QUASI–PERIODIC ATTRACTORS AND SPIN/ORBIT RESONANCES

A. CELLETTI

Dipartimento di Matematica, Università di Roma Tor Vergata
Via della Ricerca Scientifica 1, I-00133 Roma (Italy)
**E-mail: celletti@mat.uniroma2.it*
www.mat.uniroma2.it/∼celletti

L. CHIERCHIA

Dipartimento di Matematica, Università "Roma Tre"
Largo S. L. Murialdo 1, I-00146 Roma (Italy)
**E-mail: luigi@mat.uniroma3.it*
www.mat.uniroma3.it/users/chierchia

Small dissipation limits for nearly–integrable systems are considered; theorems concerning the existence of quasi–periodic attractors smoothly approaching KAM tori are presented and an application to the capture in 3:2 resonance of Mercury is discussed.

Keywords: Quasi–periodic attractors, small divisors, celestial mechanics, spin–orbit problem, spin–orbit resonances, nearly Hamiltonian systems, dissipative systems.

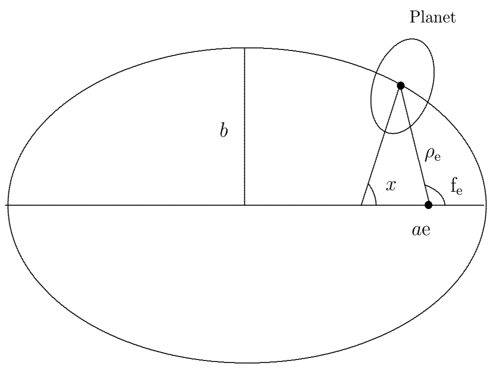
1. Introduction

Mechanical systems, in real life, are typically dissipative, and perfectly conservative systems arise as mathematical abstractions. In this lecture, we shall consider nearly–conservative mechanical systems having in mind applications to celestial mechanics. In particular we are interested in the spin–orbit model for an oblate planet (satellite) whose center of mass revolves around a "fixed" star; the planet is not completely rigid and averaged effects of tides, which bring in dissipation, are taken into account. We shall see that a mathematical theory of such systems is consistent with the strange case of Mercury, which is the only planet or satellite in the Solar system being stack in a 3:2 spin/orbit resonance (i.e., it turns three times around its rotational spin axis, while it makes one revolution around the Sun).

2. The spin–orbit model

Let us consider the dynamics of a triaxial *nearly–rigid* body (planet or satellite), having its center of mass revolving on a given (fixed) Keplerian ellipse, and subject to the gravitational attraction of a major body sitting on a focus of the ellipse. For simplicity, we consider vanishing obliquity, i.e., we assume that the satellite is symmetric with respect to an "equatorial plane" and study motions having the equatorial plane coinciding with the Keplerian orbital plane (such motions belong to the invariant submanifold of vertical spin axis).

Under such hypotheses, the motions of the satellite may be described by the angle x formed by, say, the direction of the major physical axis of the satellite (assumed to lie in the equatorial plane) with a fixed axis of the Keplerian orbit plane (say the direction of the semimajor axis of the ellipse; see figure).



We shall assume that the non–rigidity of the planet (meant to reflect the averaged effect of tides) is modeled by the *averaged MacDonald's torque*.[8] Then, the differential equation governing the motion of the satellite, in suitable units, is given by

$$\ddot{x} + K\Omega_e\,\dot{x} + \frac{\varepsilon}{\rho_e(t)^3}\sin\Big(2x - 2\,f_e(t)\Big) = KN_e \qquad (1)$$

where[a]

- $K \geq 0$ (the "dissipation parameter") is a physical constant depending on the internal (non–rigid) structure of the satellite;
- $\Omega_{\rm e} > 0$ and $N_{\rm e} > 0$ are known functions of the eccentricity $\mathrm{e} \in [0, 1)$ of the Keplerian orbit and are given by:

$$\Omega_{\rm e} := \Big(1 + 3\mathrm{e}^2 + \frac{3}{8}\mathrm{e}^4\Big)\ \frac{1}{(1-\mathrm{e}^2)^{9/2}}\ ,$$
$$N_{\rm e} := \Big(1 + \frac{15}{2}\mathrm{e}^2 + \frac{45}{8}\mathrm{e}^4 + \frac{5}{16}\mathrm{e}^6\Big)\frac{1}{(1-\mathrm{e}^2)^6}\ ; \tag{2}$$

- $\varepsilon = \dfrac{3}{2}\,\dfrac{B-A}{C}$, where $0 < A < B < C$ are the principal moments of inertia of the satellite;
- $\rho_{\rm e}(t)$ and $\mathrm{f}_{\rm e}(t)$ are, respectively, the (normalized) orbital radius and the true anomaly of the Keplerian motion, which (because of the assumed normalizations) are 2π–periodic function of time t. The explicit expression for $\rho_{\rm e}$ and $\mathrm{f}_{\rm e}$ may be described as follows. Let $\mathrm{u} = \mathrm{u}_{\rm e}(t)$ be the 2π–periodic function obtained by inverting

$$t = \mathrm{u} - \mathrm{e}\sin\mathrm{u}\ , \qquad (\text{“Kepler's equation”})\ ; \tag{3}$$

then

$$\rho_{\rm e}(t) = 1 - \mathrm{e}\cos\mathrm{u}_{\rm e}(t)$$
$$\mathrm{f}_{\rm e}(t) = 2\arctan\Big(\sqrt{\frac{1+\mathrm{e}}{1-\mathrm{e}}}\tan\frac{\mathrm{u}_{\rm e}(t)}{2}\Big)\ , \tag{4}$$

Remark 2.1. (i) For $K = 0$ the equation (1) corresponds to the Hamiltonian flow associated to the one–and–a–half degree–of–freedom Hamiltonian

$$H(y, x, t) := \frac{1}{2}y^2 - \frac{\varepsilon}{2\,\rho_{\rm e}(t)^3}\cos\Big(2x - 2\,\mathrm{f}_{\rm e}(t)\Big)\ , \tag{5}$$

(y, x) being standard symplectic variables. Such a Hamiltonian system (the "spin–orbit Hamiltonian model"), whose phase space is $\mathbb{R}\times\mathbb{T}^2$ ($\mathbb{T}^2$ being the standard flat torus $\mathbb{R}^2/(2\pi\mathbb{Z}^2)$), is non–integrable if $\varepsilon > 0$ *and*[b] $\mathrm{e} > 0$.

---

[a] The conservative equation ($K = 0$) is derived and discussed, e.g., in Ref. 4; compare, in particular, Eq. (2.2) with the normalization $n := \sqrt{(Gm)/a^3} = 1$. The dissipative term ($K \neq 0$) is derived, e.g., in Ref. 9; compare, in particular, Eq. (21), where (as above) $n = 1$ and in view of our assumption about the spin axis being vertical, one has to take vanishing $\mathbf{e}_X$ and $\mathbf{e}_Y$ components, $i = 0$ and $\psi_m = x$; K is the constant in front of the curly brackets in Eq. (21); the functions $\Omega_{\rm e}$ and $N_{\rm e}$ are denoted in Ref. 9, respectively, $f_1(e)$ and $f_2(e)$.

[b] When $\mathrm{e} = 0$, $\mathrm{u}_0(t) = t = \mathrm{f}_0(t)$, $\rho_0 = 1$ so that $H = \frac{1}{2}y^2 - \frac{\varepsilon}{2}\cos(2x - 2t)$, which is easily seen to be integrable.

(ii) For $K > 0$ the equation (1) is *dissipative* and, for $\varepsilon = 0$, the general solution is given by

$$x(t) = x_0 + \upsilon_e t + \frac{1-\exp(-\eta t)}{\eta}(v_0 - \upsilon_e)\ , \qquad \upsilon_e := \frac{N_e}{\Omega_e}\ , \quad \eta := K\Omega_e\ , \tag{6}$$

showing that the periodic (remember that x is an angle) solution $x = \mathrm{cost} + \upsilon_e t$, $\dot{x} \equiv \upsilon_e$ is a *global attractor* for the dynamics on the cylinder (phase space) $\mathbb{R}\times\mathbb{S}$, $\mathbb{S}$ denoting the circle $\mathbb{R}/(2\pi\mathbb{Z})$. The limiting frequency[c]

$$\upsilon_e := \frac{N_e}{\Omega_e} := \frac{1+\frac{15}{2}e^2+\frac{45}{8}e^4+\frac{5}{16}e^6}{(1-e)^{3/2}\left(1+3e^2+\frac{3}{8}e^4\right)} = 1+6e^2+\frac{3}{8}e^4+O(e^6) \tag{7}$$

will play an important role in the sequel; we notice, in particular, that it is a real–analytic invertible function of e mapping $(0,1)$ onto $(1,\infty)$; we denote by $\upsilon^{-1} : (1,\infty) \to (0,1)$ the inverse map (which is also real–analytic).

(iii) In many examples taken from the Solar system, *both ε and K are small*. For example, for the Earth–Moon system and for the Sun–Mercury system ε is of the order of 10^{-4}, while K is of the order of 10^{-8}.

(iv) A quasi–periodic solution $x(t)$ with frequency $\omega \in \mathbb{R}\setminus\mathbb{Q}$ of Eq. (1) is a solution of the form

$$x(t) = \omega t + u(\omega t, t) \tag{8}$$

where $u(\theta) = u(\theta_1,\theta_2)$ is a C^2 function defined on $\mathbb{T}^2$ (i.e., 2π–periodic in the variables θ_1 and θ_2). Notice that time–derivative for $x(t)$ corresponds to the directional derivative

$$\partial_\omega := \omega\frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2}\ , \tag{9}$$

for the function $u(\theta)$; since the flow $\theta \in \mathbb{T}^2 \to \theta + (\omega t, t)$ is dense in $\mathbb{T}^2$, one sees immediately that $x(t)$ is a quasi–periodic solution of (1) if and only if u solves the following quasi–linear PDE on $\mathbb{T}^2$:

$$\partial_\omega^2 u + \eta\,\partial_\omega u + \frac{\varepsilon}{\rho_e(\theta_2)^3}\sin\Big(2(\theta_1+u) - 2\,f_e(\theta_2)\Big) = \eta(\upsilon_e - \omega)\ , \tag{10}$$

where, as above $\eta := K\Omega_e$ and $\upsilon_e := N_e/\Omega_e$.

(v) The frequency ω and the function υ_e are not independent: it is not difficult to check that *if u satisfies (10) then one has necessarily*

$$\upsilon_e = \omega\Big(1 + \langle (u_{\theta_1})^2\rangle\Big)\ , \tag{11}$$

[c] As usual, $f = O(x^k)$ means that f is a smooth function of x having equal to zero the first k derivatives at $x = 0$.

where $\langle\cdot\rangle$ denotes average over $\mathbb{T}^2$. Eq. (11) may be interpreted as a compatibility condition.

(vi) The above spin–orbit model is relatively simple (since a lot of approximations have been done), nevertheless it is rather well accepted in the astronomical community: for example, it has recently been used by Correia and Laskar[7] to discuss Mercury's capture in resonance.

3. Results

Standard KAM theory (see, e.g., Ref. 1) implies that, when $K = 0$ and $\varepsilon > 0$ is small enough, (1) admits many quasi–periodic solutions as in (8) with ω Diophantine, i.e., satisfying

$$|\omega n_1 + n_2| \geq \frac{\kappa}{|n_1|^\tau} \ , \qquad \forall\ (n_1, n_2) \in \mathbb{Z}^2\ ,\ n_1 \neq 0\ , \tag{12}$$

for some $\kappa, \tau > 0$. Furthermore, such solutions are analytic in ε and are Whitney smooth in[d] ω. In the following, $\mathcal{D}_{\kappa,\tau}$ denotes the set of Diophantine numbers in $\mathbb{R}$ satisfying[e] (12).

Theorem 3.1. *Fix $\kappa, r \in (0,1)$ and $\tau \geq 1$. There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in [0, \varepsilon_0]$, any $K \in [0,1]$ and any $\omega \in \mathcal{D}_{\kappa,\tau} \cap [1+r, 1/r]$, there exist unique functions*[f]

$$\mathrm{e}_\varepsilon = \mathrm{e}_\varepsilon(K,\omega) = \upsilon^{-1}(\omega) + O(\varepsilon^2)\ , \qquad u = u_\varepsilon(\theta; K, \omega) = O(\varepsilon)\ ,$$

with $\int_{\mathbb{T}^2} u\, d\theta = 0$, satisfying (10) with $\mathrm{e} = \mathrm{e}_\varepsilon$. The functions e_ε and u_ε are smooth in the sense of Whitney in all their variables and are real–analytic in $\theta \in \mathbb{T}^2$ and ε, C^∞ in K and Whitney C^∞ in ω.

Remark 3.1. (i) Theorem 3.1 implies that the 2–torus

$$\mathcal{T}_{\varepsilon,K}(\omega) := \{(x,t) = (\theta_1 + u_\varepsilon(\theta; K, \omega), \theta_2)\ :\ \theta = (\theta_1, \theta_2) \in \mathbb{T}^2\}\ , \tag{13}$$

is a *quasi–periodic attractor* for the dynamics on the phase space $\mathbb{R} \times \mathbb{T}^2$ associated to (1) with $\mathrm{e} = \mathrm{e}_\varepsilon(K,\omega)$ and that the dynamics on $\mathcal{T}_{\varepsilon,K}(\omega)$ is analytically conjugated to the linear flow $\theta \to \theta + (\omega t, t)$.

---

[d]A function $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is Whitney C^k or C^k_W if it is the restriction on A of a $C^k(\mathbb{R}^n)$ function; for a more formal definition and for relevance in dynamical system, see, e.g., Ref. 2.

[e]Observe that if (12) holds, then $0 < \kappa < 1$ and $\tau \geq 1$. In fact, taking $n_1 = 1$ and $n_2 = -[\omega]$ ($[x]$ = integer part of x) in (12) shows that $\kappa < 1$, while the fact that $\tau \geq 1$ comes from Liouville's theorem on rational approximations ("*For any $\omega \in \mathbb{R}\backslash\mathbb{Q}$ and for any $N \geq 1$ there exist integers p and q with $|q| \leq N$ such that $|\omega q - p| < 1/N$*"). Finally, we recall that, when $\tau > 1$, $\bigcup_{\kappa>0} \mathcal{D}_{\kappa,\tau}$ is a set of full Lebesgue measure.

[f]The map υ^{-1} is the inverse map of $\mathrm{e} \to \upsilon_\mathrm{e}$ defined in (7).

(ii) The result is perturbative in ε but it is *uniform* in K. Indeed, one could replace the parameter range for K into any compact interval of $\mathbb{R}$. It is particularly noticeable the smooth dependence of u_ε on K as $K \to 0$, which shows that the invariant KAM torus $\mathcal{T}_{\varepsilon,0}(\omega)$ smoothly bifurcates into the attractor (13) as $K \neq 0$.

(iii) The invertibility of the frequency map $\upsilon_{\rm e}$ associated to the unperturbed attractor $\mathcal{T}_{0,K}(\omega)$ may be interpreted as a *nondegeneracy condition*, allowing to fix the eccentricities for which quasi–periodic attractors exist in the full dynamics. Notice that the parameter values $\mathrm{e} = 0$ and $\omega \leq 1$ are excluded.

(iv) The proof of the above theorem is based upon a "Nash–Moser" method. That is, equation (10) is rewritten as $\mathcal{F}(u; \mathrm{e}) = 0$, where $\mathcal{F}$ is a functional acting on functions on $\mathbb{T}^2$ and numbers $\mathrm{e} \in (0, 1)$; the unknowns are u *and* e, while ε, K and ω (which is taken in the Cantor set $\mathcal{D}_{\tau,\gamma}$ with γ and τ fixed) are regarded as parameters. Then, the equation $\mathcal{F}(u; \mathrm{e})$ is solved iteratively starting by the trivial approximate solution $u = 0$ and $\mathrm{e} = \upsilon^{-1}(\omega)$. To cope with the small divisors introduced by inverting the linearized functional $d\mathcal{F}$ one introduces a scale of larger and larger Banach spaces and uses a Newton method: the speed of convergence of the method is enough to beat the divergences introduced by the small divisors. With this approach the most delicate part concerns the discussion of the solution of the linearized equation, which is a degenerate linear PDE on $\mathbb{T}^2$ with nonconstant coefficients. Full details are given in Ref. 6.
Actually, the above Nash–Moser approach is rather robust and general; indeed it can be easily adapted to cover dissipative maps such as the "fattened Arnold family" studied in Ref. 3 or it could be extended to systems with more degrees of freedom.

A simple consequence of Theorem 3.1 is the following

Theorem 3.2. *For small enough oblateness ε and any rigidity parameter $K \in [0, 1]$ there exists a (Cantor) set of positive measure $\mathcal{E} \subset (0, 1)$, which depends smoothly on ε and K, such that for any $\mathrm{e} \in \mathcal{E}$ there exists a unique 2–dimensional torus, which is a quasi–periodic attractor for the dynamics governed by (1), and on which the flow is analytically conjugated to $(\theta_1, \theta_2) \to (\theta_1 + \omega t, \theta_2 + t)$ with Diophantine $\omega = \upsilon_{\rm e} + O(\varepsilon^2)$. Finally, the Lebesgue measure of $\mathcal{E}$ tends to 1 as $\varepsilon \to 0$.*

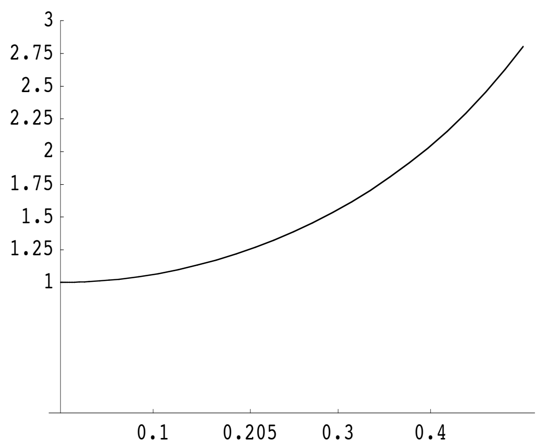
4. Mercury's capture

Many satellites in the solar system are observed in a 1:1 spin/orbit resonance, i.e., while making a revolution around their primary body, they make one turn around their rotational internal axis: in this way they show always the same "face" to their primary body. The most familiar example is our Moon; other examples are: Deimos, Phobos, Io, Europa, Ganymede, Callisto, Mimas, Enceladus, Tethys, Dione, Rhea, Titan, Janus, Epimetheus, Ariel, Umbriel,...

On the other hand only one celestial body is observed in a different spin–orbit resonance, namely, Mercury, which is observed in a 3:2 resonance. Explaining this anomaly is a very intriguing and actual problem; compare Ref. 7 and references therein.

It is a fact that all the satellites observed in the 1:1 resonance have small eccentricity (the largest is that of the Moon, which is about 0.055), while the eccentricity of Mercury's orbit is about 0.205.

Now, a look at the graph of the function $\mathrm{e} \to \upsilon_{\mathrm{e}}$



shows that the resonance 3:2 (corresponding to a frequency $\omega = 1.5$) is *above* the graph of υ in correspondence to Mercury's eccentricity $\mathrm{e} = 0.205$. Now, the phase space of our system is three dimensional and two–dimensional tori that are graphs over the angles (x, t) (such as the tori arising in Theorem 3.2) *separate* the phase space in two invariant regions. Thus a periodic or nearly–periodic orbit with frequency 1.5 would remain trapped forever above the invariant torus. In view of the fact that the measure of the set $\mathcal{E}$ of eccentricities corresponding to quasi–periodic attractors, according to Theorem 3.2, is close to 1 for small ε (which for Mercury is 10^{-4}), we see that the above analytical results might indicate a rigorous explanation to the spin–orbit trapping of Mercury[g].

References

1. V. I. Arnold (editor), *Encyclopedia of Mathematical Sciences*, Dynamical Systems III, Springer-Verlag **3**, 1988.
2. H.W. Broer, G.B. Huitema, M.B. Sevryuk, *Quasi-periodic motions in families of dynamical systems. Order amidst chaos*, Lecture Notes in Mathematics, 1645. Springer-Verlag, Berlin, 1996.
3. H.W. Broer, C. Simó, J.C. Tatjer, *Towards global models near homoclinic tangencies of dissipative diffeomorphisms*, Nonlinearity **11**, (1998) 667–770.
4. A. Celletti, *Analysis of resonances in the spin-orbit problem in Celestial Mechanics: The synchronous resonance (Part I)*, Journal of Applied Mathematics and Physics (ZAMP) **41** (1990), 174–204.
5. A. Celletti, L. Chierchia, *On the stability of realistic three-body problems*, Commun. in Math. Physics **186** (1997), 413–449.
6. A. Celletti, L. Chierchia, *Quasi–periodic attractors in celestial mechanics*, preprint April 2007, downloadable at www.mat.uniroma3.it/users/chierchia
7. A. C. M. Correia, J. Laskar: *Mercury's capture into the 3/2 spin–orbit resonance as a result of its chaotic dynamics*, Nature **429** (24 June 2004), 848–850.
8. G.J.F. MacDonald, *Tidal friction*, Rev. Geophys. **2** (1964), 467–541.
9. S.J. Peale, *The free precession and libration of Mercury*, Icarus **178** (2005), 4–18.

[g]This mechanism seems also compatible with evolutive models, which study slow changes of orbital parameters; compare Ref. 7 where it is given numerical evidence that during million of years the eccentricity of Mercury might have changed quite a bit reaching values beyond 0.3.