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# Asymptotic Expansions of Quasiperiodic Solutions

L. CHIERCHIA - E. ZEHNDER

## 1. - Introduction

We first describe the existence problem of quasiperiodic solutions in a general setting and consider a Lagrangian function  $F = F(t, x, p)$ ,

$$(1.1) \quad F(t, x, p) \text{ defined on } T^{n+1} \times \mathbb{R}^n,$$

i.e. periodic in  $(t, x) \in \mathbb{R}^{n+1}$  with periodic 1,  $T^{n+1} = \mathbb{R}^{n+1}/\mathbb{Z}^{n+1}$ . The aim is to find special solutions of the associated Euler-equations

$$(1.2) \quad \frac{d}{dt} F_p(t, x(t), \dot{x}(t)) = F_x(t, x(t), \dot{x}(t)).$$

We shall call, in the following, a solution  $x(t)$  quasiperiodic with frequencies  $\omega$ , if it is of the form

$$(1.3) \quad x(t) = U(t, \omega t),$$

where  $\omega \in \mathbb{R}^n$  is a given vector with rationally independent components, and where

$$(1.4) \quad U(t, \vartheta) - \vartheta =: u(t, \vartheta) \quad \text{is defined on } T^{n+1}$$

i.e. is periodic in  $(t, \vartheta)$ . Inserting (1.3) into (1.2), one obtains the nonlinear partial differential equation for  $U$ :

$$(1.5) \quad DF_p(t, U, DU) = F_x(t, U, DU),$$

where

$$(1.6) \quad D = D(\omega) = \sum_{j=1}^n \omega_j \frac{\partial}{\partial \vartheta_j} + \frac{\partial}{\partial t}.$$

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The differential operator  $D$  depends on the frequencies  $\omega = (\omega_1, \dots, \omega_n)$ . It is the differentiation in the direction  $(\omega, 1)$ . Restricting our attention to functions of the special form

$$(1.7) \quad F(t, x, p) = \frac{1}{2}|p|^2 + f(t, x),$$

the equation to be solved becomes

$$(1.8) \quad D^2 u = f_x(t, \vartheta + u)$$

for  $u(t, \vartheta) = U(t, \vartheta) - \vartheta$  being a function on  $T^{n+1}$ . In order to solve (1.8) we shall assume  $f$  to be analytic and the frequencies  $\omega$  to satisfy the diophantine conditions:

$$(1.9) \quad |\langle \omega, j \rangle + m| \geq \gamma(|j|)^{-\tau}$$

for two constants  $\gamma > 0$  and  $\tau \geq n$  and for all  $(j, m) \in \mathbb{Z}^n \times \mathbb{Z} \setminus \{0\}$ .

It is well known that under these conditions on  $f$  and  $\omega$  the equation (1.8) has a solution, provided  $f$  is sufficiently small (in an appropriate sense). This is a consequence of the KAM theory, and we refer to [CC], [SZ] and [M1]. However, if  $f$  is not small, then (1.8) may not admit any solutions for frequencies contained in a compact region of  $\mathbb{R}^n$ , see [Ma]. We shall not impose any smallness conditions on  $f$  in the following. Instead we shall construct quasiperiodic solutions having sufficiently large frequencies. We point out, that the system under consideration, described by a Lagrangian function in the special form of (1.7), can be viewed as being “close to an integrable system” in the region in which  $|p|$  is large. Introducing

$$(1.10) \quad \omega(\alpha) = \frac{1}{\alpha}\omega,$$

we look for quasiperiodic solutions having frequencies  $\omega(\alpha)$  for sufficiently small  $\alpha \in \mathbb{R}$  with  $\alpha \neq 0$ . We shall abbreviate

$$(1.11) \quad E(u) =: D^2 u - f_x(t, \vartheta + u),$$

with  $D = D(\omega(\alpha))$ .

In the second section we shall prove that there is unique formal powerseries expansion in  $\alpha$ :

$$(1.12) \quad \tilde{u} \sim \sum_{j=2}^{\infty} \alpha^j u_j(\vartheta, t),$$

with analytic functions  $u_j$  on  $T^{n+1}$ , which solves the equation  $E(\tilde{u}) = 0$  formally, and satisfies

$$\int_{T^{n+1}} u_j dt d\vartheta = 0, \text{ for all } j.$$

However, in general, the series diverges as it is well known, and our aim is to show that the formal series can be interpreted as an asymptotic expansion for the true quasiperiodic solutions  $u_\alpha$ , as  $\alpha$  tends to zero. For this purpose  $\alpha$  is required to belong to the subset

$$(1.13) \quad A(\omega) = \left\{ \alpha \in \mathbb{R} : \left| \frac{1}{\alpha} \langle \omega, j \rangle + m \right| \geq \gamma |j|^{-\tau}, \text{ for all } (j, m) \in \mathbb{Z}^n \times \mathbb{Z} \setminus \{0\} \right\}.$$

If  $\gamma$  is sufficiently small and  $\tau > n + 1$ , we will see that the set  $\{\alpha \in A(\omega) : |\alpha| \leq \epsilon\}$  has positive Lebesgue measure for every  $1 > \epsilon > 0$ . Setting now for every  $N \geq 2$

$$(1.14) \quad \tilde{u}_N := \sum_{j=2}^N \alpha^j u_j(t, \vartheta),$$

one concludes that, in proper norms,

$$(1.15) \quad |E(\tilde{u}_N)| \leq C_N |\alpha|^{N-1}$$

for all  $|\alpha| \leq 1$ , with a constant  $C_N$  independent of  $\alpha$ . Consequently,  $\tilde{u}$  can be interpreted as an approximate solution of  $E(u) = 0$ , if only  $\alpha$  is small. Moreover,  $\tilde{u}_N$  is stable in the sense that the matrixfunction on  $T^{n+1}$ ,

$$(1.16) \quad V_\vartheta^T F_{pp}(t, V, DV) V_\vartheta,$$

with  $V =: \vartheta + \tilde{u}_N(\vartheta, t)$ , is close to the identity matrix. Thus the assumptions of the KAM theory are met and one concludes that there is an  $\alpha^* = \alpha^*(N)$ , such that for  $\alpha \in A(\omega)$  satisfying  $|\alpha| < \alpha^*$  there is a unique analytic solution  $u_\alpha$  of (1.8) having frequencies  $\omega(\alpha)$ , hence solving

$$(1.17) \quad E(u_\alpha) = 0,$$

moreover 
$$\int_{T^{n+1}} u_\alpha = 0.$$

In addition, one has an estimate of the form

$$(1.18) \quad |u_\alpha - \tilde{u}_N| \leq C_N |E(u_N)|.$$

This establishes the existence of uncountably many quasiperiodic solutions for every analytic  $f$ . We point out again, that  $f$  is not assumed to be small. Moreover, on account of (1.18) and (1.15) one concludes that for every  $N \geq 2$  there are constants  $C_N > 0$  and  $\alpha^* = \alpha^*(N)$  such that

$$(1.19) \quad \left| u_\alpha - \sum_{j=2}^N \alpha^j u_j \right|_\infty \leq C_N |\alpha|^{N+1}$$

for all  $\alpha \in A(\omega)$  satisfying  $|\alpha| \leq \alpha^*$ . This shows that indeed the formal series (1.12) serves as an asymptotic expansion for the solutions having large frequencies  $\omega(\alpha)$ . The precise statement and the details of this argument are given in section 3. For simplicity we shall only treat the case in which  $f$  is analytic. We point out that the asymptotic expansion holds true also for  $f \in C^\infty(T^{n+1})$ , in which case also the solutions  $u_\alpha$  belong to  $C^\infty(T^{n+1})$ .

It should be mentioned that in the special case  $n = 1$  the existence of quasiperiodic solutions having large frequencies can be used in order to prove that all solutions of

$$(1.20) \quad \ddot{x} - f_x(t, x) = 0, \quad (t, x) \in T^2$$

are bounded, i.e.

$$\sup_t |\dot{x}(t)| < \infty.$$

This has already been pointed out in [M1] and we shall recall the argument. We shall write (1.20) as a system

$$(1.21) \quad \dot{x} = y, \quad \dot{y} = f_x(t, x), \quad \dot{t} = 1,$$

which is considered as a vectorfield on the phase space  $T^2 \times \mathbb{R}$ . Assume now that  $U$  is a solutions of

$$(1.22) \quad \begin{aligned} D^2U &= f_x(t, U) \\ U(t, \vartheta) - \vartheta &= u(t, \vartheta) \quad \text{on } T^2, \\ D &= \frac{\omega}{\alpha} \frac{\partial}{\partial \vartheta} + \frac{\partial}{\partial t}. \end{aligned}$$

Then the map  $\psi : T^2 \rightarrow T^2 \times \mathbb{R}$ , defined by  $(t, \vartheta) \rightarrow (t, x = U(t, \vartheta), y = DU(t, \vartheta))$ , describes an embedding of the torus  $T^2$  into the phase space. In view of (1.22), the vectorfield (1.21) is tangential to  $\psi(T^2) \subset T^2 \times \mathbb{R}$  so that its flow leaves this embedded torus invariant. If now  $a_1 = \min DU \leq DU \leq a_2 = \max DU$ , then  $\psi(T^2) \subset T^2 \times [a_1, a_2]$ , and since  $\psi(T^2)$  is invariant under the flow we conclude, for every solution  $(t, x(t), y(t))$  satisfying  $y(t^*) < a_1$  for some  $t^* \in \mathbb{R}$ , that  $y(t) < a_2$  for all  $t \in \mathbb{R}$ . Since  $DU = \frac{\omega}{\alpha} + O(\alpha)$ , we can construct for every  $C > 0$  a quasiperiodic solution  $U$  satisfying  $DU > C$  by choosing  $\alpha$  sufficiently small. This proves the claim, that all solutions are bounded. One can show that the analyticity of  $f$  is not necessary for the argument. It is sufficient to assume  $f$  to be sufficiently smooth, e.g.  $f \in C^6(T^2)$ , for the smooth case we refer to [M2]. Similar arguments allow to prove the boundedness of solutions of other equations, for example for the Euler equation associated to

$$F(t, x, p) = \frac{1}{2}p^2 + \sqrt{1 + p^2}f(t, x)$$

on  $T^2 \times \mathbb{R}$ . The above argument was used also in the more subtle proof in [DZ] of the boundedness of solutions for a nonlinear Duffing equation on  $\mathbb{R}^2 \times \mathbb{R}$ .

Observe that this note deals only with systems of very restricted nature and it is desirable to have asymptotic expansion for a more general class of Euler equations associated to

$$F(t, x, p) = g(p) + f(t, x, p)$$

on  $T^{n+1} \times \mathbb{R}^n$ , with

$$\frac{|f(t, x, p)|}{|g(p)|} \rightarrow 0, \text{ as } |p| \rightarrow \infty.$$

## 2. - The formal expansion

In order to solve  $E(u) = 0$  we set formally

$$(2.1) \quad u =: \sum_{j=0}^{\infty} \alpha^j u_j(\vartheta, t),$$

and recall that

$$(2.2) \quad E(u) := D^2 u - f_x(t, \vartheta + u)$$

contains the parameter  $\alpha$  also in the differential operator  $D$ . Introducing the operator

$$\partial =: \sum_{j=1}^n \omega_j \frac{\partial}{\partial \vartheta_j}$$

we can write

$$(2.3) \quad D^2 = \frac{1}{\alpha^2} \partial^2 + \frac{2}{\alpha} \partial D_t + D_t^2,$$

where  $D_t$  denotes partial derivative with respect to  $t$ . Expanding  $\alpha^2 E(u) = 0$  into powers of  $\alpha$  we find the following equations to be solved for the functions  $u_j$ :

$$(2.4) \quad \begin{aligned} \partial^2 u_0 &= 0 \\ \partial^2 u_1 + 2\partial D_t u_0 &= 0 \\ \partial^2 u_j + 2\partial D_t u_{j-1} + D_t^2 u_{j-2} &= \varphi_{j-2}, \end{aligned}$$

for  $j \geq 2$ , where

$$(2.5) \quad \varphi_j = \varphi_j(u_0, \dots, u_j) = \frac{1}{j!} \left( \frac{d}{d\alpha} \right)^j f_x \left( t, \vartheta + \sum_{s=0}^j \alpha^s u_s \right) \Big|_{\alpha=0}$$

is a polynomial in  $u_1, \dots, u_j$ .

We shall show that there are unique analytic solutions  $u_j$  defined on  $T^{n+1}$ , if we normalize

$$\int_{T^{n+1}} u_j d\vartheta dt = 0.$$

We first observe that the linear equation  $\partial u = g$  on  $T^{n+1}$  admits a unique analytic solution  $u$  with meanvalue zero, provided  $g$  is analytic and has vanishing meanvalue. Since we will need it we formulate this well known result in quantitative terms. Denote by  $H_\sigma$  the space of holomorphic functions  $g(t, x)$  defined in the complex strip  $\Sigma_\sigma = \{(x, t) \in C^{n+1} : |\operatorname{Im} x_i| < \sigma, |\operatorname{Im} t| < \sigma\}$  and periodic in all its variables, and abbreviate

$$|g|_\sigma =: \sup_{\Sigma_\sigma} |g|.$$

LEMMA 1. *Let  $\omega$  satisfy the diophantine conditions (1.9). Assume  $g \in H_\sigma$  satisfies  $|g|_\sigma < \infty$  and  $\int g dx = 0$ . Then there is a unique analytic and periodic solution  $u$  satisfying*

$$(2.6) \quad \partial u = g \text{ on } \Sigma_\sigma, \text{ and } \int_{T^n} u dx = 0.$$

Moreover, there is a constant  $C = C(n, \tau)$  such that

$$(2.7) \quad |u|_{\sigma-\delta} \leq \frac{1}{\gamma} \delta^{-\tau} C |f|_\sigma, \text{ for all } 0 < \delta \leq \sigma.$$

For a proof we refer e.g. to [R]. We notice that here the variable  $t$  is only a parameter. To construct the solutions one proceeds inductively.

a) First we show that  $u_0 = u_1 = 0$ . Indeed from the first two equations in (2.4) we conclude, in view of Lemma 1, that  $u_0 = u_0(t)$  and  $u_1 = u_1(t)$  are independent of the  $\vartheta$ -variable. Integration of

$$(2.8) \quad \partial^2 u_2 + 2\partial D_t u_1 + D_t u_1 + D_t^2 u_0 = f_x(t, \vartheta + u_0)$$

in the  $\vartheta$ -variable gives  $D_t^2 u_0(t) = 0$  and hence  $u_0 = 0$ , if the meanvalue should vanish. Integrating now

$$\partial^2 u_3 + 2\partial D_t u_2 + D_t^2 u_1 = f_{xx}(t, \vartheta) u_1(t)$$

in the  $\vartheta$ -variable over  $T^n$  we find  $D_t^2 u_1(t) = 0$  and hence  $u_1(t) = 0$ .

b) Next we proceed by induction and assume that

$$(2.9) \quad \partial^2 u_j + 2\partial D_t u_{j-1} + D_t^2 u_{j-1} = \varphi_{j-2},$$

$$\int_{T^n} (D_t^2 u_j - \varphi_j) d\vartheta = 0$$

hold true for  $0 \leq j \leq n$ , where quantities with negative subscripts are defined to be zero. In order to prove the statement for  $j = n + 1$  we first solve

$$(2.10) \quad \partial^2 u_{n+1} = \varphi_{n-1} - 2\partial D_t u_n - D_t^2 u_{n-1}.$$

On account of the induction assumption the meanvalue over  $T^n$  of the right hand side vanishes, and by Lemma 1 there is a solution

$$(2.11) \quad u_{n+1} = a + b,$$

where  $a = a(\vartheta, t)$  is uniquely determined, if we set

$$(2.12) \quad \int_{T^n} a(\vartheta, t) d\vartheta = 0,$$

$b = b(t)$  is arbitrary. It will be determined by the condition

$$\int_{T^n} (D_t^2 u_{n+1} - \varphi_{n+1}) d\vartheta = 0$$

or

$$(2.13) \quad D_t^2 b = \int_{T^n} (\varphi_{n+1} - D_t^2 a) \frac{d\vartheta}{(2\pi)^n} = \int_{T^n} \varphi_{n+1} \frac{d\vartheta}{(2\pi)^n}.$$

Observe that the average over  $T^n$  of  $\varphi_{n+1}$  does *not* depend on  $b$ . Indeed  $\varphi_{n+1}$  is, in view of (2.5), of the form

$$\varphi_{n+1} = f_{xx}(t, \vartheta) u_{n+1} + \tilde{\varphi},$$

where  $\tilde{\varphi}$  depends on  $u_n, u_{n-1}, \dots, u_1$  only. Therefore, since the meanvalue of  $f_{xx}(t, \vartheta) b(t)$  is zero, the meanvalue of  $\varphi_{n+1}$  is independent of  $b$ .

Now the necessary and sufficient condition for a solution of (2.13) is the vanishing of the meanvalue in the  $t$  variable:

$$(2.14) \quad \int_{T^{n+1}} \varphi_{n+1} d\vartheta dt = 0.$$



Assuming (2.14) to hold true there is a unique solution  $u$  of (2.13) having meanvalue zero and the induction is completed. It remains to prove (2.14).

c) For the proof of (2.14) we need

LEMMA 2. For every  $u \in C^2(T^{n+1})$ .

$$(2.15) \quad \int_{T^{n+1}} (1 + u_{\vartheta})^T E(u) d\vartheta dt = 0,$$

where  $u_{\vartheta}$  is the Jacobian matrix in the  $\vartheta$ -variable.

PROOF. Set  $1 + u_{\vartheta} = V$ , by integration:

$$\begin{aligned} \int_{T^{n+1}} V^T E(u) d\vartheta dt &= \int_{T^{n+1}} (V^T D F_p - V^T F_x) d\vartheta dt \\ &= - \int_{T^{n+1}} (D V^T F_p + V^T F_x) d\vartheta dt \\ &\quad - \int_{T^{n+1}} \frac{\partial}{\partial \vartheta} F(t, \vartheta + u, D(\vartheta + u)) d\vartheta dt = 0. \end{aligned}$$

Inserting the expansion for  $\alpha^2 E(u)$  into (2.15) one finds the identities

$$\int_{T^{n+1}} \varphi_j = \int_{T^{n+1}} \sum_{s+\ell=j+2} u_{s,\vartheta}^T \Phi_{\ell} d\vartheta dt,$$

where  $\Phi_{\ell} =: \partial^2 u_{\ell} + 2\partial D_t u_{\ell-1} + D_T^2 u_{\ell-2} - \varphi_{\ell-2}$ ,

for all  $j \geq 0$ , and for every formal series  $u$ . The claim (2.14) follows immediately if we set  $j = n + 1$ , since the integrand on the right hand side vanishes: indeed if  $s = 0$  and  $s = 1$ , then  $u_0 = u_1 = 0$ . If  $s \geq 2$ , then by the induction assumption and by (2.10),  $\Phi_{\ell} = 0$  for all  $\ell \leq n + 1$ . This finishes the proof of the unique formal power series.

### 3. - Existence and asymptotic character

In this section we give the necessary details in order to prove (1.17)-(1.19). First we observe that the set

$$A := \left\{ \alpha > 0 : \left| \frac{1}{\alpha} \langle \omega, j \rangle + m \right| \geq \gamma |j|^{-\tau} \text{ for all } j, m \in \mathbb{Z}^{n+1}, j \neq 0 \right\}$$

has positive Lebesgue measure  $\mu$  provided the constant  $\gamma$  is sufficiently small. Here  $\omega$  is a fixed vector with rationally independent components and  $\tau$  is a constant satisfying  $\tau > n + 1$ . More precisely:

LEMMA 3. Fix  $0 < \lambda < 1$ . Then there is a constant  $\gamma^* = \gamma^*(\lambda)$  such that for  $0 < \gamma \leq \gamma^*$

$$(3.1) \quad \mu \{ \alpha \in A \mid 0 < \alpha \leq \epsilon \} \geq \epsilon(1 - \lambda),$$

for every  $0 < \epsilon \leq 1$ .

PROOF. Assume  $\gamma \leq \frac{1}{2}$ , we prove that  $\mu(B_\epsilon) \leq \epsilon\lambda$  if  $\gamma$  is sufficiently small, where  $B_\epsilon = (0, \epsilon) \setminus A$  is the complement. We have

$$\mu(B_\epsilon) \leq \mu \left( \bigcup_{(j,m) \neq 0} A_{jm} \right)$$

where

$$A_{jm} = \left\{ 0 < \alpha \leq \epsilon : \left| \frac{1}{\alpha} - \frac{m}{\langle \omega, j \rangle} \right| < \frac{\gamma}{|\langle \omega, j \rangle| |j|^\tau} \right\}.$$

In view of  $\gamma \leq \frac{1}{2}$  one verifies readily that

$$\sum_{(j,m)} \mu(A_{jm}) \leq 4\gamma \sum_{j \neq 0} \frac{|\langle \omega, j \rangle|}{|j|^\tau} \sum_{|m| \geq \frac{1}{\epsilon} |\langle \omega, j \rangle|} \frac{1}{m^2}.$$

Since the sum over  $m$  is dominated by  $\frac{2\epsilon}{|\langle \omega, j \rangle|}$  we conclude that

$$\mu(B_\epsilon) \leq 8\gamma\epsilon \sum_{j \neq 0} \frac{1}{|j|^\tau}.$$

In view of  $\tau > n + 1$ , the right hand side is equal to  $8\gamma\epsilon C$ . Therefore, defining  $\gamma^*(\lambda) = \min \left\{ \frac{1}{2}, \frac{\lambda}{8C} \right\}$ , one concludes that  $\mu(B_\epsilon) \leq \lambda\epsilon$  as claimed.

Now, we can state our main result.

THEOREM. Assume  $\gamma < \gamma^*$ . Assume  $f$  is real analytic in the (closure of the) complex strip  $\Sigma$  for some  $1 \geq \sigma > 0$ . For every  $N \geq 2$ , there exist positive constants  $\alpha^* = \alpha^*(N)$  and  $C_N$  with the following properties:

For  $\alpha \in A(\omega)$  satisfying  $|\alpha| < \alpha^*$  there is a unique  $u_\alpha$  real-analytic in, say,  $\Sigma_{\sigma/8}$  and of mean value 0 such that

$$(3.2) \quad E(u_\alpha) = 0 \quad \text{in } \Sigma_{\sigma/8}$$

and

$$(3.3) \quad \left| u_\alpha - \sum_{j \geq 2}^N \alpha^j u_j \right|_{\sigma/8} \leq C_N |\alpha|^{N+1}.$$

The proof rests on the discussion in section 2 and on the following KAM result, for which we refer to [SZ] (Theorem 1) and [CC] (Lemma 6).

LEMMA 4. *Let  $f$  be as in the above Theorem. Let  $\omega$  satisfy (1.9) and let  $v \in H_\sigma$  with  $|v|_\sigma \leq \sigma$ ,  $|v_\emptyset|_\sigma \leq \frac{1}{2}$ . There exists a constant  $C = C(n, f, \sigma, \gamma, \tau)$  such that if*

$$(3.4) \quad C|E(v)|_\sigma \leq 1,$$

then there is a unique real analytic  $u \in U_{\sigma/2}$  satisfying

$$(3.5) \quad E(u) = 0, \quad \int (u - v) = 0, \quad |u - v|_{\sigma/2} < C|E(v)|_\sigma.$$

PROOF OF THE THEOREM. Applying iteratively Lemma 1 and the Cauchy estimates (to control derivatives in terms of functions) to the  $u_i$ 's constructed in section 2, one finds estimates of the form

$$|u_i|_{\sigma/2} \leq K_i, \quad 2 \leq i \leq N,$$

with constants  $K_i$  depending on  $n, f$  and  $\gamma, \tau$ . Thus one can find an  $\alpha_0^*$  so small, that for  $|\alpha| < \alpha_0^*$  one has

$$(3.6) \quad |\tilde{u}_N|_{\sigma/4} < \frac{\sigma}{4}, \quad |\tilde{u}_{N,\emptyset}|_{\sigma/4} \leq \frac{1}{2},$$

where, as above,  $\tilde{u}_N =: \sum_{j=2}^N \alpha^j u_j$ . Moreover, Taylor's formula leads to the bound

$$(3.7) \quad |E(\tilde{u}_N)|_{\sigma/4} \leq K_N^* |\alpha|^{N-1}.$$

Now, if we set

$$\alpha^* = \min \left\{ \alpha_0^*, (CK_N^*)^{\frac{1}{1-N}} \right\},$$

the Theorem follows from Lemma 4 simply replacing  $\omega$  by  $\frac{\omega}{\alpha}$  ( $\alpha \in A(\omega)$ ),  $\sigma$  by  $\frac{\sigma}{4}$  and  $v$  by  $\tilde{u}_N$ . In this case (3.3) holds with  $C_N =: CK_N^*$ .

This theorem gives a precise meaning to the asymptotic character of the series  $\sum \alpha^i u_i$  which, as mentioned in the introduction, is in general divergent. It would, therefore, also be desirable to have good estimates for the functions

$u_j$ . In the special case in which  $n = 1$  the operator  $\partial$  is simply the differential operator  $\omega \frac{\partial}{\partial \vartheta}$ . We may therefore assume  $\omega = 1$  and find the following estimates:

PROPOSITION. Assume  $n = 1 = \omega$ , and assume that  $f$  is analytic and bounded on the strip  $\sum_{\sigma}$  with  $0 < \sigma \leq 1$ . Then the unique formal power series in section 2 satisfies

$$|u_{j+2}|_{\sigma/2} \leq B^{j+2} j^{2j} \text{ for all } j \geq 0.$$

Here  $B = \left(\frac{30M}{\sigma}\right)^2$  with  $M = \max \{|f|_{\sigma}, |f_x|_{\sigma}, 1\}$ .

We shall use the following

LEMMA 4. For all  $j \geq 1$ :

$$\sum_{k_1+2k_2+\dots+jk_j=j} \prod_{s=1}^j \frac{1}{k_s!} < e^4.$$

PROOF. Using the generating functions, the left hand side of the inequality is equal to

$$\begin{aligned} & \frac{1}{j!} \left(\frac{d}{d\alpha}\right)^j \exp\left(\sum_1^{\infty} \alpha^s\right) \Big|_{\alpha=0} \\ &= \frac{1}{j!} \left(\frac{d}{d\alpha}\right)^j \exp\left(\frac{\alpha}{1-\alpha}\right) \Big|_{\alpha=0} = \frac{e^{-1}}{j!} \sum_{n=1}^{\infty} \frac{(n+j-1)(n+j-2)\cdots n}{n!}, \end{aligned}$$

so that the claim follows from

$$\frac{(n+j-1)(n+j-2)\cdots n}{j!} < 4^n \text{ for all } n, j \geq 1.$$

PROOF OF THE PROPOSITION. Recall that  $u_0 = u_1 = 0$ , and

$$(3.8) \quad u_j = a_j + b_j, \quad j \geq 2,$$

is determined by

$$(3.9) \quad \int a_j(\vartheta, t) d\vartheta = 0, \quad \int b_j(t) dt = 0,$$

$$(3.10) \quad \partial_{\vartheta}^2 a_{j+2} = -2\partial_{\vartheta} \partial_t a_{j+1} - \partial^2 u_j - \varphi_j$$

$$(3.11) \quad \partial_t^2 b_{j+2} = \int \varphi_{j+2} d\vartheta,$$

where  $\varphi_0 = f_x(\vartheta, t)$ ,  $\varphi_1 = 0$  and where, for  $j \geq 2$

$$(3.12) \quad \begin{aligned} \varphi_j &= \sum_{k \in P_j} \left( \partial_x^{|k|} f_x \right) \prod_{s=2}^j \frac{u_s^{k_s}}{k_s!} \\ &= \frac{1}{j!} \left( \frac{d}{d\alpha} \right)^j f_x \left( t, \vartheta + \sum_{n \geq 2} \alpha^n u_n \right) \Big|_{\alpha=0}. \end{aligned}$$

here

$$P_j = \{k_2, \dots, k_j | 2k_2 + \dots + jk_j = j\}.$$

and  $|k| = k_2 + k_3 + \dots + k_j$ . Setting

$$P_{j+2}^* = \{k_2, \dots, k_j | 2k_2 + \dots + jk_j = j + 2\}$$

we can rewrite equation (3.11) as

$$\partial_t^2 b_{j+2} = \int \varphi_{j+2} d\vartheta = \int \left\{ f_{xx} a_{j+2} + \sum_{P_{j+2}^*} \left( \partial_x^{|k|} f_x \right) \prod_{s=2}^{j+2} \frac{u_s^{k_s}}{k_s!} \right\}.$$

Integrating the first term by parts and inserting the equation (3.10) for  $a_{j+2}$  gives

$$(3.13) \quad \partial_t^2 b_{j+2} = - \int f (2\partial_\vartheta \partial_t a_{j+1} + \partial_t^2 u_j - \varphi_j) + \int \Psi_j d\vartheta,$$

where

$$(3.14) \quad \Psi_j = \Psi_j(u_j, u_{j-1}, \dots, u_2) = \sum_{P_{j+2}^*} \left( \partial_x^{|k|} f_x \right) \prod_{s=2}^j \frac{u_s^{k_s}}{k_s!}.$$

We proof first the Lemma for  $j = 0$ . From

$$\partial_t^2 b_2 = \int f f_x d\vartheta = 0$$

we conclude that  $b_2 = 0$  so that  $u_2 = a_2$ . Since the meanvalue of  $a_2$  vanishes we conclude that

$$|u_2|_\sigma \leq |\partial_\vartheta^2 u_2|_\sigma = |f_x|_\sigma \leq M,$$

which proves the Lemma for  $j = 0$ .

Assume now  $j \geq 1$ . We shall show that

$$(3.15) \quad |u_{i+2}|_\sigma \leq B^{i+1} j^{2i} \text{ for all } 0 \leq i \leq j,$$

where

$$(3.16) \quad \sigma_i = \sigma \left( 1 - \frac{i}{2j} \right).$$

The Lemma then follows by setting  $i = j$ . The estimate (3.15) will be proved by induction in  $i$ . In the case  $i = 0$ , (3.15) is already proved above for  $\sigma_0 = \sigma$  and we shall assume now that

$$(3.17) \quad |u_{s+2}|_{\sigma_s} \leq B^{s+1} j^{2s}, \quad 0 \leq s \leq i-1,$$

where, of course,  $1 \leq i \leq j$ . From (3.9), (3.10) and (3.13) we conclude

$$(3.18) \quad \begin{aligned} |u_{i+2}|_{\sigma_i} &\leq |a_{i+2}|_{\sigma_i} + |b_{i+2}|_{\sigma_i}, \\ &\leq |\partial_\vartheta^2 a_{i+2}|_{\sigma_i} + |\partial_t^2 b_{i+2}|_{\sigma_i}, \\ &\leq 4M |\partial_\vartheta \partial_t u_{i+1}|_{\sigma_i} + 2M |\partial_t^2 u_i|_{\sigma_i} + 2M |\varphi_i|_{\sigma_i} + |\Psi_i|_{\sigma_i}. \end{aligned}$$

We estimate each term separately. Using the Cauchy estimates and the induction hypothesis (3.17) one finds

$$(3.19) \quad \begin{aligned} |\partial_\vartheta \partial_t u_{i+1}|_{\sigma_i} &\leq \frac{1}{(\sigma_{i-1} - \sigma_i)^2} |u_{i+1}|_{\sigma_{i-1}} = \left( \frac{2j}{\sigma} \right)^2 |u_{i+1}|_{\sigma_i}, \\ &\leq \frac{4}{\sigma^2} B^i j^{2i}, \end{aligned}$$

similarly

$$(3.20) \quad |\partial_t^2 u_i|_{\sigma_i} \leq \left( \frac{j}{\sigma} \right)^2 |u_i|_{\sigma_{i-2}} \leq \frac{1}{\sigma^2} B^{i-1} j^{2(i-1)}.$$

Observe now that  $\sigma_{s-2} \geq \sigma_s \geq \sigma_i$ , and  $B \geq \frac{2}{\sigma}$ ,  $i \geq 1$  and that  $|k| \geq 1$  for  $k \in P_i$ , then

$$\begin{aligned} |\varphi_i|_{\sigma_i} &\leq M \sum_{P_i} \frac{1}{(\sigma - \sigma_i)^{|k|}} \prod_{s=2}^i \frac{|u_s|_{\sigma_{s-2}}^{k_s}}{k_s!} \\ &\leq M \sum_{P_i} \left( \frac{2j}{i\sigma} \right)^{|k|} \frac{B^i j^{2i}}{B^{|k|} j^{4|k|}} \prod_{s=2}^i \frac{1}{k_s!} \\ &\leq \frac{2M}{\sigma} B^{i-1} j^{2i-3} \sum_{P_i} \prod_{s=1}^i \frac{1}{k_s!}, \end{aligned}$$

so that, by Lemma 4,

$$(3.21) \quad |\varphi_i|_{\sigma_i} \leq \frac{1}{\sigma} 2e^4 M B^{i-1} j^{2i-3}.$$

Observing that, if  $k \in P_{i+2}^*$ , then  $2k_2 + \dots + jk_j = j + 2$  and  $|k| \geq 2$ , one concludes similarly

$$(3.22) \quad |\Psi_i|_{\sigma_i} \leq 4e^4 M B^i j^{2(i-1)}.$$

Adding up we find from (3.18)-(3.22) that

$$\begin{aligned} |u_{i+2}|_{\sigma_i} &\leq \frac{1}{\sigma^2} (16 + 2 + 8e^4) M^2 B^i j^{2i}, \\ &< \frac{900M^2}{\sigma^2} B^i j^{2i} = B^{i+1} j^{2i}, \end{aligned}$$

where we have used the definition of the constant  $B$ . This finishes the proof of the proposition.

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