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# *Corrigendum to: Elliptic Two-Dimensional Invariant Tori for the Planetary Three-Body Problem*

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The main result of the paper [BCV] *Elliptic two-dimensional invariant tori for the planetary three-body problem* by the authors, published in Arch. Ration. Mech. Anal. **170**, 91–135 (2003), namely, Theorem 1.2, exploits a classical result (Theorem 1.1) essentially due to Delaunay and Poincaré. In Appendix C of [BCV] a detailed proof of Theorem 1.1 is discussed. However, this proof contains a flaw. In fact, the statement following equation (C.44) in Proposition C.3, Section C.2.2, Appendix C is not correct. The $Z_j^*$'s (called in [BCV] "Poincaré integrals") are not global integrals. Therefore, Section C.2.2 requires amendment. What now follows, replaces Section C.2.2 in [BCV], correcting the above mentioned flaw. Notations are as in [BCV].

Recall that in [BCV] we consider the 10-dimensional invariant symplectic manifold, $\mathcal{M}_{\rm ver}$, defined by taking the non-vanishing total angular momentum to be vertical, i.e., $C{:=}C^{(1)} + C^{(2)} = ck_3$ where $C^{(i)}$ denotes the angular momentum of the $i^{\rm th}$ planet with respect to the origin of an inertial heliocentric frame $\{k_1, k_2, k_3\}$ and $c \in \mathcal{R}\backslash\{0\}$; compare (C.30) and (C.31). Recall, also, that in [BCV] we use the "osculating Poincaré variables" $(\Lambda^*, \eta^*, {\rm p}^*, \lambda^*, \xi^*, {\rm q}^*)$ as described in Section C.2.1 of [BCV], where $\Lambda^* = (\Lambda_1^*, \Lambda_2^*)$, $\eta^* = (\eta_1^*, \eta_2^*)$, etc., and the index $i = 1, 2$ refers to the two-body system $i^{\rm th}$ planet-star. Finally, recall that $-\zeta_i^* = \theta_i$ is the longitude of the $i^{\rm th}$ node on the plane $\{k_1, k_2\}$, while $-Z_i^* = \Theta_i = C^{(i)} \cdot k_3$. Let us now describe, analytically, $\mathcal{M}_{\rm ver}$. From the definition of the Delaunay variables $(L_i, \ell_i, G_i, g_i, \Theta_i, \theta_i)$ (compare to Section C.1.1) it follows that the relations defining $\mathcal{M}_{\rm ver}$, i.e., $C \cdot k_1 = 0 = C \cdot k_2$, are equivalent to

$$\zeta_2^* - \zeta_1^* = \pi \quad \text{and} \quad G_1^2 - \Theta_1^2 = G_2^2 - \Theta_2^2.$$

Therefore, since $G_i = \Lambda_i^* - H_i^*$, with $H_i^*{:=}(\eta_i^{*2} + \xi_i^{*2})/2$ (compare to (C.5), (C.15) and (C.17)), we find

$$Z_1^* = Z_2^* + \frac{(\Lambda_2^* - H_2^*)^2 - (\Lambda_1^* - H_1^*)^2}{c},$$

$$H_i^* := \frac{\eta_i^{*2} + \xi_i^{*2}}{2},$$
$$c := -(Z_1^* + Z_2^*).$$

**Proposition** (Reduction of the angular momentum)**.** *The motions starting on $\mathcal{M}_{\rm ver}$, with total angular momentum $C = ck_3$ ($c \neq 0$), can be described by the four-degrees-of-freedom Hamiltonian $\mathcal{H}_0^* + \varepsilon F$, with (compare to* (1.4) *of Theorem 1.1 of [BCV])*

$$\mathcal{H}_0^* := -\frac{1}{2}(\kappa_1/\Lambda_1^{*2} + \kappa_2/\Lambda_2^{*2}) ,$$
$$\varepsilon F := \varepsilon F(\Lambda^*, \lambda^*, \eta^*, \xi^*)$$
$$:= \mathcal{H}_1^*(\Lambda^*, \eta^*, (r_1^*, -r_2^*), \lambda^*, \xi^*, (0,0))$$

*where*

$$r_1^* := \sqrt{\frac{c^2 + (\Lambda_1^* - H_1^*)^2 - (\Lambda_2^* - H_2^*)^2}{c}} ,$$
$$r_2^* := \sqrt{\frac{c^2 + (\Lambda_2^* - H_2^*)^2 - (\Lambda_1^* - H_1^*)^2}{c}}, \tag{1}$$

*and $\mathcal{H}_0^* + \mathcal{H}_1^* = \mathcal{H}^*(\Lambda^*, \eta^*, \mathrm{p}^*, \lambda^*, \xi^*, \mathrm{q}^*)$ is the Hamiltonian (C.41) of the spatial three-body problem expressed in the osculating Poincaré variables.*

**Proof.** In the full-phase space, we introduce the standard symplectic variables $(\Lambda^*, \eta^*, \Psi, \lambda^*, \xi^*, \psi) = \phi(\Lambda^*, \eta^*, \mathrm{p}^*, \lambda^*, \xi^*, \mathrm{q}^*)$ defined by the relations

$$\begin{cases} (Z_1^*, Z_2^*, \zeta_1^*, \zeta_2^*) = -(\Psi_1, \Psi_2 - \Psi_1, \psi_1 + \psi_2, \psi_2), \\ (\mathrm{p}_i^*, \mathrm{q}_i^*) = (\sqrt{-2Z_i^*}\sin\zeta_i^*, \sqrt{-2Z_i^*}\cos\zeta_i^*) , \qquad i = 1, 2 . \end{cases} \tag{$*$}$$

Since the components of the total angular momentum $C_i$ Poisson-commute with $\mathcal{H}^*$, it follows that $\Psi_2 = -(Z_1^* + Z_2^*) = C_3$ is an integral for $\mathcal{H}_* := \mathcal{H}^* \circ \phi^{-1}$; hence, the conjugate angle $\psi_2$ is cyclic for $\mathcal{H}_*$ (i.e., $\mathcal{H}_*$ is constant in $\psi_2$). In these variables $\mathcal{M}_{\rm ver}$ is described by

$$\left\{\psi_1 = \pi \ , \ \Psi_1 = \hat\Psi_1(\Lambda^*, \eta^*, \xi^*; \Psi_2) := \frac{\Psi_2}{2} + \frac{(\Lambda_1^* - H_1^*)^2 - (\Lambda_2^* - H_2^*)^2}{2\Psi_2} \ , \right\} .$$

Since $\phi(\mathcal{M}_{\rm ver})$ is invariant for the flow $\phi_*^t$ of $\mathcal{H}_*$, $\dot\psi_1 \equiv 0$ and $\psi_1(t) \equiv \pi$ for motions starting on $\phi(\mathcal{M}_{\rm ver})$; this is equivalent to $(\partial_{\Psi_1}\mathcal{H}_*)|_{\phi(\mathcal{M}_{\rm ver})} = 0$. It is now easy to check that the (standard) Hamilton equations for

$$\mathcal{H}_{\rm red}^c(\Lambda^*, \lambda^*, \eta^*, \xi^*) := \mathcal{H}_*\Big(\Lambda^*, \eta^*, \big(\hat\Psi_1(\Lambda^*, \eta^*, \xi^*; c), c\big), \lambda^*, \xi^*, (\pi, \psi_2)\Big)$$

(recall that $\mathcal{H}_*$ is independent of $\psi_2$) are a subsystem of the full (standard) Hamilton equations for $\mathcal{H}_*$ when the initial data are restricted to $\phi(\mathcal{M}_{\rm ver})$ and the initial (and constant) value for $\Psi_2$ is $c$. Note that the evolution of $\Psi_1$ is then obtained via the relation $\Psi_1 = \hat\Psi_1$ while the evolution of $\psi_2$ is obtained by a quadrature ($\Psi_2 = c$ and $\psi_1 = \pi$ are constant). The claim now follows by computing explicitly $\mathcal{H}_{\rm red}^c$ via the relations in ($*$) and taking the arbitrary value for $\psi_2$ to be $\pi/2$. □

We take this opportunity to correct also the following minor points:

- Point (ii) of Remark 1.1 has to be amended according to the above discussion. In particular the sentence "called Poincaré integrals in Appendix C" at line 10 from the bottom of page 96 and the word "Poincaré" on the last line of the same page have to be disregarded. In the same spirit, the last sentence ("we also anticipate... (Poincaré integrals)") of footnote 27 should be ignored.
- In equation (C.18) sin and cos must be exchanged.
- Formula after (C.31), page 127: the suffixes 1,2,3 have to be replaced by 0,1,2.

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