentric coordinate of the asteroid and $x^{(1)}$ that of the planet (Jupiter): such function is real–analytic on $\{0<G<L\}\times\mathbb{T}^2$ and for small $\varepsilon$; for complete details, see, e.g., [6], § 3.1÷§ 3.3.

The integrable limit $\mathcal{H}^{(0)}_{\rm rcp}:=\mathcal{H}_{\rm rcp}|_{\varepsilon=0}=-\frac{1}{2L^2}-G$ has vanishing Hessian and, hence, violates Kolmogorov's nondegeneracy condition (in the sense described in item (vii) above). However, there is another nondegeneracy condition which leads to a simple variation of Kolmogorov's Theorem, as we proceed to explain briefly.

Kolmogorov's non degeneracy condition $\det^2_y H_0(I_0)\neq 0$ allows to fix $d$ parameters, namely the $d$ components of the (Diophantine) frequency vector $\omega=\partial_y H_0(I_0)$. Instead of fixing such parameters one may fix the energy $E=H_0(I_0)$ together with the direction $\{s\omega : s\in\mathbb{R}\}$ of the frequency vector: for example, in a neighborhood where $\omega_d\neq 0$, one can fix $E$ and $\omega_i/\omega_d$ for

$1 \le i \le d-1$. Notice also that if $\omega$ is Diophantine then so is $s\omega$ for any $s \neq 0$ (with same $\tau$ and rescaled $\kappa$). Now, it is easy to check that the map $I \in H_0^{-1}(E) \to \left(\frac{\omega_1}{\omega_d}, ..., \frac{\omega_{d-1}}{\omega_d}\right)$ is (at fixed energy $E$) a local diffeomorphism if and only if the $(d+1)\times(d+1)$ matrix $\begin{pmatrix} \partial_y^2 H_0 & \partial_y H_0 \\ \partial_y H_0 & 0 \end{pmatrix}$ evaluated at $I_0$ is invertible (here the vector $\partial_y H_0$ in the upper right corner has to be interpreted as a column while the vector $\partial_y H_0$ in the lower left corner has to be interpreted as a row). Such "iso–energetic nondegeneracy" condition, rephrased in terms of Kolmogorov's normal forms, becomes

$$\det \begin{pmatrix} \langle \partial_y^2 Q(0,\cdot)\rangle & \omega \\ \omega & 0 \end{pmatrix} \neq 0 \ . \tag{3.15}$$

Kolmogorov's Theorem can be easily adapted to the fixed energy case. Assuming that $\omega$ is Diophantine and that $N$ is iso–energetically non degenerate then the same conclusion as in Kolmogorov's Theorem holds with $N_\varepsilon := E + \omega_\varepsilon \cdot y' + Q_\varepsilon(y', x')$ where $\omega_\varepsilon = \alpha_\varepsilon\, \omega$ and $|\alpha_\varepsilon - 1|$ is small with $\varepsilon$.

In the RCP3BP case the iso–energetic nondegeneracy is met, since

$$\det \begin{pmatrix} \partial^2_{(L,G)} \mathcal{H}^{(0)}_{\rm rcp} & \partial_{(L,G)} \mathcal{H}^{(0)}_{\rm rcp} \\ \partial_{(L,G)} \mathcal{H}^{(0)}_{\rm rcp} & 0 \end{pmatrix} = \frac{3}{L^4} \ .$$

Therefore, one can conclude that on each negative energy level the RCP3BP admits a positive measure set of phase points, whose time evolution lies on two–dimensional invariant tori (on which the flow is analytically conjugate to linear translation by a Diophantine vector), provided the mass ratio of the primary bodies is small enough; such persistent tori are a slight deformation of the unperturbed "Keplerian" tori corresponding to the asteroid and the Sun revolving on a Keplerian ellipse on the plane where the Sun and the major planet describe a circular orbit.

In fact, one can say more. The phase space for the RCP3BP is four dimensional, the energy levels are three dimensional and Kolmogorov's invariant tori are two–dimensional. Thus, a Kolmogorov torus separates the energy level, on which it lies, in two invariant components and two Kolmogorov's tori form the boundary of a compact invariant region so that any motion starting in such region will never leave it. Thus the RCP3BP is "totally stable": in a neighborhood of any phase point of negative energy, if the mass ratio of the primary bodies is small enough, the asteroid stays for ever on a nearly Keplerian ellipse with nearly fixed orbital elements $L$ and $G$.

## 3.2 Arnold's Theorem

Let us go back to the planetary $(1 + n)$ body problem, governed by the Hamiltonian $\mathcal{H}_{\rm plt}$ in (2.5). In the integrable approximation, governed by the Hamiltonian $\mathcal{H}^{(0)}_{\rm plt}$, the $n$ planets describe Keplerian ellipses focused on the Sun. In [3], Chapt. III, V.I. Arnold stated the following

**Theorem (Arnold, 1963)** *Let $\varepsilon > 0$ be small enough. Then, there exists a bounded, $\mathcal{H}_{\rm plt}$-invariant set $\mathcal{F}(\varepsilon) \subset \mathcal{M}$ of positive Lebesgue measure corresponding to planetary motions with bounded relative distances; $\mathcal{F}(0)$ corresponds to Keplerian ellipses with small eccentricities and small relative inclinations. Furthermore, the trajectories starting in $\mathcal{F}(\varepsilon)$ are quasi–periodic with Diophantine frequencies.*

This theorem represents a major achievement in Celestial Mechanics solving a more than tri–centennial mathematical problem. In [3], Arnold gave a complete proof of this result only in the planar three–body case and gave some indications of how to extend his approach to the general situation. However, to give a full proof of Arnold's Theorem in the general case turned out to be more than a technical problem and new ideas were needed: the complete proof (due, essentially, to M. Herman) has been given only in 2004.

In the following subsections, we review briefly the history and the ideas related to the proof of Arnold's Theorem. As for credits: the proof of Arnold's Theorem in the planar 3BP case is due to Arnold himself [3]; the spatial 3BP case is due to J. Laskar and P. Robutel [13], [14]; the general case is due to M. Herman and J. Féjoz [12], [11]. The exposition that we give does not always follows the original references.

### 3.2.1 The planar 3–body problem

Recall the Hamitonian $\mathcal{H}_{\rm pln}$ of the planar $(1+n)$–body problem given in item (iii) of § 2. A convenient set of symplectic variables for nearly–circular motions are the "planar Poincaré variables". To describe such variables consider a single, planar two-body system with Hamiltonian

$$\frac{|X|^2}{2\mu} - \frac{\mu M}{|x|}\ , \quad X \in \mathbb{R}^2,\ 0 \neq x \in \mathbb{R}^2\ , \quad \Big(\text{w.r.t. } \ dX \wedge dx\Big)\ , \tag{3.16}$$

and introduce – as done before formula (3.14) for $\mathcal{H}^{(0)}_{\rm rcp}$ – planar Delaunay variables $\big((L,G),(\ell,g)\big)$ (here $g=\hat g=$ argument of the perihelion). To remove the singularity of the Delaunay variables near zero eccentricities, Poincaré introduced variables $\big((\Lambda,\eta),(\lambda,\xi)\big)$ defined by the following formulae:

$$\left\{\begin{array}{ll}\Lambda=L\,, & H=L-G\\ \lambda=\ell+g & h=-g\end{array}\right. \;,\quad \left\{\begin{array}{l}\sqrt{2H}\ \cos h=\eta\\ \sqrt{2H}\ \sin h=\xi\end{array}\right. \;. \tag{3.17}$$

As Poincaré showed, such variables are symplectic and analytic in a neighborhood of $(0,\infty)\times\mathbb{T}\times\{0,0\}$; notice that the symplectic map $\big((\Lambda,\eta),(\lambda,\xi)\big)\to(X,x)$ depends on the parameters $\mu$, $M$ and $\varepsilon$. In Poincaré variables, the 2–body Hamiltonian in (3.16) becomes $-\kappa/(2\Lambda^2)$, with $\kappa:=(\mu/m_0)^3/M$.

Now, re–insert the index $i$, let $\phi_i\,:\,\big((\Lambda_i,\eta_i),(\lambda_i,\xi_i)\big)\to(X^{(i)},x^{(i)})$ and $\phi(\Lambda,\eta,\lambda,\xi)=\big(\phi_1(\Lambda_1,\eta_1,\lambda_1,\xi_1),...,\phi_n(\Lambda_n,\eta_n,\lambda_n,\xi_n)\big)$. Then, the Hamiltonian for the planar $(1+n)$–body problem takes the form

$$\begin{aligned}&\mathcal{H}_{\rm pln}\circ\phi=\mathcal{H}_0(\Lambda)+\varepsilon\mathcal{H}_1(\Lambda,\lambda,\eta,\xi)\ ,\\ &\mathcal{H}_0:=-\frac12\sum_{i=1}^n\frac{\kappa_i}{\Lambda_i^2}\ ,\qquad \kappa_i:=\Big(\frac{\mu_i}{m_0}\Big)^3\frac{1}{\mathrm{M}_i}\ ,\\ &\mathcal{H}_1:=\mathcal{H}_1^{\rm compl}+\mathcal{H}_1^{\rm princ}\ ,\end{aligned} \tag{3.18}$$

where the so-called “complementary part” $\mathcal{H}_1^{\rm compl}$ and the “principal part” $\mathcal{H}_1^{\rm princ}$ of the perturbation are, respectively, the functions

$$\sum_{1\le i<j\le n}X^{(i)}\cdot\ X^{(j)}\qquad\text{and}\qquad\sum_{1\le i<j\le n}\frac{\mu_i\mu_j}{m_0^2}\,\frac{1}{|x^{(i)}-x^{(j)}|} \tag{3.19}$$

expressed in Poincaré variables.

The scheme of proof of Arnold’s Theorem in the planar, 3–body case (one star, $n=2$ planets) goes as follows. The Hamiltonian is given by (3.18) with $n=2$; the phase space is eight dimensional (4 degrees of freedom). This system, as mentioned several times, is properly degenerate and Kolmogorov’s Theorem cannot be applied directly; furthermore a full (four–dimensional) set of action variables needs to be identified.

A first observation is that, in the planetary model, there are “fast variables” (the $\lambda_i$’s describing the revolutions of the planets) and “secular variables” (the $\eta_i$'s and $\xi_i$'s describing the variations of position and shape of the istantaneuous Keplerian ellipses). By *averaging theory* (see, e.g., [1]) one can "neglect", in nonresonant regions, the fast–angle dependence up to high order in $\varepsilon$ obtaining an effective Hamiltonian, which, up to $O(\varepsilon^2)$, is given by the "secular" Hamiltonian

$$\mathcal{H}_{\rm sec} := \mathcal{H}_0(\Lambda) + \varepsilon\overline{\mathcal{H}}_1(\Lambda,\eta,\xi)\ , \quad \text{with} \quad \overline{\mathcal{H}}_1(\Lambda,\eta,\xi) := \int \mathcal{H}_1 \frac{d\lambda}{(2\pi)^2}\ ; \quad (3.20)$$

"nonresonant region" means, here, an open $\Lambda$–set where $\partial_\Lambda \mathcal{H}_0 \cdot k \neq 0$ for $k \in \mathbb{Z}^2$, $|k_1| + |k_2| \leq K$ and for a suitable $K \geq 1$.

In order to analyze the secular Hamiltonian, we shall briefly consider $\overline{\mathcal{H}}_1$ as a function of the symplectic variables $\eta$ and $\xi$, regarding the "slow actions" $\Lambda_i$ as parameters.

For symmetry reasons, $\overline{\mathcal{H}}_1$ is even in $(\eta,\xi)$ and the point $(\eta,\xi) = (0,0)$ is an elliptic equilibrium for $\overline{\mathcal{H}}_1$: the eigenvalues of the matrix $S\partial^2_{(\eta,\xi)}\overline{\mathcal{H}}_1(\Lambda,0,0)$, $S$ being the standard symplectic matrix, are purely immaginary numbers $\{\pm i\Omega_1, \pm i\Omega_2\}$. The real numbers $\{\Omega_i\}$ are symplectic invariants of the secular Hamiltonian and are usually called first (or linear) Birkhoff invariants. In a neighborhood of an elliptic equilibrium one can use Birkhoff's normal form theory (see, e.g., [16]): if the linear invariants $(\Omega_1,\Omega_2)$ are nonresonant up to order $r$ (i.e., if $\Omega\cdot k := \Omega_1 k_1 + \Omega_2 k_2 \neq 0$ for any $k \in \mathbb{Z}^2$ such that $|k_1| + |k_2| \leq r$), then one can find a symplectic transformation $\phi_{\rm Bir}$ so that

$$\overline{\mathcal{H}}_1 \circ \phi_{\rm Bir} = F(J_1,J_2;\Lambda) + o_r\ , \qquad J_j = \frac{\eta_j^2 + \xi_j^2}{2}\ , \qquad (3.21)$$

where $F$ is a polynomial of degree $[r/2]$ of the form $\Omega_1 J_1 + \Omega_2 J_2 + \frac{1}{2}\mathcal{M}J\cdot J + \cdots$, $\mathcal{M} = \mathcal{M}(\Lambda)$ being a $(2\times 2)$ matrix (and $o_r/|J|^{r/2} \to 0$ as $|J| \to 0$). Arnold, using computations performed by Le Verrier, checked the nonresonance condition up to order $r = 6$ in the asymptotic regime $a_1/a_2 \to 0$ (where $a_i$ denote the semi–major axes of the the approximate Keplerian ellipses of the two planets): these computations represent one of the most delicate part of the paper.

Thus, combining averaging theory and Birkhoff normal form theory one can construct a symplectic change of variables defined on an open subset of the phase space (avoiding some linear resonances) $(\Lambda,\lambda,\eta,\xi) \to (\Lambda',\lambda',J,\varphi)$, where $\eta_j + i\xi_j = \sqrt{2J_j}\exp(i\varphi_j)$, casting the three–body Hamiltonian into the form

$$\begin{aligned}&\mathcal{H}_0(\Lambda') + \varepsilon\Big(\Omega(\Lambda')\cdot J + \frac{1}{2}\mathcal{M}(\Lambda')J\cdot J\Big) + \varepsilon^2\mathcal{F}_1(\Lambda', J) + \varepsilon^p\mathcal{F}_2(\Lambda', \lambda', J, \varphi)\\ &:= \widetilde{\mathcal{H}}_0(\Lambda', J; \varepsilon) + \varepsilon^p\mathcal{F}_2(\Lambda', \lambda', J, \varphi)\ ,\end{aligned} \tag{3.22}$$

for a suitable pre–fixed order $p \geq 3$; notice that the nonresonance condition needed to apply averaging theory is not particularly hard to check since it involves the unperturbed and completely explicit Kepler Hamiltonian $\mathcal{H}_0$. The idea is now to consider $\varepsilon^p\mathcal{F}_2$ as a perturbation of the completely integrable Hamiltonian $\widetilde{\mathcal{H}}_0$ and to apply Kolmogorov's Theorem. Last thing to check is Kolmogorov's nondegenearcy condition, which since

$$\det \partial^2_{(\Lambda', J)}\widetilde{\mathcal{H}}_0(\Lambda', J'; \varepsilon) = \varepsilon^2\Big((\det \mathcal{H}_0'')\det\mathcal{M} + O(\varepsilon)\Big)\ ,$$

amounts to check the invertibility of the matrix $\mathcal{M}$. Such condition is also checked in [3] with the aids of le Verrier's tables and in the asymptotic regime $a_1/a_2 \to 0$.

### 3.2.2 The spatial 3–body problem

In order to extend the previous argument to the spatial case, Arnold suggested to connect the planar and spatial case through a limiting procedure. Such strategy presents analytical problems (the symplectic variables for the spatial case become singular in the planar limit), which have not been overcome. However, the particular structure of the three–body case allows to derive a four–degree–of–freedom Hamiltonian, to which the proof of the planar case may be easily adapted. The procedure, which we are going to describe, is based on the classical *Jacobi's reduction of the nodes*.

First, we introduce a convenient set of symplectic variables. Let, for $i = 1, 2$, $\big((L_i, G_i, \Theta_i), (\ell_i, g_i, \theta_i)\big)$ denote the Delaunay variables introduced in item (v) and (iv) above: these are the Delaunay variables associated to the two two–body system Sun–$i^{\text{th}}$ planet. Then, as Poincaré showed, the variables $\big((\Lambda_i^*, \lambda_i^*), (\eta_i^*, \xi_i^*), (\Theta_i, \theta_i)\big)$, where

$$\left\{\begin{array}{l}\Lambda_i^* = L_i\\ \lambda_i^* = \ell_i + g_i\end{array}\right. \qquad \left\{\begin{array}{l}\eta_i^* = \sqrt{2(L_i - G_i)}\cos g_i\\ \xi_i^* = -\sqrt{2(L_i - G_i)}\sin g_i\end{array}\right. \tag{3.23}$$

are symplectic and analytic near circular, non co–planar motions; for a detailed discussion of these and other sets of interesting classical variables see, e.g., [4] and references therein; the upper asterisc is introduced to avoid confusion with a closely related but different set of Poincaré variables (see below). Let us denote by

$$\mathcal{H}_{3\text{bp}} := \mathcal{H}^{(0)}(\Lambda^*) + \varepsilon \mathcal{H}^{(1)}(\Lambda^*, \lambda^*, \eta^*, \xi^*, \Theta, \theta)$$

the Hamiltonian (2.8) (with $n = 2$) expressed in terms of the symplectic variables $\big((\Lambda^*, \lambda^*), (\eta^*, \xi^*), (\Theta, \theta)\big)$, $\Lambda^* = (\Lambda_1^*, \Lambda_2^*)$, etc. Recalling the physical meaning of the Delanay variables, one realizes that $\Theta_1 + \Theta_2$ is the vertical component, $C_3 = C \cdot k_3$, of the total argument $C = C^{(1)} + C^{(2)}$, where $C^{(i)}$ denotes the angular momentum of the $i^{\text{th}}$ planet with respect to the origin of an inertial heliocentric frame $\{k_1, k_2, k_3\}$. This suggests to introduce the symplectic variables

$$(\Lambda^*, \lambda^*, \eta^*, \xi^*, \Psi, \psi) = \phi(\Lambda^*, \lambda^*, \eta^*, \xi^*, \Theta, \theta)$$

with $(\Psi_1, \Psi_2, \psi_1, \psi_2) := (\Theta_1, \Theta_1 + \Theta_2, \theta_1 - \theta_2, \theta_2)$. Let us denote by

$$\mathcal{H}^*_{3\text{bp}} := \mathcal{H}_{3\text{bp}} \circ \phi^{-1}$$

the Hamiltonian of the spatial three–body problem in these symplectic variables. Since the Poisson bracket of $\Psi_2 = \Theta_1 + \Theta_2$ and $\mathcal{H}^*_{3\text{bp}}$ vanishes ($C_3$ being an integral for the $\mathcal{H}_{3\text{bp}}$–flow), the conjugate angle $\psi_2$ is cyclic for $\mathcal{H}^*_{3\text{bp}}$, i.e.,

$$\mathcal{H}^*_{3\text{bp}} = \mathcal{H}^*_{3\text{bp}}(\Lambda^*, \lambda^*, \eta^*, \xi^*, \Psi_1, \Psi_2, \psi_1) \ .$$

Now, (because the total angular momentum $C$ is preserved) we may restrict our attention to the 10-dimensional invariant (and symplectic) submanifold $\mathcal{M}_{\text{ver}}$ defined by fixing the total angular momentum to be vertical. Such submanifold is easily described in terms of Delaunay variables; in fact, $C \cdot k_1 = 0 = C \cdot k_2$, is equivalent to

$$\theta_1 - \theta_2 = \pi \qquad \text{and} \qquad G_1^2 - \Theta_1^2 = G_2^2 - \Theta_2^2 \ . \tag{3.24}$$

Thus, we find that $\mathcal{M}^*_{\text{ver}} := \phi(\mathcal{M}_{\text{ver}})$ is given by

$$\mathcal{M}^*_{\text{ver}} = \Big\{ \psi_1 = \pi \ , \quad \Psi_1 = \widehat{\Psi}_1(\Lambda^*, \eta^*, \xi^*; \Psi_2) \Big\}$$

with

$$\widehat{\Psi}_1 := \frac{\Psi_2}{2} + \frac{(\Lambda_1^* - H_1^*)^2 - (\Lambda_2^* - H_2^*)^2}{2\Psi_2}\ , \qquad H_i^* := \frac{\eta_i^{*2} + \xi_i^{*2}}{2}\ .$$

Since $\mathcal{M}^*_{\rm ver}$ is invariant for the flow $\phi^t_*$ of $\mathcal{H}^*_{\rm 3bp}$, $\psi_1(t) \equiv \pi$ and $\dot{\psi}_1 \equiv 0$ for motions starting on $\mathcal{M}^*_{\rm ver}$, which implies that $(\partial_{\Psi_1}\mathcal{H}^*_{\rm 3bp})|_{\mathcal{M}^*_{\rm ver}} = 0$. This fact allows to introduce, for fixed values of the vertical angular momentum $\Psi_2 = c \neq 0$, the following reduced Hamiltonian:

$$\mathcal{H}^c_{\rm red}(\Lambda^*, \lambda^*, \eta^*, \xi^*) := \mathcal{H}^*_{\rm 3bp}(\Lambda^*, \lambda^*, \eta^*, \xi^*, \widehat{\Psi}_1(\Lambda^*, \eta^*, \xi^*; c), c, \pi)$$

on the 8–dimensional phase space $\mathcal{M}_{\rm red} := \{\Lambda_i^* > 0, \lambda \in \mathbb{T}^2, (\eta^*, \xi^*) \in B^4\}$ endowed with the standard symplectic form $d\Lambda^* \wedge d\lambda^* + d\eta^* \wedge d\xi^*$ ($B^4$ being a ball aroud the origin in $\mathbb{R}^4$). In fact, the (standard) Hamilton's equations for $\mathcal{H}^c_{\rm red}$ are immediately recognized to be a subsystem of the full (standard) Hamilton's equations for $\mathcal{H}_{\rm 3bp}$ when the initial data are restricted on $\mathcal{M}^*_{\rm ver}$ and the constant value of $\Psi_2$ is chosen to be $c$. More precisely, if we denote by $\phi^t_c$ the Hamiltonian flow of $\mathcal{H}^c_{\rm red}$ on $\mathcal{M}_{\rm red}$, then

$$\phi^t_*\Big(z^*, \widehat{\Psi}_1(\Lambda^*, \eta^*, \xi^*; c), c, \pi, \psi_2\Big) = \Big(\phi^t_c(z^*), \widehat{\Psi}_1(t), c, \pi, \psi_2(t)\Big)\ , \tag{3.25}$$

where we have used the short–hand notations: $z^* = (\Lambda^*, \lambda^*, \eta^*, \xi^*) \in \mathcal{M}_{\rm red}$; $\widehat{\Psi}_1(t) = \widehat{\Psi}_1 \circ \phi^t_c(z^*)$; $\psi_2(t) = \psi_2 + \int_0^t \partial_{\Psi_2}\mathcal{H}^*_{\rm 3bp}(\phi^s_c(z^*), \widehat{\Psi}_1(s), c, \pi)ds$.

At this point, the scheme used for the planar case may be easily adapted to the present situation. The nondegeneracy conditions has been checked in [14] where indications, based on a computer program, have been given for the validity of the Theorem in a wider set of initial data.

Notice that the dimension of the reduced phase space of the spatial case is eight, which is also the dimension of the phase space of the planar case. Therefore, also the Lagrangian tori obtained with this procedure have the same dimension of the tori obtained in the planar case (i.e., four) .

### 3.2.3 The general case

Let us proceed to discuss the general case following the strategy of M. Herman as presented by J. Féjoz in [11], to which we refer for complete proofs and further references.

The symplectic variables used in [11], to cope with the spatial planetary $(1+n)$–body problem (Sun and $n$ planets), are closely related to the variables defined in (3.23) above. For $1 \le i \le n$, let $\big((L_i, G_i, \Theta_i), (\ell_i, g_i, \theta_i)\big)$ denote the Delaunay variables associated to the two two–body system Sun–$i^{\rm th}$ planet. Then (as – again – Poincaré showed) the variables $\big((\Lambda_i, \lambda_i), (\eta_i, \xi_i), (p_i, q_i)\big)$, where $\Lambda_i = L_i$, $\lambda_i = \ell_i + g_i + \theta_i$ and

$$\begin{cases} \eta_i = \sqrt{2(L_i - G_i)}\ \cos(g_i + \theta_i) \\ \xi_i = -\sqrt{2(L_i - G_i)}\ \sin(g_i + \theta_i) \end{cases} \qquad \begin{cases} p_i = \sqrt{2(G_i - \Theta_i)}\ \cos\theta_i \\ q_i = -\sqrt{2(G_i - \Theta_i)}\ \sin\theta_i \end{cases} \tag{3.26}$$

are symplectic and analytic near circular, non co–planar motions; see, e.g., [4]. Let us denote by

$$\mathcal{H}_{\rm nbp} := \mathcal{H}^{(0)}(\Lambda) + \varepsilon \mathcal{H}^{(1)}(\Lambda, \lambda, \eta, \xi, p, q) \tag{3.27}$$

the Hamiltonian (2.8) expressed in terms of the Poincaré symplectic variables $\big((\Lambda, \lambda), (\eta, \xi), (p, q)\big)$, $\Lambda = (\Lambda_1, ..., \Lambda_n)$, etc.

As the number of the planets increases the degeneracies become stronger and stronger. Furthermore, a clean reduction as the one of the nodes for the three–body problem is no more available if $n > 2$. To overcome these problems Herman proposed a new approach, which we shortly proceed to describe.

Instead of Kolmogorov's non degeneracy assumption – which says that the frequency map (3.30) $I \to \omega(I)$ is a local diffeomorphism – one may consider weaker nondegeneracy conditions. In particular, in [11], one considers *nonplanar* frequency maps. A smooth curve $u \in A \to \omega(u) \in \mathbb{R}^d$, $A$ open non empty interval, is called *non planar* at $u_0 \in A$ if all the $u$–derivatives up to order $(d-1)$ at $u_0$, $\omega(u_0)$, $\omega'(u_0)$,...,$\omega^{(d-1)}(u_0)$ are linerly independent in $\mathbb{R}^d$; a smooth map $u \in A \subset \mathbb{R}^p \to \omega(u) \in \mathbb{R}^d$, $p \le d$, is called non planar at $u_0 \in A$ if there exist a smooth curve $\varphi : \hat{A} \to A$ such that $\omega \circ \varphi$ is non planar at $t_0 \in \hat{A}$ with $\varphi(t_0) = u_0$. A.S. Pyartli has proven (see, e.g., [11]) that if the map $u \in A \subset \mathbb{R}^p \to \omega(u) \in \mathbb{R}^d$ is non planar at $u_0$, then there exists a neighborhood $B \subset A$ of $u_0$ and a subset $C \subset B$ of full Lebesgue measure (i.e., ${\rm meas}\,(C) = {\rm meas}\,(A)$) such that $\omega(u)$ is Diophantine for any $u \in C$. The nonplanarity condition is weaker than Kolmogorov's nondegeneracy conditions; for example the map

$$\omega(I) := \partial_I \Big( \frac{I_1^4}{4} + I_1^2 I_2 + I_1 I_3 + I_4 \Big) = (I_1^3 + 2 I_1 I_2 + I_3, I_1^2, I_1, 1)$$

violates both Kolmogorov's nondegeneracy and the iso–energetic nondegeneracy conditions but is non planar at any point of the form $(I_1, 0, 0, 0)$, since $\omega(I_1, 0, 0, 0) = (I_1^3, I_1^2, I_1, 1)$ is a nonplanar curve (at any point).

As in the three-body case, the frequency map is that associated to the averaged secular Hamiltonian

$$\mathcal{H}_{\rm sec} := \mathcal{H}^{(0)}(\Lambda) + \varepsilon \overline{\mathcal{H}}^{(1)} \ , \quad \text{with} \quad \overline{\mathcal{H}}^{(1)}(\Lambda, \eta, \xi, p, q) := \int \mathcal{H}^{(1)} \frac{d\lambda}{(2\pi)^n} \ , \quad (3.28)$$

which has an elliptic equilibrium at $\eta = \xi = p = q = 0$ (as above, $\Lambda$ is regarded as a parameter). It is a remarkable well known fact that the quadratic part of $\overline{\mathcal{H}}^{(1)}$ does not contain "mixed terms", namely:

$$\overline{\mathcal{H}}^{(1)} = \overline{\mathcal{H}}_0^{(1)} + \varepsilon\Big(\mathcal{Q}_{\rm pln}\, \eta \cdot \eta + \mathcal{Q}_{\rm pln}\, \xi \cdot \xi + \mathcal{Q}_{\rm spt}\, p \cdot p + \mathcal{Q}_{\rm spt}\, q \cdot q + O_4\Big) \ , \quad (3.29)$$

where the function $\overline{\mathcal{H}}_0^{(1)}$ and the symmetric matrices $\mathcal{Q}_{\rm pln}$ and $\mathcal{Q}_{\rm spt}$ depend upon $\Lambda$ while $O_4$ denotes terms of order four in $(\eta, \xi, p, q)$. The eigenvalues of the matrices $\mathcal{Q}_{\rm pln}$ and $\mathcal{Q}_{\rm spt}$ are the first Birkhoff invariants of $\overline{\mathcal{H}}^{(1)}$ (with respect to the symplectic variables $(\eta, \xi, p, q)$). Let $\sigma_1, ..., \sigma_n$ and $\varsigma_1, ..., \varsigma_n$ denote, respectively the eigenvalues of $\mathcal{Q}_{\rm pln}$ and $\mathcal{Q}_{\rm spt}$, then the frequency map for the $(1 + n)$–body problem will be defined as (recall (3.18))

$$\Lambda \to (\hat{\omega}, \varepsilon\Omega) \quad (3.30)$$

with

$$\hat{\omega} := \Big(\frac{\kappa_1}{\Lambda_1^3}, ..., \frac{\kappa_n}{\Lambda_n^3}\Big), \quad \Omega := (\sigma, \varsigma) := \Big((\sigma_1, ..., \sigma_n), (\varsigma_1, , ..., \varsigma_n)\Big) \ . \quad (3.31)$$

Herman pointed out, however, that the frequencies $\sigma$ and $\varsigma$ satisfy two independent linear relations, namely (up to renumbering the indices),

$$\varsigma_n = 0 \ , \qquad \sum_{i=1}^n (\sigma_i + \varsigma_i) = 0 \ , \quad (3.32)$$

which clearly prevents the frequency map to be non planar; the second relation in (3.32) is usually called "Herman resonance" (while the first relation is a well known consequence of rotation invariance).

The degeneracy due to rotation invariance may be easily taken care of by considering (as in the three–body case) the $(6n - 2)$–dimensional invariant symplectic manifold $\mathcal{M}_{\rm ver}$, defined by taking the total angular momentum $C$ to be vertical, i.e., $C \cdot k_1 = 0 = C \cdot k_2$. But, when $n > 2$, Jacobi's reduction of the nodes is no more available and to get rid of the second degeneracy (Herman's resonance), the authors bring in a nice trick, originally due – once more! – to Poincaré. In place of considering $\mathcal{H}_{\rm nbp}$ restricted on $\mathcal{M}_{\rm ver}$, Féjoz considers the modified Hamiltonian

$$\mathcal{H}^{\delta}_{\rm nbp} := \mathcal{H}_{\rm nbp} + \delta C_3^2\ , \qquad C_3 := C \cdot k_3 = |C|\ , \tag{3.33}$$

where $\delta \in \mathbb{R}$ is an extra artificial parameter. By an analyticity argument, it is then possible to prove that the (rescaled) frequency map

$$(\Lambda, \delta) \to (\hat{\omega}, \sigma_1, ..., \sigma_n, \varsigma_1, ..., \varsigma_{n-1}) \in \mathbb{R}^{3n-1}$$

is nonplanar on an open dense set of full measure and this is enough to find a positive measure set of Lagrangian maximal $(3n-1)$–dimensional invariant tori for $\mathcal{H}^{\delta}_{\rm nbp}$; but since $\mathcal{H}^{\delta}_{\rm nbp}$ and $\mathcal{H}_{\rm nbp}$ commute a classical Lagrangian intersection argument allows to conclude that such tori are invariant also for $\mathcal{H}_{\rm nbp}$ yielding the complete proof of Arnold's theorem in the general case.

Notice that this argument yields $(3n-1)$–dimensional tori, which in the three–body case means five dimensional. Instead, the tori found in § 3.2.2 are *four* dimensional. The point is that in the reduced phase space the motion of the nodeline – denoted $\psi_2(t)$ in (3.25) – does not appear.

We conclude this discussion by mentioning that the KAM theory used in [11] is a modern and elegant function theoretic reformulation of the classical theory and is based on a $C^\infty$ local inversion theorem on "tame" Frechet spaces due to F. Sergeraert and R. Hamilton (which, in turn, is related to the Nash–Moser implicit function theorem; see [9]).

# 4 Lower dimensional tori

The maximal tori for the many–body problems described above are found near the elliptic equilibria given by the decoupled Keplerian motions. It is natural to ask what happens of such elliptic equilibria when the interaction among planets is taken into account. Even though no complete answer has yet been given to such a question, it appears that, in general, the Keplerian elliptic equilibria "bifurcate" into *elliptic $n$–dimensional tori*. We shall now give a short and non technical account of the existing results on the matter (the general theory of lower dimensional tori is, mainly, due to J.K. Moser and S.M. Graff for the hyperbolic case and to V.K. Melnikov, H. Eliasson and S.B. Kuksin for the technically more difficult elliptic case; for references, see, e.g., [7]).

The normal form of a Hamiltonian admitting an $n$–dimensional elliptic invariant torus $\mathcal{T}$ of energy $E$, proper frequencies $\hat{\omega} \in \mathbb{R}^n$ and "normal frequencies" $\Omega \in \mathbb{R}^p$ in a $2d$ dimensional phase space with $d = n + p$ is given by

$$N := E + \hat{\omega} \cdot y + \sum_{j=1}^{p} \Omega_j \frac{\eta_j^2 + \xi_j^2}{2} \; ; \tag{4.34}$$

here the symplectic form is given by $dy \wedge dx + d\eta \wedge d\xi$, $y \in \mathbb{R}^n$, $x \in \mathbb{T}^n$, $(\eta, \xi) \in \mathbb{R}^{2p}$; $\mathcal{T}$ is then given by $\mathcal{T} := \{y = 0\} \times \{\eta = \xi = 0\}$. Under suitable assumptions, a set of such tori persists under the effect of a small enough perturbation $P(y, x, \eta, \xi)$. Clearly, the union of the persistent tori (if $n < d$) form a set of zero measure in phase space; however, in general, persist, roughly speaking, $n$–parameter families.

In the many–body case considered in this article, the proper frequencies are the Keplerian frequencies $\Lambda \to \hat{\omega}(\Lambda)$ given in (3.31), which is a local diffeomorphism of $\mathbb{R}^n$. The normal frequencies $\Omega$, instead, are proportional to $\varepsilon$ and are the first Birkhoff invariants around the elliptic equilibria as discussed above. Under these circumstances, the main nondegeneracy hypothesis needed to establish the persistence of the Keplerian $n$–dimensional elliptic tori, boils down to the so–called Melnilkov condition:

$$\Omega_j \neq 0 \neq \Omega_i - \Omega_j \; , \qquad \forall \; j \neq i \; . \tag{4.35}$$

Such condition has been checked for the planar three–body case in [10], for the spatial three–body case in [4] and for the planar $n$–body case in [5]. The general spatial case is still open: in fact, while it is possible to establish lower dimensional elliptic tori for the modified Hamiltonian $\mathcal{H}^{\delta}_{\rm nbp}$ in (3.33), it is not clear how to conclude the existence of elliptic tori for the actual Hamiltonian $\mathcal{H}_{\rm nbp}$ since the argument used above works only for Lagrangian (maximal) tori; on the other hand the direct asymptotics techniques used in [4] do not extend easily to the general spatial case.

Clearly, the lower dimensional tori described in this section are not the only ones that arise in $n$–body dynamics; for more lower dimensional tori in the planar three–body case, see [10].

# 5 Towards physical applications

The above results show that, in principle, there may exist "stable planetary systems" exhibiting quasi–periodic motions around co–planar, circular Keplerian trajectories – in the Newtonian many–body approximation – provided the mass of the planets are much smaller than the mass of the central star.

A quite different question is: *In the Newtonian many–body approximation, is the Solar system or, more in general, a Solar subsystem stable?*

Clearly, even a precise mathematical reformulation of such question might be difficult. It goes, however, without saying that having a mathematical theory of important physical models allowing for observed parameter values is a primary task.

As a very preliminary step in this direction, we briefly mention a result in [6] (which is the most recent paper in a series of five computer–assisted papers of the authors on such subject begun in 1987; see [6] for references).

In [6], the (isolated) subsystem formed by the Sun, Jupiter and asteroid Victoria (one of the main objects in the Asteroidal belt) is considered. Such system is modelled by an order 10 Fourier–truncation of the restricted, planar, circular 3 body problem, whose Hamiltonian has been described in § 3.1. The Sun–Jupiter motion is therefore approximated by a circular one, the asteroid Victoria is considered massless (in the sense of § 3.1) and the motions of the three bodies are assumed to be co–planar; the remanining orbital parameters (Jupiter/Sun mass ratio, which is approximately 1/1000; eccentricity and semi–major axis of the osculating Sun–Victoria ellipse; "energy" of the system) are taken to be the actually observed values. For such system, it is proved that there exists an invariant region, on the observed fixed energy level, bounded by two maximal two–dimensional Kolmogorov tori, trapping the observed orbital parameters of the osculating Sun–Victoria ellipse.

As mentioned above, the proof of this result is computer–assisted: a long series of algebraic computations and estimates is performed on computers, keeping rigorously track of the numerical errors introduced by the machines.

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