PERIODIC SOLUTIONS OF THE PLANETARY N–BODY PROBLEM

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The closure of periodic orbits in the phase space of the spatial, planetary N–body problem (with well separated semimajor axes) has full measure in the limit of small planetary masses and small eccentricities and mutual inclinations.

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1. Introduction

In 1892 Henri Poincaré conjectured that in the restricted three–body problem, an arbitrary connected piece of a (bounded) trajectory could be approximated arbitrarily well by a periodic orbit of long period. This conjecture is still open but a "metric asymptotic version" of it was proven by Gómez and Llibre:[10]

Theorem 1.1 (Gómez, Llibre 1981). *For any fixed value of the Jacobian constant and for any* $\varepsilon > 0$*, there exists a* $\mu_0 > 0$ *such that if the mass parameter* $\mu \in [0, \mu_0]$*, then the set of bounded orbits which are not contained in the closure of the set of periodic orbits has measure of Lebesgue smaller than* ε*.*

The proof relies on the existence and abundance of KAM invariant curves and on the possibility of approximating KAM curves by periodic orbits.
As pointed out by Conley and Zehnder[7] (who combined the Birkhoff–Lewis fixed point theorem together with the quantitative version of the KAM theorem due to Pöschel to prove that, under suitable non–degeneracy assumptions, the closure of periodic orbits in a ball around an elliptic equilibrium of a smooth Hamiltonian has a density which tends to one as the radius of the ball goes to zero), such strategy can be extended to any number of degrees of freedom under suitable non–degeneracy assumptions.
However, the strong degeneracies of the planetary N–body problem have prevented, until now, to obtain results of this type in the general case.

In this paper we show how recent results by G. Pinzari and the author[4,6] allow to give a full extension of Theorem 1.1 to the general spatial planetary N–body problem in a neighborhood of co–circular ("well separated") and co–planar motions. A more precise statement is given in the following section.

2. The planetary $(1 + n)$–body problem

The planetary N–body problem consists in studying the dynamics of $N = 1 + n$ point masses interacting only through gravitational attraction with no friction or external potentials in the special case where one of the bodies (the "Sun") has mass $m_0 = 1$, while the other have masses

$$\mathrm{m_i} = \mu m_i \ll 1 \ , \qquad (1 \le i \le n)$$

("planets"); in particular, one is interested in the phase region of total negative energy near the unperturbed limit of the n–planets revolving on nearly co–planar and nearly co–circular Keplerian ellipses generated by the two–body system Sun–i^{th}–planet.

The equations of motions for $(1+n)$ gravitationally interacting bodies are described by Newton's equations:

$$\ddot{u}^{(i)} = \sum_{\substack{0 \le j \le n \\ j \ne i}} \mathrm{m}_j \frac{u^{(j)} - u^{(i)}}{|u^{(j)} - u^{(i)}|^3} \ , \qquad i = 0, 1, ..., n \tag{1}$$

where $u^{(i)} = (u_1^{(i)}, u_2^{(i)}, u_3^{(i)}) \in \mathbb{R}^3$ denotes the position in Euclidean space of the i^{th}–body, "dot" denotes derivative with respect to time $t \in \mathbb{R}$, and $|u|$ denotes the Euclidean norm $\sqrt{u \cdot u} = \sqrt{\sum_{k=1}^{3} u_k^2}$.

Equations (1) are invariant by change of "inertial frames", i.e., by change of variables of the form $u^{(i)} \to u^{(i)} - (a + ct)$ with fixed $a, c \in \mathbb{R}^3$. This allows to restrict the attention to the manifold of "initial data" given by

$$\sum_{i=0}^{n} \mathrm{m}_i u^{(i)}(0) = 0 \ , \qquad \sum_{i=0}^{n} \mathrm{m}_i \dot{u}^{(i)}(0) = 0 \ . \tag{2}$$

As well known, the total linear momentum $\mathrm{M_{tot}} := \sum_{i=0}^{n} \mathrm{m}_i \dot{u}^{(i)}$ does not change along the flow of (1), i.e., $\dot{\mathrm{M}}_{\mathrm{tot}} = 0$ along trajectories; therefore, by (2), $\mathrm{M_{tot}}(t)$ vanishes for all times. But, then, also the position of the barycenter $B(t) := \sum_{i=0}^{n} \mathrm{m}_i u^{(i)}(t)$ is constant ($\dot{B} = 0$) and, again by (2), $B(t) \equiv 0$. In other words, we can restrict our analysis, without loss of generality, to the $(6n)$–dimensional phase space

$$\mathcal{M} := \Big\{ u^{(i)} \in \mathbb{R}^3 \ , \ \dot{u}^{(i)} \in \mathbb{R}^3, 0 \le i \le n \ , \ \text{s.t.} \ \sum_{i=0}^{n} \mathrm{m}_i u^{(i)} = 0 = \sum_{i=0}^{n} \mathrm{m}_i \dot{u}^{(i)} \Big\} \ , \tag{3}$$

(or more precisely, to the open subset of it with no collisions or blowups).

Furthermore, the total angular momentum

$$\mathrm{C} := \sum_{i=0}^{n} \mathrm{m}_i \dot{u}^{(i)} \times u^{(i)}$$

is conserved and we choose a reference frame $\{k^{(1)}, k^{(2)}, k^{(3)}\}$ so that $k^{(3)}$ is parallel to C.

Now fix reference radii $r_1 < r_2 < \cdots < r_n$ ("mean distances from the Sun") and let us consider the nearly co–circular and nearly co–planar region of phase space in $\mathcal{M}$ given by

$$\mathcal{M}_\varepsilon := \Big\{ \Big| |u^{(i)} - u^{(0)}| - r_i \Big|, \ \ |u_3^{(i)} - u_3^{(0)}|, \ \ |\dot{u}_3^{(i)} - \dot{u}_3^{(0)}| < \varepsilon \ , \ \forall\, 1 \le i \le n \Big\} \quad (4)$$

and denote by $\mathcal{P}_\varepsilon$ the set of periodic orbits in $\mathcal{M}_\varepsilon$. Then we can prove

Theorem 2.1. *There exist positive numbers δ and b such that if $r_i/r_{i+1} < \delta$ and $0 < \mu < \varepsilon^b$, then*

$$\lim_{\epsilon \to 0} \frac{\text{meas closure}\,(\mathcal{P}_\epsilon)}{\text{meas}\,\mathcal{M}_\varepsilon} = 1 \ . \quad (5)$$

The proof is based on two steps:

- the first step consists in showing that $\mathcal{M}_\varepsilon$ is asymptotically filled up by Kolmogorov invarian tori (a precise definition will be given later);
- the second step consists in proving that, in general, Kolmogorov tori are accumulation manifolds for periodic orbits with longer and longer periods.

The first (hard) step has been recently proven in Ref. 4: we shall briefly illustrate it in the next section. The idea beyond the second step was remarked in Ref. 7 by Conley and Zehder in a somewhat different context (periodic orbits in a neighborhood of a non–degenerate elliptic equilibrium) and using J. Moser's version of a celebrated fixed point theorem by Birkhoff and Lewis: in the last section we shall discuss a variational proof introduced in Ref. 2.

3. Existence and abundance of non–degenerate Kolmogorov tori

The existence and abundance (positive Lebesgue measure in phase space) of Kolmogorov (or KAM) tori for the planetary problem was first stated, and proved in the particular case of $n = 2$ planets in a plane, by V.I. Arnold in his milestone paper Ref. 1. The full extension of Arnold's result to n planets in space turned out to be much more difficult than expected, essentially because of degeneracies of Birkhoff invariants of the averaged (secular) approximation (see below). The first complete proof of a somewhat weaker result was given in 2004 by M. Herman and J. Féjoz[9] and a full generalization of Arnold's results and methods has been given in Ref.s 4,6.

3.1. *The planetary Hamiltonian*

The starting point is that the space $\mathcal{M}$ in (3) is a *symplectic manifold* and the Newton's equations are equivalent to Hamiltonian equations on $\mathcal{M}$: Indeed, $\mathcal{M}$ is a $6n$–dimensional manifold and one can take as standard symplectic variables (or Darboux coordinates) the following ("heliocentric") variables:

$$x^{(i)} = u^{(i)} - u^{(0)} \ , \qquad X^{(i)} = m_i \dot{u}^{(i)} \ , \quad (1 \le i \le n) \tag{6}$$

and as Hamiltonian the function

$$\begin{aligned} \mathcal{H}_{\rm plt}(X,x) &:= \sum_{i=1}^{n} \left(\frac{|X^{(i)}|^2}{2M_i} - \frac{M_i \bar{m}_i}{|x^{(i)}|} \right) + \mu \sum_{1\le i<j\le n} \left(\frac{X^{(i)} \cdot X^{(j)}}{m_0} - \frac{m_i m_j}{|x^{(i)} - x^{(j)}|} \right) \\ &=: \mathcal{H}^{(0)}_{\rm plt}(X,x) + \mu \mathcal{H}^{(1)}_{\rm plt}(X,x) \ , \end{aligned} \tag{7}$$

where (recall that we have normalized the mass of the Sun $\mathrm{m}_0 = 1$ and set $\mathrm{m}_i = \mu m_i$ for the mass of the planets with $m_i = O(1)$ and $\mu \ll 1$)

$$M_i := \frac{m_i}{1+\mu m_i} \ , \qquad \text{and} \qquad \bar{m}_i := 1 + \mu m_i \ ;$$

the symplectic form is the standard Darboux form

$$\sum_{i=1}^{n} dX^{(i)} \wedge dx^{(i)} = \sum_{i=1}^{n} \sum_{j=1}^{3} dX^{(i)}_j \wedge dx^{(i)}_j \ .$$

The planetary Hamiltonian $\mathcal{H}_{\rm plt}$ is the sum of an *integrable part*, $\mathcal{H}^{(0)}_{\rm plt}$, corresponding to n decoupled two–body systems formed by the Sun and the $i^{\rm th}$ planet, and a perturbation term proportional to the planet/Sun mass ratio μ. The natural symplectic action variables for $\mathcal{H}^{(0)}_{\rm plt}$ are the classical *Delaunay variables*, which, however are singular exactly in the limit case of interest, namely, the co–planar and co–circular limit. The classical symplectic regularization goes back to Poincaré; however, such variables are not well suited for a KAM (or Birkhoff) analysis, and it turns out to be more convenient to use a set of variables introduced in Ref. 4 called RPS (Regularized Symplectic Planetary) variables, which are a regularization of action–angle coordinates related to certain symplectic variables introduced by Deprit[8] in 1983.

3.2. *The RPS variables*

Let us begin by recalling the definition of the *Deprit action–angle variables* $\big((\Lambda,\Gamma,\Psi),(\ell,\gamma,\psi)\big) \in \mathbb{R}^{3n} \times \mathbb{T}^{3n}$. Let $n \ge 2$. The variables Λ, Γ and ℓ are in common with the Delaunay variables: Let $\mathcal{E}_i$ be the istantaneous Keplerian ellipse associated to the two–body Hamiltonian

$$\frac{|X^{(i)}|^2}{2M_i} - \frac{M_i \bar{m}_i}{|x^{(i)}|}$$

(with a negative value); let a_i and e_i be, respectively, the semimajor axis and the eccentriity of $\mathcal{E}_i$; ℓ_i is the mean anomaly of $x^{(i)}$, i.e., the normalized area spanned by the vector $x^{(i)}$ from the perihelium P_i of $\mathcal{E}_i$, while

$$\Lambda_i := M_i\sqrt{\overline{m}_i a_i}\ , \qquad \Gamma_i := |\mathrm{C}^{(i)}| = \Lambda_i\sqrt{1-e_i^2}\ , \tag{8}$$

where $\mathrm{C}^{(i)}$ denotes the i^{th} angular momentum

$$\mathrm{C}^{(i)} := X^{(i)} \times x^{(i)}\ .$$

Introduce the "partial angular momenta"

$$S^{(i)} := \sum_{j=1}^{i} \mathrm{C}^{(j)}\ , \qquad S^{(n)} = \sum_{j=1}^{n} \mathrm{C}^{(j)} =: \mathrm{C}\ , \tag{9}$$

and define the "Deprit nodes"

$$\begin{cases} \nu_i := S^{(i)} \times \mathrm{C}^{(i)}\ , & 2 \le i \le n \\ \nu_1 := \nu_2 \\ \nu_{n+1} := k^{(3)} \times \mathrm{C} =: \bar\nu\ . \end{cases} \tag{10}$$

Finally, let us use the following notation: for $u, v \in \mathbb{R}^3$ lying in the plane orthogonal to a non–vanishing vector $w \in \mathbb{R}^3$, let $\alpha_w(u,v)$ denote the positively oriented angle (mod 2π) between u and v (orientation follows the "right hand rule"). Then,

$$\gamma_i := \alpha_{\mathrm{C}^{(i)}}(\nu_i, P_i) \qquad \Psi_i := \begin{cases} |S^{(i+1)}|\ , & 1 \le i \le n-1 \\ \mathrm{C}_3 := \mathrm{C}\cdot k^{(3)} & i = n \end{cases}$$

$$\psi_i := \begin{cases} \alpha_{S^{(i+1)}}(\nu_{i+2}, \nu_{i+1}) & 1 \le i \le n-1 \\ \zeta := \alpha_{k^{(3)}}(k^{(1)}, \bar\nu) & i = n. \end{cases} \tag{11}$$

Similarly to the case of the Delaunay variables, the Deprit action–angles variables are not defined when the Deprit nodes ν_i vanish or $e_i \notin (0,1)$ but on their domain of definition are *analytic symplectic variables* (compare Ref.s 4,5).

Now, we are ready to define the RPS variables

$$\big((\Lambda, l), z\big) := \big((\Lambda, l), (\eta, \xi, p, q)\big) \in \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{R}^{4n}\ .$$

The Λ's are again the Keplerian actions as in (8), while

$$\lambda_i = \ell_i + \gamma_i + \psi_{i-1}^n\ , \qquad \begin{cases} \eta_i = \sqrt{2(\Lambda_i - \Gamma_i)}\ \cos\big(\gamma_i + \psi_{i-1}^n\big) \\ \xi_i = -\sqrt{2(\Lambda_i - \Gamma_i)}\ \sin\big(\gamma_i + \psi_{i-1}^n\big) \end{cases}$$

$$\begin{cases} p_i = \sqrt{2(\Gamma_{i+1} + \Psi_{i-1} - \Psi_i)}\ \cos\psi_i^n \\ q_i = -\sqrt{2(\Gamma_{i+1} + \Psi_{i-1} - \Psi_i)}\ \sin\psi_i^n \end{cases} \tag{12}$$

where

$$\Psi_0 := \Gamma_1\ , \quad \Gamma_{n+1} := 0\ , \quad \psi_0 := 0\ , \quad \psi_i^n := \sum_{i \le j \le n} \psi_j\ . \tag{13}$$

On their domain of definition, the RPS variables are real analytic and symplectic (compare [4, §4]).

3.3. *Foliation of phase space (partial reduction of rotations)*

The main point here is that that

$$\begin{cases} p_n = \sqrt{2(|\mathrm{C}| - \mathrm{C}_3)}\ \cos\psi_n \\ q_n = -\sqrt{2(|\mathrm{C}| - \mathrm{C}_3)}\sin\psi_n\ , \end{cases} \tag{14}$$

showing the *the conjugated variables p_n and q_n are both integrals and hence both cyclic* for the planetary Hamiltonian, which, therefore, in such variables, will have the form

$$\mathcal{H}_{\mathrm{RPS}}(\Lambda, l, \bar z) = h_{\mathrm{K}}(\Lambda) + \mu f(\Lambda, l, \bar z)\ , \tag{15}$$

where $\bar z$ denote the set of variables

$$\bar z := (\eta, \xi, \bar p, \bar q) := \big((\eta_1, \ldots, \eta_n), (\xi_1, \ldots, \xi_n), (p_1, \ldots, p_{n-1}), (q_1, \ldots, q_{n-1})\big)\ , \tag{16}$$

and h_{K} is the integrable Hamiltonian $\mathcal{H}^{(0)}_{\mathrm{plt}}$ expressed in terms of the Keplerian actions, i.e.,

$$h_{\mathrm{K}}(\Lambda) := -\sum_{i=1}^{n} \frac{M_i^3 \bar m_i^2}{2\Lambda_i^2}\ . \tag{17}$$

In other words, *the $(6n)$–dimensional phase space where $\mathcal{H}_{\mathrm{RPS}}$ is defined is foliated by $(6n-2)$–dimensional invariant manifolds* and since the restriction of the standard symplectic form on such manifolds is symply

$$d\Lambda \wedge dl + d\eta \wedge d\xi + d\bar p \wedge d\bar q\ ,$$

such manifolds are symplectic and the planetary flow is the standard Hamiltonian flow generated by $\mathcal{H}_{\mathrm{RPS}}$ in (15). Notice, also, that the analytic expression of the planetary Hamiltonian $\mathcal{H}_{\mathrm{RPS}}$ is the same in each symplectic submanifold and that the selected submanifold of vertical angular momentum corresponds to having fixed $p_n = q_n = 0$.

3.4. *The secular Hamiltonian*

The angular variables ℓ are *fast* angles moving on a time scale of order one. It is natural to expect that an important rôle in the analysis is played by the average

$$f_{\mathrm{av}}(\Lambda, \bar z) := \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f(\Lambda, \ell, \bar z)\ d\ell$$

of the perturbation term f over such fast angles; such function is called the *secular planetary Hamiltonian*.

On the symplectic submanifolds the direction of the total angular momentum is fixed, but its Euclidean length

$$G := |\mathrm{C}|$$

is still an integral; in other words, G Poisson commutes with the planetary Hamiltonian $\mathcal{H}_{\rm RPS}$ and, hence, with the secular Hamiltonian $f_{\rm av}$, reflecting the *rotational invariance of the planetary system*. This fact accounts for the special symmetries exhibited by $f_{\rm av}$: indeed $f_{\rm av}$ is an even function $\bar z$ and

$$f_{\rm av} = c_0(\Lambda) + {\rm Q_h}(\Lambda)\cdot\frac{\eta^2+\xi^2}{2} + \bar{\rm Q}_{\rm v}(\Lambda)\cdot\frac{\bar p^2+\bar q^2}{2} + O(|\bar z|^4;\Lambda)\ , \qquad (18)$$

(where ${\rm Q}\cdot y^2 = \sum_{i,j}{\rm Q}_{ij}y_iy_j$ denotes the action of the quadratic form Q on the vector y), showing that the secular origin $\bar z = 0$ is an elliptic equilibrium for the secular Hamiltonian.

The form of $f_{\rm av}$ calls for Birkhoff analysis, which is summarized (up to order four) in the following result proven in § 7 of Ref. 4.

Proposition 3.1. *There exist $\delta, \epsilon_0 > 0$ such that if $r_i/r_{i+1} < \delta$ and $|a_i - r_i| < \epsilon_0$, then $f_{\rm av}$ can be put into Birkhoff normal form up to order four by a symplectic transformation $\phi : \tilde z \to \bar z$, parameterized by Λ and leaving the origin fixed so that*

$$f_{\rm av}\circ\phi = c_0(\Lambda) + \alpha\cdot I + \frac{1}{2}\tau\ I\cdot I + O(|I|^6);\Lambda)\ , \qquad (19)$$

where

$$I := \frac{1}{2}\big(\tilde\eta_1^2+\tilde\xi_1^2,...,\tilde\eta_n^2+\tilde\xi_n^2,\tilde p_1^2+\tilde q_1^2,...,\tilde p_{n-1}^2+\tilde q_{n-1}^2\big)$$

and the symmetric $(2n-1)\times(2n-1)$ matrix $\tau = \tau(\Lambda)$ is invertible.

Let us make a few remarks.

(i) The first order Birkhoff invariants α_j are commonly denoted $\alpha = (\sigma, \varsigma) \in \mathbb{R}^n\times\mathbb{R}^{n-1}$ and satisfy identically one, and only one, exact resonance, called *Herman resonance*, namely

$$\sum_{j=1}^{2n-1}\alpha_j = \sum_{j=1}^{n}\sigma_j + \sum_{j=1}^{n-1}\varsigma_j = 0\ . \qquad (20)$$

Notwithstanding such resonance Birkhoff normalization is possible because of the rotation invariance of the system (compare Ref 6).

(ii) The invertibility of the matrix of the second order Birkhoff invariance τ is a crucial point of the analysis; this property is usually referred to as *full torsion (or twist) of the secular Hamiltonian* and allows to apply the properly–degenerate KAM theory developed by Arnold in § 4 of Ref 1 (the "fundamental theorem") and extended in Ref. 3 (compare also the discussion, in the *totally* reduced case, about Kolmogorov tori at p. 646 of Ref. 6).

(iii) The approach in Ref. 9 is quite different and avoids the check of the twist (which indeed in the Poincaré variables used in Ref. 9 is false, as clarified in Ref. 6) but rather it is based on a "first order" KAM theory developed by Rüssmann[11] and Herman, which is based on a non–degeneracy property ("non–planarity") of the frequency map (and, still, since the frequency map of the planetary Hamiltonian *is* planar because of Herman resonance, one has to modify the planetary Hamiltonian with a term which commutes with it so as to get invariant tori for the modified Hamiltonian and then, by Lagrangian intersection theory, one can conclude that such tori are invariant also for the original Hamiltonian).

Before stating the main theorem concerning the existence of a positive measure set of maximal invariant tori, let us recall the definition of a Kolmogorov torus. An m dimensional torus $\mathcal{T}$ is a **Kolmogorov torus** for a (real–analytic) Hamiltonian H defined on a $2m$ dimensional phase space $\mathcal{M}$, if there exists a (real–analytic) symplectic diffeomorphism

$$\nu : (y, x) \in B^m \times \mathbb{T}^m \to \nu(y, x; \omega) \in \mathcal{M} \ ,$$

such that

- $H \circ \nu = E + \omega \cdot y + Q$; (Kolmogorov's normal form)
- $\omega \in \mathbb{R}^m$ is a Diophantine vector (i.e., there exist numbers $a > 0$ and $b \geq m - 1$ such that for any non vanishing integer vector $k \in \mathbb{Z}^m$, one has $|\omega \cdot k| \geq a/|k|^b$);
- $Q = O(|y|^2)$;
- $\det \int_{\mathbb{T}^m} \partial_{yy} Q(0, x)\ dx \neq 0$, (nondegeneracy)
- $\mathcal{T} = \nu(0, \mathbb{T}^m)$.

Now, as mentioned in Remark (ii) above, properly–degenerate KAM theory can be applied and one gets the following

Theorem 3.1. *Let a_i and ϵ_0 be as in Proposition 3.1, then there exists positive constants $\epsilon_* < \epsilon_0$, c_* and C_* such that the following holds. If*

$$0 < \epsilon < \epsilon_* \ , \qquad 0 < \mu < \frac{\epsilon^6}{(\log \epsilon^{-1})^{c_*}} \ , \tag{21}$$

then in each symplectic submanifold $\mathcal{M}^{6n-2}_{\epsilon_*}$ with $\{|\bar{z}| < \epsilon_*\}$ contains a positive measure set $\mathcal{K}$ ("Kolmogorov set"), which is formed by the union of $(3n-1)$–dimensional Kolmogorov tori for $\mathcal{H}_{\rm RPS}$ with $\omega_1 = O(1)$ and $\omega_2 = O(\mu)$.
Furthermore, the following measure estimates hold:

$$\mathrm{meas}\, \mathcal{M}_{\epsilon_*} \geq \mathrm{meas}\, \mathcal{K} \geq \Big(1 - C_* \sqrt{\epsilon}\Big)\, \mathrm{meas}\, \mathcal{M}_{\epsilon_*} \ . \tag{22}$$

4. Periodic orbits cumulating on Kolmogorov tori

The proof of Theorem 2.1 follows now from Theorem 3.1, from the observation at the end of the preceding section and from the following general result (compare, also, Ref. 2).

Theorem 4.1. *Each Kolmogorov torus $\mathcal{T}$ for a Hamiltonian H is a cumulation set of periodic orbits of H, i.e., for any open set $U \supseteq \mathcal{T}$,* closure$(\mathcal{P}(U)) \supseteq \mathcal{T}$*, where $\mathcal{P}(U)$ denotes the set of periodic orbits of H in U.*

Proof.

1. As a general fact, a Hamiltonian H in Kolmogorov normal form

$$H = E + \omega \cdot y + Q(y, x)$$

can be put, via a symplectic real–analytic diffeomorphism $\bar{\nu}$, into the form

$$H \circ \bar{\nu} = E + \omega \cdot y + \frac{1}{2} B_0 \; y \cdot y + f \; , \tag{23}$$

with

$$B_0 := \int_{\mathbb{T}^m} \partial_{yy} Q(0, \cdot) \; , \; f = O(|y|^3) \; . \tag{24}$$

Indeed, it is easy to check that this can be achieved taking $\bar{\nu} = \phi^1_\chi$ as the time–one map of the Hamiltonian flow generated by a quadratic–in–y Hamiltonian $\chi = \frac{1}{2} A(x) y \cdot y$ with A given by

$$A(x) := D_\omega^{-1} \; \big(B - B_0\big) \; , \quad B := \partial_{yy} Q(0, x) \, dx \, ,$$

and D_ω^{-1} denoting the "small divisor" operator acting on functions with zero average by dividing the k–Fourier coefficients by $\mathrm{i}\,\omega \cdot k$: if u is a (real–analytic) (vector/matrix valued) function on $\mathbb{T}^m$ with vanishing average $u_0 = 0$

$$D_\omega^{-1} u := \sum_{k \in \mathbb{Z}^m \setminus \{0\}} \; \frac{u_k}{\mathrm{i}\, \omega \cdot k} \; e^{\mathrm{i}\, k \cdot x} \; ,$$

u_k denoting the Fourier coefficients of u. Notice that the invertibility of B_0 is not needed for this step.

2. To prove the theorem is enough to show that in any neighborhood of the Kolmogorov torus $\{y = 0\}$ one can find a periodic orbit; compare p. 134 of Ref 7.

3. By step 1 we can assume that H is in the normal form as in the right hand side of (23) with B_0 invertible. Now, for $\epsilon > 0$, let

$$\widetilde{H}(y, x) = \frac{1}{\epsilon} H(\epsilon y, x) \; . \tag{25}$$

Then,

$$\phi^t_H(\epsilon y, x) = \; \mathrm{diag}\,(\epsilon, 1) \cdot \phi^t_{\widetilde{H}}(y, x) \; ,$$

where $\mathrm{diag}\,(\epsilon, 1) \cdot (y, x) = (\epsilon y, x)$ and ϕ^t_h denotes the (standard) Hamiltonian flow generated by h. Therefore, by step 2, the theorem holds if one can find periodic orbits for $\widetilde{H}$ for every positive ϵ small enough.

Disregarding the ininfluent constant term, the Hamiltonian $\widetilde{H}$ has the form

$$\widetilde{H}(y, x; \epsilon) = \omega \cdot y + \frac{\epsilon}{2} B_0 \, y \cdot y + \epsilon^2 \widetilde{f}(y, x; \epsilon) \tag{26}$$

with $\widetilde{f}$ real–anlytic in a neighborhood of $\{0\} \times \mathbb{T}^m$ and of $O(|y|^3)$.

4. For $T > 0$ (the period, to be determined later), let

$$\bar{y} := -\frac{2\pi}{T} B^{-1}\Big\{\frac{T\omega}{2\pi}\Big\} , \quad \bar{\omega} := \omega + B\bar{y} , \tag{27}$$

where $\{a\} := a - [a]$ denotes the fractional part of a (and $[a]$ integer part of a): in this way

$$\bar{\omega}T = 2\pi k , \qquad k \in \mathbb{Z}^m . \tag{28}$$

We shall look for T–periodic solutions of the form

$$\begin{cases} y(t) := \bar{y} + \eta(t) \\ x(t) = \theta + \bar{\omega}t + \xi(t) \end{cases} \qquad \text{with b.c.} \quad \begin{cases} \eta(0) = \eta(T) \\ \xi(0) = \xi(T) = 0 . \end{cases} \tag{29}$$

Besides the functions η and ξ, we have to fix the period T (which will turn out to be $\sim 1/\epsilon$) and to determine θ.

The looked after solution $\zeta(t) := \big(\eta(t), \xi(t)\big)$ has to satisfies the differential equation

$$L\zeta = \epsilon^2 \Phi(\zeta) , \tag{30}$$

where L is the linear operator

$$L\zeta := \begin{pmatrix} \dot{\eta} \\ \dot{\xi} - \epsilon B\eta \end{pmatrix} \tag{31}$$

and Φ is the nonlinear operator

$$\Phi(\zeta) := \begin{pmatrix} -\widetilde{f}_y(\bar{y} + \eta, \theta + \bar{\omega}t + \eta; \epsilon) \\ \widetilde{f}_x(\bar{y} + \eta, \theta + \bar{\omega}t + \eta; \epsilon) . \end{pmatrix} \tag{32}$$

Note that Φ depends also on the unknown T, θ and tr ϵ.

5. Let X denote the Banach space

$$X := \Big\{\zeta = (\eta, \xi) \in C([0,T]) : \ \xi(0) = \xi(T) = 0\Big\} , \tag{33}$$

endowed with the maximum norm, and let I denote the "right inverse" of L, namely, the linear operator acting on continuous functions $z = \big(p(t), q(t)\big)$ as

$$Iz = \begin{pmatrix} \mu + \displaystyle\int_0^t p(s)ds \\ \epsilon t B_0 \mu + \epsilon \displaystyle\int_0^t \int_0^s p(\tau)d\tau \ + \int_0^t q(s)ds \end{pmatrix} , \tag{34}$$

with

$$\mu := -\frac{1}{T}\int_0^T \int_0^s p(\tau)d\tau - \frac{1}{\epsilon T}\int_0^T B_0^{-1} q(s)ds . \tag{35}$$

Then,

$$I : C([0,T]) \to X \cap C^1([0,T]) , \qquad \text{and} \qquad LI = \text{id on } C([0,T]) . \tag{36}$$

Eq. (30) can be rewritten as

$$\zeta = F(\zeta) \ , \qquad F(\zeta) := \epsilon^2 I\Phi \ . \tag{37}$$

Observe that, if $\zeta \in X$ satisfies (37), then ζ is C^1 and, by (36), it satisfies (30). Thus, to prove the theorem we have to find solutions of (37) such that $\pi_1\zeta(T) = \pi_1\zeta(0)$ where $\pi_1(\eta, \xi) = \eta$.

6. For, $\rho > 0$, let X_ρ denote the closed, nonempty subset of X given by

$$X_\rho := \big\{\zeta \in X : \ \|\zeta\| \le \rho\big\} \ . \tag{38}$$

Let $\rho > 0$ be such that the functions $\zeta \in X_\rho$ are in the domain of definition of Φ. Then, it is easy to check that

there exist $c, \epsilon_0 > 0$ such that, for all $0 < \epsilon < \epsilon_0$ and for all $c/\epsilon \le T \le 2c/\epsilon$, F is a contraction on X_ρ and the unique fixed point $\zeta_{\epsilon,\theta}$ satisfies $\|\zeta_{\epsilon,\theta}\| \le$ const $\epsilon\rho$.

7. In this last step we see how to choose θ so that $\pi_1\zeta_{\epsilon,\theta}(T) = \pi_1\zeta_{\epsilon,\theta}(0)$. Indeed, this will follow at once from the following elementary variational principle, the proof of which is left to the reader.

Proposition 4.1. *Assume that $\big(\eta(t;\theta), \xi(t;\theta)\big) = \phi^t_{\widetilde{H}}(y, \theta)$ is defined (and smooth) for $t \in [0, T]$ and that $\xi(T, \theta) = \xi(0; \theta) = \theta$ and let*

$$J(\theta) := \int_0^T \big(\eta \cdot \dot{\xi} - \widetilde{H}(\eta, \xi)\big) dt.$$

Then $\partial_\theta J = \eta(T) - y$.

Since $J(\theta)$ is smooth on $\mathbb{T}^m$, one can take, e.g., θ as the point where the minimum of J is achieved. □

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