QUADERNI
DEL CONSIGLIO NAZIONALE DELLE RICERCHE
GRUPPO NAZIONALE DI FISICA MATEMATICA

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QUASI-PERIODIC SCHRÖDINGER OPERATORS IN ONE DIMENSION, ABSOLUTELY CONTINUOUS SPECTRA, BLOCH WAVES AND INTEGRABLE HAMILTONIAN SYSTEMS

1986

TABLE OF CONTENTS

ABSTRACT

One-dimensional quasi-periodic Schrödinger operators are studied from two different points of view. The first one deals with the construction of quasi-periodic eigensolutions and with an associated Hamiltonian interpretation. The second one concerns more general questions about absolutely continuous spectra.

In the first chapter, the eigenvalue problem for a periodic Schrodinger operator,

$$Lf \equiv \left(-\frac{d^2}{dx^2} + v\right)f = Ef,$$

is viewed as a two-dimensional Hamiltonian system which is integrable in the sense of Arnold and Liouville. With the aid of the Floquet-Bloch theory, it is shown that such a system is conjugate to two harmonic oscillators with frequencies α and ω, α being the rotation number for L and $\frac{2\pi}{\omega}$ the period of the potential v.

This picture is generalized,in the second chapter, to quasi-periodic Schrodinger operators, L_ε, with highly irrational frequencies $(\omega_1, \ldots, \omega_d)$, which are a small perturbation of periodic operators. When the parameter E belongs to a certain (explicitly constructed) large Cantor set $\mathcal{E}$, the eigenvalue problem for L_ε is embedded, via a KAM method, in a system of d+1 harmonic oscillators with frequencies $(\alpha, \omega_1, \ldots, \omega_d)$, α being the (Johnson-Moser) rotation number for L_ε. The function $E \in R \to \alpha(E)$, in general only continuous, is shown to be C^∞ on $\mathcal{E}$ in the sense of Whitney and a new proof of Moser-Deift-Simon

inequality,

$$\frac{d\alpha^2}{dE} \geq 1 ,$$

is given for E in $\mathcal{E}$ ($\frac{d}{dE} \equiv$ Whitney derivative).

A by-product of the above is that, on $\mathcal{E}$, all the eigensolutions of L_ε are quasi-periodic with frequencies $(\alpha, \omega_1, \ldots, \omega_d)$ and depend smoothly on E (in the Whitney sense). Moreover, adapting the KAM algorithm to a matrix formalism, independent eigensolutions of the form $f = e^{i\alpha x} \chi(\omega_1 x, \ldots, \omega_d x)$ and $\overline{f}$, with χ periodic in each argument, are constructed. Such functions are called Bloch waves.

In the last chapter, the absolutely continuous spectrum σ_{ac} of a general quasi-periodic Schrödinger operator is considered. The Radon-Nikodym derivatives (with respect to Lebesgue measure) of the spectral measures are computed in terms of special independent eigensolutions existing for almost every E in σ_{ac}. Such eigensolutions can be replaced, in the above Radon-Nikodym derivatives, by Bloch waves whenever these exist (as in the case treated in Chapter 2).

Finally, it is shown that weak Bloch waves always exist for almost every E in σ_{ac} and the question of the existence of genuine Bloch waves is turned into a regularity problem for a certain nonlinear partial differential equation on a d-dimensional torus.

CHAPTER 1

ONE-DIMENSIONAL PERIODIC SCHRÖDINGER OPERATORS AND INTEGRABLE HAMILTONIAN SYSTEMS

1.1 Introduction

The trivial eigenvalue problem

$$- \frac{d^2}{dx^2} q = E\, q, \quad x \in R ,$$

can be thought of as an harmonic oscillator with Hamiltonian $H(p,q) = \frac{p^2+\sqrt{E}q^2}{2}$, $(p,q) \in R^2-\{0,0\}$, the parameter x playing the role of time. It is elementary that such mechanical systems are integrable in the sense of Arnold and Liouville*: The canonical change of variables $(p,q) \to (A,\varphi) \equiv (\frac{r^2}{2\sqrt{E}},\varphi)$, where (r,φ) are polar coordinates in the $(p,\sqrt{E}q)$-plane, conjugates H to the trivial Hamiltonian $h(A,\varphi) \equiv \sqrt{E}A$, $(A,\varphi) \in R \times T$, $T \equiv R/2\pi z$.

In this chapter, we show that periodic Schrödinger operators,

$$L(v) \equiv - \frac{d^2}{dx^2} + v(x), \quad v(x + \tfrac{2\pi}{\omega}) = v(x) ,$$

carry a structure completely analogous to the one described above: For E inside the spectrum of L, the eigenvalue problem

$$L\, q = Eq$$

*We recall the mechanical language in §1.2.

can be embedded in a two-dimensional integrable system. Moreover, the integrated Hamiltonian has the form

$$h(A_1, A_2, \varphi_1, \varphi_2) = \alpha(E)A_1 + \omega A_2,$$

$$(A_1, A_2) \in R_+ \times R, (\varphi_1, \varphi_2) \in T^2 ,$$

the parameter α being the rotation number for L.

1.2 Spectrum

Let $L = L(v) \equiv -\frac{d^2}{dx^2} + v(x)$, where $v(x) = V(\omega x)$ with V a smooth real function on the circle, and $\omega > 0$ a given frequency. The abstract spectral theory for these operators is included in Weyl's "limit-point, limit-circle" theory*: Since v is real and bounded, the operator L is in the limit-point case. That is, L, considered on $C_0^\infty(R)$ ($\equiv$ the class of indefinitely differentiable functions on R with compact support) is essentially self-adjoint and admits a unique extention to a dense domain in $L^2(R)$. The resolvent set

$$\rho(L) \equiv \{E \in \mathbb{C} : (L-E)^{-1} \text{ exists and is bounded}\}$$

is characterized by the existence of two independent solutions $f_\pm(x;E)$ of

$$Lf = Ef \tag{1.1}$$

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*Weyl [1910], Stone [1932], Titchmarsh [1946], Kodaira [1949], Coddington-Levinson [1955]. See also §2, Chapter 3 for more information.

belonging to $L^2(R_\pm)$, $R_+ \equiv (0,\infty)$, $R_- \equiv (-\infty, 0)$. The <u>Green's function</u> ($\equiv$ kernel of $(L-E)^{-1}$) is given by*

$$g(x,y;E) \quad \frac{f_+(x;E)\ f_-(y;E)}{[f_+,f_-]} \qquad \text{if } x \geq y$$

and symmetrically if $x \leq y$.

The concrete analysis of (1.1) is the content of <u>Floquet theory</u>**. Let $A(x;E)$ be the fundamental matrix

$$\begin{bmatrix} f_1(x;E) & f_2(x;E) \\ f_1'(x;E) & f_2'(x;E) \end{bmatrix}$$

where $f_1(0) = f_2'(0) = 1$, $f_1'(0) = f_2(0) = 0$, and let $M(E)$ denote the monodromy matrix $A(\frac{2\pi}{\omega}; E)$. Then there is a non-trivial solution of (1.1) satisfying

$$f(x + \tfrac{2\pi}{\omega}) = \rho\ f(x) \tag{1.2}$$

if and only if the <u>Floquet multiplier</u> ρ is an eigenvalue of M, i.e., $\rho = \rho_\pm \equiv \Delta \pm \sqrt{\Delta^2-1}$ with $\Delta \equiv \frac{1}{2}$ trace M. The <u>discriminant</u> Δ is an entire function of E of order $\frac{1}{2}$, type 1, real on R and asymptotic to $\cos\sqrt{-E}$ near $-\infty$ (Magnus-Winkler [1966]). Δ is depicted in Figure 1.

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*$[f,g] \equiv f\,\frac{dg}{dx} - \frac{df}{dx}\,g \equiv fg' - f'g$.

**Floquet [1883], Bloch [1928]; see Magnus-Winkler [1966] for a review.

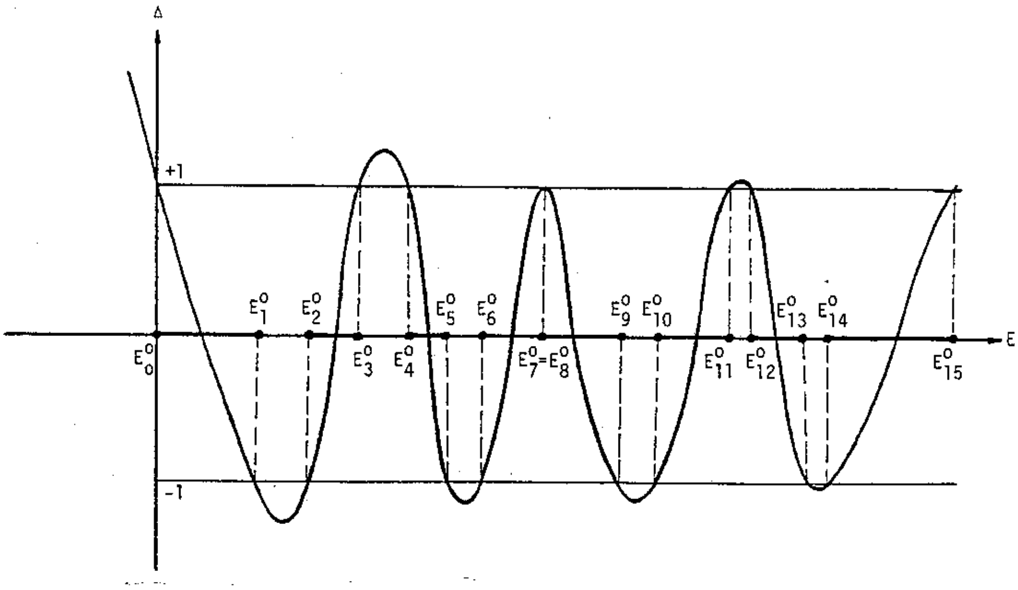

Figure 1.

We will refer to solutions having property (1.2) as <u>Floquet solutions</u>. If f,g are two Floquet solutions with the same eigenvalue and the same multiplier $\rho \neq \pm 1$, then

$$[f,g] = [f,g]\Big|_{x=\frac{2\pi}{\omega}} = \rho^2[f,g]\Big|_{x=0} = \rho^2[f,g] .$$

Thus, $[f,g] = 0$, that is, when $\rho \neq \pm 1$ the corresponding Floquet solution is determined up to a multiplicative constant. Notice also that if $\Delta^2 < 1$ (for E in R) and f is a Floquet solution with multiplier $\rho_\pm$, then $[f,\bar{f}] \neq 0$. In fact,

$$\frac{i}{2}[f,\bar{f}] = [\mathrm{Re} f, \mathrm{Im} f]$$

and $[f,\bar{f}] = 0$ would imply $f = cg$ for $c \in \mathbb{C}$ and g a real function, and taking the imaginary part of $g(x + \frac{2\pi}{\omega}) = \rho_\pm\, g(x)$, we would get

$\operatorname{Im}\rho_{\pm} = \pm\sqrt{1-\Delta^2} = 0$, a contradiction. From these trivial observations we are able to identify the spectrum of L. When $\Delta^2 > 1$, the Floquet solutions corresponding to $\rho_{\pm}$ are seen to be the functions $f_{\pm}$ of Weyl's theory*, so that $\{E:|\Delta^2| > 1\} \subset \rho(L)$. Since for E in $\{E \in R: \Delta^2 < 1\}$ we have two independent eigenfunctions of the form**

$$e^{i\beta x}\chi(\omega x),\ e^{-i\beta x}\bar{\chi}(\omega x);\quad e^{i\beta} = \rho,\ \chi \in C^\infty(T)\ ,$$

we conclude that $\sigma(L) = \bigcup_{k=0}^{\infty} [E^0_{2k}, E^0_{2k+1}]$, where $\sigma(L)$ denotes the spectrum of L and $E^0_0 < E^0_1 \leq E^0_2 < E^0_3 \leq E^0_4$ .... are the (infinitely many simple or double) roots of $\Delta^2 = 1$.

Kodaira [1949] and Gelfand [1950] gave the spectral decomposition of L(v) showing, as a byproduct, that the spectrum of L is purely absolutely continuous with double multiplicity†. This fact is also a consequence of the following representation of the spectral measures that will be a simple application of Theorem 2, §4 of Chapter 3: Let φ be any C^∞ function with compact support, $P_E = P_E(L)$ the standard spectral family of L, f_E any Floquet solution, then‡

$$d(P_E\varphi,\varphi) = \frac{1}{2\pi}\,\frac{|(\varphi,f_E)|^2 + |(\varphi,\bar{f}_E)|^2}{|[f_E,\ \bar{f}_E]|}\,dE,$$

$$E \in \sigma^*(L) \equiv \{E \in \sigma(L):\ |\Delta| < 1\}. \tag{1.3}$$

*In this case, $f_{\pm}$ decay exponentially fast at $\pm\infty$.

**$\chi \in C^\infty(T)$ means that $\varphi \in R \to \chi(\varphi)$ is a C^∞ function with period 2π.

†See Reed-Simon [1978] for a review.

‡$(\cdot,\cdot)$ denotes the usual inner product in $L^2(R)$, and dE is the Lebesgue measure.

Remark. We saw already that in σ^* the Floquet solutions corresponding to $\rho_\pm$ are one the complex conjugate of the other and that they are determined up to a constant. This makes formula (1.3) unambiguous.

Another important object associated to $L(v)-E$ is the rotation number $\alpha = \alpha(E)$*. This is a positive unbounded continuous function of the real parameter E, constant on $\rho(L) \cap R$ and strictly increasing on $\sigma(L)$. Before defining α, we introduce the winding number w: Let $g:R \to \mathbb{C}$ be a continuous curve never passing through zero. Define

$$w = w(g) \equiv \lim_{x \uparrow \infty} \frac{\arg g(x)}{x} \quad ,$$

where such a limit exists**. For example, a periodic curve g with period $\frac{2\pi}{\omega}$ will have $w(g) = n$ for some n in Z.

Now, let f be a solution of (1.1) with $[f,\overline{f}] \neq 0$, then f never vanishes and

$$\frac{d}{dx}(\arg f) = \frac{\operatorname{Im} f' \, \overline{f}}{|f|^2} = \frac{i}{2|f|^2} \; [f,\overline{f}]$$

shows that f winds around the origin (counter-)clockwise according to $i[f,\overline{f}]$ (>) < 0; furthermore w(f) exists, $|w|$ is independent of f and is the rotation number cited above. The behavior of α as a function of E can be described very precisely thanks to the analytic properties of

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*See, for example, Herman [1979], Moser [1981].

**Obviously this definition doesn't depend on the choice of the branch of arg z since constants wash out in the limit.

the discriminant: Define $\theta(E)$ by $\cos\theta(E) = \Delta(E)$. Then

$$\theta(E) = \pm i \int_{E_0^0}^{E} \frac{\frac{d\Delta}{dE}}{\sqrt{\Delta^2-1}}\, dE' \quad (\text{mod } 2\pi)$$

and

$$\begin{aligned} &\alpha(E) = 0 && E\epsilon(-\infty,\ E_0^0) \\ &\alpha(E) = \alpha(E_{2k}^0) + \frac{\omega}{2\pi}\,(-1)^{k+1}\ \mathrm{Re}\int_{E_{2k}^0}^{E} \frac{\frac{d\Delta}{dE}}{\sqrt{1-\Delta^2}} && E\epsilon[E_{2k}^0,\ E_{2k+1}^0] \end{aligned} \tag{1.4}$$

From (1.4) one sees that $\alpha(E) \sim \sqrt{E}$ for large E and that $\alpha-\alpha(E_k^0) \sim \sqrt{|E-E_k^0|}$ for E near $E_k^0 \in \partial\sigma(L)$; compare Figure 2.

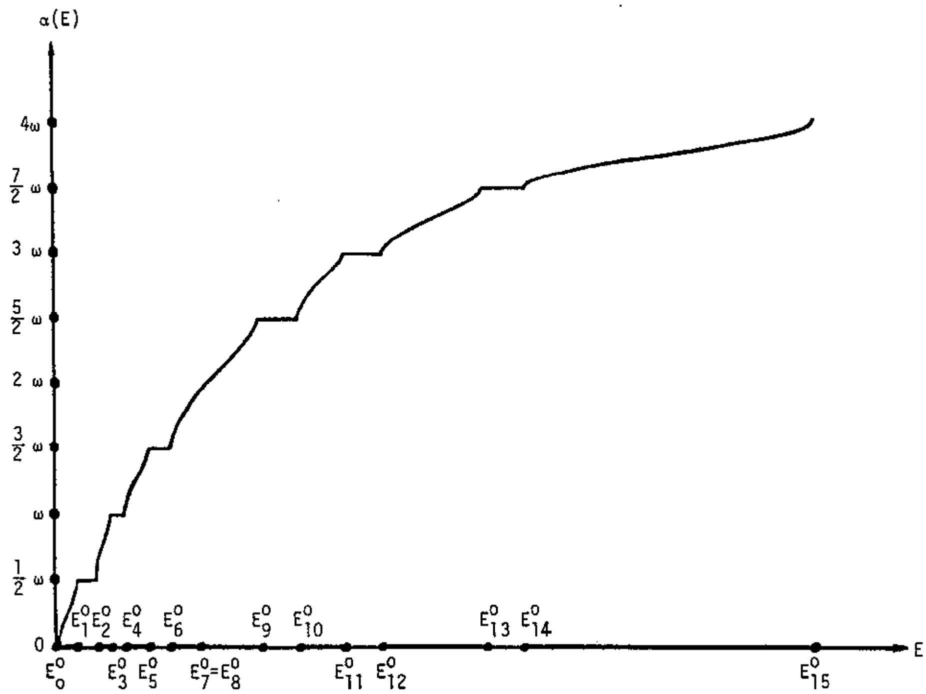

Figure 2.

Before closing this section, we point out a parametrization of the Floquet solutions for $E \in \sigma^*$ that will be useful in the following paragraphs: Let f_o be the unique Floquet solution with multiplier

$$\rho_+ = e^{i\alpha\frac{2\pi}{\omega}}$$

and $f_o(0) = 1$. Such a function has the representation

$$f_o(x) = e^{i\alpha x}\, \chi_o(\omega x) \tag{1.5}$$

where χ_o is a smooth function on the circle T with

$$\chi_o(0) = 1 \quad \text{and} \quad \lim_{x\uparrow\infty} \arg \frac{\chi_o(\omega x)}{x} = 0.$$

It is also easy to see that*

$$f_o(x) = f_1(x) + \frac{e^{i\alpha\frac{2\pi}{\omega}} - f_1(\frac{2\pi}{\omega})}{f_2(\frac{2\pi}{\omega})}\; f_2(x).$$

Thus,

$$\kappa \equiv \frac{i}{2}\,[f_o,\overline{f}_o] = \operatorname{Im} f_o'(0) = \frac{\sin(\alpha\,\frac{2\pi}{\omega})}{f_2(\frac{2\pi}{\omega})}\,.$$

Finally, the asymptotics of f_1, f_2 and (1.4) imply

$$\kappa \sim \sqrt{E} \quad \text{and} \quad \chi_o \sim 1 \quad \text{for} \quad E \uparrow +\infty \quad .$$

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*$f_2(\frac{2\pi}{\omega};E) = 0$ implies $E \in [E^o_{2k-1}, E^o_{2k}]$ for some $k \geq 2$.

1.3 Action-Angle Variables

Consider the Hamiltonian

$$H(p,B,q,\psi;E) \equiv \frac{p^2}{2} + \omega B + \frac{q^2}{2}\,(E-V(\psi)),\ (p,q) \in R^2-\{0\},$$

$$(B,\psi) \in R \times T \tag{1.6}$$

where (q,ψ) are regarded as generalized coordinates and (p,B) as the conjugate variables. The canonical equations for H are

$$q' = p \qquad\qquad p' = q(V(\psi)-E)$$
$$\psi' = \omega \qquad\qquad B' = V'(\psi)\,\frac{q^2}{2}$$

which imply

$$\frac{d^2}{dx^2}\,q = q(V(\psi_0 + \omega x)-E),\ \psi_0 = \psi\Big|_{x=0}\ .$$

Thus, for $\psi_0 = 0$, we get back (1.1) for $q(x)$.

We recall* that a Hamiltonian system on a 2n-dimensional symplectic manifold M^{2n} with Hamiltonian H is said to be <u>integrable</u> if there exist n smooth functions on M, $F_1 \equiv H, F_2, \ldots F_n$, such that

(i) the F_i's are in involution**,

(ii) the F_i's are functionally independent on $N \equiv \{F_i = \text{const},\ i=1,\ldots,n\}$,

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*See, e.g., Arnold [1978] or Gallavotti [1983].

**That is, in standard local symplectic coordinates (p,q), $\{F_i,F_j\} \equiv \sum_{k=1}^{n} \frac{\partial F_i}{\partial p_k}\frac{\partial F_j}{\partial q_k} - \frac{\partial F_i}{\partial q_k}\frac{\partial F_j}{\partial p_k} = 0$, for every choice of i and j.

(iii) the submanifold N is compact and connected.

Under these conditions, the evolution equations for H are solvable by quadratures. More precisely, there exists a cononical* change of coordinates

$$C: m \in M^{2n} \to C(m) = (A,\varphi) \in \Omega \times T^n, \ \Omega \subset R^n,$$

for which $H(C^{-1}(A,\varphi)) = h(A)$. In this case, the Hamilton equations on $\Omega \times T$ become trivial:

$$A' = 0 , \qquad \varphi' = \frac{\partial h}{\partial A} ;$$

in fact, $A(x) \equiv A(0)$ and $\varphi(x) = \varphi(0) + \frac{\partial h}{\partial A}(A(0))x \pmod{2\pi}$. The coordinates (A,φ) are called action-angle variables. This is the content of the Arnold-Liouville theorem (Arnold [1978]).

Going back to (1.6), we have the following**

Theorem 1. If $E \in \sigma^*(L)$, the Hamiltonian system (1.6) is integrable.

The proof employs new coordinates for the phase space $R^3 \times T - (0,R,0,T)$: For $E \in \sigma^*(L)$, let f_0 be the Floquet solution defined at the end of the preceding section, and set

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*That is, preserving the symplectic structure and hence the form of the Hamilton equations.

**We learned this theorem from Gallavotti [198-], see also Gallavotti [1985].

$$F_o(\varphi,\psi) = e^{i(\varphi - \frac{\alpha}{\omega}\psi)} f_o(\tfrac{\psi}{\omega}) \;, \quad (\varphi,\psi) \in T^2 \;.$$

Since f_o has Floquet multiplier $e^{i\alpha\frac{2\pi}{\omega}}$, F_o is a smooth function on T^2. Moreover

$$x \to F_o(\varphi + \alpha x,\ \psi + \omega x) = e^{i(\varphi - \frac{\alpha}{\omega}\psi)} f_o(\tfrac{\psi}{\omega} + x)$$

is easily recognized as a Floquet solution of the <u>shifted Schrödinger equation</u>

$$-f''(x) + V(\psi + \omega x)\ f(x) = E\ f(x),$$

with multiplier $e^{i\alpha\frac{2\pi}{\omega}}$.

Define

$$Q(\varphi,\psi) \equiv \mathrm{Re}\ F_o(\varphi,\psi)$$

$$P(\varphi,\psi) \equiv \mathrm{Re}\ D_\alpha F_o(\varphi,\psi) = \mathrm{Re}\ [e^{i(\varphi - \frac{\alpha}{\omega}\psi)} f_o'(\tfrac{\psi}{\omega})]$$

where D_α is the vector field $\alpha \frac{\partial}{\partial\varphi} + \omega \frac{\partial}{\partial\psi}$ and observe that $(D_\alpha F_o)(\varphi + \alpha x,\ \psi + \omega x) = \frac{d}{dx} F_o(\varphi + \alpha x,\ \psi + \omega x)$. Now consider the map

$$(r,B,\varphi,\psi) \in R_+ \times R \times T^2 \to (p,B,q,\psi) \in R^3 \times T - (0,R,0,T) \tag{1.7}$$

$$p = r\ P(\varphi,\psi) \qquad , \quad q = r\ Q(\varphi,\psi) \;.$$

<u>Lemma</u>. The map (1.7) is a diffeomorphism.

<u>Proof</u>. Let's start with the <u>Jacobian</u>. With the temporary notation $f(x;\varphi,\psi) = F_o(\varphi + \alpha x,\ \psi + \omega x)$, we have

$$\det \frac{\partial(p,B,q,\psi)}{\partial(r,B,\varphi,\psi)} = r(\frac{\partial Q}{\partial\varphi} P - Q \frac{\partial P}{\partial\varphi}) \tag{1.8}$$

$$= r[-(\mathrm{Im}\ F_0)(\mathrm{Re}\ D_\alpha F_0) + (\mathrm{Re}\ F_0)(\mathrm{Im}\ D_\alpha F_0)]$$

$$= r[\mathrm{Re}\ f,\ \mathrm{Im}\ f] = r\,\frac{i}{2}\,[f,\bar{f}] \quad (\text{at } x = 0)$$

$$= r\,\frac{i}{2}\,[f,\bar{f}] \qquad (\text{at } x = -\tfrac{\psi}{\omega})$$

$$= r\,\frac{i}{2}\,[f_0,\bar{f}_0]$$

$$= r\,\kappa > 0.$$

Next notice that, for each $\psi \in T$, the map

$$(r,\varphi) \to (p,q) = (r\,P(\varphi,\psi),\ r\,Q(\varphi,\psi))$$

can be written as

$$\begin{pmatrix} p \\ q \end{pmatrix} = T\begin{pmatrix} r\sin\varphi \\ r\cos\varphi \end{pmatrix},\quad T \equiv \begin{bmatrix} -\mathrm{Im}(e^{-\frac{\alpha}{\omega}\psi} f_0(\frac{\psi}{\omega})) & \mathrm{Re}(e^{-\frac{\alpha}{\omega}\psi} f_0(\frac{\psi}{\omega})) \\ -\mathrm{Im}(e^{-\frac{\alpha}{\omega}\psi} f_0'(\frac{\psi}{\omega})) & \mathrm{Re}(e^{-\frac{\alpha}{\omega}\psi} f_0'(\frac{\psi}{\omega})) \end{bmatrix}.$$

Now,

$$\det T = [\mathrm{Re}(e^{-\frac{\alpha}{\omega}\psi} f_0(\tfrac{\psi}{\omega}+x)),\ \mathrm{Im}(e^{-\frac{\alpha}{\omega}\psi} f_0(\tfrac{\psi}{\omega}+x))] \quad (\text{at } x = 0)$$

$$= [\mathrm{Re}(e^{-\frac{\alpha}{\omega}\psi} f_0(\tfrac{\psi}{\omega}+x)),\ \mathrm{Im}(e^{-\frac{\alpha}{\omega}\psi} f_0(\tfrac{\psi}{\omega}+x))] \quad (\text{at } x = -\tfrac{\psi}{\omega})$$

$$= e^{-2\frac{\alpha}{\omega}\psi}\,\kappa \neq 0\ .$$

The proof is finished.

The quantities (r,B,φ,ψ) will be called Floquet-variables.

Now, the integrability of (1.6) is a simple matter.

Proof of Theorem 1. We have to check (i), (ii), (iii) above. Since r is, by construction, an integral* for H, r and H are in involution, and this is (i). The level surfaces $H = C_1$, $r = C_2$ are 2-dimensional tori so (iii) is fulfilled. To check the independence of H and r, we use the Floquet-variables:

$$\frac{\partial(H,r)}{\partial(r,B,\varphi,\psi)} = \begin{bmatrix} * & \omega & * & * \\ 1 & 0 & 0 & 0 \end{bmatrix} ,$$

so we have also (ii). The proof is finished.

Now, we turn to the explicit construction of the action-angle variables.

Consider the map

$$\begin{gathered} (r,B,\varphi,\psi) \in R_+ \times R \times T^2 \to (A_1, A_2, \varphi_1, \varphi_2) \in R_+ \times R \times T^2 \\ A_1 = \frac{r^2}{2}\kappa \ , \quad A_2 = B + \frac{r^2}{2}\left(\frac{\partial Q}{\partial\psi} P - Q\frac{\partial P}{\partial\psi}\right), \ \varphi_1 = \varphi, \ \varphi_2 = \psi \ . \end{gathered} \tag{1.9}$$

Such a map is clearly one-to-one and onto, and the evaluation of the Jacobian

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*That is, a function constant on the trajectories governed by H.

$$\det \frac{\partial(A_1,A_2,\varphi_1,\varphi_2)}{\partial(r,B,\varphi,\psi)} = \begin{vmatrix} r\kappa & 0 & 0 & 0 \\ * & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = r\kappa > 0$$

shows that it is a diffeomorphism. The upshot is that $(A_1,A_2,\varphi_1,\varphi_2)$ are the required action-angle variables:

Theorem 2. The diffeomorphism

$$(p,B,q,\psi) \in R^3 \times T - (0,R,0,T) \to (A_1,A_2,\varphi_1,\varphi_2) \in R_+ \times R \times T^2$$

defined via (1.7), (1.9) is a canonical transformation, and

$$\frac{p^2}{2} + \omega B + \frac{q^2}{2}(E - V(\psi)) = \alpha A_1 + \omega A_2. \tag{1.10}$$

Proof. In (1.8), we saw that* $\kappa = Q_1P - QP_1$. Now using Floquet-variables as an intermediate step,

$$dp \wedge dq + dB \wedge d\psi$$

$$= \kappa\, r\, dr \wedge d\varphi + r(PQ_2 - QP_2)\, dr \wedge d\psi$$

$$+ r^2(P_1Q_2 - Q_1P_2)\, d \quad \wedge d\psi + dB \wedge d\psi$$

*Subscripts 1,2 mean, respectively, $\frac{\partial}{\partial\varphi}$ and $\frac{\partial}{\partial\psi}$.

$$= \kappa\, r\, dr \wedge d\varphi + r(PQ_2 - QP_2)\, dr \wedge d\psi$$

$$+ \frac{r^2}{2}(P_1Q_2 - Q_1P_2 + Q_{21}P - QP_{21})\, d\varphi \wedge d\psi + dB \wedge d\psi$$

$$= dA_1 \wedge d\varphi_1 + dA_2 \wedge d\varphi_2 \,.$$

This proves the first claim. To check (1.10), put $q_o(x) = Q(\varphi + \alpha x, \psi + \omega x)$. Then using Floquet-variables once more,

$$\begin{aligned}
&\alpha A_1 + \omega A_2 \\
&= \alpha \frac{r^2}{2}(Q_1P - QP_1) + \omega B + \omega \frac{r^2}{2}(Q_2P - QP_2) \\
&= \omega B + \frac{r^2}{2}[(\alpha Q_1 + \omega Q_2)P - (\alpha P_1 + \omega P_2)Q] \\
&= \omega B + \frac{r^2}{2}[q_o'(0)^2 + q_o''(0)\, q_o(0)] \\
&= \omega B + \frac{r^2}{2}[q_o'(0)^2 - q_o(0)^2\,(V(\psi)-E)] \\
&= \omega B + \frac{r^2}{2}[P^2 + Q^2\,(E-V)] \\
&= \frac{p^2}{2} + \omega B + \frac{q^2}{2}(E-V)\,.
\end{aligned}$$

The proof is completed.

Remark 1. The canonical transformation in Theorem 2 is produced by the generating function

$$\Phi(B,p,\varphi_1,\varphi_2) \equiv B\varphi_2 + \frac{p^2}{2}\frac{Q(\varphi_1,\varphi_2)}{P(\varphi_1,\varphi_2)}\,.$$

The Hamilton-Jacobi equation for ϕ

$$\frac{p^2}{2} + \omega B + \frac{q^2}{2}\,[E-V(\psi)] = \alpha\,\frac{\partial\phi}{\partial\varphi_1} + \omega\,\frac{\partial\phi}{\partial\varphi_2}$$

is easily seen to correspond, along $\frac{d}{dx} \equiv D_\alpha$, to a Riccati equation for

$$\frac{d}{dx}\log Q(\varphi + \alpha x,\ \psi + \omega x)\ .$$

Remark 2. The set of E's for which we carried out the integration of H excludes the double roots of $\Delta^2 = 1$ ($\equiv$ collapsed gaps). This was done for reasons of simplicity rather than for real difficulties. In fact, Floquet theory tells us that when a spectral gap collapses to a point, $E^0_{2k-1} = E^0_{2k}$, there are still two independent eigenfunctions of the form

$$f = e^{i\alpha x}\,\chi(\omega x) \quad \text{and} \quad \overline{f}$$

with χ (2π-) periodic*. Furthermore, in such a case,

$$f_2\!\left(\frac{2\pi}{\omega},\ E^0_{2k}\right) = 1$$

and κ is still different from zero. Therefore, H is actually integrable for E in

$$\sigma^*(L) \cup \{\text{collapsed gaps}\} = \text{interior of } \sigma(L).$$

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*See Magnus-Winkler [1966]. This phenomenon is called coexistence of periodic solutions since, on gaps, $\alpha = \frac{n\omega}{2}$ (compare Figure 2).

Remark 3. It is well known that the Kortweg-de Vries equation can be viewed as an isospectral flow for some Schrödinger operator L *. If L is periodic, such a flow is Hamiltonian and integrable and action-angle variables have been displayed by Flaschka-McLaughlin [1976]. It might be interesting to see if there is any relation between such variables and the ones in Theorem 2.

*Lax [1968]; see McKean [1978] for a review.

CHAPTER 2

QUASI-PERIODIC SCHRÖDINGER OPERATORS, INTEGRABLE HAMILTONIAN SYSTEMS AND KAM ITERATION SCHEMES

2.1 Introduction

Here we consider the eigenvalue problem for Schrödinger operators $L(v)$ with (real) analytic quasi-periodic potential $v(x)$.

We recall that a function $x \in R \to f(x)$ is said to be (ω-) quasi-periodic with frequencies $\omega = (\omega_1, \ldots, \omega_d)$ if one has $f(x) = F(\omega x)$, for some function F on $T^d \equiv R^d/2\pi Z^d$. The class to which the function F belongs will define the class of f *. Now, the basic question is: Do there exist quasi-periodic eigenfunctions?

The first to attack this problem were Dinaburg-Sinai [1975]. They assume that the frequencies of the potential satisfy the Diophantine condition**

$$|\omega\cdot\nu| \geq \frac{1}{c|\nu|^{d+1}}, \text{ for some } c > 0 \text{ and any } \nu \in Z^d - \{0\}. \quad (2.1)$$

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*Warning: There exist real analytic functions which are, according to this definition, merely continuous quasi-periodic (see Johnson-Moser [1982]).

**If $x,y \in C^d$, $x\cdot y \equiv \sum_{i=1}^{d} x_i\bar{y}_i$ and $|x| \equiv \sum_{i=1}^{d} |x_i|$.

Then, for a fixed $\sigma > 0$ and E_0 large enough, they construct a big nowhere dense set $\mathcal{E}(\sigma)$ in $[E_0,+\infty)$ so that, for E in $\mathcal{E}(\sigma)$, all the eigenfunctions of L are (analytic) quasi-periodic with rationally independent frequencies $(a, \omega_1, \ldots, \omega_d)$. The frequency a is a uniformly continuous function of E and verifies

$$|a - \frac{1}{\sqrt{E}}| = O(\frac{1}{\sqrt{E}}) .$$

The set $\mathcal{E}(\sigma)$ is described through its complement

$$\mathcal{R}(\sigma) \equiv \bigcup_{\substack{\nu \in Z^d \\ |\nu| \geq 2}} \left\{E \geq E_0 : |\sqrt{E} - \alpha_\nu| < K_\sigma \exp\left(- \frac{|\nu|}{\log^{1+\sigma}|\nu|}\right)\right\}$$

where K_σ is a positive constant and the numbers α_ν (<u>not</u> explicitly determined) are such that

$$|\alpha_\nu - \frac{1}{2}|\omega \cdot \nu|| < \frac{K'_\sigma}{(1 + |\omega \cdot \nu|)} .$$

Their proof is based on a <u>KAM fast iteration scheme</u>* for trace-less, two-by-two, complex matrices with certain symmetries.

*Kolmogorov [1954], Arnold [1963], Moser [1962] and [1967].

Rüssmann [1980] replaces (2.1) by the more general condition

$$|\omega \cdot \nu| \geq \frac{1}{\Omega(|\nu|)}, \quad \nu \in Z^d - \{0\} \tag{2.2}$$

where the so-called approximation function Ω satisfies*

$$\frac{r^{d-1}}{|\omega|} \leq \Omega(r) \uparrow +\infty, \quad \frac{\log \Omega(r)}{r} \downarrow 0, \quad \int^{\infty} \frac{\log \Omega(r)}{r^2}\, dr < +\infty. \tag{2.3}$$

Then, using a set of estimates from Rüssmann [1975] and [1976], he improves the Dinaburg-Sinai iteration scheme obtaining reasonable numerical bounds on its threshold of applicability.

He also gives an interpretation of the points α_ν of the Dinaburg-Sinai set that we discuss in detail later**.

The set-up in Moser-Pöschel [1984] is as in Rüssmann [1980] but two new features appear:

(i) The recently discovered properties of the rotation number α for quasi-periodic potentials allow them to work directly in the α-line instead of the $\sqrt{E}$-line.

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*The first item of (2.3) is justified by Dirichlet's theorem in the theory of Diophantine approximation:

$$\max_{0 < |\nu| \leq m} \frac{1}{|\omega \cdot \nu|} \leq \frac{m^{d-1}}{|\omega|},$$

for irrational ω and any $m \in Z_+$.

**Compare Remark 2 of §2.4 and Remark 3 of §2.9.

(ii) They construct quasi-periodic eigenfunctions with rationally dependent frequencies $(\beta, \omega_1, \ldots, \omega_d)$. In fact, β has the form $\beta = \frac{\omega \cdot \nu}{2}$, for suitable ν's in Z^d.

The rotation number is defined, as in the periodic case, by

$$\alpha(E) = \pm \lim_{x \to \infty} \frac{\arg f}{x},$$

for any complex solution of

$$L(v) f = E f$$

with $i[f, \overline{f}] \gtrless 0$*. Johnson-Moser [1982]** show that such a limit exists for any complex E and does not depend on the choice of f. Moreover, $E \in R \to \alpha(E)$ is a continuous monotone function, strictly increasing on $\sigma(L)$ and constant on the intervals of $\rho(L) \cap R$. On such intervals, $\alpha = \frac{1}{2} \omega \cdot \nu$, for some integer vector ν. This is the "gap labelling theorem" of Johnson-Moser [1982].

Going back to Moser-Pöschel [1984], they define

$$\tilde{R} = \tilde{R}(\Omega) \equiv \{\beta = \frac{\omega \cdot \nu}{2} : |\beta - \frac{\omega \cdot \mu}{2}| \geq \frac{1}{\Omega(|\mu|)}, Z^d \ni \mu \neq \nu\},$$

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*Actually, there are several equivalent ways to define α (see Johnson-Moser [1982] and Avron-Simon [1983]).

**They develop their theory more in general for almost periodic potentials; see also Avron-Simon [1983] and, for a review, Simon [1982]. For the related class of random potentials, see Pastur [1973], [1980] and Spencer [198-].

Ω being any approximation function, and prove that, if β is big enough and belongs to $\tilde{R}$, then for $E \in \alpha^{-1}(\beta)$ one has two independent eigenfunctions

$$e^{i\beta x}(\chi_1 + x\,\chi_2),\ e^{i\beta x}\,\chi_2 \quad \text{or} \quad e^{i\beta x}\,\chi_3,\ e^{-i\beta x}\,\bar{\chi}_3\ . \tag{2.4}$$

Here, the functions χ are quasi-periodic with frequencies ω and the form (2.4) depends on whether the closed interval $\alpha^{-1}(\beta)$ has positive length or not.

Also, by a limiting procedure, they prove the Dinaburg-Sinai result, replacing the set $R(\sigma)$ with the inverse image by α^{-1} of

$$\{\beta \in R:\ |\beta - \frac{\omega\cdot\mu}{2}| < \frac{3}{\Omega(|\mu|)}\ ,\quad \mu \in Z^d,\ \mu \neq 0\}\ .$$

In this chapter, we shall study the eigenvalue problem

$$\begin{aligned} &L_\varepsilon f \equiv L(v + \varepsilon w)\ f = E\ f\ , \\ &v(x) = V(\omega_1 x),\quad w(x) = W(\omega_2 x, \ldots, \omega_d x) \end{aligned} \tag{2.5}$$

for small ε*; V and W are (real) analytic on, respectively, T and T^{d-1} and $\omega \equiv (\omega_1, \ldots, \omega_d)$ subject to (2.2). We will construct a subset $E = E_\varepsilon(\Omega)$ of $\sigma(L_0) \cap \sigma(L_\varepsilon)$ and a function $E \in E \to a(E) > 0$, so that, for $E \in E$, (2.5) can be viewed as a <u>subsystem of the Hamiltonian equations</u> for (d+1) harmonic oscillators with frequencies (a,ω).

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*For E big, the smallness parameter will be $(\varepsilon/\sqrt{E})$.

In particular, the eigenfunctions of (2.5), for $E \in \mathcal{E}$, will be analytic (a,ω)-quasi-periodic. The frequencies will be seen to satisfy

$$|\nu\cdot(a,\omega)| \geq \frac{1}{\Omega(|\nu|)} , \quad \nu \in Z^{d+1} - \{0\} ,$$

and*

$$|a(E) - \alpha_0(E)| = O(\varepsilon) .$$

Our proof will make use of results from Chapter 1 and an amplification of Rüssmann's KAM scheme (see, in particular, §2.4). The set $\mathcal{E}$ will be completely specified using the smoothness (in the sense of Whitney**) of the KAM limits, especially of $E \to a(E)$.

In §2.8, the function a will be identified with Johnson-Moser rotation number α and it will be shown that the (Whitney) derivative of α^2 satisfies Moser-Deift-Simon inequality†

$$\frac{d\alpha^2}{dE} \geq 1 ,$$

for every E in $\mathcal{E}$.

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*α_0 denotes the rotation number for $L_0 \equiv L(v)$. For E big

$$|a(E) - \alpha_0(E)| = O(\frac{\varepsilon}{\sqrt{E}}).$$

**Whitney [1934]; see, also, §2.7. The idea of using Whitney's notion of smoothness in Hamiltonian perturbation theory appears in Pöschel [1982] and Chierchia-Gallavotti [1982].

†Moser [1981], Deift-Simon [1983].

Finally, adapting the Hamiltonian KAM scheme to a matrix version of (2.5), we will construct on E independent Bloch waves, i.e., solutions of the form $f = e^{i\alpha x}\chi(\omega x)$ and $\bar{f}$, and relate our situation to that one of Moser-Pöschel [1984].

2.2 Hamiltonian Framework

We start by making more precise the assumptions on the quasi-periodic potential $v + \varepsilon w$:

Assumption 1. The functions v and w, thought as functions of *, respectively, $\varphi_2 \in R$ and $(\varphi_3, \ldots, \varphi_d) \in R^{d-2}$ have holomorphic extensions to

$$S(\xi) \equiv \{z \in C: |\mathrm{Im} z| \leq \xi\} \text{ and } S^{d-2}(\xi) \equiv \{z \in C^{d-2}: |\mathrm{Im} z_i| \leq \xi\},$$

for some $\xi > 0$.

Assumption 2. The vector $\tilde{\omega} \equiv (\omega_2, \ldots, \omega_d) \in R_+^{d-1}$ is of unit length and satisfies the Diophantine condition

$$|\tilde{\omega} \cdot \nu| \geq \frac{1}{c\Omega(|\nu|)} \quad , \quad \text{any } \nu \in Z^{d-1} - \{0\} ,$$

where** c is a positive constant and

$$r^{d-2} \leq \Omega(r) \uparrow +\infty \ , \quad \text{for } 1 \leq r \uparrow +\infty \ .$$

---

*For aesthetic reasons, we make the change of notation: $(d+1) \to d$, $(\omega_1, \omega_2, \ldots, \omega_d) \to (\omega_2, \omega_3, \ldots, \omega_d)$. We will consider only $d \geq 3$.

**The role of c will be clear later (see §2.6).

The initial value problem for

$$L_\varepsilon\, q \equiv L(v + \varepsilon w)q = Eq$$

$$v(x) = V(\omega_2 x),\; w(x) = W(\omega_3 x, \ldots, \omega_d x) \tag{2.7}$$

is immediately seen to be part of the evolution equations associated with the Hamiltonian

$$\frac{p^2}{2} + \omega_2 B + \omega_3 A_3 + \ldots + \omega_d A_d + \frac{q^2}{2}\,[E - V(\psi) - \varepsilon W(\varphi_3, \ldots, \varphi_d)]$$

$$(p,q) \in R^2 - \{0\},\; (B, A_3, \ldots, A_d, \psi, \varphi_3, \ldots, \varphi_d) \in R^{d-1} \times T^{d-1} \tag{2.8}$$

with initial data

$$p(0) = q'(0),\; q(0),\; \psi(0) = \varphi_3(0) = \ldots = \varphi_d(0) = 0\ .$$

Now, consider the (surjective) canonical transformation

$$(p, B, A_3, \ldots, A_d, \varphi, \psi, \varphi_3, \ldots, \varphi_d) \in R^{d+1} \times T^{d-1} - (0, R^{d-1}, 0, T^{d-1})$$

$$\rightarrow (A, \varphi) \in M \equiv R_+ \times R^{d-1} \times T^d \tag{2.9}$$

where $(p,B,q,\psi) \rightarrow (A_1, A_2, \varphi_1, \varphi_2)$ is the map of Theorem 1.2*.

Since

$$\frac{q^2}{2} = \frac{r^2 Q^2}{2} = \frac{1}{\kappa} A_1 Q^2 \quad ,$$

---

*For facts concerning L_o we maintain, usually, the notations of Chapter 1, (with the obvious substitutions).

the transformation (2.9) conjugates the Hamiltonian (2.8) to

$$\alpha_o A_1 + \omega_2 A_2 + \ldots + \omega_d A_d - \varepsilon A_1 \frac{Q^2(\varphi_1,\varphi_2)}{\kappa} W(\varphi_3,\ldots,\varphi_d), \tag{2.10}$$

α_o being the unperturbed rotation number for L_o, κ and Q as in Chapter 1*.

Notice that the assumption on the periodic potential V implies that the periodic, complex-valued function $(\varphi_1, \varphi_2) \in R^2 \to F_o(\varphi_1, \varphi_2)$ defined in §1.3 admits a holomorphic extension to $S^2(\xi)$. Therefore, the real analytic periodic function $(\varphi_1, \varphi_2) \to Q^2(\varphi_1, \varphi_2) \equiv (\text{Re } F_o)^2$ in (2.10) has also a holomorphic extension to $S^2(\xi)$.

Remark 1. (On the role of the analyticity assumptions.) A function $\varphi \in R^n \to G(\varphi)$, (2π) - periodic in each variable, which admits a holomorphic extension to the closed strip

$$S^n(s) = \{z \in C^n : |\text{Im } z_i| \leq s\} ,$$

has a Fourier expansion

$$F(\varphi) = \sum_{\nu \in Z^n} \hat{F}_\nu e^{i\nu\cdot\varphi},$$

with coefficients

$$\hat{F}_\nu \equiv \frac{1}{(2\pi)^n} \int_{T^n} F(\varphi) e^{-i\nu\cdot\varphi} d\varphi$$

---

*Q and κ depend on E and $\frac{Q^2}{\kappa} \sim \frac{1}{\sqrt{E}}$, for E $\uparrow +\infty$ (see §1.2).

satisfying the bound

$$|\hat{F}_\nu| \leq (\max_{S^n(s)} |F|) \ e^{-s|\nu|}. \tag{2.11}$$

This will be used over and over.

Another advantage of holomorphic functions is that one can estimate derivatives in terms of the function itself and some loss in the extent of the analyticity domain. More precisely, if f is holomorphic in a (smooth) domain $D \subset C$ and D' is a subdomain of D with*

$$\mathrm{dist}(\partial D, \partial D') \equiv \rho > 0 \ ,$$

then

$$\sup_{D'} \left|\frac{df}{dz}\right| \leq \rho^{-1} \ (\sup_{D} |f|) \ . \tag{2.12}$$

With obvious changes, formula (2.12) and its proof extend to the higher dimensional case.

Remark 2. For the Dinaburg-Sinai case ($V \equiv 0$), the Hamiltonian set-up is much simpler (Gallavotti [198-]): The eigenvalue problem

$$L[\varepsilon W(\omega_3 x, \ \ldots, \ \omega_d x)] \ q = Eq$$

is embedded in the system with Hamiltonian

---

*Here "∂" denotes "the boundary of".

$$\frac{p^2}{2} + \omega_3 A_3 + \dots + \omega_d A_d + \frac{q^2}{2}\,[E - \varepsilon W(\varphi_3, \dots, \varphi_d)]\ .$$

The latter is conjugate, via a polar-coordinate transformation in the $(p, \sqrt{E}q)$-plane, to

$$\sqrt{E}\,A_1 + \omega_3 A_3 + \dots + \omega_d A_d - \frac{\varepsilon}{\sqrt{E}} A_1 \sin^2\varphi_1\, W(\varphi_3, \dots, \varphi_d)\ .$$

Notice that this system is (d-1) dimensional and that, here, $\alpha_o(E) = \sqrt{E}$.

2.3 Inductive Lemma

We will base the (Hamilton-Jacobi) integration of (2.10) on the perturbation algorithm presented in this section*.

Consider the Hamiltonian

$$H^{(j)}(A,\varphi;a,\varepsilon) \equiv \omega_1^{(j)}(a;\varepsilon)A_1 + \omega_2 A_2 + \dots + \omega_d A_d$$
$$+ \varepsilon^{2^j} A_1 F^{(j)}(\varphi;a,\varepsilon), \qquad (j \in N),$$

on the phase space $M = R_+ \times R^{d-1} \times T^d$, $0 < \varepsilon < 1$. Assume that $\omega_1^{(j)}$ and $F^{(j)}$, as functions of a, are holomorphic in

$$D_j \equiv D(\eta_j;A^{(j)}) \equiv \bigcup_{a_o \in A^{(j)}} \{a \in C: |a-a_o| < \eta_j\}$$

---

*Perturbations of harmonic oscillators were exploited by Rüssmann [1967] and Gallavotti [1982]. Their proof makes use of Moser's idea of modified systems (Moser [1967]). Our proof differs from these.

for some subset $A^{(j)}$ of the real line. As functions of $\varphi \in R^d$, $F^{(j)}$ is (2π-) periodic in each variable and admits a holomorphic extension to $S^d(\xi_j)$ with* $\|F^{(j)}\|_{\xi_j,\eta_j} \leq M_j$ independently of ε. Finally, assume $0 < \xi_j < 1$, the upper bound being imposed only for simplicity.

Now, let** $\delta_j < \frac{\xi_j}{2}$ and define

$$\zeta(s) \equiv 1 + \sum_{\nu \in Z^d - \{0\}} |\nu| \Omega(|\nu|)\, e^{-s|\nu|} \ , \ s > 0,$$

$$N_j \equiv 2^{j+1}\, \delta_j^{-1} \log \varepsilon^{-1}$$

$$F_T^{(j)}(\varphi) \equiv \sum_{|\nu| \leq N_j} \hat{F}_\nu^{(j)}\, e^{i\nu\cdot\varphi} \ , \ F_R^{(j)}(\varphi) \equiv \sum_{|\nu| > N_j} \hat{F}_\nu^{(j)}\, e^{i\nu\cdot\varphi} \ ,$$

$$\eta_{j+1} \equiv \min\left\{ \left[2cN_j \Omega(N_j) \sup_{a \in D_j} \left| \frac{d\omega_1^{(j)}}{da} \right| \right]^{-1}, \frac{\eta_j}{2} \right\} \ ,$$

$$\xi_{j+1} \equiv \xi_j - 2\delta_j \ ,$$

$$A^{(j+1)} \equiv \{ a \in A^{(j)} : |\omega^{(j)} \cdot \nu| \geq \frac{1}{c\Omega(|\nu|)} \ , \nu \in Z^d - \{0\},\ |\nu| \leq N_j \} \ ,$$

$$\omega^{(j)} \equiv (\omega_1^{(j)}, \tilde{\omega}) \ ,$$

$$D_{j+1} \equiv D(\eta_{j+1}; A^{(j+1)}) \ .$$

---

*For functions F on $S^d(s) \times D(r;I)$, we set $\|F\|_{s,r} \equiv \sup_{(z,a) \in S \times D} |F|$.

**This condition on the "analyticity loss" parameter, δ_j, is forced by the proof of the lemma.

<u>Inductive Lemma</u>. There exist two universal constants* $K_1 > K_2$ such that if $a \in A^{(j+1)}$ and

$$K_1 \, \zeta(\delta_j) \, \delta_j^{-1} \, cM_j \, \varepsilon^{2^j} \leq 1 \tag{2.13}$$

then the function

$$(A',\varphi) \in M \to A' \cdot \varphi + \varepsilon^{2^j} A_1 \, \phi_j(\varphi;a,\varepsilon) \; ,$$

$$\phi_j \equiv \sum_{0<|\nu|\leq N_j} \frac{\hat{F}_\nu^{(j)}}{-i \, \omega^{(j)} \cdot \nu} e^{i\nu\cdot\varphi}$$

is the generating function of a surjective canonical transformation, $(A,\varphi) \in M \to (A',\varphi') = (A'(A,\varphi), \varphi'(\varphi))$, that conjugates $H^j(A,\varphi)$ to

$$\begin{aligned} H^{(j+1)}(A',\varphi';a,\varepsilon) &\equiv H^{(j)}(A(A',\varphi'), \varphi(\varphi')) \\ &= \omega^{(j+1)} \cdot A' + \varepsilon^{2^{j+1}} A_1' \; F^{(j+1)}(\varphi'; a,\varepsilon) \end{aligned}$$

where

$$\omega^{(j+1)} \equiv (\omega_1^{(j)} + \varepsilon^{2^j} \hat{F}_0^{(j)} \, , \omega_2, \ldots, \omega_d) \; ,$$

$$F^{(j+1)}(\varphi'(\varphi)) \equiv \frac{\partial\phi_j}{\partial\varphi_1} (\varphi) F^{(j)}(\varphi) + \frac{F_R^{(j)}(\varphi)}{\varepsilon^{2^j}} \; .$$

---

*That is, constants depending only on the dimension d. We also indicate and will use the following bounds: $K_1 \geq 2 K_2$, $K_2 \geq 4$.

Furthermore, $a \in A^{(j+1)} \to \omega_1^{(j)}(a)$ and $(\varphi,a) \in R^d \times A^{(j+1)}$ $\to F^{(j+1)}(\varphi;a)$ have holomorphic extensions to, respectively, D_{j+1} and $S^d(\xi_{j+1}) \times D_{j+1}$ with

$$\|F^{(j+1)}\|_{\xi_{j+1},\eta_{j+1}} \leq K_2\ \zeta(\delta_j)\ \delta_j^{-d}\ c\ M_j^2 \equiv M_{j+1}\ . \tag{2.14}$$

Remark 1. It is useful to keep in mind that most of the quantities we are dealing with have physical dimensions*:

$$[\omega_i] = [\eta_j] = [M_j] = [F_j] = [\text{time}]^{-1},\ [c] = [\text{time}].$$

The angle related quantities δ_j, ξ_j are, instead, dimensionless.

Notational Warning: During this and later proofs, we indicate (sometimes different) universal constants with the same symbol "K.".

Proof. The cutoff** N_j is made so that

$$\|F_R^{(j)}\|_{\xi_j-\delta_j,\eta_j} \leq M_j \delta_j^{-d}\ \varepsilon^{2^j}. \tag{2.15}$$

In fact, for $a \in D_j$, using (2.11) to estimate $\hat{F}_\nu^{(j)}$,

---

*Here, square brackets indicate physical dimensions.

**The key idea of the cutoff goes back to Arnold [1963].

$$\sup_{z\in S^d(\xi_j-\delta_j)} \Big|\sum_{|\nu|>N_j} \hat{F}_\nu^{(j)} e^{iz\cdot\nu}\Big| \le M_j \sum_{|\nu|>N_j} e^{-\delta_j|\nu|}$$

$$\le M_j\, e^{-(\delta_j N_j)/2} \sum_{|\nu|>N_j} e^{-(\delta_j|\nu|)/2} \le K.\ M_j\ \delta_j^{-d}\ \varepsilon^{2^j} .$$

Analogously,

$$\|F\|_{\xi_j-\delta_j,\ \eta_j} \le K.\ M_j\ \delta_j^{-d} \quad . \tag{2.16}$$

Next, we show that ϕ_j has an holomorphic extension to $S^d(\xi_j-\delta_j) \times D_{j+1}$ with

$$\max\ \{\|\phi_j\|,\ \|\tfrac{\partial\phi}{\partial\varphi}\|\} \le K.\ \zeta(\delta_j)\ c\ M_j \ , \tag{2.17}$$

the norms being relative to such a set.

To prove this, we have to take care of the small denominators appearing in ϕ_j. Let $a \in D_{j+1}$. Then there is a point $a_0 \in A^{(j+1)}$ with $|a-a_0| < \eta_{j+1}$ so that, for $0 < |\nu| \le N_j$, the definitions of $A^{(j+1)}$ and η_{j+1} imply

$$|\omega^{(j)}(a)\cdot\nu| = |\omega^{(j)}(a_0)\cdot\nu + (\omega_1^{(j)}(a) - \omega_1^{(j)}(a_0))\ \nu_1|$$

$$\ge |\omega^{(j)}(a_0)\cdot\nu| \left(1 - \frac{|\omega_1^{(j)}(a) - \omega_1^{(j)}(a_0)|\,|\nu_1|}{|\omega^{(j)}(a_0)\cdot\nu|}\right)$$

$$\ge \frac{1}{c\Omega(|\nu|)} \left(1 - c\Omega(N_j)\ N_j \sup_{D_j} \Big|\frac{d\omega_1^{(j)}}{da}\Big|\ \eta_{j+1}\right) \ge \frac{1}{2c\Omega(|\nu|)} \ .$$

Now, for $(z,a) \in S^d(\xi_j - \delta_j) \times D_{j+1}$,

$$\max\{|\phi_j|, |\frac{\partial \phi_j}{\partial \varphi}|\} \le \sum_{0<|\nu|\le N_j} \frac{|\hat{F}_\nu^{(j)}|\ |\nu|}{|\omega^{(j)} \cdot \nu|} e^{(\xi_j - \delta_j)|\nu|}$$

$$\le 2cM_j \sum_{\nu \neq 0} |\nu|\ \Omega(|\nu|)\ e^{-\delta_j|\nu|} < K.\ \zeta(\delta_j)\ c\ M_j\ .$$

The function $A' \cdot \varphi + \varepsilon^{2^j} A_1\ \phi_j(\varphi)$ will generate a canonical transformation if and only if* we can invert the map $(A', \varphi) \to (A, \varphi')$ given by

$$\varphi' = \frac{\partial}{\partial A'}(A' \cdot \varphi + \varepsilon^{2^j} A_1\ \phi_j) = (\varphi_1 + \varepsilon^{2^j}\ \phi_j, \varphi_2, \ldots, \varphi_d)$$

$$A = \frac{\partial}{\partial \varphi}(A' \cdot \varphi + \varepsilon^{2^j} A_1\ \phi_j) = T_j\ A' \tag{2.18}$$

where

$$T_j = T_j(\varphi) \equiv \begin{bmatrix} 1 + \varepsilon^{2^j} \frac{\partial \phi_j}{\partial \varphi_1} & 0 \cdots 0 \\ \varepsilon^{2^j} \frac{\partial \phi_j}{\partial \varphi_2} & 1 0 \cdots 0 \\ \vdots & \ddots \\ \varepsilon^{2^j} \frac{\partial \phi_j}{\partial \varphi_d} & 0 \cdots 0 1 \end{bmatrix} .$$

*This is a standard fact in Hamiltonian mechanics. See, e.g., Arnold [1978] or Gallavotti [1983].

To confirm this, we use the following elementary version of a global implicit function theorem. We defer the proof to Appendix A.

<u>Proposition</u>. Let $z \in S^d(r) \to g(z;\sigma) \in C^d$ be a holomorphic map parametrized by $\sigma \in \Sigma \subset C^n$, and let $0 < s < 1$. There exists a universal constant $K_3 > 1$ such that if, for any $\sigma \in \Sigma$,

$$K_3 \max\left\{ \left\|\frac{\partial g}{\partial z}\right\|_r , \frac{\|g\|_r}{s} \right\} < 1 \quad , \tag{2.19}$$

then the map $z \in S^d(r) \to z + g(z;\sigma)$ is one-to-one from $S^d(r)$ onto $S^d(r-s)$. The inverse map can be written in the form

$$z' \in S^d(r-s) \to z = z' + h(z';\sigma) \in S^d(r)$$

with $z \to h(z;\sigma)$ holomorphic and $\|h\|_{r-s} \leq \|g\|_r$.

Regularity properties of h with respect to $\sigma \in \Sigma$ are the same as for g.

Finally, if g is real on R^d and periodic in each variable, so is h.

The last statement of the proposition means that the smooth map $\varphi \in T^d \to \varphi + g(\varphi;\sigma)$ is globally inverted by $\varphi' \in T^d \to \varphi' + h(\varphi';\sigma)$.

Thus, the proposition and estimate (2.17) show that we can fix K_1 so that if condition (2.13) hols then the map $\varphi \to \varphi'$ in (2.18) is globally inverted by

$$\varphi = (\varphi_1' + \varepsilon^{2^j} \Delta_j(\varphi';a,\varepsilon), \varphi_2', \ldots, \varphi_d') \; ; \qquad a \in D_{j+1}$$

with $(z,a) \to \Delta_j(z;a,\varepsilon)$ holomorphic in $S^d(\xi_{j+1}) \times D_{j+1}$ and

$$\|\Delta_j\|_{\xi_{j+1},\eta_{j+1}} \le \|\phi_j\|_{\xi_j-\delta_j,\eta_{j+1}} . \tag{2.20}$$

Also, since $\varepsilon^{2^j} \sup_{T^d} \left|\frac{\partial\phi_j}{\partial\varphi_1}\right| < 1$ because of (2.13), we have

$$A' = T_j^{-j}A \ , \quad T_j^{-1} = \begin{bmatrix} \dfrac{1}{1+\varepsilon^{2^j}\dfrac{\partial\phi_j}{\partial\varphi_1}} & 0 \cdots\cdots 0 \\ \dfrac{-\varepsilon^{2^j}\dfrac{\partial\phi_j}{\partial\varphi_2}}{1+\varepsilon^{2^j}\dfrac{\partial\phi_j}{\partial\varphi_1}} & 10 \cdots\cdots 0 \\ \vdots & \\ \dfrac{-\varepsilon^{2^j}\dfrac{\partial\phi_j}{\partial\varphi_d}}{1+\varepsilon^{2^j}\dfrac{\partial\phi_j}{\partial\varphi_1}} & 0 \cdots\cdots 01 \end{bmatrix} . \tag{2.21}$$

Notice that T_j^{-1} maps $R_+ \times R^{d-1}$ onto itself.

At this point, $H_j(A(A', \varphi'), \varphi(\varphi'))$ is readily computed and (2.15), (2.16), (2.17) and (2.20) easily imply (2.14).

The Inductive Lemma is proven.

2.4 Compatibility of Approximation Functions and Analyticity-Losses*

Our next step will be to apply the Inductive Lemma infinitely many times so to end up with an integrable system for values of a belonging to

$$A^{(\infty)} = \bigcap_{j=0}^{\infty} A^{(j)}.$$

To do this, we have to look closer at the relation between Ω and δ_j, the up-to-now arbitrary quantities appearing in condition (2.13).

First, notice that, by definition of ξ_{j+1},

$$\xi_\infty \equiv \lim_{j\uparrow\infty} \xi_j = \xi_0 - 2 \sum_{j=0}^{\infty} \delta_j ,$$

so that it is natural to require

$$\sum_{j=0}^{\infty} \delta_j < \frac{\xi_0}{2} . \tag{2.22}$$

Also, to meet condition (2.13) for any j, we need $\zeta(\delta_j) < \infty$ and since $\delta_j \downarrow 0$, this means that

$$\lim_{r\uparrow\infty} \frac{\log\Omega(r)}{r} = 0 . \tag{2.23}$$

These observations motivate

---

*This paragraph is inspired by Rüssmann [1980].

Definition 1. A sequence $\{\delta_j\}$ satisfying (2.22) will be called an analyticity-loss sequence.

A function Ω satisfying

$$r^{d-2} \leq \Omega(r) \uparrow \infty \quad \text{and} \quad \frac{\log\Omega(r)}{r} \downarrow 0 \quad (1 \leq r \uparrow \infty) \qquad (2.23)'$$

will be called an approximation function*.

Not all the approximation functions and analyticity-loss sequences will be suitable for our purpose: We will see that a necessary and sufficient condition for the application of our iterative scheme is that

$$\prod_{j=0}^{\infty} \zeta(\delta_j)^{\frac{1}{2^j}} < \infty \quad . \qquad (2.24)_{\mathrm{I}}$$

Furthermore, in order to control the set $A^{(\infty)}$, the condition $(2.24)_{\mathrm{I}}$ is not in general enough. We will use a stronger version of (2.23)':

$$\lim_{j\uparrow\infty} \frac{\log\Omega(2^j\,\delta_j^{-1})}{2^j} = 0 \ . \qquad (2.24)_{\mathrm{II}}$$

For these reasons we make the following

Definition 2. An approximation function and an analyticity-loss sequence are said to be compatible if conditions $(2.24)_{\mathrm{I}}$ and $(2.24)_{\mathrm{II}}$ hold.

---

*The nomenclature is adapted from Rüssmann [1980]. The monotonicity in (2.23)' is assumed for simplicity.

Examples . 1) $\Omega(r) = r^m$ $(m \geq d-2)$ and $\{\delta_j\}$ are compatible if and only if

$$\Sigma \frac{1}{2^j} \log\delta_j^{-1} < \infty .$$

Moreover, one has

$$(2.24)_I \Leftrightarrow \sum \frac{1}{2^j} \log\delta_j^{-1} < \infty \; ; \; (2.24)_{II} \Leftrightarrow \frac{1}{2^j} \log\delta_j^{-1} \to 0.$$

2) $\Omega(r) = e^{\sqrt{r}}$ and $\{\delta_j\}$ are compatible if and only if

$$\sum \frac{\delta_j^{-1}}{2^j} < \infty .$$

Moreover, one has

$$(2.24)_I \Leftrightarrow \sum \frac{\delta_j^{-1}}{2^j} < \infty \; ; \quad (2.24)_{II} \Leftrightarrow \frac{\delta_j^{-1}}{2^j} \to 0.$$

3) Let $\sigma > 1$ and

$$\Omega(r) = \begin{cases} \exp\left(\dfrac{r}{\log^\sigma r}\right) , & r \geq e^\sigma \\[2ex] \Omega(e^\sigma) , & 1 \leq r \leq e^\sigma . \end{cases}$$

Then, Ω and $\{\delta_j\}$ are compatible if and only if

$$\frac{\delta_j^{-1}}{j^\sigma} \to 0 .$$

Moreover, one has

$$(2.24)_I \Leftrightarrow \frac{\delta_j^{-1}}{j^\sigma} \text{ bounded } ; \quad (2.24)_{II} \Leftrightarrow \frac{\delta_j^{-1}}{j^\sigma} \to 0 \ .$$

Remark 1. In the first two examples, condition $(2.24)_I$ is stronger than $(2.24)_{II}$, while in the third one, the opposite is true.

Remark 2. Rüssmann's aptitude is slightly different: He defines

$$\Psi \equiv \inf \prod_{j=0}^{\infty} \zeta(\delta_j)^{\frac{1}{2^j}} \ , \qquad (2.24)_I'$$

where the infimum is taken over all the analyticity-loss sequences, and then shows that

$$\frac{\log\Omega(r)}{r} \downarrow 0 \ , \quad \int^{\infty} \frac{\log\Omega(r)}{r^2} < \infty$$

imply $\Psi < \infty$.

The resemblence of $(2.24)_I$ with $(2.24)_I'$ is clear, but our condition $(2.24)_{II}$, needed to control $A^{(\infty)}$, doesn't appear in Rüssmann's work.

2.5 KAM Iteration Scheme

At this point, we have to check that the Hamiltonian (2.10) associated to the Schrödinger equation (2.7) satisfies the assumptions of the Inductive Lemma.

For this purpose, it is more convenient to regard (2.10) as parametrized by $\alpha_0 = a$ rather than by E. Let*

$$a \in \bigcup_{k=0}^{\infty} \left(\frac{k}{2}\,\omega_2, \frac{k+1}{2}\,\omega_2\right) \to e_0(a) \in \sigma^*(L_0)$$

denote the inverse function of $E \in \sigma^* \to \alpha_0(E)$ (compare Figure 3), and set

$$\omega \equiv (a, \omega_2, \ldots, \omega_d),\ F(\varphi; a) \equiv -\frac{Q^2(\varphi_1, \varphi_2; e_0(a))}{\kappa(e_0(a))}\, W(\varphi_3, \ldots, \varphi_d).$$

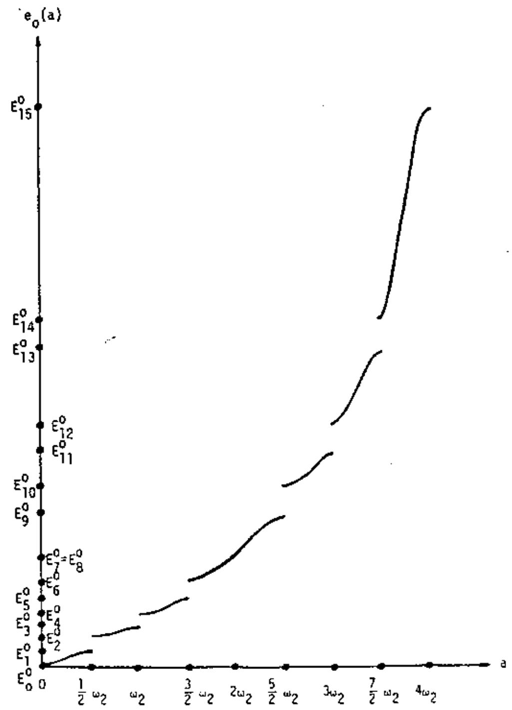

Figure 3.

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*Recall the notation from the first chapter. In particular, $\sigma^* \equiv \bigcup_{k=0}^{\infty} (E^0_{2k}, E^0_{2k+1})$.

By §2.2, F as function of φ meets the requirements of the Inductive Lemma if we put $\xi_0 \equiv \xi$. As for the a-dependence, notice that $E \to \alpha_0(E)$ has a natural holomorphic extension as soon as one stays away from $\partial\sigma^*$: Fix a point $\tilde{E}_k$ in (E^0_{2k}, E^0_{2k+1}). Then for E in any simply connected region containing E_k, the extension will be*

$$\alpha_0(E) \equiv \alpha_0(\tilde{E}_k) + (-1)^{k+1} \frac{\omega_2}{2\pi} \int_{\tilde{E}_k}^{E} \frac{\frac{d\Delta}{dE'}}{\sqrt{1-\Delta^2}} \, dE' \; .$$

Thus, it is easy to see that for any

$$\eta < \frac{\omega_2}{8}$$

and for

$$A_0 \equiv \bigcup_{k=0}^{\infty} \{a \in R: \; (\tfrac{k}{2}\,\omega_2 + 2\eta) \le a \le (\tfrac{k+1}{2}\,\omega_2 - 2\eta)\} \; ,$$

the function $a \to F(\varphi;a)$ has a holomorphic extension to the region $D(\eta;A_0)$.

Finally, we set

$$\omega^{(0)} \equiv \omega, \; F^{(0)} \equiv F, \; \xi_0 \equiv \xi, \; \eta_0 \equiv \eta, \; A^{(0)} \equiv A_0, \; D_0 \equiv D(\eta;A_0)$$

and

$$M_0 \equiv \sup_{S^d(\xi_0) \times D_0} |F| \; .$$

---

*Compare formula (1.4) of §1.2.

Now, imagine having applied the Inductive Lemma for $j = 0, \ldots, n$. Repeating estimate (2.14) (n+1) times, one gets

$$\|F^{(n+1)}\|_{\xi_{n+1},\eta_{n+1}} \leq M_{n+1}$$

$$\leq (K_2 c)^{(2^{n+1}-1)} [\zeta(\delta_n)\ \zeta(\delta_{n-1})^2 \cdots \zeta(\delta_o)^{2n}]$$

$$\times [\delta_n^{-1}\ \delta_{n-1}^{-2} \cdots \delta_o^{-2^n}]\ M_o^{2^{n+1}} \ :$$

To apply the lemma one more time ($j = n+1$), it will suffice by (2.13) to have

$$\frac{K_1}{K_2} (K_2\, c\, M_o \varepsilon)^{2^{n+1}} [(\zeta(\delta_{n+1})\delta_{n+1}^{-d})(\zeta(\delta_n)\delta_n^{-d})^2 \cdots (\zeta(\delta_o)\delta_o^{-d})^{2^{n+1}}] \leq 1. \tag{2.25}$$

But, since $\Omega(r) \geq r^{d-2}$, one has

$$\delta_j^{-d} < \zeta(\delta_j)^{\frac{1}{2}} \quad ,$$

whence, by $(2.24)_I$,

$$\Psi \equiv \prod_{j=0}^{\infty} (\zeta(\delta_j)\delta_j^{-d})^{\frac{1}{2^j}} < \left(\prod_0^{\infty} \zeta(\delta_j)^{\frac{1}{2^j}} \right)^{\frac{3}{2}} < \infty \ .$$

Thus, (2.25) is implied by

$$\left[\left(\frac{K_1}{K_2} \right)^{\frac{1}{2^{n+1}}} \Psi(K_2\, c\, M_o \varepsilon) \right]^{2^{n+1}} \leq 1 \ .$$

We see that in order to apply the Inductive Lemma, an <u>arbitrary</u> number of times, we must have

$$K_1 \Psi c M_0 \varepsilon \leq 1$$

which may be expressed as

$$\frac{K_1}{K_2} \varepsilon \tau \leq 1 , \quad \tau \equiv K_2 \Psi c M_0 . \tag{2.26}$$

Analogously, one checks the estimate

$$cM_n \leq \frac{(\varepsilon\tau)^{2^n}}{K_2} .$$

Now, the following theorem is a simple matter.

<u>Theorem</u>. If ε satisfies condition (2.26) and

$$a \in A^{(\infty)} \equiv \bigcap_{j=0}^{\infty} A^{(j)} ,$$

then the Hamiltonian $H^{(0)}$ is conjugate to the (non-resonant) system of harmonic oscillators

$$H^{(\infty)} \equiv \omega^{(\infty)} \cdot A ,$$

where $\omega^{(\infty)} \equiv (\omega_1^{(\infty)}(a,\varepsilon), \omega_2, \ldots, \omega_d)$ verifies

$$c|\omega_1^{(\infty)} - a| \leq \frac{1}{K_2} \sum_{j=0}^{\infty} (\varepsilon\tau)^{2^j} \tag{2.27}$$

$$|\omega^{(\infty)} \cdot \nu| \geq \frac{1}{c\Omega(|\nu|)}, \quad \nu \in Z^d - \{0\} .$$

The (surjective) canonical transformation conjugating $H^{(0)}$ to $H^{(\infty)}$ has the form

$$(A',\varphi') \in M \to (S(\varphi')\, A',\ \varphi_1' + \varepsilon\Delta(\varphi_1'), \varphi_2', \ldots, \varphi_d') \in M \tag{2.28}$$

with* S a (dxd)-matrix of the form

$$\begin{bmatrix} 1+\varepsilon s_1 & 0 \cdots 0 \\ \varepsilon s_2 & 1 0 \cdots 0 \\ \vdots & \ddots \\ \varepsilon s_d & 0 \cdots 0 1 \end{bmatrix} .$$

Moreover, the vector $s \equiv (s_1, \ldots, s_d)$ and Δ have holomorphic extensions to $S^d(\xi_\infty)$ and**

$$\max \{\|s\|_{\xi_\infty}, \|\Delta\|_{\xi_\infty}\} \leq K_4\tau \leq \frac{K_1}{K_2}\tau . \tag{2.29}$$

---

*S and Δ depend on $a \in A^{(\infty)}$ and ε.

**K_4 is a universal constant.

Proof. Condition (2.26) enables us to apply the Inductive Lemma an arbitrary number of times. Now, let

$$g_j(z') \equiv (z_1' + \varepsilon^{2^j} \Delta_j, z_2', \dots, z_d'),$$

$$G_j \equiv g_o \circ g_1 \circ \dots \circ g_j ,$$

$$S_j \equiv T_o(g_o \circ \dots \circ g_j)\, T_1(g_1 \circ \dots \circ g_j) \dots T_{j-1}(g_{j-1} \circ g_j) T_j(g_j).$$

We claim that for $a \in A^{(\infty)}$, $\lim_{j\uparrow\infty} G_j \equiv G$ and $\lim_{j\uparrow\infty} S_j \equiv S$ exist and that such limits are uniform on compact sets of $S^d(\xi_\infty)$. To prove this claim at this point is completely straightforward but not so short, for this reason we give the details in Appendix B.

But then, since $(A',\varphi') \in M \to (S_j(\varphi')\, A', G_j(\varphi'))$ is a canonical map and since $\lim_{j\uparrow\infty} \frac{\partial S_j}{\partial \varphi} = \frac{\partial S}{\partial \varphi}$, also the map $(A',\varphi') \to (S(\varphi')\, A', G(\varphi'))$ is canonical and

$$H(S(\varphi')\, A', G(\varphi')) = \lim_{j\uparrow\infty} H(S_j(\varphi')\, A', G_j(\varphi'))$$

$$= \lim_{j\uparrow\infty} (\omega^{(j)} \cdot A' + \varepsilon^{2^j} A_1' F^{(j)}(\varphi')) = \omega^{(\infty)} \cdot A' \equiv H^{(\infty)}(A').$$

For the last assertion of the theorem, see also Appendix B.

Remark 1. Since $a \to \omega_1^{(n)}(a)$ is continuous on $A^{(n)}$ (actually is holomorphic on $D(\eta_{n-1}; A^{(n-1)}) \supset A^{(n-1)} \supset A^{(n)}$) the set $A^{(n+1)}$ is closed, therefore also $A^{(\infty)}$ is closed.

Remark 2. A system of harmonic oscillators with Hamiltonian

$$\sum_{i=1}^{n} \beta_i I_i \quad ,$$

$(I,\psi) \in R^n \times T^n$ denoting action-angle variables, is said to be resonant if the frequencies $\beta = (\beta_1, \ldots, \beta_n)$ are rationally dependent:

$$\beta \cdot \nu_o = 0 \ , \quad \text{for some } \nu_o \in Z^n - \{0\} \ .$$

The word "resonant" comes from the fact that for such systems an arbitrarily small perturbation may produce non quasi-periodic motions*.

For such reasons, we will call, sometimes, $A^{(\infty)}$ a non-resonant set.

2.6 Structure of the Non-Resonant Set $A^{(\infty)}$

We cannot apply the KAM scheme for a in the set

$$R \equiv A^{(0)} - A^{(\infty)} = \bigcup_{j=0}^{\infty} \bigcup_{\substack{\nu \in Z^d \\ 0<|\nu|\le N_j}} R_\nu^{(j)}$$

where for $0 < |\nu| \le N_j$

$$R_\nu^{(j)} \equiv \{a \in A^{(j)} : |\omega^{(j)} \cdot \nu| < \frac{1}{c\Omega(|\nu|)} \} \quad .$$

*See Gallavotti [1983], pg. 498.

Notice that because of Assumption 2, §2.2, on $\tilde{\omega}$, $R_\nu^{(j)}$ is empty when $\nu_1 = 0$, therefore the above union is actually taken on $\nu \in Z^d$, $\nu_1 \neq 0$.

Our next task is to control the sets $R_\nu^{(j)}$. Suppose, for the moment, that we could extend the functions* $a \in A^{(j-1)} \to \omega_1^{(j)}(a)$ to R in such a way that (calling the extensions again $\omega_1^{(j)}$)

$$\sup_R \left| \frac{d\omega_1^{(j)}}{da} - 1 \right|$$

is less than, say, $\frac{2}{3}$. Then calling $b \in R \to a^{(j)}(b)$ the inverse function of the extended $\omega_1^{(j)}$ and setting**

$$a_{j,\nu} \equiv a^{(j)}\left(-\frac{\tilde{\omega}\cdot\tilde{\nu}}{\nu_1}\right), \qquad r_\nu \equiv \frac{3}{c|\nu_1|\Omega(|\nu|)}$$

for $\nu_1 \neq 0$, $|\nu| \leq N_j$, we would have

$$R_\nu^{(j)} \subset \{a \in A^{(0)} : |a - a_{j,\nu}| < r_\nu\} \equiv I_\nu^{(j)} \quad .$$

In fact, for a in $R_\nu^{(j)}$,

$$\left|\omega_1^{(j)} - \left(-\frac{\tilde{\omega}\cdot\tilde{\nu}}{\nu_1}\right)\right| < \frac{1}{c|\nu_1|\Omega(|\nu|)} \quad ,$$

which implies

---

*Since for $j = 0$ $\omega_1^{(j)}(a) = a$, we define $A^{(-1)} \equiv R$.

**Recall that for $x = (x_1, \ldots, x_d)$, $\tilde{x} \equiv (x_2, \ldots, x_d)$. Obviously $a^{(0)}(b) = b$.

$$a^{(j)}\left(-\frac{\tilde{\omega}\cdot\tilde{\nu}}{\nu_1}-\frac{1}{c|\nu_1|\Omega(|\nu|)}\right) < a < a^{(j)} \quad -\frac{\tilde{\omega}\cdot\tilde{\nu}}{\nu_1}+\frac{1}{c|\nu_1|\Omega(|\nu|)}$$

and, since $\frac{3}{5} < \frac{da^{(j)}}{db} < 3$, $a_{j,\nu} - r_\nu < a < a_{j,\nu} + r_\nu$.

In this way, we would succeed in controlling the resonant set R with exactly determined intervals.

We proceed, now, to construct extensions of the $\omega_1^{(j)}$'s satisfying

$$\sup_R \left|\frac{d\omega^{(j)}}{da} - 1\right| < \frac{2}{3}\,.$$

To do this, we have to assume that Ω and $\{\delta_j\}$ are compatible (see §2.4). We also assume, for simplicity

$$c \geq \left(\frac{3}{5}\,\frac{1}{\Omega^*(2\delta_0^{-1}\;\log\varepsilon_0^{-1})}\right)\eta_0^{-1} \tag{2.30}$$

where ε_0 satisfies the KAM-applicability condition (2.26) and $\Omega^*(r) \equiv r\Omega(r)$.

Proposition 1. If $\varepsilon \leq \varepsilon_0$ satisfies

$$2\left(\varepsilon\, M_0\, \eta_0^{-1} + (\varepsilon\tau)^2 \sum_{j=0}^{\infty} (\varepsilon\tau)^{2^j}\, \Omega^*(N_j)\right) < 1 \tag{2.31}$$

then, for $j \geq 1$,

$$\eta_j^{-1} < 4\Omega^*(N_{j-1})\, c \tag{2.32}$$

and

$$\sup_{D(\eta_{j+1}, A^{(j-1)})} \left| \frac{d\omega_1^{(j)}}{da} - 1 \right| < \frac{2}{3} \ . \tag{2.33}$$

Proof. We first show that since Ω and $\{\delta_j\}$ are compatible, condition (2.31) can be met for ε small enough. From $\frac{\log\Omega(r)}{r} \downarrow 0$ as $r \uparrow \infty$ it follows $\frac{\log\Omega^*(r)}{r} \downarrow 0$. Therefore, for any j, setting $r \equiv \log\varepsilon^{-1}$, $\alpha \equiv 2^j$, $\beta \equiv 2^j \log\tau$ and $\gamma \equiv 2^{j+1}\delta_j^{-1}$, one has

$$(\varepsilon\tau)^{2^j}\ \Omega^*(N_j) = \exp\left[-r\left(\alpha - \frac{\beta}{r} - \gamma\ \frac{\log\Omega^*(\gamma r)}{\gamma r}\right)\right] \to 0 \ (\varepsilon \downarrow 0). \tag{2.34}$$

Now, because $\varepsilon\tau < 1$, we can find $0 < \lambda < 1$ for which $\varepsilon^\lambda\tau < 1$ and since*

$$\frac{\log\Omega^*(2^j\delta_j^{-1})}{2^j} \to 0 \qquad (j \uparrow \infty)$$

there exists a j_0 (independent of ε) such that

$$\frac{\log\Omega^*(2^j\delta_j^{-1})}{2^j} \le \frac{1-\lambda}{2} , \qquad j \ge j_0 \ .$$

---

*Compatibility for Ω and $\{\delta_j\}$ implies compatibility for Ω^* and $\{\delta_j\}$.

Then, for $j \geq j_o$, using the monotonicity of $\frac{\log\Omega^*(r)}{r}$ to infer that $\log\Omega^*(rs) \leq r \log\Omega^*(s)$ $(r,s \geq 1)$, one has

$$(\varepsilon\tau)^{2^j} \Omega^*(N_j) = (\varepsilon\tau)^{2^j} e^{\log\Omega^*(N_j)}$$

$$\leq \left(\varepsilon\tau\left(\frac{1}{\varepsilon^2}\right)^{\frac{\log\Omega^*(2^j\delta_j^{-1})}{2^j}}\right)^{2^j} \leq \left(\varepsilon\tau\left(\frac{1}{\varepsilon^2}\right)^{\frac{1-\lambda}{2}}\right)^{2^j} = (\varepsilon^\lambda\tau)^{2^j} .$$

The use of (2.34) for $j \leq j_o$ shows that

$$\sum_{j=0}^{\infty} (\varepsilon\tau)^{2^j} \Omega^*(N_j) \to 0 \qquad (\varepsilon \downarrow 0) .$$

Now, let's proceed with the proof of (2.32) and (2.33). For $j = 0$, (2.33) is trivially true. Assume, inductively, that (2.33) holds for $0 \leq j \leq k$. Then for $1 \leq j \leq k$, because of $2\,\Omega^*(N_{j-1}) \leq \Omega^*(N_j)$ and (2.30), one has

$$\eta_j^{-1} \equiv \max\left\{2\eta_{j-1}^{-1}, 2\,\Omega^*(N_{j-1})\ c \left|\frac{d\omega_1^{(j-1)}}{da}\right|_{\eta_j}\right\}$$

$$\leq \max\ \{2\eta_{j-1}^{-1},\ 2\,\tfrac{5}{3}\,\Omega^*(N_{j-1})\ c\}$$

$$\leq \max\ \{2^j\ \eta_0^{-1},\ 2^j\,\tfrac{5}{3}\,\Omega^*(N_0)\ c,\ 2^{j-1}\,\tfrac{10}{3}\,\Omega^*(N_1)c,\ldots,$$

$$2\,\tfrac{10}{3}\,\Omega^*(N_{j-1})\ c\}$$

$$= \max\ \{2^j\ \eta_0^{-1},\ 2\,\tfrac{5}{3}\,\Omega^*(N_{j-1})c\}$$

$$\leq \tfrac{10}{3}\,\Omega^*(N_{j-1})c \quad .$$

Therefore*

$$\left\|\frac{d\omega^{(k+1)}}{da} - 1\right\|_{\eta_{k+2}} \leq \sum_{j=0}^{k} \varepsilon^{2^j} \left\|\frac{d\hat{F}_o^{(j)}}{da}\right\|_{\eta_{k+2}}$$

$$\leq \sum_{j=0}^{k} \varepsilon^{2^j} \frac{\|F^{(j)}\|_{\eta_j}}{\eta_j - \eta_{k+2}} \leq \frac{4}{3} \sum_{j=0}^{k} \varepsilon^{2^j} \|\hat{F}_o^{(j)}\|_{\eta_j} \eta_j^{-1}$$

$$\leq \frac{4}{3} \sum_{j=0}^{k} \varepsilon^{2^j} M_j \eta_j^{-1} \leq \frac{4}{3} [\varepsilon M_o \eta_o^{-1} + \sum_{1}^{k} \frac{10}{3} \frac{1}{K_2} (\varepsilon\tau)^{2^j} \Omega^*(N_{j-1})]$$

$$< \frac{4}{3} [\varepsilon M_o \eta_o^{-1} + (\varepsilon\tau)^2 \sum_{0}^{\infty} (\varepsilon\tau)^{2^j} \Omega^*(N_{j-1})] < \frac{2}{3} .$$

In the second inequality, we used the estimate (2.12) and in the last inequality the assumption (2.31). This concludes the proof of Proposition 1.

The function $\varepsilon^{2^j} \hat{F}_o^{(j)}$, that we shall call momentarily f, is holomorphic on $D(\eta_j; A^{(j)})$, so it is controlled, together with its derivative on, say, $D(\frac{5}{8}\eta_j; A^{(j)})$:

$$\|f\|_{\frac{5}{8}\eta_j} < \|f\|_{\eta_j} , \quad \|f'\|_{\frac{5}{8}\eta_j} \leq \frac{\|f\|_{\eta_j}}{(1 - \frac{5}{8})\eta_j} = \frac{8}{3} \|f\|_{\eta_j} \eta_j^{-1}.$$

* $\frac{\eta_{k+2}}{\eta_j} \leq \frac{1}{4}$ for $0 \leq j \leq k$. $K_2 \geq 4$.

Set $\mathcal{O} \equiv D(\frac{5}{8}\eta_j;\, A^{(j)}) \cap R$ and $\mathcal{O}' \equiv D(\frac{1}{4}\eta_j;A^{(j)}) \cap R$. We want to extend f, considered as a function on $\mathcal{O}'$, to the whole line with

$$\sup_R |f| < \|f\|_{\eta_j} \;, \quad \sup_R |f'| \leq \frac{8}{3}\|f\|_{\eta_j}\, \eta_j^{-1} \;. \tag{2.35}$$

To do this, we have to extend f to the (closed) gaps I in $R - \mathcal{O}'$. There are two different types of gaps: either $I \subset \mathcal{O}$ or $I \cap \mathcal{O} \neq I$. Notice that in the second case the length of I is at least $\frac{3}{4}\eta_j$; see Figure 4.

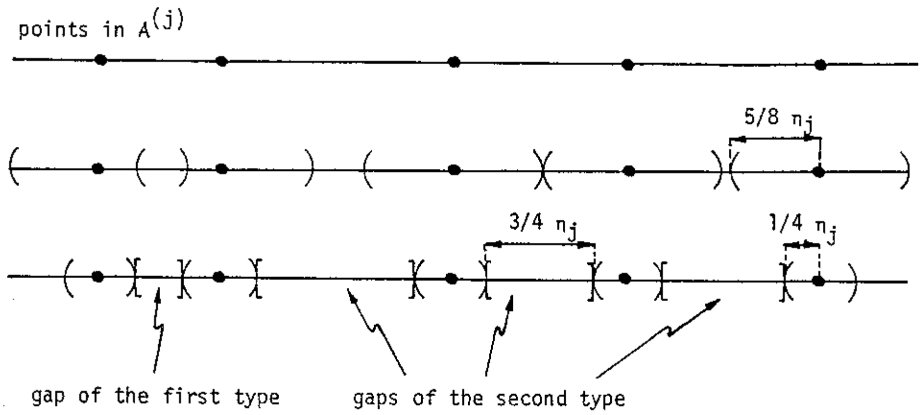

Figure 4

In the first case, we set, obviously, $f_I \equiv f_0$. For the second case, we use the following.

<u>Calculus lemma</u>. Let I be a finite closed interval contained in an open set J. Let $g \in C^1(J-I)$ with

$$\sup_{J-I} |g| < M, \quad \sup_{J-I} |g'| \leq N.$$

Then g can be extended to J so that

$$\sup_{J} |g| < M, \quad \sup_{J} |g'| \leq \max\{N, \frac{2M}{(\text{length } I)}\}.$$

The proof is synthetized in Figure 5.

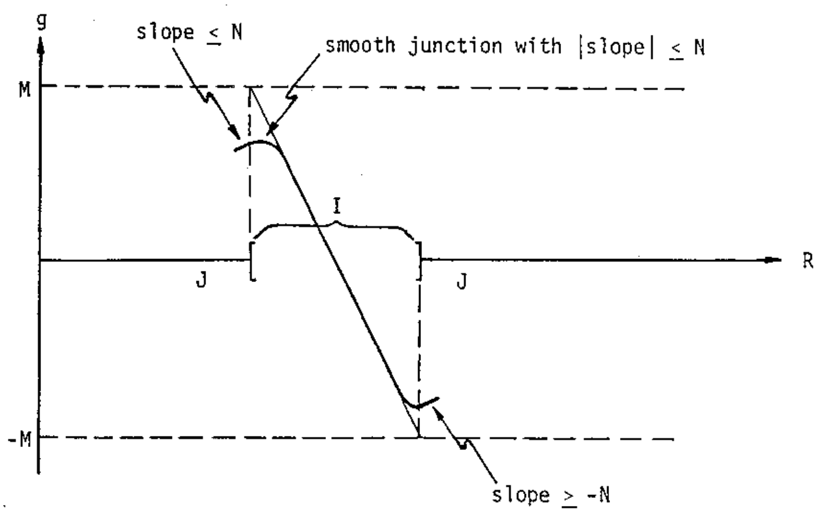

Figure 5

Thus, on a gap I of the second type, we extend f as described in the lemma so:

$$\sup_I |f| < \|f\|_{\eta_j} \ , \quad \sup_I |f'| \le \max\left\{ \frac{8}{3}\|f\|_{\eta_j}\eta_j^{-1}, \ \frac{2\,\|f\|_{\eta_j}}{\text{length}(I)} \right\}$$

$$\le \max\left\{ \frac{8}{3}\|f\|_{\eta_j}\eta_j^{-1} \ , \ \frac{2\,\|f\|_{\eta_j}}{\frac{3}{4}\ \eta_j} \right\} = \frac{8}{3}\|f\|_{\eta_j}\eta_j^{-1} \ .$$

This proves the desired estimates (2.35).

Now for any $k \ge 1$ call f_j the above extension of $\varepsilon^{2^j}\ \hat{F}_0^{(j)}$. Proposition 1 implies

$$\sup_R \left| \frac{d\omega_1^{(k)}}{da} - 1 \right| \le \sum_{j=0}^{k-1} \ \sup_R \left| \frac{df_j}{da} \right| \le \frac{8}{3}\,(\varepsilon M_0\ \eta_0^{-1} + \sum_1^{\infty} \varepsilon^{2^j}\ M_j\eta_j^{-1})$$

$$\le \frac{8}{3}\,[\varepsilon M_0\ \eta_0^{-1} + (\varepsilon\tau)^2 \sum_{j=0}^{\infty} (\varepsilon\tau)^{2^j}\ \Omega^*(N_j)] \ .$$

And if we <u>require the condition</u>, slightly stronger than (2.31),

$$4[\varepsilon M_0\eta_0^{-1} + (\varepsilon\tau)^2 \sum_{j=0}^{\infty} (\varepsilon\tau)^{2^j}\ \Omega^*(N_j)] < 1 \ , \tag{2.36}$$

we obtain, for the extension of any one of the function $\omega_1^{(k)}$,

$$\sup_R \left| \frac{d\omega_1^{(k)}}{da} - 1 \right| < \frac{2}{3} \ . \tag{2.37}$$

This concludes the proof of the inclusion $R_\nu^{(j)} \subset I_\nu^{(j)}$.

Remark 1. The set of centers $\{a_{j,\nu}\}_{\substack{j\in N \\ |\nu|\le N_j}}$ comprises all the values of a for which the j-th approximation to the integrable Hamiltonian $\omega^{(\infty)}\cdot A$, that is $\omega^{(j)}\cdot A$, becomes resonant.

Remark 2. It is easy to see that $\{a_{j,\nu}\}$ is dense in $A^{(0)}$ and to conclude that $A^{(\infty)}$ is a nowhere dense set.

Now we control the measure of R. The bound (2.37) implies

$$c \sup |a^{(j+1)} - a^{(j)}| < 3\,\lambda^{2^j}\ , \quad \lambda \equiv \frac{\varepsilon\tau}{K_2}$$

and from this there follows, for any $j \ge j_0 \ge 0$,

$$c\,|a_{j+1,\nu} - a_{j_0,\nu}| \le 3 \sum_{j_0}^{\infty} \lambda^{2^j} < (4.5)\,\lambda^{2^{j_0}}$$

where to estimate the series we have used*

$$\sum_{j=n}^{\infty} t^{2^j} < t^{2^n}\left(1 + \frac{1}{\log t^{-1}}\right)\ , \quad n \ge 1,\ t < 1\ ,$$

and $\lambda < \frac{1}{8}$ (condition (2.26)).

$$* \sum_{j=n}^{\infty} t^{2^j} = t^{2^n} + \sum_{n+1}^{\infty} t^{2^n} < t^{2^n} + \int_n^{\infty} t^{2^x}dx < t^{2^n} + \int_n^{\infty} t^{2^x}d2^x$$

$$= t^{2^n}\left(1 + \frac{1}{\log t^{-1}}\right)\ .$$

Thus, letting $j_0 = j_0(\nu) \geq 0$ be such that $\lambda^{2^{j_0-1}} > cr_\nu \geq \lambda^{2^{j_0}}$,

one has

$$I_\nu \equiv \bigcup_{j=0}^{\infty} I_\nu^{(j)} \subset \bigcup_{j=0}^{j_0-1} I_\nu^{(j)} \cup \{|a - a_{j_0,\nu}| < 6r_\nu\}.$$

Now, since $j_0 - 1 < (\log 2)^{-1} \log\left(\dfrac{\log(cr_\nu)^{-1}}{\log\lambda^{-1}}\right) < \log\log(3|\nu_1|\Omega(|\nu|))$

when $j_0 \geq 1$, we get

$$\operatorname{length}(I_\nu) \leq 2\, r_\nu(j_0+6) < \frac{14 + 2\log\log(3|\nu_1|\Omega(|\nu|))}{c|\nu_1|\Omega(|\nu|)}.$$

Finally, <u>if Ω satisfies</u>*

---

*For example, $\Omega(r) \geq r^{d-1+\sigma}$ satisfies (2.38) for any $\sigma > 0$: Let $0 < \alpha < \frac{\sigma}{\sigma+d}$ and set $\sigma' \equiv \sigma - \alpha(d-1+\sigma) > 0$. Then, for $m = m_\alpha$ big enough, $\sum_{|\nu| \geq m} \dfrac{\log\log 3|\nu_1|\Omega(|\nu|)}{|\nu_1|\Omega(|\nu|)} \leq 3^\alpha \sum_{|\nu| \geq m} \dfrac{1}{(|\nu_1|\Omega(|\nu|))^{1-\alpha}}$

$$\leq 3^\alpha \sum_{|\nu| \geq m} \frac{|\nu_1|^\alpha}{|\nu_1|\ |\nu|^{d-1+\sigma'}} \leq K. \sum \frac{|\nu_1|^\alpha}{|\nu_1|^{d+\sigma'}} \frac{1}{\left(1 + \frac{\|\tilde{\nu}\|}{|\nu_1|}\right)^{d-1+\sigma'}}$$

$$\leq K. \int_{\substack{|x| \geq 1 \\ x \in R}} \frac{|x|^\alpha}{|x|^{d+\sigma'}} \int_{R^{d-1}} \frac{dy}{\left(1 + \frac{\|y\|}{|x|}\right)^{d-1+\sigma'}}$$

$$\leq K. \left(\int_{|x| \geq 1} \frac{dx}{|x|^{1+\sigma'-\alpha}}\right)\left(\int_0^\infty \frac{r^{d-2}}{(1+r)^{d-1+\sigma'}}\right) < \infty.$$

$$\sum_{\substack{\nu\in Z^d \\ \nu_1\neq 0}} \frac{\log\log 3|\nu_1|\Omega(|\nu|)}{|\nu_1|\Omega(|\nu|)} < \infty, \tag{2.38}$$

the measure of R is bounded by (K_5/c).

Remark 3. In the smallness conditions (2.26) and (2.36), the parameter c appears in the combination $\varepsilon\tau$, $\tau = K_2\psi c M_0$, so we might set $c = c(\varepsilon) = (M_0\sqrt{\varepsilon})^{-1}$, say, and still be able to meet such conditions.

Moreover,

$$\operatorname{meas}(R) \leq \operatorname{meas}\Big(\bigcup_{\substack{\nu\in Z^d \\ \nu_1\neq 0}} I_\nu\Big) < K_5 M_0 \sqrt{\varepsilon} .$$

2.7 Whitney Smoothness of KAM Limits

In the preceding section, we constructed a differentiable extension throughout R of the function $\omega_1^{(\infty)}(a)$ originally defined on the closed nowhere dense set $A^{(\infty)}$.

Indeed, much more can be said about the smoothness of $\omega_1^{(\infty)}$: We will prove that it is indefinitely differentiable on $A^{(\infty)}$ in the sense of the following

Definition (Whitney [1934]). A function $f: A\subset R^d \to R$ is said to be C^m in the sense of Whitney if there exist, on A, functions $(f_\nu)_{\nu\in Z_+^d, |\nu|\leq m}$, $f_0 \equiv f$, with the following property: For each $x_0 \in A$, for any $\varepsilon > 0$, there is a $\delta > 0$ such that if $x, x' \in \{y \in A : |y-x_o| < \delta\}$ then*

$$|f_\nu(x) - \sum_{\substack{\mu \in Z_+^d \\ |\mu| \le m-|\nu|}} \frac{f_{\nu+\mu}(x')}{\mu!}(x-x')^\mu| \le \varepsilon \,|\, x-x'|^{m-|\nu|} \quad . \qquad (2.39)$$

Proposition. For $j \in N$, let g_j be a complex function holomorphic on $D(r_j;T)$ where $T \subset R$, $1 \ge r_j \downarrow 0$. If**

$$\sum_{j=0}^{\infty} \|g_j\|_{r_j} r_j^{-n} < \infty$$

then $g \equiv \sum_{j=0}^{\infty} g_j$ is C^n in the sense of Whitney.

Proof. Notice that for any j,m in N, x in T, $\frac{d^m g_j}{dx^m}(x)$ is well defined. Recalling the estimate (2.12) one has, for any $m \le n$,

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* $\mu! \equiv (\mu_1!)(\mu_2!)\cdots(\mu_d!)$. $x^\mu \equiv (x_1^{\mu_1})(x_2^{\mu_2})\cdots(x_d^{\mu_d})$.

** $\|g_j\|_{r_j} \equiv \sup_{D(r_j;T)} |g_j|$.

$$\sup_T \left| \sum_{j=0}^{\infty} \frac{d^m g_j}{dx^m} \right| \le \sum_{j=0}^{\infty} \left\| \frac{d^m g_j}{dx^m} \right\|_{\frac{r_j}{2}}$$

$$\le 2^m \sum_{j=0}^{\infty} \|g_j\|_{r_j} \, r_j^{-m} \le 2^m \sum_{j=0}^{\infty} \|g_j\|_{r_j} \, r_j^{-n} < \infty \, .$$

Thus, we can define, on T, the m-th Whitney derivative of g (that we will denote $\frac{d^m g}{dx^m}$) as

$$\sum_{j=0}^{\infty} \frac{d^m g_j}{dx^m} \quad .$$

Now we prove that, for $0 \le m \le n$ and $x \ne x'$ in T

$$\left| \frac{d^m g}{dx^m}(x) - \sum_{k=0}^{n-m} \frac{1}{k!} \frac{d^{m+k} g}{dx^{m+k}}(x')\,(x-x')^k \right| = o(|x-x'|^{n-m}).$$

Let $s = s(|x-x'|)$ be such that $r_{s+1} \le |x-x'| < r_s$ and call $g^{[s]} \equiv \sum_{j=0}^{s} g_j$. The function $g^{[s]}$ is C^∞ on the open set $R \cap D(r_s;T)$. Thus,

$$\left|\frac{d^m g}{dx^m}(x) - \sum_{k=0}^{n-m} \frac{1}{k!} \frac{d^{m+k} g}{dx^{m+k}}(x')(x-x')^k\right| \frac{1}{|x-x'|^{n-m}}$$

$$\leq \left|\frac{d^m g[s]}{dx^m}(x) - \sum_{k=0}^{n-m} \frac{1}{k!} \frac{d^{m+k} g[s]}{dx^{m+k}}(x')(x-x')^k\right| \frac{1}{|x-x'|^{n-m}}$$

$$+ \left|\frac{d^m g}{dx^m}(x) - \frac{d^m g[s]}{dx^m}(x)\right| \frac{1}{|x-x'|^{n-m}}$$

$$+ \sum_{k=0}^{n-m} \frac{1}{k!} \left|\frac{d^{m+k} g}{dx^{m+k}}(x') - \frac{d^{m+k} g[s]}{dx^{m+k}}(x')\right| |x-x'|^{k-(n-m)}$$

$$\leq o(1) + 2 \sum_{k=0}^{n-m} \frac{1}{k!} \sum_{j=s+1}^{\infty} \sup_{T} \left|\frac{d^{m+k}}{dx^{m+k}} g_j\right| r_{s+1}^{k-(n-m)}$$

$$\leq o(1) + 2^{m+1} \sum_{k=0}^{n-m} \frac{2^k}{k!} \left(\sum_{j=s+1}^{\infty} \|g_j\|_{r_j} r_j^{-(m+k)} \right) r_{s+1}^{k-(n-m)}$$

$$\leq o(1) + 2^{m+1} \sum_{k=0}^{n-m} \frac{2^k}{k!} \sum_{j=s+1}^{\infty} \|g_j\|_{r_j} r_j^{-(m+k)} r_j^{m+k-n}$$

$$\leq o(1) + 2^{n+1} \sum_{j=s+1}^{\infty} \|g_j\|_{r_j} r_j^{-n} .$$

The proof is finished since $s = s(|x-x'|) \uparrow \infty$ as $|x-x'| \downarrow 0$.

Remark 1. Using this proposition, one can construct non-trivial examples of functions* C_W^n on any set $T \subset R$.

---

*$C_W^n(T) \equiv$ class of functions C^n in the sense of Whitney on the set T.

The proposition and its proof extend easily to the higher dimensional case.

Now, since

$$\omega_1^{(\infty)}(a) = a + \sum_{j=0}^{\infty} f_j(a)$$

with f_j holomorphic on $D(n_j;A^{(\infty)})$, to apply the proposition we have to check that*

$$\frac{1}{c^{n-1}} \sum_{j=1}^{\infty} \|f_j\|_{n_j} \, n_j^{-n} < \infty$$

for any positive integer n. By the estimate (2.32)

$$\frac{1}{c^{n-1}} \sum_{j=1}^{\infty} \|f_j\|_{n_j} \, n_j^{-n} < K. \sum_{1}^{\infty} (\varepsilon\tau)^{2^j} \, (\Omega^*(N_j))^n$$

and, as in the proof of Proposition 1 §2.6, letting λ and j_0 be such that

$$\varepsilon^{\lambda}\tau < 1, \qquad 0 < \lambda < 1,$$

$$\frac{\log\Omega^*(2^j\delta_j^{-1})}{2^j} < \frac{1-\lambda}{2n}, \qquad j \geq j_0 \quad ,$$

*The factor $\frac{1}{c^{n-1}}$ has been introduced to fix the physical dimensions of the sum (see Remark 1 §2.3).

we obtain

$$(\varepsilon\tau)^{2^j}(\Omega^*(N_j))^n = (\varepsilon\tau)^{2^j} e^{n\log\Omega^*(N_j)}$$

$$\leq \left((\varepsilon\tau)\left(\frac{1}{\varepsilon^2}\right)^{\frac{n\log\Omega^*(2^j\delta_j^{-1})}{2^j}}\right)^{2^j} \leq \left(\varepsilon\tau\left(\frac{1}{\varepsilon^2}\right)^{\frac{1-\lambda}{2}}\right)^{2^j} = (\varepsilon^\lambda\tau)^{2^j} .$$

This concludes the proof of our claim concerning $\omega_1^{(\infty)}$.

Remark 2. It is routine, at this point, to check that also the KAM transformation (2.28), as a function of a, belongs to $C_W^\infty(A^{(\infty)})$.

Remark 3. Whitney's notion of smoothness is completely intrinsic. For example, the Whitney derivatives

$$(f_\nu)_{\nu\in Z_+^d, |\nu|\leq m}$$

of the above definition are uniquely determined on A by the property (2.39). Nonetheless, if A is closed, a function of class $C_W^m(A)$ can always be thought of as the restriction on A of a $C^m(R^d)$ function, real analytic on R^d - A. This is the content of the main theorem in Whitney [1934].

2.8 Back to the Schrödinger Equation

The flow generated on M by the integrated Hamiltonian $H^{(\infty)}$ is given simply by

$$x \in R \to (A'(x),\varphi'(x)) = (A'(0),\varphi'(0) + \omega^{(\infty)}x) \qquad (2.40)$$

with $\omega^{(\infty)} \equiv (\omega_1^{(\infty)}(a;\varepsilon),\tilde{\omega}) \equiv (\omega_1^{(\infty)},\omega_2,\ldots,\omega_d)$ satisfying

$$|\omega^{(\infty)}\cdot\nu| \geq \frac{1}{C\Omega(|\nu|)}\ , \ \nu \in Z^d,\ \nu_1 \neq 0\ . \tag{2.41}$$

Now define

$$E = E_\varepsilon \equiv \alpha_0^{-1}(A^{(\infty)})$$

and use transformations (1.7), (1.9) and (2.27) to read equation (2.40) in terms of the eigenvalue problem

$$L_\varepsilon q \equiv L[v(\omega_2 x) + \varepsilon W(\omega_3 x,\ldots,\omega_d x)]q = Eq, \qquad E \in E. \tag{2.42}$$

The result is*:

$$q(x) = r\sqrt{1+\varepsilon s_1(\varphi_1' + \varepsilon\Delta(\varphi'),\tilde{\varphi}')}\ Q(\varphi_1' + \varepsilon\Delta(\varphi'),\varphi_2'),\ r \equiv \sqrt{\frac{2A_1(0)}{\kappa}}\ , \tag{2.43}$$

where $\varphi' = \varphi'(x)$ is as above with $\varphi_1'(0) \equiv \theta, \varphi_2'(0) = \ldots = \varphi_d'(0) = 0$, and the parameter a takes back its original meaning of the unperturbed rotation number $\alpha_0(E)$.

As (r,θ) vary in $R_+ \times T$, $(p(0),q(0)) \equiv (\frac{dq}{dx}(0),q(0))$ cover $R^2 - \{0\}$. Thus, formula (2.43) gives a parameterization of all the real solutions of (2.42).

$* \tilde{\varphi}' \equiv (\varphi_2', \ldots,\varphi_d')$.

Next, we identify $\omega_1^{(\infty)}(\alpha_0(E);\varepsilon)$ with the Johnson-Moser rotation number* $\alpha(E)$ $(E \in E)$.

Recalling the definition of Q in Chapter 1, we can rewrite (2.43) as**

$$q(x) = r\ \mathrm{Re}[e^{i(\theta+bx)}\chi(\theta+bx,\tilde{\omega}x)] ,$$

with

$$\chi(\varphi) \equiv \sqrt{1+\varepsilon\ s_1(\varphi_1+\varepsilon\Delta(\varphi),\tilde{\varphi})}\ \ e^{i\varepsilon\Delta(\varphi)}\ \ \chi_0(\varphi_2).$$

Notice that†

$$\sup_{|\mathrm{Im} z_i|\leq\xi_\infty} |\sqrt{1+\varepsilon s_1(z_1+\varepsilon\Delta(z),\tilde{z})}-1| < K_4\ \varepsilon\tau < 1,$$

where, as in §2.5,

$$\tau \equiv K_2\ \Psi\ \frac{\varepsilon}{\kappa_0}\ \ c\ \|W\|_{\xi_0}\ ,\quad \frac{1}{\kappa_0} \equiv \frac{1}{\kappa}\ \|Q^2\|_{\eta_0,\xi_0}\ .$$

Thus, from‡

$$\lim_{x\uparrow\infty}\frac{\arg\ \chi_0(\omega_2 x)}{x} = 0\ ,$$

---

*See §2.1.

**b denotes, momentarily, $\omega_1^{(\infty)}(\alpha_0(E);\varepsilon)$.

†Obviously, we choose the branch of $\sqrt{z}$ where $\sqrt{1} = 1$. The inequality comes from (2.29).

‡See §1.2.

there follows

$$\lim_{x\uparrow\infty} \frac{\arg \chi(\theta+bx,\tilde{\omega}x)}{x} = 0.$$

So, we can form a complex solution of (2.42)

$$f \equiv u + iv \ ,$$

such that $\frac{i}{2}[f,\overline{f}] = [u,v] \neq 0$, with

$$u \equiv \rho_1 \cos(bx + \beta_1) \ , \quad v \equiv \rho_2 \sin(bx + \beta_2) \ .$$

Here, $\rho_i > 0$ and β_i are quasi-periodic real functions satisfying

$$\sup_{x\in R} \left|\frac{\rho_1}{\rho_2} - 1\right| < 1 \ , \quad \lim_{x\uparrow\infty} \frac{\beta_i(x)}{x} = 0.$$

Then, after a look at Figure 6, one concludes

$$\begin{aligned}
\alpha(E) &\equiv \lim_{x\uparrow\infty} \left|\frac{\arg u+iv}{x}\right| \\
&\equiv \lim_{x\uparrow\infty} \left|\frac{\arg\{\rho_1 \cos[bx(1+\frac{\beta_1}{bx})] + i\ \rho_2 \sin[bx(1+\frac{\beta_2}{bx})]\}}{x}\right| \\
&= \lim_{x\uparrow\infty} \left|\frac{\arg(\rho_1 \cos bx + i\ \rho_2 \sin bx)}{x}\right| \\
&= b \equiv \omega_1^{(\infty)}(\alpha_0(E);\varepsilon) \ .
\end{aligned}$$

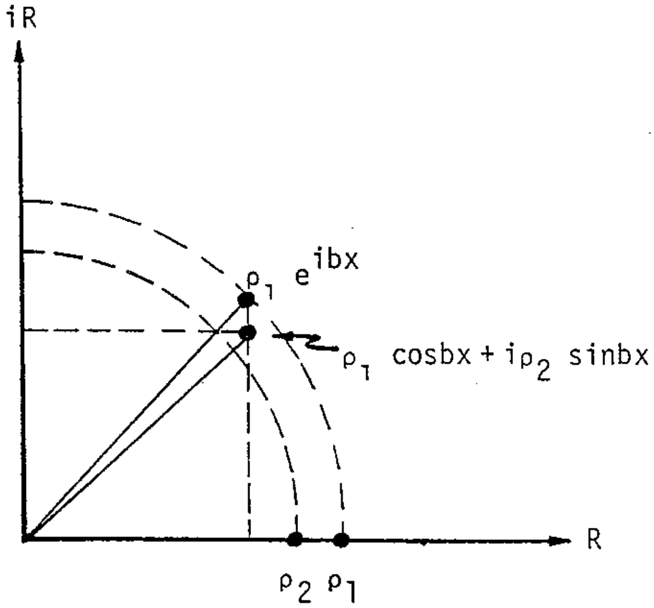

Figure 6

In short:

Theorem 1. If

$$\varepsilon\tau \equiv K_2 \, \Psi \, \frac{\varepsilon}{\kappa_0} \quad c \, ||W|| \, _{\xi_0}$$

satisfies the smallness conditions (2.26) and (2.36), then the eigen-solutions q of (2.42) are analytic quasi-periodic with frequencies $\omega^{(\infty)} = (\omega_1^{(\infty)}, \omega_2, \ldots, \omega_d)$ where $\omega_1^{(\infty)}(\alpha_0(E);\varepsilon)$ coincides with $\alpha(E)$, the Johnson-Moser rotation number for L_ε.

The vector $\omega^{(\infty)}$ satisfies the Diophantine inequalities (2.41) and*

$$\sup_{E\in E}|\alpha(E) = \alpha_0(E)| < \frac{\varepsilon\tau}{c} \quad .$$

As functions of $E \in E$, α and any eigensolution q are C^∞ in the sense of Whitney.

Finally, the Schrödinger equation (2.42) can be viewed as a piece of the Hamiltonian equations for a system of d harmonic oscillators with frequencies $(\alpha,\omega_2,\ldots,\omega_d)$.

Remark 1. The parameter κ_0, constructed upon the unperturbed operator L_0, is a (smooth) function on the interior of $\sigma(L_0)$ and, recalling the asymptotics of §1.2, one has

$$\kappa_0(E) \sim \sqrt{E} \quad \text{for} \quad E \uparrow +\infty.$$

Remark 2. From the structure of $R \equiv A^{(0)} - A^{(\infty)}$ and the invertibility of α_0 on the interior of the spectrum of L_0, there follows that the set $E(\subset\sigma(L_0))$ contains the Cantor set

$$\sigma(L_0) - \left[\bigcup_{j=0}^{\infty} \bigcup_{\substack{\nu\in Z^d \\ \nu_1\neq 0}} \alpha_0^{-1}(I_\nu^{(j)})\right],$$

see §2.7. Analogously to what was done in that paragraph, one can bound the measure of

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*This is (2.27).

$$\tilde{R} \equiv \sigma(L_0) - E$$

by c^{-1} x a constant depending only on d and L_0. But in the present case, because $\alpha_0 \sim \sqrt{E}$ as $E \uparrow +\infty$, one has to require*

$$\sum_{\substack{\nu \in Z^d \\ \nu_1 \neq 0}} \frac{|\tilde{\nu}|}{|\nu_1|^2} \frac{\log\log 3|\nu_1|\Omega(|\nu|)}{\Omega(|\nu|)} < +\infty \tag{2.44}$$

instead of (2.38).

Remark 3. Putting together the Johnson-Moser gap labelling theorem**, the strictly increase of α on $\sigma(L_\varepsilon)$, and the Diophantine inequalities (2.41) (which hold on E), one concludes that in any of the open intervals of $\sigma(L_\varepsilon) - E$ there are infinitely many points belonging to $\sigma(L_\varepsilon)$. On the other hand, Moser-Pöschel [1984] show that, for "generic" $(\omega_2,\ldots,\omega_d)$-quasi-periodic potentials, the Dinaburg-Sinai sets can be obtained as cluster points of open spectral gaps.

Remark 4. Dinaburg-Sinai [1975] construct on (their analogue of) E the densities of the absolutely continuous part of the spectral measures, showing as a by-product, that E is a subset of $\sigma_{ac}(L_\varepsilon)$. Their proof employs a KAM procedure.

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*For example, $\Omega(r) \geq r^{d+\sigma}$ satisfies (2.44) for any $\sigma > 0$. Compare last footnote in §2.6.

**On spetral gaps, $\alpha(E) = \frac{1}{2}\tilde{\omega}\cdot\tilde{\nu}$, for some $\nu \in Z^{d-1}$, $\tilde{\omega} \equiv (\omega_2,\ldots,\omega_d)$.

In Chapter 3, we generalize their formulas without reference to KAM.

We conclude this section discussing the remarkable inequality

$$\frac{d\alpha^2}{dE} \geq 1 \ . \tag{2.45}$$

Moser [1981] showed that (2.45) holds, on spectrum, for any periodic potential and, by a limiting procedure, for certain limit-periodic potentials* with nowhere dense spectrum. In this second case, α is no more differentiable in the usual sense and one has to interpret $\frac{d\alpha^2}{dE}$ as the symmetrized derivative

$$\lim_{\varepsilon \downarrow 0} \frac{\alpha^2(E+\varepsilon) - \alpha^2(E-\varepsilon)}{2\varepsilon} \ ;$$

such a limit is known to exist almost everywhere (with respect to Lebesgue measure) for any monotone function**. Deift-Simon [1983] generalized (2.45) to <u>any</u> almost-periodic potential for E almost everywhere in the support of the absolutely continuous part of the spectral measure. We are going to show that, in our case, (2.45) holds <u>everywhere on E</u>, "$\frac{d}{dE}$" representing, now, the Whitney derivative. For simplicity, we perform the computations in the (Dinaburg-Sinai) case $V \equiv 0$. We also assume (without loss of generality)

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*A limit-periodic function is a limit (in the sup-norm) of continuous periodic functions (see, e.g., Avron-Simon [1981]).

**See, e.g., Saks [1937].

$$\hat{W}_o = \frac{1}{(2\pi)^{d-1}} \int_{T^{d-1}} W(\tilde{\varphi})\, d\tilde{\varphi} = 0 \; *. \qquad (2.46)$$

In this case, the zero-order Hamiltonian is given by**

$$H^{(0)}(A,\varphi) = \sqrt{E}\, A_1 + \omega_2 A_2 + \cdots + \omega_d A_d - \frac{\varepsilon}{\sqrt{E}} A_1 \sin^2\varphi_1\, W(\tilde{\varphi})$$

$$\equiv \omega^{(0)} \cdot A + \varepsilon A_1\, F^{(0)} \;,$$

and, for $E \in \mathcal{E}$,

$$\alpha(E) = \omega_1^{(\infty)}(\alpha_o(E);\varepsilon) = \omega_1^{(\infty)}(\sqrt{E};\varepsilon)$$

$$= \sqrt{E} + \varepsilon\, \hat{F}_o^{(0)} + \varepsilon^2\, \hat{F}_o^{(1)} + O(\varepsilon^4) \;,$$

$F^{(1)}$ being as in the Inductive lemma of §2.3:

$$F^{(1)}(\varphi') \equiv \frac{\partial\phi_o}{\partial\varphi_1}(\varphi(\varphi'))\, F^{(0)}(\varphi(\varphi')) + \frac{F_R^{(0)}(\varphi(\varphi'))}{\varepsilon} \;,$$

$$\phi_o(\varphi) \equiv \sum_{0<|\nu|\leq N_o} \frac{\hat{F}_\nu^o}{-i\omega^{(0)}\cdot\nu}\, e^{i\nu\cdot\varphi}, \quad \varphi'(\varphi) \equiv (\varphi_1 + \varepsilon\phi, \varphi_2, \ldots, \varphi_d) \qquad (2.47)$$

$$F_R^{(0)}(\varphi) \equiv \sum_{N_o<|\nu|} F_\nu^{(0)}\, e^{i\nu\cdot\varphi} \;.$$

Since

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*Notice the change of notation $W(\varphi_3,\ldots,\varphi_d) \to W(\varphi_2,\ldots,\varphi_d) = W(\tilde{\varphi})$.

**See Remark 2 of §2.2.

$$F^{(0)}(\varphi) \equiv \frac{\sin^2\varphi_1 \; W(\tilde{\varphi})}{\sqrt{E}} = \frac{1}{\sqrt{E}} \left[\frac{1}{4} \left(e^{i2\varphi_1} + e^{-i2\varphi_1}\right) - \frac{1}{2}\right] W(\tilde{\varphi}), \quad (2.48)$$

(2.46) implies

$$\hat{F}_o^{(0)} = - \frac{1}{2\sqrt{E}} \hat{W}_o = 0 \; .$$

Now, reading from (2.47)

$$\hat{F}_o^{(1)} = \frac{1}{(2\pi)^d} \int_{T^d} F^{(1)}(\varphi') \; d\varphi'$$

$$= \frac{1}{(2\pi)^d} \int_{T^d} F^{(1)}(\varphi'(\varphi)) \; \left(1 + \varepsilon \frac{\partial\phi_o}{\partial\varphi_1}\right) \; d$$

$$= \frac{1}{(2\pi)^d} \int_{T^d} \frac{\partial\phi_o}{\partial\varphi_1} (\varphi) \; F^{(0)}(\varphi) + F_R^{(0)}(\varphi) \quad d\varphi$$

$$+ \varepsilon \frac{1}{(2\pi)^d} \int_{T^d} \left[\left(\frac{\partial\phi_o}{\partial\varphi_1}\right)^2 F^{(0)} + \frac{\partial\phi_o}{\partial\varphi_1} \frac{F_R^{(0)}}{\varepsilon} \right] d\varphi$$

$$= \frac{1}{(2\pi)^d} \int_{T^d} \frac{\partial\phi_o}{\partial\varphi_1} F^{(0)} \; d\varphi + \varepsilon \frac{1}{(2\pi)^d} \int_{T^d} \left(\frac{\partial\phi_o}{\partial\varphi_1}\right)^2 F^{(0)} \; d\varphi \; .$$

Besides*

---

* $\hat{F}_\nu^{(0)} = \frac{\hat{W}_\nu}{4\sqrt{E}}$ for $\nu_1 = \pm 2$ and is zero otherwise (see (2.48)).

$$\frac{1}{(2\pi)^d}\int_{T^d}\frac{\partial\phi_0}{\partial\varphi_1}F^{(0)}\,d\varphi$$

$$= \sum_{\substack{\mu,\nu\in Z^d\\ 0<|\nu|\le N_0\\ \mu+\nu=0}} \hat{\phi}_\nu(i\nu_1)\,\hat{F}^{(0)}_\mu$$

$$= -\sum_{0<|\nu|\le N_0}\frac{|\hat{F}^{(0)}_\nu|^2\,\nu_1}{\nu\cdot\omega^{(0)}}$$

$$= -\sum_{0<|\nu|\le N_0}\frac{2|\hat{F}^{(0)}_{(2,\tilde{\nu})}|^2}{2\sqrt{E}+\tilde{\nu}\cdot\tilde{\omega}} - \frac{2|\hat{F}^{(0)}_{(-2,\tilde{\nu})}|^2}{-2\sqrt{E}+\tilde{\nu}\cdot\tilde{\omega}}$$

$$= 8\sqrt{E}\sum_{|\tilde{\nu}|\le N_0}\frac{|\hat{F}^{(0)}_{(2,\tilde{\nu})}|^2}{(\tilde{\nu}\cdot\tilde{\omega})^2-4E}$$

$$= \frac{1}{2}\sum_{|\tilde{\nu}|\le N_0}\frac{|\hat{W}_{\tilde{\nu}}|^2}{(\tilde{\nu}\cdot\tilde{\omega})^2-4E} \quad .$$

Thus, we have

$$\alpha = \sqrt{E} + \varepsilon^2\,\frac{1}{2}\sum_{|\nu|\le N_0}\frac{|W_{\tilde{\nu}}|^2}{(\tilde{\nu}\cdot\tilde{\omega})^2-4E} + O(\varepsilon^3) \quad ,$$

$$\alpha^2 = E + \varepsilon^2\,\sqrt{E}\,\Sigma\,\frac{|W_{\tilde{\nu}}|^2}{(\tilde{\nu}\cdot\tilde{\omega})^2-4E} + O(\varepsilon^3) \quad .$$

Finally, notice that the ε^3-order term is Whitney differentiable in $\sqrt{E}$, $E \in \mathcal{E}$, and that such a derivative can be uniformly bounded on $\mathcal{E}$ *, so that

$$\frac{d\alpha^2}{dE} = 1 + \varepsilon^2 \sqrt{E}\,\frac{1}{2} \sum_{|\tilde{\nu}| \le N_0} \frac{|\hat{W}_{\tilde{\nu}}|^2}{[(\tilde{\nu}\cdot\tilde{\omega})^2 - 4E]^2} + \frac{1}{\sqrt{E}}\, O(\varepsilon^3).$$

The smallness of the parameter $(\varepsilon\tau)$ confirms

$$\frac{d\alpha^2}{dE} \ge 1\ , \qquad E \in \mathcal{E}.$$

2.9 Bloch Waves

Equation (2.5) is equivalent to the first order system**

$$\begin{cases} y' = \begin{bmatrix} 0 & 0 \\ V(\varphi_1)-E & 0 \end{bmatrix} y + \varepsilon W(\varphi_2, \ldots, \varphi_d) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} y \\ \varphi' = \omega \end{cases} \qquad (2.49)_\varepsilon$$

with $y = (f, f')$.

A fundamental matrix for $(2.49)_{\varepsilon=0}$ is

---

*See §2.7.

**We resume the notation of §2.1 that is $\tilde{\omega}$ and $(d+1)$ of §2.2-2.8 are replaced here by ω and d.

$$Y = \begin{bmatrix} f_o & \overline{f}_o \\ f'_o & \overline{f}'_o \end{bmatrix} ,$$

$f_o = e^{i\alpha_o x} \chi_o(\omega_1 x)$ being the Floquet solution of Chapter 1.

Setting

$$Y = T\, e^{Cx}$$

where

$$T \equiv \begin{bmatrix} \chi_o(\omega_1 x) & \overline{\chi}_o(\omega_1 x) \\ i\alpha_o \chi_o + \omega_1 \chi'_o & -i\alpha_o \overline{\chi}_o + \omega_1 \overline{\chi}'_o \end{bmatrix}, \quad C \equiv \begin{bmatrix} i\alpha_o & 0 \\ 0 & -i\alpha_o \end{bmatrix},$$

system (2.49) becomes, under the change of variable* $y = Tz$,

$$z' = Cz + \varepsilon P z , \quad \varphi' = \omega \tag{2.50}$$

with

$$P \equiv \frac{W(\varphi_2,\ldots,\varphi_d)}{2\kappa} \begin{bmatrix} -i\,|\chi_o(\varphi_1)|^2 & -i\,\overline{\chi}_o^2\,(\varphi_1) \\ i\,\chi_o^2 & i\,|\chi_o|^2 \end{bmatrix} .$$

Now, the existence of Bloch waves for (2.5), that is of independent solutions $f = e^{iax}\chi(\omega x)$ and $\overline{f}$ with χ a smooth function on T^d, is equivalent to the existence of a change of variable $z = S(\omega x)w$ that

---

* $\det T = [f_o,\overline{f}_o] \equiv \frac{\kappa}{2i}$, see Chapter 1.

transforms (2.50) into

$$w' = \begin{bmatrix} ia & 0 \\ 0 & -ia \end{bmatrix} w, \quad \varphi' = \omega .$$

The rest of this section is devoted to the construction of such an S adapting the techniques used in §2.2-2.8 to the present formalism. For ease of notation, we will indicate corresponding objects by the same symbols.

Denote by G the ring of matrix-valued functions on T^d of the form

$$G = \begin{bmatrix} g & h \\ \bar{h} & \bar{g} \end{bmatrix} ,$$

and by G_0

$$\{G \in G: \ \mathrm{tr} \int_{T^d} G = 0\} .$$

To set up the recursive scheme, let $\alpha_j \in R$ and $P^{(j)} \in G_0$ and consider the system

$$z_j' = C^{(j)}(a,\varepsilon)\, z_j + \varepsilon^{2^j}\, P^{(j)}(\varphi;a,\varepsilon)\, z_j \ , \quad \varphi' = \omega \quad (2.50)_j$$

where

$$C^{(j)} \equiv \begin{bmatrix} i\alpha_j & 0 \\ 0 & -i\alpha_j \end{bmatrix} , \quad P^{(j)} \equiv \begin{bmatrix} p^{(j)} & q^{(j)} \\ \bar{q}^{(j)} & \bar{p}^{(j)} \end{bmatrix} .$$

The regularity assumptions on $C^{(j)}$ and $P^{(j)}$ as functions of $a \in A^{(j)}$ and $\varphi \in T^d$ are as in §2.3; M_j denotes an ε-independent upper bound on $\|P^{(j)}\|_{\xi_j,\eta_j}$. Now let $\delta_j < \xi_j$ and define

$$\zeta(s) \equiv 1 + \sum_{\nu\in Z^d-\{0\}} \Omega(|\nu|)\, e^{-s|\nu|} ,$$

$$N_j \equiv 2^{j+1}\, \delta_j^{-1} \log\varepsilon^{-1} ,$$

$$P_T^{(j)}(\varphi) \equiv \sum_{|\nu|\leq N_j} \hat{P}_\nu^{(j)}\, e^{i\nu\cdot\varphi} , \quad P_R^{(j)} \equiv \sum_{|\nu|>N_j} \hat{P}_\nu^{(j)}\, e^{i\nu\cdot\varphi},$$

$$\eta_{j+1} \equiv \min\left\{\left[2c\Omega(N_j) \sup_{a\in D_j} \left\| \frac{d\alpha_j}{da} \right\|\right]^{-1}, \frac{\eta_j}{2} \right\},$$

$$\xi_{j+1} \equiv \xi_j - \delta_j ,$$

$$A^{(j+1)} \equiv \{a \in A^{(j)}: |\alpha_j(a) - \tfrac{\omega\cdot\nu}{2}| \geq \frac{1}{c\Omega(|\nu|)} , 0<|\nu|\leq N_j\}$$

$$D_{j+1} \equiv D(\eta_{j+1}, A^{(j+1)}) .$$

Inductive Lemma (II). There exist two universal constants $K_1 > K_2$ such that if $a \in A^{(j+1)}$ and

$$K_1\, \zeta(\delta_j)\, \delta_j^{-1}\, cM_j\, \varepsilon^{2^j} \leq 1$$

then, setting $z_{j+1} = S_j^{-1} z_j$ where

$$S_j = S_j(\omega x;a,\varepsilon) \;, \quad S_j(\varphi;a,\varepsilon) \equiv I + \varepsilon^{2^j} U_j(\varphi;a,\varepsilon) \quad ,$$

$$U_j \equiv \frac{1}{2i\alpha_j}\begin{bmatrix} 0 & \hat{q}_0^{(j)} \\ \hat{\bar{q}}_0^{(j)} & 0 \end{bmatrix} + \sum_{\substack{\nu\neq 0\\ |\nu|\leq N_j}} \begin{bmatrix} \dfrac{\hat{p}_\nu^{(j)}}{i\omega\cdot\nu} & \dfrac{\hat{q}_\nu^{(j)}}{2i(\frac{\omega\cdot\nu}{2}-\alpha_j)} \\ \dfrac{\hat{\bar{q}}_\nu^{(j)}}{2i(\frac{\omega\cdot\nu}{2}+\alpha_j)} & \dfrac{\hat{\bar{p}}_\nu^{(j)}}{i\omega\cdot\nu} \end{bmatrix} e^{i\nu\cdot\varphi},$$

the system $(2.50)_j$ becomes

$$z'_{j+1} = [C^{(j+1)} + \varepsilon^{2^{j+1}} P^{(j+1)}]\, z_{j+1} \;, \quad \varphi' = \omega$$

with

$$C^{(j+1)} \equiv C^{(j)} + \varepsilon^{2^j}\begin{bmatrix} \hat{p}_0^{(j)} & 0 \\ 0 & \hat{\bar{p}}_0^{(j)} \end{bmatrix} \equiv C^{(j)} + \varepsilon^{2^j}\, \tilde{C}^{(j)} \equiv \begin{bmatrix} i\alpha_{j+1} & 0 \\ 0 & -i\alpha_{j+1} \end{bmatrix}$$

and

$$P^{(j+1)} \equiv S_j^{-1}\left[P^{(j)}\, U_j - U_j \tilde{C}^{(j)} + \frac{P_R^{(j)}}{\varepsilon^{2^j}} \right] .$$

$C^{(j+1)}$ and $P^{(j+1)}$ as functions of, respectively, $a \in A^{(j+1)}$ and $(\varphi,a) \in T^d \times A^{(j+1)}$ have holomorphic extensions to D_{j+1} and $S^d(\xi_{j+1}) \times D_{j+1}$. Moreover,

$$\|P^{(j+1)}\|_{\xi_{j+1},\eta_{j+1}} \leq K_2\, \zeta(\delta_j)\, \delta_j^{-1}\, cM_j^2 \;.$$

Finally, $\alpha_{j+1} \in R$ and $P^{(j+1)} \in G_0$.

Proof. $(2.50)_{j+1}$ and the regularity properties of $C^{(j+1)}$ and $P^{(j+1)}$ are easily checked mimicking the ideas of §2.3. The only new technical element in the lemma is the last assertion. That α_{j+1} is real it follows from $P^{(j)}$ being an element of G_0. To show that $P^{(j+1)}$ belongs to G_0, notice first that $U_j \in G_0$ and that G is closed under norm limits. Then, if we put

$$S_j^{-1} \equiv I + \varepsilon^{2^j} \tilde{U}_j$$

we have

$$\tilde{U}_j = U_j \sum_{k=0}^{\infty} (-1)^{k+1} (\varepsilon^{2^j} U_j)^k \quad .$$

Thus, $\tilde{U}_j \in G$ and hence $P^{(j+1)} \in G$. Furthermore, rewriting $P^{(j+1)}$ as*

$$P^{(j+1)} = \varepsilon^{-2^{j+1}}(S_j^{-1}[-\varepsilon^{2^j} D_\omega S_j + C^{(j)} S_j + \varepsilon^{2^j} P^{(j)} S_j] - C^{(j+1)})$$

we obtain

$$\mathrm{tr}\, P^{(j+1)} = \varepsilon^{-2^j} \mathrm{tr}[S_j^{-1} D_\omega S_j] \, .$$

The claim now follows from the identity

$$\mathrm{tr}[T^{-1} D T] = \mathrm{tr}[D \log T]$$

valid for any matrix-valued $C^1(T^d)$ function T with $\|I-T\|_\infty < 1$ and any vector field D on T^d with constant coefficients. The proof is finished.

$$*D_\omega \equiv \sum_{i=1}^{d} \omega_i \frac{\partial}{\partial \varphi_i} \, .$$

At this point there are no more difficulties in following the path described in §2.4-2.8. We collect the results in the following

Theorem. Let Ω and $\{\delta_j\}$ be compatible*. If ε satisfies

$$\frac{K_1}{K_2}\,\varepsilon\tau \leq 1 \ , \quad \tau \equiv K_2\,\Psi\, cM_0$$

and E belongs to

$$E \equiv \alpha_0^{-1}\,(A^{(\infty)}), \quad A^{(\infty)} \equiv \bigcap_{j=0}^{\infty} A^{(j)} \ ,$$

then the change of variable $z = S_\infty w$,

$$S_\infty \equiv \prod_{j=0}^{\infty} S_j \equiv I + \varepsilon U_\infty \ ,$$

transforms** (2.50) into the system

$$w' = \begin{bmatrix} i\alpha_\infty & 0 \\ 0 & -i\alpha_\infty \end{bmatrix} w, \quad \alpha_\infty \equiv \lim_{j\uparrow\infty} \alpha_j \ .$$

Furthermore one has

(i) $U_\infty \in G_0$;

(ii) S_∞ as a function of $\varphi \in T^d$ is analytic with a holomorphic extension to $S^d(\xi_\infty)$ ($\xi_\infty = \lim_{j\uparrow\infty} \xi_j$) satisfying

---

*See §2.4 but notice that, because of the Inductive Lemma (II), an analyticity-loss sequence $\{\delta_j\}$ will now be required to satisfy $\sum_{j=0}^{\infty} \delta_j < \xi_0$ instead of (2.22).

**Obviously one sets $C^{(0)} \equiv C$ and $P^{(0)} \equiv P$.

$$||U||_{\xi_\infty} \leq K_4\tau \ ;$$

(iii) S_∞ and α_∞ as functions of E are C^∞ in the sense of Whitney on E;

(iv) α_∞ verifies

$$|\alpha_\infty - |\tfrac{\omega\cdot\nu}{2}|| \geq \frac{1}{c\Omega(|\nu|)} \ , \quad \text{any } \nu \in Z^d - \{0\} \ ,$$

and

$$\sup_{E\in E} |\alpha_\infty - \alpha_0| < \frac{\varepsilon\tau}{c} \ ;$$

(v) $R \equiv A^{(0)} - A^{(\infty)}$ is contained in

$$\bigcup_{j=0}^{\infty} \ \bigcup_{\substack{\nu\in Z^d \\ 0<|\nu|\leq N_j}} I_\nu^{(j)} \ ,$$

where

$$I_\nu^{(j)} \equiv \{a \in A^{(0)} : |a-a_{j,\nu}| < r_\nu\},$$

$$a_{j,\nu} \equiv \alpha^{(j)}\ (|\tfrac{\omega\cdot\nu}{2}|) \ , \quad r_\nu \equiv \frac{3}{2c\Omega(|\nu|)} \ ,$$

and $\alpha^{(j)}$ is the inverse function of* α_j;

(vi) if (2.44) holds then for $E_0 > \min v$

$$\text{meas.}(\tilde{R} \cap [E_0,\infty)) \leq \frac{K_6}{c\sqrt{E_0}} \ ,$$

where $\tilde{R} \equiv \sigma(L_0) - E$.

---

*Compare §2.6.

Remark 1. The identification of α_∞ with the Johnson-Moser rotation number is immediate: From

$$y = Tz = T\,S_\infty\,w$$

one sees that*

$$f \equiv e^{i\alpha_\infty x}\,\chi(\omega x) \equiv e^{i\alpha_\infty x}\,\chi_0(\omega_1 x)\,[1 + \varepsilon s(\omega x)]$$

and $\overline{f}$ are two (independent) solutions of (2.5). The identification is now plain from sup $|\chi - \chi_0| < 1$.

Remark 2. The resonant points of the Dinaburg-Sinai set correspond, in the present approach, to the number $a_{j,\nu}$. Notice the difference with the $a_{j,\nu}$'s of §2.6: there, because of the real formalism, we had to exclude more points**.

Remark 3. Rüssmann [1980] constructs, for E large enough, a function $a(\beta, \frac{1}{\sqrt{E}})$ analytic in $\frac{1}{\sqrt{E}}$ but (he proves) only continuous in β varying in the Cantor set

$$\mathcal{B} \equiv \bigcup_{\nu\in Z^d-\{0\}} \{\beta > 0 : |\beta - \frac{\omega\cdot\nu}{2}| \geq \frac{1}{\Omega(|\nu|)}\}.$$

---

*$s \equiv (U_\infty)_{11}$.

**For a comparison, recall the switch of notation $\omega \to \tilde{\omega}$, $d \to d+1$.

Then he identifies the resonant points α_ν of the Dinaburg-Sinai set with

$$\frac{1}{2}\,\omega\cdot\nu + a(\beta(E), \frac{1}{\sqrt{E}})$$

where $\beta(E)$ is determined implicitly by

$$\beta = \sqrt{E} - a(\beta, \frac{1}{\sqrt{E}}) \ , \ \beta \in \mathcal{B} \ . \tag{2.51}$$

To carry over such an approach, one needs to analyze further equation (2.51). This can be done along the lines of this chapter proving that $\beta \to a(\beta, \frac{1}{\sqrt{E}})$ is $C^1_W(\mathcal{B})$ and that $\frac{\partial a}{\partial \beta} \sim 1$ for large E's.

Finally, notice that in this case, α_ν are not any more numbers but functions of E (which are C^∞_W on $\beta^{-1}(\mathcal{B})$).

Remark 4. The system (2.50) meets the hypothesis of the main theorem in Moser-Pöschel [1984]. Thus, with their method, the "resonant states" described in (2.4) can be obtained also in our situation.

CHAPTER 3

ABSOLUTELY CONTINUOUS SPECTRUM OF QUASI-PERIODIC SCHRÖDINGER OPERATORS

3.1 Introduction

In this chapter, we study the absolutely continuous spectrum of a family of quasi-periodic Schrödinger operators

$$L_\theta \equiv L[V(\theta + \omega x)], \qquad \theta \in T^d ,$$

V being a real continuous function of T^d and $\omega = (\omega_1, \ldots, \omega_d)$ rationally independent frequencies. Two themes dominate: <u>spectral densities and Bloch waves</u>, that is, eigen-solutions of L_θ of the form

$$\psi = e^{i\beta x} \chi(\omega x)$$

with χ a C^2 function on T^d and β a real number.

In §4, we show that for almost all E in a minimal support S of the spectral measure*, there exist two independent solutions, u and $\overline{u}$ **, of

$$L_\theta f = E f$$

---

*For precise definitions, see §2.

**Similar functions were also exploited by Johnson-Moser [1982], Kotani [1982] and Deift-Simon [1983].

such that*

$$\frac{d(P_E^\theta\varphi,\varphi)}{dE} = \frac{1}{2\pi}\ \frac{|(\varphi,u)|^2 + |(\varphi,\overline{u})|^2}{|[u,\overline{u}]|}\ ,\ \varphi\in C_0^\infty(R). \qquad (3.1)$$

The proof of (3.1) is based on a fundamental result about S appearing in Kotani [1982] and on a characterization of the absolutely continuous spectrum for L's in the limit point case worked out in §3.

In Section 5, we show that if there exist Bloch waves ψ for E belonging to a set of positive Lebesgue measure, then, on such a set, one can replace u by ψ in the above formula (3.1). An immediate corollary of this fact, applied to the KAM set E of Chapter 2, gives a representation of the spectral densities in terms of KAM eigenfunctions.

Deift-Simon [1983] proved the existence, on S, of two independent eigenfunctions, g and $\overline{g}$, such that

$$x \to r(x) \equiv |g(x)|$$

is an L^2-quasi-periodic function:

$$r(x) = R(\theta + \omega x)\ , \qquad R \in L^2(T^d).$$

In the last section, we prove that R^{-1} belongs to $L^3(T^d)$ and that R can be thought of as a distributional solution, on T^d, of**

---

*P_E^θ denotes the standard spectral family for L_θ and $\frac{d}{dE}$ stands for the Radon-Nikodym derivative with respect to Lebesgue measure.

**$D_\omega \equiv \sum_{i=1}^{d} \omega_i \frac{\partial}{\partial\theta_i}$.

$$D_\omega^2 R = \frac{1}{R^3} + (V-E)R \ , \qquad R > 0 \text{ a.e. } .$$

Finally, if ω satisfies a Diophantine condition and if* $x \to \beta(x) + \alpha x$ denotes the phase of g, we will see that there is a natural way of associating to β a (unique) distribution on $C^\infty(T^d)$. This will allow us to interpret g as a weak Bloch wave.

3.2 Classical Background

Our analysis will be based on some classical facts that we recall here for convenience.

(1) Herglotz functions**. A function g, holomorphic in the open upper half plane

$$C_+ \equiv \{z \in C: \operatorname{Im} z > 0\} \ ,$$

is said to be a Herglotz function if $\operatorname{Im} g > 0$. The class of such functions will be denoted by H. Any g in H admits the following representation:

$$g(z) = a + bz + \int_{-\infty}^{+\infty} \left(\frac{1}{x-z} - \frac{1}{x-z_0}\right) d\mu(x) \ , \tag{3.2}$$

with $a \in R$, $b \geq 0$, $z_0 \in C_+$ and $d\mu$ a positive Borel measure subject to

---

*As usual, α is the rotation number.

**See Dym-McKean [1976] or Katznelson [1968].

$$\int \frac{d\mu(x)}{1+x^2} < \infty .$$

Viceversa, any function of the form (3.2) belongs to H.

For almost every* x in R, the boundary value of a Herglotz function g exists and is finite; we will denote it by

$$g(x) \equiv \lim_{\varepsilon \downarrow 0} g(x + i\varepsilon) .$$

Moreover, the imaginary part of g(x) coincides (almost everywhere) with the Radon-Nikodym derivative $\frac{d\mu_{ac}}{dx}$, of the absolutely continuous part of the representing measure.

(2) Limit-point theory**. A Schrödinger operator L(v) is said to be in the limit-point case (at $\pm\infty$) if, for any $E \in C - R$, there exist two solutions, $f_{\pm}(x,E)$, of

$$L(v)\, f = Ef , \tag{3.3}$$

uniquely determined up to a multiplicative constant, such that

$$x \to f_{\pm}(x) \in L^2(R_{\pm}), \quad R_+ \equiv (0,\infty), \quad R_- \equiv (-\infty,0).$$

---

*Notational warning: All measure theoretical assertions stated without specifying to which measure they refer to, have to be intended "with respect to Lebesgue measure".

**Coddington-Levinson [1955].

In this case,

(i) f_+ and f_- are independent;

(ii) the limits*

$$h_+(E) = \lim_{x\uparrow\infty} -\frac{f_1(x,E)}{f_2(x,E)}\ , \quad h_-(E) = \lim_{x\downarrow-\infty} \frac{f_1(x,E)}{f_2(x,E)}$$

exist for any $E \in C - R$;

(iii) $E \in C_+ \rightarrow h_\pm(E) \in C_+$ are Herglotz functions;

(iv) $f_\pm(x,E) = \text{const.}[f_1(x,E) \pm h_\pm(E)\ f_2(x,E)]$;

(v) $L(v)$ is essentially selfadjoint on C_0^∞ and the Green's function is given by

$$g(x,y,E) \equiv \frac{f_+(x,E)\ f_-(y,E)}{[f_+,f_-]}\ , \quad x \geq y,$$

and symmetrically for $x < y$.

Notational remark: From now on, $f_\pm$ will denote $f_1(x,E) \pm h_\pm(E)\ f_2(x,E)$.

A sufficient condition for $L(v)$ to be in the limit-point case is that

$$v(x) \geq -k\ x^2, \quad |x| \uparrow \infty$$

for some positive constant k. Thus, bounded potentials give rise to the limit-point case**.

---

*We recall that f_1 and f_2 are the solutions of (3.3) with $f_1(0) = f_2'(0) = 1$ and $f_1'(0) = f_2(0) = 0$.

**"Limit-point" is opposed to "limit-circle", characterized by the fact that all the solutions of (3.3), with $E \in C - R$ are L^2 in either R_+ or R_-.

(3) Measure theory*. All the measures that we shall consider are positive and Borel.

A support of $d\mu$ is a set A such that $\mu(R - A) = 0$. Two measures are orthogonal (or mutually singular) if some of their supports are disjoint.

A minimal (relative to Lebesgue measure) support of $d\mu$ is a support such that any smaller support $A' \subset A$ satisfies meas.$(A - A') = 0$.

Now, let

$$d\mu = d\mu_{ac} + d\mu_{s}$$

be the Lebesgue-Jordan decomposition of $d\mu$. It is a theorem by De La Vallée Pouissin that the sets

$$\{x \in R: \frac{d\mu}{dx} \text{ exists and } 0 < \frac{d\mu}{dx} < \infty\} ,$$

$$\{x \in R: \frac{d\mu}{dx} = +\infty \text{ or } \frac{d\mu}{dx} \text{ doesn't exist}\}$$

are minimal, disjoint supports of, respectively $d\mu_{ac}$ and $d\mu_{s}$. Notice that the second set, being a minimal support of a singular measure, has Lebesgue measure zero.

---

*The circle of ideas illustrated here is very close to the one found in the (spectral-theoretical) paper by Aronszajn [1957]. As a general reference, see Saks [1937].

(4) Spectral theory*. (i) For Borel sets Δ and real numbers E, let P_Δ and $P_E \equiv P_{(-\infty,E)}$ denote, respectively, the spectral projections and the spectral family of some selfadjoint operator T on $L^2(R)$. Stone's formula**,

$$\frac{1}{2}\,[P_{[a,b]} + P_{(a,b)}] = \underset{\varepsilon\downarrow 0}{\text{strong-lim}}\ \frac{1}{2\pi}\int_a^b [R_{E+i\varepsilon} - R_{E-i\varepsilon}]\ dE,$$

gives immediately

$$\mu_\varphi(\Delta) \equiv \int_\Delta d(P_E\varphi,\varphi) = \lim_{\varepsilon\downarrow 0} \frac{1}{\pi}\,\mathrm{Im}\int_\Delta (R_{E+i\varepsilon}\varphi,\varphi)\ dE\ ,$$

where Δ is any Borel set with $\mu\,(\partial\Delta) = 0$ and φ any C^∞-function with compact support. We will call $d\mu$ a spectral measure. Now, notice that

$$z \in C_+ \to (R_z\varphi,\varphi) = \int_{-\infty}^{+\infty} \frac{1}{E-z}\, d\mu_\varphi(E)$$

is Herglotz, so from points (1) and (3) one deduces easily that

$$A \equiv \{E:\lim_{\varepsilon\downarrow 0}\ \mathrm{Im}(R_{E+i\omega}\varphi,\varphi) \in (0,\infty)\}$$

is a minimal support of $d\mu_{\varphi,ac}$ and that

$$A' \equiv \{E:\lim_{\varepsilon\downarrow 0} \mathrm{Im}(R_{E+i\varepsilon}\varphi,\varphi) = 0 \ \text{ or fails to exist}\}$$

is a minimal support for $d\mu_{\varphi,s}$.

---

*Reed-Simon [1972], Simon [1982].

**R_z denotes $(T-z)^{-1}$ for $z \in \rho(T)$.

Moreover, on A

$$\frac{d\mu_{\varphi,ac}}{dE} = \lim_{\varepsilon\downarrow 0} \frac{1}{\pi}\, \mathrm{Im}(R_{E+i\varepsilon}\varphi,\varphi) \ .$$

(ii) Let $\{a_n\}$ be a sequence of positive, summable numbers and $\{\varphi_n\}$ an orthonormal basis in $L^2(R)$ and define

$$d\mu \equiv \sum_{n=0}^{\infty} a_n \, d\mu_{\varphi_n} \ .$$

Such a measure will be called a <u>spectral-class measure</u>.

Since, for any Borel set Δ,

$$\mu(\Delta) = 0 \quad \text{if and only if} \quad P_\Delta = 0 \ ,$$

one sees that the equivalence class* of $d\mu$, called <u>spectral-class</u>, does not depend on the choice of $\{a_n\}$ and $\{\varphi_n\}$.

<u>Remark</u>. A real number E is called a <u>point of increase</u> for $d\mu$ if, for any $\varepsilon > 0$,

$$\int_{(x-\varepsilon,x+\varepsilon)} d\mu > 0 \ .$$

Now, recalling the definition of absolutely continuous spectrum**,

$$\sigma_{ac}(T) \equiv \sigma(T\restriction_{L^2_{ac}}) \ ,$$

---

*The equivalence relation being "mutual absolutely continuity".

**L^2_{ac} denotes the Hilbert space $\{\varphi \in L^2 \colon d\mu_\varphi$ is purely abs. cont.$\}$.

one checks easily the relation

$$\sigma_{ac} = \{\text{points of increase of } d\mu_{ac}\} .$$

3.3 Absolutely Continuous Spectrum of Limit-Point Schrödinger Operators

Let $L(v)$ be a Schrödinger operator in the limit-point case and fix a spectral-class measure $d\mu$ based on φ_n's with compact support. Let, also, S and S' denote disjoint minimal supports of, respectively, $d\mu_{ac}$ and $d\mu_s$. Finally, define, for almost every E in R,

$$u_{\pm}(x,E) \equiv f_1(x,E) \pm h_{\pm}(E)\, f_2(x,E)$$

where $h_{\pm}(E)$ are the boundary values of $h_{\pm}(z)$ defined in (2) of §2.

Remark 1. For almost every E, the limits

$$\lim_{\varepsilon \downarrow 0} f_{\pm}(x,E+i\varepsilon) = u_{\pm}(x,E)$$

exist uniformly on compact x-sets.

Remark 2. $u_{\pm}$ are solutions of the Schrödinger equation (3.3); the evaluation

$$[u_+,u_-] = -(h_+ + h_-)$$

shows that they are independent if and only if $h_+ + h_- \neq 0$.

Theorem*. (i) $\{E: h_+ + h_- \neq 0\} = S$,

(ii) $\{E: h_+ + h_- = 0\} \subset S'$.

Proof. Let

$$A = \{E: h_{\pm}(E) \text{ exist and are finite}\}.$$

As we have already remarked** $|R-A| = 0$. For any $\varphi \in C_0^\infty$ and $E \in A$,

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \operatorname{Im} (R_{E+i\varepsilon}\varphi,\varphi) \tag{3.4}$$

$$= \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \iint_{R^2} g(x,y,E+i\varepsilon)\, \varphi(x)\, \overline{\varphi(y)}\, dxdy$$

$$= \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \operatorname{Im} \left(\frac{1}{[f_+,f_-]} \left[\iint_{x \geq y} f_+(x)f_-(y)\varphi(x)\, \overline{\varphi(y)} \right.\right.$$

$$\left.\left. + \iint_{x<y} f_+(y)f_-(x)\varphi(x)\, \overline{\varphi(y)} \right] \right)$$

$$= \frac{1}{\pi} \operatorname{Im} \left(\frac{1}{[u_+,u_-]} \left[\iint_{x \geq y} u_+(x)u_-(y)\varphi(x)\, \overline{\varphi(y)} \right.\right.$$

$$\left.\left. + \iint_{x<y} u_+(y)u_-(x)\varphi(x)\, \overline{\varphi(y)} \right] \right)$$

$$= \begin{array}{ll} +\infty & \text{if } h_+ + h_- = 0 \\ <+\infty & \text{if } h_+ + h_- \neq 0 \end{array} .$$

---

*Statements regarding $\sigma_{ac}(L)$ such as (i), should always be understood "up to set of (Lebesgue) measure zero".

**$|\Delta|$ denotes the Lebesgue measure of Δ.

The exchange of the limit with the integration is justified by Remark 1. Now by (3) and (4) of §2, we have

$$\frac{d\mu_{\varphi,ac}}{dE} = \lim_{\varepsilon\downarrow 0} \operatorname{Im}\,(R_{E+i\varepsilon}\varphi,\varphi), \qquad E \in A,$$

and since, for $E \in \{h_+ + h_- \neq 0\}$, one can choose a function $\varphi \in C_0^\infty$ such that

$$\frac{d\mu_{\varphi,ac}}{dE} > 0 \ ,$$

claims (i) and (ii) follow.

3.4 Absolutely Continuous Spectrum and Spectral Densities of Quasi-Periodic Schrödinger Operators

Now we turn to (real) potentials v which are continuous and quasi-periodic with rationally independent frequencies $(\omega_1, \ldots, \omega_d) = \omega \in R_+^d$. Consider, instead of $L(v)$, the family of shifted Schrödinger operators parametrized by $\theta \in T^d$:

$$L(v_\theta) \equiv -\frac{d^2}{dx^2} + V(\theta + \omega x) \ ,$$

where V is a continuous function on the torus T^d *.

---

*Recall that since v_θ is bounded, L_θ is in the limit-point case. See §2.

While it is easy to see that the spectrum of L_θ is independent of θ in T^d *, questions about the nature of $\sigma(L_\theta)$ are much more subtle. We start describing a basic result from Kotani [1982]. Let $\gamma_+(E,\theta)$ be the Lyapunov number for L_θ (at $+\infty$):

$$\gamma_+(E,\theta) \equiv \lim_{x\uparrow\infty} \frac{\log \|M(x,\theta,E)\|}{x}$$

when such limit exists. Here, $M(x,\theta,E)$ denotes any fundamental matrix for

$$L_\theta\, f = E\, f\ ,$$

and one checks immediately that γ_+ does not depend on the choice of M.

Since the tranlation

$$T_x : \theta \in T^d \to T_x\theta \equiv \theta + \omega x \in T^d$$

is ergodic with respect to the normalized Lebesgue measure on T^d, $đ\theta \equiv \dfrac{d\theta}{(2\pi)^d}$, the subadditive ergodic theorem proves that $\gamma_+(E,\theta)$

---

*Let $\theta,\theta' \in T^d$. By the compactness of T^d, we can pick a sequence $\{x_n\}$ such that $v_\theta(x+x_n) \to v_{\theta'}(x)$ uniformly in x. This implies that $L(v_\theta(\cdot + x\omega)$ converges to $L(v_{\theta'})$ in the norm resolvent sense. Since the spectrum is not lost under such limits and since the resolvent set is open, the claim follows. See Avron-Simon [1981], Johnson [1982].

exists for almost every θ *. Now, let $w(z)$ be the Herglotz function**

$$z \in C_+ \to w(z) \equiv \frac{1}{2} \int_{T^d} (h_+(E,\theta) + h_-(E,\theta))\, d\theta$$

and let $-\gamma(E)$ denote the real part of the boundary value of w (for almost every $E \in R$).

Theorem 1 (Kotani [1982]). (i) $\gamma_+(E,\theta) = \gamma(E)$, for almost every $(E,\theta) \in R \times T^d$;

(ii) $\int_{T^d} \frac{d\theta}{\mathrm{Im}\, h_+(\theta,E)} \in (0,\infty)$, for almost every E in $\{E:\gamma(E) = 0\}$;

(iii) $S_\theta \equiv$ a minimal support of $d\mu^\theta_{ac}$

$= \{E:\gamma(E) = 0\} \subset \{E:h_+ = -\bar{h}_-\ ,\ \mathrm{Im}\, h_+ > 0\}$,

for almost every $\theta \in T^d$.

Remark 1. Point (iii) shows that the absolutely continuous part of the spectral-class measures $d\mu^\theta$, $\theta \in T^d$, have a common minimal support $\{E:\gamma(E) = 0\}$. We will denote it by S. Notice that S is uniquely determined up to a set of measure zero.

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*For more information, see Ruelle [1979].

**Henceforth, the notation is analogous to that used before: For example, $h_\pm(z,\theta)$ denote the Herglotz functions of §2 for L_θ, $d\mu^\theta$ a spectral-class measure based on compact φ_n's, etc.

Remark 2. Recalling the definition of $u_{\pm}$, (iii) implies

$$u_+(x,\theta,E) \equiv f_1(x,\theta,E) + h_+(\theta,E)\, f_2(x,\theta,E)$$

$$= \overline{u}_-(x,\theta,E) \equiv \overline{(f_1(x,\theta,E) - h_-(\theta,E)\, f_2(x,\theta,E))}\,, \quad E \in S.$$

Moreover,

$$i[u_+,u_-] = i[u_+,\overline{u}_+] = 2\ \mathrm{Im}\ h_+ > 0\ , \qquad E \in S.$$

Now, we are ready for

Theorem 2. Let φ be any C^∞ function with compact support. Then for almost every $(\theta,E) \in T^d \times S$

$$\frac{d\mu^{\theta}_{\varphi,ac}}{dE} = \frac{1}{2\pi}\ \frac{|(\varphi,u)|^2 + |(\varphi,\overline{u})|^2}{i[u,\overline{u}]}\ , \tag{3.5}$$

where $u \equiv u_+(x,\theta,E)$.

Proof. By Remark 2, formula (3.4) and Section 2, we see that

$$\frac{d\mu_{\varphi,ac}}{dE} = \frac{1}{\pi}\,\mathrm{Im}\left[\frac{1}{[u,\bar{u}]}\iint_{x\geq y} u(x)\,\overline{u(y)}\,\varphi(x)\,\overline{\varphi(y)} + \iint_{x<y} u(y)\,\overline{u(x)}\,\varphi(x)\,\overline{\varphi(y)}\right]$$

$$= \frac{1}{\pi}\,\frac{1}{i[u,\bar{u}]}\,\mathrm{Im}\left[i\iint_{x\geq y} u(x)\,\overline{u(y)}\,\varphi(x)\,\overline{\varphi(y)} + \iint_{x<y} u(y)\,\bar{u}(x)\,\varphi(x)\,\overline{\varphi(y)}\right]$$

$$= \frac{1}{2\pi}\,\frac{1}{i[u,\bar{u}]}\left[\iint_{x\geq y} u(x)\,\overline{u(y)}\,\varphi(x)\,\overline{\varphi(y)} + \overline{u(x)}\,u(y)\,\overline{\varphi(x)}\,\varphi(y)\right.$$

$$\left. + \iint_{x<y} u(y)\,\overline{u(x)}\,\varphi(x)\,\overline{\varphi(y)} + \overline{u(y)}\,u(x)\,\overline{\varphi(x)}\,\varphi(y)\right] .$$

Now,

$$\iint_{x\geq y} u(x)\,\overline{u(y)}\,\varphi(x)\,\overline{\varphi(y)} + \overline{u(x)}\,u(y)\,\overline{\varphi(x)}\,\varphi(y)$$

$$= \iint_{x\geq y} u(x)\,\overline{u(y)}\,\varphi(x)\,\overline{\varphi(y)} + \iint_{y\geq x} \overline{u(y)}\,u(x)\,\overline{\varphi(y)}\,\varphi(x)$$

$$= \int_{-\infty}^{+\infty} dx\, u(x)\,\varphi(x)\,\overline{\int_{-\infty}^{x} u(y)\,\varphi(y)\,dy} + \int_{-\infty}^{+\infty} dx\, u(x)\,\varphi(x)\,\overline{\int_{x}^{\infty} u(y)\,\varphi(y)}$$

$$= \int u(x)\,\varphi(x)\,dx\;\overline{\int u(y)\,\varphi(y)dy} = |(\varphi,\bar{u})|^2 .$$

Analogously one checks that

$$\iint\limits_{x<y} u(y)\,\overline{u(x)}\,\varphi(x)\,\overline{\varphi(y)} + \overline{u(y)}\,u(x)\,\overline{\varphi(x)}\,\varphi(y) = |(\varphi,u)|^2 .$$

The proof is finished.

Remark 3. An easy consequence of general spectral theoretical principles* and Theorem 2 is that the absolutely continuous spectrum of L_θ is, for almost every θ, uniformly of multiplicity two.

Remark 4. If v is actually periodic, the eigenfunction u coincides with the Floquet solution f with $f(0) = 1$. Then a trivial application of Theorem 2 confirms formula (1.3) of Chapter 1.

3.5 Bloch Waves, Densities and KAM Spectrum

An eigen-solution ψ of $L(v)$** will be called a Bloch wave if it has the form

$$\psi(x) = e^{i\beta x}\,\chi(\omega x) ,$$

where β is a real number and χ is a C^2 function on T^d †. The Schrödinger equation for ψ is equivalent to

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*See, for example, Simon [1982].

**The notation is as in the preceding section.

†To avoid trivialities, we assume $\beta \neq 0$ and χ not identically zero.

$$D_\omega^2\chi + 2i\beta\, D_\omega\chi + (E-\beta^2-V)\,\chi = 0, \qquad D_\omega \equiv \sum_{i=1}^{d} \omega_i \frac{\partial}{\partial\theta_i}, \tag{3.6}$$

where everything is computed at $\theta = \omega x$. But, since $\{\theta = \omega x,\ x \in R\}$ is dense in T^d, equation (3.6) holds identically on T^d, and in particular at $\theta = \theta_0 + \omega x$ for any θ_0. This means that, for any θ,

$$e^{i\beta x}\,\chi(\theta + \omega x) \equiv \psi(x,\theta)$$

is a Bloch wave for $L(v_\theta)$.

Bloch waves with β and ω rationally independent will play a special role. In this case, we have

Lemma. (i) $[\psi,\bar{\psi}] \neq 0$;

(ii) $|\chi(\theta)| \geq \delta$, for some $\delta > 0$ and any $\theta \in T^d$;

(iii)* $\alpha = |\beta + \omega\cdot\nu_0|$, for some $\nu_0 \in Z^d$.

Proof. If $[\psi,\bar{\psi}] = 0$, then, for some complex number $a \neq 0$,

$$\chi = a e^{-i2\beta x}\,\bar{\chi},$$

and, for any $\nu \in Z^d$,

$$\frac{1}{x}\int_0^x \chi(\omega t)\, e^{-i(\nu\cdot\omega)t}\,dt = a\,\frac{1}{x}\int_0^x \bar{\chi}(\omega t)\, e^{-i(2\beta + \nu\cdot\omega)t}\,dt .$$

---

*$\alpha \equiv$ rotation number.

Now, expanding $\chi(\theta)$ in Fourier series and taking the limit as $x \uparrow \infty$ one concludes that $\hat{\chi}_\nu = 0$, and therefore $\chi \equiv 0$, a case that we ruled out in the definition of Bloch waves. This proves

$$0 \neq [\psi,\overline{\psi}] = -2i\beta|\chi|^2 + 2i \ \mathrm{Im}(\chi \ D_\omega \overline{\chi}) \ . \tag{3.7}$$

Now, if χ were not bounded away from zero, there would exist numbers $x_n \uparrow \infty$ for which $\chi(\omega x_n) \to 0$. But then (3.7) would imply $[\psi,\overline{\psi}] = 0$, contradicting (i).

As for (iii), it is a general fact that for any continuous function F on T^d, bounded away from zero, one has

$$\lim_{x\uparrow\infty} \frac{\arg F(\omega x)}{x} = \omega \cdot \nu_0 \tag{3.8}$$

for some ν_0. For completeness, we sketch the proof of (3.8): Let first F be $C^\infty(T^d)$, say. Then

$$\lim_{x\uparrow\infty} \frac{\arg F(\omega x)}{x} = \lim_{x\uparrow\infty} \frac{1}{x} \ \mathrm{Im} \log F(\omega x)$$

$$= \lim_{x\uparrow\infty} \frac{1}{x} \ \mathrm{Im} \int_0^x \frac{(D_\omega F)(\omega t)}{F(\omega t)} \, dt$$

$$= \sum_{i=1}^{d} \omega_i \, \mathrm{Im} \int_{T^d} \frac{(\partial F)/(\partial \theta_i)}{F} \frac{d\theta}{(2\pi)^d} = \omega \cdot \nu_0 \ ,$$

where the third equality is easily checked expanding in Fourier series. Formula (3.8) follows by approximating a continuous F with smooth functions. Now, (iii) is plain from the definition of α. The proof is complete.

Remark 1. The Lemma shows that, when β and ω are rationally independent, we can (and will) assume without loss of generality, that ψ has the form

$$\psi = e^{i\alpha x} \chi .$$

Moreover, by (iii), α and ω are also rationally independent.

In the next theorem, we relate Bloch waves to the spectral theory of the preceding section.

Theorem 1. Let $A \subset R$ be a set of positive Lebesgue measure and suppose that for any E in A there exist Bloch waves for L(v),

$$\psi_E = \psi = e^{i\alpha x} \chi(\omega x) ,$$

with (α,ω) rationally independent. Then*

(i) $A \subset S(\equiv\{E \in R : \gamma(E) = 0\})$;

(ii) $\psi = a\, u$, for some $a \in C$;

*Again, "almost everywhere with respect to Lebesgue measure" is intended.

(iii) for any $\varphi \in C_0^\infty(R)$

$$\frac{d\mu^{\theta}_{\varphi,ac}}{dE} = \frac{1}{2\pi} \; \frac{|(\varphi,\psi)|^2 + |(\varphi,\overline{\psi})|^2}{i[\psi,\overline{\psi}]}$$, (θ,E) almost everywhere in $T^d \times A$.

The KAM set E, constructed in Chapter 2*, satisfies the hypothesis of the theorem, so not only

$$E \subset \sigma_{ac}(L) \; ,$$

but, by (iii), one has an explicit formula for the spectral densities based upon KAM eigenfunctions. This also shows that, in the analytic quasi-periodic case, S is a big set in the sense of Lebesgue measure.

Remark 2. We recall that Dinaburg-Sinai [1975] show that the KAM spectrum is contained in $\sigma_{ac}(L)$ by constructing, using a KAM scheme, the spectral densities $\frac{d\mu_{\varphi,ac}}{dE}$ on E.

The proof of Theorem 1 will be a simple corollary of

Theorem 2 (Deift-Simon [1983]). For almost every (θ,E) in $T^d \times S$, $x \to |u(x,\theta,E)|$ is an L^2-quasi-periodic function with frequencies $\omega = (\omega_1,\dots,\omega_d)$.

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*See, especially, §2.9.

Proof. We repeat the proof by the authors. Let

$$g(x,\theta,E) \equiv \frac{u(x,\theta,E)}{\sqrt{\operatorname{Im} h_+(\theta,E)}} , \quad E \in S , \tag{3.9}$$

so that

$$[g,\overline{g}] = -2i .$$

By the uniqueness* of the f_+'s in C_+, one has

$$u(x,T_y\theta,E) \equiv \lim_{\varepsilon\downarrow 0} f_+(x,T_y\theta,E+i\varepsilon)$$

$$= \lim_{\varepsilon\downarrow 0} \frac{f_+(x+y,\theta,E+i\varepsilon)}{f_+(y,\theta,E+i\varepsilon)}$$

$$= \frac{u(x+y,\theta,E)}{u(y,\theta,E)} .$$

Thus (dropping E in the notation),

$$x \to g(x,T_y\theta) = c\, g(x+y,\theta)$$

where c depends on y, θ and E but not on x. Then,

$$[g(\cdot,T_y\theta), \overline{g}(\cdot,T_y\theta)] = -2i = [g(\cdot+y,\theta), \overline{g}(\cdot+y,\theta)]$$

implies $|c| = 1$, that is

$$|g(x+y,\theta)| = |g(x,T_y\theta)| , \text{ for any } x \in R,$$

in particular at $x = 0$

---

*Recall $f_+(0,z) = 1$, $z \in C_+$.

$$|g(y,\theta)| = |g(0,T_y\theta)| = \frac{1}{\sqrt{\operatorname{Im} h_+(T_y\theta)}} .$$

Now (iii) of Kotani's theorem* confirms the L^2-quasi-periodicity of $x \to |u(x)|$ concluding the proof.

Proof of Theorem 1. By the Lemma, we have

$$[\psi,\bar{\psi}] \neq 0,\ E \in A .$$

Thus, any fundamental matrix of the Schrödinger equation is bounded in norm and the Lyapunov number γ_+ vanishes identically on A: (i) is, now, plain by Kotani's theorem. To see (ii), fix E (a.e.) on A and let

$$x \to u(x,\theta) = a\ \psi(x,\theta) + b\ \bar{\psi}(x,\theta)\ ,\quad a,b \in C . \tag{3.10}$$

Then, we have

$$|u|^2 - (|a|^2 + |b|^2)\ |\chi|^2 = 2\ \mathrm{Re}[a\bar{b}\ e^{2i\alpha x}\ \chi^2]\ ,$$

that we rewrite as follows:

$$C_1 |g|^2(T_x\theta) + C_2\ |\chi(T_x\theta)|^2 = C_3\ e^{2i\alpha x}\ \chi^2(T_x\theta) + \bar{C}_3\ e^{-2i\alpha x}\ \bar{\chi}^2(T_x\theta), \tag{3.11}$$

where $C_1 \equiv \operatorname{Im} h_+(\theta)$ and g is as in (3.9), $C_2 \equiv -(|a|^2 + |b|^2)$, $C_3 \equiv 2a\bar{b}$ and $T_x\theta \equiv \theta + \omega x$. Notice that (3.11) holds for every x in R

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*Theorem 1 of §4.

and for almost every θ in T^d.

Thus, for any $\nu \in Z^d$ and $t \in T$, we have

$$C_1 \frac{1}{y} \int_0^y |g|^2(T_x\theta)\, e^{-2i(t+\alpha x)-i(T_x\theta)\cdot\nu} dx \tag{3.12}$$

$$+ C_2 \frac{1}{y} \int_0^y |\chi(T_x\theta)|^2\, e^{-2i(t+\alpha x)-i(T_x\theta)\cdot\nu} dx$$

$$= C_3 \frac{1}{y} \int_0^y \chi^2(T_x\theta)\, e^{-2it-i(T_x\theta)\cdot\nu}\, dx$$

$$+ \overline{C}_3 \frac{1}{y} \int_0^y \overline{\chi}^2(T_x\theta)\, e^{-2i(t+2\alpha x)-i(T_x\theta)\cdot\nu} dx\ .$$

Now, since g^2 and χ^2 belong to $L^1(T^d)$ and since the flows

$$x \to (t+\alpha x, T_x\theta) \in T^{d+1},\quad x \to T_x\theta \in T^d,\quad x \to (t+2\alpha x, T_x\theta) \in T^{d+1}$$

are all <u>ergodic</u>, we can apply the ergodic theorem and, letting $y \uparrow \infty$ in (3.12) for almost every $(t,\theta) \in T^{d+1}$, we obtain

$$C_1 \int_{T^{d+1}} |g|^2(\theta')\, e^{-2it'-\theta'\cdot\nu} \frac{dt'd\theta'}{(2\pi)^{d+1}} + C_2 \int_{T^{d+1}} |\chi(\theta')|^2 e^{-2it'-i\theta'\cdot\nu} \frac{dt'd\theta'}{(2\pi)^{d+1}}$$

$$= C_3\, e^{-2it} \int \chi^2(\theta')\, e^{-i\theta'\cdot\nu} \frac{d\theta'}{(2\pi)^d} + \overline{C}_3 \int_{T^{d+1}} |\chi(\theta')\ ^2\, e^{-2it'-i\theta'\cdot\nu} \frac{dt'd\theta'}{(2\pi)^{d+1}}\ ,$$

that is

$$0 = C_3\, e^{-2it}\, (\chi^2)_\nu\ .$$

Since ν was arbitrary and χ is not identically zero by hypothesis, we must have $C_3 = 0$, i.e., $a\bar{b} = 0$. Looking at the rotation number in (3.10), we conclude $b = 0$. This proves (ii).

Now, (iii) comes directly from Theorem 2 of §4 since the constant a washes out from formula (3.5). The proof is finished.

3.6 Weak Bloch Waves

In this section we discuss a question raised by Deift-Simon [1983]: Do there exist Bloch waves, possibly non-smooth, for almost every E in $S \equiv \{E:\gamma(E)=0\}$?

First, we summarize the situation. Let $g(x,\theta,E)$ be the eigensolution of $L(v_\theta)$ defined in (3.9) and let

$$r = r(x,\theta,E) \equiv |g(x,\theta,E)| \ .$$

In Theorem 2, §5, we saw that

$$r(x,\theta,E) = \frac{1}{\sqrt{\operatorname{Im} h_+(T_x\theta,E)}} \equiv R(T_x\theta,E) \ , \ (\theta,E) \text{ a.e. in } T^d \times S \ ,$$

and also*

$$0 < \lim_{x\uparrow\infty} \frac{1}{x}\int_0^x r^2 = \int_{T^d} R^2(\theta)\, d\theta < +\infty \ . \qquad (3.13)$$

Furthermore, since $[g,\bar{g}] = -2i$, $x \to r(x)$ is different from zero for every x and the Schrödinger equation shows that r satisfies

*Dropping the dependence on E in the notations.

$$r'' = \frac{1}{r^3} + (v_\theta - E)\, r \quad . \tag{3.14}$$

Thus, for almost every θ, $x \to R(T_x\theta)$ is at least C^2. Another important relation stems from Johnson-Moser [1982]:

$$\lim_{x\uparrow\infty} \frac{1}{x}\int_0^x \frac{1}{r^2} = \int_{T^d} \operatorname{Im} h_+ \equiv \int_{T^d} \frac{1}{R^2} = \alpha. \tag{3.15}$$

We collect a few other properties of g and r in the following

Lemma. For (θ,E) a.e. in $T^d \times S$, one has

(i) $\overline{\lim\limits_{x\uparrow\infty}}\, \frac{1}{x}\int_0^x |g'|^2 < \infty$;

(ii) $\overline{\lim\limits_{x\uparrow\infty}}\, \frac{|r'| + r + r^{-1}}{\sqrt{x}} < \infty$.

Proof. To see (i), we will need two standard inequalities that hold for arbitrary C^2-functions f *:

$$|f(x)|^2 \le K \int_0^x (|f|^2 + |f''|^2)\ , \ x \ge 1 \text{ (say) and } K \text{ a universal constant,} \tag{3.16}$$

$$|f(x)|^2 \le 2 \int_0^x (|f|^2 + |f'|^2)\ , \ x > 0. \tag{3.17}$$

---

*The first is a Sobolev inequality, the second follows immediately from the identity

$$f(x)^2 = \left[\int_{x-1}^x \left(f(y) + \int_y^x f'(z)\, dz\right) dy\right]^2.$$

Now, let f be a solution of

$$f'' = qf,$$

where q is a continuous bounded function. Then

$$\int_0^x q|f|^2 = \int_0^x \bar{f}\, f'' = (\bar{f}\, f')\Big|_0^x - \int_0^x |f'|^2 ,$$

so that

$$\int_0^x |f'|^2 \le \|q\|_\infty \int_0^x |f|^2 + |f(0)\ f'(0)| + |f(x)\ f'(x)| .$$

Moreover, by (3.16) and (3.17),

$$2|f(x)\ f'(x)| \le 2|f|^2 + \frac{1}{2}\,|f'|^2$$

$$\le 2K(1 + \|q\|_\infty) \int_0^x |f|^2 + \int_0^x |f'|^2 + \|q\|_\infty \int_0^x |f|^2 ,$$

and one concludes

$$\int_0^x |f'|^2 \le a \int_0^x |f|^2 + b \tag{3.18}$$

where $a \equiv (4K + (4K+3)\ \|q\|_\infty)$ and $b = 2|f(0)\ f'(0)|$.

So, setting $f \equiv g$ and $q \equiv v - E$, (3.15) and (3.18) show (i). From (3.14) one checks

$$\frac{d}{dx}\left[(r')^2 + \frac{1}{r^2}\right] = (v - E)(r^2)'$$

and, integrating and using (3.17),

$$(r')^2 + \frac{1}{r^2} = c + (v - E)\, r^2 - \int_0^x v'\, r^2 \tag{3.19}$$

$$\leq |c| + 2\, \|v-E\|_\infty \int_0^x (r^2 + (r')^2) + \|v'\|_\infty \int_0^x r^2 .$$

Since

$$|r'| \leq |g'|,$$

dividing by x and letting $x\uparrow\infty$, we obtain, by (i),

$$\overline{\lim}\ \frac{r' + r^{-1}}{\sqrt{x}} < \infty;$$

Now going back to (3.19), we have also

$$\overline{\lim}\ \frac{r}{\sqrt{x}} < \infty.$$

The proof is finished.

Now the upshot is the formula*

$$\int_{T^d} R\, D_\omega^2 \phi = \int_{T^d} \frac{\phi}{R^3} + \int_{T^d} (V - E)\, R\, \phi. \tag{3.20}$$

valid for any C^∞ function ϕ on T^d.

$$^*D_\omega = \sum_{i=1}^{d} \omega_i \frac{\partial}{\partial \theta_i} .$$

Proof. First, let $\phi \geq 0$ and write

$$\varphi(x) = \varphi(x,\theta) \equiv \phi(T_x\theta), \quad T_x\theta \equiv \theta + \omega x.$$

Since $R \in L^1(T^d)$ we have, by the ergodic theorem and (3.14),

$$\int_{T^d} R\, D_\omega^2\phi = \lim_{x\uparrow\infty} \frac{1}{x}\int_0^x R(T_y\theta)(D_\omega^2\phi)(T_y\theta) \tag{3.21}$$

$$= \lim_{x\uparrow\infty} \frac{1}{x}\int_0^x r\varphi''$$

$$= \lim_{x\uparrow\infty} \frac{1}{x}\left[(r\varphi')\Big|_0^x - (r'\varphi)\Big|_0^x + \int_0^x r''\varphi \right]$$

$$= \lim_{x\uparrow\infty} \frac{1}{x}\int_0^x \left(\frac{1}{r^3} + (v - E)r\right)\varphi\,.$$

In the last passage we used (ii) of the Lemma to get rid of the boundary terms. Now, another application of the ergodic theorem to positive random variables tells us that*

$$\lim \frac{1}{x}\int_0^x \frac{\varphi}{r^3} = \int_{T^d} \frac{\phi}{R^3}\,.$$

But, since

$$\lim \frac{1}{x}\int_0^x (v - E)\, r = \int_{T^d} (V - E)\, R\, \phi\,,$$

we conclude from (3.21), a fortiori,

*The right-hand side might be ∞. See, e.g., Breiman [1968].

$$\int_{T^d} \frac{\phi}{R^3} = \int_{T^d} (E - V) R \phi + \int_{T^d} R D_\omega^2 \phi < \infty.$$

In particular, for $\phi \equiv 1$, we get

$$\int_{T^d} \frac{1}{R^3} = \int_{T^d} (E - V) R ,$$

showing that $(1/R^3) \in L^1$. Now we can repeat the computation in (3.21) with an aribtrary C^∞ function ϕ and complete the proof of (3.20). A new theme has just entered: By (3.20), R, regarded as a distribution* on $C^\infty(T^d)$, <u>is a weak solution of the equation</u>

$$D_\omega^2 F = \frac{1}{F^3} + (V - E)F, \quad F > 0 \quad \text{a.e.} \; . \tag{3.22}$$

Henceforth, let us assume that V belongs to $C^\infty(T^d)$ and that ω is highly irrational:

$$|\omega\cdot\nu| \geq \frac{1}{c|\nu|^m} \quad , \quad \nu \in Z^d - \{0\} , \tag{3.23}$$

for some $c > 0$ and $m \in Z_+$.

Now, define the normalized phase of g by setting

$$g = e^{i[\alpha x + \beta(x)]} r .$$

---

*As customary, $G \in L^p(T^d)$ can be thought as a distribution on $C^\infty(T^d)$ by letting

$$\langle G,\phi\rangle \equiv \int_{T^d} G \,\overline{\phi}, \text{ for any } \phi \in C^\infty(T^d).$$

Since α is the rotation number of g, β is a real (smooth) function satisfying

$$\overline{\lim_{x\uparrow\infty}} \frac{\beta(x)}{x} = 0 .$$

Furthermore, the Schrödinger equation for $g \equiv R(\theta)\, u(x,\theta)$ implies

$$\beta' = \frac{1}{r^2} - \alpha, \quad \beta(0,\theta) = 0 \quad (\mathrm{mod}\ 2\pi) . \tag{3.24}$$

In the next theorem, we show that there is <u>a natural distribution associated to the phase</u> β.

<u>Theorem</u>. There exists a unique distribution B on $C^\infty(T^d)$ satisfying

$$\langle D_\omega B,\phi\rangle = \lim_{x\uparrow\infty} \frac{1}{x}\int_0^x \beta'(y,\theta)\, \overline{\phi}(T_y\theta)\, dy,$$

$$\langle B,1\rangle = 0 ,$$

for almost every $\theta \in T^d$ and all $\phi \in C^\infty(T^d)$. Moreover, B belongs to the Sobolev space* H_{-t} for any $t > m + \frac{d}{2}$.

<u>Proof.</u> Define B as the (formal) trygonometric series

$$B(\theta) = \sum_{\nu\neq 0} \frac{1}{i\omega\cdot\nu} \widehat{\left(\frac{1}{R^2}\right)}_\nu e^{i\nu\cdot\theta} .$$

---

*H_s, $s \in R$, is the closure of tryonometric polynomials $G = \sum \hat{G}_\nu\, e^{i\nu\cdot\theta}$ in the norm $\|G\|_s^2 \equiv (2\pi)^d \sum (1 + \nu\cdot\nu)^s\, |\hat{G}_\nu|^2$; see, e.g., Bers-John-Schechter [1964].

For $t > m + \frac{d}{2}$, (3.23) yields

$$\sum |\hat{B}_\nu|^2 (1 + \nu\cdot\nu)^{-t} \equiv \sum \frac{1}{|\omega\cdot\nu|^2} \left|\widehat{\left(\frac{1}{R^2}\right)}_\nu\right|^2 \frac{1}{(1+\nu\cdot\nu)^t}$$

$$\leq c^2 \left(\int \frac{1}{R^2}\right)^2 \sum \frac{|\nu|^{2m}}{(1+\nu\cdot\nu)^t} < \infty,$$

showing that $B \in H_{-t}$.

Next, let $\varphi(x,\theta)$ denote $\phi(T_x\theta)$. Then, $\overline{\lim}\, \frac{1}{x}\, \beta(x,\theta) = 0$, (3.15) and the ergodic theorem imply

$$\lim_{x\uparrow\infty} \frac{1}{x} \int_0^x \beta\overline{\varphi'} = \lim \frac{1}{x}\left[\beta(x)\, \overline{\varphi(x)} - \int_0^x \beta'\overline{\varphi}\right]$$

$$= -\lim \frac{1}{x} \int_0^x \left(\frac{1}{r^2} - \alpha\right)\overline{\varphi} = -\int_{T^d} \left(\frac{1}{R^2} - \alpha\right)\overline{\phi}$$

$$= \left\langle D_\omega B, \phi\right\rangle.$$

Finally, uniqueness is a trivial consequence of the irrationality of ω and the normalization $\langle B,1\rangle = 0$. The proof is finished.

Remark 1. B is a (distributional) solution of

$$D_\omega B = \frac{1}{R^2} - \alpha,$$

where such equation is the lift on T^d of (3.24).

Remark 2. If R happens to be smooth, then equation (3.22) shows that R is bounded away from zero. Thus, $\frac{1}{R^2}$ is smooth and hence B is a nice function. In such a case

$$\beta(x,\theta) \equiv B(T_x\theta) - B(\theta)$$

solves (3.24) and g is seen to be a (genuine) Bloch wave.

This last remark suggests that the question of the existence of (smooth) Bloch waves on S can be reformulated as follows: Is a weak solution R of (3.22), satisfying

$$R^2 \text{ and } \frac{1}{R^3} \in L^1(T^d)$$

smooth?

Unfortunately, regularity properties for nonlinear equations on tori are difficult to be treated with general PDE methods. On this subject we just mention a paper by Brezis-Nirenberg [1977].

We conclude indicating a possible way to attack such a problem: Assume that one is able to prove that

$$\frac{1}{R} \in L^p(T^d), \quad \text{for any } p \geq 4.$$

Then (3.22) would imply

$$D_\omega^2 R \in L^2 .$$

Applying D_ω to the equation we would see

$$D_\omega^3 R = -\frac{3}{R^4} D_\omega R + (D_\omega V)R + (V-E)\ D_\omega R$$

so that

$$D_\omega^3 R \in L^2 ,$$

and, inductively,

$$D_\omega^n R \in L^2 , \quad \text{for any } n.$$

This would mean

$$\sum |\hat{R}_\nu|^2\ |\omega\cdot\nu|^{2n} < \infty, \text{ any } n,$$

a formula that gives <u>decay of the Fourier coefficients</u> (i.e., smoothness) <u>for all</u> ν's <u>away from the small-denominator plane</u>

$$\{\nu \in \mathbb{R}^d : \omega\cdot\nu = 0\}.$$

APPENDIX A

AN ANALYTIC IMPLICIT FUNCTION THEOREM ON T^d

We sketch the proof of the Proposition of §2.3 needed in the Inductive Lemma. Injectivity is plain from

$$|z + g(z) - (z' + g(z'))| \geq |z - z'| \left(1 - K. \left\| \frac{\partial g}{\partial z} \right\|_r \right), \quad z,z' \in S^d(r).$$

Surjectivity is an elementary application of the "contraction lemma". In fact, let w be any point in $S^d(r-s)$ and consider the map $z \in \{z \in C^d : |z-w| < s\} \to j(z) \equiv w - g(z)$. If the constant K_3 is chosen suitably, j will map the sphere $\{z : |z-w| < s\}$ into itself and will be a contraction. Therefore, in such a sphere, there exists a unique fixed point z for j, that is, $z = w - f(z)$.

The remaining statements are even more obvious.

APPENDIX B

CONVERGENCE OF CHANGES OF VARIABLES IN THE KAM ITERATION

We prove the claim made in the proof of the theorem of §2.5. Let's consider first the limit of the G_j's. Recalling that $z \to G_j(z)$ is holomorphic in $S^d(\xi_{j+1})$, $\xi_{j+1} \equiv \xi_j - 2 \sum_{\ell=0}^{j} \delta_\ell$, plus the estimates on Δ_j and (2.12), we have for z,z' in a compact set of $S^d(\xi_\infty)$

$$|G_{j+m}(z) - G_j(z)| \leq \sum_{k=0}^{m-1} |G_{j+k+1}(z) - G_{j+k}(z)|$$

$$= \sum_{k=0}^{m-1} |G_{j+k}(g_{j+k+1}(z) - G_{j+k}(z)| \leq \sum_{k=0}^{m-1} K. \left\| \frac{\partial G_{j+k}}{\partial z} \right\|_{\xi_\infty} |g_{j+k+1}(z) - z|$$

$$\leq K. \sum_{k=0}^{m-1} \left[\prod_{\ell=0}^{j+k} \left\| \frac{\partial g_\ell}{\partial z} \right\|_{\xi_\infty} \right] \varepsilon^{2^{j+k+1}} \| \Delta_{j+k+1} \|_{\xi_\infty}$$

$$\leq K. \sum_{k=0}^{m-1} \prod_{\ell=0}^{j+k} \left[1 + \varepsilon^{2^\ell} \|\Delta_\ell\|_{\xi_{\ell+1}} (\xi_{\ell+1} - \xi_\infty)^{-1} \right] \varepsilon^{2^{j+k+1}} \| \Delta_{j+k+1} \|_{\xi_\infty}$$

$$\leq K. \sum_{k=0}^{m-1} \prod_{\ell=0}^{j+k} \left[1 + \frac{1}{2} \varepsilon^{2^\ell} \|\Delta_\ell\|_{\xi_{\ell+1}} \delta_{\ell+1}^{-1} \right] \varepsilon^{2^{j+k+1}} \| \Delta_{j+k+1} \|_{\xi_\infty}$$

$$\leq K. (\varepsilon\tau)^{j+1} .$$

This proves the claim concerning the G_j's.

Notice also that

$$|G(z) - z| \leq |G(z) - G_0(z)| + |G_0(z) - z|$$

$$\leq \sum_{k=0}^{\infty} |G_{j+1} - G_j| + |G_0 - z| \leq K.\ \varepsilon\tau \quad .$$

So we have $G(\varphi) \equiv (\varphi_1 + \varepsilon\Delta(\varphi;a,\varepsilon), \varphi_2, \ldots, \varphi_n)$ with Δ holomorphic in $S^d(\xi_\infty)$ and

$$||\Delta||_{\xi_\infty} \leq K_4\tau.$$

Let's turn, now, to the S_j's. First notice that S_j has the form

$$\begin{bmatrix} 1+s^{(j)} & 0 \cdots 0 \\ s_2^{(j)} & 1 0 \cdots 0 \\ \vdots & \ddots \\ s_d^{(j)} & 0 \cdots 0 1 \end{bmatrix} .$$

We want to show, by induction, that $||s_i^{(j)}||_{\xi_{j+1}} \leq 1$ for any i,j. For $j = 0, ||s_i^{(o)}||_{\xi_1} = \varepsilon\left|\left|\frac{\partial\phi}{\partial\varphi_i}\right|\right|_{\xi_1} \leq K.\ \varepsilon\tau < 1$ because of condition (2.26). Assume $||s_i^{(j)}||_{\xi_{j+1}} \leq 1$, for $0 \leq j \leq k$. Then

$$S_{k+1} = S_k \circ T_{k+1} = \begin{bmatrix} 1 + s_1^{(k)} + \varepsilon^{2^{K+1}} \frac{\partial \phi_{k+1}}{\partial \varphi_1} + s_1^{(k)} \varepsilon^{2^{k+1}} \frac{\partial \phi_{k+1}}{\partial \varphi_1} & 0 & \cdots & 0 \\ s_2^{(k)} + \varepsilon^{2^{k+1}} \frac{\partial \phi_{k+1}}{\partial \varphi_2} + s_2^{(k)} \varepsilon^{2^{k+1}} \frac{\partial \phi_{k+1}}{\partial \varphi_2} & 1 & 0 \cdots & 0 \\ \vdots & & \ddots & \\ s_d^{(k)} + \varepsilon^{2^{k+1}} \frac{\partial \phi_{k+1}}{\partial \varphi_d} + s_d^{(k)} \varepsilon^{2^{k+1}} \frac{\partial \phi_{k+1}}{\partial \varphi_d} & 0 & \cdots & 0 \; 1 \end{bmatrix}$$

and, by the inductive hypothesis (2.25),

$$\|s_i^{(k+1)}\|_{\xi_{k+2}} \equiv \left\| s_i^{(k)} + \varepsilon^{2^{k+1}} \frac{\partial \phi_{k+1}}{\partial \varphi_i} + s_i^{(k)} \varepsilon^{2^{k+1}} \frac{\partial \phi_{k+1}}{\partial \varphi_i} \right\|_{\xi_{k+2}}$$

$$\leq \|s_i^{(k)}\|_{\xi_{k+2}} + \varepsilon^{2^{k+1}} \left\| \frac{\partial \phi_{k+1}}{\partial \varphi_i} \right\|_{\xi_{k+2}} \left(\|s_i^{(k)}\|_{\xi_{k+1}} + 1 \right)$$

$$\leq \|s_i^{(k)}\|_{\xi_{k+1}} + 2\varepsilon^{2^{k+1}} \left\| \frac{\partial \phi_{k+1}}{\partial \varphi_i} \right\|_{\xi_{k+1}} \leq \|s_i^{(o)}\| + 2 \sum_{\ell=1}^{\infty} \varepsilon^{2^k} \left\| \frac{\partial \phi_\ell}{\partial \varphi} \right\|_{\xi_{\ell+1}}$$

$$\leq K.\ \varepsilon\tau + 2\, K.\ (K.\ \varepsilon\tau)^2 \leq 1 .$$

Next, we show

$$\|S_{\ell+1} - S_\ell\|_{\xi_\infty} \leq (K.\ \varepsilon\tau)^{2^{\ell+1}} .$$

In fact:

$$\|S_{\ell+1} - S_\ell\| = \|S_\ell(g_{\ell+1})\cdot T_{\ell+1}(g_{\ell+1}) - S_\ell\|$$

$$\leq \|S_\ell(g_{\ell+1}) - S_\ell\| + \|S_\ell(g_{\ell+1})\|\,\|T_{\ell+1}(g_{\ell+1}) - I\|$$

$$\leq K.\left\|\frac{\partial S_\ell}{\partial z}\right\| |g_{\ell+1} - z| + \|S_\ell\|\ (K.\ \tau\varepsilon)^{2^{\ell+1}}$$

$$\leq K.\ \frac{\|S_\ell\|_{\xi_{\ell+1}}}{\xi_{\ell+1} - \xi_\infty}(K.\ \varepsilon\tau)^{2^{\ell+1}} + \|S\|_\ell\ (K.\ \varepsilon\tau)^{2^{\ell+1}}$$

$$\leq (K.\ \varepsilon\tau)^{2^{\ell+1}}\ .$$

Now,

$$\|S_{k+m} - S_\ell\| \leq \sum_{j=0}^{m-1} \|S_{k+j+1} - S_{k+j}\|$$

$$\leq \sum_0^\infty (K.\ \varepsilon\tau)^{2^{k+j+1}} \leq K.\ (K.\ \varepsilon\tau)^{2^{k+1}}\ .$$

Thus, we can set

$$S \equiv \begin{bmatrix} 1 + \varepsilon s_1(\varphi;a,\varepsilon) & 0 \cdots 0 \\ \varepsilon s_2 & 1\,0 \cdots 0 \\ \vdots & \ddots \\ \varepsilon s_d & 0 \cdots 0\,1 \end{bmatrix}$$

with the vector s holomorphic in $S^d(\xi_\infty)$ (for each $a \in A^{(\infty)}$) and $\|s\|_{\xi_\infty} \leq K_4\tau$.

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