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Mathematical Analysis. – *Quasi-periodic motions in generic nearly-integrable mechanical systems*, by Luca Biasco and Luigi Chierchia, communicated on 22 April 2022.

Dedicated to the memory of Professor A. Ambrosetti.

Abstract. – In this note, we present and briefly discuss results, which include as a particular case the theorem announced in [Rend. Lincei Mat. Appl. 26 (2015), 423–432], concerning the typical behavior of nearly-integrable mechanical systems with generic analytic potentials.

Keywords. – Nearly-integrable Hamiltonian systems, normal forms, KAM theory.

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In 2015, encouraged by our mentor, colleague, and friend Antonio Ambrosetti, we published in *Atti della Accademia Nazionale dei Lincei* an announcement [2] concerning "typical" trajectories of nearly-integrable Hamiltonian systems. In particular, we stated a theorem [2, p. 426], which, can be roughly rephrased as follows:

In bounded regions of phase space, except for a set of measure $\varepsilon|\log\varepsilon|^{\gamma}$, trajectories of nearly-integrable mechanical systems on $\mathbb{R}^n \times \mathbb{T}^n$ with generic real-analytic potentials of size $\varepsilon \ll 1$ are quasi-periodic and span n-tori invariant for such systems.

This theorem is in agreement (up to the logarithmic correction) with a conjecture formulated by Arnold, Kozlov, and Neishtadt in the Springer Encyclopaedia of Mathematical Sciences [1, Chapter 6, p. 285].

A complete proof of the above result turned out to be much longer and more delicate than we thought and it has been completed only recently in [6, 7] which in turn exploit intermediate results published in [3, 5].

The purpose of this short note is to communicate the precise results of [7], which, as a particular case, yield the above theorem.

In order to state the main results in [7], we need to recall a few notions and give some definitions.

(a) *Hamiltonian systems on* $\mathbb{R}^n \times \mathbb{T}^n$

Given a region $B \subseteq \mathbb{R}^n$, the "phase space" $\mathcal{M} := B \times \mathbb{T}^n$ (where $\mathbb{T}^n := \mathbb{R}^n/(2\pi\mathbb{Z}^n)$), and a real analytic "Hamiltonian function" $H : \mathcal{M} \to \mathbb{R}$, we denote by $z \in \mathcal{M} \to$

$\Phi^t_H(z) \in \mathcal{M}$ the *Hamiltonian flow generated by* H, namely, the solution of the standard Hamilton equations

$$\begin{cases} \dot y = -H_x(y,x), \\ \dot x = H_y(y,x), \end{cases} \qquad (y,x)|_{t=0} = z,$$

where, as usual, "dot" denotes derivative with respect to "time" $t \in \mathbb{R}$, and H_y, H_x the gradient with respect to y, x.

A *mechanical system* on $\mathbb{R}^n \times \mathbb{T}^n$ is a Hamiltonian system with Hamiltonian

$$\mathrm{H}(y,x) = \frac{1}{2}|y|^2 + f(x), \quad \Big(\text{where } |y|^2 := y \cdot y := \sum_j |y_j|^2\Big),$$

whose evolution equations are equivalent to the Newton equation $\ddot x = -f_x(x)$; f is called the *potential* of the system; "nearly-integrable" means that the potential is of the form εf with ε a small real parameter.

(b) *Diophantine vectors*

A vector $\omega \in \mathbb{R}^n$ is called *Diophantine* if there exist $\alpha > 0$ and $\tau \geq n-1$ such that $|\omega \cdot k| \geq \alpha / |k|_1^\tau$, for any non-vanishing integer vector $k \in \mathbb{Z}^n$, where $|k|_1 := \sum |k_j|$.

(c) *Maximal KAM tori*

A set $\mathcal{T} \subseteq \mathcal{M}$ is a *maximal KAM torus* for a Hamiltonian function H if there exist a real analytic embedding $\phi : \mathbb{T}^n \to \mathcal{M}$ and a Diophantine frequency vector $\omega \in \mathbb{R}^n$ such that $\mathcal{T} = \phi(\mathbb{T}^n)$ and for each $z \in \mathcal{T}$, $\Phi^t_H(z) = \phi(x + \omega t)$, where $x = \phi^{-1}(z)$. For general information on KAM (Kolmogorov, Arnold, Moser) theory, see [1] and references therein.

(d) *Generators of* 1*d maximal lattices*

Let $\mathbb{Z}^n_\star$ be the set of integer vectors $k \neq 0$ in $\mathbb{Z}^n$ such that the first non-null component is positive:

$$\mathbb{Z}^n_\star := \big\{k \in \mathbb{Z}^n : k \neq 0 \text{ and } k_j > 0 \text{ where } j = \min\{i : k_i \neq 0\}\big\}.$$

$\mathcal{G}^n$ denotes the set of *generators of* 1*d maximal lattices* in $\mathbb{Z}^n$, namely, the set of vectors $k \in \mathbb{Z}^n_\star$ such that the greater common divisor (gcd) of their components is 1:

$$\mathcal{G}^n := \big\{k \in \mathbb{Z}^n_\star : \gcd(k_1, \dots, k_n) = 1\big\}.$$

(e) *Resonances*

A resonance $\mathcal{R}_k$ with respect to the free Hamiltonian $\frac{1}{2}|y|^2$ is the set $\{y \in \mathbb{R}^n : y \cdot k = 0\}$, where $k \in \mathcal{G}^n$. We call $\mathcal{R}_{k,\ell}$ a *double resonance* if $\mathcal{R}_{k,\ell} = \mathcal{R}_k \cap \mathcal{R}_\ell$ with k and ℓ in $\mathcal{G}^n$ linearly independent; the order of a double resonance is given by $\max\{|k|_1, |\ell|_1\}$.

(f) 1*d Fourier projectors*

Given $k \in \mathbb{Z}^n \backslash \{0\}$ and a periodic analytic function $f : \mathbb{T}^n \to \mathbb{C}$, we denote by $\pi_{\mathbb{Z}k} f$ the (analytic) periodic function of *one variable* $\theta \in \mathbb{T}$ given by

$$\theta \in \mathbb{T} \mapsto \pi_{\mathbb{Z}k} f(\theta) := \sum_{j \in \mathbb{Z}} f_{jk} e^{\mathrm{i} j\theta}.$$

Note that $f(x) = \sum_{k \in \mathcal{G}^n} \pi_{\mathbb{Z}k} f(k \cdot x)$.

(g) *Morse functions with distinct critical values*

A function $\theta \to F(\theta)$ is a Morse function if its critical points are non-degenerate, i.e., $F'(\theta_0) = 0 \implies F''(\theta_0) \neq 0$; "distinct critical values" means that if $\theta_1 \neq \theta_2$ are distinct critical points, then $F(\theta_1) \neq F(\theta_2)$.

(h) *A Banach space of real analytic functions*

Let $s > 0$. We denote by $\mathbb{B}^n_s$ the Banach space of real analytic periodic functions on $\mathbb{T}^n$ having zero average:

$$\mathbb{B}^n_s := \left\{ f = \sum_{\substack{k \in \mathbb{Z}^n \\ k \neq 0}} f_k e^{\mathrm{i} k \cdot x} \text{ s.t. } \bar{f_k} = f_{-k} \text{ and } \|f\|_s < \infty \right\},$$

where $\|f\|_s := \sup_{k \in \mathbb{Z}^n} |f_k| e^{|k|_1 s}$.

(i*) *The generic set* $\mathbb{P}^n_s$ *of potentials*

We denote by $\mathbb{P}^n_s$ the subset of the unit ball of $\mathbb{B}^n_s$ given by the set of functions $f \in \mathbb{B}^n_s$ such that the following two conditions hold:

$$\underline{\lim}_{\substack{|k|_1 \to +\infty \\ k \in \mathcal{G}^n}} |f_k| e^{|k|_1 s} |k|_1^n > 0,$$

$$\forall k \in \mathcal{G}^n, \ \pi_{\mathbb{Z}k} f \text{ is a Morse function with distinct critical values.}$$

We remark that all the above definitions are standard, except for the last one which describes the class of potentials for which our results hold.

$\mathbb{P}^n_s$ is a typical set in many ways: it contains an open and dense set (in the topology of $\mathbb{B}^n_s$), it has full measure with respect to standard probability measures on the unit ball of $\mathbb{B}^n_s$. For a detailed discussion of the properties of $\mathbb{P}^n_s$, see [5, Section 3] and [7, Appendix A.2]. We also remark that the definition given here simplifies and extends former definitions given in [2, 5].

We can now state the main results in [7].

THEOREM 1. *Let $n \geq 2$, $s > 0$, $0 < \varepsilon < 1$, $f \in \mathbb{P}^n_s$, B an open ball in $\mathbb{R}^n$, and $\mathtt{H}(y, x; \varepsilon) := \frac{1}{2}|y|^2 + \varepsilon f(x)$. Then, there exists a constant $c > 1$ such that all points in $B \times \mathbb{T}^n$ lie on a maximal KAM torus for $\mathtt{H}$, except for a subset of measure bounded by $c\, \varepsilon |\log \varepsilon|^\gamma$ with $\gamma := 11n + 4$.*

THEOREM 2. *Fix $0 < a < 1$. For any $\varepsilon > 0$, there exists an open neighborhood $\mathcal{D}^2 \subseteq B$ of double resonances of order smaller than $1/\varepsilon^b$, with $b := \frac{1-a}{\gamma}$, which satisfies* $\mathrm{meas}(\mathcal{D}^2 \times \mathbb{T}^n) \le \mathrm{c_o}\varepsilon^a$, *for a suitable constant* $\mathrm{c_o}$ *(depending only on n), such that the following holds. Under the assumptions of Theorem 1, there exists a positive constant $\hat{c}$ (independent of a) such that all points in $(B \setminus \mathcal{D}^2) \times \mathbb{T}^n$ lie on a maximal KAM torus for* $\mathtt{H}$, *except for an exponentially small subset of measure bounded by $e^{-\hat{c}/\varepsilon^b}$.*

THEOREM 3. *Let the assumptions of Theorem 1 hold and let $n = 2$. There exists a constant $\bar{c} > 0$ such that, for every $0 < a < 1$, all points in $\{y \in B : |y| > \varepsilon^{a/2}\} \times \mathbb{T}^2$ lie on a maximal KAM torus for* $\mathtt{H}$, *except for an exponentially small subset of measure bounded by $e^{-\bar{c}/\varepsilon^b}$, with $b = (1-a)/24$.*

Let us a make a few observations.

Theorem 1 – which extends the result in [2] – may be viewed as the "ultimate frontier of KAM Theory", in the sense that, as remarked by Arnold et al., near double resonances, there are regions of order ε where the dynamics of $\mathtt{H}$ is equivalent to the dynamics of the parameter-free Hamiltonian $\frac{1}{2}|y|^2 + f(x)$ and, therefore, *it is natural to expect that in a generic system with three or more degrees of freedom the measure of the "non-torus" set has order ε* [1, Remark 6.18, p. 285]. Theorem 1 provides an upper bound on the measure of the non-torus set in agreement (up to the logarithmic correction $|\log \varepsilon|^\gamma$) with this expectation. On the other hand, rigorous lower bounds on such a measure appear to be extremely hard to be proven in the analytic case; for partial results in the Gevrey case, see [8].

The KAM tori constructed in Theorem 1 are *not* uniformly distributed in phase space. Indeed, if one stays away from a finite number of double resonances, the density is exponentially small: this is the main content of Theorem 2.

Theorem 3 is a consequence of Theorem 2, since in dimension 2, the only double resonance in a mechanical system is the origin. Theorem 3 is in agreement with the conjecture formulated by Arnold et al. in [1, Remark 6.17, p. 285].

A related (weaker) result was announced in [4].

We recall that classical KAM theory yields only *primary tori* (which are graphs over $\mathbb{T}^n$) $\sqrt{\varepsilon}$-away from resonances, while the new tori constructed in the above theorems fill, with an exponential density, a neighborhood of (simple) resonances far from double resonances. Furthermore, the new KAM tori include, besides primary tori, also *secondary* tori, which exhibit different topologies and, in particular, are not graphs over $\mathbb{T}^n$.

Secondary tori close to resonances have been also investigated in [9].

To prove the above results, it is essential to study regions close to resonances $\mathcal{R}_k$ for $|k|_1 \to \infty$ as $\varepsilon \to 0$. Away from doubles resonances, the "secular" dynamics in

$\mathcal{R}_k$ is, up to exponentially long times, ruled by the integrable Hamiltonian

$$\mathtt{H}_k := \frac{1}{2}|y|^2 + \varepsilon(\pi_{\mathbb{Z}k} f)(x \cdot k).$$

Therefore, it should not surprise that one needs non-degeneracy assumptions in (i*) above; in particular, the first condition implies that for $|k|_1 \geq \mathtt{N}$ large enough, but *independent of* ε, the secular potential $\pi_{\mathbb{Z}k} f$ is essentially a rescaled and shifted cosine (as fully discussed in [5]). Notice that for low modes ($|k|_1 < \mathtt{N}$) the secular potential $\pi_{\mathbb{Z}k} f$ is a generic periodic function. In particular, the phase portrait of $\mathtt{H}_k$ is quite arbitrary and may have an arbitrary number of equilibria and separatrices. The main point here is to prove the persistence of *all* integrable tori of $\mathtt{H}_k$ up to an exponentially small set (away from double resonances).

We finally remark that one of the main issues in the proof of measure estimates is to show that the integrable secular Hamiltonian $\mathtt{H}_k$ above, *in its action variables*, is Kolmogorov non-degenerate; namely the action-to-frequency map is invertible. While $\mathtt{H}_k|_{\varepsilon=0} = \frac{1}{2}|y|^2$ is obviously non-degenerate, it is a fact that this is not always true for $\mathtt{H}_k$ (in its action variables) when $\varepsilon > 0$. Indeed, this is a *singular perturbation problem*, as suggested by the fact that the level sets of $\mathtt{H}_k$ have different topologies, due to the presence of secondary tori for $\varepsilon > 0$.

References

[1] V. I. Arnold – V. V. Kozlov – A. I. Neishtadt, *Mathematical aspects of classical and celestial mechanics. [Dynamical systems. III]. Translated from the Russian original by E. Khukhro*. 3rd edn., Encyclopaedia Math. Sci. 3, Springer, Berlin, 2006. Zbl 1105.70002 MR 2269239

[2] L. Biasco – L. Chierchia, On the measure of Lagrangian invariant tori in nearly-integrable mechanical systems. *Rend. Lincei Mat. Appl.* **26** (2015), no. 4, 423–432. Zbl 1361.37059 MR 3420497

[3] L. Biasco – L. Chierchia, Explicit estimates on the measure of primary KAM tori. *Ann. Mat. Pura Appl. (4)* **197** (2018), no. 1, 261–281. Zbl 1456.37063 MR 3747531

[4] L. Biasco – L. Chierchia, On the measure of KAM tori in two degrees of freedom. *Discrete Contin. Dyn. Syst.* **40** (2020), no. 12, 6635–6648. Zbl 1456.37064 MR 4160084

[5] L. Biasco – L. Chierchia, On the topology of nearly-integrable Hamiltonians at simple resonances. *Nonlinearity* **33** (2020), no. 7, 3526–3567. Zbl 1469.37047 MR 4105402

[6] L. Biasco – L. Chierchia, Action-angle variables of pendulum-like Hamiltonians. 2022, preprint.

[7] L. Biasco – L. Chierchia, Measure and density of KAM tori in generic nearly-integrable mechanical systems. 2022, preprint.

[8] L. Lazzarini – J.-P. Marco – D. Sauzin, Measure and capacity of wandering domains in Gevrey near-integrable exact symplectic systems. *Mem. Amer. Math. Soc.* **257** (2019), no. 1235, vi+110. Zbl 1442.37003 MR 3898992

[9] A. G. Medvedev – A. I. Neishtadt – D. V. Treschev, Lagrangian tori near resonances of near-integrable Hamiltonian systems. *Nonlinearity* **28** (2015), no. 7, 2105–2130. Zbl 1378.70021 MR 3366638

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