ISSN 1560-3547, Regular and Chaotic Dynamics, 2025, Vol. 30, No. 4, pp. 538–549. © Pleiades Publishing, Ltd., 2025.

Singular KAM Theory for Convex Hamiltonian Systems

Santiago Barbieri[1*], **Luca Biasco**[2**], **Luigi Chierchia**[2***], and **Davide Zaccaria**[3****]

[1]*Departament d'Informàtica, Matemàtica Aplicada i Estadística, Universitat de Girona, 6 Campus Montilivi, 17003 Girona, Spain*

[2]*Dipartimento di Matematica e Fisica, Università degli Studi Roma Tre, Largo San Leonardo Murialdo 1, 00146 Roma, Italy*

[3]*Department of Mathematics, University of Toronto, 40 St George St., M5S 2E4 Toronto, Canada*

Received June 18, 2025; revised July 14, 2025; accepted July 18, 2025

Abstract—In this note, we briefly discuss how the singular KAM theory of [7] — which was worked out for the mechanical case $\frac{1}{2}|y|^2 + \varepsilon f(x)$ — can be extended to *convex* real-analytic nearly integrable Hamiltonian systems with Hamiltonian in action-angle variables given by $h(y) + \varepsilon f(x)$ with h convex and f generic.

MSC2010 numbers: 37J05, 37J35, 37J40, 70H05, 70H08, 70H15

DOI: 10.1134/S1560354725040057

Keywords: nearly integrable Hamiltonian systems, convex Hamiltonians, measure of invariant tori, simple resonances, Arnold–Kozlov–Neishtadt conjecture, singular KAM theory

In the Springer *Encyclopaedia of Mathematical Sciences*, Arnold, Kozlov and Neishtadt conjectured that the relative measure of phase space free of invariant tori of a real-analytic, nearly integrable general Hamiltonian system with three or more degrees of freedom is of the *same order of the perturbation*; compare [2].

In [7], in the special case of natural systems with a Hamiltonian function given (in action-angle variables) by $H = \frac{1}{2}|y|^2 + \varepsilon f(x)$, a related result has been proven, namely, that, for generic real-analytic f, the relative measure of invariant primary and secondary tori is at least $O(1 - \varepsilon|\log\varepsilon|^c)$ for some $c > 0$, in agreement (up to the logarithmic correction) with the Arnold, Kozlov and Neishtadt conjecture[1)].

Proofs in [7] are based on a new "singular KAM theory", which extends the classical theory of Kolmogorov, Arnold and Moser [1, 8, 12], so as to deal, in particular, with primary and secondary tori arbitrarily close to action-angle singularities arising near simple resonances, *uniformly in phase space*. Let us recall that classical KAM theory predicts that the relative measure of *primary* tori is of order[2)] $O(1 - \sqrt{\varepsilon})$, and that such an estimate is optimal, as simple integrable cases show.

In this paper we briefly discuss how to extend [7] to a *convex* integrable Hamiltonian in place of the purely quadratic term $\frac{1}{2}|y|^2$.

The main issue here is geometric: one has to quantitatively describe suitable neighborhoods of simple resonances, so as to be able to put, after averaging, the secular Hamiltonians in "generic

*E-mail: santiago.barbieri@udg.edu
**E-mail: luca.biasco@uniroma3.it
***E-mail: luigi.chierchia@uniroma3.it
****E-mail: d.zaccaria@mail.utoronto.ca

1) For related partial results, see [11].
2) Compare [9, 13–15].

standard form" (as defined in [6]). More specifically, we introduce action-angle variables so that the secular Hamiltonian, after averaging over the $(n-1)$ fast angles, depends only on one slow angle. Then, we show that the level sets of the derivative of the integrable part with respect to the slow action are graphs over *uniform* domains of slow actions; see § 1.2 below. At this point, the arguments to show that the density of all KAM primary and secondary tori is at least of order $O(1-\varepsilon|\log\varepsilon|^c)$ for some $c>0$ follow, without extra difficulties, the arguments spelled out in [7], which lead to the following statement, confirming (up to the logarithmic correction) the Arnold, Kozolv, Neishdatd conjecture, namely:

For generic[3] *real-analytic potentials f on $\mathbb{T}^n$, the relative measure of invariant primary and secondary tori for the Hamiltonian $H=h(y)+\varepsilon f(x)$, with h strictly convex, is at least $O(1-\varepsilon|\log\varepsilon|^c)$ for some $c>0$.*

We also mention a technical improvement in the averaging theory discussed here, which may be useful in applications (for example, in celestial mechanics), namely, we allow for different analyticity radii in the angle variables; indeed, in singular KAM theory it is essential to have sharp control of analytic singularities in the angular variables and having a common width of analyticity strip is quite unnatural[4].

A second extension would be to replace the perturbation f with a generic function depending also on actions, but this is a much more difficult problem[5].

1. RESONANCE ANALYSIS

Consider a bounded phase space $\mathcal{M}:=B\times\mathbb{T}^n$, with $B\subset\mathbb{R}^n$ a *bounded convex open nonempty set*, and $\mathbb{T}^n$ the standard flat n-torus $\mathbb{R}^n/(2\pi\mathbb{Z}^n)$, endowed with the standard symplectic two-form $dy\wedge dx=\sum_{i=1}^n dy_i\wedge dx_i$.

Let $H(y,x)$ be a real-analytic, nearly integrable Hamiltonian on $\mathcal{M}$ given by

$$H(y,x)=H_\varepsilon(y,x)=h(y)+\varepsilon f(x);\qquad (y,x)\in\mathcal{M},\qquad 0\leqslant\varepsilon\leqslant 1. \tag{1.1}$$

Analyticity parameters

We introduce quantitative analyticity parameters as follows. Let $r>0$ and $s=(s_1,\ldots,s_n)\in\mathbb{R}^n_+$ be a vector with positive components. Denote by[6] $|z|=\sqrt{z\cdot\bar z}$ the standard Euclidean norm on vectors $z\in\mathbb{C}^n$, and define the following complex neighborhoods:

$$B_\rho:=\bigcup_{y\in B}\{z\in\mathbb{C}^n:|z-y|<\rho\};$$
$$\mathbb{T}^n_s:=\{x\in\mathbb{C}^n:|\mathrm{Im}\ x_i|<s_i\ \forall\ i=1,\ldots,n\}/(2\pi\mathbb{Z}^n).$$

Henceforth, *we assume that H is real-analytic and bounded on $B_{2r}\times\mathbb{T}^n_s$* for some $r>0$.

Remark 1. As mentioned above, the reason for allowing different analyticity widths in the angular variables is motivated by physical examples, such as the three-body problem, where this is the case. Having sharp control on the angle complex singularities is essential in analyzing fine properties of Hamiltonian systems, especially, in the context of singular KAM theory or, possibly, of Arnold diffusion.

[3] The generic class we refer to is introduced in [6, Section 1].

[4] For example, in the circular restricted three-body problem the angles have different analyticity domains; compare [10].

[5] For example, it is not obvious how to generalize the generic real-analytic class introduced in [7] in view of the zeros of f introduced, in general, by the action dependence. In this respect, as suggested to us by Laurent Niederman, it might be useful to consider the quantitative Morse – Sard theory developed by Yomdin and Comte in [16]; compare [3].

[6] As usual, "bar" denotes complex-conjugated and "dot" inner product.

Convexity assumption

In this paper *we assume that h in* (1.1) *is γ-convex* for some positive γ, i. e., we assume that

$$\partial_y^2 h(y)\,\xi\cdot\xi = \sum_{i,j=1}^n \partial_{y_iy_j} h(y)\,\xi_i\xi_j \geqslant \gamma|\xi|^2, \qquad \forall\xi\in\mathbb{R}^n,\ \forall y\in\mathrm{Re}(B_{2r}). \tag{1.2}$$

We shall also assume, without loss of generality, that *the frequency map $y\to\omega(y)=\partial_y h(y)$ is bi–Lipschitz* and satisfies for some positive constants $\bar L$, L the inequalities

$$\bar L^{-1} \leqslant \frac{|\omega(y)-\omega(y_0)|}{|y-y_0|} \leqslant L,\ \forall\ y\neq y_0\in B_{2r}, \qquad \big(\omega(y):=\omega(y)=\partial_y h(y)\big). \tag{1.3}$$

We also let $M:=\sup_{y\in B_{2r}}|\omega(y)|<+\infty$.

1.1. Geometry of Resonances and Coverings of B

In singular KAM theory [7], as well as in Nekhoroshev's theory (see, e. g., [2]), it is fundamental to consider carefully the *geometry of resonances*, which allows for high-order averaging theory.

We recall that, given an integer vector $k\in\mathbb{Z}^n$, a *resonance* $\mathcal{R}_k$ for the integrable Hamiltonian $h(y)$ is the set $\{y\in B:\omega(y)\cdot k=0\}$; a *double resonance* $\mathcal{R}_{k,\ell}$ is a set given by $\mathcal{R}_{k,\ell}=\mathcal{R}_k\cap\mathcal{R}_\ell$ with k and ℓ linearly independent integer vectors.

In these definitions, the integer vectors may be assumed to have no common divisors; indeed, it is enough to consider integer vectors in the set $\mathcal{G}^n$ defined as *the subset of integer vectors $k\in\mathbb{Z}^n$ with coprime components and with first nonnull component strictly positive*[7]; we shall also denote, for $K>0$, by $\mathcal{G}^n_K$ the vectors $k\in\mathcal{G}^n$ with 1-norm $|k|_1:=\sum|k_j|$ not exceeding K.

As is well known, not all resonances have to be taken into account, and, typically, one introduces a "Fourier cut-off" $\mathtt{K}$ corresponding to a prefixed "small-divisor threshold" $\alpha>0$, and considers resonances corresponding to Fourier modes of order less than or equal to $\mathtt{K}$, the higher ones being negligible in view of the exponentially fast decay of Fourier modes. However, in singular KAM theory a *double* scale of Fourier modes has to be taken into account, as explained, e. g., in the introduction of [4]. In the present case the definition of these thresholds has to be slightly modified (with respect to the natural systems case considered in [7]) in view of convexity and of the more general assumption on angular analyticity. We therefore give the following definitions.

Let $s\in\mathbb{R}^n_+$ be as above, and let $k\in\mathbb{Z}$; we denote

$$\underline{s}:=\min_{1\leqslant i\leqslant n} s_i, \qquad \bar s:=\max_{1\leqslant i\leqslant n} s_i, \qquad \hat s:=\bar s/\underline{s}, \qquad |k|_s=\sum_{i=1}^n s_i|k_i|. \tag{1.4}$$

Then, following [7] (compare, Eq. (20), p. 12), we introduce two Fourier scales $\mathtt{K},\mathtt{K}_{\rm o}$ (to be eventually defined as suitable functions of ε) and a small divisor threshold α satisfying[8]

$$\mathtt{K}\geqslant 6\hat s\,\mathtt{K}_{\rm o}\geqslant 6\mathtt{K}_{\rm o}\geqslant 12\,, \qquad \alpha:=\sqrt{\varepsilon}\mathtt{K}^\nu\,, \quad \nu:=\tfrac{9}{2}n+2. \tag{1.5}$$

Next, we define a covering of the action-space, according to a nonresonant zone, simply-resonant zones and a doubly-resonant zone. Indeed, denote by $\pi_k^\perp$ the orthogonal projection on the subspace perpendicular to[9] k, and let[10]

$$\mathtt{C}=\mathtt{C}(n,L,\gamma):=12\,c_1\,n\,L/\gamma\,, \quad \text{where} \quad c_1:=5n(n-1)^{n-1};$$

---

[7] This set coincides with the set of generators of maximal one-dimensional lattices in $\mathbb{Z}^n$.

[8] Here, the only (trivial) difference with [7] is the introduction of the constant $\hat s\geqslant 1$.

[9] Explicitly, $\pi_k^\perp\omega:=\omega-\frac{1}{|k|^2}(\omega\cdot k)k$.

[10] Recall (1.2) and (1.3); observe that $L\geqslant\lambda_{\max}$ and $\gamma\leqslant\lambda_{\min}$, where $\lambda_{\max}$ and $\lambda_{\min}$ are, respectively, the maximal and minimum eigenvalue of the Hessian of h.

then we define the following subsets of B:

$$\begin{aligned}
&\mathcal{R}^0 := \left\{ y \in B : |\omega(y)\cdot k| \geqslant \tfrac{\alpha}{2\mathtt{C}}, \forall\ \ 0 < |k|_1 \leqslant \mathtt{K}_{\rm o} \right\},\\
&\mathcal{R}^{1k} := \left\{ y \in B : |\omega(y)\cdot k| \leqslant \tfrac{\alpha}{\mathtt{C}}, \text{and } |\,\pi_k^\perp \omega(y)\cdot \ell| \geqslant \tfrac{3\alpha \mathtt{K}^{n+3}}{|k|}, \forall \ell \in \mathcal{G}^n_{\mathtt{K}} \setminus \mathbb{Z}k \right\},\\
&\mathcal{R}^1 := \bigcup_{k\in \mathcal{G}^n_{1,\mathtt{K}_{\rm o}}} \mathcal{R}^{1k}, \qquad\qquad (1.6)\\
&\mathcal{R}^2 := B \setminus \left(\mathcal{R}^1 \cup \mathcal{R}^0\right).
\end{aligned}$$

Remark 2. These definitions are adapted from [7]: compare, in particular, Eqs. (21)–(23) on p. 12 of [7]. Notice, first, that in the mechanical case the frequency map is simply the identity map. The second difference is the appearance here of the constant $\mathtt{C} \geqslant 1$: this is technical and will become clear below. Finally, in the lower bound on $|\,\pi_k^\perp \omega(y)\cdot \ell|$ in the definition of $\mathcal{R}^{1k}$ there appears $\mathtt{K}^{n+3}$ in place of $\mathtt{K}$ (compare (22) in [7]); the choice of the power $n+3$ here is not optimal but it is done for simplicity (as it allows for a single covering of the simply-resonant region), and does not affect in any substantial way the strategy of [7] (which is robust with respect to powers of $\mathtt{K}$, which, at the end, are chosen as suitable powers of $|\log \varepsilon|$).

Clearly, from the definition of $\mathcal{R}^2$ it follows that $\{\mathcal{R}^i\}$ is a cover of B, i. e., that

$$B = \mathcal{R}^0 \cup \mathcal{R}^1 \cup \mathcal{R}^2.$$

The set $\mathcal{R}^2$ contains all double (or higher) resonances, as well as resonances with high frequency modes. But, as remarked in [7], $\mathcal{R}^2$ is a small set (as we shall shortly see), which is out of reach of perturbation theories[11)].

Indeed, we claim that the measure of $\mathcal{R}^2$ can be bounded as

$$\text{meas}(\mathcal{R}^2) \leqslant c_\star\, \alpha^2\, \mathtt{K}^{2n+2} \leqslant c_\star\, \varepsilon\, \mathtt{K}^b, \qquad (1.7)$$

where $b = 11n+6$ and $c_\star = \frac{c}{2\cdot 3^n} M^{n-2}\, \bar{L}^n$ for a suitable $c > 0$ depending only on n. This is essentially Lemma 2.1 of [7]; for completeness we reproduce the simple geometrical proof in the convex case.

Proof (*of* (1.7)). Let $\Omega := \omega(B)$ and for $k \in \mathcal{G}^n_{\mathtt{K}_{\rm o}}$, $\ell \in \mathcal{G}^n_{\mathtt{K}} \setminus \mathbb{Z}k$, define

$$\begin{aligned}
&\Omega^2_{k,\ell} := \left\{\omega \in \Omega : |\omega\cdot k| < \tfrac{\alpha}{\mathtt{C}} \text{ and } |\,\pi_k^\perp \omega \cdot \ell| \leqslant \tfrac{3\alpha \mathtt{K}^{n+3}}{|k|}\right\};\\
&\mathcal{R}^2_{k\ell} := \left\{ y \in B : \omega \in \Omega^2_{k,\ell}\right\}.
\end{aligned} \qquad (1.8)$$

Then, one checks easily that

$$\mathcal{R}^2 \subseteq \bigcup_{k\in \mathcal{G}^n_{\mathtt{K}_{\rm o}}} \bigcup_{\substack{\ell \in \mathcal{G}^n_{\mathtt{K}} \\ \ell \notin \mathbb{Z}k}} \mathcal{R}^2_{k,\ell}. \qquad (1.9)$$

Now, denote by $v \in \mathbb{R}^n$ the projection of ω onto the plane generated by k and ℓ (recall that, by hypothesis, k and ℓ are not parallel); then,

$$|v\cdot k| = |\omega \cdot k| < \alpha, \qquad |\,\pi_k^\perp v\cdot \ell| = |\,\pi_k^\perp \omega\cdot \ell| \leqslant \tfrac{3\alpha \mathtt{K}^{n+3}}{|k|}. \qquad (1.10)$$

Set

$$\bar{\ell} := \pi_k^\perp \ell = \ell - \tfrac{\ell\cdot k}{|k|^2} k. \qquad (1.11)$$

Then, v decomposes in a unique way as $v = ak + b\bar{\ell}$ for suitable $a, b \in \mathbb{R}$. By (1.10),

$$|a| < \tfrac{\alpha}{|k|^2}, \qquad |\,\pi_k^\perp v\cdot \ell| = |b\bar{\ell}\cdot \ell| \leqslant \tfrac{3\alpha \mathtt{K}^{n+3}}{|k|}, \qquad (1.12)$$

11) Compare the Introduction in [7].

and, since $|\ell|^2|k|^2 - (\ell\cdot k)^2$ is a positive integer (recall that k and ℓ are not parallel),

$$|\bar\ell\cdot\ell| \overset{(1.11)}{=} \frac{|\ell|^2|k|^2-(\ell\cdot k)^2}{|k|^2} \geqslant \frac{1}{|k|^2}.$$

Hence,

$$|b| \leqslant \tfrac{3\alpha \mathtt{K}^{n+3}}{|k|}. \tag{1.13}$$

Then, write $\omega\in\Omega^2_{k,\ell}$ as $\omega = v+v^\perp$ with $v^\perp$ in the orthogonal complement of the plane generated by k and ℓ. Since $|v^\perp|\leqslant|\omega|<M$ and v lies in the plane spanned by k and ℓ inside a rectangle of sizes of length $2\alpha/|k|^2$ and $6\alpha\mathtt{K}^{n+3}|k|$ (compare (1.12) and (1.13)), we find that, for any $k\in\mathcal{G}^n_{\mathtt{K}_{\rm o}}$ and $\ell\in\mathcal{G}^n_{\mathtt{K}}\setminus\mathbb{Z}k$, one has

$$\mathrm{meas}(\Omega^2_{k,\ell}) \leqslant \tfrac{2\alpha}{|k|^2}\,(6\alpha\mathtt{K}^{n+3}|k|)\,(2M)^{n-2} = 3\cdot 2^n\,M^{n-2}\,\alpha^2\tfrac{\mathtt{K}^{n+3}}{|k|}.$$

Since $\sum_{k\in\mathcal{G}^n_{\mathtt{K}_{\rm o}}}|k|^{-1}\leqslant c\,\mathtt{K}_{\rm o}^{n-1}$ for a suitable $c=c(n)$, by (1.3), Eq. (1.7) follows. □

1.2. Simple Resonances as Graphs over Adiabatic Actions

As already mentioned, a key role in singular KAM theory is played by simple resonances. It is therefore important to have a precise analytic description of them. In this subsection, we express neighborhoods of simple resonances as a foliation of graphs over $(n-1)$ adiabatic actions; see Proposition 1 below. This is the first step to put the averaged secular Hamiltonian in the so-called "generic standard form"; compare § 3 in [6].

To state the results in Proposition 1, we need some preparation.

Let $k\in\mathcal{G}^n$. A simple resonance $\omega\cdot k=0$ corresponds to a "resonant angle", in the sense that one can make a linear symplectic change of variables on $B\times\mathbb{T}^n$ so that in the new variables one transforms the resonant relation $k\cdot x$ into, say, the first new angle $\mathtt{x}_1 = k\cdot x$. Indeed, one can find a matrix $\mathrm{A}\in\mathrm{SL}(n,\mathbb{Z})$ (i. e., integer-valued with determinant one) with first row given by k; such a matrix A can be chosen so that[12)]

$$\mathrm{A} := \begin{pmatrix} k\\ \hat{\mathrm{A}}\end{pmatrix},\ |\hat{\mathrm{A}}|_\infty \leqslant |k|_\infty\,,\quad |\mathrm{A}|_\infty = |k|_\infty\,,\quad |\mathrm{A}^{-1}|_\infty \leqslant (n-1)^{\frac{n-1}{2}}|k|_\infty^{n-1}. \tag{1.14}$$

Then, the linear symplectic change of variables

$$\Phi_{\rm o} : (\mathtt{y},\mathtt{x}) \mapsto (y,x) = (\mathrm{A}^T\mathtt{y}, \mathrm{A}^{-1}\mathtt{x}), \qquad (\mathtt{x}_1 = k\cdot x). \tag{1.15}$$

transforms the unperturbed Hamiltonian $h(y)$ into[13)]

$$\mathtt{h}_0^k(\mathtt{y}) := h(\mathrm{A}^T\mathtt{y}), \qquad \mathtt{y}\in\mathtt{B}^k := \mathrm{A}^{-T}B. \tag{1.16}$$

Note that $\mathtt{h}_0^k$ is holomorphic on the complex set[14)]

$$\mathtt{B}^k_{2\mathtt{r}_k}, \quad \text{with} \quad \mathtt{r}_k := \frac{r}{n|k|_\infty}. \tag{1.17}$$

Notation 1. *We shall adopt the following conventions: a vector* $\mathtt{y}\in^n$ *will be denoted*

$$\mathtt{y} = (\mathtt{y}_1,\mathtt{y}_2,\ldots,\mathtt{y}_n) = (\mathtt{y}_1,\hat{\mathtt{y}}), \qquad \hat{\mathtt{y}} = (\mathtt{y}_2,\ldots,\mathtt{y}_n)\in^{n-1},$$

namely, the hat over n-vectors denotes the projection onto the coordinates with index greater than or equal to two.

---

12) Compare Lemma 2.6, (i) in [6]; $|M|_\infty$, with M matrix (or vector), denotes the maximum norm $\max_{ij}|M_{ij}|$ (or $\max_i|M_i|$).

13) Notice that also B is convex.

14) By (1.14) $\|A\| := \max_{|v|=1}|Av| \leqslant n|A|_\infty = n|k|_\infty$.

Now, since

$$\partial_{\mathtt{y}_1}\mathtt{h}_0^k(\mathtt{y}) = \partial_y h(\mathrm{A}^T\mathtt{y})\cdot k = \omega(\mathrm{A}^T\mathtt{y})\cdot k \tag{1.18}$$

and noting that the first column of A^T is exactly k (by (1.14)), by (1.2) and (1.3) we obtain

$$|\partial_{\mathtt{y}_1}\mathtt{h}_0^k(\mathtt{y}) - \partial_{\mathtt{y}_1}\mathtt{h}_0^k(\mathtt{y}_0)| \leqslant L|k||\mathtt{y}-\mathtt{y}_0|, \qquad \forall\ \mathtt{y},\mathtt{y}_0\in \mathtt{B}^k_{2\mathtt{r}_k} \tag{1.19}$$

and, by convexity, for $(\mathtt{y}_1,\hat{\mathtt{y}}),(\mathtt{y}_1',\hat{\mathtt{y}})\in\mathtt{B}^k_{2\mathtt{r}_k}$,

$$\partial_{\mathtt{y}_1}\mathtt{h}_0^k(\mathtt{y}_1,\hat{\mathtt{y}}) - \partial_{\mathtt{y}_1}\mathtt{h}_0^k(\mathtt{y}_1',\hat{\mathtt{y}}) \geqslant \gamma|k|^2(\mathtt{y}_1-\mathtt{y}_1'), \quad \forall\ \mathtt{y}_1>\mathtt{y}_1'. \tag{1.20}$$

Indeed, by Lagrange's theorem, there exists a point $\mathtt{z}_1$ between $\mathtt{y}_1$ and $\mathtt{y}_1'$ such that[15)]

$$\begin{aligned}\partial_{\mathtt{y}_1}\mathtt{h}_0^k(\mathtt{y}_1,\hat{\mathtt{y}}) - \partial_{\mathtt{y}_1}\mathtt{h}_0^k(\mathtt{y}_1',\hat{\mathtt{y}}) \quad &= \quad \partial^2_{\mathtt{y}_1}\mathtt{h}_0^k(\mathtt{z}_1,\hat{\mathtt{y}})(\mathtt{y}_1-\mathtt{y}_1')\\ &\overset{(1.16)}{=} \quad \Big(\partial_y^2 h\big(\mathrm{A}^T(\mathtt{z}_1,\hat{\mathtt{y}})\big)k\cdot k\Big)(\mathtt{y}_1-\mathtt{y}_1')\\ &\overset{(1.2)}{\geqslant} \quad \gamma|k|^2(\mathtt{y}_1-\mathtt{y}_1').\end{aligned}$$

In these new variables the simple resonance $\omega(y)\cdot k=0$ becomes $\partial_{\mathtt{y}_1}\mathtt{h}_0^k(\mathtt{y})=0$.

We therefore define a (suitable) real neighborhood of the simple resonance in $\mathtt{B}^k$ by letting

$$\mathtt{Z}^k := \Big\{\mathtt{y}\in\mathrm{Re}\Big(\mathtt{B}^k_{\frac54\mathtt{r}_k}\Big) : \partial_{\mathtt{y}_1}\mathtt{h}_0^k(\mathtt{y})=0\Big\}. \tag{1.21}$$

We also set

$$\mathtt{Z}^k_\varpi := \big\{\mathtt{y}\in\mathrm{Re}(\mathtt{B}^k_{\mathtt{r}_k}) : \partial_{\mathtt{y}_1}\mathtt{h}_0^k(\mathtt{y})=\varpi\big\}, \tag{1.22}$$

so that $\mathtt{Z}^k\supseteq\mathtt{Z}^k_0$.

Now, for a fixed $k\in\mathcal{G}^n$, one may express (because of convexity) the resonant hyper-surface $\mathtt{Z}^k$ in (1.21) as a graph over the last $n-1$ actions $\hat{\mathtt{y}}:=(\mathtt{y}_2,\dots,\mathtt{y}_n)$, finding $\mathtt{y}_1=\eta(\hat{\mathtt{y}})$.

Let

$$\mathfrak{r}_k := \frac{\tilde\tau_k r}{8n^{3/2}|k|} \leqslant \frac{\mathtt{r}_k}{8\sqrt{n}}, \quad \tau_k := \frac{\gamma|k|}{2L}, \quad \tilde\tau_k := \min\{1,\tau_k\}. \tag{1.23}$$

and define the following $(n-1)$-dimensional cubes of edge $\mathfrak{r}_k$ and the lower left corner in $\mathfrak{r}_k j$ with $j\in\mathbb{Z}^{n-1}$:

$$\mathcal{Q}_j := \mathfrak{r}_k\cdot\big(j+[0,1)^{n-1}\big)\subset\mathbb{R}^{n-1}, \qquad j\in\mathbb{Z}^{n-1}. \tag{1.24}$$

Note that $\mathcal{Q}_j\cap\mathcal{Q}_{j'}=\emptyset$ if $j\neq j'$, while $\bigsqcup_{j\in\mathbb{Z}^{n-1}}\mathcal{Q}_j=\mathbb{R}^{n-1}$. Define the projection of a set $S\subset\mathbb{R}^n$ on a set $\hat E\subset R^{n-1}$ as

$$\Pi_{\hat E}S := \{\hat{\mathtt{y}}\in\hat E\ :\ \exists\mathtt{y}_1\in\mathbb{R} \text{ with } (\mathtt{y}_1,\hat{\mathtt{y}})\in S\}.$$

Set

$$J := \{j\in\mathbb{Z}^{n-1}\ :\ \Pi_{\mathcal{Q}_j}\mathtt{Z}^k\neq\emptyset\}. \tag{1.25}$$

Since $\mathtt{Z}^k$ is bounded, J is a finite set. Note that

$$\Pi_{\mathbb{R}^{n-1}}\mathtt{Z}^k\subset\mathcal{Q}:=\bigsqcup_{j\in J}\mathcal{Q}_j\subset\Pi_{\mathbb{R}^{n-1}}\Big(\mathrm{Re}\Big(\mathtt{B}^k_{\frac32\mathtt{r}_k}\Big)\Big). \tag{1.26}$$

15) Note that $(\mathtt{z}_1,\hat{\mathtt{y}})\in\mathtt{B}^k_{2\mathtt{r}_k}$ since $\mathtt{B}^k_{2\mathtt{r}_k}$ is a convex set.

Proposition 1. *There exists a real-analytic function*

$$\eta_0^k : [-\varpi_0^k, \varpi_0^k] \times \mathcal{Q} \subset \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}, \qquad \varpi_0^k := \sqrt{n} L |k| \mathfrak{r}_k, \tag{1.27}$$

such that

$$\partial_{\mathtt{y}_1} \mathtt{h}_0^k\big(\eta_0^k(\varpi, \hat{\mathtt{y}}), \hat{\mathtt{y}}\big) = \varpi \tag{1.28}$$

and, for any $\varpi \in [-\varpi_0^k, \varpi_0^k]$,

$$\mathtt{Z}_\varpi^k \subseteq \tilde{\mathtt{Z}}_\varpi^k := \left\{ \big(\eta_0^k(\varpi, \hat{\mathtt{y}}), \hat{\mathtt{y}}\big) \;:\; \hat{\mathtt{y}} \in \mathcal{Q} \right\} \subset \mathrm{Re}\left(\mathtt{B}^k_{\frac{3}{2}\mathtt{r}_k}\right). \tag{1.29}$$

Moreover,

$$\sup_{[-\varpi_0^k, \varpi_0^k] \times \mathcal{Q}} |\partial_\varpi \eta_0^k| \leqslant \frac{1}{\gamma |k|^2}. \tag{1.30}$$

Finally, $\eta_0^k(0, \cdot)$ *possesses a holomorphic extension on*[16] $\mathcal{Q}_{\hat{\mathtt{r}}_k}$, *with, for* $0 < \mathtt{r} \leqslant \mathtt{r}_k$,

$$\sup_{\mathcal{Q}_{\hat{\mathtt{r}}_k}} |\operatorname{Im} \eta_0^k(0, \cdot)| \leqslant \frac{1}{\tau_k} \hat{\mathtt{r}}_k, \qquad \hat{\mathtt{r}}_k := \frac{1}{2^9 n} \tilde{\tau}_k^2 \mathtt{r}. \tag{1.31}$$

Proof 1. We first construct the function η_0^k. Fix $\varpi \in [-\varpi_0^k, \varpi_0^k]$ and $\hat{\mathtt{y}} \in \mathcal{Q}_j$ for some $j \in J$ (recall (1.24)). By (1.21) and the definition of J in (1.25) we know that there exists a given

$$\mathtt{z} = (\mathtt{z}_1, \hat{\mathtt{z}}) \in \mathrm{Re}(\mathtt{B}^k_{5\mathtt{r}_k/4})$$

with $\hat{\mathtt{z}} \in \mathcal{Q}_j$ and $\partial_{\mathtt{y}_1} \mathtt{h}_0^k(\mathtt{z}) = 0$. Since $\hat{\mathtt{y}}, \hat{\mathtt{z}} \in \mathcal{Q}_j$, by (1.23) we have that

$$(\mathtt{z}_1, \hat{\mathtt{y}}) \in \mathrm{Re}(\mathtt{B}^k_{11\mathtt{r}_k/8}). \tag{1.32}$$

Setting

$$\tilde{\mathfrak{r}}_k := 2 \frac{\sqrt{n} L}{\gamma |k|} \mathfrak{r}_k \overset{(1.23)}{\leqslant} \frac{r}{8n|k|} \overset{(1.17)}{\leqslant} \frac{\mathtt{r}_k}{8}, \tag{1.33}$$

we have that the segment $[\mathtt{z}_1 - \tilde{\mathfrak{r}}_k, \mathtt{z}_1 + \tilde{\mathfrak{r}}_k] \times \{\hat{\mathtt{y}}\}$ belongs to the convex set $\mathrm{Re}\left(\mathtt{B}^k_{\frac{3}{2}\mathtt{r}_k}\right)$, namely,

$$[\mathtt{z}_1 - \tilde{\mathfrak{r}}_k, \mathtt{z}_1 + \tilde{\mathfrak{r}}_k] \times \{\hat{\mathtt{y}}\} \subset \mathrm{Re}\left(\mathtt{B}^k_{\frac{3}{2}\mathtt{r}_k}\right). \tag{1.34}$$

Since, by (1.19), we get

$$|\partial_{\mathtt{y}_1} \mathtt{h}_0^k(\mathtt{z}_1, \hat{\mathtt{y}})| = |\partial_{\mathtt{y}_1} \mathtt{h}_0^k(\mathtt{z}_1, \hat{\mathtt{y}}) - \partial_{\mathtt{y}_1} \mathtt{h}_0^k(\mathtt{z}_1, \hat{\mathtt{z}})| \leqslant L|k| |\hat{\mathtt{y}} - \hat{\mathtt{z}}| \leqslant \sqrt{n} L |k| \mathfrak{r}_k\,,$$

we have that

$$\begin{aligned} \partial_{\mathtt{y}_1} \mathtt{h}_0^k(\mathtt{z}_1 + \tilde{\mathfrak{r}}_k, \hat{\mathtt{y}}) - \varpi \quad &\overset{(1.20)}{\geqslant} \quad -\varpi_0^k + \gamma |k|^2 \tilde{\mathfrak{r}}_k + \partial_{\mathtt{y}_1} \mathtt{h}_0^k(\mathtt{z}_1, \hat{\mathtt{y}}) \\ &\geqslant \quad -\varpi_0^k + \gamma |k|^2 \tilde{\mathfrak{r}}_k - \sqrt{n} L |k| \mathfrak{r}_k = 0, \end{aligned}$$

by (1.27) and (1.33). Analogously, $\partial_{\mathtt{y}_1} \mathtt{h}_0^k(\mathtt{z}_1 - \tilde{\mathfrak{r}}_k, \hat{\mathtt{y}}) - \varpi \leqslant 0$. Note that, since the domain $\mathrm{Re}(\mathtt{B}^k_{2\mathtt{r}_k})$ is convex, the function $\mathtt{y}_1 \to \partial_{\mathtt{y}_1} \mathtt{h}_0^k(\mathtt{y}_1, \hat{\mathtt{y}})$ is defined on an interval, which contains $[\mathtt{z}_1 - \tilde{\mathfrak{r}}_k, \mathtt{z}_1 + \tilde{\mathfrak{r}}_k]$; moreover, it is continuous and strictly increasing by (1.20). As a consequence, there exists a unique value

$$a \in [\mathtt{z}_1 - \tilde{\mathfrak{r}}_k, \mathtt{z}_1 + \tilde{\mathfrak{r}}_k]$$

[16] Obviously, one could holomorphically extend η_0^k also in its first variable, but we need here only the extension in the second variable.

such that $\partial_{\mathtt{y}_1}\mathtt{h}_0^k(a,\hat{\mathtt{y}}) = \varpi$. Then we set $\eta_0^k(\varpi,\hat{\mathtt{y}}) := a$. Recollecting, we have defined η_0^k in (1.27), satisfying (1.28). Moreover, the last inclusion in (1.29) follows by (1.34). The fact that η_0^k is real-analytic follows since $\mathtt{h}_0^k$ is real-analytic.

We now prove the first inclusion in (1.29). Fix $\varpi \in [-\varpi_0^k, \varpi_0^k]$. Recalling (1.22), let us take a point

$$\mathtt{u} = (\mathtt{u}_1, \hat{\mathtt{u}}) \in \mathtt{Z}_\varpi^k,$$

namely, $\mathtt{u} \in \mathrm{Re}(\mathtt{B}^k_{\mathtt{r}_k})$ and $\partial_{\mathtt{y}_1}\mathtt{h}_0^k(\mathtt{u}) = \varpi$. If $\hat{\mathtt{u}} \in \mathcal{Q}$, by unicity $\mathtt{u}_1 = \eta_0^k(\varpi,\hat{\mathtt{u}})$, and, therefore, $\mathtt{u} \in \tilde{\mathtt{Z}}^k_\varpi$. On the other hand, it is not possible that $\hat{\mathtt{u}} \notin \mathcal{Q}$. Indeed, assume, e. g., that $\varpi = 0$; then $\mathtt{u} \in \mathtt{Z}_0^k \subseteq \mathtt{Z}^k$ and, by (1.26), $\hat{\mathtt{u}} \in \mathcal{Q}$. On the other hand, when ϖ is negative, by (1.33), we have that the segment

$$[\mathtt{u}_1, \mathtt{u}_1 + \tilde{\mathtt{r}}_k] \times \{\hat{\mathtt{u}}\}$$

belongs to the convex set $\mathrm{Re}\left(\mathtt{B}^k_{\frac{5}{4}\mathtt{r}_k}\right)$. Moreover,

$$\partial_{\mathtt{y}_1}\mathtt{h}_0^k(\mathtt{u}_1 + \tilde{\mathtt{r}}_k, \hat{\mathtt{u}}) \overset{(1.20)}{\geqslant} \gamma|k|^2\tilde{\mathtt{r}}_k + \partial_{\mathtt{y}_1}\mathtt{h}_0^k(\mathtt{u}_1,\hat{\mathtt{u}}) = \gamma|k|^2\tilde{\mathtt{r}}_k + \varpi \geqslant \gamma|k|^2\tilde{\mathtt{r}}_k - \varpi_0^k > 0,$$

by (1.27) and (1.33). Again, by continuity, there exists a value b such that

$$\partial_{\mathtt{y}_1}\mathtt{h}_0^k(b,\hat{\mathtt{u}}) = 0.$$

Since $(b,\hat{\mathtt{u}}) \in \mathrm{Re}\left(\mathtt{B}^k_{\frac{5}{4}\mathtt{r}_k}\right)$ we have that $(b,\hat{\mathtt{u}}) \in \mathtt{Z}^k$, then, by (1.26), $\hat{\mathtt{u}} \in \mathcal{Q}$. The case when ϖ is positive is analogous. We conclude that $\hat{\mathtt{u}} \in \mathcal{Q}$ in all cases. The proof of the inclusion (1.29) is completed.

We finally show that $\eta_0^k(0,\cdot)$ has a holomorphic extension on $\mathcal{Q}_{\hat{\mathtt{r}}_k}$ and satisfies the estimate in (1.31). Fix a point $\mathtt{y}^0 = (\mathtt{y}_1^0, \hat{\mathtt{y}}^0) = \left(\eta_0^k(0,\hat{\mathtt{y}}^0), \hat{\mathtt{y}}^0\right) \in \tilde{\mathtt{Z}}_0^k$ with $\hat{\mathtt{y}}^0 \in \mathcal{Q} \subset \Pi_{\mathbb{R}^{n-1}}\left(\mathrm{Re}\left(\mathtt{B}^k_{\frac{3}{2}\mathtt{r}_k}\right)\right)$ by (1.26). By construction, $\partial_{\mathtt{y}_1}\mathtt{h}_0^k(\mathtt{y}_1^0,\hat{\mathtt{y}}^0) = 0$. Let $\mathtt{Y}_1$ be the complex closed ball of radius

$$\mathtt{r}_1 := \frac{1}{\tau_k}\hat{\mathtt{r}}_k, \tag{1.35}$$

centered at $\mathtt{y}_1^0$ and let $\hat{\mathtt{Y}}$ be the complex closed ball of radius $\hat{\mathtt{r}}_k$ centered at $\hat{\mathtt{y}}^0$. Note that, by (1.31),

$$\mathtt{r}_1, \hat{\mathtt{r}}_k \leqslant \frac{1}{2^9 n}\tilde{\tau}_k\mathtt{r} \leqslant \frac{1}{2^9 n}\tilde{\tau}_k\mathtt{r}_k \leqslant \frac{1}{2^9 n}\mathtt{r}_k. \tag{1.36}$$

Since, by (1.29), $\mathtt{y}^0 \in \tilde{\mathtt{Z}}_0^k \subset \mathrm{Re}\left(\mathtt{B}^k_{\frac{3}{2}\mathtt{r}_k}\right)$, we have

$$\mathtt{Y}_1 \times \hat{\mathtt{Y}} \subset \mathtt{B}^k_{\frac{7}{4}\mathtt{r}_k}. \tag{1.37}$$

Let E be the Banach space of the continuous functions $\eta : \hat{\mathtt{Y}} \to \mathbb{C}$ that are holomorphic in the interior of $\hat{\mathtt{Y}}$, endowed with the sup-norm. Let $\mathcal{C}$ be its closed subset formed by the functions $\eta : \hat{\mathtt{Y}} \to \mathtt{Y}_1$. We claim that the map

$$\eta(\cdot) \to \eta(\cdot) - \partial_{\mathtt{y}_1}\mathtt{h}_0^k\big(\eta(\cdot),\cdot\big)/d, \qquad d := \partial^2_{\mathtt{y}_1}\mathtt{h}_0^k(\mathtt{y}^0)$$

is a contraction on $\mathcal{C}$. The fixed point of the above map is the required (local) holomorphic extension of $\eta_0^k(0,\cdot)$, proving (1.31) by the definition of $\mathtt{r}_1$. Since, by (1.20), $d \geqslant \gamma|k|^2$, it is immediate to see that the fact that the above map is a contraction on $\mathcal{C}$ follows from the following two estimates:

$$\sup_{\hat{\mathtt{y}}\in\hat{\mathtt{Y}}} |\partial_{\mathtt{y}_1}\mathtt{h}_0^k(\mathtt{y}_1^0,\hat{\mathtt{y}})| \leqslant \frac{1}{2}\gamma|k|^2\mathtt{r}_1\,, \qquad \sup_{\mathtt{y}_1\in\mathtt{Y}_1,\hat{\mathtt{y}}\in\hat{\mathtt{Y}}} |\partial^2_{\mathtt{y}_1}\mathtt{h}_0^k(\mathtt{y}) - \partial^2_{\mathtt{y}_1}\mathtt{h}_0^k(\mathtt{y}^0)| \leqslant \frac{1}{2}\gamma|k|^2. \tag{1.38}$$

Note also that the estimate in (1.31) follows since the image of η_0^k is contained, by construction, in $\mathtt{Y}_1$ and by the definition of $\mathtt{r}_1$.

It remains to prove (1.38). Since $\partial_{\mathtt{y}_1}\mathtt{h}_0^k(\mathtt{y}^0)=0$, for any $\mathtt{y}\in\mathtt{B}^k_{2\mathtt{r}_k}$ we obtain

$$|\partial_{\mathtt{y}_1}\mathtt{h}_0^k(\mathtt{y})|=|\partial_{\mathtt{y}_1}\mathtt{h}_0^k(\mathtt{y})-\partial_{\mathtt{y}_1}\mathtt{h}_0^k(\mathtt{y}^0)|\overset{(1.19)}{\leqslant}L|k||\mathtt{y}-\mathtt{y}_0|, \tag{1.39}$$

in particular,

$$\sup_{\mathtt{B}^k_{2\mathtt{r}_k}}|\partial_{\mathtt{y}_1}\mathtt{h}_0^k(\mathtt{y})|\leqslant 2L|k|\mathtt{r}_k. \tag{1.40}$$

Then the first estimate in (1.38) follows by (1.39) since, for $\hat{\mathtt{y}}\in\hat{\mathtt{Y}}$,

$$|\partial_{\mathtt{y}_1}\mathtt{h}_0^k(\mathtt{y}_1^0,\hat{\mathtt{y}})|\leqslant L|k||\hat{\mathtt{y}}-\hat{\mathtt{y}}_0|\leqslant L|k|\hat{\mathtt{r}}_k\overset{(1.31),\,(1.35)}{=}\frac12\gamma|k|^2\mathtt{r}_1.$$

Let us finally prove the second estimate in (1.38). Fix $\mathtt{y}=(\mathtt{y}_1,\hat{\mathtt{y}})$ with $\mathtt{y}_1\in\mathtt{Y}_1$ and $\hat{\mathtt{y}}\in\hat{\mathtt{Y}}$; by (1.40), (1.37) and Cauchy estimates we obtain

$$|\partial^2_{\mathtt{y}_1}\mathtt{h}_0^k(\mathtt{y})-\partial^2_{\mathtt{y}_1}\mathtt{h}_0^k(\mathtt{y}^0)|\leqslant 64nL|k|\frac{\mathtt{r}_1+\hat{\mathtt{r}}_k}{\mathtt{r}_k}\overset{(1.36)}{\leqslant}\frac18 L|k|\left(\frac{1}{\tau_k}+1\right)\tilde{\tau}_k^2\overset{(1.31)}{\leqslant}\frac12\gamma|k|^2,$$

concluding the proof of the second estimate in (1.38).

Since

$$\partial_\varpi\eta_0^k(\varpi,\hat{\mathtt{y}})=\frac{1}{\partial^2_{\mathtt{y}_1}\mathtt{h}_0^k\big(\eta_0^k(\varpi,\hat{\mathtt{y}}),\hat{\mathtt{y}}\big)}$$

and, by (1.20) $\partial^2_{\mathtt{y}_1}\mathtt{h}_0^k\geqslant\gamma|k|^2$, we get (1.30). □

Recalling (1.6) and (1.16), we have that

$$\begin{aligned} D^k &:= A^{-T}\mathcal{R}^{1k} \\ &= \big\{\mathtt{y}\in\mathtt{B}^k : |\omega(A^T\mathtt{y})\cdot k|\leqslant\tfrac{\alpha}{\mathtt{C}},\text{and }|\,\pi_k^\perp\omega(A^T\mathtt{y})\cdot\ell|\geqslant\tfrac{3\alpha\mathtt{K}^{n+3}}{|k|},\forall\ell\in\mathcal{G}^n_{\mathtt{K}}\setminus\mathbb{Z}k\big\}\end{aligned} \tag{1.41}$$

and, by using (1.18), (1.28) and (1.29), we obtain

$$\begin{aligned} D^k = \big\{\mathtt{y}\in\mathtt{B}^k : \eta_0^k(-\tfrac{\alpha}{\mathtt{C}},\hat{\mathtt{y}})\leqslant\mathtt{y}_1\leqslant\eta_0^k(\tfrac{\alpha}{\mathtt{C}},\hat{\mathtt{y}})\,,\\ \text{and }|\,\pi_k^\perp\omega(A^T\mathtt{y})\cdot\ell|\geqslant\tfrac{3\alpha\mathtt{K}^{n+3}}{|k|},\forall\ell\in\mathcal{G}^n_{\mathtt{K}}\setminus\mathbb{Z}k\big\}.\end{aligned} \tag{1.42}$$

Let us define the normal set

$$\check{D}^k:=\big\{\mathtt{y}=(\mathtt{y}_1,\hat{\mathtt{y}})\ :\ \eta_0^k(-\tfrac{\alpha}{\mathtt{C}},\hat{\mathtt{y}})\leqslant\mathtt{y}_1\leqslant\eta_0^k(\tfrac{\alpha}{\mathtt{C}},\hat{\mathtt{y}}),\ \hat{\mathtt{y}}\in\hat{D}^k\big\},\quad \hat{D}^k:=\Pi_{\mathbb{R}^{n-1}}D^k. \tag{1.43}$$

It is obvious that

$$D^k\subseteq\check{D}^k. \tag{1.44}$$

We recall that, given lattice $\Lambda\subset\mathbb{Z}^n$, one says that a set $D\subseteq^n$ is (α,K) nonresonant modulo Λ for a Hamiltonian $h(y)$ defined on D, when $|\partial_y h(y)\cdot k|\geqslant\alpha$ for every $0<|k|\leqslant K$, $k\notin\Lambda$, $y\in D$.

Lemma 1. *For* $\mathtt{K}$ *large enough, the set* $\check{D}^k$ *is* $\left(\frac{2\alpha\mathtt{K}^{n+3}}{|k|},\mathtt{K}\right)$ *nonresonant modulo* $\Lambda:=\mathbb{Z}(1,0,\ldots,0)$.

Proof. Let $\mathtt{y}=(\mathtt{y}_1,\hat{\mathtt{y}})\in\check{D}^k$. Then $\eta_0^k(-\frac{\alpha}{\mathtt{C}},\hat{\mathtt{y}})\leqslant\mathtt{y}_1\leqslant\eta_0^k(\frac{\alpha}{\mathtt{C}},\hat{\mathtt{y}})$ and there exists $\mathtt{z}_1$ with $\eta_0^k(-\frac{\alpha}{\mathtt{C}},\hat{\mathtt{y}})\leqslant\mathtt{z}_1\leqslant\eta_0^k(\frac{\alpha}{\mathtt{C}},\hat{\mathtt{y}})$ such that $\mathtt{z}=(\mathtt{z}_1,\hat{\mathtt{y}})\in D^k$. For any $\ell\in\mathcal{G}^n_{\mathtt{K}}\setminus\mathbb{Z}k$ we have that $|\,\pi_k^\perp\omega(A^T\mathtt{z})\cdot\ell|\geqslant\frac{3\alpha\mathtt{K}^{n+3}}{|k|}$. Since

$$|\mathtt{z}-\mathtt{y}|=|\mathtt{z}_1-\mathtt{y}_1|\leqslant\eta_0^k\left(\frac{\alpha}{\mathtt{C}},\hat{\mathtt{y}}\right)-\eta_0^k\left(-\frac{\alpha}{\mathtt{C}},\hat{\mathtt{y}}\right)\overset{(1.30)}{\leqslant}\frac{2\alpha}{\gamma|k|^2\mathtt{C}},$$

by (1.3) we get

$$|\,\pi_k^\perp\omega(A^T\mathtt{y})\cdot\ell|\geqslant\frac{2\alpha\mathtt{K}^{n+3}}{|k|},$$

for $\mathtt{K}$ large enough. □

2. OUTLINE OF PROOFS

We are now ready to outline the main steps needed to extend singular KAM theory, which, as a consequence, yields the announced lower bound on the measure of primary and secondary tori for generic perturbation.

(1) Applying the symplectic transformation Φ_0 in (1.15) to the Hamiltonian H in (1.1) one obtains (recall (1.16)),

$$\mathtt{H}(\mathtt{y},\mathtt{x}) = \mathtt{h}_0^k(\mathtt{y}) + \varepsilon f(\mathrm{A}^{-1}\mathtt{x}). \tag{2.1}$$

Now, one can do high-order averaging theory. More precisely, by Lemma 1, $\check{D}^k$ in (1.43) is $\big(\frac{2\alpha\mathtt{K}^{n+3}}{|k|}, \mathtt{K}\big)$ nonresonant modulo

$$\Lambda := \mathbb{Z}(1, 0, \dots, 0),$$

and, taking $\mathtt{K}$ suitably large with ε small, e. g., $\mathtt{K} \sim |\ln\varepsilon|^2$, via a close-to-the-identity symplectic transformation defined in a suitable complex neighborhood of $\check{D}^k \times \mathbb{T}^n$, one can put the Hamiltonian (2.1) in normal form:

$$\mathtt{h}_0^k(\mathtt{y}) + \varepsilon\big(g_{\mathrm{o}}^k(\mathtt{y}) + f_1^k(\mathtt{x}_1) + g^k(\mathtt{y},\mathtt{x}_1) + f^k(\mathtt{y},\mathtt{x})\big),$$

where:

- $g^k(\mathtt{y},\mathtt{x}_1) = o(1)$ as $\varepsilon \to 0$;
- $f_1^k(\mathtt{x}_1)$ is the average of $f(\mathrm{A}^{-1}\mathtt{x})$ with respect to the fast angles $\mathtt{x}_2, \dots, \mathtt{x}_n$;
- f^k is (almost[17)]) exponentially small.

For details, see, e. g., Section 2 of [6].

(2) Disregarding the exponentially small term f^k, we consider, now, the "effective Hamiltonian"

$$\mathtt{h}_0^k(\mathtt{y}) + \varepsilon\big(g_{\mathrm{o}}^k(\mathtt{y}) + f_1^k(\mathtt{x}_1) + g^k(\mathtt{y},\mathtt{x}_1)\big),$$

which is a one-degree-of-freedom Hamiltonian (in the dynamic variables $\mathtt{y}_1$ and $\mathtt{x}_1$) depending on the "dumb actions"

$$\hat{\mathtt{y}} := (\mathtt{y}_2, \dots, \mathtt{y}_n)$$

as parameters. Then, one can construct a symplectic transformation

$$\mathtt{y}_1 = \mathtt{p}_1 + \tilde{\eta}_0^k(\hat{\mathtt{p}}), \qquad \mathtt{x}_1 = \mathtt{q}_1, \qquad \hat{\mathtt{y}} = \hat{\mathtt{p}},$$

for a suitable function[18)] $\tilde{\eta}_0^k(\hat{\mathtt{p}}) = \eta_0^k(0,\hat{\mathtt{p}}) + O(\varepsilon)$, such that, in the new variables, the effective Hamiltonian takes the form

$$\mathrm{h}^k(\mathtt{p}) + \varepsilon\big(f_1^k(\mathtt{q}_1) + \mathrm{g}^k(\mathtt{p},\mathtt{q}_1)\big),$$

where[19)]

$$\partial_{\mathtt{p}_1}\mathrm{h}^k(0,\hat{\mathtt{p}}) = 0 \tag{2.2}$$

and $\mathrm{g}^k = o(1)$.

Note that the above transformation can be sympectically completed in the variables $\hat{\mathtt{x}} = \hat{\mathtt{q}} - \mathtt{q}_1\partial_{\hat{\mathtt{p}}}\tilde{\eta}_0^k(\hat{\mathtt{p}})$, however, this transformation is not well defined on $\mathbb{T}^n$, since it is obviously not periodic in $\mathtt{q}_1$. The way to overcome this problem is explained in Section 3 of [7].

---

17) I.e., smaller than any power of ε.

18) Recall Proposition 1.

19) Recall (1.28) with $\varpi = 0$.

(3) Fixing an arbitrary point $\hat{\mathtt{p}}_0 \in \hat{D}^k$ (compare (1.43)) and a Taylor series expanding the Hamiltonian at $\mathtt{p}_1 = 0$, one gets, by (2.2),

$$\mathrm{h}^k(0, \hat{\mathtt{p}}) + \mathrm{m}_k\big(1 + O(|\hat{\mathtt{p}} - \hat{\mathtt{p}}_0|) + O(|\mathtt{p}_1|)\big)\mathtt{p}^2 + \varepsilon\big(f_1^k(\mathtt{q}_1) + \mathrm{g}^k(\mathtt{p}, \mathtt{q}_1)\big), \tag{2.3}$$

where, by convexity,

$$\mathrm{m}_k := \frac{1}{2}\partial^2_{\mathtt{p}_1}\mathrm{h}^k(0, \hat{\mathtt{p}}_0) > 0.$$

It is easy to see that, up to the inessential term $\mathrm{h}^k(0, \hat{\mathtt{p}})$ and rescaling by m_k, the Hamiltonian in (2.3), for $|\hat{\mathtt{p}} - \hat{\mathtt{p}}_0|$ small, can be put into the “standard form”

$$\big(1 + \nu(\mathtt{p}, \mathtt{q}_1)\big)\mathtt{p}^2 + \varepsilon\big(G_0(\mathtt{q}_1) + G(\mathtt{p}, \mathtt{q}_1)\big),$$

where

$$G_0(\mathtt{q}_1) := f_1^k(\mathtt{q}_1)/\mathrm{m}_k$$

and ν and G are small; compare Section 2.2 of [7]. This Hamiltonian is suitable for action-angle variables as discussed in [5].

At this point no further technical difficulties arise and the singular KAM theory applies as in [7], leading to the announced measure estimates on primary and secondary tori.

FUNDING

L. Biasco and L. Chierchia were supported by grant NRR-M4C2-I1.1-PRIN 2022-PE1-Stability in Hamiltonian dynamics and beyond-F53D23002730006-Financed by E.U.–NextGenerationEU.

Santiago Barbieri was supported by the Juan de la Cierva Postdoctoral Grant JDC2023-052632-I funded by the Spanish National Agency for Research (Agencia Estatal de Investigación).

CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

REFERENCES

1. Arnol’d, V. I., Proof of a Theorem of A. N. Kolmogorov on the Invariance of Quasi-Periodic Motions under Small Perturbations of the Hamiltonian, *Russian Math. Surveys,* 1963, vol. 18, no. 5, pp. 9–36; see also: *Uspekhi Mat. Nauk,* 1963, vol. 18, no. 5, pp. 13–40.
2. Arnol’d, V. I., Kozlov, V. V., and Neĭshtadt, A. I., *Mathematical Aspects of Classical and Celestial Mechanics,* 3rd ed., Encyclopaedia Math. Sci., vol. 3, Berlin: Springer, 2006.
3. Barbieri, S., Stability in Hamiltonian Systems: Steepness and Regularity in Nekhoroshev Theory, *PhD Thesis,* Université Paris-Saclay, Paris, France, Università degli studi Roma Tre, Roma, Italy, 2023, 325 pp.
4. Biasco, L. and Chierchia, L., On the Topology of Nearly-Integrable Hamiltonians at Simple Resonances, *Nonlinearity,* 2020, vol. 33, no. 7, pp. 3526–3567.
5. Biasco, L. and Chierchia, L., Complex Arnol’d – Liouville Maps, *Regul. Chaotic Dyn.,* 2023, vol. 28, nos. 4–5, pp. 395–424.
6. Biasco, L. and Chierchia, L., Global Properties of Generic Real-Analytic Nearly-Integrable Hamiltonian Systems, *J. Differential Equations,* 2024, vol. 385, pp. 325–361.
7. Biasco, L. and Chierchia, L., Singular KAM Theory, arXiv:2309.17041 (2023).
8. Kolmogorov, A. N., Preservation of Conditionally Periodic Movements with Small Change in the Hamilton Function, in *Stochastic Behaviour in Classical and Quantum Hamiltonian Systems (Volta Memorial Conference, Como, 1977),* G. Casati, J. Ford (Eds.), Lect. Notes Phys. Monogr., vol. 93, Berlin: Springer, 1979, pp. 51–56. See also: *Dokl. Akad. Nauk SSSR (N. S.),* 1954, vol. 98, pp. 527–530 (Russian).
9. Lazutkin, V. F., On Moser’s Invariant Curves Theorem, in *Questions of the Dynamic Theory of Seismic Wave Propagation: Vol. 14,* Leningrad: Nauka, 1974, pp. 109–120 (Russian).
10. Falcolini, C. and Zaccaria, D., On the Analytic Properties of the Perturbing Function in the PCR3Body, *Preprint* (2025).

11. Medvedev, A. G., Neishtadt, A. I., and Treschev, D. V., Lagrangian Tori near Resonances of Near-Integrable Hamiltonian Systems, *Nonlinearity,* 2015, vol. 28, no. 7, pp. 2105–2130.
12. Moser, J., On Invariant Curves of Area-Preserving Mappings of an Annulus, *Nach. Akad. Wiss. Göttingen, Math. Phys. Kl. II,* 1962, vol. 1962, no. 1, pp. 1–20.
13. Neishtadt, A. I., Estimates in the Kolmogorov Theorem on Conservation of Conditionally Periodic Motions, *J. Appl. Math. Mech.,* 1981, vol. 45, no. 6, pp. 766–772; see also: *Prikl. Mat. Mekh.,* 1981, vol. 45, no. 6, pp. 1016–1025.
14. Pöschel, J., Integrability of Hamiltonian Systems on Cantor Sets, *Comm. Pure Appl. Math.,* 1982, vol. 35, no. 5, pp. 653–696.
15. Svanidze, N. V., Small Perturbations of an Integrable Dynamical System with an Integral Invariant, *Proc. Steklov Inst. Math.,* 1981, vol. 147, pp. 127–151; see also: *Trudy Mat. Inst. Steklov,* 1980, vol. 147, pp. 124–146.
16. Yomdin, Y. and Comte, G., *Tame Geometry with Application in Smooth Analysis,* Lect. Notes in Math., vol. 1834, Berlin: Springer, 2004.

Publisher's note. Pleiades Publishing remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

AI tools may have been used in the translation or editing of this article.