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Elliptic Two-Dimensional Invariant Tori for the Planetary Three-Body Problem

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Abstract

The spatial planetary three-body problem (i.e., one "star" and two "planets", modelled by three massive points, interacting through gravity in a three dimensional space) is considered. It is proved that, near the limiting stable solutions given by the two planets revolving around the star on Keplerian ellipses with small eccentricity and small non-zero mutual inclination, the system affords two-dimensional, elliptic, quasi-periodic solutions, provided the masses of the planets are small enough compared to the mass of the star and provided the osculating Keplerian major semi-axes belong to a two-dimensional set of density close to one.

1. Introduction and Results

In this paper we consider the (non-planar) planetary three-body problem, namely, the mechanical system made up of three massive points, one of which (the "star") has a significantly larger mass than the other two points (the "planets"), and which interact through a Newtonian gravitational field; the masses of the planets are regarded as small parameters. In particular, motivated by astronomical data, we are interested in nearly circular planetary motions ("small eccentricities") taking place along nearly co-planar orbits ("small mutual inclinations").

As is well known, such a system has been the source of extremely deep (and difficult) studies, among which the contributions of Charles Eugene Delaunay and, especially, of Henri Poincaré are eminent. According to Delaunay and Poincaré, the three-body problem is described by a nearly-integrable Hamiltonian system on an eight-dimensional phase space, equipped with real-analytic action-angle variables. Such a system turns out to be properly degenerate, i.e., the integrable limit (in which the three-body problem is described by two decoupled and integrable two-body systems) depends only on two (action) variables: in the integrable limit, all motions lie on two-dimensional invariant tori run by quasi-periodic motions with two frequencies (related, by Kepler's law, to the major semi-axis of the two limiting Keplerian ellipses). Furthermore, in the small-eccentricity-inclination regime, these two-tori are linearly stable (i.e., linearizing the full system along the unperturbed tori we find a linear system with purely imaginary eigenvalues).

A basic natural question is: *What is the destiny of these two-dimensional tori when the full system is considered?*

Surprisingly enough, no answer to this question has been given up to now. In a 1966 paper [JM66], JEFFERYS & MOSER established the persistence of two-dimensional invariant tori for the planetary three-body problem in the case of large mutual inclinations; in such a case the unperturbed tori (as well as the perturbed ones) are unstable (or partially hyperbolic). Actually, Jefferys and Moser considered explicitly the above question (relative to the small-eccentricity-inclination regime), but were unable to settle it, leaving the reader with the feeling that they did not believe in the persistence of two-dimensional, elliptic tori[1].

In this paper, we answer the above question, showing that, for values of the "initial" semi-axis of the osculating ellipses in a set of nearly full (two-dimensional) measure, the above described unperturbed tori do persist in the full system, provided the masses of the planets are small enough.

Let us, now, give an analytical formulation of our result. To do this we recall the classical Hamiltonian (action-angle) formulation of the planetary (non-planar) three-body problem (for small eccentricity and small mutual inclination) according to Delaunay and Poincaré. Denote the three massive points ("bodies") by P_0, P_1 , P_2 and let m_0, m_1 , m_2 be their masses interacting through gravity (with constant of gravitation 1). Assume that, for some $0 < \bar{\kappa} \leqq 1$,

$$\bar{\kappa}\varepsilon \leqq \frac{m_1}{m_0}, \frac{m_2}{m_0} \leqq \varepsilon \leqq 1.$$

The number ε is regarded as a small parameter: the point P_0 represents "the star" and the points P_1 and P_2 "the planets". For $j = 1, 2$, consider the "osculating ellipses" of the two-body problems associated with the planets P_j and the star[2] P_0 and assume that the eccentricities of such ellipses are small and that the intersection angle, $\hat{\imath}$, between the two planes containing the two osculating ellipses ("mutual

[1] "The details of the existence proof completed, it should be observed that the solutions found are of the elliptic-hyperbolic type, and hence are unstable. It would be desirable to establish similarly the existence of such solutions in the stable case. In this case, however, there is an essential difficulty,..." ([JM66], Section (7), page 575); and also: "$\cdots$ however, there are good reasons to conjecture that in general the stable solutions need not persist...." ([JM66], Section (1), page 568).

[2] The "osculating ellipses (at time t_0)" of the two-body problems associated with (P_0, P_j), ($j = 1$ or $j = 2$), are defined as follows. Let $u^{(0)}$ and $u^{(j)}$ denote the coordinates (in some reference frame) of the points P_0 and P_j at time t_0 and let $\dot{u}^{(0)}$ and $\dot{u}^{(j)}$ denote the respective velocities. The "osculating plane" is defined as the plane spanned by $(u^{(j)} - u^{(0)})$ and $(\dot{u}^{(j)} - \dot{u}^{(0)})$; the "osculating ellipse" is defined as the Keplerian ellipse (lying on the osculating plane) defined by the Kepler solution, with initial data $(u^{(0)}, u^{(j)})$ and $(\dot{u}^{(j)} - \dot{u}^{(0)})$, of the two-body problem (P_0, P_j) obtained disregarding (for $t \geqq t_0$) the third body P_i $(i \neq j)$; see Appendix C for details.

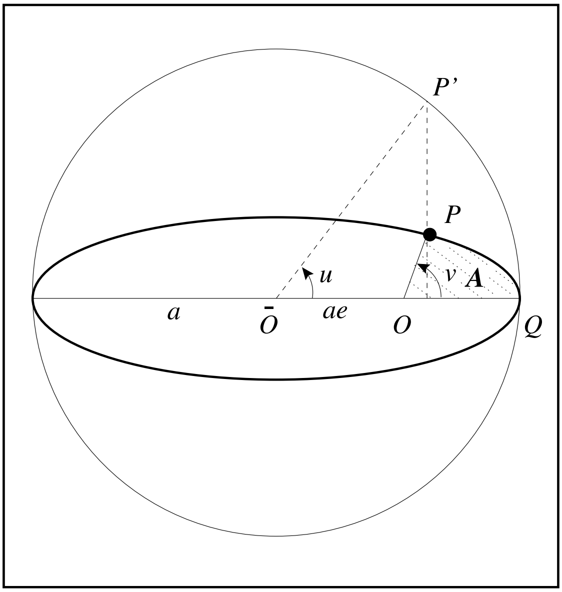

Fig. 1. (Keplerian ellipse): $\bar{O}$=center, O=focus, Q=perihelion, a=major semi-axis= dist $(\bar{O}, Q)$, e=eccentricity= dist $(\bar{O}, O)/a$, u = eccentric anomaly, v=true anomaly, A=area shaded region=$\frac{a^2\sqrt{1-e}}{2}(u - e\sin u)$.

inclination") is also small. As customary in celestial mechanics, we denote the major semi-axes of such ellipses by a_j and their eccentricity by e_j. Let us, also, denote the mean anomaly by ℓ_j and the longitude of the perihelion by g_j (see Figs. 1 and 2).

Let

$$\Lambda_j^* = \kappa_j^* \sqrt{a_j}, \qquad \kappa_j^* := \frac{m_j}{\varepsilon} \frac{1}{\sqrt{m_0(m_0 + m_j)}},$$

(κ_j^* is a dimensionless constant satisfying $\frac{\bar{\kappa}}{\sqrt{2}} < \kappa_j^* < 1$) and, following the notation common in celestial mechanics, define

$$\begin{aligned} \lambda_j^* &= \ell_j + g_j\,, \\ H_j &= \Lambda_j^*\,(1 - \sqrt{1 - e_j^2}), \\ \xi_j^* &= \sqrt{2H_j}\cos g_j\,, \\ \eta_j^* &= -\sqrt{2H_j}\sin g_j\,. \end{aligned}$$

Since we are interested in small eccentricities, collisions are avoided by requiring that the major semi-axes $a_j = a_j(\Lambda^*) := (\Lambda_j^*/\kappa_j^*)^2$ are different (and different from zero). We therefore fix, once and for all,

$$0 < a_{\min} < a_{\max} \qquad \text{and} \qquad 0 < \alpha_{\max} < 1 \tag{1.1}$$

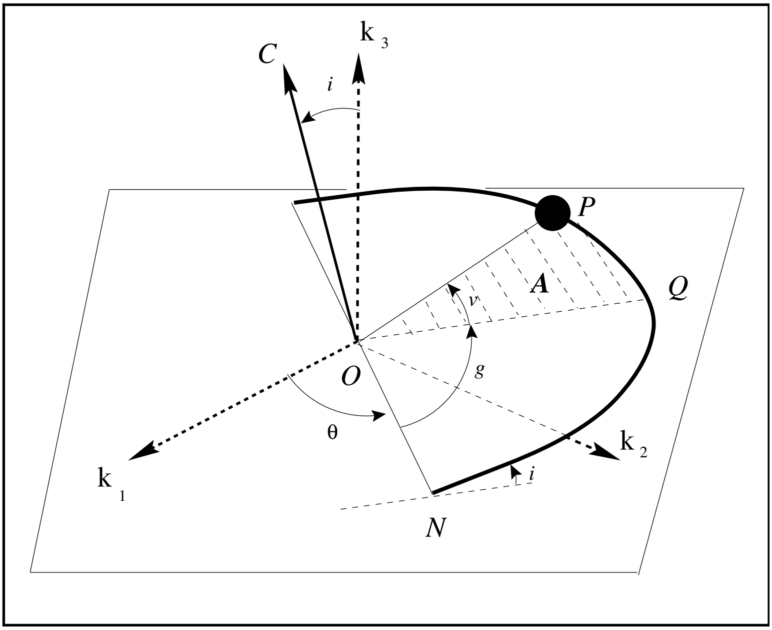

Fig. 2. (Orbital elements): $\{k_1, k_2, k_3\}$=heliocentric frame, C=angular momentum of the 2-body system, N=node $\in$ ellipse plane $\cap$ span$\{k_1, k_2\}$, i=inclination, θ=longitude of the node, g=argument of the perihelion, ℓ = mean anomaly = $2\pi \frac{A}{A_{\text{tot}}} = u - e \sin u$ w = true longitude = $\theta + g + v$, w^*=$g + v$.

and, from now on, we shall consider (attaching the index 1 to the "inner planet") values of Λ^* in the compact set

$$\{\Lambda^* \in \mathbb{R}^2 : a_{\min} \leqq a_1 < a_2 \leqq a_{\max} \text{ and } \frac{a_1}{a_2} \leqq \alpha_{\max}\}. \tag{1.2}$$

For a given set $A \subset \mathbb{R}^n$ and a given number $r > 0$, we shall denote by A_r the set in $\mathbb{C}^n$ at distance less than r from A, i.e., the set

$$A_r := \bigcup_{I \in A} D^n_r(I) \subset \mathbb{C}^n, \tag{1.3}$$

where $D^n_r(I)$ denotes the (open) complex n-ball of radius r centered at I while the real n-ball of radius r centered at I will be denoted $B^n_r(I)$; complex or real n-balls centered at the origin will be simply denoted by D^n_r or, respectively, B^n_r. The following classical result holds.

Theorem 1.1 (Delaunay, Poincaré)**.** *There exist positive constants $\bar{\varepsilon}$, $\bar{\delta}$, $\imath_{\max}$ and $e_{\max}$ such that the variables (Λ^*, λ^*) introduced above are, for $0 < \varepsilon < \bar{\varepsilon}$, standard, real-analytic symplectic variables describing all motions of the spatial three-body problem in a $O(\bar{\delta})$-neighborhood of any point Λ^*_0 in the compact set* (1.2)*, for $e_j \leqq e_{\max}$ and for non-vanishing mutual inclinations $\hat{\imath}$ not bigger than $\imath_{\max}$;*

the Hamiltonian governing such motions (with respect to the standard symplectic form $\sum_i d\Lambda_i^* \wedge d\lambda_i^* + \sum d\eta_i^* \wedge d\xi_i^*$*) is the real-analytic function*

$$-\frac{1}{2}\sum_{j=1}^{2}\frac{\kappa_j}{\Lambda_j^{*2}} + \varepsilon F(\Lambda^*, \lambda^*, \eta^*, \xi^*), \tag{1.4}$$

where $\kappa_j := \left(\frac{m_j}{\varepsilon}\right)^3 \frac{1}{m_0^2(m_0+m_j)}$ *are positive constants of order 1 (*$\frac{\bar{\kappa}^3}{2} < \kappa_j < 1$*) and* F *is a suitable function real-analytic in a neighborhood of* $B^2_{\frac{\delta}{\delta}}(\Lambda_0^*) \times \mathbb{T}^2 \times \{(0,0)\}$*. More precisely, fix* Λ_0^* *in the compact set* (1.2) *and fix* $0 < \imath_{\min} < \imath_{\max}$*, then, for any* $0 < \varepsilon < \bar{\varepsilon}$*,* $\imath_{\min} \leqq |\hat{\imath}| \leqq \imath_{\max}$*, the function* F *can be taken to be real-analytic on the complex set*

$$\mathcal{I}_{\sigma_0} \times \mathbb{T}^2_{2s_0} \times D^4_{2\rho_0} \subset \mathbb{C}^8,$$

where σ_0, s_0, ρ_0 *are suitable positive numbers and where*

$$\mathcal{I} := [\Lambda_{01}^* - \delta, \Lambda_{01}^* + \delta] \times [\Lambda_{02}^* - \delta, \Lambda_{02}^* + \delta],$$

for some $\delta \geqq \bar{\delta}$*.*

Furthermore, there exists a real-analytic, symplectic change of variables $\Psi_W : (I, \varphi, p, q) \mapsto (\Lambda^*, \lambda^*, \eta^*, \xi^*)$ *of the form*

$$\Lambda^* = I, \qquad \lambda^* = \varphi + \hat{\ell}(I, p, q), \qquad \begin{pmatrix}\eta^*\\ \xi^*\end{pmatrix} = A(I)\begin{pmatrix}p\\ q\end{pmatrix}, \tag{1.5}$$

where $\hat{\ell}$ *and* A *are real-analytic and* A *is a* (4×4) *symplectic matrix, such that, in the variables* (I, φ, p, q)*, the Hamiltonian* (1.4) *takes the form*

$$\mathcal{H}(I, \varphi, p, q) = h(I) + f(I, \varphi, p, q), \tag{1.6}$$

with

$$h := -\frac{1}{2}\sum_{i=1}^{2}\frac{\kappa_i}{I_i^2}, \quad f := \varepsilon f_1(I, p, q) + \varepsilon f_2(I, \varphi, p, q),$$

$$f_1 := f_{1,0}(I) + \sum_{i=1}^{2}\bar{\Omega}_i(I)(p_i^2 + q_i^2) + \tilde{f}_1(I, p, q),$$

$$\int_{\mathbb{T}^2} f_2\, d\varphi = 0, \qquad \sup_{\mathcal{I}_{\sigma_0}} |\tilde{f}_1| \leqq \mathrm{const}|(p,q)|^4; \tag{1.7}$$

$\hat{\ell}$*,* A *and* f_i *are real-analytic and uniformly bounded on*

$$\mathcal{D}_0 := \mathcal{I}_{\sigma_0} \times \mathbb{T}^2_{s_0} \times D^4_{\rho_0} \subset \mathbb{C}^8,$$

and $\tilde{f}$ *is even in* (p, q)*. Finally,*

$$\begin{aligned} \inf_{I\in\mathcal{I}} \bar{\Omega}_i &> \inf_{I\in\mathcal{I}_{\sigma_0}} |\bar{\Omega}_i| > \text{const} > 0, \\ \inf_{I\in\mathcal{I}} \left(\bar{\Omega}_2 - \bar{\Omega}_1\right) &> \inf_{I\in\mathcal{I}_{\sigma_0}} |\bar{\Omega}_2 - \bar{\Omega}_1| > \text{const} > 0. \end{aligned} \tag{1.8}$$

Remark 1.1. (i) Even though the results listed in this theorem are classical, the analytical formulation presented here (together with a detailed proof of it) is not easy to find in the literature. An effort to fill this gap was made at the Bureau des longitudes in Paris in the late 80's and we refer the interested reader to the Notes scientifiques et techniques du B.D.L. by CHENCINER [Ch88] and LASKAR [L88]. For completeness, we present the proof of Theorem 1.1 in[3] Appendix C.

(ii) We sketch here, very briefly, the ideas behind Theorem 1.1, referring for details to Appendix C. First of all, by elementary mechanics, we can cast the three-body problem into a nine-degree-of-freedom Hamiltonian formalism. Then, reduction of the center of mass makes it possible to lower the number of degrees of freedom to six. In the planetary case considered here (one "star" and two comparatively small "planets"), such a Hamiltonian system may be seen as a perturbation of two decoupled Kepler problems. Hence, classical Delaunay variables may be exploited to integrate the decoupled Kepler problems. Such variables present however certain singularities, which H. Poincaré showed how to avoid, by introducing an analytic set of variables, now called "Poincaré variables" (or, more precisely, "osculating Poincaré variables[4]"). Finally, the reduction of the angular momentum (also known in the literature as Jacobi's "reduction of the nodes") introduces two extra integrals of motion– called Poincaré integrals in Appendix C-which allow us to lower the degrees of freedom bringing the system into its final form of a four-degree-of-freedom Hamiltonian system. The non-planarity condition $\hat{\imath} \neq 0$ is only needed in order to define the osculating Poincaré variables, while the smallness assumption on the inclination $\hat{\imath}$ (i.e., $|\hat{\imath}| < \imath_{\max}$) is related to the linear stability of the limiting motions, which, mathematically, reflects in $\bar{\Omega}_j$ being real; compare also with item (iii) below. Finally, we stress that the function F does depend upon the chosen $O(1)$-neighborhood of Λ_0^* but obeys uniform bounds in the whole domain (1.2). Such dependence is hidden in the dependence of the perturbation function upon the above-mentioned Poincaré integrals, which,

[3] Be aware that the variables (p, q) appearing in Theorem 1.1 correspond to the variables (η', ξ') of Appendix C (and are not to be confused with the variables (p, q) used in Appendix C).

[4] In connection with three-body problems, there are two sets of Poincaré variables, differentiated in Appendix C by means of a "$*$": the "Poincaré variables" $(\Lambda, \lambda, \eta, \xi) \in \mathbb{R}^2 \times \mathbb{T}^2 \times \mathbb{R}^4$ are particularly suited to the treatment of the planar three-body problem, while the "osculating Poincaré variables" (defined only for non-vanishing mutual inclinations) $(\Lambda^*, \lambda^*, \eta^*, \xi^*) \in \mathbb{R}^2 \times \mathbb{T}^2 \times \mathbb{R}^4$ are more convenient when dealing with the spatial three-body problem; the word "osculating" refers to the fact that these variables "live" on the two planes associated with the two osculating ellipses.

for small eccentricities, are related to the mutual inclination of the osculating ellipses.

(iii) The change of variables $(\Lambda^*, \lambda^*, \eta^*, \xi^*) \mapsto (I, \varphi, p, q)$ is, simply, the (straightforward) completion of the linear symplectic diagonalization[5] of the (η^*, ξ^*)-quadratic part of the "secular term" of the Hamiltonian (1.4), i.e., of the average over the angles λ^* of εF, namely, εf_1. Under the assumption that $\hat{\imath}$ is small enough, the quadratic part turns out to be positive definite and, hence, $\pm\sqrt{-1}(2\varepsilon\bar{\Omega}_j)$ are, simply, the purely imaginary eigenvalues of the (4×4)-matrix $S_4 \mathcal{Q}''$, where S_4 denotes, here, the standard (4×4) symplectic unity matrix and $\mathcal{Q} = \mathcal{Q}(\eta^*, \xi^*; \Lambda^*)$ denotes the (positive-definite) quadratic part of $\varepsilon \int_{\mathbb{T}^2} F d\lambda^*$ (thought of as a function of (η^*, ξ^*) and parametrized by Λ^*). This diagonalization procedure, already known to Weierstrass, requires, in the case of the three-body problem, certain calculations involving Laplace coefficients (which, in turn, are simply related to the Gauss hyper-geometric function): such calculations are sketched in Appendix C (see, also, [LR95] and [R95]).

(iv) The Hamiltonian $\mathcal{H}$ describes a nearly-integrable, properly degenerate system: the integrable limit ($\varepsilon = 0$) depends only on the two action variables I_1, I_2. The frequency vector associated with the integrable limit is $\left(\kappa_1/I_1^3, \kappa_2/I_2^3\right)$, which is a vector of order one. This means that the conjugated angles φ may be regarded as fast angles and, in "first approximation", the $\mathcal{H}$-motions are governed by the averaged Hamiltonian $h + \varepsilon f_1$: such a Hamiltonian, which in case of the spatial three-body problem is non-integrable, is sometimes referred to as a "secular Hamiltonian[6]". By the discussion in the preceding item (iii) and from (1.7), it follows that $p = 0 = q$ is an elliptic equilibrium for the Hamiltonian f_1 and hence, for any I,

$$\{\varphi \in \mathbb{T}^2\} \times \{p = 0 = q\} \tag{1.9}$$

is an elliptic two-torus for the averaged Hamiltonian $h + \varepsilon f_1$ run by the linear flow $\varphi \to \varphi + \nabla(h + \varepsilon f_{1,0})t$. These are the secular motions that we shall prove to persist for $0 < \varepsilon \ll 1$ and for special, but nearly full measure, values of I.

We can, now, give a precise formulation of our main result. Let $\bar{\varepsilon}$, $\bar{\delta}$, $\imath_{\max}$, $e_{\max}$, Λ_0^*, $\mathcal{I}$, $\imath_{\min}$, σ_0, s_0, ρ_0 be as in Theorem 1.1 above and let meas_n denote the n-dimensional Lebesgue measure.

Theorem 1.2. *Fix $\tau > 1$ and pick two numbers b_i such that*

$$0 < b_1 < \frac{1}{2}, \qquad 0 < b_2 < \left(\frac{1}{2} - b_1\right)\frac{1}{\tau + 1}.$$

[5] "Symplectic diagonalization" of a quadratic Hamiltonian $\mathcal{Q}(z)$, $z \in \mathbb{R}^{2n}$, means diagonalization, by a linear symplectic map, of the constant matrix $S_{2n}\mathcal{Q}''$, S_{2n} being the standard $(2n \times 2n)$ symplectic unit matrix; $\mathcal{Q}''$ denotes the Hessian matrix of the second derivatives of $\mathcal{Q}$.

[6] A computer-assisted KAM theory for the secular Hamiltonian of the spatial three-body problem is studied in [LG00].

Then there exist ε_0 *in the range* $0 < \varepsilon_0 < \bar{\varepsilon}$ *and* $C > 1$ *such that, for any* ε *in the range* $0 < \varepsilon < \varepsilon_0$*, a Cantor set* $\mathcal{I}_* \subset \mathcal{I}$ *can be found with*

$$\mathrm{meas}_2(\mathcal{I} \setminus \mathcal{I}_*) \leqq C\varepsilon^{b_1} , \tag{1.10}$$

and the following statement holds. There exist a Lipschitz homeomorphism $\omega_* : \mathcal{I}_* \to \mathbb{R}^2$ *and a Lipschitz continuous family of tori embedding*

$$\phi : (\theta, J) \in \mathbb{T}^2 \times \mathcal{I}_* \mapsto \Big(I_\phi(\theta; J),\ \varphi_\phi(\theta; J),\ p_\phi(\theta; J),\ q_\phi(\theta; J) \Big) \in \mathcal{I} \times \mathbb{T}^2 \times B^4_{\rho_*},$$

with $\rho_* := C\varepsilon^{b_2}$*, such that, for any* $J \in \mathcal{I}_*$*,* $\phi(\mathbb{T}^2, J)$ *is a real-analytic (elliptic)* $\mathcal{H}$*-invariant torus, on which the* $\mathcal{H}$*-flow is analytically conjugated to the linear flow* $\theta \mapsto \theta + \omega_* t$*. Furthermore,* $\phi(\cdot, J)$ *is real-analytic on* $\mathbb{T}^2_{s_0/8}$ *and the following bounds hold, uniformly on* $\mathbb{T}_{s_0/8} \times \mathcal{I}_*$*:*

$$\begin{aligned} |I_\phi(\theta, J) - J| &\leqq C\varepsilon^{\frac{1}{2}+b_2} , \\ |p_\phi(\theta, J)| + |q_\phi(\theta, J)| &\leqq C\varepsilon^{b_2} , \\ |\omega_*(J) - \nabla h(J)| &\leqq C\varepsilon. \end{aligned} \tag{1.11}$$

Also, there exists a Lipschitz continuous function $\Omega_* : \mathcal{I}_* \longrightarrow \mathbb{R}^2$ *such that*

$$|\Omega_*(J) - \varepsilon\bar{\Omega}(J)| \leqq C\varepsilon^{1+b_2} , \tag{1.12}$$

and[7]

$$|\omega_* \cdot k + \Omega_* \cdot \ell| \geqq \frac{\varepsilon}{C(1 + |k|^\tau)} \tag{1.13}$$

for any $(k, \ell) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \setminus \{(0, 0)\}$ *with* $|\ell| \leqq 2$.

We make, now, a few comments and remarks:

Amplification. The proof presented below allows us to catch also the limiting case "$b_1 \to 1/2$": compare also (ii), Remark 2.1 below. In this case, b_1 and b_2 are not defined, nevertheless, the conclusion of Theorem 1.2 holds with the following estimates replacing, respectively, (1.10), (1.11) and (1.12) ((1.13) remains unchanged):

$$\mathrm{meas}_2(\mathcal{I} \setminus \mathcal{I}_*) \leqq C\varepsilon^{\frac{1}{2}} \Big(\log \frac{1}{\varepsilon} \Big)^{\tau+1} ; \tag{1.14}$$

$$\begin{aligned} |I_\phi(\theta, J) - J| &\leqq C\frac{\varepsilon^{\frac{1}{2}}}{\log \frac{1}{\varepsilon}} ; \\ |p_\phi(\theta, J)| + |q_\phi(\theta, J)| &\leqq C\frac{1}{\log \frac{1}{\varepsilon}} , \\ |\omega_*(J) - \nabla h(J)| &\leqq C\varepsilon ; \end{aligned} \tag{1.15}$$

$$|\Omega_*(J) - \varepsilon\bar{\Omega}(J)| \leqq C\frac{\varepsilon}{\log \frac{1}{\varepsilon}} . \tag{1.16}$$

[7] Dot "·" denotes the standard inner product: $x \cdot y := \sum_j x_j y_j$.

On the proof of Theorem 1.2. The proof of Theorem 1.2, presented below, rests upon a KAM (Kolmogorov Arnold Moser) theorem for lower dimensional elliptic tori, stated by MELNIKOV in a 1965 paper [M65] and proved only in 1988 by ELIASSON [E88] and, independently, by KUKSIN [K88]. In fact, we shall use a quantitative version of Melnikov's theorem adapted from a paper by PÖSCHEL [P96]. In order to apply Melnikov's theorem to the three-body problem, the difficulties introduced by the proper degeneracy of the model have to be overcome. For this purpose, we have to use a quantitatively refined version of averaging theory: the angles associated with the unperturbed two-dimensional motions are fast and the system "feels", at "low orders", only averaged effects. Averaging theory allows us to exploit this phenomenon and to "reduce" significantly the effect of the perturbation. The averaging theory we use is not standard and we present it (in a self-contained way) in Appendix A. After averaging, two more symplectic changes of variables are needed in order to cast the planetary three-body (small-eccentricity-inclinations) system in a form suitable for KAM theory.

A physical comment. The elliptic quasi-periodic orbits obtained by Theorem 1.2 are seen to be the continuation of the secular orbits $X^t_{h+\varepsilon f_1}(I, \varphi, 0, 0)$, for suitable initial values of the major semi-axis of the two osculating ellipses; compare, also, point (iv) of Remark 1.1. Eccentricities and inclinations of the persistent orbits may be described as follows:

- the eccentricities are small with ε: in fact, by the second estimate in (1.11), we will have $e_1 + e_2 \leqq \text{const}\, \varepsilon^{b_2}$;
- admissible "initial" inclinations range between inclinations of order one in ε (and close to $\imath_{\max}$) and small-with-ε inclinations: choosing b_2 small (i.e., $b_2 < \frac{1}{2(5+\tau)}$) the inclinations can be of order ε^{b_2} (compare (1.11), (2.14), points (i) and (ii) of Remark C.2 and (C.62) below);
- during the true motions, however, the inclinations vary little with ε: because of (1.11) and the relation between the Poincaré integrals and the inclinations, we find that $|i^2(t) - i^2(0)| = O(\varepsilon^{2b_2})$ (compare (2.14), Remark C.2 and (C.62)).

KAM and the n-body problem. As is well known, KAM theory has been mainly motivated (by the founding authors) by celestial mechanics, and, in particular, by the n-body problem, to which ARNOLD devoted one of the fundamental paper of this theory ([A63]). In [A63] the problem of the existence of maximal quasi-periodic solutions for the n-body problem is considered and the author proves existence of such quasi-periodic solutions for the planar, three-body problem and gives some indications about how to extend his theorem to more general situations. In 1995, LASKAR & ROBUTEL ([LR95]; see also [R95], to which we refer for further references) extended Arnold's result to the spatial three-body problem. Notice, however, that such results do not answer the question posed at the beginning of this section (as they deal with the existence of maximal invariant tori).

HERMAN announced a complete (and lengthy) proof of Arnold's theorem for the n-body problem ([H95]). Unfortunately his untimely decease deprived us of a certainly beautiful (and, probably, quite technical, as it was in Herman's style) piece of mathematics.

On the measure of the invariant tori. Clearly, the union of the two-dimensional tori described in Theorem 1.2 form an invariant subset of the (eight-dimensional) phase space of zero measure. However, it can be shown that in a full neighborhood of such two-dimensional invariant tori there exist a positive measure set of four-dimensional KAM tori (with two frequencies close to ω_* and two frequencies of order ε): in fact, such maximal tori are essentially the tori found by LASKAR & ROBUTEL in [LR95] (see also [R95]).

Incidentally, we mention that around such two-dimensional invariant tori there exist, also, plenty of periodic orbits; see [BBV].

The planar case. The methods used in this paper are also suitable for dealing with the planar case (just use planar Poincaré variables in place of osculating Poincaré variables; compare with footnote 4). However, our methods, for technical reasons, do not allow us to get the planar case as a limit for the inclination i tending to zero.

For a different approach to the planar case, we refer, also, to the recent preprint[8] [F02].

2. Proof of Theorem 1.2

The proof of Theorem 1.2 is based on three well-separated steps, which we now proceed to explain.

First of all, pick a number b so that

$$0 < b_2 < b < \left(\frac{1}{2} - b_1\right)\frac{1}{\tau + 1}. \tag{2.1}$$

Remark 2.1. (i) The estimates that we shall get in this proof (and which are expressed in terms of the constant b) are slightly better than those stated in Theorem 1.2; in the comparison keep in mind (2.1) and the fact that $\frac{1}{2} - b(\tau+1) > b_1$.

(ii) To get the limiting case "$b_1 \to 1/2$", disregard (2.1) and let, in what follows, $b := 0$ (and keep in mind that, in such a case, b_1 and b_2 are not defined).

Step 1 ("fast averaging"). The starting point is the Hamiltonian formulation given in Theorem 1.1 and the first step will consist in "removing" the angle-dependence of the perturbation function f in (1.6), (1.7) to higher order in ε. To do this, we shall make use of "averaging theory" (or "normal form theory") and, in particular, of the proposition which we shall shortly state, after we have introduced the necessary notation.

[8] After finishing our manuscript we received the preprint [F02], where an unpublished C^∞ KAM result by M. Herman (together with the inverse Nash-Moser implicit function theorem in the context of "tame Fréchet spaces") is used to derive the existence of quasi-periodic motions in the planar three-body problem. We remark that in the planar three-body problem, the averaged system ("secular Hamiltonian") is integrable, a fact that is heavily relied on in [F02]. (On the other hand, in the non-planar case, such integrability is no more available; compare, also, footnote 6.)

Let (I, φ, p, q) be standard symplectic variables in $U \times \mathbb{T}^2 \times V$ (with respect to the symplectic form $\sum_i dI_i \wedge d\varphi_i + \sum_i dp_i \wedge dq_i$), where $U \subset \mathbb{R}^2$ and $V \subset \mathbb{R}^4$; let[9]

$$W_{r,\rho,s} := U_r \times \mathbb{T}^2_s \times V_\rho := \bigcup_{I \in U} D^2_r(I) \times \mathbb{T}^2_s \times \bigcup_{(p,q)\in V} D^4_\rho((p,q)) \subset \mathbb{C}^8 \; ; \tag{2.2}$$

and for a function real-analytic on $W_{r,\rho,s}$ denote by $\|f\|_{r,\rho,s}$ its "sup-Fourier" norm given by[10]

$$\|f\|_{r,\rho,s} := \sum_{k\in\mathbb{Z}^2} \left(\sup_{(I,p,q)\in U_r \times V_\rho} |f_k(I,p,q)| \right) e^{|k|s}, \tag{2.3}$$

where $f_k(I, p, q)$ denotes the Fourier coefficient of index $k \in \mathbb{Z}^2$ of the periodic function $\varphi \to f(I, \varphi, p, q)$.

Proposition 2.1 (Averaging Theorem)**.** *Let $H := h(I) + f(I, \varphi, p, q)$ be a real-analytic Hamiltonian on $W_{r,\rho,s}$ and denote by $\omega := h' := \nabla h$ the gradient of h. Assume that there exist $\alpha, K > 0$, satisfying $Ks \geqq 6$, such that*

$$|\omega(I) \cdot k| \geqq \alpha, \qquad \forall\, k \in \{k \in \mathbb{Z}^2 : \; 0 < |k| \leqq K\}, \;\; \forall\, I \in U_r. \tag{2.4}$$

Assume, also, that, if $d := \min\{rs, \rho^2\}$, then

$$\|f\|_{r,\rho,s} < \frac{\alpha d}{c\,Ks}, \tag{2.5}$$

where $c > 1$ is a suitable (*universal*) *constant. Then, there exists a real-analytic symplectic transformation*

$$\Psi : (I', \varphi', p', q') \in W_{r/2,\rho/2,s/6} \mapsto (I, \varphi, p, q) = \Psi(I', \varphi', p', q') \in W_{r,\rho,s} \tag{2.6}$$

and a real-analytic function $g = g(I', p', q')$ such that

$$H_* := H \circ \Psi = h + g + f_* \, , \tag{2.7}$$

and the following bounds hold[11]*:*

$$\sup_{(I',p',q')\in U_{r/2}\times V_{\rho/2}} |g(I', p', q') - f_0(I', p', q')| \leqq \frac{c}{\alpha d} \, \|f\|^2_{r,\rho,s}, \tag{2.8}$$

$$\|f_*\|_{r/2,\rho/2,s/6} \leqq e^{-Ks/6} \, \|f\|_{r,\rho,s}. \tag{2.9}$$

Furthermore $(I, \varphi, p, q) = \Psi(I', \varphi', p', q')$ satisfies

$$s\,|I - I'|, \;\; r\,|\varphi - \varphi'|, \;\; \rho\,|p - p'|, \;\; \rho\,|q - q'| \leqq \frac{c\|f\|_{r,\rho,s}}{\alpha}, \tag{2.10}$$

for each $(I', \varphi', p', q') \in W_{r/2,\rho/2,s/6}$.

[9] Recall the notation in (1.3).

[10] If $k \in \mathbb{Z}^n$ $|k| := \sum_{i=1}^n |k_i|$.

[11] The 0-Fourier coefficient of f, i.e., its φ-average, is denoted by f_0.

Remark 2.2. (i) In order not to interrupt the proof of Theorem 1.2, we present the proof of (a more general form of) the Averaging Theorem in Appendix A.

(ii) Notice that, unlike the case in standard normal form theory, in the above proposition, the "dummy" symplectic variables (p, q) are also controlled.

(iii) A qualitatively similar statement can be found in [N77] (Lemma 10.3, p. 45); however, the quantitative bounds proved in [N77] are not enough for our purposes (compare, in particular, the estimates reported in Remark 10.4, p. 46 of [N77] with the stronger estimate (2.8) above).

(iv) (*Notational conventions*) In the rest of the proof, we shall denote by "const" (or c_i, c_*, C, etc.) positive constants of order one in ε, which may depend upon τ, b, b_1, b_2, $\bar\varepsilon$, $\bar\delta$, $\imath_{\max}$, $e_{\max}\sup_{\mathcal{I}_{\sigma_0}}|h''|$, and the sup-Fourier norm of f_i in their analyticity domains. Also, the expression "f is real-analytic (or, simply, analytic) on $A\subset\mathbb{C}^n$" will be short for: f is real-analytic on A with uniformly bounded sup-Fourier norm (2.3).

Let, now, $\mathcal{H}=h+f$ be as in Theorem 1.1 and let, as above, $\omega(I):=h'(I)$. Define

$$\bar\gamma = c_*\,\varepsilon^{\frac12-b(\tau+1)}\Big(\log\frac1\varepsilon\Big)^{\tau+1}, \tag{2.11}$$

with $c_*>0$ to be chosen later. Consider the set of points I in $\mathcal{I}$ for which $\omega(I)$ is $(\bar\gamma,\tau)$-Diophantine:

$$\mathcal{I}_{\bar\gamma,\tau} := \Big\{ I\in\mathcal{I} \,:\, |\omega(I)\cdot k| \geqq \frac{\bar\gamma}{|k|^\tau},\ \forall k\in\mathbb{Z}^2\setminus\{0\}\Big\}. \tag{2.12}$$

Notice that (as a standard proof shows)

$$\mathrm{meas}_2\Big(\mathcal{I}\setminus\mathcal{I}_{\bar\gamma,\tau}\Big) \leqq \mathrm{const}\,\bar\gamma = \mathrm{const}\,\varepsilon^{\frac12-b(\tau+1)}\Big(\log\frac1\varepsilon\Big)^{\tau+1}. \tag{2.13}$$

Next, let us choose the sets and the parameters involved in Proposition 2.1 as follows:

$$\begin{aligned}
&K=\frac{12}{s_0\,\varepsilon^b}\log\frac1\varepsilon, \qquad \alpha=\frac{\bar\gamma}{2\,K^\tau}, \qquad s=s_0\,,\\
&r=\frac{\bar\gamma}{2K^{\tau+1}\sup_{\mathcal{I}_{\sigma_0}}|h''|}, \qquad \rho:=C^*\frac{\varepsilon^b}{\log\frac1\varepsilon}<\rho^*\leqq\rho_0\,,\\
&U=\mathcal{I}_{\bar\gamma,\tau}, \qquad U_r=\bigcup_{I'\in\mathcal{I}_{\bar\gamma,\tau}} D_r(I'), \qquad V_\rho=D^4_\rho\subset D^4_{\rho_0}\,,
\end{aligned} \tag{2.14}$$

where C^* is a suitable large constant to be fixed later. Notice that, from these definitions, it follows (for ε small enough) that

$$\begin{aligned}
&\alpha=\mathrm{const}\,\varepsilon^{\frac12-b}\log\frac1\varepsilon, \qquad r=\mathrm{const}\,\frac{\alpha}{K}=\mathrm{const}\,\sqrt{\varepsilon}, \qquad d=\mathrm{const}\,r,\\
&\alpha d=\mathrm{const}\,\varepsilon^{1-b}\log\frac1\varepsilon, \qquad \frac{\alpha d}{cKs}=\mathrm{const}\,\varepsilon c_*^2
\end{aligned} \tag{2.15}$$

(clearly, in the last evaluation, "const" does not involve c_*).

Now, it is not difficult to check that, choosing c_* big enough and letting ε be small enough, assumptions (2.4) and (2.5) are met. In fact, observing that f in (1.6), (1.7) is such that

$$\|f\|_{r,\rho,s} \leqq \text{const}\,\varepsilon,$$

(2.5) follows from the last equality in (2.15), by choosing c_* large enough. As for (2.4), observe that for any point in $I \in U_r$ there is a point $I_0 \in \mathcal{I}_{\bar{\gamma},\tau}$ at a distance less than r from it. Hence, by (2.12), by the definitions of α and r and by Cauchy estimates[12], for any $I \in U_r$ and any $0 < |k| \leqq K$,

$$\begin{aligned} |h'(I)\cdot k| &\geqq |h'(I_0)\cdot k| - |h'(I_0) - h'(I)|\,|k| \\ &\geqq \frac{\bar{\gamma}}{K^\tau} - \sup_{\mathcal{I}_{\sigma_0}} |h''|\, r\, K \\ &= \frac{\bar{\gamma}}{2K^\tau}, \end{aligned}$$

which proves also (2.4). Thus, by Proposition 2.1, there exists a symplectic transformation

$$\Psi : (I', \varphi', p', q') \in \mathcal{D}_1 := U_{\frac{r}{2}} \times \mathbb{T}^2_{\frac{s}{6}} \times D^4_{\frac{\rho}{2}} \to (I, \varphi, p, q) \in U_r \times \mathbb{T}^2_{s_0} \times D^4_{\rho_0} \subset \mathcal{D}_0,$$

such that

$$\begin{aligned} |I' - I| &\leqq \text{const}\,\frac{\varepsilon}{\alpha} = \text{const}\,\frac{\varepsilon^{\frac{1}{2}+b}}{\log\frac{1}{\varepsilon}}, \\ |p' - p|,\ |q' - q| &\leqq \text{const}\,\frac{\varepsilon}{\alpha\rho} = \text{const}\,\sqrt{\varepsilon}, \end{aligned} \tag{2.16}$$

and which casts the Hamiltonian $\mathcal{H}$ into $\mathcal{H}' := \mathcal{H} \circ \Psi$ with

$$\mathcal{H}'(I', \varphi', p', q') := h(I') + g(I', p', q') + f_*(I', \varphi', p', q'), \tag{2.17}$$

where (since, by (1.7), f_0 coincides with $\varepsilon f_1(I, p, q)$)

$$\begin{aligned} \sup_{(I',p',q') \in U_{r/2} \times V_{\rho/2}} |g - \varepsilon f_1| &\leqq \text{const}\,\frac{\varepsilon^2}{\alpha r} = \text{const}\,\frac{\varepsilon^{1+b}}{\log(1/\varepsilon)}, \\ \|f_*\|_{r/2,\rho/2,s/6} &\leqq \text{const}\,\varepsilon e^{-Ks/6} \leqq \varepsilon^3. \end{aligned} \tag{2.18}$$

Notice that if $b > 0$ then $\|f_*\|$ is exponentially small with $1/\varepsilon$, while if $b = 0$ then the above estimates yield exactly ε^3. Thus, setting $g =: \varepsilon\bar{g}$, $f_* =: \varepsilon^3 \bar{f}$, we see that $\mathcal{H}'$ can be rewritten as

$$\begin{aligned} \mathcal{H}' &:= h(I') + \varepsilon\bar{g}(I', p', q') + \varepsilon^3 \bar{f}(I', \varphi', p', q'), \\ \bar{g} &= f_1(I', p', q') + \frac{\varepsilon^b}{\log(1/\varepsilon)} \bar{f}_1(I', p', q') \end{aligned} \tag{2.19}$$

with $\bar{f}$ and $\bar{f}_1$ real-analytic on $\mathcal{D}_1$ (compare (2.14) and recall the convention in (iv) of Remark 2.2).

12 As is well known, "Cauchy estimates" allow us to bound n-derivatives of analytic functions on a set A in terms of their sup-norm on larger domains $A' \supset A$ divided by $\text{dist}\,(\partial A, \partial A')^n$; compare, also, Lemma A.1 of Appendix A.

Step 2. We now look for elliptic equilibria of the Hamiltonian $\bar{g}$ in (2.19). Set

$$G(I', p', q') := \Big(\partial_{p'}\bar{g}(I', p', q'),\ \partial_{q'}\bar{g}(I', p', q')\Big).$$

Recalling (2.19) and the definition of f_1 in (1.7), we see that, for all $I' \in U_{r/2}$,

$$G(I', 0, 0)\Big|_{\varepsilon=0} = 0 \quad \text{and} \quad \det \partial_{(p',q')} G(I', 0, 0)\Big|_{\varepsilon=0} = 16(\bar{\Omega}_1\bar{\Omega}_2)^2 > 0.$$

Therefore, by the Implicit Function Theorem, we infer that, for any $I' \in U_{r/2}$ and for ε small enough, there exist a suitable constant C and real-analytic functions such that, choosing $C^* > C$,

$$I' \in U_{r/2} \to \Big(p'(I', \varepsilon),\ q'(I', \varepsilon)\Big) \in B_{C\varepsilon^b/\log\frac{1}{\varepsilon}} \subset B_\rho \subset B_{\rho^*},$$

and

$$\partial_{p'}\bar{g}\Big(I', p'(I', \varepsilon), q'(I', \varepsilon)\Big) = 0 = \partial_{q'}\bar{g}\Big(I', p'(I', \varepsilon), q'(I', \varepsilon)\Big). \tag{2.20}$$

For ε small enough, we can consider the following analytic symplectic transformation, which leaves fixed the I'-variable and is $O(\frac{\varepsilon^b}{\log\frac{1}{\varepsilon}})$-close to the identity[13],

$$\Phi' : (J', \psi', v', u') \in U_{r/2}\times\mathbb{T}^2_{s/7}\times D_{\rho/3} \mapsto (I', \varphi', p', q') \in U_{r/2}\times\mathbb{T}^2_{s/6}\times D^4_{\rho/2},$$

given by

$$\begin{aligned}
I' &= J',\\
\varphi' &= \psi' + p'(J', \varepsilon)\,\partial_{I'}q'(J', \varepsilon) + \partial_{I'}q'(J', \varepsilon)\,v' - \partial_{I'}p'(J', \varepsilon)\,u',\\
p' &= v' + p'(J', \varepsilon),\\
q' &= u' + q'(J', \varepsilon).
\end{aligned}$$

In view of (2.20), the new Hamiltonian $\hat{\mathcal{H}} := \mathcal{H}' \circ \Phi'$ has the form

$$\hat{\mathcal{H}}(J', \psi', v', u') = h(J') + \varepsilon\tilde{g}(J', v', u') + \varepsilon^3\tilde{f}(J', \psi', v', u'),$$

with $\tilde{f}$ and $\tilde{g}$ analytic in $U_{r/2} \times \mathbb{T}^2_{s/7} \times D_{\rho/3}$ and

$$\partial_{v',u'}\tilde{g}(J', 0, 0) = \partial_{p',q'}\bar{g}(I', p'(I', \varepsilon), q'(I', \varepsilon)) = 0 \qquad \forall\, I' \in U_{r/2}.$$

Also, the eigenvalues of the symplectic quadratic part of $\tilde{g}$ are given by $\pm i\tilde{\Omega}_j(J')$, for $j = 1, 2$, where

$$\tilde{\Omega}_j \in \mathbb{R} \qquad \text{and} \qquad |\tilde{\Omega}_j - \bar{\Omega}_j| \leqq \text{const}\frac{\varepsilon^b}{\log\frac{1}{\varepsilon}}. \tag{2.21}$$

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13 The transformation Φ' has generating function $J'\cdot\varphi' + \Big(v' + p'(J', \varepsilon)\Big)\cdot\Big(q' - q'(J', \varepsilon)\Big)$.

Thus, by a well-known result by Weierstrass on the symplectic diagonalization of quadratic Hamiltonians, we can find an analytic transformation $O\left(\frac{\varepsilon^b}{\log\frac{1}{\varepsilon}}\right)$-close to the identity

$$\widetilde{\Phi} : (\tilde{J}, \tilde{\psi}, \tilde{v}, \tilde{u}) \in U_{r/2} \times \mathbb{T}^2_{s/8} \times D^4_{\rho/4} \mapsto (J', \psi', v', u') \in U_{r/2} \times \mathbb{T}^2_{s/7} \times D^4_{\rho/3},$$

so that $J' = \tilde{J}$ and the quadratic part of $\tilde{g}$ becomes, simply, $\sum_{i=1}^2 \tilde{\Omega}_i(\tilde{J})\,(\tilde{u}_j^2 + \tilde{v}_j^2)$. Whence, the Hamiltonian $\hat{\mathcal{H}}$ takes the form $\tilde{\mathcal{H}} := \hat{\mathcal{H}} \circ \widetilde{\Phi}$, with

$$\begin{aligned}\tilde{\mathcal{H}}(\tilde{J}, \tilde{\psi}, \tilde{v}, \tilde{u}) = {} & h_0(\tilde{J}) + \varepsilon \sum_{i=1}^2 \tilde{\Omega}_i(\tilde{J})\,(\tilde{u}_i^2 + \tilde{v}_i^2) \\ & + \varepsilon \tilde{g}_0(\tilde{J}, \tilde{v}, \tilde{u}) + \varepsilon^3 \tilde{f}_0(\tilde{J}, \tilde{\psi}, \tilde{v}, \tilde{u}),\end{aligned} \tag{2.22}$$

where

$$h_0(\tilde{J}) := h(\tilde{J}) + \varepsilon \tilde{g}(\tilde{J}, 0, 0), \tag{2.23}$$

$\tilde{g}_0$, $\tilde{f}_0$, $\tilde{\Omega}_j$ are real-analytic for $(\tilde{J}, \tilde{\psi}, \tilde{v}, \tilde{u})$ in

$$\mathcal{D}_2 := U_{r/2} \times \mathbb{T}^2_{s/8} \times D^4_{\rho/4} \tag{2.24}$$

and

$$\sup_{\tilde{J} \in U_{r/2}} |\tilde{g}_0(\tilde{J}, \tilde{v}, \tilde{u})| \leqq \text{const}\,|(\tilde{v}, \tilde{u})|^3. \tag{2.25}$$

Finally, because of (2.21), the non-degeneracy condition (1.8) implies (for ε small enough)

$$\begin{aligned}\inf_{\tilde{J} \in U} \tilde{\Omega}_i &> \inf_{\tilde{J} \in U_{r/2}} |\tilde{\Omega}_i| > \text{const} > 0, \\ \inf_{\tilde{J} \in U} \left(\tilde{\Omega}_2 - \tilde{\Omega}_1\right) &> \inf_{\tilde{J} \in U_{r/2}} |\tilde{\Omega}_2 - \tilde{\Omega}_1| > \text{const} > 0.\end{aligned} \tag{2.26}$$

Step 3 (KAM). We are now in a position to apply a KAM result in order to find two-dimensional elliptic tori. The KAM Theorem we shall use is, basically, the version in [P96] of a result first proved by ELIASSON ([E88]) and KUKSIN ([K88]) about the conservation of lower dimensional invariant elliptic tori[14]. To state the KAM theorem, we need a bit of preparation.

Consider a Hamiltonian system with symplectic variables $(y, x, v, u) \in \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{R}^{2m}$ (endowed with the standard symplectic form $\sum dy_j \wedge dx_j + \sum dv_i \wedge du_i$) and consider a real-analytic Hamiltonian function of the form

$$H(y, x, v, u; \xi) = N(y, v, u; \xi) + P(y, x, v, u; \xi), \tag{2.27}$$

[14] Actually, [K88] and [P96] cover also infinite-dimensional (in (v, u)) cases.

where ξ is a parameter running over a compact set $\Pi \subset \mathbb{R}^n$ of positive Lebesgue measure, N is in normal form,

$$N(y, v, u; \xi) = e(\xi) + \omega(\xi) \cdot y + \sum_{j=1}^{m} \Omega_j(\xi)(u_j^2 + v_j^2), \tag{2.28}$$

and P is a small perturbation.

Note that the Hamiltonian $N(\cdot; \xi)$ affords, for any given value of the parameter $\xi \in \Pi$, the n-dimensional elliptic torus

$$\{y = 0\} \times \mathbb{T}^n \times \{v = u = 0\},$$

which is invariant for the Hamiltonian vector field X_N, the flow being, simply, $x \mapsto x + \omega(\xi)t$.

Assume that P is real-analytic on

$$D(\bar r, \bar s) := \{(y, x, v, u) \in \mathbb{C}^{2(n+m)} \ : \ |y| < \bar r^2\, , x \in \mathbb{T}^n_{\bar s}, \quad |v| + |u| < \bar r\}, \tag{2.29}$$

and that the dependence of ω, Ω_j and P is Lipschitz in $\xi \in \Pi$.

Let $\tau > n - 1$ and, as in [P96], let us introduce the following weighted norms:

$$\begin{aligned} |\omega|^{\mathrm{Lip}}_{\Pi} &:= \sup_{\xi \neq \xi' \in \Pi} \frac{|\omega(\xi) - \omega(\xi')|}{|\xi - \xi'|}, \\ \|X_H\|_{\bar r, D(\bar r, \bar s)} &:= \sup_{D(\bar r, \bar s) \times \Pi} \Big(|\partial_y H| + \frac{1}{\bar r^2} |\partial_x H| + \frac{1}{\bar r}(|\partial_v H| + |\partial_u H|) \Big), \\ \|X_H\|^{\mathrm{Lip}}_{\bar r, D(\bar r, \bar s)} &:= \sup_{D(\bar r, \bar s)} \Big(|\partial_y H|^{\mathrm{Lip}}_{\Pi} + \frac{1}{\bar r^2} |\partial_x H|^{\mathrm{Lip}}_{\Pi} + \frac{1}{\bar r}(|\partial_v H|^{\mathrm{Lip}}_{\Pi} + |\partial_u H|^{\mathrm{Lip}}_{\Pi}) \Big). \end{aligned}$$

Proposition 2.2 (KAM Theorem for elliptic tori)**.** *Assume that ω is a Lipschitz homeomorphism onto its image. Let L and M be such that*[15]

$$|\omega|^{\mathrm{Lip}}_{\Pi} + |\Omega|^{\mathrm{Lip}}_{\Pi} \leqq M, \qquad |\omega^{-1}|^{\mathrm{Lip}}_{\Pi} \leqq L.$$

Assume that there exists $\gamma_0 > 0$ such that

$$\min_{\xi \in \Pi,\, i \neq j} \{|\Omega_i(\xi)|\, ,\ |\Omega_i(\xi) - \Omega_j(\xi)|\} \geqq \gamma_0, \tag{2.30}$$

$$|\omega(\xi) \cdot k + \Omega(\xi) \cdot \ell| \geqq \gamma_0 \quad \forall\, 0 < |k| \leqq K_0\, ,\ |\ell| \leqq 2, \tag{2.31}$$

where K_0 is a suitable constant (depending, also, on n and τ), which is assumed to be bigger than $16LM$. Let $\gamma \in (0, \gamma_0/2]$ and define

$$|||X_H|||_{\bar r, \bar s, \gamma} := \|X_H\|_{\bar r, D(\bar r, \bar s)} + \frac{\gamma}{M}\, \|X_H\|^{\mathrm{Lip}}_{\bar r, D(\bar r, \bar s)}.$$

15 Define $\Omega := (\Omega_1, \dots, \Omega_m)$

Then, there exist suitable constants $c = c(\bar{s}, \tau, n)$ and $a = a(\tau, n) > 1$ such that, if

$$\frac{c}{\gamma}\,(LM)^a\,\|X_P\|_{\bar{r},\bar{s},\gamma} \leqq 1, \tag{2.32}$$

the following holds. There exists a Cantor set of parameters $\Pi_* \subset \Pi$ and a Lipschitz continuous family of toris embedding

$$\Phi : (\theta,\xi) \in \mathbb{T}^n \times \Pi_* \longrightarrow \Big(y_\Phi(\theta;\xi),\ x_\Phi(\theta;\xi),\ v_\Phi(\theta;\xi), u_\Phi(\theta;\xi) \Big),$$
$$\Phi(\mathbb{T}^n \times \Pi_*) \subset \{|y| < \bar{r}^2\} \times \mathbb{T}^n \times \{|v| + |u| < \bar{r}\},$$

a Lipschitz homeomorphism ω_* on Π_* and a Lipschitz continuous function Ω_* on Π_*, such that, for any $\xi \in \Pi_*$, $\Phi(\mathbb{T}^n, \xi)$ is a real-analytic (elliptic) H-invariant n-dimensional torus, on which the flow X_H is analytically conjugated to the linear flow $\theta \mapsto \theta + \omega_* t$. Furthermore, $\Phi(\cdot, \xi)$ is real-analytic on $\mathbb{T}^n_{\bar{s}/2}$ and the following bounds hold:

$$\frac{1}{\bar{r}^2}|y_\Phi| + |x_\Phi - \mathrm{Id}| + \frac{1}{\bar{r}}(|v_\Phi| + |u_\Phi|)$$
$$+\frac{\gamma}{M}\Big(\frac{1}{\bar{r}^2}|y_\Phi|^{\mathrm{Lip}}_{\Pi_*} + |x_\Phi|^{\mathrm{Lip}}_{\Pi_*} + \frac{1}{\bar{r}}(|v_\Phi|^{\mathrm{Lip}}_{\Pi_*} + |u_\Phi|^{\mathrm{Lip}}_{\Pi_*})\Big) \leqq c\,\frac{\|X_P\|_{\bar{r},\bar{s},\gamma}}{\gamma}, \tag{2.33}$$

$$|\omega_* - \omega| + \frac{\gamma}{M}|\omega_* - \omega|^{\mathrm{Lip}}_{\Pi_*} \leqq c\,\|X_P\|_{\bar{r},\bar{s},\gamma}, \tag{2.34}$$

$$|\Omega_* - \Omega| + \frac{\gamma}{M}|\Omega_* - \Omega|^{\mathrm{Lip}}_{\Pi_*} \leqq c\,\|X_P\|_{\bar{r},\bar{s},\gamma}, \tag{2.35}$$

$$|\omega_* \cdot k + \Omega_* \cdot \ell| \geqq \frac{\gamma}{1+|k|^\tau} \quad \forall (k,\ell) \neq (0,0),\ |\ell| \leqq 2, \tag{2.36}$$

$$\mathrm{meas}_n(\Pi \setminus \Pi_*) \leqq c\,\frac{\gamma}{M}\,(LM\ \mathrm{diam}\,\Pi)^{n-1}. \tag{2.37}$$

Remark 2.3. This Theorem is a summary (in the finite-dimensional case) of Theorems A and B and Corollary C of [P96], to which we refer for the proof[16]. Notice that (2.30) and (2.31) here play the role of the "Non-degeneracy Assumption A" of [P96], while Assumptions B and C of [P96] are trivially satisfied in the finite dimensional case. Assumptions (2.30) and (2.31) imply the measure estimate (2.37), as briefly shown in Appendix B.

To apply Proposition 2.2 to our case, we let

$$n = 2,\ \omega_0 = h_0',\ \Omega_i = \varepsilon\tilde{\Omega}_i\,,\ L = \text{const}\,,\ M = \text{const}\,,\ K_0 := 16LM,$$
$$J = \xi \in \Pi = U = \mathcal{I}_{\bar{\gamma},\tau}, \quad e(\xi) = h_0(\xi), \quad \bar{r} = \varepsilon^{3/4}, \quad \bar{s} = \frac{s}{8},$$

[16] For comparison purposes, we have kept the notation as close as possible to the notation in [P96]; notice, however, that (in order to avoid confusion with other parameters introduced in our paper) r and s in [P96] are denoted here by $\bar{r}$ and $\bar{s}$.

$$\begin{aligned} N(y, v, u; \xi) &= e(\xi) + \omega_0(\xi) \cdot y + \sum_{i=1}^{2} \Omega_i(\xi)(u_i^2 + v_i^2), \ (x, v, u) = (\tilde{\psi}, \tilde{v}, \tilde{u}), \\ H(y, x, v, u; \xi) &= \tilde{\mathcal{H}}(J + y, \tilde{\psi}, \tilde{v}, \tilde{u}). \end{aligned} \tag{2.38}$$

Recall that $r \sim \sqrt{\varepsilon}$, which is much larger than $\bar{r} = \varepsilon^{3/4}$, so that

$$\Pi_{\bar{r}^2} := \bigcup_{\xi \in \Pi} D_{\bar{r}^2}(\xi) \subset U_{r/2},$$

and also $\bar{r} \ll \rho/4$ so that the Hamiltonian H in (2.38) is real-analytic for $|y| < \bar{r}^2$, $|u| + |v| < \bar{r}$ and $|\operatorname{Im} x| < \bar{s}$, for any $\xi \in \Pi$.

Next, we observe that the perturbation P may be written as

$$P = \sum_{1 \leqq k \leqq 4} P_k ,$$

with

$$\begin{aligned} P_1 &= h_0(\xi + y) - h_0(\xi) - \omega_0(\xi) \cdot y, \\ P_2 &= \sum_{i=1}^{2} \Big(\Omega_i(\xi + y) - \Omega_i(\xi) \Big) (u_i^2 + v_i^2) , \\ P_3 &= \varepsilon \tilde{g}_0(\xi + y, v, u), \\ P_4 &= \varepsilon^3 \tilde{f}_0(\xi + y, x, v, u). \end{aligned}$$

By (2.26) and (2.38), we see that (2.30) holds true, provided $\gamma_0 = \text{const}\, \varepsilon$.

To check (2.31), take $0 < |k| \leqq K_0$ and $|\ell| \leqq 2$. Then, observing that $|\Omega_j| \leqq \text{const}\, \varepsilon$, recalling the definitions of $U = \Pi = \mathcal{I}_{\bar{\gamma}, \tau}$ and $\bar{\gamma}$ in (2.14), (2.12), (2.11) and (2.38), for any $0 < |k| \leqq K_0$ and $|\ell| \leqq 2$, we have

$$\begin{aligned} |\omega_0(\xi) \cdot k + \Omega(\xi) \cdot \ell| &\geqq |\omega_0(\xi) \cdot k| - \text{const}\, \varepsilon \\ &\geqq \frac{\bar{\gamma}}{K_0^\tau} - \text{const}\, \varepsilon > \varepsilon, \end{aligned}$$

proving (2.31). Finally, recalling the definition of the weighted norms introduced before Proposition 2.2, (2.25) and (2.38), we infer that

$$\begin{aligned} \| X_P \|_{\bar{r}, \bar{s}, \gamma} &\leqq \text{const} \left(\bar{r}^2 + \varepsilon \bar{r} + \frac{\varepsilon^3}{\bar{r}^2} \right) = \text{const} \left(\varepsilon^{3/2} + \varepsilon^{7/4} + \varepsilon^{3/2} \right) \\ &\leqq \text{const}\, \varepsilon^{3/2}, \end{aligned}$$

so that, letting (say)

$$\gamma := \gamma_0 / 2 = \text{const}\, \varepsilon, \tag{2.39}$$

we find

$$\frac{\| X_P \|_{\bar{r}, \bar{s}, \gamma}}{\gamma} \leqq \text{const} \sqrt{\varepsilon}. \tag{2.40}$$

Therefore, the assumptions of Proposition 2.2 are fulfilled and the existence of the elliptic quasi-periodic orbits follows from Proposition 2.2: the parametrization ϕ in Theorem 1.2 of the elliptic tori is given by

$$\begin{aligned}&\Big(I_\phi(\theta;\, J),\ \varphi_\phi(\theta;\, J),\ p_\phi(\theta;\, J),\ q_\phi(\theta;\, J) \Big)\\ &\quad := \Psi \circ \Phi' \circ \widetilde{\Phi}\Big(J + y_\Phi(\theta;\, J),\ x_\Phi(\theta;\, J),\ v_\Phi(\theta;\, J)\, , u_\Phi(\theta;\, J) \Big), \end{aligned} \tag{2.41}$$

where the parameter $J = \xi$ varies in

$$J \in \mathcal{I}_* := \Pi_*. \tag{2.42}$$

The estimates (1.11), (1.12) (or (1.15), (1.16)) and (1.13) follow easily from (2.16), the fact that Φ' and $\widetilde{\Phi}$ leave fixed the variables J' and $\tilde{J}$ and are $\varepsilon^b/\log\frac{1}{\varepsilon}$-close to the identity in the other variables, (2.33)÷(2.36), (2.16), (2.21) and (2.23).

Finally, we turn to the measure estimates (1.10). It follows from (2.37) and (2.39) that the 2-dimensional elliptic tori are described by a set of parameters Π_*, with

$$\text{meas}_2(\Pi \setminus \Pi_*) \leqq \text{const}\, \varepsilon. \tag{2.43}$$

Thus, from (2.13), (2.42) and (2.43), there follows

$$\begin{aligned}\text{meas}_2(\mathcal{I} \setminus \mathcal{I}_*) &\leqq \text{meas}_2(\mathcal{I} \setminus \mathcal{I}_{\bar\gamma,\tau}) + \text{meas}_2(\mathcal{I}_{\bar\gamma,\tau} \setminus \mathcal{I}_*)\\ &:= \text{meas}_2(\mathcal{I} \setminus \mathcal{I}_{\bar\gamma,\tau}) + \text{meas}_2(\Pi \setminus \Pi_*)\\ &\leqq \text{const}\left(\varepsilon^{\frac{1}{2}-b(\tau+1)} \left(\log\frac{1}{\varepsilon}\right)^{\tau+1} + \varepsilon \right)\\ &\leqq \text{const}\, \varepsilon^{\frac{1}{2}-b(\tau+1)} \left(\log\frac{1}{\varepsilon}\right)^{\tau+1},\end{aligned}$$

completing the proof of Theorem 1.2. □

Appendix A. Averaging Theory

In this appendix, we prove a general result (Proposition A.1 below) in averaging theory, which will immediately imply Proposition 2.1. The techniques used here are similar to techniques used in [P93].

Let us first fix some notation. As above, (I, φ) and (p, q) denote sets of standard symplectic conjugate variables. We will use for $I \in \mathbb{R}^n$ the usual Euclidean norm $|I| := |I|_2 := (\sum_{i=1}^n |I_i|^2)^{1/2}$, but for $p, q \in \mathbb{R}^m$ or $\varphi \in \mathbb{T}^n$ we will use the norm $|p| := |p|_\infty := \max_{1\leqq i\leqq m} |p_i|$, $|q| := |q|_\infty := \max_{1\leqq i\leqq m} |q_i|$, $|\varphi| := |\varphi|_\infty := \max_{1\leqq i\leqq n} |\varphi_i|$, (mod 2π). If such variables are considered in complex domains, we shall use the corresponding conventions. If $d > 0$ and A is a subset of $\mathbb{R}^l$ with $l = n$ or $l = m$ we define

$$A_d := \{z \in \mathbb{C}^l, \quad \text{such that} \quad \exists\, x \in A \quad \text{with} \quad |z - x|_j < d\},$$

where $j = 2$ or $j = \infty$ (according to whether the set is considered in the space of the actions I's or in the space of the other symplectic variables p, q or φ). Let

$$D \subset \mathbb{R}^n, \quad E, F \subset \mathbb{R}^m, \quad U := D \times E \times F, \quad W := U \times \mathbb{T}^n,$$

and let $r, r_p, r_q, s > 0$ and $v := (r, r_p, r_q)$. For a function

$$f(u,\varphi) = \sum_{k\in\mathbb{Z}^n} f_k(u) e^{ik\cdot\varphi}, \qquad u := (I, p, q),$$

real-analytic for $(u,\varphi) \in W_{v,s} := U_v \times \mathbb{T}^n_s$, with $U_v := D_r \times E_{r_p} \times F_{r_q}$, we shall use the norm[17]

$$\|f\|_{v,s} := \sum_{k\in\mathbb{Z}^n} \sup_{u\in U_v} |f_k(u)| e^{|k|s}.$$

Finally, we let Λ be a sub-lattice of $\mathbb{Z}^n$ and, if $f = \sum f_k e^{ik\cdot\varphi}$, we set

$$T_K f := \sum_{|k|\leqq K} f_k e^{ik\cdot\varphi}, \qquad P_\Lambda f := \sum_{k\in\Lambda} f_k e^{ik\cdot\varphi}.$$

Proposition A.1. *Let $H := h(I) + f(u,\varphi)$ be a real-analytic Hamiltonian on $W_{v,s}$. Denoting $\omega := h'$ and $c_m := e(1+em)/2$, suppose that*

$$|\omega(I)\cdot k| \geqq \alpha > 0, \qquad \forall\,|k| \leqq K, \quad k \notin \Lambda, \ \ \forall\, I \in D_r, \tag{A.1}$$

$Ks \geqq 6$ and

$$\|f\|_{v,s} =: \varepsilon < \frac{\alpha d}{2^7 c_m K s} \qquad \text{where} \qquad d := \min\{rs, r_p r_q\}. \tag{A.2}$$

Then, there exists a real-analytic symplectic transformation

$$\Psi : (u', f') \in W_{v_*,s_*} \longrightarrow (u,\varphi) \in W_{v,s}$$

with $v_* := v/2$, $s_* := s/6$, such that

$$H_* := H \circ \Psi = h + g + f_*, \tag{A.3}$$

with g in normal form:

$$g = \sum_{k\in\Lambda} g_k(u') e^{ik\cdot\varphi'}. \tag{A.4}$$

Moreover, when the projection of is denoted $\Psi(I', p', q', \varphi')$ onto the I-variables by $I(I', p', q', \varphi')$, etc.,

$$\|g - P_\Lambda T_K f\|_{v_*,s_*} \leqq \frac{12}{11}\,\frac{2^7 c_m \varepsilon}{\alpha d}\,\varepsilon \leqq \frac{1}{4}\varepsilon, \tag{A.5}$$

$$\|f_*\|_{v_*,s_*} \leqq \frac{2^9 c_m \varepsilon}{\alpha d} e^{-Ks/6}\varepsilon \leqq e^{-Ks/6}\varepsilon, \tag{A.6}$$

$$s\,|I(u',f') - I'|,\ r_q\,|p(u',f') - p'|,\ r_p\,|q(u',f') - q'|,\ r\,|\varphi(u',f') - f'| \leqq 9\varepsilon/\alpha. \tag{A.7}$$

[17] If $k \in \mathbb{Z}^n$ $|k| := \sum_{i=1}^n |k_i|$.

The proof of this result rests upon a series of technical elementary lemmata, which we now proceed to state:

Lemma A.1. *Let*[18] $0 < \nu < v$ *and* $0 < \sigma < s$, *where* $\nu := (\rho, \rho_p, \rho_q)$. *Then*

$$\sum_{1 \leqq i \leqq n} \| \frac{\partial f}{\partial \varphi_i} \|_{v,s-\sigma} \leqq \frac{1}{e\sigma} \| f \|_{v,s}, \qquad \max_i \| \frac{\partial f}{\partial I_i} \|_{r-\rho,r_p,r_q,s} \leqq \frac{1}{\rho} \| f \|_{r,r_p,r_q,s}$$

$$\max_l \| \frac{\partial f}{\partial p_l} \|_{r,r_p-\rho_p,r_q,s} \leqq \frac{1}{\rho_p} \| f \|_{v,s}, \qquad \max_l \| \frac{\partial f}{\partial q_l} \|_{r,r_p,r_q-\rho_q,s} \leqq \frac{1}{\rho_q} \| f \|_{v,s}\,.$$

This Lemma is a precise version of classical Cauchy estimates. We omit the well-known proof; just notice that the estimates relative to φ-derivatives are a consequence of the choice of the (Fourier) norm.

An immediate corollary of Lemma A.1 is the following Lemma on estimates for Poisson's brackets[19].

Lemma A.2. *Let* $0 < v - \nu < \tilde{v}$, *where* $\tilde{v} := (r_0, \tilde{r}_p, \tilde{r}_q)$. *Then,*

$$\|\{f, g\}\|_{v-\nu,s-\sigma} \leqq \left[\frac{1}{e(r_0 - r + \rho)\sigma} + \frac{1}{e(\tilde{s} - s + \sigma)\rho} + \frac{m}{(\tilde{r}_p - r_p + \rho_p)\rho_q} + \frac{m}{(\tilde{r}_q - r_q + \rho_q)\rho_p} \right] \| f \|_{\tilde{v},\tilde{s}} \ \| g \|_{v,s}.$$

If ϕ is a Hamiltonian function, we denote by X^t_ϕ the Hamiltonian flow of ϕ at the time t. If f and ϕ are analytic, expanding in Taylor series in time the function $f \circ X^t_\phi$, we get

$$f \circ X^t_\phi = \sum_j \frac{t^j}{j!} L^j_\phi f,$$

where $L_\phi f := \{f, \phi\}$, $L^0_\phi :=$ Id and

$$L^j_\phi := \overbrace{L_\phi \circ \cdots \circ L_\phi}^{j \text{ times}}.$$

Lemma A.3. *Let* $0 < \nu < v \leqq \tilde{v} - \nu$, $0 < \sigma < s \leqq \tilde{s} - \sigma$ *and*

$$\|\phi\|_{\tilde{v},\tilde{s}} < G := \frac{2}{e} \left(\frac{\rho \rho_p \rho_q \sigma}{\rho_p \rho_q + e m \rho \sigma} \right). \tag{A.8}$$

Then

$$\| f \circ X^1_\phi \|_{v-\nu,s-\sigma} \leqq \left(1 - \frac{\|\phi\|_{\tilde{v},\tilde{s}}}{G} \right)^{-1} \| f \|_{v,s}.$$

[18] If $a := (a_1, \ldots, a_j)$, $b := (b_1, \ldots, b_j) \in \mathbb{R}^j$, then $a < b$ means $a_i < b_i$ for all $1 \leqq i \leqq j$.

[19] The Poisson brackets are $\{f, g\} := \sum_j \partial_{\varphi_j} f \ \partial_{I_j} g - \partial_{I_j} f \ \partial_{\varphi_j} g + \sum_i \partial_{q_i} f \ \partial_{p_i} g - \partial_{p_i} f \ \partial_{q_i} g$.

Proof. Fix $h \geqq 1$ and let $\bar{\nu} := \nu/h$, $\bar{\sigma} := \sigma/h \| \cdot \|_i := \| \cdot \|_{v-i\bar{\nu}, s-i\bar{\sigma}}$, for all i with $1 \leqq i \leqq h$. We will use Lemma A.2 with $\nu = \bar{\nu}$ and $\sigma = \bar{\sigma}$. Then, $v - i\bar{\nu} = [v - (i-1)\bar{\nu}]$ and $s - i\bar{\sigma} = [s - (i-1)\bar{\sigma}]$. Hence,

$$\begin{aligned}
\|L^i_\phi f\|_i &= \|\{L^{i-1}_\phi, \phi\}\|_i \\
&\leqq \Bigg[\frac{1}{e(r_0 - r + i\bar{\rho})\bar{\sigma}} + \frac{1}{e(\tilde{s} - s + i\bar{\sigma})\bar{\rho}} \\
&\quad + \frac{m}{(\tilde{r}_p - r_p + i\bar{\gamma})\bar{\kappa}} + \frac{m}{(\tilde{r}_q - r_q + i\bar{\kappa})\bar{\gamma}} \Bigg] \|\phi\|_{\tilde{v},\tilde{s}} \; \|L^{i-1}_\phi\|_{i-1} \\
&\leqq \left[\frac{2}{e\bar{\rho}\bar{\sigma}} + \frac{2m}{\bar{\gamma}\bar{\kappa}} \right] \frac{1}{h+i} \|\phi\|_{\tilde{v},\tilde{s}} \|L^{i-1}_\phi\|_{i-1} \,.
\end{aligned}$$

Iterating h times the previous estimate we obtain

$$\begin{aligned}
\|L^h_\phi\|_h = \|L^h_\phi\|_{v-\nu,s-\sigma} &\leqq \left[\frac{2}{e\bar{\rho}\bar{\sigma}} + \frac{2m}{\bar{\gamma}\bar{\kappa}} \right]^h \frac{h!}{(2h)!} \|\phi\|^h_{\tilde{v},\tilde{s}} \|f\|_{v,s} \\
&= \left[\frac{2}{e\rho\sigma} + \frac{2m}{\rho_p\rho_q} \right]^h \frac{h^{2h} h!}{(2h)!} \|\phi\|^h_{\tilde{v},\tilde{s}} \|f\|_{v,s} \\
&\leqq \left[\frac{e}{2\rho\sigma} + \frac{e^2 m}{2\rho_p\rho_q} \right]^h h! \|\phi\|^h_{\tilde{v},\tilde{s}} \|f\|_{v,s} = \frac{1}{G} h! \|\phi\|^h_{\tilde{v},\tilde{s}} \|f\|_{v,s} \,.
\end{aligned}$$

Finally, summing over h, we get

$$\begin{aligned}
\|f \circ X^1_\phi\|_{v-\nu,s-\sigma} &= \| \sum_h \frac{1}{h!} L^h_\phi f \|_{v-\nu,s-\sigma} \leqq \sum_h \frac{1}{h!} \|L^h_\phi f\|_{v-\nu,s-\sigma} \\
&\leqq \|f\|_{v,s} \sum_h \frac{1}{G^h} \|\phi\|^h_{\tilde{v},\tilde{s}} = \left(1 - \frac{\|\phi\|_{\tilde{v},\tilde{s}}}{G} \right)^{-1} \|f\|_{v,s}. \quad \square
\end{aligned}$$

The next lemma is an immediate consequence of Hamilton equations (and the trivial proof is omitted).

Lemma A.4. *Let* $0 < v - \nu < v$, $0 < s - \sigma < s$ *and suppose that*

$$\max_{1 \leqq i \leqq n} \| \frac{\partial \phi}{\partial I_i} \|_{v,s} \leqq \bar{\sigma} \leqq \sigma, \qquad \sum_{i=1}^n \| \frac{\partial \phi}{\partial \varphi_i} \|_{v,s} \leqq \bar{\rho} \leqq \rho,$$

$$\max_{1 \leqq l \leqq m} \| \frac{\partial \phi}{\partial p_l} \|_{v,s} \leqq \bar{\rho_q} \leqq \rho_q, \quad \max_{1 \leqq l \leqq m} \| \frac{\partial \phi}{\partial q_l} \|_{v,s} \leqq \bar{\rho_p} \leqq \rho_p \,.$$

Let $(u_0, \varphi_0) := (u(0), \varphi(0)) \in W_{v-\nu, s-\sigma}$. *Then* $X^t_\phi(u(0), \varphi(0)) = (u(t), \varphi(t)) \in W_{v,s}$ *for all* t *with* $0 \leqq t \leqq 1$. *More precisely,*

$$|I(t) - I(0)| \leqq \sum_{i=1}^n |I_i(t) - I_i(0)| \leqq \bar{\rho}, \qquad \max_i |\varphi_i(t) - \varphi_i(0)| \leqq \bar{\sigma},$$

$$\max_{1 \leqq l \leqq m} |p_l(t) - p_l(0)| \leqq \bar{\rho_p}, \quad \max_{1 \leqq l \leqq m} |q_l(t) - q_l(0)| \leqq \bar{\rho_q} \,.$$

Lemma A.5. *Let* $H(u,\varphi) := h(I) + g(u,\varphi) + f(u,\varphi)$ *be real-analytic on* $W_{v,s}$ *with* $g = \sum_{k\in\Lambda} g_k(u)e^{ik\cdot\varphi}$. *Let* $\nu < v/2$ *and* $\sigma < s/2$. *Suppose that*

$$|\omega(I)\cdot k| \geqq \alpha > 0 \qquad \forall\, |k| \leqq K, \quad k \notin \Lambda\ , \ \ \forall\, I \in D_r\ , \tag{A.9}$$

and

$$\|f\|_{v,s} < \alpha\delta/c_m \qquad \text{where} \qquad \delta := \min\{\rho\sigma, \rho_p\rho_q\}\,. \tag{A.10}$$

Then, there exists a real-analytic symplectic transformation

$$\Phi : (\tilde{u}, \tilde{\varphi}) \in W_{v-2\nu, s-2\sigma} \longrightarrow (u,\varphi) \in W_{v,s}$$

such that

$$H \circ \Phi = h + g_+ + f_+, \qquad g_+ - g = P_\Lambda T_K f. \tag{A.11}$$

Here, $\Phi := X^1_\phi$ *for a suitable* $\phi : W_{v,s} \to \mathbb{C}$ *such that*

$$\|f_+\|_{v-2\nu,s-2\sigma} \leqq \left(1 - \frac{c_m}{\alpha\delta}\|f\|_{v,s}\right)^{-1}\left[\frac{c_m}{\alpha\delta}\|f\|^2_{v,s} + \|\{g,\phi\}\|_{v-\nu,s-\sigma} + e^{-K\sigma}\|f\|_{v,s}\right]. \tag{A.12}$$

Furthermore,

$$\sigma\ |I(\tilde{u},\tilde{\varphi}) - \tilde{I}|,\ \ \rho_q\ |p(\tilde{u},\tilde{\varphi}) - \tilde{p}|,\ \ \rho_p\ |q(\tilde{u},\tilde{\varphi}) - \tilde{q}|,\ \ \rho\ |\varphi(\tilde{u},\tilde{\varphi}) - \tilde{\varphi}| \leqq \|f\|_{v,s}/\alpha. \tag{A.13}$$

Proof. We have $H \circ \Phi = h \circ \Phi + (g + T_K f) \circ \Phi + (f - T_K f) \circ \Phi$ and we can write

$$h \circ \Phi = h + \{h,\phi\} + \int_0^1 (1-t)\{\{h,\phi\},\phi\} \circ X^t_\phi\, dt,$$

$$(g + T_K f) \circ \Phi = (g + T_K f) + \int_0^1 \{(g + T_K f),\phi\} \circ X^t_\phi\, dt.$$

Since we want $g_+ = [\{h,\phi\} + g + T_K f]$ and $g_+ - g = P_\Lambda T_K f$ we have to solve $\{h,\phi\} + T_K f = P_\Lambda T_K f$, that is

$$\{h,\phi\} = \sum_{|k|\leqq K, k\notin\Lambda} f_k(u)e^{ik\cdot\varphi}.$$

The solution of such an equation is explicitly given by

$$\phi(u,\varphi) = \sum_{|k|\leqq K, k\notin\Lambda} \frac{f_k(u)}{ik\cdot\omega(I)}e^{ik\cdot\varphi} \qquad \text{with} \qquad \|\phi\|_{v,s} \leqq \frac{1}{\alpha}\|f\|_{v,s}. \tag{A.14}$$

From (A.14), Lemma A.1 and Lemma A.4, we obtain (A.13) and the inclusion

$$\Phi : W_{v-2\nu,s-2\sigma} \longrightarrow W_{v,s}\ .$$

If $f_t := (1-t)(g_+ - g) + tT_K f$, we have

$$f_+ = \int_0^1 \{(g+f_t), \phi\} \circ X_\phi^t \, dt + (f - T_K f) \circ \Phi.$$

We can estimate G in (A.8) with $G \geqq \delta/c_m$. Then, for all F and for all t with $0 \leqq t \leqq 1$, substituting $\tilde{v} \to v$, $v \to v - \nu$, $\nu \to \nu$, we have (by Lemma A.3)

$$\|F \circ X_\phi^t\|_{v-2\nu, s-2\sigma} \leqq C\|F\|_{v-\nu, s-\sigma},$$

with $C := (1 - c_m\|\phi\|_{v,s}/\delta)^{-1}$. Then, choosing $F := \{f_t, \phi\}$, and using Lemma A.2 we have

$$\begin{aligned}\left\|\int_0^1 \{f_t, \phi\} \circ X_\phi^t \, dt\right\|_{v-2\nu, s-2\sigma} &\leqq \int_0^1 \|\{f_t, \phi\} \circ X_\phi^t\|_{v-2\nu, s-2\sigma} \, dt \\ &\leqq C \int_0^1 \|\{f_t, \phi\}\|_{v-\nu, s-\sigma} \, dt \\ &\leqq C \int_0^1 \left(\frac{2}{e\rho\sigma} + \frac{2m}{\rho_p \rho_q}\right) \|f_t\|_{v,s} \|\phi\|_{v,s} \, dt \\ &\leqq C \frac{c_m}{\delta} \int_0^1 \|f_t\|_{v,s} \|\phi\|_{v,s} \, dt \\ &\leqq C \frac{c_m}{\alpha\delta} \|f\|_{v,s}^2, \end{aligned} \tag{A.15}$$

where in the last inequality we have used the simple fact that $\|f_t\|_{v,s} \leqq \|f\|_{v,s}$. Similarly we obtain

$$\left\|\int_0^1 \{g, \phi\} \circ X_\phi^t \, dt\right\|_{v-2\nu, s-2\sigma} \leqq C\|\{g, \phi\}\|_{v-\nu, s-\sigma} \tag{A.16}$$

and

$$\begin{aligned}&\|(f - T_K f) \circ X_\phi^1\|_{v-2\nu, s-2\sigma} \\ &\quad \leqq C\|f - T_K f\|_{v-\nu, s-\sigma} \leqq C\, e^{-K\sigma} \|f\|_{v,s}. \end{aligned} \tag{A.17}$$

Collecting (A.15), (A.16), (A.17) we have (A.12). □

We are now ready for the

Proof of Proposition A.1. Let $\varepsilon_0 := \varepsilon$, $\nu_0 := v/8$, $\sigma_0 := s/6$, $\delta_0 := \min\{\rho_0\sigma_0, \rho_{p0}\rho_{q0}\}$. Suppose that[20]

$$e^{-Ks/6} \leqq 32 c_m \varepsilon/\alpha d \tag{A.18}$$

[20] The case in which in (A.18) holds ">" is even simpler. In fact it is sufficient to apply Lemma A.5 with $\nu := \frac{v}{4}$, $\sigma := \frac{s}{3}$, $v - 2\nu = v_*$, $s - 2\sigma = s_*$, $g = 0$, having $\Psi := \Phi$, $g_+ = P_\Lambda T_K f$, $f_* := f_+$. It is easy to verify that (A.6) and (A.7) follow from (A.2), (A.12) and from $Ks \geqq 6$.

Substituting $\nu \to \nu_0$, $\sigma \to \sigma_0$, $\delta \to \delta_0$, we can use Lemma A.5 since $\delta_0 \geqq d/64$ and (A.2) implies (A.10). Defining $W_1 := W_{v_1,s_1}$ with $v_1 := v - 2\nu_0 = 3/4v$ and $s_1 := s - 2\sigma_0 = 2/3s$, we obtain an analytic symplectic transformation $\Phi_0 : W_1 \to W_{v,s}$ with $H \circ \Phi_0 = h + g_0 + f_1$, where $g_0 = P_\Lambda T_K f$. Moreover from (A.2), (A.18), (A.12) we obtain

$$\begin{aligned} \|f_1\|_1 := \|f_1\|_{v_1,s_1} =: \varepsilon_1 &\leqq \left(1 - \frac{64c_m\varepsilon}{\alpha d}\right)^{-1}\left[\frac{64c_m\varepsilon}{\alpha d} + e^{-Ks}\right]\varepsilon \\ &\leqq \frac{9}{11}\,\frac{2^7c_m\varepsilon}{\alpha d}\varepsilon \\ &\leqq \frac{1}{6}\varepsilon. \end{aligned} \tag{A.19}$$

Letting $(u, \varphi) = \Phi_0(u^{(1)}, \varphi^{(1)})$, from (A.13) we have, for all $(u, \varphi) \in W_1$,

$$s\,|I^{(1)} - I|,\ r_p\,|p^{(1)} - p|,\ r_q\,|q^{(1)} - q|,\ r\,|\varphi^{(1)} - \varphi|,\ \leqq 8\varepsilon/\alpha. \tag{A.20}$$

Let $L \in \mathbb{N}$ be such that

$$L \leqq \frac{Ks}{12\ln 2} < L + 1, \qquad \Big(\implies Ks \geqq 8L\Big). \tag{A.21}$$

Let $\nu := v/8L$, $\sigma := s/4L$, and define for $1 \leqq i \leqq L$, $v_i := v_{i-1} - 2\nu = v_1 - 2(i-1)\nu$, $s_i := s_{i-1} - 2\sigma = s_1 - 2(i-1)\sigma$, $\Phi_i : W_{i+1} \to W_i := W_{v_i,s_i}$ with $H_i := H \circ \Phi_{i-1} =: h + g_{i-1} + f_i$ on W_i and $\varepsilon_i := \|f_i\|_{W_i} =: \|f_i\|_i$. Observing that $W_{i+1} \subset W_i$, we can iterate Lemma A.5 with $v \to v_i$, $s \to s_i$, after verifying by induction (see below) that

$$\varepsilon_i \leqq \varepsilon_1 \qquad \forall\, 1 \leqq i \leqq L. \tag{A.22}$$

In fact, in order to apply Lemma A.5 we have to verify (A.10), which is implied, for any $1 \leqq i \leqq L$, by (A.22) and the estimate[21]

$$\varepsilon_i \leqq \varepsilon_1 \leqq 2^{-6}(\alpha d/64c_m L^2), \tag{A.23}$$

which follows directly from (A.19), (A.2) and (A.21). We observe that, for all $1 \leqq i \leqq L$,

$$\|g_i - g_{i-1}\|_i = \|P_\Lambda T_K f_i\|_i \leqq \|f_i\|_i = \varepsilon_i\,. \tag{A.24}$$

We now prove (A.22). In order to estimate f_{i+1}, we evaluate $g_{i-1} = \sum_{j=0}^{i-1} \tilde g_j$ where $\tilde g_0 := g_0$ and $\tilde g_j := (g_j - g_{j-1})$ are defined on W_j. Since from (A.14) we have $\|\phi_i\|_i \leqq \|f_i\|_i/\alpha = \varepsilon_i/\alpha$, and

$$\|\{g_{i-1}, \phi_i\}\|_{v_i-\nu,s_i-\sigma}$$

[21] We observe that $\delta \geqq d/64$.

$$
\begin{aligned}
&\leqq \sum_{j=0}^{i-1} \|\{\tilde{g}_j, \phi_i\}\|_{v_i-\nu, s_i-\sigma} \\
&\leqq \sum_{j=0}^{i-1} \bigg[\frac{1}{e(r_j - r_i + \rho)\sigma} + \frac{1}{e(s_j - s_i + \sigma)\rho} \\
&\quad + \frac{m}{(r_{pj} - r_{pi} + \rho_p)\rho_q} + \frac{m}{(r_{qj} - r_{qi} + \rho_q)\rho_p} \bigg] \|\tilde{g}_j\|_j \|\phi_i\|_i \\
&\leqq \frac{2}{3}\bigg[\frac{1}{e\rho\sigma} + \frac{m}{\rho_p\rho_q}\bigg] \sum_{j=1}^{i-1} \|P_\Lambda T_K f_j\|_j \frac{1}{\alpha}\varepsilon_i + \frac{\varepsilon}{\alpha L}\varepsilon_i \bigg[\frac{1}{e\rho\sigma} + \frac{m}{\rho_q\rho_p}\bigg] \\
&\leqq \frac{64 c_m L^2 \varepsilon_i}{\alpha d}\bigg[\frac{2}{3}\sum_{j=1}^{i-1}\varepsilon_j + \frac{\varepsilon}{L}\bigg] \\
&\leqq \frac{64 c_m L^2 \varepsilon_i}{\alpha d}\bigg[\varepsilon_1 + \frac{\varepsilon}{L}\bigg], \qquad \text{(A.25)}
\end{aligned}
$$

where we have used Lemma A.2 and the fact that $\tilde{g}_j = P_\Lambda T_K f_j$, considered separately the case $j = 0$ from $j > 0$, and observed that, if $j > 0$, then

$$
\begin{aligned}
v - v_j + \nu &\geqq v - v_1 + \nu \geqq \nu + v/4 = (2L+1)\nu, \\
s - s_j + \sigma &\geqq s - s_1 + \sigma \geqq \sigma + s/3 = (4L/3 + 1)\sigma .
\end{aligned}
$$

Using (A.21), (A.19), (A.23) and $\varepsilon_i \leqq \varepsilon_1$ we obtain from Lemma A.5

$$
\begin{aligned}
\varepsilon_{i+1} &= \|f_{i+1}\|_{i+1} \\
&\leqq \bigg(1 - \frac{64 c_m L^2 \varepsilon_i}{\alpha d}\bigg)^{-1} \bigg[\frac{64 c_m L^2 \varepsilon_i}{\alpha d}(\varepsilon_i + \varepsilon_1 + \varepsilon/L) + e^{-Ks}\bigg]\varepsilon_i \\
&\leqq \frac{\varepsilon_i}{4}. \qquad \text{(A.26)}
\end{aligned}
$$

Moreover, from (A.19), there follows

$$
\begin{aligned}
\|f_*\|_{v_*, s_*} = \varepsilon_{L+1} &\leqq 4^{-L}\varepsilon_1 \leqq \frac{2^9 c_m \varepsilon}{\alpha d} 4^{-(L+1)}\varepsilon \\
&\leqq \frac{2^9 c_m \varepsilon}{\alpha d} 4^{-\frac{Ks}{12\ln 2}}\varepsilon = \frac{2^9 c_m \varepsilon}{\alpha d} e^{-Ks/6}\varepsilon,
\end{aligned}
$$

and

$$
\|g - g_0\|_{v_*, s_*} \leqq \sum_{i=1}^{L} \|g_i - g_{i-1}\|_{v_i, s_i} \leqq \sum_{i=1}^{L} \varepsilon_i \leqq 4 \sum_{i=1}^{L} \bigg(\frac{1}{4}\bigg)^i \varepsilon_1 = \frac{4}{3}\varepsilon_1 ,
$$

from which (A.5) and (A.6) follow.

Let, now, $\Psi := \phi_0 \circ \cdots \circ \phi_L$ and $(u^{(i)}, \varphi^{(i)}) \in W_i$. Using (A.21), (A.20) and Lemma A.5 we have

$$\begin{aligned} |I - I'| &= |I^{(L+1)} - I^{(0)}| \\ &\leqq |I^{(1)} - I^{(0)}| + \sum_{i=1}^{L} |I^{(i+1)} - I^{(i)}| \leqq \frac{8\varepsilon}{\alpha s} + \frac{4L}{\alpha s}\sum_{i=1}^{L} \varepsilon_i \leqq \frac{9}{\alpha s}\varepsilon. \end{aligned}$$

The estimates for $|p - p'|$, $|q - q'|$ and $|\varphi - \varphi'|$ are analogous. □

Appendix B. KAM measure estimates

Here, we show how assumptions (2.30) and (2.31) imply the measure estimate (2.37).

Indeed, the set Π_* is obtained as $\Pi_* = \bigcap_{\nu\in\mathbb{N}} \Pi_\nu$, where $\Pi_0 := \Pi$ and, recursively,

$$\Pi_{\nu+1} := \Pi_\nu \setminus \Big(\bigcup_{\substack{(k,\ell)\in\mathbb{Z}^{n+m}\setminus\{0\} \\ |\ell|\leqq 2,\ |k|>K_\nu}} \mathcal{R}_{k\ell}^{\nu+1} \Big),$$

with $K_\nu := K_0\, 2^\nu$, and $\mathcal{R}_{k\ell}^{\nu+1}$ is a suitable "resonant set" to be discarded at the ν^{th} step of the KAM iteration (compare with the Iterative Lemma in Section 4 of [P96]). The sets $\mathcal{R}_{k\ell}^{\nu+1}$ satisfy the measure estimate

$$\mathrm{meas}_n(\mathcal{R}_{kl}^{\nu+1}) \leqq \frac{\lambda}{|k|^{\tau+1}}, \qquad \lambda := \text{ const } (LM)^n \frac{\gamma}{M}(\text{ diam } \Pi)^{n-1}, \tag{B.1}$$

for any $|k| \geqq K_0$, $\nu \geqq 0$, $|\ell| \leqq 2$ (see Lemma 5 in [P96]). Therefore,

$$\begin{aligned} \mathrm{meas}_n(\Pi_{\nu+1}) &\geqq \ \mathrm{meas}_n(\Pi_\nu) - \ \text{const}\,\lambda \sum_{|k|>K_\nu} |k|^{-(\tau+1)} \\ &\geqq \ \mathrm{meas}_n(\Pi_\nu) - \ \text{const}\,\lambda \frac{1}{K_\nu^{\tau-n+1}}. \end{aligned}$$

Iterating this relation and using the definition of K_ν, we get

$$\mathrm{meas}_n(\Pi_{\nu+1}) \geqq \ \mathrm{meas}_n(\Pi) - \ \text{const}\ \frac{\gamma}{M}(LM\ \ \text{diam } \Pi)^{n-1},$$

which proves (2.37).

Appendix C. The Delaunay-Poincaré theory of the planetary three-body problem

In this appendix, following [Ch88] and [L88], we discuss, in a self-contained way, the Hamiltonian formulation of the planetary (non-planar) three-body problem, discussing, in particular, the classical Delaunay-Poincaré Theorem 1.1 and its proof. The appendix is divided in two parts dealing, respectively, with the canonical treatment of the two-body problem and with the (partial) extension of such theory to the three-body problem.

C.1. Canonical variables for the two-body problem

C.1.1. Integration of the Kepler problem. Consider two bodies $\mathrm{P}_0,\ \mathrm{P}_1$ of masses $\mathrm{m}_0,\ \mathrm{m}_1$ and spatial position $u^{(0)},\ u^{(1)} \in \mathbb{R}^3$, interacting through gravity, with gravitational constant 1; the (inertial) frame $\mathbb{R}^3$ is chosen so that its origin coincides with the center of mass. Let

$$\mathrm{M} := \mathrm{m}_0 + \mathrm{m}_1, \quad \mathrm{m} := \frac{\mathrm{m}_0\mathrm{m}_1}{\mathrm{M}}, \quad x := u^{(1)} - u^{(0)}, \quad X := \mathrm{m}\dot{x}. \tag{C.1}$$

Then, the motion of the two bodies is governed by the Hamiltonian

$$\mathcal{K}(X, x) = \frac{1}{2\mathrm{m}}|X|^2 - \frac{\mathrm{mM}}{|x|}, \tag{C.2}$$

with $(X, x) \in \mathbb{R}^3 \times \mathbb{R}^3$ conjugate variables[22], i.e., the equations of motion are $\dot{x} = \partial_X \mathcal{K}$, $\dot{X} = -\partial_x \mathcal{K}$.

As is well known, such a system is integrable and for $\mathcal{K} < 0$ the (x-projection of the) orbits are ellipses. More precisely, we have

Proposition C.1. *Fix* $\Lambda_- > 0 > \mathcal{K}_0$ *and let* $\Lambda_+ := \left(\frac{\mathrm{m}^3\mathrm{M}^2}{-2\mathcal{K}_0}\right)^{\frac{1}{2}}$. *Then, there exist* $\hat{\rho} > 0$ *and a real-analytic symplectic transformation*[23]

$$\begin{aligned}\Psi_{\mathrm{DP}} : \Big((\Lambda, \eta, \mathrm{p}), (\lambda, \xi, \mathrm{q})\Big) &\in \Big([\Lambda_-, \Lambda_+] \times B^2_{\hat{\rho}}\Big) \times \Big(\mathbb{T} \times B^2_{\hat{\rho}}\Big) \\ &\mapsto (X, x) \in \{|x| \geqq \frac{\hat{\rho}^2}{\mathrm{m}^2\mathrm{M}}\},\end{aligned}$$

casting (C.2) *into the integrable Hamiltonian* $(-\mathrm{m}^3\mathrm{M}^2)/(2\Lambda^2)$.

This classical proposition is due to POINCARÉ ([Poi1905], Chapter III) and the variables $(\Lambda, \eta, \mathrm{p}, \lambda, \xi, \mathrm{q})$ are, usually, called "Poincaré variables". The proof of Proposition C.1 is particularly interesting from the physical point of view and rests upon the introduction of three different (famous) changes of variables, which we, now, proceed to describe briefly (for more details, see [Ch88]).

Recall that ℓ, θ and g denote, respectively, the mean anomaly, the longitude of the (ascending) node and the argument of the perihelion (see Fig. 2).

[22] Often, in this appendix, upper/lower case letters indicate couples of standard symplectic conjugate momentum-coordinate variables.

[23] Recall that B^n_r, D^n_r, $B^n_r(x_0)$ and $D^n_r(x_0)$ denote, respectively, the real n-ball of radius r centered at the origin, the complex n-ball of radius r centered at the origin, the real n-ball of radius r centered at x_0 and the complex n-ball of radius r centered at x_0.

Step 1. The system is set in *"symplectic" spherical polar variables*: namely, we consider the symplectic map $\Psi_{\rm spc} : \Big((R, \Omega, \Phi), (r, \omega, \varphi)\Big) \mapsto (X, x)$ (where $r > 0$, $0 < \omega < \pi$ and $0 \leqq \varphi < 2\pi$) given by[24]

$$\begin{cases} x_1 = r \sin\omega \ \cos\varphi \\ x_2 = r \sin\omega \ \sin\varphi \ , \\ x_3 = r \cos\omega \end{cases} \quad X = \begin{pmatrix} \sin\omega \ \cos\varphi & \frac{\cos\omega\cos\varphi}{r} & -\frac{\sin\varphi}{r\sin\omega} \\ \sin\omega \ \sin\varphi & \frac{\cos\omega\sin\varphi}{r} & \frac{\cos\varphi}{r\sin\omega} \\ \cos\omega & -\frac{\sin\omega}{r} & 0 \end{pmatrix} \begin{pmatrix} R \\ \Omega \\ \Phi \end{pmatrix} \quad \text{(C.3)}$$

and consider the new Hamiltonian $\mathcal{K}_{\rm spc} := \mathcal{K} \circ \Psi_{\rm spc}$.

Step 2. Using the Hamilton-Jacobi, method we can find a symplectic map $\Psi_{\rm D} : \Big((L, G, \Theta), (\ell, g, \theta)\Big) \mapsto \Big((R, \Omega, \Phi), (r, \omega, \varphi)\Big)$ that integrates the system: $\Psi_{\rm D}$ is the symplectic transformation with generating function

$$S(L, G, \Theta, r, \omega, \varphi) = \int \sqrt{-\frac{{\rm m}^4{\rm M}^2}{L^2} + \frac{2{\rm m}^2{\rm M}}{r} - \frac{G^2}{r^2}}\, dr + \int \sqrt{G^2 - \frac{\Theta^2}{\sin^2\omega}}\, d\omega + \Theta\varphi. \quad \text{(C.4)}$$

The variables $\Big((L, G, \Theta), (\ell, g, \theta)\Big)$ are known as "Delaunay variables". In such variables, the new Hamiltonian becomes

$$\mathcal{K}_{\rm D} := \mathcal{K}_{\rm spc} \circ \Psi_{\rm D} = -\frac{{\rm m}^3{\rm M}^2}{2L^2}.$$

Let C be the angular momentum of the planet, let a be the major semi-axis and let i be its inclination, i.e., the angle between a fixed reference plane and the Keplerian ellipse plane; compare Fig. 2 (later, such a reference plane will be taken to be the "total angular momentum plane"). By construction, the following relations hold:

$$G = |C|, \quad \Theta = G \cos i \quad \text{and} \quad L = {\rm m}\sqrt{{\rm M}a}.$$

Step 3. To remove singularities, following Poincaré, we proceed as follows. First, we introduce *Poincaré action-angle variables* by means of the linear symplectic transformation

$$\Psi_{\rm P_{aa}} : \Big((\Lambda, H, Z), (\lambda, h, \zeta)\Big) \mapsto \Big((L, G, \Theta), (\ell, g, \theta)\Big)$$

given by

$$\Psi_{\rm P_{aa}} : \quad \begin{cases} \Lambda = L, \quad H = L - G, \quad Z = G - \Theta, \\ \lambda = \ell + g + \theta, \quad h = -g - \theta, \quad \zeta = -\theta. \end{cases} \quad \text{(C.5)}$$

[24] The matrix in (C.3) is the transpose of the inverse of the Jacobian $\frac{\partial x}{\partial(r,\omega,\varphi)}$.

Then, we let[25] $\Psi_{\mathrm{P}} : \Big((\Lambda, \eta, \mathrm{p}), (\lambda, \xi, \mathrm{q})\Big) \mapsto \Big((\Lambda, H, Z), (\lambda, h, \zeta)\Big)$ be the symplectic map defined by the relations

$$\begin{array}{lll} H = \frac{\eta^2+\xi^2}{2}, & \sqrt{2H}\,\cos h = \eta, & \sqrt{2H}\,\sin h = \xi, \\ Z = \frac{p^2+q^2}{2}, & \sqrt{2Z}\,\cos\zeta = \mathrm{p}, & \sqrt{2Z}\,\sin\zeta = \mathrm{q}. \end{array} \tag{C.6}$$

As Poincaré showed (see below), *the symplectic map*

$$\Psi_{\mathrm{DP}} : \Big((\Lambda, \eta, \mathrm{p}), (\lambda, \xi, \mathrm{q})\Big) \mapsto (X, x)$$

with

$$\Psi_{\mathrm{DP}} := \Psi_{\mathrm{spc}} \circ \Psi_{\mathrm{D}} \circ \Psi_{\mathrm{P_{aa}}} \circ \Psi_{\mathrm{P}} \tag{C.7}$$

is real-analytic in a neighborhood of $\Big([\Lambda_-, \Lambda_+] \times \{(0,0)\}\Big) \times \Big(\mathbb{T} \times \{(0,0)\}\Big)$, (and the two-body Hamiltonian, in Poincaré variables, is $\mathcal{K} \circ \Psi = -\frac{\mathrm{m}^3\mathrm{M}^2}{2\Lambda^2}$).

Remark C.1. (i) If we define $(X, x) = \Phi_{\mathrm{DP}}\Big((\Lambda, \eta, \mathrm{p}), (\lambda, \xi, \mathrm{q})\Big)$, then[26]

$$X = \frac{\mathrm{m}^4\mathrm{M}^2}{\Lambda^3}\frac{\partial x}{\partial \lambda}.$$

(ii) Let us collect, here, some important relations among the quantities introduced above. Let, as usual, e denote the eccentricity of the Keplerian ellipse and let a and i denote the major semi-axis and the inclination. Then, by construction, we see that

$$\begin{aligned} \Lambda &= \mathrm{m}\sqrt{\mathrm{M}a}, \\ \sqrt{\xi^2+\eta^2} &= \sqrt{\Lambda}\, e\,(1 + O(e^2)), \\ \sqrt{\mathrm{p}^2+\mathrm{q}^2} &= \sqrt{\Lambda}\, i\,(1 + O(e^2) + O(i^2)). \end{aligned} \tag{C.8}$$

A more explicit link between H, the eccentricity and the major semi-axis is given by

$$H = \Lambda\,(1 - \sqrt{1-e^2}) = \Lambda\,\frac{e^2}{2}(1 + O(e^2)), \tag{C.9}$$

$$e(H, \Lambda) = \sqrt{\frac{H}{\Lambda}\Big(2 - \frac{H}{\Lambda}\Big)}. \tag{C.10}$$

[25] Do not confuse the variables (p, q) here with the variables (p, q) used in the text (and, in particular, in Theorem 1.2, where the variables (p, q) correspond to the variables (η', ξ') introduced below).

[26] By Hamilton equations it can be seen that $\dot\lambda = \partial_\Lambda\Big(-\frac{\mathrm{m}^3\mathrm{M}^2}{2\Lambda^2}\Big) = \frac{\mathrm{m}^3\mathrm{M}^2}{\Lambda^3}$, and $\dot\Lambda = \dot\xi = \dot\eta = \dot{\mathrm{p}} = \dot{\mathrm{q}} = 0$. Thus, by the chain rule, $X = \mathrm{m}\dot x = \mathrm{m}(\partial_\lambda x)\,\dot\lambda = \frac{\mathrm{m}^4\mathrm{M}^2}{\Lambda^3}\frac{\partial x}{\partial\lambda}$.

Also, if C is the angular momentum of the system, we infer that

$$|C| = \Lambda\sqrt{1-e^2} = \Lambda(1+O(e^2)), \tag{C.11}$$

$$\begin{aligned} Z &= |C|\,(1-\cos i) = \Lambda\sqrt{1-e^2}\,(1-\cos i) \\ &= |C|\,\frac{i^2}{2}(1+O(i^2)). \end{aligned} \tag{C.12}$$

Poincaré's argument ([Poi1905]) for proving the analyticity of Ψ_{DP} goes as follows:

Let us define $\alpha := g + v$, $\psi := \varphi - \theta$, $\mathcal{X} := r\cos(v-\ell)$, $\mathcal{Y} := r\sin(v-\ell)$. Let us also denote by "trig" sin or cos and by $\mathcal{Z}$ the vector $(\mathcal{X}, \mathcal{Y})$. By (C.3), we get an analytic expression of x in terms of

$$\Big(\mathcal{Z},\ \cos^2\frac{i}{2}\ \mathrm{trig}\,\lambda,\ \sin^2\frac{i}{2}\ \mathrm{trig}\,(\lambda+2\zeta),\ \sin i\ \mathrm{trig}\,(\lambda+\zeta)\Big). \tag{C.13}$$

By geometric considerations, $r\cos v = a(\cos u - e)$ and $r\sin v = a\sqrt{1-e^2}\sin u$. Thus, we get an analytic expression of $\mathcal{Z}$ of the form

$$\mathcal{Z} = \mathcal{Z}(a,\ e^2,\ \mathrm{trig}\,(u-\ell),\ e^2\,\mathrm{trig}\,(u+\ell),\ e\,\mathrm{trig}\,\ell). \tag{C.14}$$

By geometric considerations, $u - \ell = e\sin u$, from which $u-\ell$ results to be an analytic function of $e\,\mathrm{trig}\,\ell$. Hence, by standard trigonometric computations, $e^2\,\mathrm{trig}\,(u+\ell)$ is proved to be an analytic function of $e\,\mathrm{trig}\,\ell$.

Thus, from (C.14), we get an analytic representation $\mathcal{Z} = \mathcal{Z}(a,\ e^2,\ e\,\mathrm{trig}\,\ell)$. By $e^2 = (e\sin\ell)^2 + (e\cos\ell)^2$ and the first of (C.8), it follows that $\mathcal{Z} = \mathcal{Z}(\Lambda,\ e\,\mathrm{trig}\,\ell)$.

From (C.5), (C.6) and (C.9), $\mathcal{Z} = \mathcal{Z}(\Lambda,\ \lambda,\ \eta,\ \xi)$.

Hence, we come back to the expression in (C.13). To complete the proof of the analytic dependence of x with respect to the Poincaré variables, we need to find an expression of $\sin\frac{i}{2}\ \mathrm{trig}\,\zeta$ and $\mathrm{trig}\,\frac{i}{2}$. From (C.6),

$$\sin\frac{i}{2}\cos\zeta = \frac{p}{2\sqrt{\Lambda - H}},\qquad \sin\frac{i}{2}\sin\zeta = \frac{q}{2\sqrt{\Lambda - H}},\qquad H = \frac{\xi^2+\eta^2}{2}.$$

Using again trigonometric relations, we see that $\sin i\ \mathrm{trig}\,\zeta$ is an analytic function of $(\Lambda, \lambda, \eta, \xi, p, q)$. Moreover, by (C.6), (C.11) and (C.12), we have

$$1-\cos i = \frac{p^2+q^2}{2\Lambda - (\eta^2+\xi^2)},$$

and an analytic expression of $\mathrm{trig}\,\frac{i}{2}$ in terms of $(\Lambda, \eta, \xi, p, q)$ easily follows. Hence, we obtain an analytic expression of x in terms of $(\Lambda, \lambda, \eta, \xi, p, q)$.

From point (i) of Remark C.1, we finally show that X is analytic in $(\Lambda, \lambda, \eta, \xi, p, q)$.

C.I.2. "Osculating" Poincaré variables. Following Poincaré, we introduce a new set of action-angle variables (linearly related to the Delaunay variables), $\Big((\Lambda^*, H^*, Z^*), (\lambda^*, h^*, \zeta^*)\Big) \in (\mathbb{R}^3 \times \mathbb{T}^3)$, by letting

$$\Psi_{\mathrm{P}^*_{\mathrm{aa}}} : \quad \begin{cases} \Lambda^* = \Lambda, & H^* = H, \quad Z^* = Z - \Lambda + H, \\ \lambda^* = \lambda + \zeta, & h^* = h - \zeta, \quad \zeta^* = \zeta. \end{cases} \tag{C.15}$$

The physical interpretation of these variables follows from the above construction. In particular,

$$\lambda^* = \ell + g, \quad h^* = -g, \quad \zeta^* = -\theta. \tag{C.16}$$

Observe that the angles (λ^*, h^*) are defined in the orbital plane: for this reason, we shall call the set of variables $\Big((\Lambda^*, H^*, Z^*), (\lambda^*, h^*, \zeta^*)\Big)$ "osculating action-angle variables"[27]. Notice that, by (C.11),

$$Z^* = Z - \Lambda\sqrt{1 - e^2} = -\Lambda + O(e^2) + O(i^2)$$

is negative for small eccentricity and inclination.

In a way analogous to (C.6), we can introduce symplectic "osculating Poincaré variables",

$$\Psi_{\mathrm{P}^*} : \Big((\Lambda^*, \eta^*, \mathrm{p}^*), (\lambda^*, \xi^*, \mathrm{q}^*)\Big) \mapsto \Big((\Lambda^*, H^*, Z^*), (\lambda^*, h^*, \zeta^*)\Big),$$

through the relations

$$H^* = \frac{\eta^{*2} + \xi^{*2}}{2}, \quad \sqrt{2H^*}\cos h^* = \eta^*, \quad \sqrt{2H^*}\sin h^* = \xi^*, \tag{C.17}$$

$$-Z^* = \frac{\mathrm{p}^{*2} + \mathrm{q}^{*2}}{2}, \quad \sqrt{-2Z^*}\cos\zeta^* = \mathrm{p}^*, \quad \sqrt{-2Z^*}\sin\zeta^* = \mathrm{q}^*. \tag{C.18}$$

Notice that (C.17) and (C.18) are singular for $H^* = 0$ and $Z^* = 0$, respectively[28]. However, the Hamiltonian formalism in osculating Poincaré variables is analytic for $\mathrm{p}^2 + \mathrm{q}^2 > 0$ (which, for small eccentricities and in view of (C.8), means *for non-zero inclinations*):

[27] Obviously, in the two-body problem the "osculating plane" coincides with the orbital plane, but we shall use these symplectic variables also for the spatial three-body problem, where the two two-body systems considered (star+planet P_j) will not move on fixed planes and, in such a case, it makes sense to speak about "osculating planes"; we also anticipate that, choosing as reference plane the "total angular momentum plane", ζ^* will turn out to be a cyclic variable and, hence, Z^*_j will be integrals of the motions (Poincaré integrals).

[28] The singularity $Z^* = 0$ would not actually be a problem since in our case $Z^* \sim -\Lambda$ which is bounded away from 0.

Proposition C.2. *There exists* $0 < \rho_0 < \sqrt{2\min\{\Lambda_-, \Lambda_+ - \Lambda_-\}}$, *such that the symplectic transformation* $\Psi_{\rm DP^*} : \Big((\Lambda^*, \eta^*, {\rm p}^*), (\lambda^*, \xi^*, {\rm q}^*)\Big) \mapsto (X, x) \in \{|x| \geqq \frac{\hat\rho^2}{{\rm m}^2{\rm M}}\}$ *given by*

$$\Psi_{\rm DP^*} = \Psi_{\rm spc} \circ \Psi_{\rm D} \circ \Psi_{{\rm P}^*_{\rm aa}} \circ \Psi_{\rm P^*}$$

is real-analytic for $\Big((\Lambda^*, \lambda^*), (\eta^*, \xi^*), ({\rm p}^*, {\rm q}^*)\Big) \in \mathcal{C}$ *where* $\mathcal{C} = \mathcal{C}(\Lambda_-, \Lambda_+, \rho_0)$ *is the "conical" region defined as*

$$\mathcal{C} := \bigcup_{\substack{\Lambda_- \leqq \tilde\Lambda \leqq \Lambda_+ \\ 0 \leqq \tilde\rho < \rho_0}} \Big(\{\tilde\Lambda\} \times \mathbb{T}\Big) \times \Big\{|(\eta^*, \xi^*)| = \tilde\rho\Big\}$$
$$\times \Big\{({\rm p}^*, {\rm q}^*) : \ \tilde\Lambda - \frac{\tilde\rho^2 + \rho_0^2}{2} < -Z^* < \tilde\Lambda - \frac{\tilde\rho^2}{2}\Big\}. \qquad \text{(C.19)}$$

Furthermore, there exist positive numbers $\beta, \bar\beta, \rho^*$ *satisfying*

$$2\beta + \bar\beta < \frac{\rho_0^2}{2}, \qquad \rho^{*2} < \rho_0^2 - 2(\bar\beta + 2\beta), \qquad \text{(C.20)}$$

such that, for any $\Lambda_0^* \in [\Lambda_- + \beta, \Lambda_+ - \beta]$,

$$([\Lambda_0^* - \beta, \Lambda_0^* + \beta] \times \mathbb{T}) \times B^2_{\rho^*}$$
$$\times \Big\{({\rm p}^*, {\rm q}^*) : \ \Lambda_0^* - \beta - \frac{\rho^{*2}}{2} - \bar\beta < -Z^* < \Lambda_0^* - \beta - \frac{\rho^{*2}}{2}\Big\} \subset \mathcal{C}. \quad \text{(C.21)}$$

For a suitable $s > 0$ *and for any* $({\rm p}^*, {\rm q}^*)$ *such that* $\Lambda_0^* - \beta - \frac{\rho^{*2}}{2} - \bar\beta < -Z^* < \Lambda_0^* - \beta - \frac{\rho^{*2}}{2}$, *the map* $((\Lambda^*, \lambda^*), (\eta^*, \xi^*)) \to \Psi_{\rm DP^*}\Big((\Lambda^*, \eta^*, {\rm p}^*), (\lambda^*, \xi^*, {\rm q}^*)\Big)$ *is analytic on the complex domain*

$$((\Lambda^*, \lambda^*), (\eta^*, \xi^*)) \in (\mathcal{E}_{\rho^{*2}} \times \mathbb{T}_s) \times D^2_{\rho^*}, \qquad \text{(C.22)}$$

where

$$\mathcal{E} := \Big[\Lambda_0^* - \beta + \rho^{*2}, \Lambda_0^* + \beta - \rho^{*2}\Big], \qquad \mathcal{E}_{\rho^{*2}} := \bigcup_{\tilde\Lambda \in \mathcal{E}} D^1_{\rho^{*2}}(\tilde\Lambda). \qquad \text{(C.23)}$$

In the osculating variables the two-body Hamiltonian is given by $\mathcal{K} \circ \Psi_{\rm DP^*} = -\frac{{\rm m}^3{\rm M}^2}{2(\Lambda^*)^2}$.

Proof. By (C.17), (C.15) and (C.6),

$$\eta^* = \frac{\eta{\rm p} + \xi{\rm q}}{\sqrt{{\rm p}^2 + {\rm q}^2}} \qquad \text{and} \qquad \xi^* = \frac{\xi{\rm p} - \eta{\rm q}}{\sqrt{{\rm p}^2 + {\rm q}^2}}.$$

Also, (C.6) is regular for $({\rm p}, {\rm q}) \neq 0$, so that, for non-zero inclination, we can express $Z = Z({\rm p}, {\rm q}) = ({\rm p}^2 + {\rm q}^2)/2$ and $\zeta = \zeta({\rm p}, {\rm q})$ as analytic functions (more precisely,

$\zeta(\mathrm{p},\mathrm{q})$ is analytic on the pinched complex torus $\{\mathbb{C}/(2\pi\mathbb{Z})\}\setminus 2\pi\mathbb{Z})$. Thus, in light of (C.15) and (C.18),

$$\begin{aligned}\lambda^* &= \lambda + \zeta(\mathrm{p},\mathrm{q}),\\ \mathrm{p}^* &= \sqrt{2\Lambda - \mathrm{p}^2 - \mathrm{q}^2 - \eta^2 - \xi^2}\,\sin\zeta(\mathrm{p},\mathrm{q}),\\ \mathrm{q}^* &= \sqrt{2\Lambda - \mathrm{p}^2 - \mathrm{q}^2 - \eta^2 - \xi^2}\,\cos\zeta(\mathrm{p},\mathrm{q}).\end{aligned}$$

Therefore, the diffeomorphism

$$\begin{aligned}\Psi := \ &\Psi_{\mathrm{P}_*}^{-1}\circ\Psi_{\mathrm{P}_{\mathrm{aa}}^*}^{-1}\circ\Psi_{\mathrm{P}_{\mathrm{aa}}}\circ\Psi_{\mathrm{P}} : \Big((\Lambda,\eta,\mathrm{p}),(\lambda,\xi,\mathrm{q})\Big)\\ &\mapsto \Big((\Lambda^*,\eta^*,\mathrm{p}^*),(\lambda^*,\xi^*,\mathrm{q}^*)\Big),\end{aligned}$$

which maps the Poincaré variables into the osculating Poincaré variables is analytic for $\Lambda_- < \Lambda < \Lambda_+$, $\lambda \in \mathbb{T}$, $(\eta,\xi)\in B^2_{\rho_0}$, $(\mathrm{p},\mathrm{q})\in B^2_{\rho_0}\setminus\{0\}$, for a suitably small ρ_0.

By construction, $\Psi_{\mathrm{DP}^*} = \Psi_{\mathrm{DP}}\circ\Psi^{-1}$, where Ψ_{DP} has been defined in (C.7), and, as shown in Fig. 3,

$$\begin{aligned}&\Psi\Big(\{\Lambda=\tilde\Lambda,\ \lambda\in\mathbb{T},\ |(\eta,\xi)|=\tilde\rho,\ |(\mathrm{p},\mathrm{q})|=\tilde r\}\Big)\\ &\quad=\left\{\Lambda^*=\tilde\Lambda,\ \lambda^*\in\mathbb{T},\ |(\eta^*,\xi^*)|=\tilde\rho,\ |(\mathrm{p}^*,\mathrm{q}^*)|=\sqrt{2\tilde\Lambda-\tilde\rho^2-\tilde r^2}\right\}.\end{aligned}$$

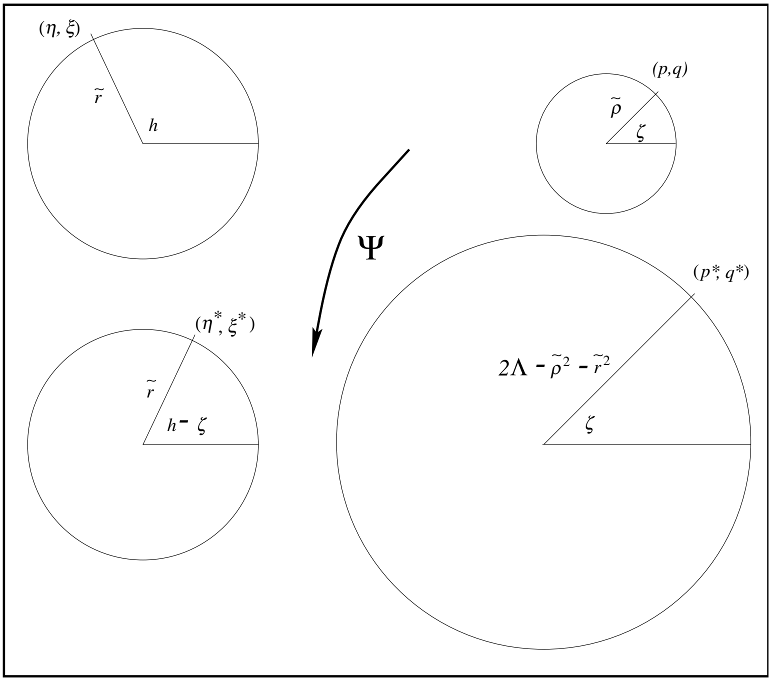

Fig. 3.

Now, observe that, in view of the above relation,

$$\begin{aligned}
&\Psi\Big(\Big\{\Lambda \in [\Lambda_-, \Lambda_+],\ \lambda \in \mathbb{T},\ (\eta, \xi) \in B^2_{\rho_0},\ (\mathrm{p}, \mathrm{q}) \in B^2_{\rho_0} \setminus \{0\}\Big\}\Big) \\
&= \bigcup_{\substack{\tilde\Lambda \in [\Lambda_-, \Lambda_+] \\ \tilde\rho \in [0, \rho_0) \\ \tilde r \in (0, \rho_0)}} \Big\{\Lambda^* = \tilde\Lambda,\ \lambda^* \in \mathbb{T},\ |(\eta^*, \xi^*)| \\
&\qquad\qquad = \tilde\rho,\ |(\mathrm{p}^*, \mathrm{q}^*)| = \sqrt{2\tilde\Lambda - \tilde\rho^2 - \tilde r^2}\Big\} \\
&= \bigcup_{\substack{\tilde\Lambda \in [\Lambda_-, \Lambda_+] \\ \tilde\rho \in [0, \rho_0)}} \Big\{\Lambda^* = \tilde\Lambda, \lambda^* \in \mathbb{T}\,,\ |(\eta^*, \xi^*)| = \tilde\rho, \\
&\qquad\qquad \sqrt{2\tilde\Lambda - \tilde\rho^2 - \rho_0^2} < |(\mathrm{p}^*, \mathrm{q}^*)| < \sqrt{2\tilde\Lambda - \tilde\rho^2}\Big\},
\end{aligned}$$

which proves the real-analyticity of Ψ_{DP^*} on $\mathcal{C}$. Any choice of β, $\bar\beta$, ρ^* satisfying (C.20) implies easily the inclusion (C.21) and (C.22). □

Remark C.2. (i) The Poincaré integral Z^* is (in the small inclination-eccentricity regime considered here), for fixed values of H^* and Λ^*, in one-to-one correspondence with the squared inclination i^2. In fact, by (C.9)÷(C.12) and (C.15),

$$\begin{aligned}
\frac{i^2}{2}\Big(1 + O(i^2)\Big) &= 1 - \cos i \\
&= \frac{Z^* + \Lambda^*\sqrt{1 - e(H^*, \Lambda^*)^2}}{\Lambda^*\sqrt{1 - e(H^*, \Lambda^*)^2}} \\
&= 1 + \frac{Z^*}{\Lambda^*} + O(e^2).
\end{aligned} \tag{C.24}$$

(ii) In view of the preceding remark, the set in (C.21) corresponds to absolute values of the inclination between $O(\rho^*)$ and $O\Big((\bar\beta)^{\frac{1}{2}}\Big)$: indeed, choosing, for $\beta' \in (0, \bar\beta)$,

$$-Z^* = \Lambda_0^* - \beta - \frac{\rho^{*2}}{2} - \beta', \quad \Lambda^* = \Lambda_0^* - \beta + \rho^{*2}, \quad H^* = 0,$$

we have (by (C.24))

$$i^2 = O\Big(\frac{Z^* + \Lambda^*}{\Lambda^*}\Big) = O\Big(\rho^{*2} + \beta'\Big).$$

(iii) Finally, we point out that, as above (compare Remark C.1 and the relative proof in the footnote), if $(X, x) = \Phi_{\mathrm{DP}^*}\Big((\Lambda^*, \eta^*, \mathrm{p}^*), (\lambda^*, \xi^*, \mathrm{q}^*)\Big)$ is as in Proposition C.2, then

$$X = \frac{\mathrm{m}^4 \mathrm{M}^2}{\Lambda^{*3}} \frac{\partial x}{\partial \lambda^*}.$$

(iv) The use of osculating variables will turn out to be particularly useful in the reduction of the angular momentum (see Section C.2, below).

C.1.3. Some orbital elements in terms of osculating Poincaré variables. In this subsubsection we show a way to express some of the classical orbital elements as functions of the osculating Poincaré variables $(\Lambda^*, \lambda^*, \eta^*, \xi^*)$. Denote the eccentric anomaly by u and the true anomaly by v (compare Fig. 1); denote, also, by d the distance between the planet and its star, by w the true longitude $w := v + g + \theta$ and let $w^* := w - \theta = v + g$. The target here is, also, to find (for later use) analytic expressions d and w^* as functions of $(\Lambda^*, \lambda^*, \eta^*, \xi^*)$. From the definition of ℓ and (C.16) we have

$$\lambda^* + h^* = \ell = u - e \sin u. \tag{C.25}$$

Inverting (C.25), we find $u = u(\Lambda^*, \lambda^*, h^*, e)$. Then we obtain u in terms of $(\Lambda^*, \lambda^*, H^*, h^*)$ from (C.10). Furthermore, we have

$$d = a\,(1 - e\ \cos u) \tag{C.26}$$

$$= \frac{a\,(1-e^2)}{1 + e\ \cos v}. \tag{C.27}$$

In light of (C.26), (C.10), the first of (C.8) and the above expression of u, we get a representation d in terms of $(\Lambda^*, \lambda^*, H^*, h^*)$. Therefore, via (C.17), we obtain a representation of d in terms of osculating Poincaré variables $(\Lambda^*, \lambda^*, \eta^*, \xi^*)$.

Let us now express w^* in osculating Poincaré variables. From (C.26) and (C.27),

$$\cos v = \frac{\cos u - e}{1 - e \cos u}. \tag{C.28}$$

Hence, recalling (C.10) and the expressions of u and v above, we can readily express v as a function of $(\Lambda^*, \lambda^*, H^*, h^*)$. Making use of (C.16), we infer that $w^* = v - h^*$. Hence, from the above construction and (C.17), we obtain a representation of w^* in terms of osculating Poincaré variables $(\lambda^*, \eta^*, \xi^*)$.

C.2. Canonical variables for the three-body problem

C.2.1. The Poincaré Hamiltonian of the planetary three-body problem. The discussion of this subsection follows [Ch88] and [L88]. We consider three bodies $P_0,\ P_1\ , P_2$ of mass $m_0,\ m_1\ , m_2$ interacting through gravity (with constant of gravitation 1). Assume that

$$\bar\kappa \varepsilon \leqq \frac{m_1}{m_0}\,,\ \frac{m_2}{m_0} \leqq \varepsilon < 1, \tag{C.29}$$

for a fixed constant $0 < \bar\kappa \leqq 1$ and a small parameter $\varepsilon > 0$. We consider a (inertial) frame $\{k_1, k_2, k_3\}$ in $\mathbb{R}^3$ with origin in the center of mass of the system and with vertical axis parallel to the (conserved) total angular momentum. Thus, if $u^{(i)}$ denotes the position of P_i, $U^{(i)} := m_i \dot u_i$ denotes the momentum of P_i and C is the total angular momentum,

$$C = \sum_{i=0}^{2} u^{(i)} \times U^{(i)}, \tag{C.30}$$

our choices imply

$$\sum_{i=0}^{2} m_i u^{(i)} = 0, \qquad \frac{C}{|C|} = k_3. \tag{C.31}$$

Newton's laws imply that the three-body problem is governed by the nine-degree-of-freedom Hamiltonian

$$\bar{\mathcal{H}}^{\mathrm{tb}}(U^{(1)}, U^{(2)}, U^{(3)}, u^{(1)}, u^{(2)}, u^{(3)}) := \sum_{i=0}^{2} \frac{1}{2m_i}|U^{(i)}|^2 - \sum_{0\leqq i<j\leqq 2} \frac{m_i m_j}{|u^{(i)} - u^{(j)}|},$$

where $U = (U^{(0)}, U^{(1)}, U^{(2)}) \in \mathbb{R}^9$ and $u = (u^{(0)}, u^{(1)}, u^{(2)}) \in \mathbb{R}^9$ are conjugate symplectic variables.

We, now, introduce *canonical heliocentric variables* $(R^{(0)}, R^{(1)}, R^{(2)}, r^{(0)}, r^{(1)}, r^{(2)})$ by means of the linear symplectic transformation

$$\begin{aligned} &u^{(0)} = r^{(0)}, \quad u^{(1)} = r^{(0)} + r^{(1)}, \quad u^{(2)} = r^{(0)} + r^{(2)}, \\ &U^{(0)} = R^{(0)} - R^{(1)} - R^{(2)}, \quad U^{(1)} = R^{(1)}, \quad U^{(2)} = R^{(2)}. \end{aligned} \tag{C.32}$$

The angular momentum is preserved by this transformation:

$$\sum_{i=0}^{2} r^{(i)} \times R^{(i)} = C. \tag{C.33}$$

By means of (C.31), $R^{(0)} = \sum_{i=0}^{2} U^{(i)} = 0$. Hence, the three-body Hamiltonian $\bar{\mathcal{H}}^{\mathrm{tb}}$ in $(R^{(i)}, r^{(i)})$-variables takes the form

$$\begin{aligned} \tilde{\mathcal{H}}^{\mathrm{tb}}(R^{(1)}, R^{(2)}, r^{(1)}, r^{(2)}) := &\sum_{i=1}^{2} \Big(\frac{m_0 + m_i}{2m_0 m_i}|R^{(i)}|^2 - \frac{m_0 m_i}{|r^{(i)}|} \Big) \\ &+ \frac{R^{(1)} \cdot R^{(2)}}{m_0} - \frac{m_1 m_2}{|r^{(1)} - r^{(2)}|}. \end{aligned}$$

We have, therefore, obtained a six-degree-of-freedom Hamiltonian of conjugated variables $(R^{(1)}, R^{(2)}) \in \mathbb{R}^6$ and $(r^{(1)}, r^{(2)}) \in \mathbb{R}^6$: the number of degrees of freedom decreased by three units because of the above *reduction of the center of mass*.

The masses m_j appear in the definition of the momenta $R^{(1)}$ and $R^{(2)}$, which are both of order ε. In order to remove this singularity (as $\varepsilon \to 0$), we introduce new symplectic variables[29]

$$X^{(i)} = \frac{R^{(i)}}{\varepsilon m_0^{5/3}}, \qquad x^{(i)} = \frac{r^{(i)}}{m_0^{2/3}}. \tag{C.34}$$

[29] Recall that, if $\alpha > 0$ and $\beta > 0$ are two "rescaling factors", the Hamiltonian flow governed by a Hamiltonian function $h(X, x)$ (with respect to the symplectic form $dX \wedge dx$) coincides with the Hamiltonian flow governed by the Hamiltonian $\frac{1}{\alpha\beta}h(\alpha\tilde{X}, \beta\tilde{x})$ (with respect to the symplectic form $d\tilde{X} \wedge d\tilde{x}$) with initial data $\tilde{X}(0) = \frac{1}{\alpha}X(0)$ and $\tilde{x}(0) = \frac{1}{\beta}x(0)$.

In such variables, the $\tilde{\mathcal{H}}^{\rm tb}$-flow is equivalent to the $\mathcal{H}^{\rm tb}$-flow with

$$\begin{aligned}\mathcal{H}^{\rm tb}(X^{(1)}, X^{(2)}, x^{(1)}, x^{(2)}) := {}& \mathcal{H}_0^{\rm tb}(X^{(1)}, X^{(2)}, x^{(1)}, x^{(2)}) \\ & + \mathcal{H}_1^{\rm tb}(X^{(1)}, X^{(2)}, x^{(1)}, x^{(2)}),\end{aligned} \tag{C.35}$$

where

$$\begin{aligned}\mathcal{H}_0^{\rm tb} &:= \sum_{i=1}^{2}\Big(\frac{1}{2\mathrm{m_i}}|X^{(i)}|^2 - \frac{\mathrm{m}_i \mathrm{M}_i}{|x^{(i)}|}\Big), \\ \mathcal{H}_1^{\rm tb} &:= \varepsilon\, X^{(1)} \cdot X^{(2)} - \frac{m_1 m_2}{\varepsilon\, m_0^2}\, \frac{1}{|x^{(1)} - x^{(2)}|}, \\ \mathrm{m}_i &:= \frac{m_i}{\varepsilon}\, \frac{1}{m_0 + m_i}, \qquad \mathrm{M}_i := 1 + \frac{m_i}{m_0}.\end{aligned} \tag{C.36}$$

Recalling (C.29), it follows that m_i and M_i are bounded and bounded away from zero (uniformly in ε):

$$\frac{\bar{\kappa}}{1+\bar{\kappa}} \leqq \mathrm{m}_i \leqq 1, \qquad 1 \leqq \mathrm{M}_i \leqq 2. \tag{C.37}$$

Notice that the Hamiltonian $\mathcal{H}_0^{\rm tb}$, which is of order one, is simply the sum of two uncoupled Kepler problems, while $\mathcal{H}_1^{\rm tb}$ is of order ε and will be considered as a perturbation.

We can, therefore, introduce osculating Poincaré variables associated with the osculating orbital elements relative to (P_0, P_i) with $i = 1, 2$ and masses m_0, m_i. More precisely, let

$$(\Lambda_i^*, \eta_i^*, \mathrm{p}_i^*, \lambda_i^*, \xi_i^*, \mathrm{q}_i^*) := \Psi_{\rm DP^*}^{-1}(X^{(i)}, x^{(i)}),$$

where $\Psi_{\rm DP^*}$ is defined in Proposition C.2 (and the reference plane for computing the orbital elements relative to (P_0, P_i) is the plane spanned by $\{k_1, k_2\}$, i.e., the total-angular-momentum plane). For example, the expression of Λ_i^* with respect to the major semi-axis of the planet P_i is

$$\Lambda_i^* = \kappa_i^* \sqrt{a_i}\,, \qquad \kappa_i^* := \frac{m_i}{\varepsilon}\, \frac{1}{\sqrt{m_0(m_0+m_i)}}. \tag{C.38}$$

In osculating Poincaré variables

$$\begin{aligned}&(\Lambda^*, \eta^*, \mathrm{p}^*, \lambda^*, \xi^*, \mathrm{q}^*) \\ &\quad := \Big((\Lambda_1^*, \Lambda_2^*), (\eta_1^*, \eta_2^*), (\mathrm{p}_1^*, \mathrm{p}_2^*), (\lambda_1^*, \lambda_2^*), (\xi_1^*, \xi_2^*), (\mathrm{q}_1^*, \mathrm{q}_2^*)\Big)\end{aligned} \tag{C.39}$$

the unperturbed Hamiltonian $\mathcal{H}_0^{\rm tb}$ becomes, simply,

$$\mathcal{H}_0^*(\Lambda^*) := -\sum_{i=1}^{2} \frac{\kappa_i}{2(\Lambda_i^*)^2}\,, \qquad \kappa_i := \Big(\frac{m_i}{\varepsilon}\Big)^3 \frac{1}{m_0^2(m_0+m_i)} \tag{C.40}$$

(with κ_i of order one), and the full three-body Hamiltonian becomes

$$\mathcal{H}^*(\Lambda^*, \eta^*, \mathrm{p}^*, \lambda^*, \xi^*, \mathrm{q}^*) = \mathcal{H}_0^*(\Lambda^*) + \mathcal{H}_1^*(\Lambda^*, \eta^*, \mathrm{p}^*, \lambda^*, \xi^*, \mathrm{q}^*), \qquad \text{(C.41)}$$

with $\mathcal{H}_1^*$ real-analytic on $\mathcal{C} \times \mathcal{C}$ (compare Proposition C.2). It is customary to split the $O(\varepsilon)$-perturbation $\mathcal{H}_1^*$ as the sum of two functions:

$$\mathcal{H}_1^* := \mathcal{H}_1^{*,\mathrm{compl}} + \mathcal{H}_1^{*,\mathrm{princ}}, \qquad \text{(C.42)}$$

where $\mathcal{H}_1^{*,\mathrm{compl}}$, called the "complementary part" of the perturbation, is the function $\varepsilon\, X^{(1)} \cdot X^{(2)}$ expressed in Poincaré osculating variables (C.39), while $\mathcal{H}_1^{*,\mathrm{princ}}$, called the "principal part" of the perturbation, is the function $\frac{m_1 m_2}{\varepsilon m_0^2} \frac{1}{|x^{(1)} - x^{(2)}|}$ expressed in Poincaré osculating variables (C.39).

Notice that, because of point (iii) of Remark C.2, the λ^*-average of $\mathcal{H}_1^{*,\mathrm{compl}}$ vanishes: let $x^{(i)} = x^{(i)}(\Lambda_i^*, \eta_i^*, \mathrm{p}_i^*, \lambda_i^*, \xi_i^*, \mathrm{q}_i^*)$ and $X^{(i)} = X^{(i)}(\Lambda_i^*, \eta_i^*, \mathrm{p}_i^*, \lambda_i^*, \xi_i^*, \mathrm{q}_i^*)$. Then

$$\begin{aligned} &\int_0^{2\pi} \int_0^{2\pi} \mathcal{H}_1^{*,\mathrm{compl}}(\Lambda^*, \eta^*, \mathrm{p}^*, \lambda^*, \xi^*, \mathrm{q}^*)\, d\lambda_1^*\, d\lambda_2^* \\ &\quad = \varepsilon \int_0^{2\pi} \int_0^{2\pi} X^{(1)} \cdot X^{(2)}\, d\lambda_1^*\, d\lambda_2^* \\ &\quad = \varepsilon \,\mathrm{const} \int_0^{2\pi} \int_0^{2\pi} \partial_{\lambda_1^*} x^{(1)} \cdot \partial_{\lambda_2^*} x^{(2)}\, d\lambda_1^*\, d\lambda_2^* = 0. \end{aligned} \qquad \text{(C.43)}$$

C.2.2. Reduction of the angular momentum. The conservation of the total angular momentum allows us to lower by two more units the number of degrees of freedom ("reduction of the angular momentum"). Recall that we are *excluding planar motions*, i.e., motions with vanishing mutual inclinations.

Proposition C.3. *The function* $\mathcal{H}_1^*$ *in* (C.41) *is independent of* ζ^*. *More precisely, let* $r_j^* := \sqrt{\mathrm{p}^{*2}_j + \mathrm{q}^{*2}_j}$ *and* $r^* := (r_1^*, r_2^*)$. *Then, for any* $\zeta_j^* \in \mathbb{T}$ *and for each* $r_j^* > 0$ (*for which* $(\mathrm{p}^*, \mathrm{q}^*)$ *belong to the real part of their domain of definition; see Proposition C.2*),

$$\begin{aligned} &\mathcal{H}_1^*\Big(\Lambda^*, \eta^*, (r_1^* \cos \zeta_1^*, r_2^* \cos \zeta_2^*), \lambda^*, \xi^*, (r_1^* \sin \zeta_1^*, r_2^* \sin \zeta_2^*)\Big) \\ &\quad = \mathcal{H}_1^*(\Lambda^*, \eta^*, r^*, \lambda^*, \xi^*, (0, 0)). \end{aligned} \qquad \text{(C.44)}$$

In particular, $Z_j^* := -\frac{r_j^{*2}}{2}$ *are analytic integrals:* $\{Z_j^*,\ \mathcal{H}_1^*\} = 0$.

We shall call the Z_j^*'s the "Poincaré integrals" of the non-planar three-body problem.

Proof. Looking at the force field, we see that, if at some time t_0 the mutual inclination of the planets vanishes, it vanishes at every instant of time: therefore the motions of the three-body problem are either *planar*, so that the mutual inclination is identically zero, or *non-planar*, so that the mutual inclination is always non-zero.

Now, recall our choice of reference plane as the total-angular-momentum plane, fix a time t_0 and consider the inclination $i_j = i_j(t_0)$ of the instant orbital plane ("osculating plane") associated with the planet P_j (i.e., the plane spanned by the position and velocity of P_j at time t_0): $i_j(t_0)$ is the angle between the $\{k_1, k_2\}$-plane and the plane spanned by the position and velocity of P_j at time t_0. If $i_1 \neq i_2$, we can define the line of the nodes as the intersection of the two osculating planes; let $N \neq P_0$ be a node, i.e., a point in the line of the nodes. Let, also, the vector V be the difference between N and the position of the star P_0. By construction, V lies in the intersection of the osculating planes spanned by $(r_j(t_0), R_j(t_0))$, $j = 1, 2$. Therefore, recalling (C.33), $V \cdot C = 0$. Whence, the difference between the longitudes of the ascending nodes of the planets is constant: indeed,

$$\theta_1 - \theta_2 = \pi. \tag{C.45}$$

Since $i_1 \neq i_2$ in a neighborhood of t_0, we can perform the transformation (C.18): we obtain a Hamiltonian $\mathcal{H}^{**}(\Lambda^*, \lambda^*, \eta^*, \xi^*, Z^*, \zeta^*)$, with

$$\zeta_2^* - \zeta_1^* = \theta_1 - \theta_2 = \pi. \tag{C.46}$$

Let us consider a rotation $\mathcal{R}_\vartheta$ of an angle ϑ around C. By construction (see the relations (C.8), (C.9), (C.11) and (C.16) above), the variables $(\Lambda^*, \lambda^*, \eta^*, \xi^*, Z^*)$ are *preserved* by $\mathcal{R}_\vartheta$, while (ζ_1^*, ζ_2^*) is sent into $(\zeta_1^* + \vartheta, \zeta_2^* + \vartheta)$. Since the energy of the system is also *preserved* by $\mathcal{R}_\vartheta$, we find that $\mathcal{H}^{**}(\dots, \zeta_1^*, \zeta_2^*) = \mathcal{H}^{**}(\dots, \zeta_1^* + \vartheta, \zeta_2^* + \vartheta)$. Thus, the Hamiltonian $\mathcal{H}^{**}$ (and, hence, the function $\mathcal{H}_1^*$) does not depend on ζ_1^* and ζ_2^* separately, but *only on their difference*. The thesis follows, now, from (C.46). □

Hence, we can consider the Hamiltonian $\mathcal{H}^*$ as depending only on $(\Lambda^*, \lambda^*, \eta^*, \xi^*)$ and on the initial value of Poincaré integral Z^*. We, therefore, let[30]

$$\begin{aligned} \varepsilon F &:= \varepsilon F(\Lambda^*, \eta^*, \lambda^*, \xi^*; Z^*) \\ &:= \mathcal{H}_1^*\left(\Lambda^*, \eta^*, \left(\sqrt{-2Z_1^*}, \sqrt{-2Z_2^*}\right), \lambda^*, \xi^*, (0,0)\right), \\ Z^* &= (Z_1^*, Z_2^*), \quad Z_j^* := -\frac{\mathrm{p}_j^{*2} + \mathrm{q}_j^{*2}}{2}. \end{aligned} \tag{C.47}$$

Analogously, we set

$$\begin{aligned} &\mathcal{H}_1^{\mathrm{compl}}(\Lambda^*, \eta^*, \lambda^*, \xi^*; Z^*) \\ &\quad := \mathcal{H}_1^{*,\mathrm{compl}}\left(\Lambda^*, \eta^*, \left(\sqrt{-2Z_1^*}, \sqrt{-2Z_2^*}\right), \lambda^*, \xi^*, (0,0)\right), \\ &\mathcal{H}_1^{\mathrm{princ}}(\Lambda^*, \eta^*, \lambda^*, \xi^*; Z^*) \\ &\quad := \mathcal{H}_1^{*,\mathrm{princ}}\left(\Lambda^*, \eta^*, \left(\sqrt{-2Z_1^*}, \sqrt{-2Z_2^*}\right), \lambda^*, \xi^*, (0,0)\right). \end{aligned} \tag{C.48}$$

From now on the values of the Poincaré integrals Z_j^* will be taken to be real in the domains described in Proposition C.2. Notice that, physically, changing values of

[30] Compare (1.4) in Theorem 1.1.

the Poincaré integrals corresponds to considering different ranges of mutual inclinations; compare (C.9), (C.12), (C.15), In fact, choosing Z_j^* as in (C.21), (C.22) implies that $\imath_{\min} \sim \rho^*$.

Often, however, the value of the Poincaré integral will be omitted from the notation.

C.3.3. The principal part of the perturbation Here we will obtain an "explicit" representation in terms of osculating Poincaré variables of the principal part of the perturbation $\mathcal{H}_1^{\mathrm{princ}}$.

Let, as above, $w_i := \theta_i + g_i + v_i$ be the true longitude of the planet P_i and $\hat{S}$ the angle between the planets P_1 and P_2 (see Fig. 4).

Let $\hat{\imath}$ be the mutual inclination of the planets and $w_i^* = w_i - \theta_i$. Notice that, by (C.45), $w_2^* - w_1^* = w_2 - w_1 + \pi$. By elementary trigonometry,

$$\cos \hat{S} = -\cos w_1^* \cos w_2^* - \sin w_1^* \sin w_2^* \cos \hat{\imath}. \tag{C.49}$$

Denote the angular momentum of the planet P_i by $C^{(i)}$, so that $C = C^{(1)} + C^{(2)}$. Then, using (C.11), we infer that

$$\begin{aligned} |C|^2 &= |C^{(1)}|^2 + |C^{(2)}|^2 + 2\,|C^{(1)}|\,|C^{(2)}|\,\cos\hat{\imath} \\ &= \Lambda_1^2\,(1-e_1^2) + \Lambda_2^2\,(1-e_2^2) + 2\Lambda_1\Lambda_2\sqrt{(1-e_1^2)(1-e_2^2)}\,\cos\hat{\imath}. \end{aligned} \tag{C.50}$$

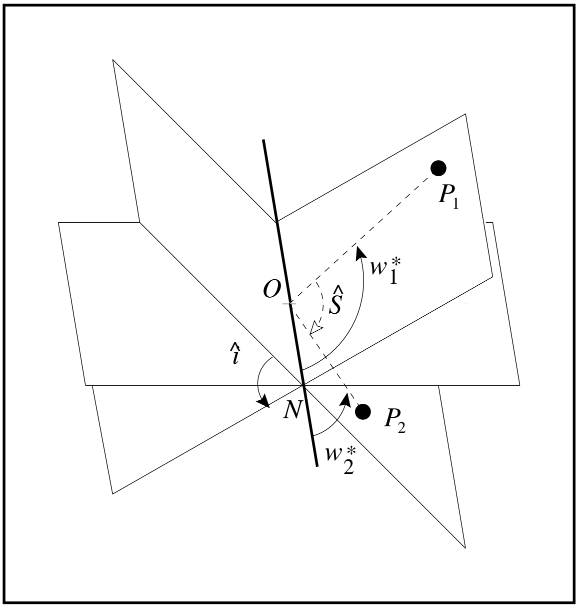

Fig. 4.

Using (C.50) and (C.10), we can express $\cos\hat{\imath}$ in terms of the osculating Poincaré variables (and the initial value parameter C). Explicitly:

$$\begin{aligned}\cos\hat{\imath} &= \frac{|C|^2 - \sum_{i=1}^{2}\left(\Lambda_i^* - H_i^*\right)^2}{2\prod_{i=1}^{2}(\Lambda_i^* - H_i^*)} \\ &= \frac{|C|^2 - \Lambda_1^{*2} + \Lambda_2^{*2}}{2\Lambda_1^*\Lambda_2^*} + O(e_1^2) + O(e_2^2)\,. \end{aligned} \tag{C.51}$$

Also, as proved in Section C.1.3 $w_i^* = w_i^*(\Lambda_i^*, \lambda_i^*, \eta_i^*, \xi_i^*)$. Therefore, by (C.49), the expression of $\cos\hat{S}$ in terms of osculating Poincaré variables follows.

Also,

$$|r^{(1)} - r^{(2)}|^2 = |r^{(1)}|^2 + |r^{(2)}|^2 - 2\,|r^{(1)}|\,|r^{(2)}|\,\cos\hat{S}.$$

Thus, from the definition of $x^{(i)}$, we have

$$\frac{m_0^2}{|x^{(1)} - x^{(2)}|} = \frac{1}{|r^{(1)} - r^{(2)}|} = \frac{1}{\sqrt{|r^{(1)}|^2 + |r^{(2)}|^2 - 2\,|r^{(1)}|\,|r^{(2)}|\,\cos\hat{S}}} \tag{C.52}$$

Since we expressed $\cos\hat{S}$ and $|r^{(i)}|$ in osculating Poincaré variables, the expression of the principal part of the perturbation in osculating Poincaré variables $(\Lambda^*, \eta^*, \lambda^*, \xi^*)$ readily follows.

Also, we can see, by (C.18) and a 180-degree rotation of the perihelia, that the λ^*-average of $\mathcal{H}_1^{\text{princ}}$ in (C.48) is even in (η^*, ξ^*):

Proposition C.4. *Let*

$$f_1^*(\Lambda^*, \eta^*, \xi^*) := \frac{1}{\varepsilon}\int_{\mathbb{T}^2} \mathcal{H}_1^{\text{princ}}(\Lambda^*, \eta^*, \lambda^*, \xi^*; Z^*)\, d\lambda^*.$$

Then, $f_1^*(\Lambda^*, -\eta^*, -\xi^*) = f_1^*(\Lambda^*, \eta^*, \xi^*)$.

The rescaling by $\frac{1}{\varepsilon}$ is made so that f_1^* is a (real-analytic) uniformly bounded (by an order-one constant) function.

Proof. The system is invariant under the map $\mathcal{R}(\Lambda^*, \eta^*, \lambda^*, \xi^*) = (\Lambda^*, -\eta^*, \lambda^* + \pi, -\xi^*)$ (usually referred to as "space inversion"). Thus, the thesis follows by observing that $\int_{\mathbb{T}^2} \mathcal{H}^* \circ \mathcal{R}\, d\lambda^* = \int_{\mathbb{T}^2} \mathcal{H}^*\, d\lambda^*$, and making use of (C.43). □

Thus, dropping the explicit dependence upon the Poincaré integrals, the perturbation function εF (see (1.4) and (C.47)), has the form

$$F(\Lambda^*, \eta^*, \lambda^*, \xi^*) =: f_1^*(\Lambda^*, \eta^*, \xi^*) + f_2^*(\Lambda^*, \eta^*, \lambda^*, \xi^*), \tag{C.53}$$

where

$$\frac{1}{4\pi^2}\int_{\mathbb{T}^2} f_2^*\, d\lambda^* = 0 \tag{C.54}$$

and f_1^* is even in (η^*, ξ^*) (Proposition C.4). Thus, we can split f_1^* as

$$f_1^* = f_{1,0}^*(\Lambda^*) + f_{1,2}^*(\Lambda^*, \eta^*, \xi^*) + \tilde{f}_1^*(\Lambda^*, \eta^*, \xi^*),$$

with $f_{1,2}^*$ quadratic in (η^*, ξ^*) and, uniformly in Λ^* (complex),

$$|\tilde{f}_1^*(\Lambda^*, \eta^*, \xi^*)| \leqq \text{const}\,|(\eta^*, \xi^*)|^4.$$

C.2.4. Symplectic diagonalization and conclusion. The final step consists in showing that $f_{1,2}^*$ is a positive definite quadratic form and in finding "explicitly" the (purely imaginary) eigenvalues of

$$Q := S_4 \partial^2_{\eta^*,\xi^*} f_{1,2}^*(\Lambda^*, 0, 0) \tag{C.55}$$

(S_4 being the standard (4×4)-symplectic matrix). This calculation has been performed in detail in [R95] (§ 3.4, § 3.5), which, we follow here[31].

Let us introduce some notation (which we shall keep similar to that used in [R95]): let α denote the ratio of the planetary semi-axis, $\alpha := a_1/a_2$ (recall that by our assumptions $\alpha \leqq \alpha_{\max} < 1$); let

$$h := \frac{m_1}{m_2}\sqrt{\frac{m_0 + m_2}{m_0 + m_1}}$$

(which is close to the planetary mass ratio for small ε and is a quantity of order one); let

$$D := \frac{(\Lambda_1^* + \Lambda_2^*)^2 - |C|^2}{\Lambda_1^* \Lambda_2^*} = \hat{\imath}^2 + O(\hat{\imath}^4) + O(e_1^2) + O(e_2^2)$$

(where the asymptotic evaluation is a consequence of (C.51)); let $\pm\sqrt{-1}(2\varepsilon\bar{\Omega}_j)$ denote the eigenvalues of the matrix Q in (C.55) and let

$$\mathcal{L}_i := \frac{\Lambda_1}{c}\bar{\Omega}_i\,, \qquad c := -\frac{2m_1 m_2^3}{\varepsilon^4 m_0^3 (m_0 + m_2)\,\Lambda_2^2} \tag{C.56}$$

(notice that c is a quantity of order one). Finally, recall the well-known definition of Laplace coefficients for $0 \leqq \alpha < 1$:

$$b_s^{(k)}(\alpha) = 2\,\frac{s(s+1)\ldots(s+k-1)}{k!}\,\alpha^k$$
$$* \left(1 + \sum_{\ell \geqq 0} \Big(\frac{s\ldots(s+\ell)}{\ell!}\;\frac{(s+k)\ldots(s+k+\ell)}{(k+1)\ldots(k+1+\ell)}\,\alpha^{2(\ell+1)}\Big)\right).$$

After the above preparation (and after quite a bit of algebra), we find that (compare with [R95])

$$\mathcal{L}_i = \mathcal{L}_i^{(0)}(\alpha) + O(D),$$

---

31 Notational remark: the Hamiltonian $\mathcal{H}^*$ here differs from the one in [R95] by a scaling factor of size $\varepsilon^3 m_0^5$.

($O(D)$ being a real quantity of order D) with

$$\begin{aligned}\mathcal{L}_1^{(0)}(\alpha) &= \frac{\alpha}{32}\Big[3(1+h\sqrt{\alpha})b_{3/2}^{(1)}(\alpha)-\sqrt{\mathcal{D}(\alpha)}\Big],\\ \mathcal{L}_2^{(0)}(\alpha) &= \frac{\alpha}{32}\Big[3(1+h\sqrt{\alpha})b_{3/2}^{(1)}(\alpha)+\sqrt{\mathcal{D}(\alpha)}\Big],\\ \mathcal{D}(\alpha) &:= (1-h\sqrt{\alpha})^2\Big(b_{3/2}^{(1)}(\alpha)\Big)^2+4h\sqrt{\alpha}\Big(b_{3/2}^{(2)}(\alpha)\Big)^2,\end{aligned} \tag{C.57}$$

showing, in particular, that $\mathcal{L}_i$ and hence $\bar{\Omega}_i$ are real. Furthermore, from the definition of Laplace coefficients, it follows that $b_{3/2}^{(2)}(\alpha) < b_{3/2}^{(1)}(\alpha)$ for $\alpha \in (0, 1)$. Thus, because of (C.57), there exist suitable (order-one) constants $\bar{D} > 0$ and $\bar{c} > 0$ such that, if $D \leqq \bar{D}$ (i.e., if the mutual inclination is sufficiently small), then, uniformly in Λ^*,

$$\inf \mathcal{L}_i > \bar{c} > 0, \qquad \inf\Big(\mathcal{L}_2 - \mathcal{L}_1\Big) > \bar{c} > 0. \tag{C.58}$$

Finally, by a standard argument going back to Weierstrass, we can find, for any *fixed* Λ^*, a linear symplectic transformation

$$\begin{pmatrix}\eta'\\ \xi'\end{pmatrix} \mapsto \begin{pmatrix}\eta^*\\ \xi^*\end{pmatrix} = A(\Lambda^*)\begin{pmatrix}\eta'\\ \xi'\end{pmatrix}, \tag{C.59}$$

which sends $f_{1,2}^*$ into

$$f_{1,2} := \frac{c}{\Lambda_1}\Big(\mathcal{L}_1(\Lambda^*)\cdot((\eta_1')^2+(\xi_1')^2)+\mathcal{L}_2(\Lambda^*)\cdot((\eta_2')^2+(\xi_2')^2)\Big). \tag{C.60}$$

By classical generating function theory, it is easy to see that the transformation (C.59) can be extended to a symplectic transformation on the whole phase space $\Psi_{\mathrm{W}} : (\Lambda', \eta', \lambda', \xi') \mapsto (\Lambda^*, \eta^*, \lambda^*, \xi^*)$ with

$$\Lambda^* = \Lambda' \qquad \text{and} \qquad \lambda^* = \lambda' + \hat{\ell}(\Lambda', \eta', \xi'), \tag{C.61}$$

for a suitable function $\hat{\ell}$.

Letting $(I, \varphi) := (\Lambda', \lambda')$, $(p, q) := (\eta', \xi')$, $f_1 := f_1^* \circ \Psi_{\mathrm{W}}$, $f_2 := f_2^* \circ \Psi_{\mathrm{W}}$,

$$\sigma_0 := \rho^{*2}, \qquad \delta := \beta - \rho^{*2}, \qquad \rho_0 := \rho^*, \tag{C.62}$$

we realize that the proof of the Delaunay-Poincaré Theorem 1.1 is completed.

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