The steep Nekhoroshev's Theorem and optimal stability exponents (an announcement)

Nota di Massimiliano Guzzo, Luigi Chierchia e Giancarlo Benettin
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M. Guzzo
Dipartimento di Matematica
Università degli Studi di Padova
Via Trieste 63 - 35121 Padova, Italy
guzzo@math.unipd.it

L. Chierchia (corresponding author)
Dipartimento di Matematica e Fisica
Università degli Studi Roma Tre
Largo San L. Murialdo 1 - 00146 Roma, Italy
luigi@mat.uniroma3.it

G. Benettin
Dipartimento di Matematica
Università degli Studi di Padova
Via Trieste 63 - 35121 Padova, Italy
benettin@math.unipd.it

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Abstract

A new statement of Nekhoroshev's theorem in the general steep case with stability exponents, conjectured to be optimal, is presented.

Si presenta un nuovo enunciato del teorema di Nekhoroshev nel caso ripido ("steep") generico, con migliori esponenti di stabilità, che si congetturano essere ottimali.

Nekhoroshev's Theorem concerns one of the main results in the modern theory of perturbative Hamiltonian systems. Indeed, KAM theory (see [1] for general infor-

mation) ensures that the majority (in measure theoretic sense) of initial data in the phase space of a general, real–analytic (or smooth enough) nearly–integrable Hamiltonian system evolve in a regular (quasi–periodic) way with a small displacement of the action variables ("metric stability of non–degenerate nearly–integrable Hamiltonian systems"). Nekhoroshev's Theorems deals with what happens in the complementary region of phase space where, in general, unstable motions may occur with a large variation of the actions (Arnold diffusion). N.N. Nekhoroshev, in the late 1970's ([11, 12]), proved that all motions of a real–analytic, "steep" Hamiltonian system are stable (in the sense that the action variables stay close to their initial values) for *exponentially long times* (exponential stability). "Steepness" is a generic non–degeneracy condition, which is the main natural hypothesis for Nekhoroshev's theorem (in general, non–steep systems are not exponentially stable).
It is important – both from a purely mathematical and an applicative point of view – to compute the stability exponents, which gives the leading order for the stability time ("effective stability").

Here, we announce a version of Nekhoroshev's Theorem in the general steep case with improved stability exponents and conjecture that such formulation is "sharp" (i.e., the stability exponents are optimal).

Let us proceed, now, with more formal statements. Consider a real–analytic Hamiltonian system with Hamiltonian given, in standard action–angle coordinates, by

$$H(I,\varphi) = h(I) + \varepsilon f(I,\varphi), \qquad (I,\varphi) \in B \times \mathbb{T}^n \ , \tag{1}$$

where: $B \subseteq \mathbb{R}^n$ is open, $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$, and ε is a small parameter. We recall that a C^1 function $h(I)$ is steep in $B \subseteq \mathbb{R}^n$ with steepness indices $\alpha_1, \ldots, \alpha_{n-1} \geq 1$ and (strictly positive) steepness coefficients $C_1, \ldots, C_{n-1}$ and r, if $\inf_{I\in U} \|\omega(I)\| > 0$ and, for any $I \in B$, for any j–dimensional linear subspace $\Lambda \subseteq \mathbb{R}^n$ orthogonal to $\omega(I)$ with $1 \leq j \leq n-1$, one has (for any vector $u \in \mathbb{R}^n$ we denote by $\|u\| := \sqrt{\sum_i |u_i|^2}$ its euclidean norm):

$$\max_{0\leq\eta\leq\xi} \ \min_{u\in\Lambda:\|u\|=\eta} \|\pi_\Lambda \omega(I+u)\| \geq C_j \xi^{\alpha_j} \qquad \forall\ \xi \in (0,r], \tag{2}$$

where π_Λ denotes the orthogonal projection over Λ. The simplest steep functions are the convex, or quasi–convex ones, in which case the steepness indices have the minimum values $\alpha_1 = \ldots = \alpha_{n-1} = 1$.

The following statement can be found in [11, p. 4 and p. 8] and [11, p. 30]:

Theorem (Nekhoroshev) *Let H in* (1) *be real–analytic with h steep. Then, there exist positive constants a, b and ε_0 such that for any $0 \leq \varepsilon < \varepsilon_0$ any solution (I_t, φ_t) of the Hamilton equations for $H(I,\varphi)$ satisfies*

$$|I_t - I_0| \leq \varepsilon^b$$

for any time t satisfying

$$|t| \;\le\; \frac{1}{\varepsilon}\;\exp\left(\frac{1}{\varepsilon^a}\right)\;.$$

Furthermore, a and b can be taken as follows:

$$a = \frac{2}{12\zeta + 3n + 14} \quad , \qquad b = \frac{3a}{2\alpha_{n-1}} \tag{3}$$

where

$$\zeta = \Big[\alpha_1 \cdot \Big(\alpha_2\Big(\dots\big(\alpha_{n-3}(n\alpha_{n-2} + n - 2) + n - 3\big) + \dots\Big) + 2\Big) + 1\Big] - 1,$$

and α_i are the steepness indices of h.

The steepness hypothesis on the integrable limit $h(I)$ was introduced by Nekhoroshev (compare [10]; conditions close to steepness appeared earlier in [6]) and can be viewed as a "minimal transversality" condition, which Nekhoroshev proved to be generic in C^∞ class.
The original papers by Nekhoroshev are quite hard to read and the exponents (3) not optimal. A lot of effort has been put in simplifying the proof and in trying to obtain better exponents. However, this has been done essentially only under much stronger (and non–generic) hypothesis, namely, that $h(I)$ is convex or quasi–convex. In this case, conservation of energy, may be used to essentially simplify the action confinement mechanism (see [5, 2, 3]). The only alternative proof in general steep case, to the best of our knowledge, has been given in [14] using simultaneous Diophantine approximations (but the stability exponents are worst than Nekhoroshev's ones).
In the quasi–convex case it is usually believed that the stability exponent $a = 1/(2n)$ is sharp. Such value has been obtained, with different techniques, by P. Lochak and A. Neishtadt, [9], [8], on one side (who introduced the idea of covering the phase space with neighborhoods of periodic orbits using the theory of Diophantine approximations) and by J. Pöschel, [16], on the other side (who simplified significantly Nekhoroshev's geometric construction and introduced the idea of taking into account the geometry of the resonance lattices to weight differently small divisors).
Obviously the stability exponents a and b are related to each other and one can improve one at the expense of worsening the other; this was pointed out, e.g., in [16]; for a more refined discussion see [4].
A confirmation, based on Arnold diffusion, towards the optimality of $a = 1/(2n)$ in the convex case is given in [17].

On the other hand, no improvements on the original Nekhoroshev's stability exponents, in the general steep case, are yet available (in [13], there is a statement concerning improved values for the stability exponents, however, the proof appears to have a serious gap and such values are not justified, compare [15]).
At this regard, we can prove (the proof will appear elsewhere, [7]):

Theorem 1. *Let H in (1) be real–analytic with h steep in B with steepness indices α_1,...,α_{n-1} and let*

$$p_1 := \prod_{k=1}^{n-2} \alpha_k \ , \qquad a := \frac{1}{2np_1} \quad , \qquad b := \frac{a}{\alpha_{n-1}} \ .$$

Then, there exist positive constants $\varepsilon_0, R_0, T, c > 0$ such that for any $0 \leq \varepsilon < \varepsilon_0$ the solution (I_t, φ_t) of the Hamilton equations for $H(I, \varphi)$ with initial data (I_0, φ_0) with $I_0 \in U - 2R_0\varepsilon^b$ satisfies

$$\|I_t - I_0\| \leq R_0\varepsilon^b \tag{4}$$

for any time t satisfying:

$$|t| \leq \frac{T}{\sqrt{\varepsilon}} \exp\left(\frac{c}{\varepsilon^a}\right) . \tag{5}$$

Let us make a few remarks.

- Notice that the gradient of a steep functions does not yield in general an invertible map, a fact, which makes the analysis of resonances much more complicate (since the frequency space is not diffeomorphic to the action space).
- The constants $\varepsilon_0, R_0, T, c > 0$ are explicitly computed in [7].
- In the quasi–convex case $\alpha_k = 1$ for all k and the sharp statements in [9] and [16] are recovered (but with a substantially different proof).
- The conjecture that the exponents are optimal is also based on relatively simple heuristic arguments, thoroughly discussed in [7]. Here we only mention that the root $1/p_1$ comes in by a natural (and necessary) iterative dimensional argument (related to a power–law scaling of the amplitudes of the resonance domains) and the tangential non–degeneracies given by the steepness property (2). The constant in front of p_1 in the simplest and less degenerate case is the $2n$ coming from the quasi–convex case. Putting these two things together one sees that the exponent $1/(2np_1)$ is, roughly speaking, "necessary".
- The proof of Nekhoroshev's theorem, in its various settings, is clearly split into three parts: (a) a *geometric part*, devoted to the analysis of distribution of small divisors in action–space; (b) an *analytic part*, devoted to the construction of normal forms, which is obtained by adapting averaging methods to an analytic setting; (c) a *stability argument* yielding the confinement of the actions.
  As for (b), we follow the nice and effective normal form theory as given by Pöschel in [16]. Part (c) is short and simple and based on the so–called

"trapping in (lower dimensional) resonances". The geometric part (a) is the real heart of the whole proof and is where all improvements are realized. Of course this part is highly technical, expanding, among other things, some key ideas of Nekhoroshev (e.g., non–overlapping of resonant zones) and of Pöschel (e.g., weighting small divisors with the volume of the resonant lattice). Few more details are discussed below.

On the proof of Theorem 1. As mentioned above, the core of the argument is based on a subtle analysis of resonances which are responsible of the small divisors appearing in averaging or normal form theory. To carry out such analysis one has to divide the action space into non–resonant and resonant regions. In particular it is necessary to provide a hierarchy of resonances (simple, double, etc.) and to introduce suitable weighted neighborhoods of them.

Let us now give a precise description of the "geography of resonances". First, we introduce a "Fourier cut–off" $K \sim 1/\varepsilon^a$ needed to truncate the Fourier series of the perturbing function so as to carry out the analytic part and construct the normal forms up to an exponentially small remainder. Therefore one needs to consider only resonances due to integer vectors (Fourier modes) up to order K. As in [16], we consider only resonances defined by integer vectors k in some maximal K–lattice $\Lambda \subseteq \mathbb{Z}^n$. We recall that a "maximal K–lattice" Λ is a lattice which admits a basis of vectors $\tilde{k} \in \mathbb{Z}^n$ with $|\tilde{k}| := \sum_{i=1}^n |\tilde{k}_i| \le K$, and it is not properly contained in any other lattice of the same dimension; the volume $|\Lambda|$ of the lattice Λ is defined as the euclidean volume of the parallelepiped spanned by a basis for Λ; (see [16]).

We define the following sets in action space (the explicit values of the parameters appearing in these definitions are listed at the end of the note).

Resonant zones (neighborhoods of exact resonances $\omega(I) \cdot k = 0$ with k in some K–maximal lattice Λ):

$$\mathcal{Z}_\Lambda := \{I \in B : \quad \|\pi_{\langle\Lambda\rangle}\omega(I)\| < \delta_\Lambda\}, \tag{6}$$

where $\langle\Lambda\rangle$ denotes the real vector space spanned by the lattice Λ, and δ_Λ is set equal to $\delta_\Lambda = \lambda_j/|\Lambda|$, where λ_j depends only on K and on $j = \dim \Lambda$;

resonant blocks (region of resonances of order j but not of higher order): $B_\Lambda := \mathcal{Z}_\Lambda \backslash \mathcal{Z}_{j+1}$, where $j = \dim \Lambda$ and

$$\mathcal{Z}_i := \cup_{\{\Lambda' : \ \dim \Lambda' = i\}} \mathcal{Z}_{\Lambda'} \ ;$$

non–resonant block (non–resonant region): $B_0 := \mathcal{Z}_0 \backslash \mathcal{Z}_1$, where $\mathcal{Z}_0 := B$; by requiring (because of steepness) that $\inf_{I \in B} \|\omega(I)\| > \delta_\Lambda$, the completely resonant zone $\mathcal{Z}_{\mathbb{Z}^n}$ is empty and so is $\mathcal{Z}_n$;

discs (neighborhoods of a given fast drifts plane passing through a point I within a given resonant zone):

$$\mathcal{D}^{\rho}_{\Lambda,\eta}(I) := \left(\left(\bigcup_{I'\in I+\langle\Lambda\rangle} B(I',\eta)\right)\cap \mathcal{Z}_\Lambda \cap (B-\rho)\right)^{I} \subseteq \mathcal{Z}_\Lambda \cap (B-\rho), \quad (7)$$

where $B(I',\eta)$ denotes the euclidean ball centered in I of radius η, $I + \langle\Lambda\rangle$ (called by Nekhoroshev, "fast drift plane") denotes the plane through I parallel to $\langle\Lambda\rangle$, $(C)^I$ denotes the connected component of a set C which contains I, η is any positive number less or equal than ρ;

extended resonant blocks (open neighborhood of all fast drift planes passing through an arbitrary point of a resonant block):

$$B^{\rho}_{\Lambda,r_\Lambda} := \bigcup_{I\in B_\Lambda\cap(B-\rho)} \mathcal{D}^{\rho}_{\Lambda,r_\Lambda}(I) \subseteq \mathcal{Z}_\Lambda \cap (B-\rho), \quad (8)$$

extended non–resonant block: $B^{\rho}_0 := B_0 \cap (B-\rho)$.

The strategy is now the following:

Step 1: Compute the diameter of the intersection of a fast drift plane with a resonant zone (more precisely the maximum distance between a point $I \in B_\Lambda \cap (B-\rho)$ and a point in $\mathcal{D}^{\rho}_{\Lambda,r_\Lambda}(I)$: such diameter is (roughly speaking) given by $\delta_\Lambda^{(1/\alpha_j)}$. Here is the only place where steepness enters directly.

Step 2: Give small divisor estimates on blocks and extended blocks: a resonant block B_Λ is γ_Λ–K non–resonant (meaning that $|\omega(I)\cdot k| \geq \gamma_\Lambda$ for I in the block, all k outside the resonant modulus Λ and $|k| \leq K$); B_0 is λ_1–K non–resonant; extended resonant blocks satisfy similar properties.

Step 3: Prove non–overlapping of resonant regions of the same dimension: more precisely, the closure of $B^{\rho}_{\Lambda,r_\Lambda}$ does not intersect the resonant zone $\mathcal{Z}_{\Lambda'}$ if Λ and Λ' have the same dimension.

From these steps (and the normal form theory in [16]) it follows easily

Step 4: the "resonant trap argument", which, roughly speaking, says that a motion starting in a certain resonant zone either stays there for exponentially long times or goes into lower order resonant regions ending, eventually, in the non–resonant block where it remains for exponentially long times.

In order for the first three crucial steps to hold and fit in an "optimal way" it is crucial to optimize the choice of parameters defining the geography of resonances.

This is done by defining:

$$K := \left(\frac{\varepsilon_*}{\varepsilon}\right)^a \tag{9}$$

$$\rho := \frac{R_0}{n}\varepsilon^b \tag{10}$$

$$\widehat{\omega} := \frac{1}{2\sqrt{2}} \inf_{I\in B} \|\omega(I)\| \tag{11}$$

$$\lambda_j := \frac{\widehat{\omega}}{(AK)^{q_j}} \ , \qquad \text{where , and } A := 6E \tag{12}$$

$$r_\Lambda := \frac{\delta_\Lambda}{M} \tag{13}$$

$$\tag{14}$$

with suitable $R_0, \epsilon_* > 0$, $E \geq 4$, while M is a Lipschitz constant for $\omega(I)$ and, for any $1 \leq j \leq n-2$:

$$p_j := \prod_{k=j}^{n-2} \alpha_k \ , \quad q_j := np_j - j \ , \quad q_n := 0 \ , \quad q_{n-1} := 1 \ ,$$

$$a_{n-1} := 1 \ , \quad a_j := q_j - q_{j+1} \quad (1 \leq j \leq n-2).$$

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