Quasi–periodic attractors in celestial mechanics

Alessandra Celletti
Dipartimento di Matematica
Università di Roma Tor Vergata
I-00133 Roma (Italy)
(celletti@mat.uniroma2.it)

Luigi Chierchia
Dipartimento di Matematica
Università "Roma Tre"
I-00146 Roma (Italy)
(luigi@mat.uniroma3.it)

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Abstract

We prove that KAM tori smoothly bifurcate into quasi–periodic attractors in dissipative mechanical models, provided external parameters are tuned with the frequency of the motion. An application to the dissipative "spin–orbit model" of celestial mechanics (which actually motivated the analysis in this paper) is presented.

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Contents

1 Introduction and results

In physical examples, Hamiltonian dynamics, typically, arises through dissipative systems with very small dissipation. It is therefore natural to ask which part of the Hamiltonian dynamics smoothly persists when the dissipation is turned on. In particular, if the reference Hamiltonian system is nearly–integrable, a basic question is to discuss the fate of quasi–periodic trajectories, which, as KAM theory shows (see, e.g., [1]), are common in the purely Hamiltonian nearly–integrable regime.

In this paper we show that – under suitable assumptions relating external (physical) parameters with the motion frequencies – KAM tori smoothly bifurcate into quasi–periodic attractors when dissipative effects are taken into account.

1.1 Dissipative nearly–integrable flows on $\mathbb{R} \times \mathbb{T}^2$

Motivated by the "dissipative spin–orbit model" of celestial mechanics (see § 1.2 below), we first consider dissipative, nearly–integrable flows on $\mathbb{R} \times \mathbb{T}^2$, where $\mathbb{T}^2$ denotes the standard flat 2–torus $\mathbb{R}^2/(2\pi\mathbb{Z}^2)$. Namely, we consider the differential equation

$$\ddot{x} + \eta(\dot{x} - \upsilon) + \varepsilon f_x(x,t) = 0 \ , \tag{1.1}$$

where:

- dot stands for time derivative and $x = x(t)$;
- x and t are periodic variables: $(x,t) \in \mathbb{T}^2$, while the velocity $\dot{x} \in \mathbb{R}$;
- η is the "dissipation parameter": for $\eta > 0$ the system is dissipative, while for $\eta = 0$ the system is conservative (Hamiltonian); values $\eta < 0$ are also allowed;
- υ is an "external parameter" (in the spin–orbit problem it will be related to the eccentricity of the reference Keplerian orbit);
- ε measures the size of the perturbation (for $\varepsilon = 0$ the system is integrable);

- "the potential" f is a given real–analytic function[1] on $\mathbb{T}^2$.

As mentioned above, for $\eta = 0$, (1.1) is the Lagrange equation for the nearly–integrable Lagrangian

$$\mathcal{L}_\varepsilon(\dot{x}, x, t) := \frac{\dot{x}^2}{2} - \varepsilon f(x,t) \ , \qquad (\dot{x}, x, t) \in \mathbb{R} \times \mathbb{T}^2 \ ,$$

or, equivalently, corresponds to the Hamiltonian equation

$$\dot{y} = -\partial_x H_\varepsilon \ , \qquad \dot{x} = \partial_y H_\varepsilon \ ,$$

for the nearly–integrable Hamiltonian $H_\varepsilon(y, x, t) := \frac{y^2}{2} + \varepsilon f(x,t)$ defined on $\mathbb{R} \times \mathbb{T}^2$.
In the conservative case $\eta = 0$, standard KAM theory (see, e.g., [1]) implies that[2], if ε is small enough, (1.1) admits many quasi–periodic solutions, i.e., solutions of the form

$$x(t) = \omega t + u(\omega t, t) \tag{1.2}$$

with $u(\theta) = u(\theta_1, \theta_2)$ real–anlytic on $\mathbb{T}^2$ and ω *Diophantine*:

$$|\omega n_1 + n_2| \geq \frac{\kappa}{|n_1|^\tau} \ , \qquad \forall \ (n_1, n_2) \in \mathbb{Z}^2 \ , \ n_1 \neq 0 \ , \tag{1.3}$$

for some $\kappa, \tau > 0$. Furthermore, such solutions are analytic in ε and are Whitney smooth in[3] ω. In the following, $\mathcal{D}_{\kappa,\tau}$ denotes the set of Diophantine numbers in $\mathbb{R}$ satisfying[4] (1.3).
On the other hand, when $\eta \neq 0$ and $\varepsilon = 0$, the general solution of (1.1) is given by

$$x(t) = x_0 + \upsilon \, t + \frac{1 - \exp(-\eta t)}{\eta} (v_0 - \upsilon) \ ,$$

showing that the periodic (remember that x is an angle) solution $x = k + \upsilon t$ (with k constant) and $\dot{x} \equiv \upsilon$ is a *global attractor* for the dynamics .

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[1] "Finitely differentiable function" suffices, but we focus on the real–analytic case in view of our application to celestial mechanics (and for simplicity).

[2] Note that, for $\varepsilon = 0$, the Hamiltonian $H_0 = \frac{y^2}{2}$ is non–degenerate in the sense of KAM theory; compare [1].

[3] A function $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is Whitney C^k or C^k_W if it is the restriction on A of a $C^k(\mathbb{R}^n)$ function; for the original definition by H. Whitney and for relevance in dynamical system, see, e.g., [2]. Incidentally, we mention that Whitney smoothness was discussed for the first time in the framework of conservative dynamical systems in 1982 in [7] and, independently, in [11].

[4] Observe that if (1.3) holds, then $0 < \kappa < 1$ and $\tau \geq 1$. In fact, taking $n_1 = 1$ and $n_2 = -[\omega]$ ($[x] =$ integer part of x) in (1.3) shows that $\kappa < 1$, while the fact that $\tau \geq 1$ comes from Liouville's theorem on rational approximations ("*For any $\omega \in \mathbb{R} \backslash \mathbb{Q}$ and for any $N \geq 1$ there exist integers p and q with $|q| \leq N$ such that $|\omega q - p| < 1/N$*"). Finally, we recall that, when $\tau > 1$, $\bigcup_{\kappa > 0} \mathcal{D}_{\kappa,\tau}$ is a set of full Lebesgue measure.

The *leitmotiv* is that KAM quasi–periodic solutions, for $\eta \neq 0$, smoothly bifurcate into quasi–periodic attractors[5] provided the "external frequency" $\boldsymbol{\upsilon}$ is tuned with the "internal frequency" ω in (1.2). This is the content of the first result:

Theorem 1 *Fix $0 < \kappa < 1 \leq \tau$ and $\eta_0 > 0$. Then, there exists $0 < \varepsilon_0 < 1$ such that for any $\varepsilon \in [0, \varepsilon_0]$, any $\eta \in I_0 := [-\eta_0, \eta_0]$ and any $\omega \in \mathcal{D}_{\kappa,\tau}$, there exists a unique function*[6]

$$u = u_\varepsilon(\theta; \eta, \omega) = O(\varepsilon) \ , \qquad \langle u \rangle := \int_{\mathbb{T}^2} u \, \frac{d\theta}{(2\pi)^2} = 0 \ ,$$

such that $x(t)$ in (1.2) solves (1.1) with

$$\boldsymbol{\upsilon} = \omega \Big(1 + \langle (u_{\theta_1})^2 \rangle \Big) \ . \tag{1.4}$$

Furthermore, the function u_ε is smooth in the sense of Whitney in all its variables, is real–analytic in $\theta \in \mathbb{T}^2$ and ε, C^∞ in η and Whitney C^∞ in ω.

Remark 1 (i) Uniqueness has to be understood in the following sense: if one is given a second solution $\tilde{x}(t) = \omega t + \tilde{u}(\omega t, t)$ of (1.1) for some $\boldsymbol{\upsilon} \in \mathbb{R}$, with $\tilde{u} = \tilde{u}_\varepsilon(\theta; \eta, \omega) = O(\varepsilon)$ real–analytic in (θ, ε) and having vanishing average over $\mathbb{T}^2$, then $\tilde{u} \equiv u$ and $\boldsymbol{\upsilon}$ is as in (1.4).
(ii) Time–derivative for $x(t)$ corresponds to the directional derivative

$$\partial_\omega := \omega \frac{\partial}{\partial \theta_1} + \frac{\partial}{\partial \theta_2} \ , \tag{1.5}$$

for the function $u(\theta)$, so that, being the flow $\theta \in \mathbb{T}^2 \to \theta + (\omega t, t)$ dense in $\mathbb{T}^2$, one sees that $x(t)$ *as in* (1.2) *is a quasi–periodic solution of* (1.1) *if and only if* u *solves the following PDE on* $\mathbb{T}^2$:

$$\partial_\omega^2 u + \eta \, \partial_\omega u + \varepsilon \, f_x(\theta_1 + u, \theta_2) + \gamma = 0 \ , \qquad \gamma = \eta(\omega - \boldsymbol{\upsilon}) \ . \tag{1.6}$$

This equation will actually be the main object of investigation of this paper.
(iii) Theorem 1 implies that the 2–torus

$$\mathcal{T}_{\varepsilon,\eta}(\omega) := \{(x,t) = (\theta_1 + u_\varepsilon(\theta; \eta, \omega), \theta_2) \ : \ \theta \in \mathbb{T}^2\} \ , \tag{1.7}$$

[5] In general, in the non–integrable regime, such attractors will be only *local attractors*.

[6] As usual, $f = O(x^k)$ means that f is a smooth function of x having equal to zero the first k derivatives at $x = 0$; here $\theta \in \mathbb{T}^2$ corresponds to the variables (x,t), or, more precisely, θ denotes the variables in which the (x,t) motion linearizes.

is a *quasi–periodic attractor* for the dynamics associated to (1.1) with υ as in (1.4) and that the dynamics on $\mathcal{T}_{\varepsilon,\eta}(\omega)$ is analytically conjugated to the linear flow $\theta \to \theta + (\omega t, t)$; compare, also, point (i) of Remark 3 below.

(iv) The result is perturbative in ε but it is *uniform* in η. It is particularly noticeable the smooth dependence of u_ε on η as $\eta \to 0$, which shows that the invariant KAM torus $\mathcal{T}_{\varepsilon,0}(\omega)$ smoothly bifurcates into the attractor (1.7) as $\eta \neq 0$.

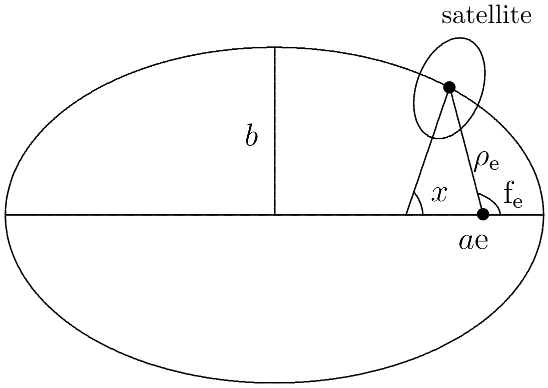
(v) Theorem 1 will be obtained as a corollary of a "dissipative Nash–Moser" theorem (see § 1.3 below), after having rewritten Eq. (1.6) as a functional equation. Indeed, the method of proof is rather robust and general and could be easily adapted to cover dissipative maps such as the "fattened Arnold family" studied in [3] or it could be extended to systems with more degrees of freedom. In this second case, however, it would be important, in our opinion, to motivate physically the form of the dissipation, a problem in itself difficult and intriguing.

1.2 The dissipative spin–orbit model

We turn, now, to the mechanical problem that motivated this paper, namely, the dissipative spin–orbit problem. Such problem consists in studying the rotations of a triaxial non–rigid body (satellite), having its center of mass revolving on a given Keplerian ellipse, and subject to the gravitational attraction of a major body sitting on a focus of the ellipse.

To simplify the analysis, we assume that the satellite is symmetric with respect to an "equatorial plane" and study motions having the equatorial plane coinciding with the Keplerian orbit plane (because of the assumed symmetry of the satellite, such motions belong to an invariant submanifold of the phase space). Furthermore, following Correia and Laskar [8], we consider a "viscous tidal model, with a linear dependence on the tidal frequency". The dissipation, in such model, is meant to reflect the averaged effect of tides on the motion (see, also, Remark 2 (i) below).

Under such hypotheses, the motions of the satellite are described by the angle x formed by, say, the direction of the major physical axis of the satellite (assumed to lie in the equatorial plane) with a fixed axis of the Keplerian orbit plane (say the direction of the semimajor axis of the ellipse),



and the differential equation governing the motion of the satellite, in suitable units, is given by (1.1) with

$$\begin{cases} \eta = K\Omega_{\rm e}\ , \quad \upsilon = \upsilon_{\rm e}\ , \quad \varepsilon = \frac{3}{2}\,\frac{B-A}{C}\ , \\ f = f(x,t;{\rm e}) := -\dfrac{1}{2\,\rho_{\rm e}(t)^3}\cos\Big(2x - 2\,{\rm f}_{\rm e}(t)\Big)\ , \end{cases} \tag{1.8}$$

where:

- ${\rm e} \in [0,1)$ is the eccentricity of the Keplerian orbit on which the center of mass of the satellite is revolving;
- $K \geq 0$ is a physical constant depending on the internal (non–rigid) structure of the satellite;
- $\Omega_{\rm e} > 0$, $N_{\rm e} > 0$ and $\upsilon_{\rm e} > 1$ are known functions of the eccentricity ${\rm e} \in [0,1)$ and are given by:

$$\Omega_{\rm e} := \Big(1 + 3{\rm e}^2 + \frac{3}{8}{\rm e}^4\Big)\,\frac{1}{(1-{\rm e}^2)^{9/2}}\ , \tag{1.9}$$

$$N_{\rm e} := \Big(1 + \frac{15}{2}{\rm e}^2 + \frac{45}{8}{\rm e}^4 + \frac{5}{16}{\rm e}^6\Big)\frac{1}{(1-{\rm e}^2)^6}\ ;$$

$$\upsilon_{\rm e} := \frac{N_{\rm e}}{\Omega_{\rm e}} = 1 + 6{\rm e}^2 + \frac{3}{8}{\rm e}^4 + O({\rm e}^6)\ . \tag{1.10}$$

- $0 < A < B < C$ are the principal moments of inertia of the satellite;

- $\rho_e(t)$ and $f_e(t)$ are, respectively, the (normalized) orbital radius and the true anomaly of the Keplerian motion, which (because of the assumed normalizations) are 2π–periodic function of time t. The explicit expression for ρ_e and f_e may be described as follows. Let $u = u_e(t)$ be the 2π–periodic function obtained by inverting

$$t = u - e \sin u \ , \qquad (\text{“Kepler's equation''}) \ ; \tag{1.11}$$

then

$$\begin{aligned} \rho_e(t) &= 1 - e \cos u_e(t) \\ f_e(t) &= 2 \arctan\Big(\sqrt{\frac{1+e}{1-e}} \tan \frac{u_e(t)}{2}\Big) \ . \end{aligned} \tag{1.12}$$

Remark 2 (i) The derivation of the conservative spin–orbit model (i.e., Eq.'s (1.1)&(1.8) with $K = 0$) is classical and can be found, e.g., in [4]: compare Eq. (2.2) of [4] with the normalization $n := \sqrt{(Gm)/a^3} = 1$.
The derivation of the dissipative contribution is less straightforward and, as mentioned above, we follow Correia and Laskar [8]: compare (3) and (4) in [8], where Ω_e and N_e are denoted, respectively, $\Omega(e)$ and $N(e)$. Essentially, the dissipative term is given by the average over one revolution period (i.e., 2π with our normalization) of the so–called MacDonald's torque [9]; compare [10], where the functions Ω_e and N_e are denoted, respectively, by $f_1(e)$ and $f_2(e)$.

(ii) For $K = 0$, equations (1.1)&(1.8) correspond to the Hamiltonian flow associated to the one–and–a–half degree–of–freedom Hamiltonian

$$H(y, x, t) := \frac{1}{2} y^2 - \frac{\varepsilon}{2\, \rho_e(t)^3} \cos\Big(2x - 2\, f_e(t)\Big) \ ,$$

(y, x) being standard symplectic variables. The associated “spin–orbit” Hamiltonian system is non–integrable if $\varepsilon > 0$ *and*[7] $e > 0$.

(iii) The external frequency υ_e is a real–analytic invertible function of e mapping $(0, 1)$ onto $(1, \infty)$; we denote by $\upsilon^{-1} : (1, \infty) \to (0, 1)$ the inverse map (which is also real–analytic). The invertibility of the frequency map υ_e will play the rôle of a *nondegeneracy condition*, allowing to fix the eccentricities for which quasi–periodic attractors exist in the full dynamics.

(iv) In many examples taken from the Solar system, *both* ε *and* K *are small*. For example, for the Earth–Moon system and for the Sun–Mercury system ε is of the order of 10^{-4}, while K is of the order of 10^{-8}.

[7]When $e = 0$, $u_0(t) = t = f_0(t)$, $\rho_0 = 1$ so that $H = \frac{1}{2} y^2 - \frac{\varepsilon}{2} \cos(2x - 2t)$, which is easily seen to be integrable.

(v) Mercury is observed in a nearly 3:2 spin–orbit resonance (i.e., it rotates three times on its spin axis, while making two orbital revolutions around the Sun) and is moving on a nearly Keplerian orbit with eccentricity $e \simeq 0.2$. However, Correia and Laskar's numerical investigations [8] show that "the chaotic evolution of Mercury's orbit can drive its eccentricity beyond 0.325 during the planet's history". Now, if υ^{-1} is the function introduced in point (iii) above, $\upsilon^{-1}(3/2) = 0.284....$ Thus, Theorem 1 for the dissipative spin–orbit model (spelled out in Theorem 2 below) might give new mathematical insight into the question of the so–called "capture in the 3:2 resonance" of Mercury (see [8]) suggesting that such capture is related to the existence of an underlying attractor with Diophantine frequency close to 3/2. For more information about this point, see [6].

(vi) As mentioned in point (ii), Remark 1, quasi–periodic trajectories for (1.1)&(1.8) correspond to solutions of the PDE

$$\partial_\omega^2 u + K\Omega_{\rm e}\, \partial_\omega u + \frac{\varepsilon}{\rho_{\rm e}(\theta_2)^3} \sin\Big(2(\theta_1 + u(\theta)) - 2\,{\rm f}_{\rm e}(\theta_2)\Big) = KN_{\rm e} - K\Omega_{\rm e}\omega \ . \tag{1.13}$$

Next result translates Theorem 1 in the case of the dissipative spin–orbit model.

Theorem 2 *Fix $\kappa, r \in (0,1)$ and $\tau \geq 1$. There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in [0, \varepsilon_0]$, any $K \in [0,1]$ and any $\omega \in \mathcal{D}_{\kappa,\tau} \cap [1+r, 1/r]$, there exist unique functions*[8]

$$\mathrm{e}_\varepsilon = \mathrm{e}_\varepsilon(K,\omega) = \upsilon^{-1}(\omega) + O(\varepsilon^2) \ , \qquad u = u_\varepsilon(\theta; K, \omega) = O(\varepsilon) \ ,$$

with $\int_{\mathbb{T}^2} u \, d\theta = 0$, satisfying (1.13) with $\mathrm{e} = \mathrm{e}_\varepsilon$. The functions e_ε and u_ε are smooth in the sense of Whitney in all their variables and are real–analytic in $\theta \in \mathbb{T}^2$ and ε, C^∞ in K and Whitney C^∞ in ω.

1.3 A dissipative Nash–Moser Theorem

We, now, state the main technical result, namely, an existence and uniqueness[9] Nash–Moser (or KAM) theorem for dissipative/conservative flows on a two torus. In this theorem the "potential" is not assumed to be small but, rather, we assume to start with a good enough approximate solution. Special care is devoted to the dependence of solutions upon the dissipative parameter, which appears explicitly in the small divisor problems involved.

More specifically, Theorem 3 below deals with finding real–analytic "local" solutions $u : \mathbb{T}^2 \to \mathbb{R}$ and a number γ such that[10]

[8] The map υ^{-1} is the inverse map of $\mathrm{e} \to \upsilon_{\mathrm{e}}$ defined in (1.10).

[9] While, KAM procedures are abundant in literature, local uniqueness is seldom discussed.

[10] ∂_ω is defined in (1.5) and $\langle u \rangle := \int_{\mathbb{T}^2} u(\theta) \frac{d\theta}{(2\pi)^2}$.

$$\begin{cases} \partial_\omega^2 u + \eta\, \partial_\omega u + g_x(\theta_1 + u, \theta_2) + \gamma = 0 \ , \\ \\ \langle u \rangle = 0 \ , \quad 1 + u_{\theta_1} \neq 0 \ . \end{cases} \tag{1.14}$$

Remark 3 (i) While the first condition in the second line of (1.14) is just a normalization condition (needed for uniqueness), the second one implies that the map

$$\theta \in \mathbb{T}^2 \to (\theta_1 + u(\theta), \theta_2) \in \mathbb{T}^2$$

is a diffeomorphism and, therefore, if u solves (1.14), the set

$$\Big\{ (y, (x,t)) = \Big(\omega + \partial_\omega u, (\theta_1 + u, \theta_2) \Big) \ : \ \theta \in \mathbb{T}^2 \Big\}$$

is a 2–dimensional torus embedded in the 3–dimensional phase space $\mathbb{R} \times \mathbb{T}^2$, which is invariant for the dynamics generated by the differential equation

$$\ddot{x} + \eta\, (\dot{x} - \omega) + g_x(x,t) + \gamma = 0 \ , \tag{1.15}$$

meaning that, for each $\theta \in \mathbb{T}^2$,

$$t \to x(t;\theta) := \theta_1 + \omega t + u(\theta_1 + \omega t, \theta_2 + t)$$

is a solution of (1.15).

(ii) As mentioned above, the unknowns of (1.14) are u *and* $\gamma \in \mathbb{R}$. Indeed, we shall see below that u and γ are *not independent*, as they satisfy the relation

$$\eta\, \omega\, \langle (u_{\theta_1})^2 \rangle + \gamma = 0 \ . \tag{1.16}$$

On the other hand, ω and η are regarded as external parameters taken, respectively in $\mathcal{D}_{\kappa,\tau}$ (for some prefixed $0 < \kappa < 1 \leq \tau$) and in a compact interval $[-\eta_0, \eta_0]$ (for some prefixed $\eta_0 > 0$).

It will be useful to rewrite the differential equation in (1.14) as a functional equation involving parameters. For $\eta \in \mathbb{R}$ we define

$$\begin{aligned} & D_\eta : \ v \in C^1(\mathbb{T}^2) \to D_\eta v = \partial_\omega v + \eta v \ , \\ & \Delta_\eta := D_\eta \partial_\omega = \partial_\omega D_\eta \ , \\ & \mathcal{F}_\eta : (v, \gamma) \in C^2(\mathbb{T}^2) \times \mathbb{R} \to \mathcal{F}_\eta(v;\gamma) := \Delta_\eta v + g_x(\theta_1 + v, \theta_2) + \gamma \end{aligned} \tag{1.17}$$

(notice that $D_0 = \partial_\omega$). Then, problem (1.14) may be rewritten as

$$\begin{cases} \mathcal{F}_\eta(u;\gamma) = 0 \ , \\ \langle u \rangle = 0 \ , \qquad 1 + u_{\theta_1} \neq 0 \ . \end{cases} \tag{1.18}$$

In order to state existence and uniqueness theorems for (1.18), we need to introduce suitable function spaces, which, because of our motivating model, will be spaces of real–analytic functions.

Let us denote by $\mathcal{H}^\xi$ the Banach space of continuous functions $u : \mathbb{T}^2 \to \mathbb{R}$ with finite norm[11]

$$\|u\|_\xi := \sum_{n\in\mathbb{Z}^2} |u_n| \ \mathbb{e}^{|n|\xi} \ ,$$

where u_n denotes the n–th Fourier coefficient and $|n| = |n_1| + |n_2|$.

Let us also denote by $\mathcal{H}^\xi_0$ the closed subspace of $\mathcal{H}^\xi$ formed by functions with zero average:

$$\mathcal{H}^\xi_0 := \{u \in \mathcal{H}^\xi \ \text{s.t.} \ u_0 = \langle u \rangle = 0\} \ .$$

Remark 4 (i) The spaces $\{\mathcal{H}^\xi : \ \xi \geq 0\}$ form a *nested family of Banach spaces* as

$$0 \leq \xi' < \xi \qquad \Longrightarrow \qquad \mathcal{H}^\xi \subsetneqq \mathcal{H}^{\xi'} \quad \text{and} \quad \|u\|_{\xi'} \leq \|u\|_\xi \ .$$

(ii) Clearly, functions in $\mathcal{H}^\xi$ can be analytically extended to the complex multi–strip

$$\mathbb{T}^2_\xi := \{\theta \in \mathbb{C}^2 : |\operatorname{Im}\theta_i| < \xi \ , \quad i = 1, 2\}$$

and

$$\sup_{\mathbb{T}^2_\xi} |u| \leq \|u\|_\xi \ .$$

Viceversa, if u is a real–analytic function on $\mathbb{T}^2$, then its Fourier coefficients decay exponentially fast and therefore there exists $\xi > 0$ such that $u \in \mathcal{H}^\xi$.

(iii) From the definition of the norm $\|\cdot\|_\xi$, it follows that if $u \in \mathcal{H}^\xi_0$ then[12]

$$\|\partial^h_\theta u\|_\xi \leq \|\partial^k_\theta u\|_\xi \ , \qquad \forall \ h, k \in \mathbb{N}^2 : \quad h_i \leq k_i \ ;$$

the same inequalities hold also for $u \in \mathcal{H}^\xi$ when $h \neq 0$.

[11] The letter $\mathbb{e}$ denotes the Neper number $\exp(1) = 2.71828...$; do not get confused with the letter e used for eccentricity and the letter e, which will be used, below, for the "error" function $e(\theta)$.

[12] As usual, if $h \in \mathbb{N}^2$, $\partial^h_\theta := \partial^{h_1}_{\theta_1}\partial^{h_2}_{\theta_2}$.

(iv) D_η is a "diagonal" operator on Fourier spaces mapping $\mathcal{H}^\xi$ into $\mathcal{H}^{\xi'}$ for any[13] $0 \le \xi' < \xi$:

$$D_\eta u = D_\eta\Big(\sum_{n\in\mathbb{Z}^2} u_n \, \mathrm{e}^{in\cdot\theta}\Big) = \sum_{n\in\mathbb{Z}^2} \lambda_{\eta,n} \, u_n \, \mathrm{e}^{in\cdot\theta} \ ,$$

where

$$\lambda_{\eta,n} := i(\omega n_1 + n_2) + \eta \ . \tag{1.19}$$

If $\eta \ne 0$, D_η is bounded (since $|\lambda_{\eta,n}| \ge |\eta|$), invertible and its inverse, D_η^{-1}, maps $\mathcal{H}^\xi$ onto itself:

$$\begin{aligned} &D_\eta^{-1} : \mathcal{H}^\xi \to \mathcal{H}^\xi \ , \qquad \forall\, \eta \ne 0 \ , \ \xi \ge 0 \\ &D_\eta^{-1} u = \sum_{n\in\mathbb{Z}^2} \lambda_{\eta,n}^{-1} \, u_n \ \mathrm{e}^{in\cdot\theta} \ . \end{aligned}$$

Notice, however, that the limit $\eta \to 0$, which is of particular interest to us, is singular (compare, also, next point).

(v) If $\eta = 0$, then

$$D_0 = \partial_\omega : \mathcal{H}^\xi \to \mathcal{H}_0^{\xi'} \ , \qquad\qquad (0 \le \xi' < \xi) \ ;$$

since $\omega \in \mathcal{D}_{\kappa,\tau}$, D_0 is invertible on $\mathcal{H}_0^\xi$:

$$D_0^{-1} : u \in \mathcal{H}_0^\xi \quad \longrightarrow \quad \sum_{n\in\mathbb{Z}^2\setminus\{0\}} \lambda_{\eta,n}^{-1} u_n \, \mathrm{e}^{in\cdot\theta} \in \mathcal{H}_0^{\xi'} \ , \qquad (0 \le \xi' < \xi) \ .$$

(vi) We, finally, recall two elementary properties concerning the product and the composition of functions in $\mathcal{H}^\xi$. If $u, v \in \mathcal{H}^\xi$, then $uv \in \mathcal{H}^\xi$ and

$$\|uv\|_\xi \le \|u\|_\xi \, \|v\|_\xi \ .$$

As for composition, if $0 \le \xi < \bar\xi$ and $f \in \mathcal{H}^{\bar\xi}$, $v \in \mathcal{H}^\xi$ with $\|v\|_\xi \le \bar\xi - \xi$, then one has $\theta \to F(\theta) := f(\theta_1 + v(\theta), \theta_2) \in \mathcal{H}^\xi$ and

$$\|F\|_\xi \le \|f\|_{\bar\xi} \ .$$

For the proof, see, e.g., [5], p. 426, 427.

We are now ready to formulate the main technical result.

[13] As usual $n \cdot \theta$ denotes the usual inner product $n_1\theta_1 + n_2\theta_2$.

Theorem 3 *Let* $0 < \xi_* < \xi < \bar{\xi} \le 1$*; let* $0 < \kappa < 1 \le \tau$*; let* $I_0 := [-\eta_0, \eta_0]$ *for some* $\eta_0 > 0$*; let* $\omega \in \mathcal{D}_{\kappa,\tau}$ *and* $(x,t) \in \mathbb{T}^2 \to g(x,t) \in \mathcal{H}^{\bar{\xi}}$*; let* $M > 0$ *be such that*

$$\|\partial_x^3 g\|_{\bar{\xi}} \le M \ ; \tag{1.20}$$

let, also, $0 < \nu < \bar{\xi} - \xi$*,* $0 < \alpha < 1$ *and* $0 < \sigma < 1$*.*
Then, there exist a constant $k = k(\xi, \xi_*, \kappa, \tau, \eta_0, M, \nu, \alpha, \sigma) > 1$ *such that the following holds.*
Assume that there exist functions $v = v(\theta;\eta) \in C^\infty(\mathbb{T}^2 \times I_0)$ *and* $\beta = \beta(\eta) \in C^\infty(I_0)$ *("initial approximate solution") satisfying the following hypotheses:*

(H1) *for each* $\eta \in I_0$*, the function* $\theta \to v(\theta;\eta)$ *belongs to* $\mathcal{H}_0^{\xi}$*;*

(H2) $\|v_{\theta_1}\|_\xi \le \sigma\nu$ *for any* $\eta \in I_0$*;*

(H3) *define*[14]

$$V := 1 + v_{\theta_1} \ , \qquad W := V^2 \ , \qquad \rho := \eta \, \frac{\langle W^{-1} D_\eta^{-1} v_{\theta_1} \rangle}{\langle W^{-1} \rangle} \ , \tag{1.21}$$

and assume that, for any $\eta \in I_0$*,*

$$|\rho| \le \sigma\alpha \ ; \tag{1.22}$$

(H4) *assume that, for any* $\eta \in I_0$*, the "error function*[15]*"*

$$e = e(\theta;\eta) := \mathcal{F}_\eta(v;\beta) \ , \tag{1.23}$$

satisfies the smallness condition

$$k \, \|e\|_\xi \le 1 \ . \tag{1.24}$$

Under the hypotheses (H1)÷(H4)*, there exist functions* $u = u(\theta;\eta) \in C^\infty(\mathbb{T}^2 \times I_0)$ *and* $\gamma = \gamma(\eta) \in C^\infty(I_0)$ *such that* (1.18) *holds. Furthermore, for each* $\eta \in I_0$*,* $u(\cdot;\eta) \in \mathcal{H}_0^{\xi_*}$ *and, if* V_**,* W_* *and* ρ_* *are defined as in* (1.21) *with* v *replaced by* u*, then, for each* $\eta \in I_0$*, one has*

$$\|u_{\theta_1}\|_{\xi_*} \le \nu \ , \qquad |\rho_*| \le \alpha \ , \tag{1.25}$$
$$|\gamma - \beta| \, , \, \|u_{\theta_1} - v_{\theta_1}\|_{\xi_*} \, , \, |\rho - \rho_*| \, , \, \|W - W_*\|_{\xi_*} \le k \, \|e\|_\xi \ , \tag{1.26}$$

[14] Since $\sigma\nu < 1$, hypothesis (H2) implies that $V > 0$ (and hence $W > 0$) for all θ and η.
[15] Recall the definitions given in (1.17).

The functions u and γ are unique in the following sense. If $u' = u'(\theta;\eta) \in C^\infty(\mathbb{T}^2 \times I_0)$ and $\gamma' = \gamma'(\eta) \in C^\infty(I_0)$ are also solutions of (1.18)*, i.e., $\mathcal{F}_\eta(u';\gamma') = 0$ and u' is such that, for any $\eta \in I_0$,*

$$\begin{aligned} &\theta \to u'(\theta;\eta) \in \mathcal{H}_0^{\xi_*} \ , \\ &\|u'_{\theta_1}\|_{\xi_*} \le \nu \ , \\ &k\,\|u' - v\|_{\xi_*} \le 1 \end{aligned} \tag{1.27}$$

then $u' = u$ and $\gamma' = \gamma$.

Remark 5 (i) The proof of this theorem is fully constructive: the solution is gotten as a limit $(u,\gamma) = \lim(v_j,\beta_j)$ where $(v_0,\beta_0) = (v,\beta)$ is the starting approximate solution and (v_j,β_j) are quadratically better and better approximations to the solution (u,γ). Details on how to evaluate the constant k are given along the proof.

(ii) The assumptions that all the ξ's are smaller than one and that $\nu < 1$ are not needed and are made only to simplify the exposition. It would suffice to assume that $1 + v_{\theta_1}$ never vanishes.

(iii) The smooth dependence upon η around $\eta = 0$ (the conservative case) is one of the main points of the theorem: it shows that KAM tori in the conservative case bifurcate smoothly into dissipative attractors, keeping the dynamics conjugated to the linear flow $\theta \to \theta + (\omega t, t)$. Indeed, also the dependence upon the frequencies $\omega \in \mathcal{D}_{\gamma,\tau}$ is smooth, as explained briefly in Remark 7 below. The rôle of the bifurcation parameter is played by γ and in the application to the spin–orbit problem by the eccentricity e of the reference Keplerian orbit.

2 Proofs

2.1 Preliminaries

Here, we discuss some more properties of the small divisor operators D_η and Δ_η and prove the compatibility condition (1.16).

Lemma 1 (i) *If $u \in C^2(\mathbb{T}^2)$, then*

$$\langle D_\eta u\rangle = \eta\langle u\rangle \ , \tag{2.28}$$

$$\langle u_{\theta_1} \Delta_\eta u\rangle = \eta\omega\langle (u_{\theta_1})^2\rangle \ . \tag{2.29}$$

(ii) *If $v, V \in C^2(\mathbb{T}^2)$ and $V(\theta) \neq 0\ \forall\ \theta \in \mathbb{T}^2$ then*

$$V\Delta_\eta v - v\Delta_\eta V = D_\eta\Big(V^2 D_0(\frac{v}{V})\Big) \ . \tag{2.30}$$

(iii) *Let* $\omega \in \mathcal{D}_{\kappa,\tau}$*; let* $\xi > \xi' \geq 0$*; let* $\eta \in \mathbb{R}$ *and let* p *and* s *be non negative integers. Then, for any* $u \in \mathcal{H}_0^\xi$*,*

$$
\begin{aligned}
&D_\eta^{-1} : u \in \mathcal{H}_0^\xi \quad \longrightarrow \sum_{n\in\mathbb{Z}^2\setminus\{0\}} \lambda_{\eta,n}^{-1} u_n \,\mathrm{e}^{in\cdot\theta} \in \mathcal{H}_0^{\xi'} \ , \\
&\|D_\eta^{-s}\,\partial_{\theta_1}^p u\|_{\xi'} \leq \sigma_{p,s}(\xi-\xi')\,\|u\|_\xi \ ,
\end{aligned}
\tag{2.31}
$$

where $\lambda_{\eta,n} = i(\omega n_1 + n_2) + \eta$ *(compare* (1.19)*) and*

$$
\sigma_{p,s}(\delta) := \sigma_{p,s}(\delta;\omega,\eta) := \sup_{n\in\mathbb{Z}^2\setminus\{0\}} \Big(|i(\omega n_1+n_2)+\eta|^{-s}\,|n_1|^p\,\mathrm{e}^{-\delta|n|}\Big) \ .
$$

Furthermore,

$$
\sigma_{p,s}(\delta) \leq \frac{1}{\kappa^s\,\delta^{s\tau+p}} \Big(\frac{s\tau+p}{\mathrm{e}}\Big)^{s\tau+p} \ . \tag{2.32}
$$

Finally, if $p>0$*,* (2.31) *holds for any* $u \in \mathcal{H}^\xi$*.*

(iv) *Let* $B := \mathbb{T}^2 \times I_0$*. Let* $(\theta,\eta) \in B \to u(\theta;\eta)$ *belong to* $C^\infty(B)$ *and* $\theta \to u(\theta;\eta)$ *belong to* $\mathcal{H}^\xi$ *for some* $\xi > 0$ *and any* $\eta \in I_0$*. Assume that*

$$
u_0(0) := \langle u(\cdot;0)\rangle = 0 \ ,
$$

and let $\omega \in \mathcal{D}_{\kappa,\tau}$*. Then,* $D_\eta^{-1}u \in C^\infty(B)$*;* $\theta \to D_\eta^{-1}u(\theta;\eta)$ *belongs to* $\mathcal{H}^{\xi'}$ *for any* $0 \leq \xi' < \xi$ *and any* $\eta \in I_0$*. Furthermore,*

$$
\begin{aligned}
&D_\eta^{-1}u = \frac{u_0}{\eta} + D_\eta^{-1}(u-u_0) \ , \\
&D_\eta^{-1}u(\theta;0) = -\partial_\eta u_0(0) + D_0^{-1}u(\theta;0) \ , \\
&\|D_\eta^{-1}u(\cdot;\eta)\|_{\xi'} \leq \Big|\frac{u_0(\eta)}{\eta}\Big| + \sigma_{0,1}(\xi-\xi')\,\|u(\cdot,\cdot;\eta)-u_0(\eta)\|_\xi \ .
\end{aligned}
\tag{2.33}
$$

Proof Equality (2.28) is obvious.

The operator $D_0^2\partial_{\theta_1}$ is skew–symmetric, hence, by integration by parts,

$$
\begin{aligned}
\Big\langle u_{\theta_1}\Delta_\eta u\Big\rangle &= \langle u_{\theta_1} D_0^2 u\rangle + \eta\langle u_{\theta_1} D_0 u\rangle \\
&= \langle D_0^2 u_{\theta_1} u\rangle + \eta\langle u_{\theta_1} D_0 u\rangle \\
&= \eta\langle u_{\theta_1} D_0 u\rangle \\
&= \eta\omega\langle (u_{\theta_1})^2\rangle \ ,
\end{aligned}
$$

which is (2.29).

(ii) Relation (2.30) follows from the definitions of D_η and Δ_η:

$$\begin{aligned} V\Delta_\eta v - v\Delta_\eta V &= VD_\eta D_0 v - vD_\eta D_0 V \\ &= VD_0^2 v - vD_0^2 V + \eta(VD_0 v - vD_0 V) \ ; \end{aligned}$$

on the other hand one has

$$\begin{aligned} D_\eta(V^2 D_0(\frac{v}{V})) &= D_\eta(VD_0 v - vD_0 V) \\ &= VD_0^2 v - vD_0^2 V + \eta(VD_0 v - vD_0 V) \ , \end{aligned}$$

from which (2.30) follows.

(iii) (2.31) follows immediately from the definitions of D_η^{-1} and of $\sigma_{p,s}$:

$$\begin{aligned} \|\partial_{\theta_1}^p D_\eta^{-s} u\|_{\xi'} &= \sum_{n\in\mathbb{Z}^2\setminus\{0\}} \Big(|n_1|^p \, |i(\omega n_1 + n_2) + \eta|^{-s} \ \mathrm{e}^{-|n|(\xi-\xi')} \Big) \ |u_n| \ \mathrm{e}^{|n|\xi} \\ &\le \ \sigma_{p,s}(\xi - \xi') \ \|u\|_\xi \ . \end{aligned}$$

Since

$$|i(\omega n_1 + n_2) + \eta| \ge |\omega n_1 + n_2| \ ,$$

(2.32) follows at once from the Diophantine estimate (1.3) and from the evaluation

$$\sup_{x>0} \Big(x^a \ \mathrm{e}^{-x} \Big) = \Big(\frac{a}{\mathrm{e}} \Big)^a \ ,$$

valid for any $a \ge 0$.

(iv): The Fourier coefficients

$$u_n(\eta) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} u(\theta;\eta) \ \mathrm{e}^{-in\cdot\theta} \, d\theta$$

are $C^\infty(I_0)$ and, by assumption,

$$\|u(\cdot;\eta)\|_\xi := \sum_{n\in\mathbb{Z}^2} |u_n(\eta)| \ \mathrm{e}^{|n|\xi} < \infty \ , \qquad \forall \eta \in I_0 \ .$$

Therefore, by the assumption on ω, it follows immediately that $D_\eta^{-1}u$ belongs to $C^\infty(B)$ and that $D_\eta^{-1}u(\cdot,\cdot)$ belongs to $\mathcal{H}^{\xi'}$ for any $\eta \ne 0$ and $\xi' < \xi$; the evaluations in the first two lines of (2.33) show that the same is true *for any* $\eta \in I_0$ (and any $\xi' < \xi$). Last estimate in (2.33) follows at once from point (iii). ∎

Corollary 1 *Let $\mathcal{F}_\eta$ be as in (1.17). If $u \in C^2(\mathbb{T}^2)$ and $\eta \in \mathbb{R}$ then*

$$\Big\langle (1+u_{\theta_1})\mathcal{F}_\eta(u;\gamma)\Big\rangle \;=\; \eta\omega\,\langle (u_{\theta_1})^2\rangle + \gamma\ . \tag{2.34}$$

In particular, if $\mathcal{F}_\eta(u;\gamma)=0$, then (1.16) holds.

Proof First observe that by (2.28)

$$\langle \Delta_\eta u\rangle = \langle D_0(D_\eta u)\rangle = 0\ .$$

Observe also that

$$\langle (1+u_{\theta_1})g_x(\theta_1+u,\theta_2)\rangle = \langle \partial_{\theta_1}\cdot g(\theta_1+u,\theta_2)\rangle = 0\ .$$

By these observations and (2.29),

$$\begin{aligned}\Big\langle (1+u_{\theta_1})\mathcal{F}_\eta(u;\gamma)\Big\rangle &= \Big\langle (1+u_{\theta_1})\Delta_\eta u + (1+u_{\theta_1})g_x(\theta_1+u,\theta_2) + (1+u_{\theta_1})\gamma\Big\rangle\\ &= \Big\langle u_{\theta_1}\Delta_\eta u + \partial_{\theta_1}\cdot g(\theta_1+u,\theta_2)\Big\rangle + \gamma\\ &= \Big\langle u_{\theta_1}\Delta_\eta u\Big\rangle + \gamma\\ &= \eta\omega\,\langle (u_{\theta_1})^2\rangle + \gamma\ ,\end{aligned}$$

proving the claims. ∎

2.2 Newton scheme

We describe, now, the Newton (KAM) scheme, on which the proof of Theorem 3 is based. We start with a simple lemma on the differential $d\mathcal{F}$ of the operator

$$\mathcal{F}_\eta(v;\beta) = \Delta_\eta v + g_x(\theta_1+v,\theta_2) + \beta\ ,$$

which is given by[16]

$$d\mathcal{F}_{\eta,v} = \Delta_\eta + g_{xx}(\theta_1+v,\theta_2)\ . \tag{2.35}$$

[16]By definition $d\mathcal{F}_{\eta,v}(w) := \lim_{\tau\to 0}\dfrac{\mathcal{F}_\eta(v+\tau w;\beta)-\mathcal{F}_\eta(v;\beta)}{\tau}$; notice that since β appears as an additive constant in the expression for $\mathcal{F}_\eta$, the differential of $\mathcal{F}_\eta$ is independent of β.

Lemma 2 *Let $v \in C^2(\mathbb{T}^2)$ and $\beta \in \mathbb{R}$. Assume that*

$$V := 1 + v_{\theta_1} \neq 0 \ , \qquad \forall\ \theta \in \mathbb{T}^2 \ , \tag{2.36}$$

and define

$$\begin{aligned} & e := e(\theta;\eta) := \mathcal{F}_\eta(v;\beta) \ , \\ & W := V^2 \ , \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.37) \\ & A_{\eta,v} : w \in C^2(\mathbb{T}^2) \to A_{\eta,v} w := V^{-1} D_\eta \Big(W D_0 \left(V^{-1} w \right) \Big) \in C(\mathbb{T}^2) \ . \qquad (2.38) \end{aligned}$$

Then, for every $w \in C^2(\mathbb{T}^2)$ and any $\eta \in \mathbb{R}$,

$$d\mathcal{F}_{\eta,v}(w) = A_{\eta,v} w + V^{-1} e_{\theta_1}\ w \ . \tag{2.39}$$

Notice that from (2.30) it follows that

$$A_{\eta,v} w = \Delta_\eta w - V^{-1} \Delta_\eta V\ w \ . \tag{2.40}$$

Proof (of Lemma 2) From the definition of $e(\theta)$ it follows that

$$e_{\theta_1} = \Delta_\eta V + g_{xx}(\theta_1 + v, \theta_2) V \ . \tag{2.41}$$

Then, by (2.35), (2.40) and (2.41) one sees that

$$\begin{aligned} d\mathcal{F}_{\eta,v}(w) &= \Delta_\eta w + g_{xx}(\theta_1 + v, \theta_2) w \\ &= A_{\eta,v} w + V^{-1} \Big(\Delta_\eta V + g_{xx}(\theta_1 + v, \theta_2) V \Big) w \\ &= A_{\eta,v} w + V^{-1} e_{\theta_1}\ w \ . \quad \blacksquare \end{aligned}$$

The idea of a Newton scheme is to start with an *approximate solution* of the equation $\mathcal{F}_\eta(u;\gamma) = 0$, namely, a function $v : \mathbb{T}^2 \to \mathbb{R}$ *and* a number β such that

$$e \ := \ \mathcal{F}_\eta(v;\beta) := \Delta_\eta v + g_x(\theta_1 + v, \theta_2) + \beta$$

is small and, then, to find a "quadratically better approximation"

$$v' = v + w \qquad \text{and} \qquad \beta' = \beta + \hat\beta$$

satisfying

$$w = O_1 = \hat\beta \qquad \text{and} \qquad \mathcal{F}_\eta(v';\beta') = O_2 \quad \text{where} \quad O_k = O(\|e\|^k) \ ,$$

To find w and $\hat\beta$, we define

$$Q_1 := g_x(\theta_1+v+w,\theta_2) - g_x(\theta_1+v,\theta_2) - g_{xx}(\theta_1+v,\theta_2)w\ , \qquad Q_2 := V^{-1}e_{\theta_1}\, w\ ; \tag{2.42}$$

notice that the Q_i's are quadratic in w and e. Then, by Lemma 2, one has

$$\begin{aligned} \mathcal{F}_\eta(v';\beta') &:= \mathcal{F}_\eta(v+w;\beta+\hat\beta) \\ &= \mathcal{F}_\eta(v;\beta)+\hat\beta+d\mathcal{F}_{\eta,v}(w)+Q_1 \\ &= e+\hat\beta+A_{\eta,v}w+Q_1+Q_2\ . \end{aligned} \tag{2.43}$$

Next result shows how to solve the equation

$$e+\hat\beta+A_{\eta,v}w=0$$

under suitable conditions upon the function v; in such equation v and (hence) e are given, while w and $\hat\beta$ are unknowns.

Proposition 1 *Let g, ω and I_0 be as in Theorem 3; let $B := \mathbb{T}^2\times I_0$. Let $v\in C^\infty(B)$ and let $\theta\to v(\theta;b)$ belong to $\mathcal{H}_0^\xi$ for some $\xi>0$ and any $\eta\in I_0$; let $\beta\in C^\infty(I_0)$. Finally, let V and W be defined, respectively, as in* (2.36) *and* (2.37) *and assume that $V(\theta)\neq 0$ for any $\theta\in\mathbb{T}^2$, $\langle W^{-1}\rangle\neq 0$ and that*

$$\xi+\|v_{\theta_1}\|_\xi<\bar\xi\ , \tag{2.44}$$

$$|\eta\,\langle W^{-1}D_\eta^{-1}v_{\theta_1}\rangle| < |\langle W^{-1}\rangle|\ . \tag{2.45}$$

Define:

$$\begin{aligned} E(\theta) &:= E(\theta;\eta) := Ve := V\mathcal{F}_\eta(v;\beta)\ , \\ \overline{E} &:= \overline{E}(\eta) := \langle E\rangle\ , \\ \widetilde{E}(\theta) &:= \widetilde{E}(\theta;\eta) := E-\overline{E}\ , \\ a &:= a(\eta) := \frac{\langle W^{-1}D_\eta^{-1}\widetilde{E}\rangle - \overline{E}\langle W^{-1}D_\eta^{-1}v_{\theta_1}\rangle}{\langle W^{-1}\rangle+\eta\langle W^{-1}D_\eta^{-1}v_{\theta_1}\rangle}\ , \\ \hat\beta &:= \hat\beta(\eta) := -(\overline{E}+\eta)\,a = -\frac{\langle W^{-1}\rangle\overline{E}+\eta\,\langle W^{-1}D_\eta^{-1}\widetilde{E}\rangle}{\langle W^{-1}\rangle+\eta\langle W^{-1}D_\eta^{-1}v_{\theta_1}\rangle}\ , \\ E_1(\theta) &:= E_1(\theta;b) := D_\eta^{-1}\widetilde{E} - a\left(1+\eta D_\eta^{-1}v_{\theta_1}\right) - \overline{E}\,D_\eta^{-1}v_{\theta_1}\ . \end{aligned} \tag{2.46}$$

Then:

(i) all functions in (2.46) *are* $C^\infty(B)$ *(or* $C^\infty(I_0)$ *if do not depend on* θ *explicitly) and, for any* $\xi' < \xi$*, they belong to* $\mathcal{H}^{\xi'}$*, for all* $\eta \in I_0$*;*

(ii) the following equalities hold:

$$D_\eta E_1 = E + \hat\beta V \ , \tag{2.47}$$
$$\langle W^{-1} E_1 \rangle = 0 \ . \tag{2.48}$$

(iii) If we define

$$\hat w(\theta;\eta) := -V D_0^{-1}(W^{-1}E_1) \ , \qquad w := \hat w - V\langle \hat w\rangle \tag{2.49}$$

then these functions are $C^\infty(B)$ *(or* $C^\infty(I_0)$ *if do not depend on* θ *explicitly) and, for any* $\xi' < \xi$*, they belong to* $\mathcal{H}^{\xi'}$*, for all* $\eta \in I_0$*. Furthermore, the following identities hold:*

$$\langle w \rangle = 0 \ , \tag{2.50}$$
$$e + \hat\beta + A_{\eta,v} w = 0 \ . \tag{2.51}$$

From this statement[17], the definitions in (2.42) and from (2.43), there follows immediately the following

Corollary 2 *Under the assumptions of Proposition 1, if* $\hat\beta$ *and* w *are defined as in* (2.46) *and* (2.49)*, then*

$$e' := \mathcal{F}_\eta(v + w; \beta + \hat\beta) = Q_1 + Q_2 \ .$$

Proof (of Proposition 1)

(i): The regularity properties of the functions defined in (2.46) follow from the assumption (2.45) and point (iv) of Lemma 1. Notice that, in view of point (vi) of Remark 4, assumption (2.44) implies that $\theta \to g_x(\theta_1 + v(\theta), \theta_2)$ belongs to $\mathcal{H}^\xi$ (and, hence, so does $e(\theta_1)$).

(ii): From the definitions of E_1 and $\hat\beta$ (and of the operator D_η), there follows

$$\begin{aligned} D_\eta E_1 &= \widetilde{E} - D_\eta\, a - \eta\, a\, v_{\theta_1} - \overline{E} v_{\theta_1} \\ &= E - (\overline{E} + \eta a) V \\ &= E + \hat\beta V \ , \end{aligned}$$

proving (2.47). Again, from the definitions of E_1 and a, there follows

$$\langle W^{-1}E_1\rangle = \langle W^{-1} D_\eta^{-1} \widetilde{E}\rangle - a\Big(\langle W^{-1}\rangle + \eta \langle W^{-1} D_\eta^{-1} v_{\theta_1}\rangle\Big) - \overline{E}\langle W^{-1} D_\eta^{-1} v_{\theta_1}\rangle = 0 \ ,$$

[17] Compare, especially, (2.51).

proving (2.48).

(iii): The regularity claim is handled as above. Equation (2.50) follows at once from the definition of w (noting that $\langle V\rangle = 1$). To check (2.51), first, observe that the definition of w implies that

$$\begin{aligned} D_0(V^{-1}w) &= D_0\Big(- D_0^{-1}(W^{-1}E_1) - \langle \hat{w}\rangle\Big) \\ &= -W^{-1}E_1 \ . \end{aligned}$$

Whence, recalling (2.38) and (2.47) we find

$$\begin{aligned} e + \hat{\beta} + A_{\eta,v} w &= V^{-1}\Big[Ve + \hat{\beta}V + VA_{\eta,v}w\Big] \\ &= V^{-1}\Big[E + \hat{\beta}V + D_\eta\Big(WD_0(V^{-1}w)\Big)\Big] \\ &= V^{-1}\Big[E + \hat{\beta}V - D_\eta E_1\Big] \\ &= 0 \ . \qquad \blacksquare \end{aligned}$$

Remark 6 (i) Notice that $a, \hat{\beta} = O(\|e\|)$ so that, also, $E_1, w = O(\|e\|)$. Thus, $Q_i, e' = O(\|e\|^2)$. Explicit estimates will be provided in the next paragraph.
(ii) From (2.34) it follows that

$$\overline{E} := \langle Ve\rangle = \langle (1 + v_{\theta_1})\mathcal{F}_\eta(v;\beta)\rangle = \beta + \eta\,\omega \langle (v_{\theta_1})^2\rangle. \tag{2.52}$$

(iii) In the conservative case ($\eta = 0$) we have that $\overline{E} = \beta$ and

$$E = V\mathcal{F}_0(v;\beta) = E_0 + \beta V$$

with (compare (2.52))

$$E_0 := V\mathcal{F}_0(v;0) \ , \qquad \langle E_0\rangle = 0 \ .$$

Thus, from the definitions given in (2.46) and (2.49), there follows that

$$\begin{aligned} &\widetilde{E} = E - \beta = E_0 + \beta v_{\theta_1} \ , \\ &a = \frac{\langle W^{-1}D_0^{-1}E_0\rangle}{\langle W^{-1}\rangle} \ , \\ &\hat{\beta} = -\beta \ , \\ &E_1 = D_0^{-1}E_0 - a \ , \end{aligned}$$

and w as in (2.49). Thus, w *is independent of* β *and so is the new approximate solution*

$$e' = \mathcal{F}_0(v + w; 0) \ ,$$

(recall that $\beta' = \beta + \hat{\beta} = 0$ in this case). This shows that in the conservative case *one can always take* $\beta = 0$.

2.3 KAM Estimates

Here we collect the main estimates for the KAM algorithm described in Proposition 1. We start with the following

Definition 1 *Let $\bar{\xi} > \xi > 0$. Then,*

- $\mathcal{V}_\xi$ *denotes the set of functions* $v \in C^\infty(\mathbb{T}^2 \times I_0)$ *such that, for all* $\eta \in I_0$, $\theta \to v(\theta;\eta) \in \mathcal{H}_0^\xi$.
- $\mathcal{V}_\xi^{\bar{\xi}}$ *denotes the subset of* $v \in \mathcal{V}_\xi$ *verifying, for all* $\eta \in I_0$, $V(\theta) := 1 + v_{\theta_1}(\theta) \neq 0$ *for all* $\theta \in \mathbb{T}^2$, $\langle W^{-1}\rangle \neq 0$ *where* $W := V^2$ *and*
$$\|v_{\theta_1}\|_\xi < \bar{\xi} - \xi \ , \qquad |\eta \langle W^{-1} D_\eta^{-1} v_{\theta_1}\rangle| < |\langle W^{-1}\rangle| \ .$$
- $\mathcal{W}_\xi$ *denotes the set* $\mathcal{V}_\xi \times C^\infty(I_0)$.
- $\mathcal{W}_\xi^{\bar{\xi}}$ *denotes the set* $\mathcal{V}_\xi^{\bar{\xi}} \times C^\infty(I_0)$.

If $(v,\beta) \in \mathcal{W}_\xi^{\bar{\xi}}$, *we define*

$$\mathcal{K}(v,\beta) := (v',\beta') := (v + w, \beta + \hat{\beta}) \ ,$$

where w *and* $\hat{\beta}$ *are defined as in Proposition 1.*

Thus, by Proposition 1,

$$\mathcal{K} : \mathcal{W}_\xi^{\bar{\xi}} \to \mathcal{W}_{\xi'} \ , \qquad\qquad \forall\, \bar{\xi} > \xi > \xi' > 0 \ .$$

Lemma 3 *(i) Let* $(v,\beta) \in \mathcal{W}_\xi^{\bar{\xi}}$; *let* $(w,\hat{\beta}) := \mathcal{K}(v,\beta) - (v,\beta)$ *(compare Definition 1) and define*

$$V := 1 + v_{\theta_1} \ , \qquad W := V^2 \ , \qquad \rho := \eta \, \frac{\langle W^{-1} D_\eta^{-1} v_{\theta_1}\rangle}{\langle W^{-1}\rangle} \ . \tag{2.53}$$

Assume that there exist ν, α *such that, for any* $\eta \in I_0$,

$$\|v_{\theta_1}\|_\xi < \nu < \bar{\xi} - \xi \ , \tag{2.54}$$
$$|\rho| < \alpha < 1 \ . \tag{2.55}$$

Then, there exists $s := s(\tau) > 1$ *and*[18] $c := c(\kappa, \tau, M, \nu, \alpha, \eta_0) > 1$ *such that, for any* $0 < \delta < \xi$ *and any* $\eta \in I_0$, *the following estimates hold:*

$$|\hat{\beta}| \, , \, \|w_{\theta_1}\|_{\xi-\delta} \, , \, \|D_\eta^{-1} w_{\theta_1}\|_{\xi-\delta} \ \leq c\, \delta^{-s}\, \|e\|_\xi \ . \tag{2.56}$$

[18]Recall the definition of M in (1.20).

(ii) Let W' and ρ' be defined as, respectively, W and ρ with v replaced by $v' := (v+w)$. If

$$\|w_{\theta_1}\|_{\xi-\delta} \le \nu - \|v_{\theta_1}\|_\xi \ , \tag{2.57}$$

then

$$\|W' - W\|_{\xi-\delta} \le c\,\delta^{-s}\,\|e\|_\xi \ ; \tag{2.58}$$

$e' := \mathcal{F}_\eta(v';\beta') := \mathcal{F}_\eta(\mathcal{K}(v,\beta))$ *belongs to* $\mathcal{H}^{\xi-\delta}$*, for any* $\eta \in I_0$*, and satisfies*

$$\|e'\|_{\xi-\delta} \le c\,\delta^{-s}\,\|e\|_\xi^2 \ . \tag{2.59}$$

Finally, if

$$|W' - W|_{\mathbb{T}^2} \le \frac{1}{4}\frac{(1-\nu)^4}{(1+\nu)^2} \ , \tag{2.60}$$

then

$$\inf_{\mathbb{T}^2} W' > \frac{(1-\nu)^2}{2} \ , \qquad \langle (W')^{-1}\rangle > \frac{(1+\nu)^2}{2} \tag{2.61}$$

and

$$\sup_{\mathbb{T}^2} |(W')^{-1} - W^{-1}|\,,\ |\rho' - \rho| \ \le c\,\delta^{-s}\,\|e\|_\xi \ . \tag{2.62}$$

Proof The proof of this Lemma is (KAM) routine and it is based on a systematic use of point (iii) of Lemma 1 and of point (vi) of Remark 4.
We begin by observing that (2.54) implies that

$$\begin{aligned}
&\mu_0 := (1+\nu)^{-2} < |W^{-1}(\theta)| < \mu_1 := (1-\nu)^{-2} \ , \quad \forall\,\theta \in \mathbb{S}^2_\xi \ ,\\
&\mu_0 < W^{-1}(\theta) < \mu_1 \ , \quad \forall\,\theta \in \mathbb{T}^2 \ .
\end{aligned} \tag{2.63}$$

Next, from the definitions in (2.46) and (2.54) it follows immediately that

$$|\overline{E}|\,,\ \|\widetilde{E}\|_\xi\,,\ \|E\|_\xi \le (1+\nu)\,\|e\|_\xi < 2\|e\|_\xi \ .$$

From (2.31) it follows

$$\begin{aligned}
&\sup_{\mathbb{T}^2} |D_\eta^{-1} v_{\theta_1}| \le \|D_\eta^{-1} v_{\theta_1}\|_0 \le \sigma_{0,1}(\xi)\|v_{\theta_1}\|_\xi < \sigma_{0,1}(\xi)\,\nu\\
&\sup_{\mathbb{T}^2} |D_\eta^{-1}\widetilde{E}| \le \|D_\eta^{-1}\widetilde{E}\|_0 \le \sigma_{0,1}(\xi)\|e\|_\xi \ .
\end{aligned}$$

From (2.63) one has

$$\inf_{\mathbb{T}^2} W > \frac{1}{\mu_1} \ , \qquad \langle W^{-1}\rangle > \mu_0 \ ,$$

so that

$$\begin{aligned}\langle W^{-1}\rangle + \eta\langle W^{-1}D_\eta^{-1}v_{\theta_1}\rangle &\geq \langle W^{-1}\rangle(1-|\rho|)\\ &> \langle W^{-1}\rangle(1-\alpha)\\ &> \mu_0\,(1-\alpha)\ .\end{aligned}$$

We are, now, ready to estimate $|\hat\beta|$: in view of (2.32) and the above estimates, we get

$$|\hat\beta| \leq c_1\,\xi^{-\tau}\,\|e\|_\xi < c_1\,\delta^{-\tau}\,\|e\|_\xi$$

with

$$c_1 := \frac{1}{1-\alpha}\,\frac{\mu_1}{\mu_0}\left(2+\frac{\eta_0}{\kappa}\,(\tau/\,\mathrm{e})^\tau\right) = \frac{1}{1-\alpha}\,\frac{(1+\nu)^2}{(1-\nu)^2}\left(2+\frac{\eta_0}{\kappa}\,(\tau/\,\mathrm{e})^\tau\right)\ .$$

Analogously, we find

$$\begin{aligned}|a| &\leq c_2\,\xi^{-\tau}\|e\|_\xi < c_2\,\delta^{-\tau}\|e\|_\xi\ ,\\[1em] \|E_1\|_{\xi-\frac{\delta}{3}} &\leq \|D_\eta^{-1}\widetilde E\|_{\xi-\frac{\delta}{3}} + |a|(1+|\eta|\|D_\eta^{-1}v_{\theta_1}\|_{\xi-\frac{\delta}{3}}) + |\overline{E}|\ \|D_\eta^{-1}v_{\theta_1}\|_{\xi-\frac{\delta}{3}}\\ &\leq \sigma_{0,1}(\delta/3)\|\widetilde E\|_\xi + |a|(1+|\eta|\sigma_{0,1}(\delta/3)\nu) + |\overline{E}|\sigma_{0,1}(\delta/3)\nu\\ &\leq c_3\,\delta^{-2\tau}\|e\|_\xi\ \ ,\end{aligned}\tag{2.64}$$

for suitable constants $c_i = c_i(\kappa,\tau,\eta_0,\nu,\alpha)$ (which, being clear the ideas, we do not compute any more); notice that $\|E_1\|$ has been estimated on an intermediate space $\mathcal{H}^{\xi-\frac{\delta}{3}}$ so as to be able to control further applications of the operator D_η^{-1} or ∂_{θ_1}.

Now, recalling the definition of w and $\hat w$ in (2.49), by point (iii) of Lemma 1, by (2.64), one gets

$$\begin{aligned}\|\hat w\|_{\xi-\frac{2}{3}\delta} &\leq (1+\|v_{\theta_1}\|_\xi)\,\|D_0^{-1}(W^{-1}E_1)\|_{\xi-\frac{2}{3}\delta}\\ &\leq (1+\nu)\sigma_{0,1}(\delta/3)\,(1-\nu)^{-2}\,\|E_1\|_{\xi-\frac{\delta}{3}}\\ &\leq c_4\,\delta^{-3\tau}\,\|e\|_\xi\ .\end{aligned}$$

Thus, since $|\langle\hat w\rangle| \leq \|\hat w\|_{\xi-\frac{2}{3}\delta}$, we find

$$\|w\|_{\xi-\frac{2}{3}\delta} \leq c_5\delta^{-3\tau}\|e\|_\xi\ ,\tag{2.65}$$

and

$$\begin{aligned}&\|w_{\theta_1}\|_{\xi-\delta} \leq \sigma_{1,0}(\delta/3)\,\|w\|_{\xi-\frac{2}{3}\delta} \leq c_6\,\delta^{-(3\tau+1)}\,\|e\|_\xi\ ,\\ &\|D_\eta^{-1}w_{\theta_1}\|_{\xi-\delta} \leq \sigma_{0,1}(\delta/3)\,\|w_{\theta_1}\|_{\xi-\frac{2}{3}\delta} \leq c_7\,\delta^{-(4\tau+1)}\,\|e\|_\xi\ ,\end{aligned}\tag{2.66}$$

which proves part (i) of the Lemma.
Now, by (2.57) and (2.66),

$$\begin{aligned}\|W' - W\|_{\xi-\delta} &= \|2w_{_{\theta_1}} V + w_{_{\theta_1}}^2\|_{\xi-\delta} \\ &\leq 2\nu\|w_{_{\theta_1}}\|_{\xi-\delta} + \|w_{_{\theta_1}}\|_{\xi-\delta}^2 \leq 3\nu\|w_{_{\theta_1}}\|_{\xi-\delta} \\ &\leq c_8\delta^{-(3\tau+1)}\,\|e\|_\xi\ . \end{aligned} \tag{2.67}$$

Inequality (2.57) guarantees, also, that $e' \in \mathcal{H}^{\xi-\delta}$; compare point (vi), Remark 4. Then, by Corollary 2 and (2.65), one gets (for a suitable $c_9 > 1$)

$$\begin{aligned}\|e'\|_{\xi-\delta} &\leq \|Q_1\|_{\xi-\delta} + \|Q_2\|_{\xi-\delta} \leq M\,\|w\|_{\xi-\delta}^2 + (1-\nu)^{-1}\sigma_{1,0}(\delta)\|e\|_\xi\|w\|_{\xi-\delta} \\ &\leq c_9\,\delta^{-6\tau}\,\|e\|_\xi^2\ .\end{aligned}$$

Let $\widetilde{W} := W' - W$. From (2.60) it follows that

$$\sup_{\mathbb{T}^2}|\widetilde{W}| \leq \frac{1}{4}\frac{(1-\nu)^4}{(1+\nu)^2} = \frac{1}{4}\frac{\mu_0}{\mu_1^2} < \frac{1}{2\mu_1}\ , \tag{2.68}$$

so that, on $\mathbb{T}^2$ one has (recall (2.63))

$$\frac{1}{2} \leq 1 + W^{-1}\widetilde{W} \leq \frac{3}{2}\ , \qquad W' = W + \widetilde{W} > \frac{1}{\mu_1} - \frac{1}{2\mu_1} = \frac{(1-\nu)^2}{2} > 0\ .$$

Let now $Z := (W')^{-1} - W^{-1}$. Then, by (2.67) and (2.68) one finds (on $\mathbb{T}^2$)

$$|Z| = \frac{\widetilde{W}}{W^2(1+W^{-1}\widetilde{W})} \leq 2\mu_1^2\,|\widetilde{W}| \leq c_{10}\min\{\delta^{-(3\tau+1)}\,\|e\|_\xi\,,\,1\} \tag{2.69}$$

(for a suitable $c_{10} > 1$), proving also the first inequality in (2.62). Furthermore, by the first inequality in (2.68),

$$\langle (W')^{-1}\rangle = \langle W^{-1}\rangle + \langle Z\rangle \geq \langle W^{-1}\rangle - |\langle Z\rangle| \geq \mu_0 - 2\mu_1^2\,|\widetilde{W}| \geq \frac{\mu_0}{2} = \frac{1}{2(1+\nu)^2}\ .$$

Finally, using (2.69), (2.66) and (2.57) (in order to estimate $\|w_{_{\theta_1}}\|$ in terms of $\nu < 1$), one obtains

$$|\rho' - \rho| \leq \left|\eta\frac{(W^{-1}+Z)(D_\eta^{-1}v_{_{\theta_1}} + D_\eta^{-1}w_{_{\theta_1}})}{\langle W^{-1}\rangle + \langle Z\rangle} - \eta\frac{W^{-1}D_\eta^{-1}v_{_{\theta_1}}}{\langle W^{-1}\rangle}\right| \leq c_{11}\,\delta^{-(4\tau+1)}\,\|e\|_\xi\ ,$$

proving also the second inequality in (2.62).
The theses of the Lemma follow, now, by taking $c := \max_i c_i$ and $s = 6\tau$. ∎

2.4 Convergence of the Nash–Moser algorithm

Here we complete the quantitative description of the KAM procedure giving a sufficient condition in order for the algorithm to converge.
For any $i \geq 0$ and for $0 < \xi_* < \xi < \bar{\xi}$, we let

$$\xi_i := \xi_* + \frac{\xi - \xi_*}{2^i} \ , \qquad \delta_{i+1} := \xi_i - \xi_{i+1} = \frac{\xi - \xi_*}{2^{i+1}} \ , \qquad (\forall\ i \geq 0) \ ;$$

as above (compare (2.63)) we let $\mu_0 := (1+\nu)^{-2}$ and $\mu_1 := (1-\nu)^{-2}$; we let

$$0 < \mu_0 < \bar{\mu}_0 := \inf_{\mathbb{T}^2} W^{-1} \leq \bar{\mu}_1 := \sup_{\mathbb{T}^2} W^{-1} < \mu_1 \ ; \tag{2.70}$$

finally, we let (recall the definition of ρ in (1.21))

$$\mu := \min \left\{ \nu - \|v_{\theta_1}\|_\xi \ , \ \frac{\mu_0}{2\mu_1^2} \ , \ \mu_1 - \bar{\mu}_1 \ , \ \bar{\mu}_0 - \mu_0 \ , \ \alpha - |\rho| \right\} > 0 \ . \tag{2.71}$$

Proposition 2 *Under the same assumptions and notations of part (i) of Lemma 3, let $0 < \xi_* < \xi$ and let ξ_i, μ_i, $\bar{\mu}_i$ and μ be as above; let C and m be positive numbers such that*

$$C \geq c\, 4^s\, (\xi - \xi_*)^{-s} \ , \qquad m \leq 2^{s-1}\, \mu \ , \tag{2.72}$$

and assume that

$$C\, \|e\|_\xi \leq \mathrm{e}^{-\frac{1}{\mathrm{e}\, m}} \ . \tag{2.73}$$

Then, $(v_i, \beta_i) := \mathcal{K}^i(v, \beta) = (v_{i-1} + w_i, \beta_{i-1} + \hat{\beta}_i) \in \mathcal{W}^{\bar{\xi}}_{\xi_i}$ for all $i \geq 1$; the sequences $\{v_i\}$ and $\{\beta_i\}$ converge uniformly on, respectively, $\mathbb{S}^2_{\xi_*} \times I_0$ and I_0, defining a limit

$$(u, \gamma) := \lim_{j \to \infty} (v_j, \beta_j) = \Big(v + \sum_{i=1}^\infty w_i \ , \ \beta + \sum_{i=1}^\infty \hat{\beta}_i \Big) \in \mathcal{W}^{\bar{\xi}}_{\xi_*} \ ,$$

which is a solution of (1.18), i.e., $\mathcal{F}_\eta(u; \gamma) = 0$ for all $\eta \in I_0$. Furthermore, if W_i and ρ_i are defined as, respectively, W and ρ in (1.21) with v replaced by v_i, then $W_* := \lim W_i \in \mathcal{W}^{\bar{\xi}}_{\xi_*}$, $\rho_* := \lim \rho_i \in C^\infty(I_0)$, and (for all $\theta \in \mathbb{T}^2$ and any $\eta \in I_0$)

$$\|u_{\theta_1}\|_{\xi_*} \leq \nu \ , \qquad |\rho_*| \leq \alpha \ , \tag{2.74}$$

$$|\gamma - \beta| \, , \, \|u_{\theta_1} - v_{\theta_1}\|_{\xi_*} \, , \, |\rho - \rho_*| \, , \, \|W - W_*\|_{\xi_*} \leq C_*\, \|e\|_\xi \ , \tag{2.75}$$

where

$$C_* := \frac{C}{2^s} \left(1 - \mathrm{e}^{-\frac{1}{\mathrm{e}\, m}} \right)^{-1} \ .$$

Proof We claim that (1.24) implies that, for any $i \geq 1$,

$$(v_i, \beta_i) := \mathcal{K}^i(v, \beta) \in \mathcal{W}_{\xi_i}^{\bar{\xi}} \tag{2.76}$$

$$\max\{|\hat{\beta}_i|, \|\partial_{\theta_1} w_i\|_{\xi_i}, |\rho_i - \rho_{i-1}|, \sup_{\mathbb{T}^2} |W_i - W_{i-1}|, \sup_{\mathbb{T}^2} |W_i^{-1} - W_{i-1}^{-1}|\} \leq \frac{(C\|e\|_\xi)^{2^{i-1}}}{2^s} \tag{2.77}$$

$$\theta \to e_i(\theta) := \mathcal{F}_\eta(v_i; \beta_i) \in \mathcal{H}^{\xi_i} \quad \text{and} \quad \|e_i\|_{\xi_i} \leq \frac{(C\|e\|_\xi)^{2^i}}{C\, 2^{s\, i}} \tag{2.78}$$

where $W_0 := W$, $\rho_0 := \rho$. We prove the claim by induction. First of all, observe that (1.24) implies immediately that[19]

$$2^k (C\|e\|_\xi)^{2^k} \leq m , \qquad \forall\, k \geq 0 . \tag{2.79}$$

Now, let us check (2.76)÷(2.78) for $i = 1$. By Lemma 3, part (i), with $\delta := \delta_1$, $\xi - \delta = \xi_1$, $(v', \beta') = (v_1, \beta_1) = \mathcal{K}(v, \beta) = (v + w_1, \beta + \hat{\beta}_1)$, by definition of C, δ_1, m and μ and by (2.79) (with $k = 0$), we have

$$|\hat{\beta}_1| , \|\partial_{\theta_1} w_1\|_{\xi_1} \leq c\delta_1^{-s} \|e\|_\xi = \frac{C\|e\|_\xi}{2^s} \leq \frac{\mu}{2} . \tag{2.80}$$

In particular

$$\|\partial_{\theta_1} w_1\|_{\xi_1} \leq \frac{\nu - \|v_{\theta_1}\|_\xi}{2} ,$$

which allows to apply part (ii) of Lemma 3, with $W' = W_1$, $e' = e_1$, $\rho' = \rho_1$, and to obtain, as above,

$$\|W_1 - W\|_{\xi_1} \leq \frac{C\|e\|_\xi}{2^s} \leq \frac{\mu}{2} , \qquad \|e_1\|_{\xi_1} \leq \frac{C}{2^s} \|e\|_\xi^2 = \frac{(C\|e\|_\xi)^2}{C\, 2^s} , \tag{2.81}$$

and, since $\mu \leq \mu_0/(2\mu_1^2)$, by (2.62) (recall the identity in (2.68)), we have also

$$\inf_{\mathbb{T}^2} W_1 > \frac{1}{2\mu_1} , \qquad \langle W_1^{-1} \rangle > \frac{\mu_0}{2}$$
$$\sup_{\mathbb{T}^2} |W_1^{-1} - W^{-1}| , |\rho_1 - \rho| \leq \frac{C\|e\|_\xi}{2^s} ,$$

[19] If x and y are positive numbers such that $x \leq \exp(-1/(\mathrm{e}y))$, then $tx^t \leq y$ for any $t > 0$. In fact, let $\lambda := \log x^{-1}$ and observe that the hypothesis is equivalent to $\frac{1}{\mathrm{e}\,\lambda} \leq y$. Then, $tx^t = \frac{1}{\lambda}(\lambda t)\exp(-\lambda t) \leq \frac{1}{\mathrm{e}\lambda} \leq y$.

which, together with (2.80) and (2.81), prove (2.77) and (2.78) for $i = 1$. Finally, from the above estimates and definitions, there follows

$$\|\partial_{\theta_1} v_1\|_{\xi_1} \le \|v_{\theta_1}\|_\xi + \|\partial_{\theta_1} w_1\|_{\xi_1} \le \|v_{\theta_1}\|_\xi + \frac{\nu - \|v_{\theta_1}\|_\xi}{2} < \nu < \bar\xi - \xi \ ,$$
$$|\rho_1| \le |\rho| + |\rho_1 - \rho| \le |\rho| + \frac{\mu}{2} \le |\rho| + \frac{\alpha - |\rho|}{2} < \alpha < 1 \ ,$$

showing that $(v_1, \beta_1) \in \mathcal{W}^{\bar\xi}_{\xi_1}$, proving also (2.76) for $i = 1$.
Let, now, $j \ge 1$ and assume that (2.76)÷(2.78) hold true for $1 \le i \le j$; we want to prove (2.76)÷(2.78) for $i = j + 1$. Let $1 \le i \le j$; then, by definition of v_i, by (2.77) and (2.79), there follows

$$\begin{aligned}
\|\partial_{\theta_1} v_i\|_{\xi_i} &= \|v_{\theta_1} + \sum_{k=1}^{i} \partial_{\theta_1} w_k\|_{\xi_i} \\
&\le \|v_{\theta_1}\|_\xi + \sum_{k=1}^{i} \|\partial_{\theta_1} w_k\|_{\xi_k} \\
&\le \|v_{\theta_1}\|_\xi + \sum_{k=1}^{i} \frac{(C\|e\|_\xi)^{2^{k-1}}}{2^s} \\
&\le \|v_{\theta_1}\|_\xi + \frac{m}{2^{s-1}} \sum_{k=1}^{i} \frac{1}{2^k} \\
&= \|v_{\theta_1}\|_\xi + \mu \sum_{k=1}^{i} \frac{1}{2^k} \\
&< \|v_{\theta_1}\|_\xi + \mu < \nu \ .
\end{aligned} \tag{2.82}$$

Analogously, for any $1 \le i \le j$,

$$\begin{aligned}
|\rho_i| &= |\rho + \sum_{k=1}^{i} \rho_k - \rho_{k-1}| \\
&\le |\rho| + \sum_{k=1}^{i} |\rho_k - \rho_{k-1}| \\
&< |\rho| + \mu < \alpha \ .
\end{aligned} \tag{2.83}$$

Thus, we can apply Lemma 3, part (i) (with $\delta := \delta_{j+1}$, $\xi = \xi_j$, v, β, ρ replaced by v_j, β_j, ρ_j, $(v', \beta') = (v_{j+1}, \beta_{j+1}) = \mathcal{K}(v_j, \beta_j) = (v_j + w_{j+1}, \beta_j + \hat\beta_{j+1})$), which, in view of (2.56), the identity

$$c\,\delta_{i+1}^{-s} = \frac{C}{2^s}\,2^{s\,i}\ , \tag{2.84}$$

and (2.78), yields

$$\begin{aligned} |\hat\beta_{j+1}|\,,\ \|\partial_{\theta_1} w_{j+1}\|_{\xi_{j+1}} &\le c\,\delta_{j+1}^{-s}\ \|e_j\|_{\xi_j} \\ &= \frac{C}{2^s}\,2^{s\,j}\,\|e_j\|_{\xi_j} \\ &\le \frac{(C\|e\|_\xi)^{2^j}}{2^s}\ , \end{aligned} \tag{2.85}$$

which proves the first two bounds in (2.77) with $i=j+1$. To apply part (ii) of Lemma 3, we have to check (2.57) with w and v replaced, respectively, by w_{j+1} and v_j. Since

$$\|\partial_{\theta_1} v_j\|_{\xi_j} + \|\partial_{\theta_1} w_{j+1}\|_{\xi_{j+1}} \le \|v_{\theta_1}\|_\xi + \sum_{k=1}^{i+1} \|\partial_{\theta_1} w_k\|_{\xi_k}\ ,$$

(2.85) shows that the inequalities in (2.82) hold also for $i=j+1$ so that (2.57) is satisfied. Thus, by (2.58) (with W' and W corresponding, respectively, to W_{j+1} and W_j), (2.84) and by (2.78) with $i=j$, we see that

$$\|W_{j+1}-W_j\|_{\xi_{j+1}} \le c\,\delta_{i+1}^{-s}\|e_j\|_{\xi_j} = \frac{C}{2^s}\,2^{s\,i}\|e_j\|_{\xi_j} \le \frac{(C\|e\|_\xi)^{2^j}}{2^s}\ , \tag{2.86}$$

showing that (2.77) for $\|W_i - W_{i-1}\|$ holds also with $i=j+1$. Now, by (2.59), (2.84) and (2.78) with $i=j$, we find

$$\|e_{j+1}\|_{\xi_{j+1}} \le \frac{C}{2^s}\,2^{s\,j}\,\frac{(C\|e\|_\xi)^{2^{j+1}}}{C^2\,2^{2s\,j}} = \frac{(C\|e\|_\xi)^{2^{j+1}}}{C\,2^{s(j+1)}}\ , \tag{2.87}$$

i.e., (2.78) with $i=j+1$. Next, from (2.86), (2.79), the fact that $j\ge 1$ and the definition of the μ's, there follows

$$\begin{aligned} \|W_{j+1}-W_j\|_{\xi_{j+1}} &\le \frac{(C\|e\|_\xi)^{2^j}}{2^s} \\ &\le \frac{m}{2^s\,2^j} = \frac{\mu}{2^{j+1}} \\ &\le \frac{\mu}{4} \le \frac{\mu_0}{4\mu_1^2} \\ &= \frac{1}{4}\,\frac{(1-\nu)^4}{(1+\nu)^2}\ , \end{aligned}$$

showing that (2.60) holds in the present case. Therefore, by (2.61),

$$\inf_{\mathbb{T}^2} W_{j+1} > \frac{(1-\nu)^2}{2} \ , \qquad \langle W_{j+1}^{-1} \rangle > \frac{(1+\nu)^2}{2}$$

and

$$\sup_{\mathbb{T}^2} |W_{j+1}^{-1} - W_j^{-1}| \ , \ |\rho_{j+1} - \rho_j| \le c\, \delta_{j+1}^{-s} \|e_j\|_{\xi_j} \le \frac{(C\|e\|_\xi)^{2^j}}{2^s} \ , \tag{2.88}$$

showing that (2.77) holds for $i = j+1$. Finally, by (2.88), one sees that (2.83) holds also for $i = j+1$, implying that

$$|\rho_{j+1}| < \alpha < 1 \ ,$$

which shows that $(v_{j+1}, \beta_{j+1}) \in \mathcal{W}_{\xi_{j+1}}^{\bar\xi}$. The claim has been completely proven.

The (fast) decay of $(C\|e\|)^{2^j}$ implies that (v_j, β_j) converge uniformly to $(u, \gamma) \in \mathcal{W}_{\xi_*}^{\bar\xi}$ and that, for any $\eta \in I_0$,

$$\mathcal{F}_\eta(u; \gamma) = \lim_{j\to\infty} \mathcal{F}_\eta(v_j; \beta_j) = \lim_{j\to\infty} e_i = 0 \ ,$$

showing that (u, γ) is a solution of (1.18) for any $\eta \in I_0$. Furthermore, since

$$\|\partial_{\theta_1} v_i\|_{\xi_i} < \nu \ , \qquad |\rho_i| < \alpha \ ,$$

we see, by taking limits, that (1.25) holds. As for the bounds in (1.26), we have, for example, that

$$\begin{aligned} \|u_{\theta_1} - v_{\theta_1}\|_{\xi_*} &\le \sum_{i\ge 1} \|\partial_{\theta_1} w_i\|_{\xi_i} \\ &\le \frac{1}{2^s} \sum_{i\ge 1} (C\|e\|_\xi)^{2^i} \\ &\le \frac{1}{2^s} \sum_{i\ge 1} (C\|e\|_\xi)^i \\ &\le C_* \|e\|_\xi \ . \end{aligned}$$

which implies the second inequality in (1.26); the other inequalities are obtained in exactly the same way. ∎

2.5 Local uniqueness

Next, we prove a local uniqueness result for the solutions of (1.18). Such result is based upon the following simple observation on analytic functions.

Lemma 4 *Let $w \in \mathcal{H}^{\xi_*}$ and assume that there exist $c, s > 0$ such that*

$$\|w\|_{\xi-\delta} \le c\,\|w\|_\xi^2\ \delta^{-s}\ , \qquad \forall\ 0 < \delta < \xi \le \xi_*\ , \tag{2.89}$$

$$(c\,4^s\,\xi_*^{-s})\,\|w\|_{\xi_*} \le 1\ . \tag{2.90}$$

Then $w \equiv 0$.

Proof Let, for all $i \ge 0$, $\xi_i := \xi_*/2^i$ and $\delta_i := \xi_*/2^{i+1}$. Then, by (2.89), one has

$$\|w\|_{\xi_{i+1}} \le B_0\,B_1^i\ \|w\|_{\xi_i}^2\ , \qquad B_0 := c\ 2^s\xi_*^{-s}\ , \quad B_1 := 2^s\ .$$

Iterating this relation[20] one gets, for all $i \ge 0$,

$$\|w\|_0 \le \|w\|_{\xi_i} \le \frac{\Big(C\|w\|_{\xi_*}\Big)^{2^i}}{C\ 2^{s\,i}}\ , \qquad C := c\,4^s\,\xi_*^{-s}\ ,$$

which, by (2.90), implies that $\|w\|_0 := \sum |w_n| = 0$ so that $w \equiv 0$. ∎

Proposition 3 *Let $0 < \xi_* < \bar\xi$ and let $(u,\gamma) \in \mathcal{W}^{\bar\xi}_{\xi_*}$ be a solution of* (1.18) *(i.e., $\mathcal{F}_\eta(u;\gamma) = 0$), satisfying also*

$$\|u_{\theta_1}\|_{\xi_*} \le \nu \le \bar\xi - \xi_*\ , \qquad |\rho| \le \alpha < 1\ ,$$

for some $\nu, \alpha \in (0,1)$. Then, there exist $\hat s := \hat s(\tau) > 1$ and $\widehat C = \widehat C(\kappa,\tau,\eta_0,M,\nu,\alpha) > 1$ such that, if $(u',\gamma') \in \mathcal{W}^{\bar\xi}_{\xi_*}$ is also a solution of (1.18) *(i.e. $\mathcal{F}_\eta(u';\gamma') = 0$) satisfying*

$$\|u'_{\theta_1}\|_{\xi_*} \le \nu\ , \qquad \widehat C\,\xi_*^{-\hat s}\ \|u' - u\|_\xi \le 1 \tag{2.91}$$

then $u \equiv u'$ and $\gamma = \gamma'$.

[20] If $y_i > 0$ and $\{x_i\}_{i\ge0}$ is a sequence of positive numbers satisfying

$$x_{i+1} \le y_0 y_1^i x_i^2\ ,$$

then one also has $x_i \le (y_0y_1x_0)^{2^i}/(y_0y_1^{i+1})$, as it follows multiplying both sides of the above inequality by $y_0y_1^{i+2}$ so as to obtain $z_{i+1} \le z_i^2$ with $z_i := y_0y_1^{i+1}x_i$.

Proof Let

$$w := u' - u \ , \qquad \hat{\gamma} := \gamma' - \gamma \ ,$$

and observe that, since $e := \mathcal{F}_\eta(u;\gamma) = 0$, then, by (2.35) and (2.39), one has the following identities:

$$A_{\eta,u} w := V^{-1} D_\eta \Big(W D_0 \left(V^{-1} w \right) \Big) = d\mathcal{F}_{\eta,u}(w) = \Delta_\eta w + g_{xx}(\theta_1 + u, \theta_2) w \ , \tag{2.92}$$

where $V := 1 + u_{\theta_1}$ and $W := V^2$. Thus, (since also $\mathcal{F}_\eta(u';\gamma') = 0$)

$$\begin{aligned} 0 &= \mathcal{F}_\eta(u';\gamma') = \mathcal{F}_\eta(u;\gamma) + \Delta_\eta w + g_x(\theta_1 + u', \theta_2) - g_x(\theta_1 + u, \theta_2) + \hat{\gamma} \\ &= \Delta_\eta w + g_{xx}(\theta_1 + u, \theta_2)\, w + \hat{\gamma} \\ &\quad + \Big(g_x(\theta_1 + u + w, \theta_2) - g_x(\theta_1 + u, \theta_2) - g_{xx}(\theta_1 + u, \theta_2) w \Big) \\ &=: A_{\eta,u} w + \hat{\gamma} + Q_1(w) \ . \end{aligned} \tag{2.93}$$

We, now, claim that, if we define

$$E := V Q_1 \ , \qquad \overline{E} := \langle E \rangle \ , \qquad \widetilde{E} := E - \overline{E} \ , \tag{2.94}$$

then

$$\hat{\gamma} = - \frac{\langle W^{-1} \rangle \overline{E} + \eta \, \langle W^{-1} D_\eta^{-1} \widetilde{E} \rangle}{\langle W^{-1} \rangle + \eta \langle W^{-1} D_\eta^{-1} u_{\theta_1} \rangle} \ . \tag{2.95}$$

In fact, from (2.92) and (2.93), there follows

$$D_\eta \Big(W D_0 (V^{-1} w) \Big) + \hat{\gamma} + u_{\theta_1} \hat{\gamma} + \widetilde{E} + \overline{E} = 0 \ . \tag{2.96}$$

If $\eta = 0$, then taking average in (2.96) yields $\hat{\gamma} = -\overline{E}$, which is (2.95) when $\eta = 0$. If $\eta \neq 0$, then, observing that $D_\eta^{-1} 1 = \frac{1}{\eta}$, from (2.96) one obtains

$$D_0(V^{-1} w) + \frac{W^{-1} \hat{\gamma}}{\eta} + \hat{\gamma} W^{-1} D_\eta^{-1} u_{\theta_1} + W^{-1} D_\eta^{-1} \widetilde{E} + \frac{W^{-1} \overline{E}}{\eta} = 0 \ ;$$

multiplying by η and taking average in the latter relation, yields, upon solving for $\hat{\gamma}$, (2.95).

Thus, w satisfies the equation

$$A_{\eta,u} w = Q \ , \qquad Q := -\hat{\gamma} - Q_1 \ , \tag{2.97}$$

with Q “quadratic” in w. In fact, observing that

$$\|Q_1\|_\xi \le M \, \|w\|_\xi^2 \ , \qquad \forall \, \xi \le \xi_* \ ,$$

(M being as in (1.20)), by the relations in (2.97), (2.95) and (2.94), one finds that

$$|\hat{\gamma}|\,,\;\|Q\|_\xi \le c_{11}\,\xi^{-\tau}\;\|w\|_\xi^2\;, \tag{2.98}$$

for a suitable $c_{11} = c_{11}(\kappa, \tau, \eta_0, M, \nu, \alpha) > 1$ and for any $\xi \le \xi_*$. Next, from the definition of $A_{\eta,u}$, one finds the identity

$$D_0(V^{-1}w) = W^{-1}D_\eta^{-1}(VQ)\;, \tag{2.99}$$

and, since $\langle w\rangle = 0$ (as it follows from $u, u' \in \mathcal{H}_0^{\xi_*}$), one sees that (2.99) is equivalent to

$$w = \hat{w} - V\langle\hat{w}\rangle\;, \qquad \hat{w} := VD_0^{-1}\Big(W^{-1}D_\eta^{-1}(VQ)\Big)\;.$$

By Lemma 1, one gets the estimate

$$\|w\|_{\xi-\delta} \le c_{12}\delta^{-2\tau}\;\|Q\|_\xi\;, \qquad (0<\delta<\xi\le\xi_*)\;,$$

which, by (2.98), implies

$$\|w\|_{\xi-\delta} \le c_{13}\,\delta^{-3\tau}\|w\|_\xi^2\;,\;, \qquad (0<\delta<\xi\le\xi_*)$$

showing that w satisfies the estimate (2.89). Letting $\widehat{C} := c_{13}\,4^{3\tau}$ and $\hat{s} := 3\tau$, the thesis follows from Lemma 4. ▮

The above analysis shows the smooth (C^∞) dependence upon the "dissipation" parameter η. We close this section with a brief remark on how solutions depend upon other eventual "external" parameters.

Remark 7 (i) If the function g in (1.18) depends also in a real–analytic way on one (or more) external parameters $\wp \in J \subset \mathbb{C}^m$, then so do KAM solutions $u(\theta;\eta,\wp)$ *provided the smallness condition* (1.24) *holds uniformly in* $\wp$, i.e., provided such conditions holds with the $\|\cdot\|_\xi$ norm redefined as the norm

$$\|e\|_\xi := \sum_{n\in\mathbb{Z}} \Big(\sup_{\wp\in J}|e_n(\wp)|\Big)\;\mathrm{e}^{|n|\xi}\;.$$

This claim follows from the uniform convergence of the KAM scheme and Weierstrass theorem on analytic limits of holomorphic function; for more details compare, e.g, with [5].
(ii) The dependence upon the frequency ω, as well known, is more delicate since it involves the small divisors $\lambda_{\eta,n}$: it is, however, standard to check that this dependence is C^∞ in the sense of Whitney on a bounded set of Diophantine numbers, say, $\mathcal{D}_{\kappa,\tau}\cap[1+r,1/r]$ for any prefixed $0<r<0$; for more details on Whitney smoothness and proofs we refer the reader to [7], [11] and [2].

2.6 Proof of Theorem 3

We start by observing that the hypotheses (H1)÷(H3) of Theorem 3 imply that $(v, \beta) \in \mathcal{W}_{\xi}^{\bar{\xi}}$ (recall Definition 1) and that (2.54) and (2.55) hold.
Next, because of (H2) and (H3), we have that, on $\mathbb{T}^2$, one has

$$\frac{1}{(1-\sigma\nu)^2} \geq W^{-1} \geq \frac{1}{(1+\sigma\nu)^2} \ ,$$

so that (recall the definitions of μ_0, μ_1, $\bar{\mu}_0$ and $\bar{\mu}_1$ in (2.63) and (2.70)) we find

$$\begin{aligned}
\bar{\mu}_0 - \mu_0 &\geq \frac{(1+\nu)^2 - (1+\sigma\nu)^2}{(1+\nu)^2(1+\sigma\nu)^2} =: \hat{\mu}_0 > 0 \ , \\
\mu_1 - \bar{\mu}_1 &\geq \frac{(1-\sigma\nu)^2 - (1-\nu)^2}{(1-\nu)^2(1-\sigma\nu)^2} =: \hat{\mu}_1 > 0 \ .
\end{aligned}$$

Thus, the number μ defined in (2.71) is bounded below by

$$\mu \geq \mu_* = \mu_*(\nu, \alpha, \sigma) := \min\left\{(1-\sigma)\nu \ , \ \frac{(1-\nu)^2}{2(1+\nu)^2} \ , \ \hat{\mu}_0 \ , \ \hat{\mu}_1 \ , \ (1-\sigma)\alpha\right\} > 0 \ .$$

Therefore, taking

$$m = m_* := 2^{s-1}\mu_*$$

we see that condition (2.73) may be rewritten as

$$k_1 \, \|e\|_\xi \leq 1 \ ,$$

with

$$k_1 = k_1(\xi, \xi_*, \kappa, \tau, \eta_0, M, \nu, \alpha, \sigma) := C \ \mathrm{e}^{\frac{1}{\mathrm{e}\, m_*(\nu,\alpha,\sigma)}} \ ,$$

while (2.75) holds with C_* equal to k_2 with

$$k_2 = k_2(\xi, \xi_*, \kappa, \tau, \eta_0, M, \nu, \alpha, \sigma) := \frac{C}{2^s} \left(1 - \mathrm{e}^{-\frac{1}{\mathrm{e}\, m_*(\nu,\alpha,\sigma)}}\right)^{-1} \ . \tag{2.100}$$

Thus, if k is taken to be not smaller than $\max\{k_1, k_2\}$, we see that (1.24) implies (2.73) so that, by Proposition 2, the claims about existence of the solution (u, γ) and the estimates (1.25) and (1.26) hold.
Let us turn to uniqueness. Let $\hat{s}$ and $\widehat{C}$ as in Proposition 3 and define

$$\begin{aligned}
k_3 &= k_3(\xi_*, \kappa, \tau, \eta_0, M, \nu, \alpha) := 2\widehat{C}\xi_*^{-\hat{s}} \ , \\
k_4 &:= k_2 \ k_3 \ ,
\end{aligned}$$

and assume that

$$k_4 \, \|e\|_{\xi_*} \leq 1 \; . \tag{2.101}$$

Then, if u' and γ' solve (1.18), i.e., $\mathcal{F}_\eta(u';\gamma') = 0$ for each $\eta \in I_0$, and if

$$k_3 \, \|u' - v\|_{\xi_*} \leq 1 \; , \tag{2.102}$$

then, by (2.102), (2.75), the definition of k_2 in (2.100), (2.101) , we see that

$$\|u' - v\|_{\xi_*} \leq \|u' - u\|_{\xi_*} + \|u - v\|_{\xi_*} \leq \frac{1}{k_3} + k_2 \, \|e\|_{\xi_*} \leq \frac{1}{k_3} + \frac{k_2}{k_4} = \frac{2}{k_3} = \frac{1}{\widehat{C}\xi_*^{-\hat{s}}} \; ,$$

showing that (2.91) is satisfied so that, by Proposition 3, $u' = u$ and $\gamma' = \gamma$.
Thus (since k_4 is greater than k_2 and k_3), we see that all claims in Theorem 3 follow by taking $k := \max\{k_1 \, , \, k_4\}$. ∎

2.7 Proof of Theorem 1

We now show how Theorem 1 can be obtained as a corollary of Theorem 3.
Since f in (1.6) is assumed to be real–analytic, there exists a $\bar{\xi} > 0$ such that $f \in \mathcal{H}^{\bar{\xi}}$ (point (ii) of Remark 4). Assuming, as we shall henceforth do, that

$$|\varepsilon| \leq \varepsilon_0 < 1$$

we can take the constant M in Theorem 3 to be

$$M := \|\partial_x^3 f\|_{\bar{\xi}} \; .$$

In this section, $\|\cdot\|_\xi$ denotes the norm (compare (i), Remark 7)

$$\|h\|_\xi := \sum_{n\in\mathbb{Z}} \Big(\sup_{\varepsilon \in J} |h_n(\varepsilon)| \Big) \, e^{|n|\xi} \; , \qquad J := \{\varepsilon \in \mathbb{C} \; , \; |\varepsilon| \leq \varepsilon_0\} \; .$$

The numbers ξ_*, ξ, ν, α and σ can be chosen arbitrarily as long as they satisfy

$$0 < \xi_* < \xi < \bar{\xi} \; , \quad 0 < \nu < \bar{\xi} - \xi \; , \quad 0 < \alpha < 1 \; , \quad 0 < \sigma < 1 \; . \tag{2.103}$$

Finally, we choose, as initial approximate solution, the trivial couple

$$(v, \beta) := (0, 0) \; .$$

Then, the error function e defined in (1.23) is simply given by

$$e = e(\theta;\varepsilon) := \mathcal{F}_\eta(0;0) = \varepsilon\,\partial_x f(\theta)\ , \qquad\qquad \|e\|_\xi \le \varepsilon_0 M \tag{2.104}$$

and the functions defined in (1.21) are given by

$$V = 1\ , \qquad W = 1\ , \qquad \rho = 0\ .$$

Thus, (H1)÷(H3) of Theorem 3 are trivially satisfied and in order to meet (H4), i.e., the smallness condition (1.24), it suffices to require

$$\varepsilon_0 \le \varepsilon_* := \min\Big\{1\,,\ \frac{1}{k\,M}\Big\}\ . \tag{2.105}$$

Thus, if (2.105) holds, by Theorem 3 and Remark 7, there exist unique functions $u = u(\theta;\eta) = u_\varepsilon(\theta;\eta,\omega)$ and $\gamma = \gamma(\eta) = \gamma_\varepsilon(\eta,\omega)$ such that $\mathcal{F}_\eta(u;\gamma) = 0$, for all $\eta \in I_0$, and $\theta \to u(\theta;\eta) \in \mathcal{H}_0^{\xi_*}$. Furthermore, u and γ are Whitney C^∞ in all their variables $(\theta,\eta,\varepsilon,\omega)$ in the domain

$$\mathbb{T}^2_{\xi_*} \times I_0 \times J \times \mathcal{D}_{\kappa,\tau}\ ,$$

they are C^∞ in $(\theta,\eta,\varepsilon)$ and real–analytic in $(\theta;\varepsilon) \in \mathbb{T}^2_{\xi_*} \times J$.
The solution (u,γ) satisfies the bounds (1.25) and (1.26). In particular (by (1.26) with $v = 0$), it holds

$$\|u_{\theta_1}\|_{\xi_*} \le \varepsilon_0\, kM\ ,$$

which, together with analyticity in ε, implies that $u = O(\varepsilon)$, i.e., $u|_{\varepsilon=0} = 0$. Finally, the relation between γ and ω in (1.6) and Eq. (1.16) imply (1.4), completing the proof of Theorem 1. ∎

2.8 Proof of Theorem 2

Also the proof of Theorem 2 is based upon Theorem 3 along the lines of § 2.7, but, first, we have to investigate the analytical properties of the spin–orbit potential defined in (1.8). For this purpose, we denote[21]

$$\mathrm{e}_1 := \upsilon^{-1}(1+r)\ , \qquad \mathrm{e}_2 := \upsilon^{-1}\Big(\frac{1}{r}\Big)\ , \tag{2.106}$$

where $0 < r < 1$ is a prefixed number as in Theorem 2 and υ^{-1} is the real–analytic function (inverse of $\mathrm{e} \to \upsilon_{\mathrm{e}}$) defined in point (iii) of Remark 2. Clearly,

$$0 < \mathrm{e}_1 < \mathrm{e}_2 < 1\ .$$

[21] Again: do not confuse the letter e, which stands for eccentricity, with the letter e, which denotes the error function.

It is also clear that $\rho_{\rm e}(t)$ and $\mathrm{f}_{\rm e}(t)$ (defined in (1.11) and (1.12)) are real–analytic function of $({\rm e},t) \in (0,1)\times\mathbb{S}^1$, where $\mathbb{S}^1 := \mathbb{R}/(2\pi\mathbb{Z})$. Thus, there exist positive numbers

$$0<\bar\xi<1\ ,\qquad 0<d<\min\{{\rm e}_1\ ,\ 1-{\rm e}_2\}\ , \tag{2.107}$$

such that the functions $\rho_{\rm e}(t)$ and $\mathrm{f}_{\rm e}(t)$ may be analytically continued into the complex domain $\mathcal{E}_{r,d}\times\mathbb{S}^1_{\bar\xi}$, where

$$\mathcal{E}_{r,d} := \bigcup_{{\rm e}'\in[{\rm e}_1,{\rm e}_2]} \{{\rm e}\in\mathbb{C}\ :\ |{\rm e}-{\rm e}'|\le d\}\ ,\qquad \mathbb{S}^1_{\bar\xi} := \{t\in\mathbb{C}\ :\ |\operatorname{Im} t|<\bar\xi\}\ . \tag{2.108}$$

Therefore, for any $\varepsilon_0>0$, which will be henceforth assumed to be smaller or equal than 1, the function

$$g(x,t;\varepsilon,{\rm e}) := \varepsilon f(x,t;{\rm e}) \tag{2.109}$$

is real–analytic for

$$((x,t),(\varepsilon,{\rm e}))\in\mathbb{T}^2_{\bar\xi}\times J \tag{2.110}$$

where, now,

$$J := \{\varepsilon\in\mathbb{C}\ :\ |\varepsilon|\le\varepsilon_0\}\times\mathcal{E}_{r,d}\subset\mathbb{C}^2\ . \tag{2.111}$$

Clearly, in the present situation the $\|\cdot\|_\xi$ denotes the norm

$$\|h\|_\xi := \sum_{n\in\mathbb{Z}}\Big(\sup_{(\varepsilon,{\rm e})\in J}|h_n(\varepsilon,{\rm e})|\Big)\,\mathrm{e}^{|n|\xi}\ .$$

Finally, we choose η_0 as[22]

$$\eta_0 := \Omega_{{\rm e}_2}\ ,$$

and let

$$\omega\in\mathcal{D}_{\kappa,\tau}\cap\Big[1+r,\frac{1}{r}\Big]\ ,$$

which guarantees that, as ${\rm e}\in[{\rm e}_1,{\rm e}_2]$ then $\boldsymbol{\upsilon}_{\rm e}\in[1+r,1/r]$.

At this point, we can proceed as in the previous section (with the same choices of M, ξ_*, ξ, ν, α and σ, (v,β)) and deduce from Theorem 3 the existence and uniqueness of functions $u=u(\theta;\eta)=u(\theta;\eta,\varepsilon,{\rm e},\omega)$ and $\gamma=\gamma(\eta)=\gamma(\eta,\varepsilon,{\rm e},\omega)$ such that $\mathcal{F}_\eta(u;\gamma)=0$, for all $\eta\in I_0$, and $\theta\to u(\theta;\eta)\in\mathcal{H}_0^{\xi_*}$.

As above, u and γ are Whitney C^∞ in all their variables $(\theta,\eta,(\varepsilon,{\rm e}),\omega)$ in the domain

$$\mathbb{T}^2_{\xi_*}\times I_0\times J\times\Big(\mathcal{D}_{\kappa,\tau}\cap\Big[1+r,\frac{1}{r}\Big]\Big)\ ,$$

[22] Recall the definition of $\Omega_{\rm e}$ in (1.9) and note that K will be taken in the interval $[-1,1]$.

they are C^∞ in $(\theta, \eta, (\varepsilon, \mathrm{e}))$ and real–analytic in $(\theta; (\varepsilon, \mathrm{e})) \in \mathbb{T}^2_{\xi_*} \times J$; (u, γ) satisfies the bounds (1.25), (1.26) and $\|u_{\theta_1}\|_{\xi_*} \leq \varepsilon_0\, kM$, which, together with analyticity in ε, implies that $u = O(\varepsilon)$, or $u|_{\varepsilon=0} = 0$. Therefore, relation (1.16) implies that $\gamma = O(\varepsilon^2)$ and we can write

$$\gamma =: -\eta\omega\, \varepsilon^2\, \tilde\gamma(\eta, \varepsilon, \mathrm{e}, \omega) \ ,$$

with $\tilde\gamma$ Whitney C^∞ in all its variables, C^∞ in η and real–analytic in $(\varepsilon, \mathrm{e})$; the minus sign accounts for the fact that $\tilde\gamma \geq 0$ for real values of its arguments.

To finish the proof of Theorem 2 we have to discuss the parameter relations (compare (1.13))

$$\eta = K\Omega_{\mathrm{e}} \ , \qquad \gamma = K\Omega_{\mathrm{e}}\omega - KN_{\mathrm{e}} \ . \tag{2.112}$$

By definition of υ_{e} and $\tilde\gamma$, we can rewrite (2.112) as

$$\eta = K\Omega_{\mathrm{e}} \ , \qquad \omega\varepsilon^2\, \tilde\gamma(\eta, \varepsilon, \mathrm{e}, \omega) = \upsilon_{\mathrm{e}} - \omega \ . \tag{2.113}$$

Letting

$$\hat\gamma(K, \varepsilon, \mathrm{e}, \omega) := \tilde\gamma(K\Omega_{\mathrm{e}}, \varepsilon, \mathrm{e}, \omega) \ ,$$

the second relation in (2.113) can be rewritten as

$$h(\mathrm{e}, \varepsilon, K, \omega) := \upsilon_{\mathrm{e}} - \omega\Big(1 + \varepsilon^2 \hat\gamma(K, \varepsilon, \mathrm{e}, \omega)\Big) = 0 \ .$$

This last equation may be solved by the standard Implicit Function Theorem: Let $\mathrm{e}_0(\omega) := \upsilon^{-1}(\omega)$, then

$$h(\mathrm{e}_0(\omega), 0, K, \omega) = 0 \ , \qquad h_{\mathrm{e}}(\mathrm{e}_0(\omega), 0, K, \omega) = \partial_{\mathrm{e}} \upsilon_{\mathrm{e}}|_{\mathrm{e}=\mathrm{e}_0(\omega)} > 0 \ .$$

Thus, there exists a unique

$$\mathrm{e}_\varepsilon(K, \omega) = \mathrm{e}_0(\omega) + O(\varepsilon^2) = \upsilon^{-1}(\omega) + O(\varepsilon^2) \ ,$$

which is Whitney C^∞ in all its variables, C^∞ in $K \in [-1, 1]$ and real–analytic in ε such that

$$h(\mathrm{e}_\varepsilon(K, \omega), \varepsilon, K, \omega) \equiv 0 \ ,$$

implying that the parameter relations (2.113) are satisfied for $\eta = K\Omega_{\mathrm{e}}$, $\mathrm{e} = \mathrm{e}_\varepsilon(K, \omega)$. The proof of Theorem 2 is finished upon the identification

$$u = u_\varepsilon(\theta; K, \omega) := u(\theta; K\Omega_{\mathrm{e}_\varepsilon(K,\omega)}, \varepsilon, \mathrm{e}_\varepsilon(K, \omega), \omega) \ . \quad \blacksquare$$

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