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Quasi-Periodic Attractors in Celestial Mechanics

Alessandra Celletti & Luigi Chierchia

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Abstract

We prove that KAM tori smoothly bifurcate into quasi-periodic attractors in dissipative mechanical models, provided external parameters are tuned with the frequency of the motion. An application to the dissipative spin–orbit model of celestial mechanics (which actually motivated the analysis in this paper) is presented.

Contents

1. Introduction and results

In physical examples, Hamiltonian dynamics typically arise through dissipative systems with very small dissipation. It is therefore natural to ask which part of the Hamiltonian dynamics smoothly persists when the dissipation is turned on. In particular, if the reference Hamiltonian system is nearly integrable, a basic question is to discuss the fate of quasi-periodic trajectories, which, as KAM theory shows (see, for example, [1]), are common in the purely Hamiltonian nearly integrable regime.

In this paper we show that—under suitable assumptions relating external (physical) parameters with the motion frequencies—KAM tori smoothly bifurcate into quasi-periodic attractors when dissipative effects are taken into account.

1.1. Dissipative nearly integrable flows on $\mathbb{R} \times \mathbb{T}^2$

Motivated by the "dissipative spin–orbit model" of celestial mechanics (see Sect. 1.2 below), we first consider dissipative nearly integrable flows on $\mathbb{R} \times \mathbb{T}^2$, where $\mathbb{T}^2$ denotes the standard flat two-torus $\mathbb{R}^2/(2\pi\mathbb{Z}^2)$. Namely, we consider the differential equation

$$\ddot{x} + \eta(\dot{x} - \upsilon) + \varepsilon f_x(x, t) = 0, \tag{1.1}$$

where

- Dot stands for time derivative and $x = x(t)$.
- x and t are periodic variables: $(x, t) \in \mathbb{T}^2$, while the velocity $\dot{x} \in \mathbb{R}$.
- η is the "dissipation parameter": for $\eta > 0$ the system is dissipative, while for $\eta = 0$ the system is conservative (Hamiltonian); values $\eta < 0$ are also allowed.
- υ is an "external parameter" (in the spin–orbit problem it will be related to the eccentricity of the reference Keplerian orbit).
- ε measures the size of the perturbation (for $\varepsilon = 0$ the system is integrable).
- The potential f is a given real-analytic function[1] on $\mathbb{T}^2$.

As mentioned above, for $\eta = 0$, (1.1) is the Lagrange equation for the nearly integrable Lagrangian

$$\mathcal{L}_\varepsilon(\dot{x}, x, t) := \frac{\dot{x}^2}{2} - \varepsilon f(x, t), \quad (\dot{x}, x, t) \in \mathbb{R} \times \mathbb{T}^2,$$

or, equivalently, corresponds to the Hamiltonian equation

$$\dot{y} = -\partial_x H_\varepsilon, \quad \dot{x} = \partial_y H_\varepsilon,$$

for the nearly integrable Hamiltonian $H_\varepsilon(y, x, t) := \frac{y^2}{2} + \varepsilon f(x, t)$ defined on $\mathbb{R} \times \mathbb{T}^2$.

In the conservative case $\eta = 0$, standard KAM theory (see for example, [1]) implies that[2] if ε is small enough, (1.1) admits many quasi-periodic solutions, that is, solutions of the form

$$x(t) = \omega t + u(\omega t, t) \tag{1.2}$$

with $u(\theta) = u(\theta_1, \theta_2)$ real-analytic on $\mathbb{T}^2$ and ω *Diophantine*:

$$|\omega n_1 + n_2| \geqq \frac{\kappa}{|n_1|^\tau}, \quad \forall\, (n_1, n_2) \in \mathbb{Z}^2,\ n_1 \neq 0, \tag{1.3}$$

[1] "Finitely differentiable function" suffices, but we focus on the real-analytic case in view of our application to celestial mechanics (and for simplicity).

[2] Note that for $\varepsilon = 0$, the Hamiltonian $H_0 = \frac{y^2}{2}$ is non-degenerate in the sense of KAM theory; compare [1].

for some $\kappa, \tau > 0$. Furthermore, such solutions are analytic in ε and are Whitney smooth in[3] ω. In the following, $\mathcal{D}_{\kappa,\tau}$ denotes the set of Diophantine numbers in $\mathbb{R}$ satisfying[4] (1.3).

On the other hand, when $\eta \neq 0$ and $\varepsilon = 0$, the general solution of (1.1) is given by

$$x(t) = x_0 + \upsilon\, t + \frac{1 - \exp(-\eta t)}{\eta}\,(v_0 - \upsilon),$$

showing that the periodic solution (remember that x is an angle) $x = k + \upsilon t$ (with k constant) and $\dot{x} \equiv \upsilon$ is a *global attractor* for the dynamics.

The *leitmotif* is that KAM quasi-periodic solutions for $\eta \neq 0$, smoothly bifurcate into quasi-periodic attractors[5] provided the "external frequency" υ is tuned with the "internal frequency" ω in (1.2). This is the content of the first result:

Theorem 1. *Fix $0 < \kappa < 1 \leqq \tau$ and $\eta_0 > 0$. Then there exists $0 < \varepsilon_0 < 1$ such that for any $\varepsilon \in [0, \varepsilon_0]$, any $\eta \in I_0 := [-\eta_0, \eta_0]$ and any $\omega \in \mathcal{D}_{\kappa,\tau}$ there exists a unique function*[6]

$$u = u_\varepsilon(\theta; \eta, \omega) = O(\varepsilon), \quad \langle u \rangle := \int_{\mathbb{T}^2} u \,\frac{d\theta}{(2\pi)^2} = 0,$$

such that $x(t)$ in (1.2) solves (1.1) with

$$\upsilon = \omega\,\Big(1 + \big\langle (u_{\theta_1})^2 \big\rangle\Big)\,. \tag{1.4}$$

Furthermore, the function u_ε is smooth in the sense of Whitney in all its variables, is real-analytic in $\theta \in \mathbb{T}^2$ and ε, C^∞ in η and Whitney C^∞ in ω.

Remark 1. (i) Uniqueness has to be understood in the following sense: if one is given a second solution $\tilde{x}(t) = \omega t + \tilde{u}(\omega t, t)$ of (1.1) for some $\upsilon \in \mathbb{R}$, with $\tilde{u} = \tilde{u}_\varepsilon(\theta; \eta, \omega) = O(\varepsilon)$ real-analytic in (θ, ε) and having vanishing average over $\mathbb{T}^2$, then $\tilde{u} \equiv u$ and υ is as in (1.4).

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[3] A function $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is Whitney C^k or C^k_W if it is the restriction on A of a $C^k(\mathbb{R}^n)$ function; for the original definition by H. Whitney and for relevance in dynamical system, see for example, [2]. Incidentally, we mention that Whitney smoothness was discussed for the first time in the framework of conservative dynamical systems in [9] and later in [7] and independently in [12].

[4] Observe that if (1.3) holds, then $0 < \kappa < 1$ and $\tau \geqq 1$. In fact, taking $n_1 = 1$ and $n_2 = -[\omega]$ ($[x]$ = integer part of x) in (1.3) shows that $\kappa < 1$, while the fact that $\tau \geqq 1$ comes from Liouville's theorem on rational approximations ("*For any $\omega \in \mathbb{R}\backslash\mathbb{Q}$ and for any $N \geqq 1$ there exist integers p and q with $|q| \leqq N$ such that $|\omega q - p| < 1/N$*"). Finally, we recall that when $\tau > 1$, $\bigcup_{\kappa>0} \mathcal{D}_{\kappa,\tau}$ is a set of full Lebesgue measure.

[5] In general, in the non-integrable regime, such attractors will be only *local attractors*.

[6] As usual, $f = O(x^k)$ means that f is a smooth function of x having equal to zero the first k derivatives at $x = 0$; here $\theta \in \mathbb{T}^2$ corresponds to the variables (x, t), or more precisely, θ denotes the variables in which the (x, t) motion linearizes.

(ii) The time-derivative for $x(t)$ corresponds to the directional derivative

$$\partial_\omega := \omega \frac{\partial}{\partial \theta_1} + \frac{\partial}{\partial \theta_2}, \tag{1.5}$$

for the function $u(\theta)$, so that being the flow $\theta \in \mathbb{T}^2 \to \theta + (\omega t, t)$ dense in $\mathbb{T}^2$, one sees that $x(t)$ *as in* (1.2) *is a quasi-periodic solution of* (1.1) *if and only if u solves the following PDE* on $\mathbb{T}^2$:

$$\partial_\omega^2 u + \eta\, \partial_\omega u + \varepsilon\, f_x(\theta_1 + u, \theta_2) + \gamma = 0, \quad \gamma = \eta(\omega - \upsilon). \tag{1.6}$$

This equation will actually be the main object of investigation of this paper.

(iii) Theorem 1 implies that the two-torus

$$\mathcal{T}_{\varepsilon,\eta}(\omega) := \{(x,t) = (\theta_1 + u_\varepsilon(\theta; \eta, \omega), \theta_2) \ : \ \theta \in \mathbb{T}^2\}, \tag{1.7}$$

is a *quasi-periodic attractor* for the dynamics associated to (1.1) with υ as in (1.4) and that the dynamics on $\mathcal{T}_{\varepsilon,\eta}(\omega)$ is analytically conjugated to the linear flow $\theta \to \theta + (\omega t, t)$; compare also point (i) of Remark 3 below.

(iv) The result is perturbative in ε but it is *uniform* in η. It is particularly noticeable the smooth dependence of u_ε on η as $\eta \to 0$, which shows that the invariant KAM torus $\mathcal{T}_{\varepsilon,0}(\omega)$ smoothly bifurcates into the attractor (1.7) as $\eta \neq 0$.

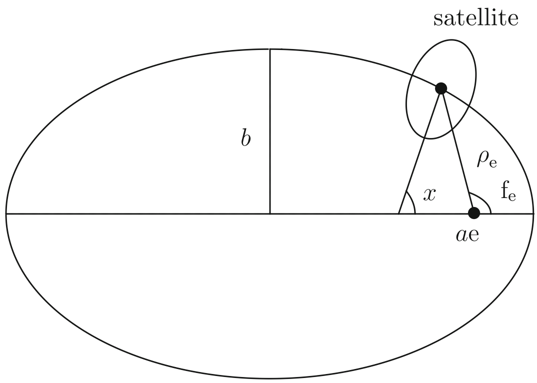
(v) Theorem 1 will be obtained as a corollary of a "dissipative Nash–Moser" theorem (see Section 1.3 below), after having rewritten Equation (1.6) as a functional equation. Indeed, the method of proof is rather robust and general and could be easily adapted to cover dissipative maps such as the "fattened Arnold family" studied in [3] or it could be extended to systems with more degrees of freedom. In this second case however, it would be important, in our opinion, to motivate physically the form of the dissipation, a problem in itself difficult and intriguing.

1.2. The dissipative spin–orbit model

We turn now to the mechanical problem that motivated this paper, namely, the dissipative spin–orbit problem. Such a problem consists of studying the rotations of a triaxial non-rigid body (satellite), having the satellite's center of mass revolving on a given Keplerian ellipse, and subject to the gravitational attraction of a major body sitting on a focus of the ellipse.

To simplify the analysis, we assume that the satellite is symmetric with respect to an "equatorial plane" and study motions having the equatorial plane coinciding with the Keplerian orbit plane. Because of the assumed symmetry of the satellite, such motions belong to an invariant submanifold of the phase space. Furthermore, following Correia and Laskar [8], we consider a "viscous tidal model, with a linear dependence on the tidal frequency". The dissipation in such model is meant to reflect the averaged effect of tides on the motion (see also, Remark 2 (i) below).

Under such hypotheses, the motions of the satellite are described by the angle x formed by, for example, the direction of the major physical axis of the satellite (assumed to lie in the equatorial plane) with a fixed axis of the Keplerian orbit plane (for example the direction of the semimajor axis of the ellipse),



and the differential equation governing the motion of the satellite in suitable units is given by (1.1) with

$$\begin{cases} \eta = K\Omega_{\rm e}, \quad \upsilon = \upsilon_{\rm e}, \quad \varepsilon = \frac{3}{2}\,\frac{B-A}{C}, \\ f = f(x,t;{\rm e}) := -\dfrac{1}{2\,\rho_{\rm e}(t)^3}\cos\left(2x - 2\,{\rm f_e}(t)\right), \end{cases} \tag{1.8}$$

where

- e $\in [0, 1)$ is the eccentricity of the Keplerian orbit on which the center of mass of the satellite is revolving;
- $K \geqq 0$ is a physical constant depending on the internal (non-rigid) structure of the satellite;
- $\Omega_{\rm e} > 0$, $N_{\rm e} > 0$ and $\upsilon_{\rm e} > 1$ are known functions of the eccentricity e $\in [0, 1)$ and are given by:

$$\Omega_{\rm e} := \left(1 + 3{\rm e}^2 + \frac{3}{8}{\rm e}^4\right)\frac{1}{(1-{\rm e}^2)^{9/2}}, \tag{1.9}$$

$$N_{\rm e} := \left(1 + \frac{15}{2}{\rm e}^2 + \frac{45}{8}{\rm e}^4 + \frac{5}{16}{\rm e}^6\right)\frac{1}{(1-{\rm e}^2)^6};$$

$$\upsilon_{\rm e} := \frac{N_{\rm e}}{\Omega_{\rm e}} = 1 + 6{\rm e}^2 + \frac{3}{8}{\rm e}^4 + O({\rm e}^6). \tag{1.10}$$

- $0 < A < B < C$ are the principal moments of inertia of the satellite;
- $\rho_{\rm e}(t)$ and ${\rm f_e}(t)$ are, respectively, the (normalized) orbital radius and the true anomaly of the Keplerian motion, which (because of the assumed normalizations) are 2π-periodic functions of time t. The explicit expression for $\rho_{\rm e}$ and ${\rm f_e}$ may

be described as follows. Let $\mathrm{u} = \mathrm{u_e}(t)$ be the 2π-periodic function obtained by inverting

$$t = \mathrm{u} - \mathrm{e}\sin\mathrm{u}, \quad (\text{“Kepler's equation”}); \tag{1.11}$$

then

$$\begin{aligned} \rho_{\mathrm{e}}(t) &= 1 - \mathrm{e}\cos\mathrm{u_e}(t) \\ \mathrm{f_e}(t) &= 2\arctan\left(\sqrt{\frac{1+\mathrm{e}}{1-\mathrm{e}}}\tan\frac{\mathrm{u_e}(t)}{2}\right). \end{aligned} \tag{1.12}$$

Remark 2. (i) The derivation of the conservative spin–orbit model (that is, Equations (1.1) & (1.8) with $K = 0$) is classical and can be found, for example, in [4]: compare Equation (2.2) of [4] with the normalization $n := \sqrt{(Gm)/a^3} = 1$.
The derivation of the dissipative contribution is less straightforward and, as mentioned above, we follow Correia and Laskar [8]: compare (3) and (4) in [8], where Ω_{e} and N_{e} are denoted, respectively, $\Omega(e)$ and $N(e)$. Essentially, the dissipative term is given by the average over one revolution period (that is, 2π with our normalization) of the so-called MacDonald's torque [10]; compare [11], where the functions Ω_{e} and N_{e} are denoted, respectively, by $f_1(e)$ and $f_2(e)$.

(ii) For $K = 0$, equations (1.1) & (1.8) correspond to the Hamiltonian flow associated to the one-and-a-half degree-of-freedom Hamiltonian

$$H(y, x, t) := \frac{1}{2}y^2 - \frac{\varepsilon}{2\,\rho_{\mathrm{e}}(t)^3}\cos\left(2x - 2\,\mathrm{f_e}(t)\right),$$

(y, x) being standard symplectic variables. The associated “spin–orbit” Hamiltonian system is non-integrable if $\varepsilon > 0$ *and*[7] $\mathrm{e} > 0$.

(iii) The external frequency υ_{e} is a real-analytic invertible function of e mapping $(0, 1)$ onto $(1, \infty)$; we denote by $\upsilon^{-1} : (1, \infty) \to (0, 1)$ the inverse map (which is also real-analytic). The invertibility of the frequency map υ_{e} will play the rôle of a *nondegeneracy condition*, allowing us to fix the eccentricities for which quasi-periodic attractors exist in the full dynamics.

(iv) In many examples taken from the Solar system, both ε and K are small. For example, for the Earth–Moon system and for the Sun–Mercury system ε is of the order of 10^{-4}, while K is of the order of 10^{-8}.

(v) Mercury is observed in a nearly 3:2 spin–orbit resonance (that is, it rotates three times on its spin axis, while making two orbital revolutions around the Sun) and is moving on a nearly Keplerian orbit with eccentricity $e \simeq 0.2$. However, Correia and Laskar's numerical investigations [8] show that “the chaotic evolution of Mercury's orbit can drive its eccentricity beyond 0.325 during the planet's history”. Now, if υ^{-1} is the function introduced in point

[7] When $\mathrm{e} = 0$, $\mathrm{u}_0(t) = t = \mathrm{f}_0(t)$, $\rho_0 = 1$ so that $H = \frac{1}{2}y^2 - \frac{\varepsilon}{2}\cos(2x - 2t)$, which is easily seen to be integrable.

(iii) above, $\upsilon^{-1}(3/2) = 0.284\ldots$. Thus, Theorem 1 for the dissipative spin–orbit model (spelled out in Theorem 2 below) might give new mathematical insight into the question of the so-called "capture in the 3:2 resonance" of Mercury (see [8]) suggesting that such capture is related to the existence of an underlying attractor with Diophantine frequency close to 3/2. For more information about this point, see [6].

(vi) As mentioned in point (ii), Remark 1, quasi-periodic trajectories for (1.1) & (1.8) correspond to solutions of the PDE

$$\partial_\omega^2 u + K\Omega_{\rm e}\,\partial_\omega u + \frac{\varepsilon}{\rho_{\rm e}(\theta_2)^3}\sin\left(2(\theta_1 + u(\theta)) - 2\,{\rm f}_{\rm e}(\theta_2)\right) = KN_{\rm e} - K\Omega_{\rm e}\omega. \tag{1.13}$$

The next result translates Theorem 1 in the case of the dissipative spin–orbit model.

Theorem 2. *Fix $\kappa, r \in (0,1)$ and $\tau \geqq 1$. There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in [0, \varepsilon_0]$, any $K \in [0,1]$ and any $\omega \in \mathcal{D}_{\kappa,\tau} \cap [1+r, 1/r]$, there exist unique functions*[8]

$$ {\rm e}_\varepsilon = {\rm e}_\varepsilon(K,\omega) = \upsilon^{-1}(\omega) + O(\varepsilon^2), \quad u = u_\varepsilon(\theta; K, \omega) = O(\varepsilon),$$

with $\int_{\mathbb{T}^2} u\, d\theta = 0$, satisfying (1.13) with ${\rm e} = {\rm e}_\varepsilon$. The functions ${\rm e}_\varepsilon$ and u_ε are smooth in the sense of Whitney in all their variables and are real-analytic in $\theta \in \mathbb{T}^2$ and ε, C^∞ in K and Whitney C^∞ in ω.

1.3. A dissipative Nash–Moser theorem

We, now, state the main technical result, namely, an existence and uniqueness[9] Nash–Moser (or KAM) theorem for dissipative/conservative flows on a two-torus. In this theorem the potential is not assumed to be small but rather, we assume to start with a good enough approximate solution. Special care is devoted to the dependence of solutions upon the dissipative parameter which appears explicitly in the small divisor problems involved.

More specifically, Theorem 3 below deals with finding real-analytic "local" solutions $u : \mathbb{T}^2 \to \mathbb{R}$ and a number γ such that[10]

$$\begin{cases} \partial_\omega^2 u + \eta\,\partial_\omega u + g_x(\theta_1 + u, \theta_2) + \gamma = 0, \\ \langle u \rangle = 0, \quad 1 + u_{\theta_1} \neq 0. \end{cases} \tag{1.14}$$

[8] The map υ^{-1} is the inverse map of ${\rm e} \to \upsilon_{\rm e}$ defined in (1.10).

[9] While KAM procedures are abundant in literature, local uniqueness is seldom discussed.

[10] ∂_ω is defined in (1.5) and $\langle u \rangle := \int_{\mathbb{T}^2} u(\theta) \frac{d\theta}{(2\pi)^2}$.

Remark 3. (i) While the first condition in the second line of (1.14) is just a normalization condition needed for uniqueness, the second one implies that the map

$$\theta \in \mathbb{T}^2 \to (\theta_1 + u(\theta), \theta_2) \in \mathbb{T}^2$$

is a diffeomorphism and therefore, if u solves (1.14), the set

$$\{(y, (x, t)) = (\omega + \partial_\omega u, (\theta_1 + u, \theta_2)) \; : \; \theta \in \mathbb{T}^2\}$$

is a two-dimensional torus embedded in the three-dimensional phase space $\mathbb{R} \times \mathbb{T}^2$ which is invariant for the dynamics generated by the differential equation

$$\ddot{x} + \eta\,(\dot{x} - \omega) + g_x(x, t) + \gamma = 0, \tag{1.15}$$

meaning that for each $\theta \in \mathbb{T}^2$,

$$t \to x(t; \theta) := \theta_1 + \omega t + u(\theta_1 + \omega t, \theta_2 + t)$$

is a solution of (1.15).

(ii) As mentioned above, the unknowns of (1.14) are u and $\gamma \in \mathbb{R}$. Indeed, we shall see below that u and γ are *not independent* as they satisfy the relation

$$\eta\,\omega \left\langle \left(u_{\theta_1}\right)^2 \right\rangle + \gamma = 0. \tag{1.16}$$

On the other hand, ω and η are regarded as external parameters taken, respectively in $\mathcal{D}_{\kappa,\tau}$ (for some prefixed $0 < \kappa < 1 \leqq \tau$) and in a compact interval $[-\eta_0, \eta_0]$ (for some prefixed $\eta_0 > 0$).

It will be useful to rewrite the differential equation in (1.14) as a functional equation involving parameters. For $\eta \in \mathbb{R}$ we define

$$\begin{aligned} &D_\eta : \; v \in C^1(\mathbb{T}^2) \to D_\eta v = \partial_\omega v + \eta v, \\ &\Delta_\eta := D_\eta \partial_\omega = \partial_\omega D_\eta, \\ &\mathcal{F}_\eta : (v, \gamma) \in C^2(\mathbb{T}^2) \times \mathbb{R} \to \mathcal{F}_\eta(v; \gamma) \\ &\quad := \Delta_\eta v + g_x(\theta_1 + v, \theta_2) + \gamma \end{aligned} \tag{1.17}$$

(notice that $D_0 = \partial_\omega$). Then problem (1.14) may be rewritten as

$$\begin{cases} \mathcal{F}_\eta(u; \gamma) = 0, \\ \langle u \rangle = 0, \quad 1 + u_{\theta_1} \neq 0. \end{cases} \tag{1.18}$$

In order to state existence and uniqueness theorems for (1.18), we need to introduce suitable function spaces which, because of our motivating model, will be spaces of real-analytic functions.

Let us denote by $\mathcal{H}^\xi$ the Banach space of continuous functions $u : \mathbb{T}^2 \to \mathbb{R}$ with finite norm[11]

$$\|u\|_\xi := \sum_{n\in\mathbb{Z}^2} |u_n| \; \mathbb{e}^{|n|\xi},$$

where u_n denotes the nth Fourier coefficient and $|n| = |n_1| + |n_2|$.

Let us also denote by $\mathcal{H}_0^\xi$ the closed subspace of $\mathcal{H}^\xi$ formed by functions with zero average:

$$\mathcal{H}_0^\xi := \{u \in \mathcal{H}^\xi \text{ s.t. } u_0 = \langle u \rangle = 0\}.$$

Remark 4. (i) The spaces $\{\mathcal{H}^\xi : \xi \geqq 0\}$ form a nested family of Banach spaces as

$$0 \leqq \xi' < \xi \Longrightarrow \mathcal{H}^\xi \subsetneqq \mathcal{H}^{\xi'} \quad \text{and} \quad \|u\|_{\xi'} \leqq \|u\|_\xi.$$

(ii) Clearly, functions in $\mathcal{H}^\xi$ can be analytically extended to the complex multi-strip

$$\mathbb{T}_\xi^2 := \{\theta \in \mathbb{C}^2 : |\operatorname{Im}\theta_i| < \xi, \quad i = 1, 2\}$$

and

$$\sup_{\mathbb{T}_\xi^2} |u| \leqq \|u\|_\xi.$$

Conversely, if u is a real-analytic function on $\mathbb{T}^2$, then its Fourier coefficients decay exponentially fast and therefore there exists $\xi > 0$ such that $u \in \mathcal{H}^\xi$.

(iii) From the definition of the norm $\|\cdot\|_\xi$, it follows that if $u \in \mathcal{H}_0^\xi$ then[12]

$$\|\partial_\theta^h u\|_\xi \leqq \|\partial_\theta^k u\|_\xi, \quad \forall\, h, k \in \mathbb{N}^2 : \; h_i \leqq k_i;$$

the same inequalities hold also for $u \in \mathcal{H}^\xi$ when $h \neq 0$.

(iv) D_η is a "diagonal" operator on Fourier spaces mapping $\mathcal{H}^\xi$ into $\mathcal{H}^{\xi'}$ for any[13] $0 \leqq \xi' < \xi$:

$$D_\eta u = D_\eta \left(\sum_{n\in\mathbb{Z}^2} u_n \; \mathbb{e}^{in\cdot\theta} \right) = \sum_{n\in\mathbb{Z}^2} \lambda_{\eta,n}\, u_n \; \mathbb{e}^{in\cdot\theta},$$

where

$$\lambda_{\eta,n} := i(\omega n_1 + n_2) + \eta. \tag{1.19}$$

[11] The letter $\mathbb{e}$ denotes the Neper number $\exp(1) = 2.71828\ldots$; do not get confused with the letter e used for eccentricity and the letter e, which will be used below for the "error" function $e(\theta)$.

[12] As usual, if $h \in \mathbb{N}^2$, $\partial_\theta^h := \partial_{\theta_1}^{h_1} \partial_{\theta_2}^{h_2}$.

[13] As usual $n \cdot \theta$ denotes the usual inner product $n_1\theta_1 + n_2\theta_2$.

If $\eta \neq 0$, D_η is bounded (since $|\lambda_{\eta,n}| \geqq |\eta|$), invertible and its inverse, D_η^{-1}, maps $\mathcal{H}^\xi$ onto itself:

$$D_\eta^{-1} : \mathcal{H}^\xi \to \mathcal{H}^\xi, \quad \forall\, \eta \neq 0,\ \xi \geqq 0$$
$$D_\eta^{-1} u = \sum_{n\in\mathbb{Z}^2} \lambda_{\eta,n}^{-1}\, u_n\ \mathrm{e}^{in\cdot\theta}.$$

Notice, however, that the limit $\eta \to 0$, which is of particular interest to us, is singular (compare also, the next point).

(v) If $\eta = 0$, then

$$D_0 = \partial_\omega : \mathcal{H}^\xi \to \mathcal{H}_0^{\xi'}, \quad (0 \le \xi' < \xi);$$

since $\omega \in \mathcal{D}_{\kappa,\tau}$, D_0 is invertible on $\mathcal{H}_0^\xi$:

$$D_0^{-1} : u \in \mathcal{H}_0^\xi \quad \longrightarrow \quad \sum_{n\in\mathbb{Z}^2\setminus\{0\}} \lambda_{0,n}^{-1} u_n\, \mathrm{e}^{in\cdot\theta} \in \mathcal{H}_0^{\xi'}, \quad (0 \le \xi' < \xi).$$

(vi) We finally recall two elementary properties concerning the product and the composition of functions in $\mathcal{H}^\xi$. If $u, v \in \mathcal{H}^\xi$, then $uv \in \mathcal{H}^\xi$ and

$$\|uv\|_\xi \leqq \|u\|_\xi\ \|v\|_\xi.$$

As for composition, if $0 \leqq \xi < \bar\xi$ and $f \in \mathcal{H}^{\bar\xi}$, $v \in \mathcal{H}^\xi$ with $\|v\|_\xi \leqq \bar\xi - \xi$, then one has $\theta \to F(\theta) := f(\theta_1 + v(\theta), \theta_2) \in \mathcal{H}^\xi$ and

$$\|F\|_\xi \leqq \|f\|_{\bar\xi}.$$

For the proof, see for example, [5], p. 426, 427.

We are now ready to formulate the main technical result.

Theorem 3. *Let* $0 < \xi_* < \xi < \bar\xi \leqq 1$*; let* $0 < \kappa < 1 \leqq \tau$*; let* $I_0 := [-\eta_0, \eta_0]$ *for some* $\eta_0 > 0$*; let* $\omega \in \mathcal{D}_{\kappa,\tau}$ *and* $(x,t) \in \mathbb{T}^2 \to g(x,t) \in \mathcal{H}^{\bar\xi}$*; let* $M > 0$ *be such that*

$$\|\partial_x^3 g\|_{\bar\xi} \leqq M; \tag{1.20}$$

let, also, $0 < \nu < \bar\xi - \xi$, $0 < \alpha < 1$ *and* $0 < \sigma < 1$.

Then there exist a constant $k = k(\xi, \xi_*, \kappa, \tau, \eta_0, M, \nu, \alpha, \sigma) > 1$ *such that the following holds.*

Assume that there exist functions $v = v(\theta; \eta) \in C^\infty(\mathbb{T}^2 \times I_0)$ *and* $\beta = \beta(\eta) \in C^\infty(I_0)$ *("initial approximate solution") satisfying the following hypotheses:*

(H1) *for each* $\eta \in I_0$*, the function* $\theta \to v(\theta;\eta)$ *belongs to* $\mathcal{H}_0^{\xi}$*;*
(H2) $\|v_{\theta_1}\|_{\xi} \leqq \sigma\nu$ *for any* $\eta \in I_0$*;*
(H3) *define*[14]

$$V := 1 + v_{\theta_1}, \quad W := V^2, \quad \rho := \eta \frac{\left\langle W^{-1} D_\eta^{-1} v_{\theta_1} \right\rangle}{\left\langle W^{-1} \right\rangle}, \tag{1.21}$$

and assume that for any $\eta \in I_0$*,*

$$|\rho| \leqq \sigma\alpha; \tag{1.22}$$

(H4) *assume that for any* $\eta \in I_0$*, the "error function*[15]*"*

$$e = e(\theta;\eta) := \mathcal{F}_\eta(v;\beta), \tag{1.23}$$

satisfies the smallness condition

$$k\,\|e\|_{\xi} \leqq 1. \tag{1.24}$$

Under the hypotheses (H1)–(H4)*, there exist functions* $u = u(\theta;\eta) \in C^\infty(\mathbb{T}^2 \times I_0)$ *and* $\gamma = \gamma(\eta) \in C^\infty(I_0)$ *such that* (1.18) *holds. Furthermore, for each* $\eta \in I_0$*,* $u(\cdot;\eta) \in \mathcal{H}_0^{\xi_*}$ *and, if* V_**,* W_* *and* ρ_* *are defined as in* (1.21) *with* v *replaced by* u*, then for each* $\eta \in I_0$*, one has*

$$\|u_{\theta_1}\|_{\xi_*} \leqq \nu, \quad |\rho_*| \leqq \alpha, \tag{1.25}$$

$$|\gamma - \beta|\,,\ \|u_{\theta_1} - v_{\theta_1}\|_{\xi_*}\,,\ |\rho - \rho_*|\,,\ \|W - W_*\|_{\xi_*} \leqq k\,\|e\|_{\xi}\,, \tag{1.26}$$

The functions u *and* γ *are unique in the following sense. If* $u' = u'(\theta;\eta) \in C^\infty(\mathbb{T}^2 \times I_0)$ *and* $\gamma' = \gamma'(\eta) \in C^\infty(I_0)$ *are also solutions of* (1.18)*, that is,* $\mathcal{F}_\eta(u';\gamma') = 0$ *and* u' *is such that for any* $\eta \in I_0$*,*

$$\begin{aligned} &\theta \to u'(\theta;\eta) \in \mathcal{H}_0^{\xi_*}, \\ &\|u'_{\theta_1}\|_{\xi_*} \leqq \nu, \\ &k\,\|u' - v\|_{\xi_*} \leqq 1 \end{aligned} \tag{1.27}$$

then $u' = u$ *and* $\gamma' = \gamma$*.*

Remark 5. (i) The proof of this theorem is fully constructive: the solution is obtained as a limit $(u,\gamma) = \lim(v_j,\beta_j)$ where $(v_0,\beta_0) = (v,\beta)$ is the starting approximate solution and (v_j,β_j) are quadratically better and better approximations to the solution (u,γ). Details on how to evaluate the constant k are given along the proof.
(ii) The assumptions that all the ξ's are smaller than one and that $\nu < 1$ are not needed and are made only to simplify the exposition. It would suffice to assume that $1 + v_{\theta_1}$ never vanishes.

[14] Since $\sigma\nu < 1$, hypothesis (H2) implies that $V > 0$ (and hence $W > 0$) for all θ and η.
[15] Recall the definitions given in (1.17).

(iii) The smooth dependence upon η around $\eta = 0$ (the conservative case) is one of the main points of the theorem: it shows that KAM tori in the conservative case bifurcate smoothly into dissipative attractors, keeping the dynamics conjugated to the linear flow $\theta \to \theta + (\omega t, t)$. Indeed, also the dependence upon the frequencies $\omega \in \mathcal{D}_{\gamma,\tau}$ is smooth, as explained briefly in Remark 7 below. The rôle of the bifurcation parameter is played by γ and in the application to the spin–orbit problem by the eccentricity e of the reference Keplerian orbit.

2. Proofs

2.1. Preliminaries

Here we discuss some more properties of the small divisor operators D_η and Δ_η and prove the compatibility condition (1.16).

Lemma 1. (i) *If $u \in C^2(\mathbb{T}^2)$, then*

$$\langle D_\eta u\rangle = \eta\langle u\rangle, \tag{2.28}$$

$$\Big\langle u_{\theta_1}\,\Delta_\eta u\Big\rangle = \eta\omega\Big\langle\Big(u_{\theta_1}\Big)^2\Big\rangle. \tag{2.29}$$

(ii) *If $v, V \in C^2(\mathbb{T}^2)$ and $V(\theta) \neq 0\ \forall\,\theta \in \mathbb{T}^2$ then*

$$V\Delta_\eta v - v\Delta_\eta V = D_\eta\left(V^2 D_0(\frac{v}{V})\right). \tag{2.30}$$

(iii) *Let $\omega \in \mathcal{D}_{\kappa,\tau}$; let $\xi > \xi' \geqq 0$; let $\eta \in \mathbb{R}$ and let p and s be non negative integers. Then, for any $u \in \mathcal{H}_0^\xi$,*

$$D_\eta^{-1} : u \in \mathcal{H}_0^\xi \longrightarrow \sum_{n\in\mathbb{Z}^2\setminus\{0\}} \lambda_{\eta,n}^{-1} u_n\, e^{in\cdot\theta} \in \mathcal{H}_0^{\xi'},$$

$$\|D_\eta^{-s}\,\partial_{\theta_1}^p u\|_{\xi'} \leqq \sigma_{p,s}(\xi-\xi')\,\|u\|_\xi, \tag{2.31}$$

where $\lambda_{\eta,n} = i(\omega n_1 + n_2) + \eta$ (compare (1.19)) and

$$\sigma_{p,s}(\delta) := \sigma_{p,s}(\delta;\omega,\eta) := \sup_{n\in\mathbb{Z}^2\setminus\{0\}} \left(|i(\omega n_1+n_2)+\eta|^{-s}\,|n_1|^p\, e^{-\delta|n|}\right).$$

Furthermore,

$$\sigma_{p,s}(\delta) \leqq \frac{1}{\kappa^s\,\delta^{s\tau+p}}\left(\frac{s\tau+p}{e}\right)^{s\tau+p}. \tag{2.32}$$

Finally, if $p > 0$, (2.31) holds for any $u \in \mathcal{H}^\xi$.

(iv) *Let* $B := \mathbb{T}^2 \times I_0$. *Let* $(\theta, \eta) \in B \to u(\theta; \eta)$ *belong to* $C^\infty(B)$ *and* $\theta \to u(\theta; \eta)$ *belong to* $\mathcal{H}^\xi$ *for some* $\xi > 0$ *and any* $\eta \in I_0$. *Assume that*

$$u_0(0) := \langle u(\cdot; 0)\rangle = 0,$$

and let $\omega \in \mathcal{D}_{\kappa,\tau}$. *Then,* $D_\eta^{-1}u \in C^\infty(B)$*;* $\theta \to D_\eta^{-1}u(\theta; \eta)$ *belongs to* $\mathcal{H}^{\xi'}$ *for any* $0 \leqq \xi' < \xi$ *and any* $\eta \in I_0$. *Furthermore,*

$$\begin{aligned} &D_\eta^{-1}u = \frac{u_0}{\eta} + D_\eta^{-1}(u - u_0),\\ &D_\eta^{-1}u(\theta; 0) = -\partial_\eta u_0(0) + D_0^{-1}u(\theta; 0), \qquad (2.33)\\ &\|D_\eta^{-1}u(\cdot; \eta)\|_{\xi'} \leqq \left|\frac{u_0(\eta)}{\eta}\right| + \sigma_{0,1}(\xi - \xi') \left\|u(\cdot, \cdot; \eta) - u_0(\eta)\right\|_\xi. \end{aligned}$$

Proof. Equality (2.28) is obvious.

The operator $D_0^2\partial_{\theta_1}$ is skew-symmetric, hence, by integration by parts,

$$\begin{aligned} \Big\langle u_{\theta_1}\, \Delta_\eta u\Big\rangle &= \Big\langle u_{\theta_1}\, D_0^2 u\Big\rangle + \eta\Big\langle u_{\theta_1}\, D_0 u\Big\rangle\\ &= \Big\langle D_0^2 u_{\theta_1}\, u\Big\rangle + \eta\Big\langle u_{\theta_1}\, D_0 u\Big\rangle\\ &= \eta\Big\langle u_{\theta_1}\, D_0 u\Big\rangle\\ &= \eta\omega\Big\langle (u_{\theta_1})^2\Big\rangle, \end{aligned}$$

which is (2.29).

(ii) Relation (2.30) follows from the definitions of D_η and Δ_η:

$$\begin{aligned} V\Delta_\eta v - v\Delta_\eta V &= VD_\eta D_0 v - vD_\eta D_0 V\\ &= VD_0^2 v - vD_0^2 V + \eta(VD_0 v - vD_0 V); \end{aligned}$$

on the other hand one has

$$\begin{aligned} D_\eta\left(V^2 D_0\left(\frac{v}{V}\right)\right) &= D_\eta(VD_0 v - vD_0 V)\\ &= VD_0^2 v - vD_0^2 V + \eta(VD_0 v - vD_0 V), \end{aligned}$$

from which (2.30) follows.

(iii) (2.31) follows immediately from the definitions of D_η^{-1} and of $\sigma_{p,s}$:

$$\begin{aligned} \|\partial_{\theta_1}^p D_\eta^{-s} u\|_{\xi'} &= \sum_{n\in\mathbb{Z}^2\setminus\{0\}} \left(|n_1|^p\, |i(\omega n_1 + n_2) + \eta|^{-s}\ \mathrm{e}^{-|n|(\xi-\xi')}\right)\ |u_n|\ \mathrm{e}^{|n|\xi}\\ &\leqq \sigma_{p,s}(\xi - \xi')\ \|u\|_\xi. \end{aligned}$$

Since

$$|i(\omega n_1 + n_2) + \eta| \geqq |\omega n_1 + n_2|,$$

(2.32) follows at once from the Diophantine estimate (1.3) and from the evaluation

$$\sup_{x>0}\left(x^a\ \mathrm{e}^{-x}\right) = \left(\frac{a}{\mathrm{e}}\right)^a,$$

valid for any $a \geqq 0$.

(iv): The Fourier coefficients

$$u_n(\eta) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} u(\theta; \eta) \; e^{-in\cdot\theta} \, d\theta$$

are $C^\infty(I_0)$ and by assumption,

$$\left\| u(\cdot; \eta) \right\|_\xi := \sum_{n\in\mathbb{Z}^2} |u_n(\eta)| \; e^{|n|\xi} < \infty, \quad \forall \eta \in I_0.$$

Therefore, by the assumption on ω, it follows immediately that $D_\eta^{-1}u$ belongs to $C^\infty(B)$ and that $D_\eta^{-1}u(\cdot, \cdot)$ belongs to $\mathcal{H}^{\xi'}$ for any $\eta \neq 0$ and $\xi' < \xi$; the evaluations in the first two lines of (2.33) show that the same is true *for any* $\eta \in I_0$ (and any $\xi' < \xi$). Last estimate in (2.33) follows at once from point (iii). □

Corollary 1. *Let* $\mathcal{F}_\eta$ *be as in* (1.17). *If* $u \in C^2(\mathbb{T}^2)$ *and* $\eta \in \mathbb{R}$, *then*

$$\left\langle \left(1 + u_{\theta_1}\right) \mathcal{F}_\eta(u; \gamma) \right\rangle = \eta\omega \left\langle \left(u_{\theta_1}\right)^2 \right\rangle + \gamma. \tag{2.34}$$

In particular, if $\mathcal{F}_\eta(u; \gamma) = 0$, *then* (1.16) *holds.*

Proof. First observe that by (2.28)

$$\langle \Delta_\eta u \rangle = \langle D_0(D_\eta u) \rangle = 0.$$

Observe also that

$$\left\langle (1 + u_{\theta_1}) g_x(\theta_1 + u, \theta_2) \right\rangle = \left\langle \partial_{\theta_1} \cdot g(\theta_1 + u, \theta_2) \right\rangle = 0.$$

By these observations and (2.29),

$$\begin{aligned}
\left\langle \left(1 + u_{\theta_1}\right) \mathcal{F}_\eta(u; \gamma) \right\rangle &= \left\langle \left(1 + u_{\theta_1}\right) \Delta_\eta u + \left(1 + u_{\theta_1}\right) g_x\left(\theta_1 + u, \theta_2\right) + \left(1 + u_{\theta_1}\right) \gamma \right\rangle \\
&= \left\langle u_{\theta_1} \Delta_\eta u + \partial_{\theta_1} \cdot g\left(\theta_1 + u, \theta_2\right) \right\rangle + \gamma \\
&= \left\langle u_{\theta_1} \Delta_\eta u \right\rangle + \gamma \\
&= \eta\omega \left\langle \left(u_{\theta_1}\right)^2 \right\rangle + \gamma,
\end{aligned}$$

proving the claims. □

2.2. Newton scheme

We describe now the Newton (KAM) scheme, on which the proof of Theorem 3 is based.

We start with a simple lemma on the differential $d\mathcal{F}$ of the operator

$$\mathcal{F}_\eta(v;\beta) = \Delta_\eta v + g_x(\theta_1 + v, \theta_2) + \beta,$$

which is given by[16]

$$d\mathcal{F}_{\eta,v} = \Delta_\eta + g_{xx}(\theta_1 + v, \theta_2). \tag{2.35}$$

Lemma 2. *Let $v \in C^2(\mathbb{T}^2)$ and $\beta \in \mathbb{R}$. Assume that*

$$V := 1 + v_{\theta_1} \neq 0, \quad \forall\, \theta \in \mathbb{T}^2, \tag{2.36}$$

and define

$$e := e(\theta;\eta) := \mathcal{F}_\eta(v;\beta),$$

$$W := V^2, \tag{2.37}$$

$$A_{\eta,v} : w \in C^2(\mathbb{T}^2) \to A_{\eta,v} w := V^{-1} D_\eta \left(W D_0 \left(V^{-1} w\right)\right) \in C(\mathbb{T}^2). \tag{2.38}$$

Then for every $w \in C^2(\mathbb{T}^2)$ and any $\eta \in \mathbb{R}$,

$$d\mathcal{F}_{\eta,v}(w) = A_{\eta,v} w + V^{-1} e_{\theta_1}\, w. \tag{2.39}$$

Notice that from (2.30) it follows that

$$A_{\eta,v} w = \Delta_\eta w - V^{-1} \Delta_\eta V\, w. \tag{2.40}$$

Proof of Lemma 2. From the definition of $e(\theta)$ it follows that

$$e_{\theta_1} = \Delta_\eta V + g_{xx}(\theta_1 + v, \theta_2) V. \tag{2.41}$$

Then by (2.35), (2.40) and (2.41) one sees that

$$\begin{aligned} d\mathcal{F}_{\eta,v}(w) &= \Delta_\eta w + g_{xx}(\theta_1 + v, \theta_2) w \\ &= A_{\eta,v} w + V^{-1}\left(\Delta_\eta V + g_{xx}(\theta_1 + v, \theta_2) V\right) w \\ &= A_{\eta,v} w + V^{-1} e_{\theta_1}\, w. \end{aligned}$$

□

The idea of a Newton scheme is to start with an approximate solution of the equation $\mathcal{F}_\eta(u;\gamma) = 0$, namely, a function $v : \mathbb{T}^2 \to \mathbb{R}$ and a number β such that

$$e := \mathcal{F}_\eta(v;\beta) := \Delta_\eta v + g_x(\theta_1 + v, \theta_2) + \beta$$

is small, and then to find a "quadratically better approximation"

$$v' = v + w \quad \text{and} \quad \beta' = \beta + \hat{\beta}$$

[16] By definition $d\mathcal{F}_{\eta,v}(w) := \lim_{\tau\to 0} \dfrac{\mathcal{F}_\eta(v + \tau w;\beta) - \mathcal{F}_\eta(v;\beta)}{\tau}$; notice that since β appears as an additive constant in the expression for $\mathcal{F}_\eta$, the differential of $\mathcal{F}_\eta$ is independent of β.

satisfying

$$w = O_1 = \hat{\beta} \quad \text{and} \quad \mathcal{F}_\eta(v'; \beta') = O_2 \text{ where } O_k = O(\|e\|^k).$$

To find w and $\hat{\beta}$, we define

$$Q_1 := g_x(\theta_1 + v + w, \theta_2) - g_x(\theta_1 + v, \theta_2) - g_{xx}(\theta_1 + v, \theta_2)w, \quad Q_2 := V^{-1} e_{\theta_1} w; \tag{2.42}$$

notice that the Q_i's are quadratic in w and e. Then, by Lemma 2, one has

$$\begin{aligned} \mathcal{F}_\eta(v'; \beta') &:= \mathcal{F}_\eta(v + w; \beta + \hat{\beta}) \\ &= \mathcal{F}_\eta(v; \beta) + \hat{\beta} + d\mathcal{F}_{\eta,v}(w) + Q_1 \\ &= e + \hat{\beta} + A_{\eta,v} w + Q_1 + Q_2. \end{aligned} \tag{2.43}$$

The next result shows how to solve the equation

$$e + \hat{\beta} + A_{\eta,v} w = 0$$

under suitable conditions upon the function v; in this equation, v and, hence, e are given, while w and $\hat{\beta}$ are unknowns.

Proposition 1. *Let g, ω and I_0 be as in Theorem 3; let $B := \mathbb{T}^2 \times I_0$. Let $v \in C^\infty(B)$ and let $\theta \to v(\theta; b)$ belong to $\mathcal{H}_0^\xi$ for some $\xi > 0$ and any $\eta \in I_0$; let $\beta \in C^\infty(I_0)$. Finally, let V and W be defined, respectively, as in (2.36) and (2.37) and assume that $V(\theta) \neq 0$ for any $\theta \in \mathbb{T}^2$, $\langle W^{-1} \rangle \neq 0$ and that*

$$\xi + \|v_{\theta_1}\|_\xi < \bar{\xi}, \tag{2.44}$$

$$\left| \eta \left\langle W^{-1} D_\eta^{-1} v_{\theta_1} \right\rangle \right| < \left| \left\langle W^{-1} \right\rangle \right|. \tag{2.45}$$

Define:

$$\begin{aligned} E(\theta) &:= E(\theta; \eta) := Ve := V\mathcal{F}_\eta(v; \beta), \\ \overline{E} &:= \overline{E}(\eta) := \langle E \rangle, \\ \widetilde{E}(\theta) &:= \widetilde{E}(\theta; \eta) := E - \overline{E}, \\ a &:= a(\eta) := \frac{\left\langle W^{-1} D_\eta^{-1} \widetilde{E} \right\rangle - \overline{E} \left\langle W^{-1} D_\eta^{-1} v_{\theta_1} \right\rangle}{\left\langle W^{-1} \right\rangle + \eta \left\langle W^{-1} D_\eta^{-1} v_{\theta_1} \right\rangle}, \\ \hat{\beta} &:= \hat{\beta}(\eta) := -(\overline{E} + \eta)\, a = - \frac{\langle W^{-1} \rangle \overline{E} + \eta \left\langle W^{-1} D_\eta^{-1} \widetilde{E} \right\rangle}{\left\langle W^{-1} \right\rangle + \eta \left\langle W^{-1} D_\eta^{-1} v_{\theta_1} \right\rangle}, \\ E_1(\theta) &:= E_1(\theta; \beta) := D_\eta^{-1} \widetilde{E} - a \left(1 + \eta D_\eta^{-1} v_{\theta_1} \right) - \overline{E}\, D_\eta^{-1} v_{\theta_1}. \end{aligned} \tag{2.46}$$

Then:

(i) *All functions in (2.46) are $C^\infty(B)$ (or $C^\infty(I_0)$ if do not depend on θ explicitly) and, for any $\xi' < \xi$, they belong to $\mathcal{H}^{\xi'}$, for all $\eta \in I_0$;*

(ii) *The following equalities hold:*

$$D_\eta E_1 = E + \hat{\beta} V, \tag{2.47}$$

$$\left\langle W^{-1} E_1 \right\rangle = 0. \tag{2.48}$$

(iii) *If we define*

$$\hat{w}(\theta;\eta) := -V D_0^{-1}(W^{-1}E_1), \quad w := \hat{w} - V\langle\hat{w}\rangle \tag{2.49}$$

then these functions are $C^\infty(B)$ (or $C^\infty(I_0)$ if they do not depend on θ explicitly) and for any $\xi' < \xi$, they belong to $\mathcal{H}^{\xi'}$, for all $\eta \in I_0$. Furthermore, the following identities hold:

$$\langle w \rangle = 0, \tag{2.50}$$

$$e + \hat{\beta} + A_{\eta,v} w = 0. \tag{2.51}$$

From this statement,[17] the definitions in (2.42) and from (2.43), there follows immediately the following:

Corollary 2. *Under the assumptions of Proposition 1, if $\hat{\beta}$ and w are defined as in (2.46) and (2.49), then*

$$e' := \mathcal{F}_\eta(v + w; \beta + \hat{\beta}) = Q_1 + Q_2.$$

Proof of Proposition 1. (i) The regularity properties of the functions defined in (2.46) follow from the assumption (2.45) and point (iv) of Lemma 1. Notice that in view of point (vi) of Remark 4, assumption (2.44) implies that $\theta \to g_x(\theta_1 + v(\theta), \theta_2)$ belongs to $\mathcal{H}^\xi$ (and hence, so does $e(\theta_1)$).

(ii) From the definitions of E_1 and $\hat{\beta}$ (and of the operator D_η), there follows

$$\begin{aligned} D_\eta E_1 &= \widetilde{E} - D_\eta\, a - \eta\, a\, v_{\theta_1} - \overline{E} v_{\theta_1} \\ &= E - (\overline{E} + \eta a) V \\ &= E + \hat{\beta} V, \end{aligned}$$

proving (2.47). Again, from the definitions of E_1 and a, there follows

$$\left\langle W^{-1} E_1 \right\rangle = \left\langle W^{-1} D_\eta^{-1} \widetilde{E} \right\rangle - a \left(\left\langle W^{-1} \right\rangle + \eta \left\langle W^{-1} D_\eta^{-1} v_{\theta_1} \right\rangle \right) - \overline{E} \left\langle W^{-1} D_\eta^{-1} v_{\theta_1} \right\rangle = 0,$$

proving (2.48).

(iii) The regularity claim is handled as above. Equation (2.50) follows at once from the definition of w (noting that $\langle V \rangle = 1$). To check (2.51), first observe that the definition of w implies that

$$\begin{aligned} D_0(V^{-1} w) &= D_0 \left(-D_0^{-1}(W^{-1} E_1) - \langle \hat{w} \rangle \right) \\ &= -W^{-1} E_1. \end{aligned}$$

[17] Compare, especially, (2.51).

Recalling (2.38) and (2.47), we find

$$\begin{aligned} e + \hat{\beta} + A_{\eta,v} w &= V^{-1}[Ve + \hat{\beta} V + V A_{\eta,v} w] \\ &= V^{-1}[E + \hat{\beta} V + D_\eta (W D_0 (V^{-1} w))] \\ &= V^{-1}[E + \hat{\beta} V - D_\eta E_1] \\ &= 0. \end{aligned}$$

□

Remark 6. (i) Notice that $a, \hat{\beta} = O(\|e\|)$ so that also, $E_1, w = O(\|e\|)$. Thus, $Q_i, e' = O(\|e\|^2)$. Explicit estimates will be provided in the next paragraph.

(ii) From (2.34) it follows that

$$\overline{E} := \langle Ve \rangle = \Big\langle (1 + v_{\theta_1}) \mathcal{F}_\eta(v; \beta) \Big\rangle = \beta + \eta\, \omega \Big\langle (v_{\theta_1})^2 \Big\rangle. \tag{2.52}$$

(iii) In the conservative case ($\eta = 0$) we have that $\overline{E} = \beta$ and

$$E = V \mathcal{F}_0(v; \beta) = E_0 + \beta V$$

with (compare (2.52))

$$E_0 := V \mathcal{F}_0(v; 0), \quad \langle E_0 \rangle = 0.$$

Thus, from the definitions given in (2.46) and (2.49), there follows that

$$\begin{aligned} \widetilde{E} &= E - \beta = E_0 + \beta v_{\theta_1}, \\ a &= \frac{\Big\langle W^{-1} D_0^{-1} E_0 \Big\rangle}{\big\langle W^{-1} \big\rangle}, \\ \hat{\beta} &= -\beta, \\ E_1 &= D_0^{-1} E_0 - a, \end{aligned}$$

and w as in (2.49). Thus, *w is independent of β and so is the new approximate solution*

$$e' = \mathcal{F}_0(v + w; 0),$$

(recall that $\beta' = \beta + \hat{\beta} = 0$ in this case). This shows that in the conservative case *one can always take* $\beta = 0$.

2.3. KAM estimates

Here we collect the main estimates for the KAM algorithm described in Proposition 1. We start with the following

Definition 1. Let $\bar{\xi} > \xi > 0$. Then,

- $\mathcal{V}_\xi$ denotes the set of functions $v \in C^\infty\left(\mathbb{T}^2 \times I_0\right)$ such that for all $\eta \in I_0$, $\theta \to v(\theta; \eta) \in \mathcal{H}_0^\xi$.

- $\mathcal{V}_\xi^{\bar\xi}$ denotes the subset of $v \in \mathcal{V}_\xi$ verifying that for all $\eta \in I_0$, $V(\theta) := 1 + v_{\theta_1}(\theta) \neq 0$ for all $\theta \in \mathbb{T}^2$, $\langle W^{-1} \rangle \neq 0$ where $W := V^2$ and

$$\|v_{\theta_1}\|_\xi < \bar\xi - \xi, \quad \left|\eta \left\langle W^{-1} D_\eta^{-1} v_{\theta_1} \right\rangle\right| < |\langle W^{-1} \rangle|.$$

- $\mathcal{W}_\xi$ denotes the set $\mathcal{V}_\xi \times C^\infty(I_0)$.
- $\mathcal{W}_\xi^{\bar\xi}$ denotes the set $\mathcal{V}_\xi^{\bar\xi} \times C^\infty(I_0)$.

If $(v, \beta) \in \mathcal{W}_\xi^{\bar\xi}$, we define

$$\mathcal{K}(v, \beta) := (v', \beta') := (v + w, \beta + \hat\beta),$$

where w and $\hat\beta$ are defined as in Proposition 1.

Thus, by Proposition 1,

$$\mathcal{K} : \mathcal{W}_\xi^{\bar\xi} \to \mathcal{W}_{\xi'}, \quad \forall\, \bar\xi > \xi > \xi' > 0.$$

Lemma 3. (i) *Let* $(v, \beta) \in \mathcal{W}_\xi^{\bar\xi}$*; let* $(w, \hat\beta) := \mathcal{K}(v, \beta) - (v, \beta)$ (*compare Definition* 1) *and define*

$$V := 1 + v_{\theta_1}, \quad W := V^2, \quad \rho := \eta \frac{\left\langle W^{-1} D_\eta^{-1} v_{\theta_1} \right\rangle}{\langle W^{-1} \rangle}. \tag{2.53}$$

Assume that there exist ν, α *such that for any* $\eta \in I_0$*,*

$$\|v_{\theta_1}\|_\xi < \nu < \bar\xi - \xi, \tag{2.54}$$

$$|\rho| < \alpha < 1. \tag{2.55}$$

Then, there exists $s := s(\tau) > 1$ *and*[18] $c := c(\kappa, \tau, M, \nu, \alpha, \eta_0) > 1$ *such that for any* $0 < \delta < \xi$ *and any* $\eta \in I_0$*, the following estimates hold:*

$$|\hat\beta|, \|w_{\theta_1}\|_{\xi-\delta}, \|D_\eta^{-1} w_{\theta_1}\|_{\xi-\delta} \leqq c\,\delta^{-s}\,\|e\|_\xi. \tag{2.56}$$

(ii) *Let* W' *and* ρ' *be defined as, respectively,* W *and* ρ *with* v *replaced by* $v' := (v + w)$*. If*

$$\|w_{\theta_1}\|_{\xi-\delta} \leqq \nu - \|v_{\theta_1}\|_\xi, \tag{2.57}$$

then

$$\|W' - W\|_{\xi-\delta} \leqq c\,\delta^{-s}\,\|e\|_\xi; \tag{2.58}$$

$e' := \mathcal{F}_\eta(v'; \beta') := \mathcal{F}_\eta(\mathcal{K}(v, \beta))$ *belongs to* $\mathcal{H}^{\xi-\delta}$*, for any* $\eta \in I_0$*, and satisfies*

$$\|e'\|_{\xi-\delta} \leqq c\,\delta^{-s}\,\|e\|_\xi^2. \tag{2.59}$$

[18] Recall the definition of M in (1.20).

Finally, if

$$|W' - W|_{\mathbb{T}^2} \leqq \frac{1}{4}\,\frac{(1-\nu)^4}{(1+\nu)^2}, \tag{2.60}$$

then

$$\inf_{\mathbb{T}^2} W' > \frac{(1-\nu)^2}{2}, \quad \Big\langle (W')^{-1} \Big\rangle > \frac{(1+\nu)^2}{2} \tag{2.61}$$

and

$$\sup_{\mathbb{T}^2} |(W')^{-1} - W^{-1}|, |\rho' - \rho| \leqq c\, \delta^{-s}\, \|e\|_\xi. \tag{2.62}$$

Proof. The proof of this Lemma is (KAM) routine and it is based on a systematic use of point (iii) of Lemma 1 and of point (vi) of Remark 4.

We begin by observing that (2.54) implies

$$\begin{aligned} &\mu_0 := (1+\nu)^{-2} < |W^{-1}(\theta)| < \mu_1 := (1-\nu)^{-2}, \quad \forall\, \theta \in \mathbb{S}^2_\xi, \\ &\mu_0 < W^{-1}(\theta) < \mu_1, \quad \forall\, \theta \in \mathbb{T}^2. \end{aligned} \tag{2.63}$$

Next, from the definitions in (2.46) and (2.54) it follows immediately that

$$|\overline{E}|,\ \|\widetilde{E}\|_\xi,\ \|E\|_\xi \leqq (1+\nu)\,\|e\|_\xi < 2\|e\|_\xi.$$

From (2.31) it follows

$$\sup_{\mathbb{T}^2} |D_\eta^{-1} v_{\theta_1}| \leqq \|D_\eta^{-1} v_{\theta_1}\|_0 \leqq \sigma_{0,1}(\xi)\|v_{\theta_1}\|_\xi < \sigma_{0,1}(\xi)\,\nu$$
$$\sup_{\mathbb{T}^2} |D_\eta^{-1}\widetilde{E}| \leqq \|D_\eta^{-1}\widetilde{E}\|_0 \leqq \sigma_{0,1}(\xi)\|e\|_\xi.$$

From (2.63) one has

$$\inf_{\mathbb{T}^2} W > \frac{1}{\mu_1}, \quad \Big\langle W^{-1} \Big\rangle > \mu_0,$$

so that

$$\begin{aligned} \Big\langle W^{-1}\Big\rangle + \eta \Big\langle W^{-1} D_\eta^{-1} v_{\theta_1}\Big\rangle &\geqq \Big\langle W^{-1}\Big\rangle (1 - |\rho|) \\ &> \Big\langle W^{-1}\Big\rangle (1-\alpha) \\ &> \mu_0\,(1-\alpha). \end{aligned}$$

We are now ready to estimate $|\hat{\beta}|$: in view of (2.32) and the above estimates, we get

$$|\hat{\beta}| \leqq c_1\, \xi^{-\tau}\, \|e\|_\xi < c_1\, \delta^{-\tau}\, \|e\|_\xi$$

with

$$c_1 := \frac{1}{1-\alpha}\,\frac{\mu_1}{\mu_0}\left(2 + \frac{\eta_0}{\kappa}\,(\tau/\mathrm{e})^\tau\right) = \frac{1}{1-\alpha}\,\frac{(1+\nu)^2}{(1-\nu)^2}\left(2 + \frac{\eta_0}{\kappa}\,(\tau/\mathrm{e})^\tau\right).$$

Analogously, we find

$$|a| \leq c_2\,\xi^{-\tau}\|e\|_\xi < c_2\,\delta^{-\tau}\|e\|_\xi,$$

$$\begin{aligned}\|E_1\|_{\xi-\frac{\delta}{3}} &\leq \|D_\eta^{-1}\widetilde{E}\|_{\xi-\frac{\delta}{3}} + |a|\left(1+|\eta|\|D_\eta^{-1}v_{\theta_1}\|_{\xi-\frac{\delta}{3}}\right) + |\overline{E}|\,\|D_\eta^{-1}v_{\theta_1}\|_{\xi-\frac{\delta}{3}}\\ &\leq \sigma_{0,1}(\delta/3)\|\widetilde{E}\|_\xi + |a|\left(1+|\eta|\sigma_{0,1}(\delta/3)\nu\right) + |\overline{E}|\sigma_{0,1}(\delta/3)\nu\\ &\leqq c_3\,\delta^{-2\tau}\|e\|_\xi, \end{aligned} \tag{2.64}$$

for suitable constants $c_i = c_i(\kappa,\tau,\eta_0,\nu,\alpha)$ (since the ideas are clear, it is not necessary to compute further); notice that $\|E_1\|$ has been estimated on an "intermediate" space $\mathcal{H}^{\xi-\frac{\delta}{3}}$ so as to be able to control further applications of the operator D_η^{-1} or ∂_{θ_1}.

Now, recalling the definition of w and $\hat{w}$ in (2.49), by point (iii) of Lemma 1, by (2.64), one gets

$$\begin{aligned}\|\hat{w}\|_{\xi-\frac{2}{3}\delta} &\leq (1+\|v_{\theta_1}\|_\xi)\,\|D_0^{-1}(W^{-1}E_1)\|_{\xi-\frac{2}{3}\delta}\\ &\leq (1+\nu)\sigma_{0,1}(\delta/3)\,(1-\nu)^{-2}\,\|E_1\|_{\xi-\frac{\delta}{3}}\\ &\leq c_4\,\delta^{-3\tau}\,\|e\|_\xi. \end{aligned}$$

Thus, since $|\langle\hat{w}\rangle| \leqq \|\hat{w}\|_{\xi-\frac{2}{3}\delta}$, we find

$$\|w\|_{\xi-\frac{2}{3}\delta} \leqq c_5\delta^{-3\tau}\|e\|_\xi, \tag{2.65}$$

and

$$\begin{aligned}&\|w_{\theta_1}\|_{\xi-\delta} \leqq \sigma_{1,0}(\delta/3)\,\|w\|_{\xi-\frac{2}{3}\delta} \leqq c_6\,\delta^{-(3\tau+1)}\,\|e\|_\xi,\\ &\|D_\eta^{-1}w_{\theta_1}\|_{\xi-\delta} \leqq \sigma_{0,1}(\delta/3)\,\|w_{\theta_1}\|_{\xi-\frac{2}{3}\delta} \leqq c_7\,\delta^{-(4\tau+1)}\,\|e\|_\xi, \end{aligned} \tag{2.66}$$

which proves part (i) of the Lemma.

Now, by (2.57) and (2.66),

$$\begin{aligned}\|W'-W\|_{\xi-\delta} &= \left\|2w_{\theta_1}V + w_{\theta_1}^2\right\|_{\xi-\delta}\\ &\leqq 2\nu\left\|w_{\theta_1}\right\|_{\xi-\delta} + \left\|w_{\theta_1}\right\|_{\xi-\delta}^2 \leqq 3\nu\left\|w_{\theta_1}\right\|_{\xi-\delta}\\ &\leq c_8\delta^{-(3\tau+1)}\,\|e\|_\xi. \end{aligned} \tag{2.67}$$

Inequality (2.57) also guarantees that $e' \in \mathcal{H}^{\xi-\delta}$; compare point (vi), Remark 4. Then by Corollary 2 and (2.65), one gets (for a suitable $c_9 > 1$)

$$\begin{aligned}\|e'\|_{\xi-\delta} &\leq \|Q_1\|_{\xi-\delta} + \|Q_2\|_{\xi-\delta} \leqq M\,\|w\|_{\xi-\delta}^2 + (1-\nu)^{-1}\sigma_{1,0}(\delta)\|e\|_\xi\|w\|_{\xi-\delta}\\ &\leq c_9\,\delta^{-6\tau}\,\|e\|_\xi^2. \end{aligned}$$

Let $\widetilde{W} := W' - W$. From (2.60) it follows that

$$\sup_{\mathbb{T}^2}|\widetilde{W}| \leqq \frac{1}{4}\frac{(1-\nu)^4}{(1+\nu)^2} = \frac{1}{4}\frac{\mu_0}{\mu_1^2} < \frac{1}{2\mu_1}, \tag{2.68}$$

so that on $\mathbb{T}^2$ one has (recall (2.63))

$$\frac{1}{2} \leqq 1 + W^{-1}\widetilde{W} \leqq \frac{3}{2}, \quad W' = W + \widetilde{W} > \frac{1}{\mu_1} - \frac{1}{2\mu_1} = \frac{(1-\nu)^2}{2} > 0.$$

Let now $Z := (W')^{-1} - W^{-1}$. Then by (2.67) and (2.68) one finds (on $\mathbb{T}^2$)

$$|Z| = \frac{\widetilde{W}}{W^2(1 + W^{-1}\widetilde{W})} \leqq 2\mu_1^2\, |\widetilde{W}| \leqq c_{10} \min\{\delta^{-(3\tau+1)}\, \|e\|_\xi, 1\} \tag{2.69}$$

(for a suitable $c_{10} > 1$), proving also the first inequality in (2.62). Furthermore, by the first inequality in (2.68),

$$\left\langle (W')^{-1}\right\rangle = \left\langle W^{-1}\right\rangle + \langle Z\rangle \geqq \left\langle W^{-1}\right\rangle - |\langle Z\rangle| \geq \mu_0 - 2\mu_1^2\, |\widetilde{W}| \geqq \frac{\mu_0}{2} = \frac{1}{2(1+\nu)^2}.$$

Finally, using (2.69), (2.66) and (2.57) (in order to estimate $\|w_{\theta_1}\|$ in terms of $\nu < 1$), one obtains

$$|\rho' - \rho| \leqq \left| \eta \frac{(W^{-1} + Z)(D_\eta^{-1} v_{\theta_1} + D_\eta^{-1} w_{\theta_1})}{\left\langle W^{-1}\right\rangle + \langle Z\rangle} - \eta \frac{W^{-1} D_\eta^{-1} v_{\theta_1}}{\left\langle W^{-1}\right\rangle} \right| \leqq c_{11}\, \delta^{-(4\tau+1)}\, \|e\|_\xi,$$

proving also the second inequality in (2.62).

The theses of the Lemma follow now by taking $c := \max_i c_i$ and $s = 6\tau$. $\square$

2.4. Convergence of the Nash–Moser algorithm

Here we complete the quantitative description of the KAM procedure giving a sufficient condition in order for the algorithm to converge.

For any $i \geqq 0$ and for $0 < \xi_* < \xi < \bar{\xi}$, we let

$$\xi_i := \xi_* + \frac{\xi - \xi_*}{2^i}, \quad \delta_{i+1} := \xi_i - \xi_{i+1} = \frac{\xi - \xi_*}{2^{i+1}}, \quad (\forall\, i \geqq 0);$$

as above (compare (2.63)) we let $\mu_0 := (1+\nu)^{-2}$ and $\mu_1 := (1-\nu)^{-2}$; we let

$$0 < \mu_0 < \bar{\mu}_0 := \inf_{\mathbb{T}^2} W^{-1} \leqq \bar{\mu}_1 := \sup_{\mathbb{T}^2} W^{-1} < \mu_1; \tag{2.70}$$

finally, we let (recall the definition of ρ in (1.21))

$$\mu := \min\left\{ \nu - \|v_{\theta_1}\|_\xi\, ,\, \frac{\mu_0}{2\mu_1^2}, \mu_1 - \bar{\mu}_1\, ,\, \bar{\mu}_0 - \mu_0\, ,\, \alpha - |\rho| \right\} > 0. \tag{2.71}$$

Proposition 2. *Under the same assumptions and notations of part* (i) *of Lemma 3, let $0 < \xi_* < \xi$ and let ξ_i, μ_i, $\bar{\mu}_i$ and μ be as above; let C and m be positive numbers such that*

$$C \geqq c\, 4^s\, (\xi - \xi_*)^{-s}, \quad m \leqq 2^{s-1}\, \mu, \tag{2.72}$$

and assume that

$$C\, \|e\|_\xi \leqq \mathrm{e}^{-\frac{1}{\mathrm{e}m}}. \tag{2.73}$$

Then, $(v_i, \beta_i) := \mathcal{K}^i(v, \beta) = (v_{i-1} + w_i, \beta_{i-1} + \hat{\beta}_i) \in \mathcal{W}^{\bar{\xi}}_{\xi_i}$ *for all* $i \geqq 1$*; the sequences* $\{v_i\}$ *and* $\{\beta_i\}$ *converge uniformly on, respectively,* $\mathbb{S}^2_{\xi_*} \times I_0$ *and* I_0*, defining a limit*

$$(u, \gamma) := \lim_{j\to\infty} (v_j, \beta_j) = \left(v + \sum_{i=1}^{\infty} w_i, \beta + \sum_{i=1}^{\infty} \hat{\beta}_i \right) \in \mathcal{W}^{\bar{\xi}}_{\xi_*},$$

which is a solution of (1.18)*, that is,* $\mathcal{F}_\eta(u; \gamma) = 0$ *for all* $\eta \in I_0$*. Furthermore, if* W_i *and* ρ_i *are defined as, respectively,* W *and* ρ *in* (1.21) *with* v *replaced by* v_i*, then* $W_* := \lim W_i \in \mathcal{W}^{\bar{\xi}}_{\xi_*}$*,* $\rho_* := \lim \rho_i \in C^\infty(I_0)$*, and (for all* $\theta \in \mathbb{T}^2$ *and any* $\eta \in I_0$*)*

$$\|u_{\theta_1}\|_{\xi_*} \leqq \nu, \quad |\rho_*| \leqq \alpha, \tag{2.74}$$

$$|\gamma - \beta|, \; \|u_{\theta_1} - v_{\theta_1}\|_{\xi_*}, \; |\rho - \rho_*|, \; \|W - W_*\|_{\xi_*} \leqq C_* \|e\|_\xi, \tag{2.75}$$

where

$$C_* := \frac{C}{2^s} \left(1 - \mathrm{e}^{-\frac{1}{\mathrm{e}m}}\right)^{-1}.$$

Proof. We claim that (1.24) implies that for any $i \geqq 1$,

$$(v_i, \beta_i) := \mathcal{K}^i(v, \beta) \in \mathcal{W}^{\bar{\xi}}_{\xi_i} \tag{2.76}$$

$$\max\left\{ |\hat{\beta}_i|, \|\partial_{\theta_1} w_i\|_{\xi_i}, |\rho_i - \rho_{i-1}|, \sup_{\mathbb{T}^2} |W_i - W_{i-1}|, \sup_{\mathbb{T}^2} \left| W_i^{-1} - W_{i-1}^{-1} \right| \right\} \leqq \frac{(C\|e\|_\xi)^{2^{i-1}}}{2^s} \tag{2.77}$$

$$\theta \to e_i(\theta) := \mathcal{F}_\eta(v_i; \beta_i) \in \mathcal{H}^{\xi_i} \quad \text{and} \quad \|e_i\|_{\xi_i} \leqq \frac{(C\|e\|_\xi)^{2^i}}{C\, 2^{s\,i}}, \tag{2.78}$$

where $W_0 := W$, $\rho_0 := \rho$. We prove the claim by induction. First of all, observe that (1.24) implies immediately that[19]

$$2^k (C\|e\|_\xi)^{2^k} \leqq m, \quad \forall\, k \geqq 0. \tag{2.79}$$

Now, let us check (2.76)÷(2.78) for $i = 1$. By Lemma 3, part (i), with $\delta := \delta_1$, $\xi - \delta = \xi_1$, $(v', \beta') = (v_1, \beta_1) = \mathcal{K}(v, \beta) = (v + w_1, \beta + \hat{\beta}_1)$, by definition of C, δ_1, m and μ and by (2.79) (with $k = 0$), we have

$$|\hat{\beta}_1|, \|\partial_{\theta_1} w_1\|_{\xi_1} \leqq c\delta_1^{-s} \|e\|_\xi = \frac{C\|e\|_\xi}{2^s} \leqq \frac{\mu}{2}. \tag{2.80}$$

---

[19] If x and y are positive numbers such that $x \leqq \exp(-1/(\mathrm{e}y))$, then $tx^t \leqq y$ for any $t > 0$. In fact, let $\lambda := \log x^{-1}$ and observe that the hypothesis is equivalent to $\frac{1}{\mathrm{e}\lambda} \leqq y$. Then, $tx^t = \frac{1}{\lambda}(\lambda t)\exp(-\lambda t) \leqq \frac{1}{\mathrm{e}\lambda} \leqq y$.

In particular

$$\|\partial_{\theta_1} w_1\|_{\xi_1} \leqq \frac{\nu - \|v_{\theta_1}\|_\xi}{2},$$

which allows us to apply part (ii) of Lemma 3, with $W' = W_1$, $e' = e_1$, $\rho' = \rho_1$, and to obtain, as above,

$$\|W_1 - W\|_{\xi_1} \leqq \frac{C\|e\|_\xi}{2^s} \leqq \frac{\mu}{2}, \quad \|e_1\|_{\xi_1} \leqq \frac{C}{2^s}\|e\|_\xi^2 = \frac{(C\|e\|_\xi)^2}{C\,2^s}, \tag{2.81}$$

and, since $\mu \leqq \mu_0/(2\mu_1^2)$, by (2.62) (recall the identity in (2.68)), we have also

$$\inf_{\mathbb{T}^2} W_1 > \frac{1}{2\mu_1}, \quad \left\langle W_1^{-1} \right\rangle > \frac{\mu_0}{2}$$
$$\sup_{\mathbb{T}^2} |W_1^{-1} - W^{-1}|, |\rho_1 - \rho| \leqq \frac{C\|e\|_\xi}{2^s},$$

which, together with (2.80) and (2.81), prove (2.77) and (2.78) for $i = 1$. Finally, from the above estimates and definitions, there follows

$$\|\partial_{\theta_1} v_1\|_{\xi_1} \leqq \|v_{\theta_1}\|_\xi + \|\partial_{\theta_1} w_1\|_{\xi_1} \leqq \|v_{\theta_1}\|_\xi + \frac{\nu - \|v_{\theta_1}\|_\xi}{2} < \nu < \bar{\xi} - \xi,$$
$$|\rho_1| \leqq |\rho| + |\rho_1 - \rho| \leqq |\rho| + \frac{\mu}{2} \leqq |\rho| + \frac{\alpha - |\rho|}{2} < \alpha < 1,$$

showing that $(v_1, \beta_1) \in \mathcal{W}_{\xi_1}^{\bar{\xi}}$, proving also (2.76) for $i = 1$.

Let $j \geqq 1$ and assume that (2.76)÷(2.78) hold true for $1 \leqq i \leqq j$; we want to prove (2.76)–(2.78) for $i = j + 1$. Let $1 \leqq i \leqq j$; then, by definition of v_i, by (2.77) and (2.79), there follows

$$\begin{aligned}
\|\partial_{\theta_1} v_i\|_{\xi_i} &= \left\| v_{\theta_1} + \sum_{k=1}^{i} \partial_{\theta_1} w_k \right\|_{\xi_i} \\
&\leq \|v_{\theta_1}\|_\xi + \sum_{k=1}^{i} \|\partial_{\theta_1} w_k\|_{\xi_k} \\
&\leq \|v_{\theta_1}\|_\xi + \sum_{k=1}^{i} \frac{(C\|e\|_\xi)^{2^{k-1}}}{2^s} \\
&\leq \|v_{\theta_1}\|_\xi + \frac{m}{2^{s-1}} \sum_{k=1}^{i} \frac{1}{2^k} \\
&= \|v_{\theta_1}\|_\xi + \mu \sum_{k=1}^{i} \frac{1}{2^k} \\
&< \|v_{\theta_1}\|_\xi + \mu < \nu.
\end{aligned} \tag{2.82}$$

Analogously, for any $1 \leqq i \leqq j$,

$$
\begin{aligned}
|\rho_i| &= \left| \rho + \sum_{k=1}^{i} \rho_k - \rho_{k-1} \right| \\
&\leq |\rho| + \sum_{k=1}^{i} |\rho_k - \rho_{k-1}| \\
&< |\rho| + \mu < \alpha.
\end{aligned} \tag{2.83}
$$

Thus, we can apply Lemma 3, part (i) (with $\delta := \delta_{j+1}$, $\xi = \xi_j$, v, β, ρ replaced by v_j, β_j, ρ_j, $(v', \beta') = (v_{j+1}, \beta_{j+1}) = \mathcal{K}(v_j, \beta_j) = (v_j + w_{j+1}, \beta_j + \hat{\beta}_{j+1})$), which, in view of (2.56), the identity

$$
c\, \delta_{i+1}^{-s} = \frac{C}{2^s}\, 2^{s\,i}, \tag{2.84}
$$

and (2.78), yields

$$
\begin{aligned}
|\hat{\beta}_{j+1}|, \|\partial_{\theta_1} w_{j+1}\|_{\xi_{j+1}} &\leqq c\, \delta_{j+1}^{-s}\, \|e_j\|_{\xi_j} \\
&= \frac{C}{2^s}\, 2^{s\,j}\, \|e_j\|_{\xi_j} \\
&\leqq \frac{(C\|e\|_\xi)^{2^j}}{2^s},
\end{aligned} \tag{2.85}
$$

which proves the first two bounds in (2.77) with $i = j + 1$. To apply part (ii) of Lemma 3, we have to check (2.57) with w and v replaced, respectively, by w_{j+1} and v_j. Since

$$
\|\partial_{\theta_1} v_j\|_{\xi_j} + \|\partial_{\theta_1} w_{j+1}\|_{\xi_{j+1}} \leq \|v_{\theta_1}\|_\xi + \sum_{k=1}^{i+1} \|\partial_{\theta_1} w_k\|_{\xi_k},
$$

(2.85) shows that the inequalities in (2.82) hold also for $i = j + 1$ so that (2.57) is satisfied. Thus, by (2.58) (with W' and W corresponding, respectively, to W_{j+1} and W_j), (2.84) and by (2.78) with $i = j$, we see that

$$
\|W_{j+1} - W_j\|_{\xi_{j+1}} \leqq c\, \delta_{i+1}^{-s} \|e_j\|_{\xi_j} = \frac{C}{2^s}\, 2^{s\,i} \|e_j\|_{\xi_j} \leqq \frac{(C\|e\|_\xi)^{2^j}}{2^s}, \tag{2.86}
$$

showing that (2.77) for $\|W_i - W_{i-1}\|$ holds also with $i = j + 1$. Now, by (2.59), (2.84) and (2.78) with $i = j$, we find

$$
\|e_{j+1}\|_{\xi_{j+1}} \leqq \frac{C}{2^s}\, 2^{s\,j}\, \frac{(C\|e\|_\xi)^{2^{j+1}}}{C^2\, 2^{2s\,j}} = \frac{(C\|e\|_\xi)^{2^{j+1}}}{C\, 2^{s(j+1)}}, \tag{2.87}
$$

that is, (2.78) with $i = j + 1$. Next, from (2.86), (2.79), the fact that $j \geqq 1$ and the definition of the μ's, there follows

$$\begin{aligned} \|W_{j+1} - W_j\|_{\xi_{j+1}} &\leqq \frac{(C\|e\|_\xi)^{2^j}}{2^s} \\ &\leq \frac{m}{2^s\, 2^j} = \frac{\mu}{2^{j+1}} \\ &\leq \frac{\mu}{4} \leqq \frac{\mu_0}{4\mu_1^2} \\ &= \frac{1}{4}\,\frac{(1-\nu)^4}{(1+\nu)^2}, \end{aligned}$$

showing that (2.60) holds in the present case. Therefore, by (2.61),

$$\inf_{\mathbb{T}^2} W_{j+1} > \frac{(1-\nu)^2}{2}, \quad \left\langle W_{j+1}^{-1} \right\rangle > \frac{(1+\nu)^2}{2}$$

and

$$\sup_{\mathbb{T}^2} |W_{j+1}^{-1} - W_j^{-1}|,\ |\rho_{j+1} - \rho_j| \leqq c\,\delta_{j+1}^{-s} \|e_j\|_{\xi_j} \leqq \frac{(C\|e\|_\xi)^{2^j}}{2^s}, \tag{2.88}$$

showing that (2.77) holds for $i = j + 1$. Finally, by (2.88), one sees that (2.83) holds also for $i = j + 1$, implying that

$$|\rho_{j+1}| < \alpha < 1,$$

which shows that $(v_{j+1}, \beta_{j+1}) \in \mathcal{W}_{\xi_{j+1}}^{\bar{\xi}}$. The claim has been completely proven.

The (fast) decay of $(C\|e\|)^{2^j}$ implies that (v_j, β_j) converge uniformly to $(u, \gamma) \in \mathcal{W}_{\xi_*}^{\bar{\xi}}$ and that, for any $\eta \in I_0$,

$$\mathcal{F}_\eta(u; \gamma) = \lim_{j\to\infty} \mathcal{F}_\eta(v_j; \beta_j) = \lim_{j\to\infty} e_j = 0,$$

showing that (u, γ) is a solution of (1.18) for any $\eta \in I_0$. Furthermore, since

$$\|\partial_{\theta_1} v_i\|_{\xi_i} < \nu, \quad |\rho_i| < \alpha,$$

by taking limits, we see that (1.25) holds. As for the bounds in (1.26), we have, for example, that

$$\begin{aligned} \|u_{\theta_1} - v_{\theta_1}\|_{\xi_*} &\leqq \sum_{i \geqq 1} \|\partial_{\theta_1} w_i\|_{\xi_i} \\ &\leq \frac{1}{2^s} \sum_{i \geqq 1} (C\|e\|_\xi)^{2^i} \\ &\leq \frac{1}{2^s} \sum_{i \geqq 1} (C\|e\|_\xi)^{i} \\ &\leq C_*\, \|e\|_\xi \end{aligned}$$

which implies the second inequality in (1.26); the other inequalities are obtained in exactly the same way. □

2.5. Local uniqueness

Next, we prove a local uniqueness result for the solutions of (1.18). Such result is based upon the following simple observation on analytic functions.

Lemma 4. *Let* $w \in \mathcal{H}^{\xi_*}$ *and assume that there exist* $c, s > 0$ *such that*

$$\|w\|_{\xi-\delta} \leqq c\,\|w\|_{\xi}^{2}\,\delta^{-s}, \quad \forall\, 0 < \delta < \xi \leqq \xi_*, \tag{2.89}$$

$$\big(c\,4^{s}\,\xi_*^{-s}\big)\,\|w\|_{\xi_*} \leqq 1. \tag{2.90}$$

Then $w \equiv 0$.

Proof. Let $\xi_i := \xi_*/2^i$ and $\delta_i := \xi_*/2^{i+1}$ for all $i \geqq 0$. Then, by (2.89), one has

$$\|w\|_{\xi_{i+1}} \leqq B_0\,B_1^i\,\|w\|_{\xi_i}^2, \quad B_0 := c\,2^s\xi_*^{-s}, \quad B_1 := 2^s.$$

Iterating this relation[20] one gets

$$\|w\|_0 \leqq \|w\|_{\xi_i} \leqq \frac{\big(C\|w\|_{\xi_*}\big)^{2^i}}{C\,2^{s\,i}}, \quad C := c\,4^s\,\xi_*^{-s},$$

for all $i \geqq 0$ which, by (2.90), implies that $\|w\|_0 := \sum |w_n| = 0$ so that $w \equiv 0$. □

Proposition 3. *Let* $0 < \xi_* < \bar\xi$ *and let* $(u, \gamma) \in \mathcal{W}_{\xi_*}^{\bar\xi}$ *be a solution of* (1.18) (*that is,* $\mathcal{F}_\eta(u; \gamma) = 0$)*, satisfying also*

$$\Big\|u_{\theta_1}\Big\|_{\xi_*} \leqq \nu \leqq \bar\xi - \xi_*, \quad |\rho| \le \alpha < 1,$$

for some $\nu, \alpha \in (0, 1)$. *Then, there exist* $\hat s := \hat s(\tau) > 1$ *and* $\widehat C = \widehat C(\kappa, \tau, \eta_0, M, \nu, \alpha) > 1$ *such that if* $(u', \gamma') \in \mathcal{W}_{\xi_*}^{\bar\xi}$ *is also a solution of* (1.18) *(that is* $\mathcal{F}_\eta(u'; \gamma') = 0$*) satisfying*

$$\Big\|u'_{\theta_1}\Big\|_{\xi_*} \leqq \nu, \quad \widehat C\,\xi_*^{-\hat s}\,\|u' - u\|_{\xi} \leqq 1 \tag{2.91}$$

then $u \equiv u'$ *and* $\gamma = \gamma'$.

---

[20] If $y_i > 0$ and $\{x_i\}_{i \geqq 0}$ is a sequence of positive numbers satisfying

$$x_{i+1} \leqq y_0 y_1^i x_i^2,$$

then one also has $x_i \leqq (y_0 y_1 x_0)^{2^i}/(y_0 y_1^{i+1})$, as it follows multiplying both sides of the above inequality by $y_0 y_1^{i+2}$ so as to obtain $z_{i+1} \leqq z_i^2$ with $z_i := y_0 y_1^{i+1} x_i$.

Proof. Let

$$w := u' - u, \qquad \hat{\gamma} := \gamma' - \gamma,$$

and observe that, since $e := \mathcal{F}_\eta(u;\gamma) = 0$, then, by (2.35) and (2.39), one has the following identities:

$$A_{\eta,u} w := V^{-1} D_\eta \left(W D_0 \left(V^{-1} w\right)\right) = d\mathcal{F}_{\eta,u}(w) = \Delta_\eta w + g_{xx}(\theta_1 + u, \theta_2) w, \tag{2.92}$$

where $V := 1 + u_{\theta_1}$ and $W := V^2$. Thus, (since also $\mathcal{F}_\eta(u';\gamma') = 0$)

$$\begin{aligned} 0 &= \mathcal{F}_\eta(u';\gamma') = \mathcal{F}_\eta(u;\gamma) + \Delta_\eta w + g_x(\theta_1 + u', \theta_2) - g_x(\theta_1 + u, \theta_2) + \hat{\gamma} \\ &= \Delta_\eta w + g_{xx}(\theta_1 + u, \theta_2)\, w + \hat{\gamma} \\ &\quad + (g_x(\theta_1 + u + w, \theta_2) - g_x(\theta_1 + u, \theta_2) - g_{xx}(\theta_1 + u, \theta_2) w) \\ &=: A_{\eta,u} w + \hat{\gamma} + Q_1(w). \end{aligned} \tag{2.93}$$

We now claim that, if we define

$$E := V Q_1, \quad \overline{E} := \langle E \rangle, \quad \widetilde{E} := E - \overline{E}, \tag{2.94}$$

then

$$\hat{\gamma} = -\frac{\langle W^{-1}\rangle \overline{E} + \eta \left\langle W^{-1} D_\eta^{-1} \widetilde{E} \right\rangle}{\left\langle W^{-1}\right\rangle + \eta \left\langle W^{-1} D_\eta^{-1} u_{\theta_1} \right\rangle}. \tag{2.95}$$

In fact, from (2.92) and (2.93), there follows

$$D_\eta \left(W D_0 (V^{-1} w)\right) + \hat{\gamma} + u_{\theta_1} \hat{\gamma} + \widetilde{E} + \overline{E} = 0. \tag{2.96}$$

If $\eta = 0$, then taking the average in (2.96) yields $\hat{\gamma} = -\overline{E}$, which is (2.95) when $\eta = 0$. If $\eta \neq 0$, then, observing that $D_\eta^{-1} 1 = \frac{1}{\eta}$, from (2.96) one obtains

$$D_0(V^{-1} w) + \frac{W^{-1}\hat{\gamma}}{\eta} + \hat{\gamma} W^{-1} D_\eta^{-1} u_{\theta_1} + W^{-1} D_\eta^{-1} \widetilde{E} + \frac{W^{-1}\overline{E}}{\eta} = 0;$$

multiplying by η and taking the average in the latter relation, yields, upon solving for $\hat{\gamma}$, (2.95).

Thus, w satisfies the equation

$$A_{\eta,u} w = Q, \quad Q := -\hat{\gamma} - Q_1, \tag{2.97}$$

with Q "quadratic" in w. In fact, observing that

$$\|Q_1\|_\xi \leqq M \, \|w\|_\xi^2, \quad \forall\, \xi \leqq \xi_*,$$

(M being as in (1.20)), by the relations in (2.97), (2.95) and (2.94), one finds that

$$|\hat{\gamma}|, \|Q\|_\xi \leqq c_{11}\, \xi^{-\tau} \, \|w\|_\xi^2, \tag{2.98}$$

for a suitable $c_{11} = c_{11}(\kappa, \tau, \eta_0, M, \nu, \alpha) > 1$ and for any $\xi \le \xi_*$. Next, from the definition of $A_{\eta,u}$, one finds the identity

$$D_0(V^{-1}w) = W^{-1}D_\eta^{-1}(VQ), \tag{2.99}$$

and since $\langle w \rangle = 0$ (as it follows from $u, u' \in \mathcal{H}_0^{\xi_*}$), one sees that (2.99) is equivalent to

$$w = \hat{w} - V\langle \hat{w} \rangle, \quad \hat{w} := VD_0^{-1}\left(W^{-1}D_\eta^{-1}(VQ)\right).$$

By Lemma 1, one gets the estimate

$$\|w\|_{\xi-\delta} \leqq c_{12}\delta^{-2\tau}\,\|Q\|_\xi, \quad (0 < \delta < \xi \le \xi_*),$$

which, by (2.98), implies

$$\|w\|_{\xi-\delta} \leqq c_{13}\,\delta^{-3\tau}\|w\|_\xi^2, \quad (0 < \delta < \xi \le \xi_*)$$

showing that w satisfies the estimate (2.89). Letting $\widehat{C} := c_{13}\,4^{3\tau}$ and $\hat{s} := 3\tau$, the thesis follows from Lemma 4. □

The above analysis shows the smooth (C^∞) dependence upon the "dissipation" parameter η. We close this section with a brief remark on how solutions depend upon other eventual "external" parameters.

Remark 7. (i) If the function g in (1.18) depends also in a real-analytic way on one (or more) external parameters $\wp \in J \subset \mathbb{C}^m$, then so do KAM solutions $u(\theta; \eta, \wp)$ *provided the smallness condition* (1.24) *holds uniformly in* $\wp$, that is, provided such conditions holds with the $\|\cdot\|_\xi$ norm redefined as the norm

$$\|e\|_\xi := \sum_{n\in\mathbb{Z}} \left(\sup_{\wp\in J} |e_n(\wp)| \right) \mathrm{e}^{|n|\xi}.$$

This claim follows from the uniform convergence of the KAM scheme and Weierstrass theorem on analytic limits of holomorphic function; for more details compare, for example, with [5].

(ii) The dependence upon the frequency ω, as is well known, is more delicate since it involves the small divisors $\lambda_{\eta,n}$: it is, however, standard to check that this dependence is C^∞ in the sense of Whitney on a bounded set of Diophantine numbers, say, $\mathcal{D}_{\kappa,\tau} \cap [1 + r, 1/r]$ for any prefixed $0 < r < 0$; for more details on Whitney smoothness and proofs we refer the reader to [7], [12] and [2].

2.6. *Proof of Theorem 3*

We start by observing that the hypotheses (H1)÷(H3) of Theorem 3 imply that $(v, \beta) \in \mathcal{W}_{\xi}^{\bar{\xi}}$ (recall Definition 1) and that (2.54) and (2.55) hold.

Next, because of (H2) and (H3), we have that, on $\mathbb{T}^2$, one has

$$\frac{1}{(1-\sigma\nu)^2} \geqq W^{-1} \geqq \frac{1}{(1+\sigma\nu)^2},$$

so that (recall the definitions of μ_0, μ_1, $\bar{\mu}_0$ and $\bar{\mu}_1$ in (2.63) and (2.70)) we find

$$\begin{aligned}\bar{\mu}_0 - \mu_0 &\geqq \frac{(1+\nu)^2 - (1+\sigma\nu)^2}{(1+\nu)^2(1+\sigma\nu)^2} =: \hat{\mu}_0 > 0,\\ \mu_1 - \bar{\mu}_1 &\geqq \frac{(1-\sigma\nu)^2 - (1-\nu)^2}{(1-\nu)^2(1-\sigma\nu)^2} =: \hat{\mu}_1 > 0.\end{aligned}$$

Thus, the number μ defined in (2.71) is bounded below by

$$\mu \geq \mu_* = \mu_*(\nu, \alpha, \sigma) := \min\left\{(1-\sigma)\nu\,,\ \frac{(1-\nu)^2}{2(1+\nu)^2}\,,\ \hat{\mu}_0\,,\ \hat{\mu}_1\,,\ (1-\sigma)\alpha\right\} > 0.$$

Therefore, taking

$$m = m_* := 2^{s-1}\mu_*$$

we see that condition (2.73) may be rewritten as

$$k_1\, \|e\|_{\xi} \leqq 1,$$

with

$$k_1 = k_1(\xi, \xi_*, \kappa, \tau, \eta_0, M, \nu, \alpha, \sigma) := C\ \mathrm{e}^{\frac{1}{\mathrm{e}\, m_*(\nu,\alpha,\sigma)}},$$

while (2.75) holds with C_* equal to k_2 with

$$k_2 = k_2(\xi, \xi_*, \kappa, \tau, \eta_0, M, \nu, \alpha, \sigma) := \frac{C}{2^s}\ \left(1 - \mathrm{e}^{-\frac{1}{\mathrm{e}\, m_*(\nu,\alpha,\sigma)}}\right)^{-1}. \tag{2.100}$$

Thus, if k is taken to be not smaller than $\max\{k_1, k_2\}$, we see that (1.24) implies (2.73) so that by Proposition 2, the claims about existence of the solution (u, γ) and the estimates (1.25) and (1.26) hold.

Let us turn to uniqueness. Let us take $\hat{s}$ and $\widehat{C}$ as in Proposition 3 and define

$$\begin{aligned}k_3 &= k_3(\xi_*, \kappa, \tau, \eta_0, M, \nu, \alpha) := 2\widehat{C}\xi_*^{-\hat{s}},\\ k_4 &:= k_2\, k_3,\end{aligned}$$

and assume that

$$k_4\, \|e\|_{\xi_*} \leqq 1. \tag{2.101}$$

Then, if u' and γ' solve (1.18), that is, $\mathcal{F}_\eta(u'; \gamma') = 0$ for each $\eta \in I_0$, and if

$$k_3\, \|u' - v\|_{\xi_*} \leqq 1, \tag{2.102}$$

then, by (2.102), (2.75), the definition of k_2 in (2.100), (2.101), we see that

$$\|u'-v\|_{\xi_*} \leqq \|u'-u\|_{\xi_*} + \|u-v\|_{\xi_*} \leqq \frac{1}{k_3} + k_2\,\|e\|_{\xi_*} \leqq \frac{1}{k_3} + \frac{k_2}{k_4} = \frac{2}{k_3} = \frac{1}{\widehat{C}\xi_*^{-\hat{s}}},$$

showing that (2.91) is satisfied so that, by Proposition 3, $u' = u$ and $\gamma' = \gamma$.

Thus (since k_4 is greater than k_2 and k_3), we see that all claims in Theorem 3 follow by taking $k := \max\{k_1, k_4\}$.

2.7. *Proof of Theorem 1*

We now show how Theorem 1 can be obtained as a corollary of Theorem 3.

Since f in (1.6) is assumed to be real-analytic, there exists a $\bar{\xi} > 0$ such that $f \in \mathcal{H}^{\bar{\xi}}$ (point (ii) of Remark 4). Assuming, as we shall henceforth do, that

$$|\varepsilon| \leqq \varepsilon_0 < 1$$

we can take the constant M in Theorem 3 to be

$$M := \|\partial_x^3 f\|_{\bar{\xi}}.$$

In this section, $\|\cdot\|_\xi$ denotes the norm (compare (i), Remark 7)

$$\|h\|_\xi := \sum_{n\in\mathbb{Z}} \left(\sup_{\varepsilon\in J} |h_n(\varepsilon)|\right) \mathrm{e}^{|n|\xi}, \quad J := \{\varepsilon \in \mathbb{C},\ |\varepsilon| \leqq \varepsilon_0\}.$$

The numbers ξ_*, ξ, ν, α and σ can be chosen arbitrarily as long as they satisfy

$$0 < \xi_* < \xi < \bar{\xi}, \quad 0 < \nu < \bar{\xi} - \xi, \quad 0 < \alpha < 1, \quad 0 < \sigma < 1. \tag{2.103}$$

Finally, we choose as the initial approximate solution, the trivial couple

$$(v, \beta) := (0, 0).$$

Then the error function e defined in (1.23) is simply given by

$$e = e(\theta;\varepsilon) := \mathcal{F}_\eta(0;0) = \varepsilon\,\partial_x f(\theta), \qquad \|e\|_\xi \leqq \varepsilon_0 M \tag{2.104}$$

and the functions defined in (1.21) are given by

$$V = 1, \quad W = 1, \quad \rho = 0.$$

Thus, (H1)÷(H3) of Theorem 3 are trivially satisfied and in order to meet (H4), that is, the smallness condition (1.24), it suffices to require

$$\varepsilon_0 \leqq \varepsilon_* := \min\left\{1\,,\ \frac{1}{k\,M}\right\}. \tag{2.105}$$

Thus, if (2.105) holds, by Theorem 3 and Remark 7, there exist unique functions $u = u(\theta;\eta) = u_\varepsilon(\theta;\eta,\omega)$ and $\gamma = \gamma(\eta) = \gamma_\varepsilon(\eta,\omega)$ such that $\mathcal{F}_\eta(u;\gamma) = 0$, for all $\eta \in I_0$, and $\theta \to u(\theta;\eta) \in \mathcal{H}_0^{\xi_*}$. Furthermore, u and γ are Whitney C^∞ in all their variables $(\theta, \eta, \varepsilon, \omega)$ in the domain

$$\mathbb{T}^2_{\xi_*} \times I_0 \times J \times \mathcal{D}_{\kappa,\tau},$$

they are C^∞ in $(\theta, \eta, \varepsilon)$ and real-analytic in $(\theta; \varepsilon) \in \mathbb{T}^2_{\xi_*} \times J$.

The solution (u, γ) satisfies the bounds (1.25) and (1.26). In particular (by (1.26) with $v = 0$), it holds

$$\|u_{\theta_1}\|_{\xi_*} \leqq \varepsilon_0 kM,$$

which, together with analyticity in ε, implies that $u = O(\varepsilon)$, that is, $u|_{\varepsilon=0} = 0$. Finally, the relation between γ and ω in (1.6) and Equation (1.16) imply (1.4), completing the proof of Theorem 1. □

2.8. *Proof of Theorem 2*

The proof of Theorem 2 is also based upon Theorem 3 along the lines of Section 2.7, but first we have to investigate the analytical properties of the spin–orbit potential defined in (1.8). For this purpose, we denote[21]

$$\mathrm{e}_1 := \upsilon^{-1}(1+r), \quad \mathrm{e}_2 := \upsilon^{-1}\left(\frac{1}{r}\right), \tag{2.106}$$

where $0 < r < 1$ is a prefixed number as in Theorem 2 and υ^{-1} is the real-analytic function (inverse of $\mathrm{e} \to \upsilon_{\mathrm{e}}$) defined in point (iii) of Remark 2. Clearly,

$$0 < \mathrm{e}_1 < \mathrm{e}_2 < 1.$$

It is also clear that $\rho_{\mathrm{e}}(t)$ and $\mathrm{f}_{\mathrm{e}}(t)$ (defined in (1.11) and (1.12)) are real-analytic function of $(\mathrm{e}, t) \in (0, 1) \times \mathbb{S}^1$, where $\mathbb{S}^1 := \mathbb{R}/(2\pi\mathbb{Z})$. Thus, there exist positive numbers

$$0 < \bar{\xi} < 1, \quad 0 < d < \min\{\mathrm{e}_1,\ 1 - \mathrm{e}_2\}, \tag{2.107}$$

such that the functions $\rho_{\mathrm{e}}(t)$ and $\mathrm{f}_{\mathrm{e}}(t)$ may be analytically continued into the complex domain $\mathcal{E}_{r,d} \times \mathbb{S}^1_{\bar{\xi}}$, where

$$\mathcal{E}_{r,d} := \bigcup_{\mathrm{e}' \in [\mathrm{e}_1, \mathrm{e}_2]} \{\mathrm{e} \in \mathbb{C} : |\mathrm{e} - \mathrm{e}'| \leqq d\}, \quad \mathbb{S}^1_{\bar{\xi}} := \{t \in \mathbb{C} : |\operatorname{Im} t| < \bar{\xi}\}. \tag{2.108}$$

Therefore, for any $\varepsilon_0 > 0$, which will be henceforth assumed to be smaller than or equal to one, the function

$$g(x, t; \varepsilon, \mathrm{e}) := \varepsilon f(x, t; \mathrm{e}) \tag{2.109}$$

is real-analytic for

$$((x, t), (\varepsilon, \mathrm{e})) \in \mathbb{T}^2_{\bar{\xi}} \times J, \tag{2.110}$$

[21] Again: do not confuse the letter e, which stands for eccentricity, with the letter e, which denotes the error function.

where,

$$J := \{\varepsilon \in \mathbb{C} : |\varepsilon| \leqq \varepsilon_0\} \times \mathcal{E}_{r,d} \subset \mathbb{C}^2. \tag{2.111}$$

Clearly, in the present situation the $\|\cdot\|_\xi$ denotes the norm

$$\|h\|_\xi := \sum_{n\in\mathbb{Z}} \left(\sup_{(\varepsilon,\mathrm{e})\in J} |h_n(\varepsilon, \mathrm{e})| \right) e^{|n|\xi}.$$

Finally, we choose η_0 as[22]

$$\eta_0 := \Omega_{\mathrm{e}_2},$$

and let

$$\omega \in \mathcal{D}_{\kappa,\tau} \cap \left[1 + r, \frac{1}{r}\right],$$

which guarantees that, as $\mathrm{e} \in [\mathrm{e}_1, \mathrm{e}_2]$, then $\upsilon_\mathrm{e} \in [1 + r, 1/r]$.

At this point, we can proceed as in the previous section (with the same choices of M, ξ_*, ξ, ν, α and σ, (v, β)) and deduce from Theorem 3 the existence and uniqueness of functions $u = u(\theta; \eta) = u(\theta; \eta, \varepsilon, \mathrm{e}, \omega)$ and $\gamma = \gamma(\eta) = \gamma(\eta, \varepsilon, \mathrm{e}, \omega)$ such that $\mathcal{F}_\eta(u; \gamma) = 0$, for all $\eta \in I_0$, and $\theta \to u(\theta; \eta) \in \mathcal{H}_0^{\xi_*}$.

As above, u and γ are Whitney C^∞ in all their variables $(\theta, \eta, (\varepsilon, \mathrm{e}), \omega)$ in the domain

$$\mathbb{T}^2_{\xi_*} \times I_0 \times J \times \left(\mathcal{D}_{\kappa,\tau} \cap \left[1 + r, \frac{1}{r}\right]\right),$$

they are C^∞ in $(\theta, \eta, (\varepsilon, \mathrm{e}))$ and real-analytic in $(\theta; (\varepsilon, \mathrm{e})) \in \mathbb{T}^2_{\xi_*} \times J$; (u, γ) satisfies the bounds (1.25), (1.26) and $\|u_{\theta_1}\|_{\xi_*} \leqq \varepsilon_0 kM$, which, together with analyticity in ε, implies that $u = O(\varepsilon)$, or $u|_{\varepsilon=0} = 0$. Therefore, relation (1.16) implies that $\gamma = O(\varepsilon^2)$ and we can write

$$\gamma =: -\eta\omega\varepsilon^2\,\tilde{\gamma}(\eta, \varepsilon, \mathrm{e}, \omega),$$

with $\tilde{\gamma}$ Whitney C^∞ in all its variables, C^∞ in η and real-analytic in $(\varepsilon, \mathrm{e})$; the minus sign accounts for the fact that $\tilde{\gamma} \geqq 0$ for real values of its arguments.

To finish the proof of Theorem 2, we have to discuss the parameter relations (compare (1.13))

$$\eta = K\Omega_\mathrm{e}, \quad \gamma = K\Omega_\mathrm{e}\omega - KN_\mathrm{e}. \tag{2.112}$$

By definition of υ_e and $\tilde{\gamma}$, we can rewrite (2.112) as

$$\eta = K\Omega_\mathrm{e}, \quad \omega\varepsilon^2\,\tilde{\gamma}(\eta, \varepsilon, \mathrm{e}, \omega) = \upsilon_\mathrm{e} - \omega. \tag{2.113}$$

Letting

$$\hat{\gamma}(K, \varepsilon, \mathrm{e}, \omega) := \tilde{\gamma}(K\Omega_\mathrm{e}, \varepsilon, \mathrm{e}, \omega),$$

the second relation in (2.113) can be rewritten as

$$h(\mathrm{e}, \varepsilon, K, \omega) := \upsilon_\mathrm{e} - \omega\left(1 + \varepsilon^2\hat{\gamma}(K, \varepsilon, \mathrm{e}, \omega)\right) = 0.$$

[22] Recall the definition of Ω_e in (1.9) and note that K will be taken in the interval $[-1, 1]$.

This last equation may be solved by the standard Implicit Function Theorem: Let $e_0(\omega) := \upsilon^{-1}(\omega)$, then

$$h(e_0(\omega), 0, K, \omega) = 0, \quad h_e(e_0(\omega), 0, K, \omega) = \partial_e \upsilon_e|_{e=e_0(\omega)} > 0.$$

Thus, there exists a unique

$$e_\varepsilon(K, \omega) = e_0(\omega) + O(\varepsilon^2) = \upsilon^{-1}(\omega) + O(\varepsilon^2),$$

which is Whitney C^∞ in all its variables, C^∞ in $K \in [-1, 1]$ and real-analytic in ε such that

$$h(e_\varepsilon(K, \omega), \varepsilon, K, \omega) \equiv 0,$$

implying that the parameter relations (2.113) are satisfied for $\eta = K\Omega_e$, $e = e_\varepsilon(K, \omega)$. The proof of Theorem 2 is finished upon the identification

$$u = u_\varepsilon(\theta; K, \omega) := u(\theta; K\Omega_{e_\varepsilon(K,\omega)}, \varepsilon, e_\varepsilon(K, \omega), \omega).$$

□

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Dipartimento di Matematica,
Università di Roma Tor Vergata,
00133 Rome,
Italy.
e-mail: celletti@mat.uniroma2.it

and

Dipartimento di Matematica,
Università "Roma Tre",
00146 Rome,
Italy.
e-mail: luigi@mat.uniroma3.it

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