THE PLANETARY N–BODY PROBLEM

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Summary

Planetary systems, under suitable general assumptions, admit positive measure sets of "initial data" whose evolution give rise to the planets revolving on nearly circular and nearly co–planar orbits around their star. This statement (or more primitive formulations) challenged astronomers, physicists and mathematicians for centuries. In this article we shall review the mathematical theory (with particular attention to recent developments) needed to prove the above statement.

1 The N–body problem: a continuing mathematical challenge

The problem of the motion of $N \geq 2$ point–masses (i.e., ideal bodies with no physical dimension identified with points in the Euclidean three–dimensional space) interacting only through Newton's law of mutual gravitational attraction, has been a central issue in astronomy, physics and mathematics since the early developments of modern calculus. When $N = 2$ the problem has been completely solved ("integrated") by Newton: the motion take place on conics, whose focus is occupied by the center of mass of the two bodies; but for $N \geq 3$ a complete understanding of the problem is still far away.

While the original impulse, coming from astronomy, has been somehow shaded by the massive use of machines for computing orbits of celestial bodies or satellites, the mathematical richness and beauty of the N–body problem has retained most of its original attraction; for a selection of recent contributions, see, e.g., [Chenciner and Montgomery, 2000], [Ferrario and Terracini, 2004], [Hampton and Moeckel, 2006], [Chen, 2007], [Fusco, Gronchi and Negrini, 2011], [Chierchia and Pinzari, 2011 (c)].

Here, we will be concerned with the *planetary N–body problem*, which, as the name says, deals with the case of one body (the "Sun" or the "Star") having mass much bigger than the remaining bodies ("planets"). The main question is then to determine "general" conditions under which the planets revolve around the Sun without collisions and in a "regular way" so that, in particular, no planet crashes onto another planet or onto the Sun, nor it escapes away from such "solar system".

Despite the efforts of Newton, Euler, d'Alembert, Lagrange, Laplace and, especially, Henri Poincaré and G.D. Birkhoff, such question remained essentially unanswered for centuries. It is only with the astonishing work of a 26–year–old mathematician, V.I. Arnold (1937–2010), that a real breakthrough was achieved. Arnold, continuing and extending fundamental analytical discoveries of his advisor A.N. Kolmogorov on the so called "small divisors" (singularities appearing in the perturbative expansions of orbital trajectories), stated in 1963 [Arnold, 1963] a result, which may be roughly formulated as follows (*verbatim* formulations are given in § 3.1 below).

If the masses of the planets are small enough compared to the mass of the Sun, there exists, in the phase space of the planetary N–body problem, a bounded set of positive Lebesgue measure corresponding to planetary motions with bounded *relative distances; such motions are well approximated by Keplerian ellipses with small eccentricities and small relative inclinations.*

Arnold gave a brilliant proof in a special case, namely, the planar three–body problem (two planets), giving some suggestions on how to generalize his proof to the general case (arbitrary number of planets in space). However, a complete generalization of his proof turned out to be quite a difficult task, which took nearly another fifty years to be completed: the first complete proof, based on work by M.R. Herman, appeared in [Féjoz, 2004] and a full generalization of Arnold's approach in [Chierchia and Pinzari, 2011 (c)].

The main reason beyond the difficulties which arise in the general spatial case, is related to the presence of certain "secular degeneracies" which do not allow a *tout court* application of Arnold's "fundamental theorem" (see § 3.2 below) to the general planetary case.

In this article we shall give a brief account (avoiding computations) of these results trying to explain the main ideas and technical tools needed to overcome the difficulties involved.

2 The classical Hamiltonian structure

2.1 Newton equations and their Hamiltonian version

The starting point are the Newton's equations for $1 + n$ bodies (point masses), interacting only through gravitational attraction:

$$\ddot{u}^{(i)} = \sum_{\substack{0 \le j \le n \\ j \ne i}} \mathrm{m}_j \frac{u^{(j)} - u^{(i)}}{|u^{(i)} - u^{(j)}|^3} \ , \qquad i = 0, 1, ..., n \ , \tag{1}$$

where $u^{(i)} = \big(u_1^{(i)}, u_2^{(i)}, u_3^{(i)}\big) \in \mathbb{R}^3$ are the cartesian coordinates of the i^{th} body of mass $\mathrm{m}_i > 0$, $|u| = \sqrt{u \cdot u} = \sqrt{\sum_i u_i^2}$ is the standard Euclidean norm, "dots" over functions denote time derivatives, and the gravitational constant has been set to one (which is possible by rescaling time t).

Equations (1) are invariant by change of "inertial frames", i.e., by change of variables of the form $u^{(i)} \to u^{(i)} - (a + ct)$ with fixed $a, c \in \mathbb{R}^3$. This allows to restrict the attention to the manifold of "initial data" given by

$$\sum_{i=0}^{n} \mathrm{m}_i u^{(i)}(0) = 0 \ , \qquad \sum_{i=0}^{n} \mathrm{m}_i \dot{u}^{(i)}(0) = 0 \ ; \tag{2}$$

indeed, just replace the coordinates $u^{(i)}$ by $u^{(i)} - (a + ct)$ with

$$a := \mathrm{m}_{\mathrm{tot}}^{-1} \sum_{i=0}^{n} \mathrm{m}_i u^{(i)}(0) \qquad \text{and} \qquad c := \mathrm{m}_{\mathrm{tot}}^{-1} \sum_{i=0}^{n} \mathrm{m}_i \dot{u}^{(i)}(0) \ , \qquad \mathrm{m}_{\mathrm{tot}} := \sum_{i=0}^{n} \mathrm{m}_i \ .$$

The total linear momentum $\mathrm{M}_{\rm tot} := \sum_{i=0}^n \mathrm{m}_i \dot u^{(i)}$ does not change along the flow of (1), i.e., $\dot{\mathrm{M}}_{\rm tot} = 0$ along trajectories; therefore, by (2), $\mathrm{M}_{\rm tot}(t)$ vanishes for all times. But, then, also the position of the barycenter $B(t) := \sum_{i=0}^n \mathrm{m}_i u^{(i)}(t)$ is constant ($\dot B = 0$) and, again by (2), $B(t) \equiv 0$. In other words, the manifold of initial data (2) is invariant under the flow (1).

Equations (1) may be seen as the Hamiltonian equations associated to the Hamiltonian function

$$\widehat{\mathcal{H}}_{\rm N} := \sum_{i=0}^n \frac{|U^{(i)}|^2}{2\mathrm{m}_i} - \sum_{0\le i<j\le n} \frac{\mathrm{m}_i \mathrm{m}_j}{|u^{(i)} - u^{(j)}|} \ ,$$

where $(U^{(i)}, u^{(i)})$ are standard symplectic variables ($U^{(i)} = \mathrm{m}_i \dot u^{(i)}$ is the momentum conjugated to $u^{(i)}$) and the phase space is the "collisionless" open domain in $\mathbb{R}^{6(n+1)}$ given by

$$\widehat{\mathcal{M}} := \{U^{(i)}, u^{(i)} \in \mathbb{R}^3 : \ u^{(i)} \neq u^{(j)} \ , \ 0 \le i \neq j \le n\}$$

endowed with the standard symplectic form

$$\sum_{i=0}^n dU^{(i)} \wedge du^{(i)} := \sum_{\substack{0\le i\le n\\ 1\le k\le 3}} dU_k^{(i)} \wedge du_k^{(i)} \ . \tag{3}$$

We recall that the Hamiltonian equations associated to a Hamiltonian function $H(p,q) = H(p_1, ..., p_n, q_1, ..., q_n)$, where (p,q) are standard symplectic variables (i.e., the associated symplectic form is $dp \wedge dq = \sum_{i=1}^n dp_i \wedge dq_i$) are given by

$$\begin{cases} \dot p = -\partial_q H \\ \dot q = \partial_p H \end{cases} \quad \text{i.e.} \quad \begin{cases} \dot p_i = -\partial_{q_i} H \ , \\ \dot q_i = \partial_{p_i} H \ , \qquad (1 \le i \le n) \ . \end{cases} \tag{4}$$

We shall denote the standard Hamiltonian flow, namely, the solution of (4) with initial data p_0 and q_0, by $\phi_H^t(p_0, q_0)$. For general information, see [Arnold, Kozlov and Neishtadt, 2006].

2.2 The Linear momentum reduction

In view of the invariance properties discussed above, it is enough to consider the submanifold

$$\widehat{\mathcal{M}}_0 := \{(U,u) \in \widehat{\mathcal{M}} : \ \sum_{i=0}^n \mathrm{m}_i u^{(i)} = 0 = \sum_{i=0}^n U^{(i)}\} \ ,$$

which corresponds to the manifold described in (2).

The submanifold $\widehat{\mathcal{M}}_0$ is symplectic, i.e., the restriction of the form (3) to $\widehat{\mathcal{M}}_0$ is again a symplectic form; indeed:

$$\big(\sum_{i=0}^n dU^{(i)} \wedge du^{(i)}\big)\big|_{\widehat{\mathcal{M}}_0} = \sum_{i=1}^n \frac{\mathrm{m}_0 + \mathrm{m}_i}{\mathrm{m}_0} \ dU^{(i)} \wedge du^{(i)} \ .$$

Following Poincaré, one can perform a *symplectic reduction* ("reduction of the linear momentum") allowing to lower the number of degrees of freedom by three units; recall that the number of degree of freedom of an autonomous Hamiltonian system is half of the dimension of the phase space (classically, the dimension of the configuration space). Indeed, let $\phi_{\rm he} : (R, r) \to (U, u)$ be the linear transformation given by

$$\phi_{\rm he} : \left\{ \begin{array}{lll} u^{(0)} = r^{(0)} \ , & u^{(i)} = r^{(0)} + r^{(i)} \ , & (i = 1, ..., n) \\ U^{(0)} = R^{(0)} - \sum_{i=1}^n R^{(i)} \ , & U^{(i)} = R^{(i)} \ , & (i = 1, ..., n) \ ; \end{array} \right. \tag{5}$$

such transformation is symplectic, i.e.,

$$\sum_{i=0}^n dU^{(i)} \wedge du^{(i)} = \sum_{i=0}^n dR^{(i)} \wedge dr^{(i)} \ ;$$

recall that this means, in particular, that in the new variables the Hailtonian flow is again standard: more precisely, one has that $\phi^t_{\widehat{\mathcal{H}}_{\rm N}} \circ \phi_{\rm he} = \phi_{\rm he} \circ \phi^t_{\widehat{\mathcal{H}}_{\rm N} \circ \phi}$.

Letting

$$\mathrm{m_{tot}} := \sum_{i=0}^n \mathrm{m}_i$$

one sees that, in the new variables, $\widehat{\mathcal{M}}_0$ reads

$$\Big\{ (R, r) \in \mathbb{R}^{6(n+1)} : \ R^{(0)} = 0 \ , \ \ r^{(0)} = -\mathrm{m_{tot}^{-1}} \sum_{i=1}^n \mathrm{m}_i r^{(i)} \\ \text{and} \quad 0 \neq r^{(i)} \neq r^{(j)} \ \forall \ 1 \leq i \neq j \leq n \Big\} \ .$$

The restriction of the 2–form (3) to $\widehat{\mathcal{M}}_0$ is simply $\displaystyle\sum_{i=1}^n dR^{(i)} \wedge dr^{(i)}$ and

$$\begin{aligned} (\widehat{\mathcal{H}}_{\rm N} \circ \phi_{\rm he})|_{\mathcal{M}_0} &= \sum_{i=1}^n \Big(\frac{|R^{(i)}|^2}{2 \frac{\mathrm{m}_0 \mathrm{m}_i}{\mathrm{m}_0 + \mathrm{m}_i}} - \frac{\mathrm{m}_0 \mathrm{m}_i}{|r^{(i)}|} \Big) + \sum_{1 \leq i < j \leq n} \Big(\frac{R^{(i)} \cdot R^{(j)}}{\mathrm{m}_0} - \frac{\mathrm{m}_i \mathrm{m}_j}{|r^{(i)} - r^{(j)}|} \Big) \\ &=: \mathcal{H}_{\rm N} . \end{aligned}$$

Thus, the dynamics generated by $\widehat{\mathcal{H}}_{\rm N}$ on $\widehat{\mathcal{M}}_0$ is equivalent to the dynamics generated by the Hamiltonian $(R, r) \in \mathbb{R}^{6n} \to \mathcal{H}_{\rm N}(R, r)$ on

$$\begin{aligned} \mathcal{M}_0 \ := \ \Big\{ & (R, r) = (R^{(1)}, ..., R^{(n)}, r^{(1)}, ..., r^{(n)}) \in \mathbb{R}^{6n} : \\ & 0 \neq r^{(i)} \neq r^{(j)} \ \forall \ 1 \leq i \neq j \leq n \Big\} \end{aligned}$$

with respect to the standard symplectic form $\sum_{i=1}^n dR^{(i)} \wedge dr^{(i)}$; to recover the full dynamics on $\widehat{\mathcal{M}}_0$ from the dynamics on $\mathcal{M}_0$ one will simply set $R^{(0)}(t) \equiv 0$ and $r^{(0)}(t) := -\mathrm{m_{tot}^{-1}} \displaystyle\sum_{i=1}^n \mathrm{m}_i r^{(i)}(t)$.

Since we are interested in the planetary case, we perform the trivial rescaling by a small positive parameter μ:

$$
\begin{aligned}
& m_0 := \mathrm{m}_0\ ,\ \ \mathrm{m}_i = \mu m_i\ (i \geq 1)\ , \quad X^{(i)} := \frac{R^{(i)}}{\mu}\ ,\ x^{(i)} := r^{(i)}\ , \\
& \mathcal{H}_{\rm plt}(X,x) := \frac{1}{\mu}\mathcal{H}_{\rm N}(\mu X, x)\ ,
\end{aligned}
$$

which leaves unchanged Hamilton's equations. Explicitly, if

$$
M_i := \frac{m_0 m_i}{m_0 + \mu m_i}\ , \qquad \text{and} \qquad \bar{m}_i := m_0 + \mu m_i\ ,
$$

then

$$
\begin{aligned}
\mathcal{H}_{\rm plt}(X,x) \ &:= \ \sum_{i=1}^{n} \left(\frac{|X^{(i)}|^2}{2M_i} - \frac{M_i \bar{m}_i}{|x^{(i)}|} \right) + \mu \sum_{1 \leq i < j \leq n} \left(\frac{X^{(i)} \cdot X^{(j)}}{m_0} - \frac{m_i m_j}{|x^{(i)} - x^{(j)}|} \right) \\
&=: \ \mathcal{H}^{(0)}_{\rm plt}(X,x) + \mu \mathcal{H}^{(1)}_{\rm plt}(X,x)\ , \qquad (6)
\end{aligned}
$$

the phase space being

$$
\begin{aligned}
\mathcal{M} \ := \ \Big\{ &(X,x) = (X^{(1)}, ..., X^{(n)}, x^{(1)}, ..., x^{(n)}) \in \mathbb{R}^{6n} : \\
& 0 \neq x^{(i)} \neq x^{(j)}\ \forall\ 1 \leq i \neq j \leq n \Big\}\ ,
\end{aligned}
$$

endowed with the standard symplectic form $\sum_{i=1}^n dX^{(i)} \wedge dx^{(i)}$.
Recall that $F(X,x)$ is an integral for $\mathcal{H}(X,x)$ if $\{F,\mathcal{H}\} = 0$ where $\{F,G\} = F_X \cdot G_x - F_x \cdot G_X$ denotes the (standard) Poisson bracket. Now, observe that while $\sum_{i=1}^n X^{(i)}$ is obviously *not* an integral for $\mathcal{H}_{\rm plt}$, the transformation (5) does preserve the total *angular momentum* $\sum_{i=0}^n U^{(i)} \times u^{(i)}$, " $\times$ " denoting the standard vector product in $\mathbb{R}^3$, so that the total angular momentum

$$
\mathrm{C} = (\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3) := \sum_{i=1}^{n} \mathrm{C}_i\ , \qquad \mathrm{C}_i := X^{(i)} \times x^{(i)}\ , \qquad (7)
$$

is still a (vector–valued) integral for $\mathcal{H}_{\rm plt}$. The integrals C_i, however, do not commute (i.e., their Poisson brackets do not vanish):

$$
\{\mathrm{C}_1, \mathrm{C}_2\} = \mathrm{C}_3\ , \qquad \{\mathrm{C}_2, \mathrm{C}_3\} = \mathrm{C}_1\ , \qquad \{\mathrm{C}_3, \mathrm{C}_1\} = \mathrm{C}_2\ ,
$$

but, for example, $|\mathrm{C}|^2$ and C_3 are two commuting, independent integrals.

2.3 Delaunay variables

The Hamiltonian $\mathcal{H}^{(0)}_{\rm plt}$ in (6) governes the motion of n decoupled two–body problems with Hamiltonian

$$h^{(i)}_{\rm 2B} = \frac{|X^{(i)}|^2}{2M_i} - \frac{M_i \bar m_i}{|x^{(i)}|}\ , \qquad (X^{(i)}, x^{(i)}) \in \mathbb{R}^3 \times \mathbb{R}^3_* := \mathbb{R}^3 \times (\mathbb{R}^3 \backslash \{0\})\ .$$

Such two–body sytems are, as well known, integrable. The explicit "symplectic integration" is done by means of the *Delaunay variables*, whose construction we, now, briefly, recall (for full details and proofs, see, e.g., [Celletti and Chierchia, 2007]). Assume that $h^{(i)}_{\rm 2B}(X^{(i)}, x^{(i)}) < 0$ so that the Hamiltonian flow $\phi^t_{h^{(i)}_{\rm 2B}}(X^{(i)}, x^{(i)})$ evolves on a Keplerian ellipse $\mathfrak{E}_i$ and assume that the eccentricity $e_i \in (0, 1)$.
Let a_i, P_i denote, respectively, the *semimajor axis* and the *perihelion* of $\mathfrak{E}_i$.
Let $\mathrm{C}^{(i)}$ denote the $i^{\rm th}$ angular momentum $\mathrm{C}^{(i)} := x^{(i)} \times y^{(i)}$.

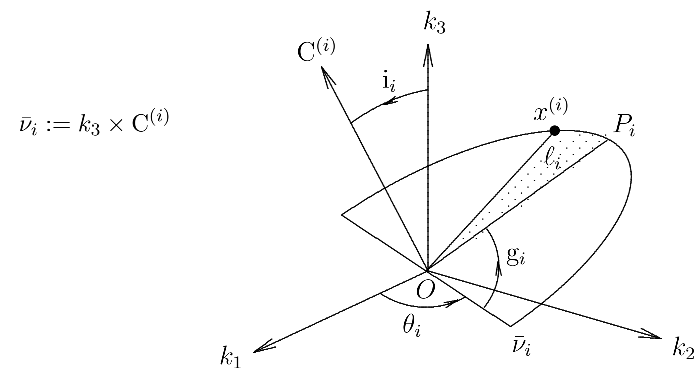

Figure 1: Spatial Delaunay angle variables.

Let us, also, introduce the "Delaunay nodes"

$$\bar\nu_i := k^{(3)} \times \mathrm{C}^{(i)} \quad 1 \le i \le n\ , \tag{8}$$

where $(k^{(1)}, k^{(2)}, k^{(3)})$ is the standard orthonormal basis in $\mathbb{R}^3$. Finally, for $u, v \in \mathbb{R}^3$ lying in the plane orthogonal to a non–vanishing vector w, let $\alpha_w(u, v)$ denote the positively oriented angle (mod 2π) between u and v (orientation follows the "right hand rule").

The *Delaunay action–angle variables* $(\Lambda_i, \Gamma_i, \Theta_i, \ell_i, \mathrm{g}_i, \theta_i)$ are, then, defined as

$$\begin{cases} \Lambda_i := M_i\sqrt{\bar{m}_i a_i} \\ \ell_i := \text{mean anomaly of } x^{(i)} \text{ on } \mathfrak{E}_i \end{cases}$$
$$\begin{cases} \Gamma_i := |\mathrm{C}^{(i)}| = \Lambda_i\sqrt{1-e_i^2} \\ \mathrm{g}_i := \alpha_{\mathrm{C}^{(i)}}(\bar{\nu}_i, P_i) \end{cases} \tag{9}$$
$$\begin{cases} \Theta_i := \mathrm{C}^{(i)} \cdot k^{(3)} \\ \theta_i := \alpha_{k^{(3)}}(k^{(1)}, \bar{\nu}_i) \end{cases}$$

Notice that the Delaunay variables are defined on an open set of full measure of the Cartesian phase space $\mathbb{R}^{3n} \times \mathbb{R}^{3n}_*$, namely, on the set where $e_i \in (0,1)$ and the nodes $\bar{\nu}_i$ in (8) are well defined; on such set the "Delaunay inclinations" i_i defined through the relations

$$\cos \mathrm{i}_i := \frac{\mathrm{C}^{(i)} \cdot k^{(3)}}{|\mathrm{C}^{(i)}|} = \frac{\Theta_i}{\Gamma_i} \ , \tag{10}$$

are well defined and we choose the branch of $\cos^{-1}$ so that $\mathrm{i}_i \in (0,\pi)$.

The Delaunay variables become singular when $\mathrm{C}^{(i)}$ is vertical (the Delaunay node is no more defined) and in the circular limit (the perihelion is not unique). In these cases different variables have to been used (see below).

On the set where the Delaunay variables are well posed, they define a *symplectic set of action–angle variables*, meaning that

$$\sum_{i=1}^{n} dX^{(i)} \wedge dx^{(i)} = \sum_{i=1}^{n} d\Lambda_i \wedge d\ell_i + d\Gamma_i \wedge d\mathrm{g}_i + d\Theta_i \wedge d\theta_i \ ;$$

for a proof, see §3.2 of [Celletti and Chierchia, 2007].

In Delaunay action–angle variables $((\Lambda, \Gamma, \Theta), (\ell, \mathrm{g}, \theta))$ the Hamiltonian $\mathcal{H}^{(0)}_{\mathrm{plt}}$ takes the form

$$-\sum_{i=1}^{n} \frac{M_i^3 \bar{m}_i^2}{2\Lambda_i^2} =: h_{\mathrm{K}}(\Lambda) \ . \tag{11}$$

We shall restrict our attention to the collisionless phase space

$$\begin{aligned} \mathcal{M}_{\mathrm{plt}} \ := \ \Big\{ & (\Lambda, \Gamma, \Theta) \in \mathbb{R}^{3n} : \\ & \Lambda_i > \Gamma_i > \Theta_i > 0 \ , \quad \frac{\Lambda_i}{M_i\sqrt{\bar{m}_i}} \neq \frac{\Lambda_j}{M_j\sqrt{\bar{m}_j}} \ , \ \forall\, i \neq j \Big\} \times \mathbb{T}^{3n} \ , \end{aligned}$$

endowed with the standard symplectic form

$$\sum_{i=1}^{n} d\Lambda_i \wedge d\ell_i + d\Gamma_i \wedge d\mathrm{g}_i + d\Theta_i \wedge d\theta_i \ .$$

Notice that the $6n$–dimensional phase space $\mathcal{M}_{\rm plt}$ is foliated by $3n$–dimensional $\mathcal{H}^{(0)}_{\rm plt}$–invariant tori $\{\Lambda, \Gamma, \Theta\} \times \mathbb{T}^3$, which, in turn, are foliated by n–dimensional tori $\{\Lambda\} \times \mathbb{T}^n$, expressing geometrically the degeneracy of the integrable Keplerian limit of the $(1+n)$–body problem.

2.4 Poincaré variables and the truncated secular dynamics

A regularization of the Delaunay variables in their singular limit was introduced by Poincaré, in such a way that the set of action–angle variables $((\Gamma, \Theta), (\mathrm{g}, \theta))$ is mapped onto cartesian variables regular near the origin, which corresponds to co–circular and co–planar motions, while the angles conjugated to Λ_i, which remains invariant, are suitably shifted.

More precisely, the *Poincaré variables* are given by

$$(\Lambda, \lambda, \mathrm{z}) := (\Lambda, \lambda, \eta, \xi, \mathrm{p}, \mathrm{q}) \in \mathbb{R}^n_+ \times \mathbb{T}^n \times \mathbb{R}^{4n} \ ,$$

with the Λ's as in (9) and

$$\lambda_i = \ell_i + \mathrm{g}_i + \theta_i$$

$$\begin{cases} \eta_i = \sqrt{2(\Lambda_i - \Gamma_i)}\ \cos(\theta_i + \mathrm{g}_i) \\ \xi_i = -\sqrt{2(\Lambda_i - \Gamma_i)}\ \sin(\theta_i + \mathrm{g}_i) \end{cases}$$

$$\begin{cases} \mathrm{p}_i = \sqrt{2(\Gamma_i - \Theta_i)}\ \cos\theta_i \\ \mathrm{q}_i = -\sqrt{2(\Gamma_i - \Theta_i)}\ \sin\theta_i \end{cases}$$

Notice that $e_i = 0$ corresponds to $\eta_i = 0 = \xi_i$, while $\mathrm{i}_i = 0$ corresponds to $\mathrm{p}_i = 0 = \mathrm{q}_i$; compare (9) and (10).

On the domain of definition, the Poincaré variables are symplectic

$$\sum_{i=1}^n d\Lambda_i \wedge d\ell_i + d\Gamma_i \wedge d\mathrm{g}_i + d\Theta_i \wedge d\theta_i = \sum_{i=1}^n d\Lambda_i \wedge d\lambda_i + d\eta_i \wedge d\xi_i + d\mathrm{p}_i \wedge d\mathrm{q}_i \ ;$$

for a proof, see Appendix C of [Biasco, Chierchia and Valdinoci, 2003]. As phase space, we shall consider a collisionless domain around the "secular origin" $\mathrm{z} = 0$ (which correspond to co–planar, co–circular motions) of the form

$$(\Lambda, \lambda, \mathrm{z}) \in \mathcal{M}^{6n}_{\rm P} := \mathcal{A} \times \mathbb{T}^n \times B^{4n} \tag{12}$$

where $\mathcal{A}$ is a set of well separated semimajor axes

$$\mathcal{A} := \big\{\Lambda : \ \underline{a}_j < a_j < \overline{a}_j \quad \text{for} \quad 1 \le j \le n\big\} \tag{13}$$

where $\underline{a}_1, \cdots, \underline{a}_n, \overline{a}_1, \cdots, \overline{a}_n$, are positive numbers verifying $\underline{a}_j < \overline{a}_j < \underline{a}_{j+1}$ for any $1 \le j \le n$, $\overline{a}_{n+1} := \infty$, and B^{4n} is a small $4n$–dimensional ball around the secular origin $\mathrm{z} = 0$.

In Poincaré coordinates, the Hamiltonian $\mathcal{H}_{\rm plt}$ (6) takes the form

$$\mathcal{H}_{\rm P}(\Lambda,\lambda,{\rm z}) = h_{\rm K}(\Lambda) + \mu f_{\rm P}(\Lambda,\lambda,{\rm z}) \ , \quad {\rm z} := (\eta,{\rm p},\xi,{\rm q}) \in \mathbb{R}^{4n} \tag{14}$$

where the "Kepler" unperturbed term $h_{\rm K}$ is as above; compare (11).
Because of rotation (with respect the $k^{(3)}$–axis) and reflection invariance of the Hamiltonian (6) (with respect to the coordinate planes), the perturbation $f_{\rm P}$ in (14) satisfies well known symmetry relations called *d'Alembert rules* i.e., $f_{\rm P}$ is invariant under the following transformations:

$$\begin{cases} (\eta,\xi,{\rm p},{\rm q}) \to (-\xi,-\eta,{\rm q},{\rm p}) \\ (\eta,\xi,{\rm p},{\rm q}) \to (\eta,\xi,-{\rm p},-{\rm q}) \\ (\eta,\xi,{\rm p},{\rm q}) \to (-\eta,\xi,{\rm p},-{\rm q}) \\ (\eta,\xi,{\rm p},{\rm q}) \to (\eta,-\xi,-{\rm p},{\rm q}) \\ (\Lambda,\lambda,{\rm z}) \to (\Lambda,\lambda+g,\mathcal{S}^g({\rm z})) \end{cases} \tag{15}$$

where, for any $g \in \mathbb{T}$, $\mathcal{S}^g$ acts as synchronous clock–wise rotation by the angle g in the symplectic ${\rm z}_i$–planes:

$$\mathcal{S}^g : {\rm z} \to \mathcal{S}^g({\rm z}) = \Big(\mathcal{S}_g({\rm z}_1), ..., \mathcal{S}_g({\rm z}_{2n})\Big) \ , \qquad \mathcal{S}_g := \begin{pmatrix} \cos g & \sin g \\ -\sin g & \cos g \end{pmatrix} \ . \tag{16}$$

compare (3.26)–(3.31) in [Chierchia and Pinzari, 2011 (b)]. By such symmetries, in particular, the averaged perturbation

$$f_{\rm P}^{\rm av}(\Lambda,{\rm z}) := \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} f_{\rm P}(\Lambda,\lambda,{\rm z})d\lambda \ , \tag{17}$$

which is called the *secular Hamiltonian*, is even around the origin ${\rm z} = 0$ and its expansion in powers of z has the form

$$f_{\rm P}^{\rm av} = C_0(\Lambda) + \mathcal{Q}_h(\Lambda)\cdot\frac{\eta^2+\xi^2}{2} + \mathcal{Q}_v(\Lambda)\cdot\frac{{\rm p}^2+{\rm q}^2}{2} + {\rm O}(|{\rm z}|^4) \ , \tag{18}$$

where $\mathcal{Q}_h$, $\mathcal{Q}_v$ are suitable quadratic forms and $\mathcal{Q}\cdot u^2$ denotes the 2–indices contraction $\sum_{i,j}\mathcal{Q}_{ij}u_iu_j$ ($\mathcal{Q}_{ij}$, u_i denoting the entries of $\mathcal{Q}$, u). This shows that ${\rm z} = 0$ *is an elliptic equilibrium for the secular dynamics* (i.e, the dynamics generated by $f_{\rm P}^{\rm av}$). The explicit expression of such quadratic forms can be found, *e.g.*, in (36), (37) of [Féjoz, 2004] (revised version).
The *truncated averaged Hamiltonian*

$$h_{\rm K} + \mu\Big(C_0(\Lambda) + \mathcal{Q}_h(\Lambda)\cdot\frac{\eta^2+\xi^2}{2} + \mathcal{Q}_v(\Lambda)\cdot\frac{{\rm p}^2+{\rm q}^2}{2}\Big)$$

is integrable, with $3n$ commuting integrals given by

$$\Lambda_i \ , \qquad \rho_i = \frac{{\eta_i}^2+{\xi_i}^2}{2} \ , \qquad {\rm r}_i = \frac{{{\rm p}_i}^2+{{\rm q}_i}^2}{2} \ , \qquad (1\le i\le n) \ ;$$

the general trajectory fills a $3n$–dimensional torus with n fast frequencies $\partial_{\Lambda_i} h_{\rm K}(\Lambda_i)$ and $2n$ slow frequencies given by

$$\mu\Omega = \mu(\sigma, \varsigma) = \mu(\sigma_1, \cdots, \sigma_n, \varsigma_1, \cdots, \varsigma_n) \ , \tag{19}$$

σ_i and ς_i being the real eigenvalues of $\mathcal{Q}_h(\Lambda)$ and $\mathcal{Q}_v(\Lambda)$, respectively; such tori surround the n–dimensional elliptic tori given by $\{\Lambda\} \times \{{\rm z} = 0\}$, corresponding to n–coplanar and co–circular planets rotating around the Sun with Keplerian frequencies $\partial_{\Lambda_i} h_{\rm K}(\Lambda_i)$.

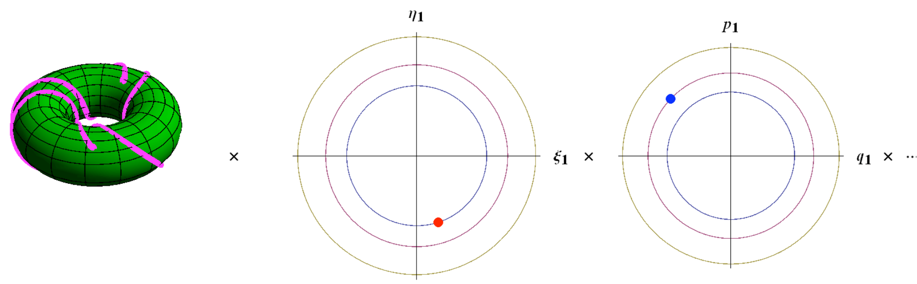

Figure 2: The truncated averaged planetary dynamics

3 Arnold's planetary Theorem

In the following section, we report some of Arnold's statements concerning the existence of regular quasi–periodic motions for the planetary $(1 + n)$–body problem. We recall that, in general, a "quasi–periodic" (or "conditionally periodic") orbit $\zeta(t)$ with (rationally independent) frequencies $(\omega_1, ..., \omega_d) = \omega \in \mathbb{R}^d$ is a solution of the Hamilton equations of the form $\zeta(t) = Z(\omega_1 t, ..., \omega_d t)$ for a suitable smooth function $Z(\theta_1, ..., \theta_d)$ 2π–periodic in each variable θ_i.

3.1 Arnold's Statements (1963)

At p. 87 of [Arnold, 1963] Arnold says:

Conditionally periodic motions in the many–body problem have been found. If the masses of n "planets" are sufficiently small in comparison with the mass of the central body, the motion is conditionally periodic for the majority of initial conditions for which the eccentricities and inclinations of the Kepler ellipses are small. Further, the major semiaxis perpetually remain close to their original values and the eccentricities and inclinations remain small.

Later, p. 125 of [Arnold, 1963]:

With the help of the fundamental theorem of Chapter IV, we investigate in this chapter the class of "planetary" motions in the three–body and many–body problems. We show that, for the majority of initial conditions under which the instantaneous

orbits of the planets are close to circles lying in a single plane, perturbation of the planets on one another produces, in the course of an infinite interval of time, little change on these orbits provided the masses of the planets are sufficiently small.
In particular, it follows from our results that *in the n-body problem there exists a set of initial conditions having a positive Lebesgue measure and such that, if the initial positions and velocities of the bodies belong to this set, the distances of the bodies from each other will remain perpetually bounded.*

The "fundamental theorem" to which Arnold refers is a KAM (Kolmogorov–Arnold–Moser) theorem for properly–degenerate nearly–integrable Hamiltonian systems: it will be discussed in § 3.2 below. For generalities on KAM theory, see, e.g., [Arnold, Kozlov and Neishtadt, 2006] or [Chierchia, 2009].

Finally, [p. 127 of [Arnold, 1963]:

Our basic result is that *if the masses, eccentricities and inclinations of the planets are sufficiently small, then for the majority of initial conditions the true motion is conditionally periodic and differs little from Lagrangian motion with suitable initial conditions throughout an infinite interval of time $-\infty < t < +\infty$.*

Arnold defines the "Lagrangian motions", at p. 127 as follows: *the Lagrangian motion is conditionally periodic and to the n "rapid" frequencies of the Kepler motion are added n (in the planar problem) or $2n-1$ (in the space problem) "slow" frequencies of the secular motions.* This dynamics corrosponds, essentially, to the above "truncated integrable planetary dynamics"; the missing frequency in the space problem is related to the fact that one of the spatial secular frequency, say, ς_n vanishes identically; compare § 3.5 below.

As mentioned in the introduction, Arnold provides a full detailed proof, checking the applicability (non–degeneracy conditions) of his fundamental theorem, only for the two–planet model ($n = 2$) in the planar regime. As for generalizations, he states (p. 139 of [Arnold, 1963]):

The plane problem of $n > 2$ planets. *The arguments of §2 and 3 easily carry over to the case of more than two planets. $[\cdots]$ We shall not dwell on the details of the calculations which lead to the results of §1, 4.*

As for the spatial general case (p. 142 of [Arnold, 1963]):

The rather lengthy calculations involved in the solution of (3.5.9), the construction of variables satisfying conditions 1)–4), and the verification of non–degeneracy conditions analogous to the arguments of § 4 will not be discussed here.

In the next section we shall discuss Arnold's strategy.

3.2 Proper degeneracies and the "Fundamental Theorem"

The main technical tool is a KAM theorem for properly degenerate systems.

A nearly–integrable system with Hamiltonian

$$H_\mu(I,\varphi) := h(I) + \mu f(I,\varphi) \ , \qquad (I,\varphi) \in \mathbb{R}^d \times \mathbb{T}^d,$$

for which h does not depend upon all the actions I_1,...,I_d is called *properly degenerate*. This is the case of the many–body problem since $h_{\rm K}(\Lambda)$ in (11) depends only on n actions Λ_1,...,Λ_n, while the number of degrees of freedom is $d = 3n$.

In general, maximal quasi–periodic solutions (i.e., quasi–periodic solutions with d rationally–independent frequencies) for properly degenerate systems do not exist: trivially, any unperturbed properly–degenerate system on a $2d$ dimensional phase space with $d \geq 2$ will have motions with frequencies not rationally independent over $\mathbb{Z}^d$. But they may exist under further conditions on the perturbation f.

In Chapter IV of [Arnold, 1963] Arnold overcome for the first time this problem proving the following result which he called "the fundamental theorem".

Let $\mathcal{M}$ denote the phase space

$$\mathcal{M} := \Big\{(I,\varphi,p,q) : \ (I,\varphi) \in V \times \mathbb{T}^n \text{ and } (p,q) \in B\Big\} \ ,$$

where V is an open bounded region in $\mathbb{R}^n$ and B is a ball around the origin in $\mathbb{R}^{2m}$; $\mathcal{M}$ is equipped with the standard symplectic form

$$dI \wedge d\varphi + dp \wedge dq = \sum_{i=1}^n dI_i \wedge d\varphi_i + \sum_{i=1}^m dp_i \wedge dq_i \ .$$

Let, also, H_μ be a real analytic Hamiltonian on $\mathcal{M}$ of the form

$$H_\mu(I,\varphi,p,q) := h(I) + \mu f(I,\varphi,p,q) \ ,$$

and denote by $f^{\rm av}$ the average of f over the "fast angles" φ:

$$f^{\rm av}(I,p,q) := \int_{\mathbb{T}^n} f(I,\varphi,p,q) \frac{d\varphi}{(2\pi)^n} \ .$$

Theorem 3.1 (Arnold 1963) *Assume that $f^{\rm av}$ is of the form*

$$f^{\rm av} = f_0(I) + \sum_{j=1}^m \Omega_j(I) r_j + \frac{1}{2}\tau(I) r\cdot r + o_4 \ , \qquad r_j := \frac{p_j^2+q_j^2}{2} \ , \tag{20}$$

where τ is a symmetric $(m \times m)$–matrix and $\lim_{(p,q)\to 0} |o_4|/|(p,q)|^4 = 0$. Assume, also, that $I_0 \in V$ is such that

$$\det h''(I_0) \neq 0 \ , \tag{21}$$

$$\det \tau(I_0) \neq 0 \ . \tag{22}$$

Then, in any neighborhood of $\{I_0\} \times \mathbb{T}^d \times \{(0,0)\} \subseteq \mathcal{M}$ there exists a positive measure set of phase points belonging to analytic "KAM tori" spanned by maximal quasi–periodic solutions with $n + m$ rationally–independent (Diophantine) frequencies, provided μ is small enough.

Recall that $\omega \in \mathbb{R}^d$ is Diophantine if there exist positive constants γ and c such that

$$|\omega \cdot k| \geq \frac{\gamma}{|k|^c}\ , \qquad\qquad \forall\ k \in \mathbb{Z}^d \backslash \{0\}\ .$$

Let us make some remarks.

(i) Actually, Arnold requires that $f^{\rm av}$ is in Birkhoff normal form up to order 6, which means that

$$f^{\rm av} = f_0(I) + \sum_{j=1}^{m} \Omega_j(I) r_j + \frac{1}{2}\tau(I) r \cdot r + P_3(r; I) + o_6$$

where P_3 is a homogeneous polynomial of degree 3 in the variables r_i (with I–dependent coefficients); but such condition can be relaxed and (20) is sufficient: compare [Chierchia and Pinzari, 2010], where Arnold's properly degenerate KAM theory is revisited and various improvements obtained.

(ii) Condition (21) is immediately seen to be satisfied in the general planetary problem; the correspondence with the planetary Hamiltonian in Poincaré variables (14) being the following: $m = 2n$, $I = \Lambda$, $\varphi = \lambda$, $\mathrm{z} = (p, q)$, $h = h_{\rm K}$, $f = f_{\rm P}$.

(iii) Condition (22) is a "twist" or "torsion" condition. It is actually possible to develop a weaker KAM theory where no torsion is required. This theory is due to Rüssmann [Rüßmann, 2001], Herman and Féjoz [Féjoz, 2004], where $f^{\rm av}$ is assumed to be in Birkhoff normal form up to order 2, $f^{\rm av} = f_0(I) + \sum_{j=1}^{m} \Omega_j(I) r_j + o_2$, and the secular frequency map $I \to \Omega(I)$ is assumed to be *non–planar*, meaning that no neighborhood of I_0 is mapped into an hyperplane.

(iv) Indeed, the torsion assumption (22) implies stronger results. First, it is possible to give explicit bounds on the measure of the "Kolmogorov set", i.e., the set covered by the closure of quasi–periodic motions; see [Chierchia and Pinzari, 2010].
Furthermore, the quasi–periodic motions found belong to a smooth family of *non–degenerate Kolmogorov tori*, which means, essentially, that the dynamics can be linearized in a neighborhood of each torus; see § 6.1 for more information.

On the base of Theorem 3.1, Arnold's strategy is to compute the Birkhoff normal form (20) of the secular Hamiltonian $f^{\rm av}_{\rm P}$ in (17) and to check the non–vanishing of the torsion (22).

3.3 Birkhoff normal forms

Before proceeding, let us recall a few known and less known facts about the general theory of Birkhoff normal forms.

Consider as phase space a $2m$ ball B^{2m}_δ around the origin in $\mathbb{R}^{2m}$ and a real–analytic Hamiltonian of the form

$$H(w) = c_0 + \Omega \cdot \mathrm{r} + o(|w|^2) \ ,$$

where

$$\begin{cases} w = (u_1, \ldots, u_m, v_1, \ldots, v_m) \in \mathbb{R}^{2m} \ , \\ \mathrm{r} = (\mathrm{r}_1, \ldots, \mathrm{r}_m) \ , \quad \mathrm{r}_j = \frac{u_j^2 + v_j^2}{2} \ . \end{cases}$$

The components Ω_j of Ω are called the first order Birkhoff invariants. The following is a classical by G.D. Birkhoff.

Proposition 3.1 *Assume that the first order Birkhoff invariants Ω_j verify, for some $a > 0$ and integer s,*

$$|\Omega \cdot k| \geq a > 0, \quad \forall\ k \in \mathbb{Z}^m : \ 0 < |k|_1 := \sum_{j=1}^{m} |k_j| \leq 2s \ . \tag{23}$$

Then, there exists $0 < \delta' \leq \delta$ and a symplectic transformation $\breve\phi : \breve w \in B^{2m}_{\delta'} \to w \in B^{2m}_\delta$ which puts H into Birkhoff normal form up to the order $2s$, i.e.,

$$H \circ \breve\phi = c_0 + \Omega \cdot \breve{\mathrm{r}} + \sum_{2 \leq h \leq s} P_h(\breve{\mathrm{r}}) + o(|\breve w|^{2s})] \ , \tag{24}$$

where P_h are homogeneous polynomials in $\breve{\mathrm{r}}_j = |\breve w_j|^2/2 := (\breve u_j^2 + \breve v_j^2)/2$ of degree h.

Less known is that the hypotheses of this theorem may be loosened in the case of *rotation invariant Hamiltonians*: this fact, for example, has not been used neither in [Arnold, 1963] nor in [Féjoz, 2004].

First, let us generalize the class of Hamiltonian function so as to include the secular Hamiltonian (18): let us consider an open, bounded, connected set $U \subseteq \mathbb{R}^n$ and consider the phase space $\mathcal{D} := U \times \mathbb{T}^n \times B^{2m}_\delta$, endowed with the standard symplectic form $dI \wedge d\varphi + du \wedge dv$.

We say that *a Hamiltonian $H(I, \varphi, w)$ on $\mathcal{D}$ is rotation invariant* if $H \circ \mathcal{R}^g = H$ for any $g \in \mathbb{T}$, where $\mathcal{R}^g$ is a *symplectic rotation by an angle $g \in \mathbb{T}$ on $\mathcal{D}$*, i.e., a symplectic map of the form

$$\mathcal{R}^g : (I, \varphi, w) \to (I', \varphi', w') \quad \text{with} \quad I'_i = I_i, \ \varphi'_i = \varphi_i + g, \ w' = \mathcal{S}^g(w) \ ,$$

with $\mathcal{S}^g$ definined in (16).

Now, consider a φ–independent real–analytic Hamiltonian $H:\ (I,\varphi,w)\in\mathcal{D}\to H(I,w)\in\mathbb{R}$ of the form

$$H(I,w)=c_0(I)+\Omega(I)\cdot \mathrm{r}+o(|w|^2;I)\ ;$$

by $f=o(|w|^2;I)$ we mean that $f=f(I,w)$ and $|f|/|w|^2\to 0$ as $w\to 0$.
Then, it can be proven the following

Proposition 3.2 *Assume that H is rotation–invariant and that the first order Birkhoff invariants Ω_j verify, for all $I\in U$, for some $a>0$ and integer s*

$$|\Omega\cdot k|\geq a>0,\quad \forall\ 0\neq k\in\mathbb{Z}^m:\sum_{i=1}^n k_i=0\ \ \text{and}\ \ |k|_1\leq 2s\ . \tag{25}$$

Then, there exists $0<\delta'\leq\delta$ and a symplectic transformation $\breve\phi:(I,\breve\varphi,\breve w)\in\breve{\mathcal{D}}:=U\times\mathbb{T}^n\times B^{2m}_{\delta'}\to(I,\varphi,w)\in\mathcal{D}$ which puts H into Birkhoff normal form up to the order $2s$ as in (24) with the coefficients of P_h and the reminder depending also on I. Furthermore, $\breve\phi$ leaves the I–variables fixed, acts as a $\breve\varphi$–independent shift on $\breve\varphi$, is $\breve\varphi$–independent on the remaining variables and is such that

$$\breve\phi\circ\mathcal{R}^g=\mathcal{R}^g\circ\breve\phi\ . \tag{26}$$

We shall call (23) the Birkhoff non–resonance condition (up to order s) and (25) the "reduced" Birkhoff non–resonance condition. The proof of Proposition 3.2 may be found in §7.2 in [Chierchia and Pinzari, 2011 (c)].

3.4 The planar three–body case (1963)

In the planar case the Poincaré variables become simply

$$(\Lambda,\lambda,\mathrm{z}):=(\Lambda,\lambda,\eta,\xi)\in\mathbb{R}^n_+\times\mathbb{T}^n\times\mathbb{R}^{2n}\ ,$$

with the Λ's as in (9) and

$$\lambda_i=\ell_i+\mathrm{g}_i\ ,\qquad \begin{cases}\eta_i=\sqrt{2(\Lambda_i-\Gamma_i)}\ \cos\mathrm{g}_i\\ \xi_i=-\sqrt{2(\Lambda_i-\Gamma_i)}\ \sin\mathrm{g}_i\end{cases}\ .$$

The planetary, planar Hamiltonian, is then given by

$$\mathcal{H}_{\rm P,pln}(\Lambda,\lambda,\mathrm{z})=h_{\rm K}(\Lambda)+\mu f_{\rm P,pln}(\Lambda,\lambda,\mathrm{z})\ ,\quad \mathrm{z}:=(\eta,\mathrm{p},\xi)\in\mathbb{R}^{2n}$$

and

$$\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}f_{\rm P,pln}=:f^{\rm av}_{\rm P,pln}=C_0(\Lambda)+\mathcal{Q}_h(\Lambda)\cdot\frac{\eta^2+\xi^2}{2}+\mathrm{O}(|\mathrm{z}|^4)\ .$$

In Eq. (3.4.31), p.138 of [Arnold, 1963], Arnold computed the first and second order Birkhoff invariants finding, in the asymptotics $a_1 \ll a_2$:

$$\begin{cases} \Omega_1 = -\frac{3}{4} m_1 m_2 \Big(\frac{a_1}{a_2}\Big)^2 \frac{1}{a_2 \Lambda_1} \Big(1 + \mathrm{O}\Big(\frac{a_1}{a_2}\Big)\Big) \\ \\ \Omega_2 = -\frac{3}{4}\ m_2^2 \frac{1}{a_2 \Lambda_2} \Big(1 + \mathrm{O}\Big(\frac{a_1}{a_2}\Big)^2\Big) \end{cases}$$

$$\tau = m_1 m_2 \frac{a_1^2}{a_2^3} \begin{pmatrix} \frac{3}{4\Lambda_1^2} & -\frac{9}{4\Lambda_1 \Lambda_2} \\ -\frac{9}{4\Lambda_1\Lambda_2} & -\frac{3}{\Lambda_2^2} \end{pmatrix} (1 + \mathrm{O}(a_2^{-5/4}))\ ,$$

which shows that the Ω_j's are non resonant up to any finite order (in a suitable Λ–domain), so that the planetary, planar Hamiltonian can be put in Birkhoff normal form up to order 4 and that the second order Birkhoff invariants are non–degenerate in the sense that

$$\begin{aligned} \det \tau &= -(m_1 m_2)^2 \frac{117}{16} \frac{a_1^4}{a_2^6 (\Lambda_1 \Lambda_2)^2} (1 + o(1)) \\ &= -\frac{117}{16} \frac{1}{m_0^2} \frac{a_1^3}{a_2^7} (1 + o(1)) \neq 0\ ; \end{aligned}$$

actually, in [Arnold, 1963] the τ_{ij} are defined as 1/2 of the ones defined here; moreover, a_2^4 in Eq. (3.4.31) of [Arnold, 1963] should be replaced a_2^7.

This allow to apply Theorem 3.1 and to prove Arnold's planetary theorem in the planar three–body ($n = 2$) case.

An extension of this method to the *spatial three–body problem*, exploiting Jacobi's reduction of the nodes and its symplectic realization, is due to P. Robutel [Robutel, 1995].

3.5 Secular Degeneracies

In the general spatial case it is custumary to call σ_i the eigenvalues of $\mathcal{Q}_h(\Lambda)$ and ς_i the eigenvalues of and $\mathcal{Q}_v(\Lambda)$, so that $\Omega = (\sigma, \varsigma)$; compare (19).

It turns out that such invariants satisfy identically the following two *secular resonances*

$$\varsigma_n = 0\ , \qquad \sum_{i=1}^n (\sigma_i + \varsigma_i) = 0 \tag{27}$$

and, actually, it can be shown that *these are the only resonances identically satisfied by the first order Birkhoff invariants*; compare Proposition 78, p. 1575 of [Féjoz, 2004].

The first resonance was well known to Arnold, while the second one was apparently discovered by M. Herman in the 90's and is now known as *Herman resonance*.

Notice that both resonances violate the usual Birkhoff non–resonance condition (23) but *do not violate* the reduced Birkhoff condition (25).

What is a more serious problem for Arnold's approach is that the matrix τ indeed is degenerate, as clarified in [Chierchia and Pinzari, 2011 (b)], since

$$\tau = \begin{pmatrix} \bar\tau & 0 \\ 0 & 0 \end{pmatrix} \tag{28}$$

$\bar\tau$ being a matrix of order $(2n-1)$.

3.6 Herman–Fejóz proof (2004)

In 2004 J. Fejóz published the first complete proof of a general version of Arnold's planetary theorem [Féjoz, 2004]. As mentioned above (remark (ii), §3.2), in order to avoid fourth order computations (and also because M. Herman seemed to suspect the degeneracy of the matrix of the second order Birkhoff invariant; compare the Remark towards the end of p. 24 of [Herman, 2009]), Herman's approach was to use a first order KAM condition based on the non–planarity of the frequency map. But, the resonances (27) show that the frequency map lies in the intersection of two planes, violating the non–planarity condition. To overcome this problem Herman and Féjoz use a trick by Poincarè, consisting in modifying the Hamiltonian by adding a commuting Hamiltonian, so as to remove the degeneracy. By a Lagrangian intersection theory argument, if two Hamiltonian commute and $\mathcal{T}$ is a Lagrangian invariant transitive torus for one of them, then $\mathcal{T}$ is invariant (but not necessarly transitive) also for the other Hamiltonian; compare Lemma 82, p. 1578] of [Féjoz, 2004]. Thus, the KAM tori constructed for the modified Hamiltonian are indeed invariant tori also for the original system. Now, the expression of the vertical component of the total angular momentum C_3 has a particular simple expression in Poincaré variables, since

$$\mathrm{C}_3 := \sum_{j=1}^{n} \Big(\Lambda_j - \frac{1}{2}(\eta_j^2 + \xi_j^2 + \mathrm{p}_j^2 + \mathrm{q}_j^2) \Big) \ ,$$

so that the modified Hamiltonian

$$\mathcal{H}_\delta := \mathcal{H}_{\mathrm{P}}(\Lambda, \lambda, \mathrm{z}) + \delta \mathrm{C}_3$$

is easily seen to have a non–planar frequency map (first order Birlhoff invariants), and the above abstract remark applies.

3.7 Chierchia–Pinzari proof (2011)

In [Chierchia and Pinzari, 2011 (c)] Arnold's original strategy is reconsidered and full torsion of the planetary problem is shown by introducing new symplectic variables (called RPS–variables standing for Regularized Planetary Symplectic variables; see

§ 4.1 below), which allow for a symplectic reduction of rotations eliminating one degree of freedom (i.e., lowering by two units the dimension of the phase space). In such reduced setting the first resonance in (27) disappears and the question about the torsion is reduced to study the determinant of $\bar{\tau}$ in (28), which, in fact, is shown to be non–singular; compare §8 of [Chierchia and Pinzari, 2011 (c)] and [Chierchia and Pinzari, 2011 (b)] (where a precise connection is made between the Poincaré and the RPS–variables).

The rest of this article is devoted to explain the main ideas beyond this approach.

4 Symplectic reduction of rotations

We start by describing the new set of symplectic variables, which allow to have a new insight on the symplectic structure of the phase space of the planetary model, or, more in general, of any rotation invariant model.

The idea is to start with action–angle variables having, among the actions, two independent commuting integrals related to rotations, for example, the Euclidean length of the total angular momentum C and its vertical component C_3, and then (imitating Poincaré) to regularize around co–circular and co–planar configurations.

The variables that do the job are an action–angle version of certain variables introduced by A. Deprit in 1983 [Deprit, 1983] (see also [Chierchia and Pinzari, 2011 (a)]), which generalize to an arbitrary number of bodies Jacobi's reduction of the nodes; the regularization has been done in [Chierchia and Pinzari, 2011 (c)].

4.1 The Regularized Planetary Symplectic (RPS) variables

Let $n \geq 2$ and consider the "partial angular momenta"

$$S^{(i)} := \sum_{j=1}^{i} \mathrm{C}^{(j)} \ , \qquad S^{(n)} = \sum_{j=1}^{n} \mathrm{C}^{(j)} =: \mathrm{C} \ ;$$

and define the "Deprit nodes"

$$\begin{cases} \nu_i := S^{(i)} \times \mathrm{C}^{(i)} \ , & \qquad 2 \leq i \leq n \\ \nu_1 := \nu_2 \\ \nu_{n+1} := k^{(3)} \times \mathrm{C} =: \bar{\nu} \ ; \end{cases}$$

(recall the definition of the “individual” and total angular momenta in (7)).

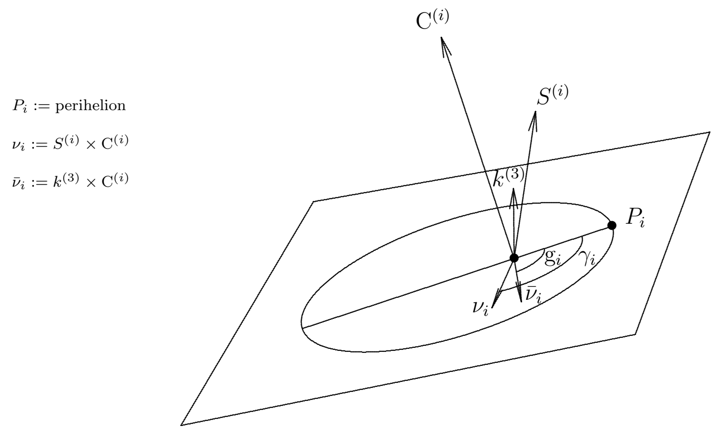

Figure 1: The angle γ_i

The *Deprit action–angle variables* $(\Lambda, \Gamma, \Psi, \ell, \gamma, \psi)$ are defined as follows. The variables Λ, Γ and ℓ are in common with the Delaunay variables (9), while

$$\gamma_i := \alpha_{\mathrm{C}^{(i)}}(\nu_i, P_i) \qquad \Psi_i := \begin{cases} |S^{(i+1)}|\,, & 1 \le i \le n-1 \\ \mathrm{C}_3 := \mathrm{C} \cdot k^{(3)} & i = n \end{cases}$$

$$\psi_i := \begin{cases} \alpha_{S^{(i+1)}}(\nu_{i+2}, \nu_{i+1}) & 1 \le i \le n-1 \\ \zeta := \alpha_{k^{(3)}}(k^{(1)}, \bar\nu) & i = n. \end{cases}$$

Define also $G := |\mathrm{C}| = |S^{(n)}|$.

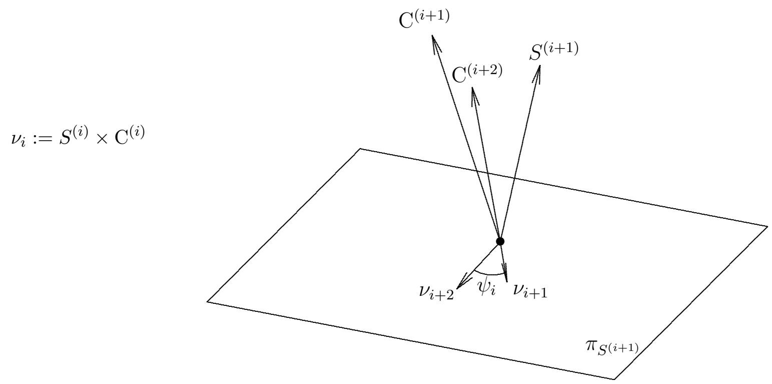

Figure 1: The angle ψ_i for $1 \le i \le n-2$

The "Deprit inclinations" ι_i are defined through the relations

$$\cos \iota_i := \begin{cases} \dfrac{\mathrm{C}^{(i+1)} \cdot S^{(i+1)}}{|\mathrm{C}^{(i+1)}||S^{(i+1)}|} \ , & \qquad 1 \le i \le n-1 \ , \\[3ex] \dfrac{\mathrm{C} \cdot k^{(3)}}{|\mathrm{C}|} \ , & \qquad i = n \ . \end{cases}$$

Similarly to the case of the Delaunay variables, the Deprit action–angle variables are not defined when the Deprit nodes ν_i vanish or $e_i \notin (0,1)$; on the domain where they are well defined they define a real–analytic set of symplectic variables, i.e.,

$$\sum_{i=1}^{n} dX^{(i)} \wedge dx^{(i)} = \sum_{i=1}^{n} d\Lambda_i \wedge d\ell_i + d\Gamma_i \wedge d\gamma_i + d\Psi_i \wedge d\psi_i \ ;$$

for a proof, see [Chierchia and Pinzari, 2011 (a)] or §3 of [Chierchia and Pinzari, 2011 (c)].

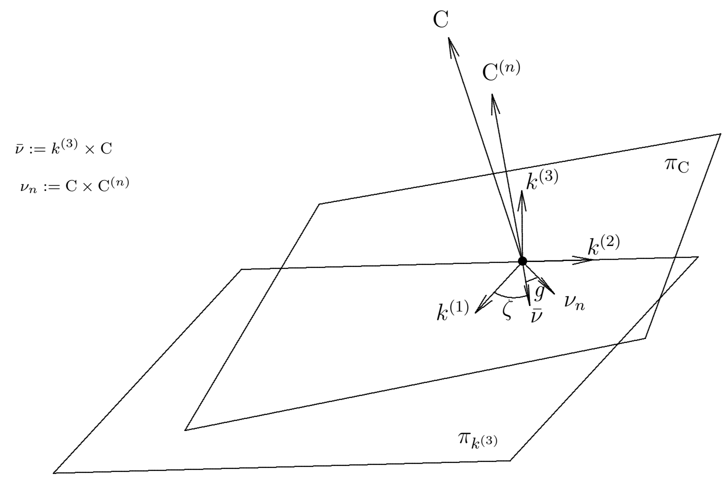

Figure 1: The angles $\psi_{n-1} =: g$ and $\psi_n =: \zeta$

The RPS variables are given by $(\Lambda, \lambda, z) := (\Lambda, \lambda, \eta, \xi, p, q)$ with (again) the Λ's as in (9) and

$$\lambda_i = \ell_i + \gamma_i + \psi^n_{i-1} \qquad \begin{cases} \eta_i = \sqrt{2(\Lambda_i - \Gamma_i)}\ \cos\left(\gamma_i + \psi^n_{i-1}\right) \\ \xi_i = -\sqrt{2(\Lambda_i - \Gamma_i)}\ \sin\left(\gamma_i + \psi^n_{i-1}\right) \end{cases}$$

$$\begin{cases} p_i = \sqrt{2(\Gamma_{i+1} + \Psi_{i-1} - \Psi_i)}\ \cos\psi^n_i \\ q_i = -\sqrt{2(\Gamma_{i+1} + \Psi_{i-1} - \Psi_i)}\ \sin\psi^n_i \end{cases}$$

where

$$\Psi_0 := \Gamma_1\ , \quad \Gamma_{n+1} := 0\ , \quad \psi_0 := 0\ , \quad \psi^n_i := \sum_{i \le j \le n} \psi_j\ .$$

On the domain of definition, the RPS variables are symplectic

$$\sum_{i=1}^n d\Lambda_i \wedge d\ell_i + d\Gamma_i \wedge d\gamma_i + d\Psi_i \wedge d\psi_i = \sum_{i=1}^n d\Lambda_i \wedge d\lambda_i + d\eta_i \wedge d\xi_i + dp_i \wedge dq_i\ ;$$

for a proof, see §4 of [Chierchia and Pinzari, 2011 (c)]. As phase space, we shall consider a collisionless domain around the "secular origin" $\mathrm{z} = 0$ (which correspond to co–planar, co–circular motions) of the form

For the planetary problem, we shall resctrict the phase space in the RPS variables to a set of the same form as in (12), (13), namely

$$(\Lambda, \lambda, z) \in \mathcal{M}^{6n}_{\rm RPS} := \mathcal{A} \times \mathbb{T}^n \times B^{4n} \tag{29}$$

with B a $4n$–dimensional ball around the origin.

The relation between Poincaré variables and the RPS variables is rather simple. Indeed, if we denote by

$$\phi^{\rm RPS}_{\rm P} : \quad (\Lambda, \lambda, z) \to (\Lambda, \mathrm{\lambda}, \mathrm{z}) \tag{30}$$

the symplectic trasformation between the RPS and the Poincaré variables, one has the following

Theorem 4.1 (Chierchia and Pinzari, 2011 (b)) *The symplectic map* $\phi^{\rm RPS}_{\rm P}$ *in* (30) *has the form*

$$\mathrm{\lambda} = \lambda + \varphi(\Lambda, z) \qquad \mathrm{z} = \mathcal{Z}(\Lambda, z)$$

where $\varphi(\Lambda, 0) = 0$ *and, for any fixed* Λ*, the map* $\mathcal{Z}(\Lambda, \cdot)$ *is 1:1, symplectic (i.e., it preserves the two form* $d\eta \wedge d\xi + dp \wedge dq$*) and its projections verify, for a suitable* $\mathcal{V} = \mathcal{V}(\Lambda) \in \mathrm{SO}(n)$*, with* $O_3 = O(|z|^3)$*,*

$$\Pi_{\mathrm{\eta}}\mathcal{Z} = \eta + O_3 \ , \ \Pi_{\mathrm{\xi}}\mathcal{Z} = \xi + O_3 \ , \ \Pi_{\mathrm{p}}\mathcal{Z} = \mathcal{V}p + O_3 \ , \ \Pi_{\mathrm{q}}\mathcal{Z} = \mathcal{V}q + O_3 \ .$$

4.2 Partial reduction of rotations

Recalling that

$$\Gamma_{n+1} = 0 \ , \quad \Psi_{n-1} = |S^{(n)}| = |\mathrm{C}| \ , \quad \Psi_n = \mathrm{C}_3 \ , \quad \psi_n = \alpha_{k^{(3)}}(k^{(1)}, k_3 \times \mathrm{C})$$

one sees that

$$\begin{cases} p_n = \sqrt{2(|\mathrm{C}| - \mathrm{C}_3)} \ \cos\psi_n \\ q_n = -\sqrt{2(|\mathrm{C}| - \mathrm{C}_3)} \sin\psi_n \ , \end{cases}$$

showing that *the conjugated variables* p_n *and* q_n *are both integrals and hence both cyclic* for the planetary Hamiltonian, which, therefore, in such variables, will have the form

$$\mathcal{H}_{\rm RPS}(\Lambda, \lambda, \bar z) = h_{\rm K}(\Lambda) + \mu f_{\rm RPS}(\Lambda, \lambda, \bar z) \ , \tag{31}$$

where $\bar z$ denotes the set of variables

$$\bar z := (\eta, \xi, \bar p, \bar q) := \big((\eta_1, \ldots, \eta_n), (\xi_1, \ldots, \xi_n), (p_1, \ldots, p_{n-1}), (q_1, \ldots, q_{n-1})\big) \ .$$

In other words, *the phase space* $\mathcal{M}^{6n}_{\rm RPS}$ *in* (29) *is foliated by* $(6n-2)$*–dimensional invariant manifolds*

$$\mathcal{M}^{6n-2}_{p_n, q_n} := \mathcal{M}^{6n}_{\rm RPS}|_{p_n, q_n = \rm const} \ , \tag{32}$$

and since the restriction of the standard symplectic form on such manifolds is simply

$$d\Lambda \wedge d\lambda + d\eta \wedge d\xi + d\bar{p} \wedge d\bar{q} \ ,$$

such manifolds are symplectic and the planetary flow is the standard Hamiltonian flow generated by $\mathcal{H}_{\text{RPS}}$ in (31). We shall call the symplectic, invariant submanifolds $\mathcal{M}^{6n-2}_{p_n,q_n}$ "symplectic leaves". They depend upon a particular orientation of the total angular momentum: in particular, the leaf $\mathcal{M}^{6n-2}_0$ correspond to the total angular momentum parallel to the vertical k_3–axis. Notice, also, that *the analytic expression of the planetary Hamltonian $\mathcal{H}_{\text{RPS}}$ is independent of the leaves.*

In view of thiese observations, it is enough to study the planetary flow of $\mathcal{H}_{\text{RPS}}$ on, say, the vertical leaf $\mathcal{M}^{6n-2}_0$.

5 Planetary Birkhoff normal forms and torsion

The RPS variables share with Poincaré variables classical *D'Alembert symmetries*, i.e., $\mathcal{H}_{\text{RPS}}$ is invariant under the transformations (15) $\mathcal{S}$ being as in (16); compare Remark 3.3 of [Chierchia and Pinzari, 2011 (b)].

This implies that the averaged perturbation

$$f^{\text{av}}_{\text{RPS}} := \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f_{\text{RPS}} \ d\lambda$$

also enjoys D'Alembert rules and thus has an expansion analogue to (18), but independent of (p_n, q_n):

$$f^{\text{av}}_{\text{RPS}}(\Lambda, \bar{z}) = C_0(\Lambda) + \mathcal{Q}_h(\Lambda) \cdot \frac{\eta^2 + \xi^2}{2} + \bar{\mathrm{Q}}_{\mathrm{v}}(\Lambda) \cdot \frac{\bar{p}^2 + \bar{q}^2}{2} + \mathrm{O}(|\bar{z}|^4) \tag{33}$$

with $\mathcal{Q}_h$ of order n and $\bar{\mathrm{Q}}_{\mathrm{v}}$ of order $(n-1)$. Notice that the matrix $\mathcal{Q}_h$ in (33) is the same as in (18), since, when $p = (\bar{p}, p_n) = 0$ and $q = (\bar{q}, q_n) = 0$, Poincaré and RPS variables coincide.

Using Theorem 4.1, one can also show that

$$\mathrm{Q}_{\mathrm{v}} := \begin{pmatrix} \bar{\mathrm{Q}}_{\mathrm{v}} & 0 \\ 0 & 0 \end{pmatrix}$$

is conjugated (by a unitary matrix) to $\mathcal{Q}_v$ in (18), so that the eigenvalues $\bar{\varsigma}_i$ of $\bar{\mathrm{Q}}_{\mathrm{v}}$ coincide con $(\varsigma_1, ..., \varsigma_{n-1})$, as one naively would expect.

In view of the remark after (27), and of the rotation–invariant Birkhoff theory (Proposition 3.2), one sees that one can construct, in an open neighborhood of co–planar and co–circular motions, the Birkhoff normal form of $f^{\text{av}}_{\text{RPS}}$ at any finite order.

More precisely, for $\epsilon > 0$ small enough, denoting

$$\mathcal{P}_\epsilon := \mathcal{A} \times \mathbb{T}^n \times B^{4n-2}_\epsilon \ , \qquad B^{4n-2}_\epsilon := \{\bar{z} \in \mathbb{R}^{4n-2} : |\bar{z}| < \epsilon\} \ ,$$

an ϵ–neighborhood of the co–circular, co–planar region, one can find an real–analytic symplectic transformation

$$\phi_\mu : (\Lambda, \breve\lambda, \breve z) \in \mathcal{P}_\epsilon \to (\Lambda, \lambda, \bar z) \in \mathcal{P}_\epsilon$$

such that

$$\breve{\mathcal{H}} := \mathcal{H}_{\rm RPS} \circ \phi_\mu = h_{\rm K}(\Lambda) + \mu f(\Lambda, \breve\lambda, \breve z)$$

with

$$\breve f_{\rm av}(\Lambda, \breve z) := \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} f \, d\breve\lambda = C_0(\Lambda) + \Omega \cdot \breve{\rm R} + \frac{1}{2}\, \bar\tau \; \breve{\rm R} \cdot \breve{\rm R} + \breve{\mathcal{P}}(\Lambda, \breve z)$$

where

$$\begin{cases} \Omega = (\sigma, \bar\varsigma) \\ \breve z := (\breve\eta, \breve\xi, \breve p, \breve q) \ , \quad \breve{\rm R} = (\breve\rho, \breve r) \ , \quad \breve{\mathcal{P}}(\Lambda, \breve z) = O(|\breve z|^6) \ , \\ \breve\rho = (\breve\rho_1, \cdots, \breve\rho_n) \ , \quad \breve r = (\breve r_1, \cdots, \breve r_{n-1}) \ , \\ \breve\rho_i := \frac{\breve\eta_i^2 + \breve\xi_i^2}{2} \ , \quad \breve r_i = \frac{\breve p_i^2 + \breve q_i^2}{2} \end{cases}$$

With straightforward (but not trivial!) computations, one can then show full torsion for the planetary problem. More precisely, one finds (Proposition 8.1 of [Chierchia and Pinzari, 2011 (c)])

Proposition 5.1 *For $n \ge 2$ and $0 < \delta_\star < 1$ there exist $\bar\mu > 0$,*

$$0 < \underline{a}_1 < \overline{a}_1 < \cdots < \underline{a}_n < \overline{a}_n$$

such that, on the set $\mathcal{A}$ defined in (13) and for $0 < \mu < \bar\mu$, the matrix $\bar\tau = (\tau_{ij})$ is non–singular:

$$\det \bar\tau = d_n(1 + \delta_n) \ ,$$

where $|\delta_n| < \delta_\star$ with

$$d_n = (-1)^{n-1} \frac{3}{5} \Big(\frac{45}{16} \, \frac{1}{m_0^2}\Big)^{n-1} \frac{m_2}{m_1 m_0} a_1 \Big(\frac{a_1}{a_n}\Big)^3 \prod_{2 \le k \le n} \Big(\frac{1}{a_k}\Big)^4 \ . \tag{34}$$

Incidentally, we remark that $\bar\mu$ is taken small only to simplify (34), but a similar evaluation hold with $\bar\mu = 1$.

6 Dynamical consequences

6.1 Kolmogorov tori for the planetary problem

At this point one can apply to the planetary Hamiltonian in normalized variables $\breve{\mathcal{H}}(\Lambda, \breve\lambda, \breve z)$ Arnold's Theorem 3.1 above completing Arnold's project on the planetary N–body problem.

Indeed, by using the refinements of Theorem 3.1 as given in [Chierchia and Pinzari, 2010], from Proposition 5.1 there follows

Theorem 6.1 *There exists positive constants ϵ_*, c_* and C_* such that the following holds. If*

$$0 < \epsilon < \epsilon_* \ , \qquad 0 < \mu < \frac{\epsilon^6}{(\log \epsilon^{-1})^{c_*}} \ ,$$

then each symplectic leaf $\mathcal{M}^{6n-2}_{p_n,q_n}$ (32) contains a positive measure $\mathcal{H}_{\rm RPS}$–invariant Kolmogorov set $\mathcal{K}_{p_n,q_n}$, which is actually the suspension of the same Kolmogorov set $\mathcal{K} \subseteq \mathcal{P}_\epsilon$, which is $\breve{\mathcal{H}}$–invariant.
Furthermore, $\mathcal{K}$ is formed by the union of $(3n-1)$–dimensional Lagrangian, real–analytic tori on which the $\breve{\mathcal{H}}$–motion is analytically conjugated to linear Diophantine quasi–periodic motions with frequencies $(\omega_1, \omega_2) \in \mathbb{R}^n \times \mathbb{R}^{2n-1}$ with $\omega_1 = O(1)$ and $\omega_2 = O(\mu)$.
Finally, $\mathcal{K}$ satisfies the bound

$$\operatorname{meas} \mathcal{P}_\epsilon \geq \operatorname{meas} \mathcal{K} \geq \Big(1 - C_* \sqrt{\epsilon}\Big) \operatorname{meas} \mathcal{P}_\epsilon \ .$$

In particular, $\operatorname{meas} \mathcal{K} \simeq \epsilon^{4n-2} \simeq \operatorname{meas} \mathcal{P}_\epsilon$.

6.2 Conley–Zehnder stable periodic orbits

Indeed, the tori $\mathcal{T} \in \mathcal{K}$ form a (Whitney) smooth family of *non–degenerate Kolmogorov tori*, which means the following. The tori in $\mathcal{K}$ can be parameterized by their frequency $\omega \in \mathbb{R}^{3n-1}$ (i.e., $\mathcal{T} = \mathcal{T}_\omega$) and there exist a real–analytic symplectic diffeomorphism

$$\nu : (y, x) \in B^m \times \mathbb{T}^m \to \nu(y, x; \omega) \in \mathcal{P}_\epsilon \ , \qquad m := 3n - 1 \ ,$$

uniformly Lipschitz in ω (actually C^∞ in the sense of Whitney) such that, for each ω

a) $\breve{\mathcal{H}} \circ \nu = E + \omega \cdot y + Q$; (Kolmogorov's normal form)

b) $E \in \mathbb{R}$ (the energy of the torus); $\omega \in \mathbb{R}^m$ is a Diophantine vector;

c) $Q = O(|y|^2)$

d) $\det \displaystyle\int_{\mathbb{T}^m} \partial_{yy} Q(0, x) \ dx \neq 0$, (nondegeneracy)

e) $\mathcal{T}_\omega = \nu(0, \mathbb{T}^m)$.

Now, in the first paragraph of [Conley and Zehnder, 1983] Conley and Zehnder, putting together KAM theory (and in particular exploiting Kolmogorv's normal form for KAM tori) together with Birkhoff–Lewis fixed–point theorem show that long–period periodic orbits cumulate densely on Kolmogorov tori so that, in particular, *the Lebesgue measure of the closure of the periodic orbits can be bounded below by the measure of the Kolmogorov set.* Notwithstanding the proper degeneracy, this remark applies also in the present situation and as a consequence of Theorem 6.1

and of the fact that the tori in $\mathcal{K}$ are non–degenerate Kolmogorov tori it follows that

in the planetary model the measure of the closure of the periodic orbits in $\mathcal{P}_\epsilon$ can be bounded below by a constant times ϵ^{4n-2}.

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Glossary

Averaging: In a nearly–integrable Hamiltonian system (i.e., a small Hamiltonian perturbation of a completely integrable Hamiltonian system) the procedure of averaging over fast angle variables.
Action–angle variables: A particular set of symplectic variables: half of them have the physical dimension of an action (energy times time).
Birkhoff normal forms Normal form in which, under suitable assumptions, a Hamiltonian having an elliptic equilibrium can be transformed, through a symplectic map, into a Hamiltonian depending only on polar action variables.
Degenerate Hamiltonian systems: Integrable Hamiltonian systems that, when expressed in action variables, do not depend in a "general" way on the actions.
Diophantine numbers: Irrational numbers or set of numbers badly (in a specific quantitative way) approximated by rational numbers.
Elliptic equilibrium: An equilibrium point for a Hamiltonian (i.e., a point where the gradient of the Hamiltonian vanishes)
Hamiltonian equations: First order evolution equations ruling the dynamics of a conservative system.
Integrable Hamiltonian system: A Hamiltonian systems whose evolutions can be solved in terms of integrals ("quadratures") and when all solutions are bounded can be put, through a symplectic transformation of action–angle variables, into a system with Hamiltonian depending only upon action variables.
Invariant tori: Tori embedded in a phase space which are invariant under the Hamiltonian evolution.
KAM Theory: The bulk of techniques and theorems, beginning with the contribution of Kolmogorov, Arnold and Moser, dealing with the problem of the stability of quasi–periodic motion in perturbation of integrable Hamiltonian systems.
Keplerian ellipses: Ellipses described by a two–body system with negative energy.
Kolmogorov tori: Invariant tori on which the Hamiltonian flow is symplectically conjugated to a Diophantine linear flow and which are non–degenerate in a suitable sense.

Linear and angular momentum: Physical characteristics of a system of many bodies.
N**–body problem:** The mathematical problem of studying the motion of an N body system.
N**–body system:** N point–masses (“bodies”) mutually interacting only through gravitational attraction.
Newton’s equations: The fundamental evolution equations of classical mechanics expressing the proportionality between forces and accelerations of a given body (the proportionality constant being the mass of the body).
Node lines (or simply “nodes”): Intersections of relevant planes (e.g., the intersection of the plane orthogonal to the total angular momentum and a fixed reference plane).
Phase space: Classically, the space of positions and corresponding velocities (times masses); in modern and more general terms, the “symplectic manifold” (see below) of a Hamiltonian system.
Planetary systems: A system of N–bodies where one body has mass much larger than the other bodies studied in nearly–coplanar and nearly co–circular regime.
Quasi–periodic motions: Motions that can be described by a linear flow on a torus with incommensurate frequencies.
Resonances: Commensurate relations.
Secular Hamiltonian and secular degeneracies: In the planetary Hamiltonian problem, the Hamiltonian obtained by averaging over the mean anomalies.
Small divisors: linear combination (with rational coefficients) of frequencies appearing in the denominator of expansions arising in averaging theory.
Symplectic coordinates: Coordinates on a symplectic manifold allowing to express in a standard way the associated symplectic form.
Symplectic manifold: An even dimensional manifold endowed with a symplectic form, i.e., a non–degenerate, closed differential 2–form.
Symplectic reduction: A mathematical process allowing to lower the dimension of the phase space, which amount to simplify significantly the original set of differential equations.
Symplectic transformation: a diffeomorfism on a symplectic manifold preserving the symplectic form.
Twist or torsion : A non–degeneracy condition of integrable Hamiltonian systems expressing the fact that the map between actions and frequencies is a local diffeomorfism.

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