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**QUASI-PERIODIC SCHRÖDINGER OPERATORS  
IN ONE DIMENSION,  
ABSOLUTELY CONTINUOUS SPECTRA, BLOCH WAVES  
AND INTEGRABLE HAMILTONIAN SYSTEMS**

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## TABLE OF CONTENTS

	Page
ABSTRACT. . . . .	ii
1. ONE-DIMENSIONAL PERIODIC SCHRÖDINGER OPERATORS AND INTEGRABLE HAMILTONIAN SYSTEMS. . . . .	1
1.1 Introduction . . . . .	1
1.2 Spectrum . . . . .	2
1.3 Action-Angle Variables . . . . .	9
2. QUASI-PERIODIC SCHRÖDINGER OPERATORS, INTEGRABLE HAMILTONIAN SYSTEMS AND KAM ITERATION SCHEMES . . . . .	18
2.1 Introduction . . . . .	18
2.2 Hamiltonian Framework. . . . .	24
2.3 Inductive Lemma. . . . .	28
2.4 Compatibility of Approximation Functions and Analyticity-Losses . . . . .	36
2.5 KAM Iteration Scheme . . . . .	39
2.6 Structure of the Non-Resonant Set $A^{(\infty)}$ . . . . .	46
2.7 Whitney Smoothness of KAM Limits . . . . .	57
2.8 Back to the Schrödinger Equation . . . . .	62
2.9 Bloch Waves. . . . .	73
3. ABSOLUTELY CONTINUOUS SPECTRUM OF QUASI-PERIODIC SCHRÖDINGER OPERATORS . . . . .	83
3.1 Introduction . . . . .	83
3.2 Classical Background . . . . .	85
3.3 Absolutely Continuous Spectrum of Limit- Point Schrödinger Operators. . . . .	91
3.4 Absolutely Continuous Spectrum and Spectral Densities of Quasi-Periodic Schrödinger Operators. . . . .	93
3.5 Bloch Waves, Densities and KAM Spectrum. . . . .	98
3.6 Weak Bloch Waves . . . . .	106
APPENDIX A: AN ANALYTIC IMPLICIT FUNCTION THEOREM ON $T^d$ . . . . .	116
APPENDIX B: CONVERGENCE OF CHANGES OF VARIABLES IN THE KAM ITERATION. . . . .	117
REFERENCES. . . . .	121

## ABSTRACT

One-dimensional quasi-periodic Schrödinger operators are studied from two different points of view. The first one deals with the construction of quasi-periodic eigensolutions and with an associated Hamiltonian interpretation. The second one concerns more general questions about absolutely continuous spectra.

In the first chapter, the eigenvalue problem for a periodic Schrodinger operator,

$$Lf \equiv \left( -\frac{d^2}{dx^2} + v \right) f = Ef,$$

is viewed as a two-dimensional Hamiltonian system which is integrable in the sense of Arnold and Liouville. With the aid of the Floquet-Bloch theory, it is shown that such a system is conjugate to two harmonic oscillators with frequencies  $\alpha$  and  $\omega$ ,  $\alpha$  being the rotation number for  $L$  and  $\frac{2\pi}{\omega}$  the period of the potential  $v$ .

This picture is generalized, in the second chapter, to quasi-periodic Schrodinger operators,  $L_\epsilon$ , with highly irrational frequencies  $(\omega_1, \dots, \omega_d)$ , which are a small perturbation of periodic operators. When the parameter  $E$  belongs to a certain (explicitly constructed) large Cantor set  $E$ , the eigenvalue problem for  $L_\epsilon$  is embedded, via a KAM method, in a system of  $d+1$  harmonic oscillators with frequencies  $(\alpha, \omega_1, \dots, \omega_d)$ ,  $\alpha$  being the (Johnson-Moser) rotation number for  $L_\epsilon$ . The function  $E \in \mathbb{R} \rightarrow \alpha(E)$ , in general only continuous, is shown to be  $C^\infty$  on  $E$  in the sense of Whitney and a new proof of Moser-Deift-Simon

inequality,

$$\frac{d\alpha^2}{dE} \geq 1,$$

is given for  $E$  in  $E$  ( $\frac{d}{dE} \equiv$  Whitney derivative).

A by-product of the above is that, on  $E$ , all the eigensolutions of  $L_\varepsilon$  are quasi-periodic with frequencies  $(\alpha, \omega_1, \dots, \omega_d)$  and depend smoothly on  $E$  (in the Whitney sense). Moreover, adapting the KAM algorithm to a matrix formalism, independent eigensolutions of the form  $f = e^{i\alpha x} \chi(\omega_1 x, \dots, \omega_d x)$  and  $\bar{f}$ , with  $\chi$  periodic in each argument, are constructed. Such functions are called Bloch waves.

In the last chapter, the absolutely continuous spectrum  $\sigma_{ac}$  of a general quasi-periodic Schrödinger operator is considered. The Radon-Nikodym derivatives (with respect to Lebesgue measure) of the spectral measures are computed in terms of special independent eigensolutions existing for almost every  $E$  in  $\sigma_{ac}$ . Such eigensolutions can be replaced, in the above Radon-Nikodym derivatives, by Bloch waves whenever these exist (as in the case treated in Chapter 2).

Finally, it is shown that weak Bloch waves always exist for almost every  $E$  in  $\sigma_{ac}$  and the question of the existence of genuine Bloch waves is turned into a regularity problem for a certain nonlinear partial differential equation on a  $d$ -dimensional torus.

## CHAPTER 1

### ONE-DIMENSIONAL PERIODIC SCHRÖDINGER OPERATORS AND INTEGRABLE HAMILTONIAN SYSTEMS

#### 1.1 Introduction

The trivial eigenvalue problem

$$-\frac{d^2}{dx^2} q = E q, \quad x \in \mathbb{R},$$

can be thought of as an harmonic oscillator with Hamiltonian  $H(p,q) = \frac{p^2 + \sqrt{E}q^2}{2}$ ,  $(p,q) \in \mathbb{R}^2 - \{0,0\}$ , the parameter  $x$  playing the role of time.

It is elementary that such mechanical systems are integrable in the sense of Arnold and Liouville\*: The canonical change of variables  $(p,q) \rightarrow (A,\varphi) \equiv (\frac{r^2}{2\sqrt{E}}, \varphi)$ , where  $(r,\varphi)$  are polar coordinates in the  $(p, \sqrt{E}q)$ -plane, conjugates  $H$  to the trivial Hamiltonian  $h(A,\varphi) \equiv \sqrt{E}A$ ,  $(A,\varphi) \in \mathbb{R} \times \mathbb{T}$ ,  $\mathbb{T} \equiv \mathbb{R}/2\pi\mathbb{Z}$ .

In this chapter, we show that periodic Schrödinger operators,

$$L(v) \equiv -\frac{d^2}{dx^2} + v(x), \quad v(x + \frac{2\pi}{\omega}) = v(x),$$

carry a structure completely analogous to the one described above: For  $E$  inside the spectrum of  $L$ , the eigenvalue problem

$$L q = E q$$

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\*We recall the mechanical language in §1.2.

can be embedded in a two-dimensional integrable system. Moreover, the integrated Hamiltonian has the form

$$h(A_1, A_2, \varphi_1, \varphi_2) = \alpha(E)A_1 + \omega A_2,$$

$$(A_1, A_2) \in \mathbb{R}_+ \times \mathbb{R}, (\varphi_1, \varphi_2) \in \mathbb{T}^2,$$

the parameter  $\alpha$  being the rotation number for  $L$ .

### 1.2 Spectrum

Let  $L = L(v) \equiv -\frac{d^2}{dx^2} + v(x)$ , where  $v(x) = V(\omega x)$  with  $V$  a smooth real function on the circle, and  $\omega > 0$  a given frequency. The abstract spectral theory for these operators is included in Weyl's "limit-point, limit-circle" theory\*: Since  $v$  is real and bounded, the operator  $L$  is in the limit-point case. That is,  $L$ , considered on  $C_0^\infty(\mathbb{R})$  ( $\equiv$  the class of indefinitely differentiable functions on  $\mathbb{R}$  with compact support) is essentially self-adjoint and admits a unique extension to a dense domain in  $L^2(\mathbb{R})$ . The resolvent set

$$\rho(L) \equiv \{E \in \mathbb{C} : (L-E)^{-1} \text{ exists and is bounded}\}$$

is characterized by the existence of two independent solutions  $f_\pm(x; E)$  of

$$Lf = Ef \tag{1.1}$$

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\*Weyl [1910], Stone [1932], Titchmarsh [1946], Kodaira [1949], Coddington-Levinson [1955]. See also §2, Chapter 3 for more information.

belonging to  $L^2(\mathbb{R}_\pm)$ ,  $\mathbb{R}_+ \equiv (0, \infty)$ ,  $\mathbb{R}_- \equiv (-\infty, 0)$ . The Green's function ( $\equiv$  kernel of  $(L-E)^{-1}$ ) is given by\*

$$g(x,y;E) = \frac{f_+(x;E) f_-(y;E)}{[f_+, f_-]} \quad \text{if } x \geq y$$

and symmetrically if  $x \leq y$ .

The concrete analysis of (1.1) is the content of Floquet theory\*\* . Let  $A(x;E)$  be the fundamental matrix

$$\begin{bmatrix} f_1(x;E) & f_2(x;E) \\ f_1'(x;E) & f_2'(x;E) \end{bmatrix}$$

where  $f_1(0) = f_2'(0) = 1$ ,  $f_1'(0) = f_2(0) = 0$ , and let  $M(E)$  denote the monodromy matrix  $A(\frac{2\pi}{\omega}; E)$ . Then there is a non-trivial solution of (1.1) satisfying

$$f(x + \frac{2\pi}{\omega}) = \rho f(x) \quad (1.2)$$

if and only if the Floquet multiplier  $\rho$  is an eigenvalue of  $M$ , i.e.,  $\rho = \rho_\pm \equiv \Delta \pm \sqrt{\Delta^2 - 1}$  with  $\Delta \equiv \frac{1}{2}$  trace  $M$ . The discriminant  $\Delta$  is an entire function of  $E$  of order  $\frac{1}{2}$ , type 1, real on  $\mathbb{R}$  and asymptotic to  $\cos\sqrt{-E}$  near  $-\infty$  (Magnus-Winkler [1966]).  $\Delta$  is depicted in Figure 1.

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$$*[f,g] \equiv f \frac{dg}{dx} - \frac{df}{dx} g \equiv fg' - f'g.$$

\*\*Floquet [1883], Bloch [1928]; see Magnus-Winkler [1966] for a review.

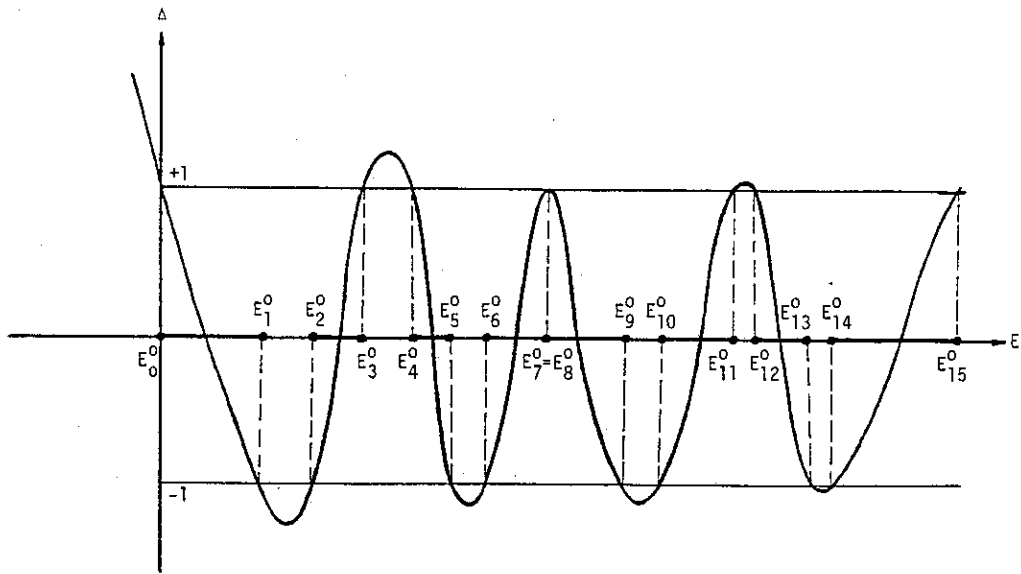


Figure 1.

We will refer to solutions having property (1.2) as Floquet solutions. If  $f, g$  are two Floquet solutions with the same eigenvalue and the same multiplier  $\rho \neq \pm 1$ , then

$$[f, g] = [f, g] \Big|_{x = \frac{2\pi}{\omega}} = \rho^2 [f, g] \Big|_{x=0} = \rho^2 [f, g].$$

Thus,  $[f, g] = 0$ , that is, when  $\rho \neq \pm 1$  the corresponding Floquet solution is determined up to a multiplicative constant. Notice also that if  $\Delta^2 < 1$  (for  $E$  in  $R$ ) and  $f$  is a Floquet solution with multiplier  $\rho_{\pm}$ , then  $[f, \bar{f}] \neq 0$ . In fact,

$$\frac{i}{2} [f, \bar{f}] = [\operatorname{Re} f, \operatorname{Im} f]$$

and  $[f, \bar{f}] = 0$  would imply  $f = cg$  for  $c \in \mathbb{C}$  and  $g$  a real function, and taking the imaginary part of  $g(x + \frac{2\pi}{\omega}) = \rho_{\pm} g(x)$ , we would get



$\text{Im } \rho_{\pm} = \pm\sqrt{1-\Delta^2} = 0$ , a contradiction. From these trivial observations we are able to identify the spectrum of  $L$ . When  $\Delta^2 > 1$ , the Floquet solutions corresponding to  $\rho_{\pm}$  are seen to be the functions  $f_{\pm}$  of Weyl's theory\*, so that  $\{E: |\Delta^2| > 1\} \subset \rho(L)$ . Since for  $E$  in  $\{E \in \mathbb{R}: \Delta^2 < 1\}$  we have two independent eigenfunctions of the form\*\*

$$e^{i\beta x} \chi(\omega x), e^{-i\beta x} \bar{\chi}(\omega x); e^{i\beta} = \rho, \chi \in C^{\infty}(T),$$

we conclude that  $\sigma(L) = \bigcup_{k=0}^{\infty} [E_{2k}^0, E_{2k+1}^0]$ , where  $\sigma(L)$  denotes the spectrum of  $L$  and  $E_0^0 < E_1^0 \leq E_2^0 < E_3^0 \leq E_4^0 \dots$  are the (infinitely many simple or double) roots of  $\Delta^2 = 1$ .

Kodaira [1949] and Gelfand [1950] gave the spectral decomposition of  $L(v)$  showing, as a byproduct, that the spectrum of  $L$  is purely absolutely continuous with double multiplicity<sup>†</sup>. This fact is also a consequence of the following representation of the spectral measures that will be a simple application of Theorem 2, §4 of Chapter 3: Let  $\varphi$  be any  $C^{\infty}$  function with compact support,  $P_E = P_E(L)$  the standard spectral family of  $L$ ,  $f_E$  any Floquet solution, then<sup>‡</sup>

$$d(P_E \varphi, \varphi) = \frac{1}{2\pi} \frac{|(\varphi, f_E)|^2 + |(\varphi, \bar{f}_E)|^2}{|[f_E, \bar{f}_E]|} dE,$$

$$\underline{E \in \sigma^*(L) \equiv \{E \in \sigma(L): |\Delta| < 1\}. \quad (1.3)}$$

\*In this case,  $f_{\pm}$  decay exponentially fast at  $\pm\infty$ .

\*\* $\chi \in C^{\infty}(T)$  means that  $\varphi \in \mathbb{R} \rightarrow \chi(\varphi)$  is a  $C^{\infty}$  function with period  $2\pi$ .

†See Reed-Simon [1978] for a review.

‡ $(\cdot, \cdot)$  denotes the usual inner product in  $L^2(\mathbb{R})$ , and  $dE$  is the Lebesgue measure.

Remark. We saw already that in  $\sigma^*$  the Floquet solutions corresponding to  $\rho_{\pm}$  are one the complex conjugate of the other and that they are determined up to a constant. This makes formula (1.3) unambiguous.

Another important object associated to  $L(v)$ -E is the rotation number  $\alpha = \alpha(E)^*$ . This is a positive unbounded continuous function of the real parameter E, constant on  $\rho(L) \cap \mathbb{R}$  and strictly increasing on  $\sigma(L)$ . Before defining  $\alpha$ , we introduce the winding number  $w$ : Let  $g: \mathbb{R} \rightarrow \mathbb{C}$  be a continuous curve never passing through zero. Define

$$w = w(g) \equiv \lim_{x \uparrow \infty} \frac{\arg g(x)}{x},$$

where such a limit exists\*\*. For example, a periodic curve  $g$  with period  $\frac{2\pi}{\omega}$  will have  $w(g) = n$  for some  $n$  in  $\mathbb{Z}$ .

Now, let  $f$  be a solution of (1.1) with  $[f, \bar{f}] \neq 0$ , then  $f$  never vanishes and

$$\frac{d}{dx} (\arg f) = \frac{\operatorname{Im} f' \bar{f}}{|f|^2} = \frac{i}{2|f|^2} [f, \bar{f}]$$

shows that  $f$  winds around the origin (counter-)clockwise according to  $i[f, \bar{f}]$  ( $>$ )  $<$  0; furthermore  $w(f)$  exists,  $|w|$  is independent of  $f$  and is the rotation number cited above. The behavior of  $\alpha$  as a function of  $E$  can be described very precisely thanks to the analytic properties of

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\*See, for example, Herman [1979], Moser [1981].

\*\*Obviously this definition doesn't depend on the choice of the branch of  $\arg z$  since constants wash out in the limit.

the discriminant: Define  $\theta(E)$  by  $\cos\theta(E) = \Delta(E)$ . Then

$$\theta(E) = \pm i \int_{E_0^0}^E \frac{d\Delta}{\sqrt{\Delta^2-1}} dE' \quad (\text{mod } 2\pi)$$

and

$$\alpha(E) = 0 \quad E \in (-\infty, E_0^0) \quad (1.4)$$

$$\alpha(E) = \alpha(E_{2k}^0) + \frac{\omega}{2\pi} (-1)^{k+1} \operatorname{Re} \int_{E_{2k}^0}^E \frac{d\Delta}{\sqrt{1-\Delta^2}} \quad E \in [E_{2k}^0, E_{2k+1}^0]$$

From (1.4) one sees that  $\alpha(E) \sim \sqrt{E}$  for large  $E$  and that  $\alpha - \alpha(E_k^0) \sim \sqrt{|E - E_k^0|}$  for  $E$  near  $E_k^0 \in \partial\sigma(L)$ ; compare Figure 2.

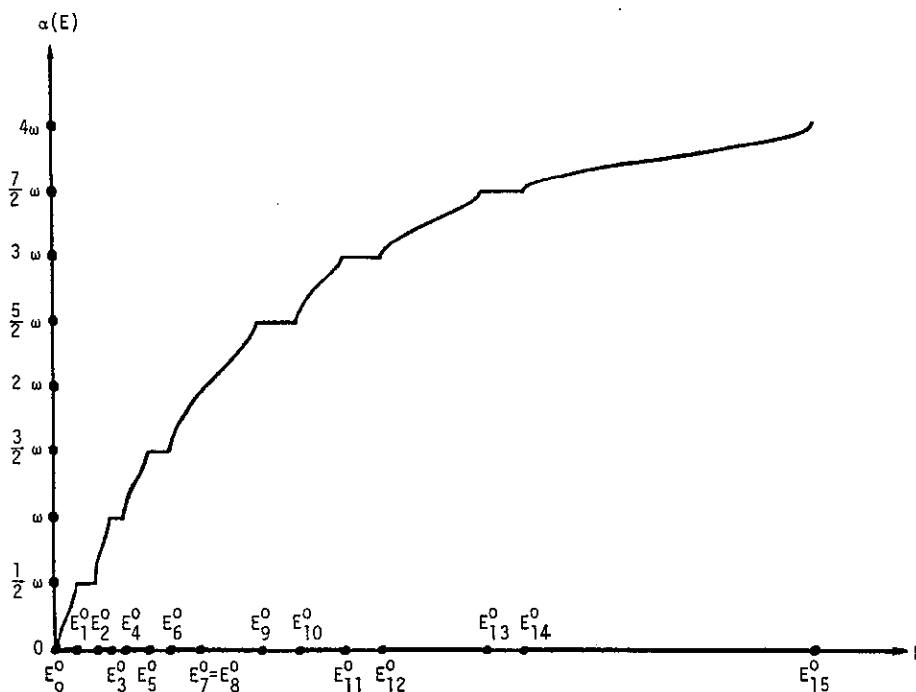


Figure 2.

Before closing this section, we point out a parametrization of the Floquet solutions for  $E \in \sigma^*$  that will be useful in the following paragraphs: Let  $f_0$  be the unique Floquet solution with multiplier

$$\rho_+ = e^{i\alpha \frac{2\pi}{\omega}}$$

and  $f_0(0) = 1$ . Such a function has the representation

$$f_0(x) = e^{i\alpha x} \chi_0(\omega x) \quad (1.5)$$

where  $\chi_0$  is a smooth function on the circle  $T$  with

$$\chi_0(0) = 1 \quad \text{and} \quad \lim_{x \rightarrow \infty} \arg \frac{\chi_0(\omega x)}{x} = 0.$$

It is also easy to see that\*

$$f_0(x) = f_1(x) + \frac{e^{i\alpha \frac{2\pi}{\omega}} - f_1(\frac{2\pi}{\omega})}{f_2(\frac{2\pi}{\omega})} f_2(x).$$

Thus,

$$\kappa \equiv \frac{i}{2} [f_0, \bar{f}_0] = \text{Im } f_0'(0) = \frac{\sin(\alpha \frac{2\pi}{\omega})}{f_2(\frac{2\pi}{\omega})}.$$

Finally, the asymptotics of  $f_1, f_2$  and (1.4) imply

$$\kappa \sim \sqrt{E} \quad \text{and} \quad \chi_0 \sim 1 \quad \text{for} \quad E \rightarrow +\infty.$$

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\* $f_2(\frac{2\pi}{\omega}; E) = 0$  implies  $E \in [E_{2k-1}^0, E_{2k}^0]$  for some  $k \geq 2$ .

### 1.3 Action-Angle Variables

Consider the Hamiltonian

$$H(p, B, q, \psi; E) \equiv \frac{p^2}{2} + \omega B + \frac{q^2}{2} (E - V(\psi)), \quad (p, q) \in \mathbb{R}^2 - \{0\},$$

$$(B, \psi) \in \mathbb{R} \times \mathbb{T} \quad (1.6)$$

where  $(q, \psi)$  are regarded as generalized coordinates and  $(p, B)$  as the conjugate variables. The canonical equations for  $H$  are

$$\begin{aligned} q' &= p & p' &= q(V(\psi) - E) \\ \psi' &= \omega & B' &= V'(\psi) \frac{q^2}{2} \end{aligned}$$

which imply

$$\frac{d^2}{dx^2} q = q(V(\psi_0 + \omega x) - E), \quad \psi_0 = \psi \Big|_{x=0}.$$

Thus, for  $\psi_0 = 0$ , we get back (1.1) for  $q(x)$ .

We recall\* that a Hamiltonian system on a  $2n$ -dimensional symplectic manifold  $M^{2n}$  with Hamiltonian  $H$  is said to be integrable if there exist  $n$  smooth functions on  $M$ ,  $F_1 \equiv H, F_2, \dots, F_n$ , such that

- (i) the  $F_i$ 's are in involution\*\*,
- (ii) the  $F_i$ 's are functionally independent on  $N \equiv \{F_i = \text{const}, i=1, \dots, n\}$ ,

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\*See, e.g., Arnold [1978] or Gallavotti [1983].

\*\*That is, in standard local symplectic coordinates  $(p, q)$ ,

$$\{F_i, F_j\} \equiv \sum_{k=1}^n \frac{\partial F_i}{\partial p_k} \frac{\partial F_j}{\partial q_k} - \frac{\partial F_i}{\partial q_k} \frac{\partial F_j}{\partial p_k} = 0, \text{ for every choice of } i \text{ and } j.$$

(iii) the submanifold  $N$  is compact and connected.

Under these conditions, the evolution equations for  $H$  are solvable by quadratures. More precisely, there exists a cononical\* change of coordinates

$$C: m \in M^{2n} \rightarrow C(m) = (A, \varphi) \in \Omega \times T^n, \quad \Omega \subset \mathbb{R}^n,$$

for which  $H(C^{-1}(A, \varphi)) = h(A)$ . In this case, the Hamilton equations on  $\Omega \times T$  become trivial:

$$A' = 0, \quad \varphi' = \frac{\partial h}{\partial A};$$

in fact,  $A(x) \equiv A(0)$  and  $\varphi(x) = \varphi(0) + \frac{\partial h}{\partial A}(A(0))x \pmod{2\pi}$ . The coordinates  $(A, \varphi)$  are called action-angle variables. This is the content of the Arnold-Liouville theorem (Arnold [1978]).

Going back to (1.6), we have the following\*\*

Theorem 1. If  $E \in \sigma^*(L)$ , the Hamiltonian system (1.6) is integrable.

The proof employs new coordinates for the phase space  $\mathbb{R}^3 \times T - (0, R, 0, T)$ : For  $E \in \sigma^*(L)$ , let  $f_0$  be the Floquet solution defined at the end of the preceding section, and set

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\*That is, preserving the symplectic structure and hence the form of the Hamilton equations.

\*\*We learned this theorem from Gallavotti [198-], see also Gallavotti [1985].

$$F_0(\varphi, \psi) = e^{i(\varphi - \frac{\alpha}{\omega} \psi)} f_0\left(\frac{\psi}{\omega}\right), \quad (\varphi, \psi) \in \mathbb{T}^2.$$

Since  $f_0$  has Floquet multiplier  $e^{i\alpha \frac{2\pi}{\omega}}$ ,  $F_0$  is a smooth function on  $\mathbb{T}^2$ .

Moreover

$$x \rightarrow F_0(\varphi + \alpha x, \psi + \omega x) = e^{i(\varphi - \frac{\alpha}{\omega} \psi)} f_0\left(\frac{\psi}{\omega} + x\right)$$

is easily recognized as a Floquet solution of the shifted Schrödinger equation

$$-f''(x) + V(\psi + \omega x) f(x) = E f(x),$$

with multiplier  $e^{i\alpha \frac{2\pi}{\omega}}$ .

Define

$$Q(\varphi, \psi) \equiv \operatorname{Re} F_0(\varphi, \psi)$$

$$P(\varphi, \psi) \equiv \operatorname{Re} D_\alpha F_0(\varphi, \psi) = \operatorname{Re} \left[ e^{i(\varphi - \frac{\alpha}{\omega} \psi)} f_0'\left(\frac{\psi}{\omega}\right) \right]$$

where  $D_\alpha$  is the vector field  $\alpha \frac{\partial}{\partial \varphi} + \omega \frac{\partial}{\partial \psi}$  and observe that

$(D_\alpha F_0)(\varphi + \alpha x, \psi + \omega x) = \frac{d}{dx} F_0(\varphi + \alpha x, \psi + \omega x)$ . Now consider the map

$$(r, B, \varphi, \psi) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{T}^2 \rightarrow (p, B, q, \psi) \in \mathbb{R}^3 \times \mathbb{T} - (0, \mathbb{R}, 0, \mathbb{T}) \quad (1.7)$$

$$p = r P(\varphi, \psi), \quad q = r Q(\varphi, \psi).$$

Lemma. The map (1.7) is a diffeomorphism.

Proof. Let's start with the Jacobian. With the temporary notation

$$f(x; \varphi, \psi) = F_0(\varphi + \alpha x, \psi + \omega x), \text{ we have}$$

$$\begin{aligned}
\det \frac{\partial(p, B, q, \psi)}{\partial(r, B, \varphi, \psi)} &= r \left( \frac{\partial Q}{\partial \varphi} P - Q \frac{\partial P}{\partial \varphi} \right) \quad (1.8) \\
&= r [-(\operatorname{Im} F_0)(\operatorname{Re} D_\alpha F_0) + (\operatorname{Re} F_0)(\operatorname{Im} D_\alpha F_0)] \\
&= r [\operatorname{Re} f, \operatorname{Im} f] = r \frac{i}{2} [f, \bar{f}] \quad (\text{at } x = 0) \\
&= r \frac{i}{2} [f, \bar{f}] \quad (\text{at } x = -\frac{\psi}{\omega}) \\
&= r \frac{i}{2} [f_0, \bar{f}_0] \\
&= r \kappa > 0.
\end{aligned}$$

Next notice that, for each  $\psi \in T$ , the map

$$(r, \varphi) \rightarrow (p, q) = (r P(\varphi, \psi), r Q(\varphi, \psi))$$

can be written as

$$\begin{pmatrix} p \\ q \end{pmatrix} = T \begin{pmatrix} r \sin \varphi \\ r \cos \varphi \end{pmatrix}, \quad T \equiv \begin{bmatrix} -\operatorname{Im}(e^{-\frac{\alpha}{\omega} \psi} f_0(\frac{\psi}{\omega})) & \operatorname{Re}(e^{-\frac{\alpha}{\omega} \psi} f_0(\frac{\psi}{\omega})) \\ -\operatorname{Im}(e^{-\frac{\alpha}{\omega} \psi} f_0'(\frac{\psi}{\omega})) & \operatorname{Re}(e^{-\frac{\alpha}{\omega} \psi} f_0'(\frac{\psi}{\omega})) \end{bmatrix}.$$

Now,

$$\begin{aligned}
\det T &= [\operatorname{Re}(e^{-\frac{\alpha}{\omega} \psi} f_0(\frac{\psi}{\omega} + x)), \operatorname{Im}(e^{-\frac{\alpha}{\omega} \psi} f_0(\frac{\psi}{\omega} + x))] \quad (\text{at } x = 0) \\
&= [\operatorname{Re}(e^{-\frac{\alpha}{\omega} \psi} f_0(\frac{\psi}{\omega} + x)), \operatorname{Im}(e^{-\frac{\alpha}{\omega} \psi} f_0(\frac{\psi}{\omega} + x))] \quad (\text{at } x = -\frac{\psi}{\omega}) \\
&= e^{-\frac{2\alpha}{\omega} \psi} \kappa \neq 0.
\end{aligned}$$

The proof is finished.



The quantities  $(r, B, \varphi, \psi)$  will be called Floquet-variables.

Now, the integrability of (1.6) is a simple matter.

Proof of Theorem 1. We have to check (i), (ii), (iii) above. Since  $r$  is, by construction, an integral\* for  $H$ ,  $r$  and  $H$  are in involution, and this is (i). The level surfaces  $H = C_1$ ,  $r = C_2$  are 2-dimensional tori so (iii) is fulfilled. To check the independence of  $H$  and  $r$ , we use the Floquet-variables:

$$\frac{\partial(H, r)}{\partial(r, B, \varphi, \psi)} = \begin{bmatrix} * & \omega & * & * \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

so we have also (ii). The proof is finished.

Now, we turn to the explicit construction of the action-angle variables.

Consider the map

$$(r, B, \varphi, \psi) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{T}^2 \rightarrow (A_1, A_2, \varphi_1, \varphi_2) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{T}^2 \quad (1.9)$$

$$A_1 = \frac{r^2}{2} \kappa, \quad A_2 = B + \frac{r^2}{2} \left( \frac{\partial Q}{\partial \psi} P - Q \frac{\partial P}{\partial \psi} \right), \quad \varphi_1 = \varphi, \quad \varphi_2 = \psi.$$

Such a map is clearly one-to-one and onto, and the evaluation of the Jacobian

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\*That is, a function constant on the trajectories governed by  $H$ .

$$\det \frac{\partial(A_1, A_2, \varphi_1, \varphi_2)}{\partial(r, B, \varphi, \psi)} = \begin{vmatrix} r\kappa & 0 & 0 & 0 \\ * & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = r\kappa > 0$$

shows that it is a diffeomorphism. The upshot is that  $(A_1, A_2, \varphi_1, \varphi_2)$  are the required action-angle variables:

Theorem 2. The diffeomorphism

$$(p, B, q, \psi) \in \mathbb{R}^3 \times T - (0, R, 0, T) \rightarrow (A_1, A_2, \varphi_1, \varphi_2) \in \mathbb{R}_+ \times \mathbb{R} \times T^2$$

defined via (1.7), (1.9) is a canonical transformation, and

$$\frac{p^2}{2} + \omega B + \frac{q^2}{2} (E - V(\psi)) = \alpha A_1 + \omega A_2. \quad (1.10)$$

Proof. In (1.8), we saw that  $\kappa = Q_1 P - Q P_1$ . Now using Floquet-variables as an intermediate step,

$$\begin{aligned} & dp \wedge dq + dB \wedge d\psi \\ &= \kappa r dr \wedge d\varphi + r(PQ_2 - QP_2) dr \wedge d\psi \\ & \quad + r^2(P_1Q_2 - Q_1P_2) d\varphi \wedge d\psi + dB \wedge d\psi \end{aligned}$$

---

\*Subscripts 1,2 mean, respectively,  $\frac{\partial}{\partial \varphi}$  and  $\frac{\partial}{\partial \psi}$ .

$$\begin{aligned}
&= \kappa r dr \wedge d\varphi + r(PQ_2 - QP_2) dr \wedge d\psi \\
&\quad + \frac{r^2}{2} (P_1Q_2 - Q_1P_2 + Q_{21}P - QP_{21}) d\varphi \wedge d\psi + dB \wedge d\psi \\
&= dA_1 \wedge d\varphi_1 + dA_2 \wedge d\varphi_2 .
\end{aligned}$$

This proves the first claim. To check (1.10), put  $q_0(x) = Q(\varphi + \alpha x, \psi + \omega x)$ . Then using Floquet-variables once more,

$$\begin{aligned}
&\alpha A_1 + \omega A_2 \\
&= \alpha \frac{r^2}{2} (Q_1P - QP_1) + \omega B + \omega \frac{r^2}{2} (Q_2P - QP_2) \\
&= \omega B + \frac{r^2}{2} [(\alpha Q_1 + \omega Q_2)P - (\alpha P_1 + \omega P_2)Q] \\
&= \omega B + \frac{r^2}{2} [q_0'(0)^2 + q_0''(0) q_0(0)] \\
&= \omega B + \frac{r^2}{2} [q_0'(0)^2 - q_0(0)^2 (V(\psi) - E)] \\
&= \omega B + \frac{r^2}{2} [P^2 + Q^2 (E - V)] \\
&= \frac{p^2}{2} + \omega B + \frac{q^2}{2} (E - V) .
\end{aligned}$$

The proof is completed.

Remark 1. The canonical transformation in Theorem 2 is produced by the generating function

$$\Phi(B, p, \varphi_1, \varphi_2) \equiv B\varphi_2 + \frac{p^2}{2} \frac{Q(\varphi_1, \varphi_2)}{P(\varphi_1, \varphi_2)} .$$

The Hamilton-Jacobi equation for  $\phi$

$$\frac{p^2}{2} + \omega B + \frac{q^2}{2} [E - V(\psi)] = \alpha \frac{\partial \phi}{\partial \varphi_1} + \omega \frac{\partial \phi}{\partial \varphi_2}$$

is easily seen to correspond, along  $\frac{d}{dx} \equiv D_\alpha$ , to a Riccati equation for

$$\frac{d}{dx} \log Q(\varphi + \alpha x, \psi + \omega x) .$$

Remark 2. The set of E's for which we carried out the integration of H excludes the double roots of  $\Delta^2 = 1$  ( $\equiv$  collapsed gaps). This was done for reasons of simplicity rather than for real difficulties. In fact, Floquet theory tells us that when a spectral gap collapses to a point,  $E_{2k-1}^0 = E_{2k}^0$ , there are still two independent eigenfunctions of the form

$$f = e^{i\alpha x} \chi(\omega x) \quad \text{and} \quad \bar{f}$$

with  $\chi(2\pi)$  periodic\*. Furthermore, in such a case,

$$f_2\left(\frac{2\pi}{\omega}, E_{2k}^0\right) = 1$$

and  $\kappa$  is still different from zero. Therefore, H is actually integrable for E in

$$\sigma^*(L) \cup \{\text{collapsed gaps}\} = \text{interior of } \sigma(L).$$

---

\*See Magnus-Winkler [1966]. This phenomenon is called coexistence of periodic solutions since, on gaps,  $\alpha = \frac{n\omega}{2}$  (compare Figure 2).

Remark 3. It is well known that the Kortweg-de Vries equation can be viewed as an isospectral flow for some Schrödinger operator  $L^*$ . If  $L$  is periodic, such a flow is Hamiltonian and integrable and action-angle variables have been displayed by Flaschka-McLaughlin [1976]. It might be interesting to see if there is any relation between such variables and the ones in Theorem 2.

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\*Lax [1968]; see McKean [1978] for a review.

## CHAPTER 2

### QUASI-PERIODIC SCHRÖDINGER OPERATORS, INTEGRABLE HAMILTONIAN SYSTEMS AND KAM ITERATION SCHEMES

#### 2.1 Introduction

Here we consider the eigenvalue problem for Schrödinger operators  $L(v)$  with (real) analytic quasi-periodic potential  $v(x)$ .

We recall that a function  $x \in \mathbb{R} \rightarrow f(x)$  is said to be  $(\omega-)$  quasi-periodic with frequencies  $\omega = (\omega_1, \dots, \omega_d)$  if one has  $f(x) = F(\omega x)$ , for some function  $F$  on  $T^d \equiv \mathbb{R}^d / 2\pi\mathbb{Z}^d$ . The class to which the function  $F$  belongs will define the class of  $f$  \*. Now, the basic question is: Do there exist quasi-periodic eigenfunctions?

The first to attack this problem were Dinaburg-Sinai [1975]. They assume that the frequencies of the potential satisfy the Diophantine condition\*\*

$$|\omega \cdot v| \geq \frac{1}{c|v|^{d+1}}, \text{ for some } c > 0 \text{ and any } v \in \mathbb{Z}^d - \{0\}. \quad (2.1)$$

---

\*Warning: There exist real analytic functions which are, according to this definition, merely continuous quasi-periodic (see Johnson-Moser [1982]).

\*\*If  $x, y \in \mathbb{C}^d$ ,  $x \cdot y \equiv \sum_{i=1}^d x_i \bar{y}_i$  and  $|x| \equiv \sum_{i=1}^d |x_i|$ .

Then, for a fixed  $\sigma > 0$  and  $E_0$  large enough, they construct a big nowhere dense set  $E(\sigma)$  in  $[E_0, +\infty)$  so that, for  $E$  in  $E(\sigma)$ , all the eigenfunctions of  $L$  are (analytic) quasi-periodic with rationally independent frequencies  $(a, \omega_1, \dots, \omega_d)$ . The frequency  $a$  is a uniformly continuous function of  $E$  and verifies

$$\left| a - \frac{1}{\sqrt{E}} \right| = O\left(\frac{1}{\sqrt{E}}\right).$$

The set  $E(\sigma)$  is described through its complement

$$R(\sigma) \equiv \bigcup_{\substack{v \in \mathbb{Z}^d \\ |v| \geq 2}} \left\{ E \geq E_0 : |\sqrt{E} - \alpha_v| < K_\sigma \exp\left(-\frac{|v|}{\log^{1+\sigma}|v|}\right) \right\}$$

where  $K_\sigma$  is a positive constant and the numbers  $\alpha_v$  (not explicitly determined) are such that

$$\left| \alpha_v - \frac{1}{2} |\omega \cdot v| \right| < \frac{K'_\sigma}{(1 + |\omega \cdot v|)}.$$

Their proof is based on a KAM fast iteration scheme\* for trace-less, two-by-two, complex matrices with certain symmetries.

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\*Kolmogorov [1954], Arnold [1963], Moser [1962] and [1967].

Rüssmann [1980] replaces (2.1) by the more general condition

$$|\omega \cdot v| \geq \frac{1}{\Omega(|v|)}, \quad v \in \mathbb{Z}^d - \{0\} \quad (2.2)$$

where the so-called approximation function  $\Omega$  satisfies\*

$$\frac{r^{d-1}}{|\omega|} \leq \Omega(r) < +\infty, \quad \frac{\log \Omega(r)}{r} \rightarrow 0, \quad \int_0^\infty \frac{\log \Omega(r)}{r^2} dr < +\infty. \quad (2.3)$$

Then, using a set of estimates from Rüssmann [1975] and [1976], he improves the Dinaburg-Sinai iteration scheme obtaining reasonable numerical bounds on its threshold of applicability.

He also gives an interpretation of the points  $\alpha_v$  of the Dinaburg-Sinai set that we discuss in detail later\*\*.

The set-up in Moser-Pöschel [1984] is as in Rüssmann [1980] but two new features appear:

- (i) The recently discovered properties of the rotation number  $\alpha$  for quasi-periodic potentials allow them to work directly in the  $\alpha$ -line instead of the  $\sqrt{E}$ -line.

---

\*The first item of (2.3) is justified by Dirichlet's theorem in the theory of Diophantine approximation:

$$0 < \max_{|v| \leq m} \frac{1}{|\omega \cdot v|} \leq \frac{m^{d-1}}{|\omega|},$$

for irrational  $\omega$  and any  $m \in \mathbb{Z}_+$ .

\*\*Compare Remark 2 of §2.4 and Remark 3 of §2.9.



- (ii) They construct quasi-periodic eigenfunctions with rationally dependent frequencies  $(\beta, \omega_1, \dots, \omega_d)$ . In fact,  $\beta$  has the form  $\beta = \frac{\omega \cdot \nu}{2}$ , for suitable  $\nu$ 's in  $Z^d$ .

The rotation number is defined, as in the periodic case, by

$$\alpha(E) = \pm \lim_{x \rightarrow \infty} \frac{\arg f}{x},$$

for any complex solution of

$$L(\nu) f = E f$$

with  $i[f, \bar{f}] \geq 0^*$ . Johnson-Moser [1982]\*\* show that such a limit exists for any complex  $E$  and does not depend on the choice of  $f$ . Moreover,  $E \in R \rightarrow \alpha(E)$  is a continuous monotone function, strictly increasing on  $\sigma(L)$  and constant on the intervals of  $\rho(L) \cap R$ . On such intervals,  $\alpha = \frac{1}{2} \omega \cdot \nu$ , for some integer vector  $\nu$ . This is the "gap labelling theorem" of Johnson-Moser [1982].

Going back to Moser-Pöschel [1984], they define

$$\tilde{R} = \tilde{R}(\omega) \equiv \left\{ \beta = \frac{\omega \cdot \nu}{2} : \left| \beta - \frac{\omega \cdot \mu}{2} \right| \geq \frac{1}{\Omega(|\mu|)} \right\}, \quad Z^d \ni \mu \neq \nu,$$

---

\*Actually, there are several equivalent ways to define  $\alpha$  (see Johnson-Moser [1982] and Avron-Simon [1983]).

\*\*They develop their theory more in general for almost periodic potentials; see also Avron-Simon [1983] and, for a review, Simon [1982]. For the related class of random potentials, see Pastur [1973], [1980] and Spencer [198-].

$\Omega$  being any approximation function, and prove that, if  $\beta$  is big enough and belongs to  $\tilde{R}$ , then for  $E \in \alpha^{-1}(\beta)$  one has two independent eigenfunctions

$$e^{i\beta x}(\chi_1 + x \chi_2), e^{i\beta x} \chi_2 \text{ or } e^{i\beta x} \chi_3, e^{-i\beta x} \frac{-}{\chi_3}. \quad (2.4)$$

Here, the functions  $\chi$  are quasi-periodic with frequencies  $\omega$  and the form (2.4) depends on whether the closed interval  $\alpha^{-1}(\beta)$  has positive length or not.

Also, by a limiting procedure, they prove the Dinaburg-Sinai result, replacing the set  $R(\sigma)$  with the inverse image by  $\alpha^{-1}$  of

$$\{\beta \in \mathbb{R}: |\beta - \frac{\omega \cdot \mu}{2}| < \frac{3}{\Omega(|\mu|)}\}, \mu \in \mathbb{Z}^d, \mu \neq 0\}.$$

In this chapter, we shall study the eigenvalue problem

$$L_\epsilon f \equiv L(V + \epsilon W) f = E f, \quad (2.5)$$

$$v(x) = V(\omega_1 x), \quad w(x) = W(\omega_2 x, \dots, \omega_d x)$$

for small  $\epsilon^*$ ;  $V$  and  $W$  are (real) analytic on, respectively,  $T$  and  $T^{d-1}$  and  $\omega \equiv (\omega_1, \dots, \omega_d)$  subject to (2.2). We will construct a subset  $E = E_\epsilon(\Omega)$  of  $\sigma(L_0) \cap \sigma(L_\epsilon)$  and a function  $E \in E \rightarrow a(E) > 0$ , so that, for  $E \in E$ , (2.5) can be viewed as a subsystem of the Hamiltonian equations for  $(d+1)$  harmonic oscillators with frequencies  $(a, \omega)$ .

---

\*For  $E$  big, the smallness parameter will be  $(\epsilon/\sqrt{E})$ .

In particular, the eigenfunctions of (2.5), for  $E \in E$ , will be analytic  $(a, \omega)$ -quasi-periodic. The frequencies will be seen to satisfy

$$|v \cdot (a, \omega)| \geq \frac{1}{\Omega(|v|)}, \quad v \in \mathbb{Z}^{d+1} - \{0\},$$

and\*

$$|a(E) - \alpha_0(E)| = O(\varepsilon).$$

Our proof will make use of results from Chapter 1 and an amplification of Rüssmann's KAM scheme (see, in particular, §2.4). The set  $E$  will be completely specified using the smoothness (in the sense of Whitney\*\*) of the KAM limits, especially of  $E \rightarrow a(E)$ .

In §2.8, the function  $a$  will be identified with Johnson-Moser rotation number  $\alpha$  and it will be shown that the (Whitney) derivative of  $\alpha^2$  satisfies Moser-Deift-Simon inequality<sup>†</sup>

$$\frac{d\alpha^2}{dE} \geq 1,$$

for every  $E$  in  $E$ .

\* $\alpha_0$  denotes the rotation number for  $L_0 \equiv L(v)$ . For  $E$  big

$$|a(E) - \alpha_0(E)| = O\left(\frac{\varepsilon}{\sqrt{E}}\right).$$

\*\*Whitney [1934]; see, also, §2.7. The idea of using Whitney's notion of smoothness in Hamiltonian perturbation theory appears in Pöschel [1982] and Chierchia-Gallavotti [1982].

<sup>†</sup>Moser [1981], Deift-Simon [1983].

Finally, adapting the Hamiltonian KAM scheme to a matrix version of (2.5), we will construct on  $E$  independent Bloch waves, i.e., solutions of the form  $f = e^{i\alpha x} \chi(\omega x)$  and  $\bar{f}$ , and relate our situation to that one of Moser-Pöschel [1984].

## 2.2 Hamiltonian Framework

We start by making more precise the assumptions on the quasi-periodic potential  $v + \varepsilon w$ :

Assumption 1. The functions  $v$  and  $w$ , thought as functions of  $*$ , respectively,  $\varphi_2 \in \mathbb{R}$  and  $(\varphi_3, \dots, \varphi_d) \in \mathbb{R}^{d-2}$  have holomorphic extensions to

$$S(\xi) \equiv \{z \in \mathbb{C} : |\operatorname{Im} z| \leq \xi\} \text{ and } S^{d-2}(\xi) \equiv \{z \in \mathbb{C}^{d-2} : |\operatorname{Im} z_j| \leq \xi\},$$

for some  $\xi > 0$ .

Assumption 2. The vector  $\tilde{\omega} \equiv (\omega_2, \dots, \omega_d) \in \mathbb{R}_+^{d-1}$  is of unit length and satisfies the Diophantine condition

$$|\tilde{\omega} \cdot v| \geq \frac{1}{c\Omega(|v|)} \quad , \quad \text{any } v \in \mathbb{Z}^{d-1} - \{0\} \quad ,$$

where\*\*  $c$  is a positive constant and

$$r^{d-2} \leq \Omega(r) \uparrow \uparrow \infty \quad , \quad \text{for } 1 \leq r \uparrow \uparrow \infty \quad .$$

\*For aesthetic reasons, we make the change of notation:  $(d+1) \rightarrow d$ ,  $(\omega_1, \omega_2, \dots, \omega_d) \rightarrow (\omega_2, \omega_3, \dots, \omega_d)$ . We will consider only  $d \geq 3$ .

\*\*The role of  $c$  will be clear later (see §2.6).

The initial value problem for

$$L_\varepsilon q \equiv L(v + \varepsilon w)q = Eq \quad (2.7)$$

$$v(x) = V(\omega_2 x), w(x) = W(\omega_3 x, \dots, \omega_d x)$$

is immediately seen to be part of the evolution equations associated with the Hamiltonian

$$\frac{p^2}{2} + \omega_2 B + \omega_3 A_3 + \dots + \omega_d A_d + \frac{q^2}{2} [E - V(\psi) - \varepsilon W(\varphi_3, \dots, \varphi_d)] \quad (2.8)$$

$$(p, q) \in \mathbb{R}^2 - \{0\}, (B, A_3, \dots, A_d, \psi, \varphi_3, \dots, \varphi_d) \in \mathbb{R}^{d-1} \times \mathbb{T}^{d-1}$$

with initial data

$$p(0) = q'(0), q(0), \psi(0) = \varphi_3(0) = \dots = \varphi_d(0) = 0.$$

Now, consider the (surjective) canonical transformation

$$\begin{aligned} (p, B, A_3, \dots, A_d, \varphi, \psi, \varphi_3, \dots, \varphi_d) &\in \mathbb{R}^{d+1} \times \mathbb{T}^{d-1} - (0, \mathbb{R}^{d-1}, 0, \mathbb{T}^{d-1}) \\ &\rightarrow (A, \varphi) \in M \equiv \mathbb{R}_+ \times \mathbb{R}^{d-1} \times \mathbb{T}^d \end{aligned} \quad (2.9)$$

where  $(p, B, q, \psi) \rightarrow (A_1, A_2, \varphi_1, \varphi_2)$  is the map of Theorem 1.2\*.

Since

$$\frac{q^2}{2} = \frac{r^2 Q^2}{2} = \frac{1}{\kappa} A_1 Q^2,$$

---

\*For facts concerning  $L_\varepsilon$  we maintain, usually, the notations of Chapter 1, (with the obvious substitutions).

the transformation (2.9) conjugates the Hamiltonian (2.8) to

$$\alpha_0 A_1 + \omega_2 A_2 + \dots + \omega_d A_d - \varepsilon A_1 \frac{Q^2(\varphi_1, \varphi_2)}{\kappa} W(\varphi_3, \dots, \varphi_d), \quad (2.10)$$

$\alpha_0$  being the unperturbed rotation number for  $L_0$ ,  $\kappa$  and  $Q$  as in Chapter 1\*.

Notice that the assumption on the periodic potential  $V$  implies that the periodic, complex-valued function  $(\varphi_1, \varphi_2) \in \mathbb{R}^2 \rightarrow F_0(\varphi_1, \varphi_2)$  defined in §1.3 admits a holomorphic extension to  $S^2(\xi)$ . Therefore, the real analytic periodic function  $(\varphi_1, \varphi_2) \rightarrow Q^2(\varphi_1, \varphi_2) \equiv (\operatorname{Re} F_0)^2$  in (2.10) has also a holomorphic extension to  $S^2(\xi)$ .

Remark 1. (On the role of the analyticity assumptions.) A function  $\varphi \in \mathbb{R}^n \rightarrow G(\varphi)$ ,  $(2\pi)$ -periodic in each variable, which admits a holomorphic extension to the closed strip

$$S^n(s) = \{z \in \mathbb{C}^n: |\operatorname{Im} z_i| \leq s\},$$

has a Fourier expansion

$$F(\varphi) = \sum_{\nu \in \mathbb{Z}^n} \hat{F}_\nu e^{i\nu \cdot \varphi},$$

with coefficients

$$\hat{F}_\nu \equiv \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} F(\varphi) e^{-i\nu \cdot \varphi} d\varphi$$

---

\* $Q$  and  $\kappa$  depend on  $E$  and  $\frac{Q^2}{\kappa} \sim \frac{1}{\sqrt{E}}$ , for  $E \rightarrow +\infty$  (see §1.2).

satisfying the bound

$$|\hat{F}_v| \leq \left( \max_{S^n(s)} |F| \right) e^{-s|v|}. \quad (2.11)$$

This will be used over and over.

Another advantage of holomorphic functions is that one can estimate derivatives in terms of the function itself and some loss in the extent of the analyticity domain. More precisely, if  $f$  is holomorphic in a (smooth) domain  $D \subset \mathbb{C}$  and  $D'$  is a subdomain of  $D$  with\*

$$\text{dist}(\partial D, \partial D') \equiv \rho > 0,$$

then

$$\sup_{D'} \left| \frac{df}{dz} \right| \leq \rho^{-1} \left( \sup_D |f| \right). \quad (2.12)$$

With obvious changes, formula (2.12) and its proof extend to the higher dimensional case.

Remark 2. For the Dinaburg-Sinai case ( $V \equiv 0$ ), the Hamiltonian set-up is much simpler (Gallavotti [198-]): The eigenvalue problem

$$L[\epsilon W(\omega_3^x, \dots, \omega_d^x)] q = E q$$

is embedded in the system with Hamiltonian

---

\*Here " $\partial$ " denotes "the boundary of".

$$\frac{p^2}{2} + \omega_3 A_3 + \dots + \omega_d A_d + \frac{q^2}{2} [E - \varepsilon W(\varphi_3, \dots, \varphi_d)] .$$

The latter is conjugate, via a polar-coordinate transformation in the  $(p, \sqrt{\varepsilon}q)$ -plane, to

$$\sqrt{\varepsilon} A_1 + \omega_3 A_3 + \dots + \omega_d A_d - \frac{\varepsilon}{\sqrt{\varepsilon}} A_1 \sin^2 \varphi_1 W(\varphi_3, \dots, \varphi_d) .$$

Notice that this system is  $(d-1)$  dimensional and that, here,  $\alpha_0(E) = \sqrt{\varepsilon}$  .

### 2.3 Inductive Lemma

We will base the (Hamilton-Jacobi) integration of (2.10) on the perturbation algorithm presented in this section\*.

Consider the Hamiltonian

$$H^{(j)}(A, \varphi; a, \varepsilon) \equiv \omega_1^{(j)}(a; \varepsilon) A_1 + \omega_2 A_2 + \dots + \omega_d A_d \\ + \varepsilon^{2j} A_1 F^{(j)}(\varphi; a, \varepsilon), \quad (j \in \mathbb{N}),$$

on the phase space  $M = \mathbb{R}_+ \times \mathbb{R}^{d-1} \times \mathbb{T}^d$ ,  $0 < \varepsilon < 1$ . Assume that  $\omega_1^{(j)}$  and  $F^{(j)}$ , as functions of  $a$ , are holomorphic in

$$D_j \equiv D(\eta_j; A^{(j)}) \equiv \bigcup_{a_0 \in A^{(j)}} \{a \in \mathbb{C} : |a - a_0| < \eta_j\}$$

---

\*Perturbations of harmonic oscillators were exploited by Rüssmann [1967] and Gallavotti [1982]. Their proof makes use of Moser's idea of modified systems (Moser [1967]). Our proof differs from these.



for some subset  $A^{(j)}$  of the real line. As functions of  $\varphi \in \mathbb{R}^d$ ,  $F^{(j)}$  is  $(2\pi-)$  periodic in each variable and admits a holomorphic extension to  $S^d(\varepsilon_j)$  with\*  $\|F^{(j)}\|_{\varepsilon_j, \eta_j} \leq M_j$  independently of  $\varepsilon$ . Finally, assume  $0 < \varepsilon_j < 1$ , the upper bound being imposed only for simplicity.

Now, let\*\*  $\delta_j < \frac{\varepsilon_j}{2}$  and define

$$\zeta(s) \equiv 1 + \sum_{v \in \mathbb{Z}^d - \{0\}} |v| \Omega(|v|) e^{-s|v|}, \quad s > 0,$$

$$N_j \equiv 2^{j+1} \delta_j^{-1} \log \varepsilon^{-1}$$

$$F_T^{(j)}(\varphi) \equiv \sum_{|v| \leq N_j} \hat{F}_v^{(j)} e^{iv \cdot \varphi}, \quad F_R^{(j)}(\varphi) \equiv \sum_{|v| > N_j} \hat{F}_v^{(j)} e^{iv \cdot \varphi},$$

$$\eta_{j+1} \equiv \min \left\{ \left[ 2cN_j \Omega(N_j) \sup_{a \in D_j} \left| \frac{d\omega_1^{(j)}}{da} \right| \right]^{-1}, \frac{\eta_j}{2} \right\},$$

$$\varepsilon_{j+1} \equiv \varepsilon_j - 2\delta_j,$$

$$A^{(j+1)} \equiv \{a \in A^{(j)} : |\omega^{(j)} \cdot v| \geq \frac{1}{c\Omega(|v|)}, v \in \mathbb{Z}^d - \{0\}, |v| \leq N_j\},$$

$$\omega^{(j)} \equiv (\omega_1^{(j)}, \tilde{\omega}),$$

$$D_{j+1} \equiv D(\eta_{j+1}; A^{(j+1)}).$$

\*For functions  $F$  on  $S^d(s) \times D(r; I)$ , we set  $\|F\|_{s,r} \equiv \sup_{(z,a) \in S \times D} |F|$ .

\*\*This condition on the "analyticity loss" parameter,  $\delta_j$ , is forced by the proof of the lemma.

Inductive Lemma. There exist two universal constants\*  $K_1 > K_2$  such that if  $a \in A^{(j+1)}$  and

$$K_1 \zeta(\delta_j) \delta_j^{-1} c M_j \varepsilon^{2^j} \leq 1 \quad (2.13)$$

then the function

$$(A', \varphi) \in M \rightarrow A' \cdot \varphi + \varepsilon^{2^j} A_1 \phi_j(\varphi; a, \varepsilon),$$

$$\phi_j \equiv \sum_{0 < |v| \leq N_j} \frac{\hat{F}_v^{(j)}}{-i \omega^{(j)} \cdot v} e^{i v \cdot \varphi}$$

is the generating function of a surjective canonical transformation,

$(A, \varphi) \in M \rightarrow (A', \varphi') = (A'(A, \varphi), \varphi'(\varphi))$ , that conjugates  $H^j(A, \varphi)$  to

$$H^{(j+1)}(A', \varphi'; a, \varepsilon) \equiv H^{(j)}(A(A', \varphi'), \varphi(\varphi'))$$

$$= \omega^{(j+1)} \cdot A' + \varepsilon^{2^{j+1}} A_1' F^{(j+1)}(\varphi'; a, \varepsilon)$$

where

$$\omega^{(j+1)} \equiv (\omega_1^{(j)} + \varepsilon^{2^j} \hat{F}_0^{(j)}, \omega_2, \dots, \omega_d),$$

$$F^{(j+1)}(\varphi'(\varphi)) \equiv \frac{\partial \phi_j}{\partial \varphi_1}(\varphi) F^{(j)}(\varphi) + \frac{F_R^{(j)}(\varphi)}{\varepsilon^{2^j}}.$$

---

\*That is, constants depending only on the dimension  $d$ . We also indicate and will use the following bounds:  $K_1 \geq 2 K_2$ ,  $K_2 \geq 4$ .

Furthermore,  $a \in A^{(j+1)} \rightarrow \omega_1^{(j)}(a)$  and  $(\varphi, a) \in \mathbb{R}^d \times A^{(j+1)} \rightarrow F^{(j+1)}(\varphi; a)$  have holomorphic extensions to, respectively,  $D_{j+1}$  and  $S^d(\xi_{j+1}) \times D_{j+1}$  with

$$\|F^{(j+1)}\|_{\xi_{j+1}, \eta_{j+1}} \leq K_2 \zeta(\delta_j) \delta_j^{-d} c M_j^2 \equiv M_{j+1}. \quad (2.14)$$

Remark 1. It is useful to keep in mind that most of the quantities we are dealing with have physical dimensions\*

$$[\omega_j] = [\eta_j] = [M_j] = [F_j] = [\text{time}]^{-1}, \quad [c] = [\text{time}].$$

The angle related quantities  $\delta_j, \xi_j$  are, instead, dimensionless.

Notational Warning: During this and later proofs, we indicate (sometimes different) universal constants with the same symbol "K".

Proof. The cutoff\*\*  $N_j$  is made so that

$$\|F_R^{(j)}\|_{\xi_j - \delta_j, \eta_j} \leq M_j \delta_j^{-d} \varepsilon^{2^j}. \quad (2.15)$$

In fact, for  $a \in D_j$ , using (2.11) to estimate  $\hat{F}_v^{(j)}$ ,

---

\*Here, square brackets indicate physical dimensions.

\*\*The key idea of the cutoff goes back to Arnold [1963].

$$\begin{aligned} \sup_{z \in S^d(\xi_j - \delta_j)} \left| \sum_{|v| > N_j} \hat{F}_v^{(j)} e^{iz \cdot v} \right| &\leq M_j \sum_{|v| > N_j} e^{-\delta_j |v|} \\ &\leq M_j e^{-(\delta_j N_j)/2} \sum_{|v| > N_j} e^{-(\delta_j |v|)/2} \leq K \cdot M_j \delta_j^{-d} \epsilon^{2j}. \end{aligned}$$

Analogously,

$$\|F\|_{\xi_j - \delta_j, \eta_j} \leq K \cdot M_j \delta_j^{-d}. \quad (2.16)$$

Next, we show that  $\phi_j$  has an holomorphic extension to  $S^d(\xi_j - \delta_j) \times D_{j+1}$  with

$$\max \{ \|\phi_j\|, \|\frac{\partial \phi}{\partial \varphi}\| \} \leq K \cdot \tau(\delta_j) \subset M_j, \quad (2.17)$$

the norms being relative to such a set.

To prove this, we have to take care of the small denominators appearing in  $\phi_j$ . Let  $a \in D_{j+1}$ . Then there is a point  $a_0 \in A^{(j+1)}$  with  $|a - a_0| < \eta_{j+1}$  so that, for  $0 < |v| \leq N_j$ , the definitions of  $A^{(j+1)}$  and  $\eta_{j+1}$  imply

$$\begin{aligned} |\omega^{(j)}(a) \cdot v| &= |\omega^{(j)}(a_0) \cdot v + (\omega_1^{(j)}(a) - \omega_1^{(j)}(a_0)) \cdot v| \\ &\geq |\omega^{(j)}(a_0) \cdot v| \left( 1 - \frac{|\omega_1^{(j)}(a) - \omega_1^{(j)}(a_0)| |v|}{|\omega^{(j)}(a_0) \cdot v|} \right) \\ &\geq \frac{1}{c\Omega(|v|)} \left( 1 - c\Omega(N_j) N_j \sup_{D_j} \left| \frac{d\omega_1^{(j)}}{da} \right| \eta_{j+1} \right) \geq \frac{1}{2c\Omega(|v|)}. \end{aligned}$$

Now, for  $(z, a) \in S^d(\varepsilon_j - \delta_j) \times D_{j+1}$ ,

$$\begin{aligned} \max \{ |\phi_j|, \left| \frac{\partial \phi_j}{\partial \varphi} \right| \} &\leq \sum_{0 < |v| \leq N_j} \frac{|\hat{F}_v^{(j)}| |v| e^{(\varepsilon_j - \delta_j)|v|}}{|\omega^{(j)} \cdot v|} \\ &\leq 2cM_j \sum_{v \neq 0} |v| \Omega(|v|) e^{-\delta_j |v|} < K \cdot \varepsilon(\delta_j) c M_j. \end{aligned}$$

The function  $A' \cdot \varphi + \varepsilon^{2j} A_1 \phi_j(\varphi)$  will generate a canonical transformation if and only if\* we can invert the map  $(A', \varphi) \rightarrow (A, \varphi')$  given by

$$\varphi' = \frac{\partial}{\partial A'} (A' \cdot \varphi + \varepsilon^{2j} A_1 \phi_j) = (\varphi_1 + \varepsilon^{2j} \phi_j, \varphi_2, \dots, \varphi_d) \quad (2.18)$$

$$A = \frac{\partial}{\partial \varphi} (A' \cdot \varphi + \varepsilon^{2j} A_1 \phi_j) = T_j A'$$

where

$$T_j = T_j(\varphi) \equiv \begin{bmatrix} 1 + \varepsilon^{2j} \frac{\partial \phi_j}{\partial \varphi_1} & 0 \dots 0 \\ \varepsilon^{2j} \frac{\partial \phi_j}{\partial \varphi_2} & 1 \ 0 \dots 0 \\ \vdots & \vdots \\ \varepsilon^{2j} \frac{\partial \phi_j}{\partial \varphi_d} & 0 \dots 0 \ 1 \end{bmatrix}.$$

---

\*This is a standard fact in Hamiltonian mechanics. See, e.g., Arnold [1978] or Gallavotti [1983].

To confirm this, we use the following elementary version of a global implicit function theorem. We defer the proof to Appendix A.

Proposition. Let  $z \in S^d(r) \rightarrow g(z; \sigma) \in \mathbb{C}^d$  be a holomorphic map parametrized by  $\sigma \in \Sigma \subset \mathbb{C}^n$ , and let  $0 < s < 1$ . There exists a universal constant  $K_3 > 1$  such that if, for any  $\sigma \in \Sigma$ ,

$$K_3 \max \left\{ \left\| \frac{\partial g}{\partial z} \right\|_r, \frac{\|g\|_r}{s} \right\} < 1, \quad (2.19)$$

then the map  $z \in S^d(r) \rightarrow z + g(z; \sigma)$  is one-to-one from  $S^d(r)$  onto  $S^d(r-s)$ . The inverse map can be written in the form

$$z' \in S^d(r-s) \rightarrow z = z' + h(z'; \sigma) \in S^d(r)$$

with  $z \rightarrow h(z; \sigma)$  holomorphic and  $\|h\|_{r-s} \leq \|g\|_r$ .

Regularity properties of  $h$  with respect to  $\sigma \in \Sigma$  are the same as for  $g$ .

Finally, if  $g$  is real on  $\mathbb{R}^d$  and periodic in each variable, so is  $h$ .

The last statement of the proposition means that the smooth map  $\varphi \in T^d \rightarrow \varphi + g(\varphi; \sigma)$  is globally inverted by  $\varphi' \in T^d \rightarrow \varphi' + h(\varphi'; \sigma)$ .

Thus, the proposition and estimate (2.17) show that we can fix  $K_1$  so that if condition (2.13) holds then the map  $\varphi \rightarrow \varphi'$  in (2.18) is globally inverted by

$$\varphi = (\varphi_1' + \varepsilon^{2^j} \Delta_j(\varphi'; a, \varepsilon), \varphi_2', \dots, \varphi_d') ; \quad a \in D_{j+1}$$

with  $(z, a) \rightarrow \Delta_j(z; a, \varepsilon)$  holomorphic in  $S^d(\varepsilon_{j+1}) \times D_{j+1}$  and

$$\|\Delta_j\|_{\varepsilon_{j+1}, n_{j+1}} \leq \|\phi_j\|_{\varepsilon_j - \delta_j, n_{j+1}} \quad (2.20)$$

Also, since  $\varepsilon^{2^j} \sup_{T^d} \left| \frac{\partial \phi_j}{\partial \varphi_1} \right| < 1$  because of (2.13), we have

$$A' = T_j^{-j} A, \quad T_j^{-1} = \begin{bmatrix} \frac{1}{1 + \varepsilon^{2^j} \frac{\partial \phi_j}{\partial \varphi_1}} & 0 & \dots & 0 \\ -\varepsilon^{2^j} \frac{\partial \phi_j}{\partial \varphi_2} & 1 & 0 & \dots & 0 \\ \frac{-\varepsilon^{2^j} \frac{\partial \phi_j}{\partial \varphi_2}}{1 + \varepsilon^{2^j} \frac{\partial \phi_j}{\partial \varphi_1}} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\varepsilon^{2^j} \frac{\partial \phi_j}{\partial \varphi_d} & 0 & \dots & 0 \\ \frac{-\varepsilon^{2^j} \frac{\partial \phi_j}{\partial \varphi_d}}{1 + \varepsilon^{2^j} \frac{\partial \phi_j}{\partial \varphi_1}} & 0 & \dots & 0 & 1 \end{bmatrix} \quad (2.21)$$

Notice that  $T_j^{-1}$  maps  $R_+ \times R^{d-1}$  onto itself.

At this point,  $H_j(A(A', \varphi'), \varphi(\varphi'))$  is readily computed and (2.15), (2.16), (2.17) and (2.20) easily imply (2.14).

The Inductive Lemma is proven.

### 2.4 Compatibility of Approximation Functions and Analyticity-Losses\*

Our next step will be to apply the Inductive Lemma infinitely many times so to end up with an integrable system for values of  $a$  belonging to

$$A^{(\infty)} = \bigcap_{j=0}^{\infty} A^{(j)}.$$

To do this, we have to look closer at the relation between  $\Omega$  and  $\delta_j$ , the up-to-now arbitrary quantities appearing in condition (2.13).

First, notice that, by definition of  $\xi_{j+1}$ ,

$$\xi_{\infty} \equiv \lim_{j \rightarrow \infty} \xi_j = \xi_0 - 2 \sum_{j=0}^{\infty} \delta_j,$$

so that it is natural to require

$$\sum_{j=0}^{\infty} \delta_j < \frac{\xi_0}{2}. \quad (2.22)$$

Also, to meet condition (2.13) for any  $j$ , we need  $\zeta(\delta_j) < \infty$  and since  $\delta_j \downarrow 0$ , this means that

$$\lim_{r \rightarrow \infty} \frac{\log_{\Omega}(r)}{r} = 0. \quad (2.23)$$

These observations motivate

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\*This paragraph is inspired by Rüssmann [1980].



Definition 1. A sequence  $\{\delta_j\}$  satisfying (2.22) will be called an analyticity-loss sequence.

A function  $\Omega$  satisfying

$$r^{d-2} \leq \Omega(r) \uparrow \infty \quad \text{and} \quad \frac{\log \Omega(r)}{r} \rightarrow 0 \quad (1 \leq r \uparrow \infty) \quad (2.23)'$$

will be called an approximation function\*.

Not all the approximation functions and analyticity-loss sequences will be suitable for our purpose: We will see that a necessary and sufficient condition for the application of our iterative scheme is that

$$\prod_{j=0}^{\infty} \zeta(\delta_j)^{\frac{1}{2^j}} < \infty . \quad (2.24)_I$$

Furthermore, in order to control the set  $A^{(\infty)}$ , the condition (2.24)<sub>I</sub> is not in general enough. We will use a stronger version of (2.23)':

$$\lim_{j \uparrow \infty} \frac{\log \Omega(2^j \delta_j^{-1})}{2^j} = 0 . \quad (2.24)_{II}$$

For these reasons we make the following

Definition 2. An approximation function and an analyticity-loss sequence are said to be compatible if conditions (2.24)<sub>I</sub> and (2.24)<sub>II</sub> hold.

---

\*The nomenclature is adapted from Rüssmann [1980]. The monotonicity in (2.23)' is assumed for simplicity.

Examples. 1)  $\Omega(r) = r^m$  ( $m \geq d-2$ ) and  $\{\delta_j\}$  are compatible if and only if

$$\sum \frac{1}{2^j} \log \delta_j^{-1} < \infty .$$

Moreover, one has

$$(2.24)_I \Leftrightarrow \sum \frac{1}{2^j} \log \delta_j^{-1} < \infty ; (2.24)_{II} \Leftrightarrow \frac{1}{2^j} \log \delta_j^{-1} \rightarrow 0 .$$

2)  $\Omega(r) = e^{\sqrt{r}}$  and  $\{\delta_j\}$  are compatible if and only if

$$\sum \frac{\delta_j^{-1}}{2^j} < \infty .$$

Moreover, one has

$$(2.24)_I \Leftrightarrow \sum \frac{\delta_j^{-1}}{2^j} < \infty ; (2.24)_{II} \Leftrightarrow \frac{\delta_j^{-1}}{2^j} \rightarrow 0 .$$

3) Let  $\sigma > 1$  and

$$\Omega(r) = \begin{cases} \exp\left(\frac{r}{\log^\sigma r}\right) , & r \geq e^\sigma \\ \Omega(e^\sigma) , & 1 \leq r \leq e^\sigma . \end{cases}$$

Then,  $\Omega$  and  $\{\delta_j\}$  are compatible if and only if

$$\frac{\delta_j^{-1}}{j^\sigma} \rightarrow 0 .$$

Moreover, one has

$$(2.24)_{\text{I}} \Leftrightarrow \frac{\delta_j^{-1}}{j^\sigma} \text{ bounded ; } (2.24)_{\text{II}} \Leftrightarrow \frac{\delta_j^{-1}}{j^\sigma} \rightarrow 0 .$$

Remark 1. In the first two examples, condition  $(2.24)_{\text{I}}$  is stronger than  $(2.24)_{\text{II}}$ , while in the third one, the opposite is true.

Remark 2. Rüssmann's aptitude is slightly different: He defines

$$\Psi \equiv \inf_{\prod_{j=0}^{\infty} \zeta(\delta_j)^{2^j}} \frac{1}{2^j} , \quad (2.24)'_{\text{I}}$$

where the infimum is taken over all the analyticity-loss sequences, and then shows that

$$\frac{\log \Omega(r)}{r} \rightarrow 0 , \quad \int \frac{\log \Omega(r)}{r^2} < \infty$$

imply  $\Psi < \infty$ .

The resemblance of  $(2.24)_{\text{I}}$  with  $(2.24)'_{\text{I}}$  is clear, but our condition  $(2.24)_{\text{II}}$ , needed to control  $A^{(\infty)}$ , doesn't appear in Rüssmann's work.

## 2.5 KAM Iteration Scheme

At this point, we have to check that the Hamiltonian (2.10) associated to the Schrödinger equation (2.7) satisfies the assumptions of the Inductive Lemma.

For this purpose, it is more convenient to regard (2.10) as parametrized by  $\alpha_0 = a$  rather than by  $E$ . Let\*

$$a \in \bigcup_{k=0}^{\infty} \left( \frac{k}{2} \omega_2, \frac{k+1}{2} \omega_2 \right) \rightarrow e_0(a) \in \sigma^*(L_0)$$

denote the inverse function of  $E \in \sigma^* \rightarrow \alpha_0(E)$  (compare Figure 3), and set

$$\omega \equiv (a, \omega_2, \dots, \omega_d), \quad F(\varphi; a) \equiv - \frac{Q^2(\varphi_1, \varphi_2; e_0(a))}{\kappa(e_0(a))} W(\varphi_3, \dots, \varphi_d).$$

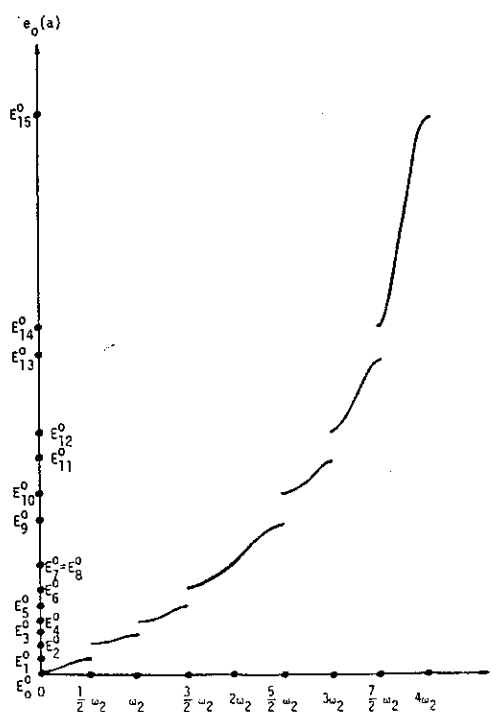


Figure 3.

\*Recall the notation from the first chapter. In particular,

$$\sigma^* \equiv \bigcup_{k=0}^{\infty} (E_{2k}^0, E_{2k+1}^0).$$

By §2.2,  $F$  as function of  $\varphi$  meets the requirements of the Inductive Lemma if we put  $\xi_0 \equiv \xi$ . As for the  $a$ -dependence, notice that  $E \rightarrow \alpha_0(E)$  has a natural holomorphic extension as soon as one stays away from  $\partial\sigma^*$ : Fix a point  $\tilde{E}_k$  in  $(E_{2k}^0, E_{2k+1}^0)$ . Then for  $E$  in any simply connected region containing  $E_k$ , the extension will be\*

$$\alpha_0(E) \equiv \alpha_0(\tilde{E}_k) + (-1)^{k+1} \frac{\omega_2}{2\pi} \int_{\tilde{E}_k}^E \frac{d\Delta}{\sqrt{1-\Delta^2}} dE' .$$

Thus, it is easy to see that for any

$$n < \frac{\omega_2}{8}$$

and for

$$A_0 \equiv \bigcup_{k=0}^{\infty} \{a \in \mathbb{R} : (\frac{k}{2} \omega_2 + 2n) \leq a \leq (\frac{k+1}{2} \omega_2 - 2n)\} ,$$

the function  $a \rightarrow F(\varphi; a)$  has a holomorphic extension to the region  $D(n; A_0)$ .

Finally, we set

$$\omega^{(0)} \equiv \omega, F^{(0)} \equiv F, \xi_0 \equiv \xi, \eta_0 \equiv \eta, A^{(0)} \equiv A_0, D_0 \equiv D(n; A_0)$$

and

$$M_0 \equiv \sup_{S^d(\xi_0) \times D_0} |F| .$$

---

\*Compare formula (1.4) of §1.2.

Now, imagine having applied the Inductive Lemma for  $j = 0, \dots, n$ . Repeating estimate (2.14)  $(n+1)$  times, one gets

$$\begin{aligned} \|F^{(n+1)}\|_{\xi_{n+1}, \eta_{n+1}} &\leq M_{n+1} \\ &\leq (K_2 c)^{(2^{n+1}-1)} [\zeta(\delta_n) \zeta(\delta_{n-1})^2 \dots \zeta(\delta_0)^{2n}] \\ &\quad \times [\delta_n^{-1} \delta_{n-1}^{-2} \dots \delta_0^{-2^n}] M_0^{2^{n+1}}. \end{aligned}$$

To apply the lemma one more time ( $j = n+1$ ), it will suffice by (2.13) to have

$$\frac{K_1}{K_2} (K_2 c M_0 \varepsilon)^{2^{n+1}} [(\zeta(\delta_{n+1}) \delta_{n+1}^{-d}) (\zeta(\delta_n) \delta_n^{-d})^2 \dots (\zeta(\delta_0) \delta_0^{-d})^{2^{n+1}}] \leq 1. \quad (2.25)$$

But, since  $\Omega(r) \geq r^{d-2}$ , one has

$$\delta_j^{-d} < \zeta(\delta_j)^{\frac{1}{2}},$$

whence, by (2.24)<sub>I</sub>,

$$\Psi \equiv \prod_{j=0}^{\infty} (\zeta(\delta_j) \delta_j^{-d})^{\frac{1}{2^j}} < \left( \prod_{j=0}^{\infty} \zeta(\delta_j)^{\frac{1}{2^j}} \right)^{\frac{3}{2}} < \infty.$$

Thus, (2.25) is implied by

$$\left[ \left( \frac{K_1}{K_2} \right)^{\frac{1}{2^{n+1}}} \Psi(K_2 c M_0 \varepsilon) \right]^{2^{n+1}} \leq 1.$$

We see that in order to apply the Inductive Lemma, an arbitrary number of times, we must have

$$K_1 \Psi C M_0 \varepsilon \leq 1$$

which may be expressed as

$$\frac{K_1}{K_2} \varepsilon \tau \leq 1, \quad \tau \equiv K_2 \Psi C M_0. \quad (2.26)$$

Analogously, one checks the estimate

$$C M_n \leq \frac{(\varepsilon \tau)^{2^n}}{K_2}.$$

Now, the following theorem is a simple matter.

Theorem. If  $\varepsilon$  satisfies condition (2.26) and

$$a \in A^{(\infty)} \equiv \bigcap_{j=0}^{\infty} A^{(j)},$$

then the Hamiltonian  $H^{(0)}$  is conjugate to the (non-resonant) system of harmonic oscillators

$$H^{(\infty)} \equiv \omega^{(\infty)} \cdot A,$$

where  $\omega^{(\infty)} \equiv (\omega_1^{(\infty)}(a, \varepsilon), \omega_2, \dots, \omega_d)$  verifies

$$c|\omega_1^{(\infty)} - a| \leq \frac{1}{K_2} \sum_{j=0}^{\infty} (\varepsilon\tau)^{2^j} \quad (2.27)$$

$$|\omega^{(\infty)} \cdot v| \geq \frac{1}{c\Omega(|v|)}, \quad v \in \mathbb{Z}^d - \{0\}.$$

The (surjective) canonical transformation conjugating  $H^{(0)}$  to  $H^{(\infty)}$  has the form

$$(A', \varphi') \in M \rightarrow (S(\varphi') A', \varphi'_1 + \varepsilon\Delta(\varphi'_1), \varphi'_2, \dots, \varphi'_d) \in M \quad (2.28)$$

with\*  $S$  a  $(dx)$ -matrix of the form

$$\begin{bmatrix} 1 + \varepsilon s_1 & 0 & \dots & 0 \\ \varepsilon s_2 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \varepsilon s_d & 0 & \dots & 0 & 1 \end{bmatrix}.$$

Moreover, the vector  $s \equiv (s_1, \dots, s_d)$  and  $\Delta$  have holomorphic extensions to  $S^d(\varepsilon_\infty)$  and\*\*

$$\max \{ \|s\|_{\varepsilon_\infty}, \|\Delta\|_{\varepsilon_\infty} \} \leq K_4 \tau \leq \frac{K_1}{K_2} \tau. \quad (2.29)$$

---

\* $S$  and  $\Delta$  depend on  $a \in A^{(\infty)}$  and  $\varepsilon$ .

\*\* $K_4$  is a universal constant.



Proof. Condition (2.26) enables us to apply the Inductive Lemma an arbitrary number of times. Now, let

$$g_j(z') \equiv (z'_1 + \varepsilon^{2^j} \Delta_j, z'_2, \dots, z'_d),$$

$$G_j \equiv g_0 \circ g_1 \circ \dots \circ g_j,$$

$$S_j \equiv T_0(g_0 \circ \dots \circ g_j) T_1(g_1 \circ \dots \circ g_j) \dots T_{j-1}(g_{j-1} \circ g_j) T_j(g_j).$$

We claim that for a  $\in A^{(\infty)}$ ,  $\lim_{j \rightarrow \infty} G_j \equiv G$  and  $\lim_{j \rightarrow \infty} S_j \equiv S$  exist and that such limits are uniform on compact sets of  $S^{(j)}(\varepsilon_\infty)$ . To prove this claim at this point is completely straightforward but not so short, for this reason we give the details in Appendix B.

But then, since  $(A', \varphi') \in M \rightarrow (S_j(\varphi') A', G_j(\varphi'))$  is a canonical map and since  $\lim_{j \rightarrow \infty} \frac{\partial S_j}{\partial \varphi} = \frac{\partial S}{\partial \varphi}$ , also the map  $(A', \varphi') \rightarrow (S(\varphi') A', G(\varphi'))$  is canonical and

$$\begin{aligned} H(S(\varphi') A', G(\varphi')) &= \lim_{j \rightarrow \infty} H(S_j(\varphi') A', G_j(\varphi')) \\ &= \lim_{j \rightarrow \infty} (\omega^{(j)} \cdot A' + \varepsilon^{2^j} A'_1 F^{(j)}(\varphi')) = \omega^{(\infty)} \cdot A' \equiv H^{(\infty)}(A'). \end{aligned}$$

For the last assertion of the theorem, see also Appendix B.

Remark 1. Since  $a \rightarrow \omega_1^{(n)}(a)$  is continuous on  $A^{(n)}$  (actually is holomorphic on  $D(\eta_{n-1}; A^{(n-1)}) \supset A^{(n-1)} \supset A^{(n)}$ ) the set  $A^{(n+1)}$  is closed, therefore also  $A^{(\infty)}$  is closed.

Remark 2. A system of harmonic oscillators with Hamiltonian

$$\sum_{i=1}^n \beta_i I_i \quad ,$$

$(I, \psi) \in \mathbb{R}^n \times \mathbb{T}^n$  denoting action-angle variables, is said to be resonant if the frequencies  $\beta = (\beta_1, \dots, \beta_n)$  are rationally dependent:

$$\beta \cdot v_0 = 0 \quad , \quad \text{for some } v_0 \in \mathbb{Z}^n - \{0\} \quad .$$

The word "resonant" comes from the fact that for such systems an arbitrarily small perturbation may produce non quasi-periodic motions\*.

For such reasons, we will call, sometimes,  $A^{(\infty)}$  a non-resonant set.

### 2.6 Structure of the Non-Resonant Set $A^{(\infty)}$

We cannot apply the KAM scheme for a in the set

$$R \equiv A^{(0)} - A^{(\infty)} = \bigcup_{j=0}^{\infty} \bigcup_{\substack{v \in \mathbb{Z}^d \\ 0 < |v| \leq N_j}} R_v^{(j)}$$

where for  $0 < |v| \leq N_j$

$$R_v^{(j)} \equiv \left\{ a \in A^{(j)} : |\omega^{(j)} \cdot v| < \frac{1}{c\Omega(|v|)} \right\} \quad .$$

---

\*See Gallavotti [1983], pg. 498.