EFFECTIVE HAMILTONIAN FOR THE D'ALEMBERT PLANETARY MODEL NEAR A SPIN/ORBIT RESONANCE*

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Abstract. The D'Alembert model for the spin/orbit problem in celestial mechanics is considered. Using a Hamiltonian formalism, it is shown that in a small neighborhood of a $p : q$ spin/orbit resonance with (p, q) different from (1, 1) and (2, 1) the 'effective' D'Alembert Hamiltonian is a completely integrable system with phase space foliated by maximal invariant curves; instead, in a small neighborhood of a $p : q$ spin/orbit resonance with (p, q) equal to (1, 1) or (2, 1) the 'effective' D'Alembert Hamiltonian has a phase portrait similar to that of the standard pendulum (elliptic and hyperbolic equilibria, separatrices, invariant curves of different homotopy). A fast averaging with respect to the 'mean anomaly' is also performed (by means of Nekhoroshev techniques) showing that, up to exponentially small terms, the resonant D'Alembert Hamiltonian is described by a two-degrees-of-freedom, properly degenerate Hamiltonian having the lowest order terms corresponding to the 'effective' Hamiltonian mentioned above.

Key words: Hamiltonian systems, D'Alembert model, spin/orbit resonances, fast averaging, Nekhoroshev normal forms, proper degeneracies, stability, effective Hamiltonian

1. Introduction

In this paper, we consider the Hamiltonian version of the D'Alembert model for the planetary spin/orbit problem. The model may be described as follows. Let a planet be modeled by a rotational ellipsoid slightly flattened along the symmetry axis (called 'north–south' direction); assume that the center of mass of such planet revolves on a slightly eccentric Keplerian ellipse around a *fixed* star occupying one of the foci of the ellipse: the planet is subject to the gravitational attraction of the star and the problem is to study the relative position of the planet and, most notably, the time evolution of its angular momentum. Such a model may be described using Hamiltonian formalism, as done, for example, in Chierchia and Gallavotti (1998) where action–angle symplectic variables are used (in §2, we recall, in a self-contained way, the main definitions and properties relative to such Hamiltonian formulation). The Hamiltonian system describing the D'Alembert model is a two-degrees-of-freedom system depending explicitly and periodically on time (the period being the year of the planet); furthermore, such a Hamiltonian system is

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nearly–integrable (with *two* smallness parameters: the flatness of the planet and the eccentricity of the Keplerian ellipse) and *properly degenerate.*[⋆]

In particular, we are interested in studying the D'Alembert model in the vicinity of a *spin/orbit resonance*, that is, in a phase space region where the period of revolution of the planet around the star (the 'year') and the period of the rotation of the planet around its spin axis (the 'day') are in a close-to-exact rational relation. If such rational relation is p/q (p and q positive, co-prime integers) we shall speak of a $p : q$ (spin/orbit) resonance.

The degeneracy of the system implies that the time variable (better: the angle–variable corresponding to time, that is, the so-called 'mean anomaly') may be considered a *fast variable* with respect to the two ('symplectic') angles describing the relative position of the planet. Thus, the explicit dependence of the system upon time may be *averaged out*. In more precise mathematical terms, one can use Nekhoroshev theory (Nekhoroshev, 1977) to show that the D'Alembert resonant Hamiltonian is equivalent, *up to an exponentially small term*, to a two-degrees-of-freedom properly degenerate Hamiltonian. Here, the main 'smallness parameter' will be the flatness of the planet and the eccentricity will be taken to be a power of the flatness: if ε measures the flatness and μ measures the eccentricity, we shall take $\mu = \varepsilon^c$ with $c > 0$.

Properly degenerate systems have three 'intrinsic' scales: a scale of order one describing the typical time scale of the unperturbed system; an 'intermediate' scale of order, say, $\varepsilon \ll 1$ describing the 'effective' effects of the perturbation; and 'higher order terms.' Such 'higher order terms' are measured by a power ε^a with $a > 1$ (and are not to be confused with the exponentially, in $1/\varepsilon$, small term mentioned above).

Now, it turns out that *near a* $p : q$ *spin/orbit resonance with* (p, q) *different from* $(1, 1)$ *and* $(2, 1)$*, the intermediate system (and hence the 'effective' Hamiltonian) is independent of any angle variable*: thus the integrable system obtained dropping (besides the exponential remainder) the higher order term is a completely integrable system with phase space entirely foliated by (maximal) invariant curves. On the other hand, *near a* $p : q$ *spin/orbit resonance with* (p, q) *equal to* $(1, 1)$ *or* $(2, 1)$*, the intermediate system (and hence the 'effective' Hamiltonian) does depend on one (and only one) angle variable*: in such 'exceptional' case, the system obtained by dropping the higher order terms is still integrable (being, effectively, a one-degree-of-freedom system) but its phase space presents a structure similar to that of a standard pendulum (i.e. elliptic and hyperbolic equilibria, separatrices, invariant curves of different homotopy).

Thus, the effective Hamiltonian associated to the 1:1 or 2:1 spin–orbit resonance exhibits instability phase–space zones that are not present in the general

[⋆]Roughly speaking, 'properly degenerate' means that in the integrable limit (i.e. when the perturbative parameters are set to zero) the Hamiltonian does not depend on the action–variables in a 'general' way.

case, a phenomenon which may be, perhaps, exploited in the understanding of the exceptional role played by such resonances in our solar system and in its evolution.

We also mention that such peculiarity of the 1:1 and 2:1 spin/orbit resonance is *intrinsic* in the model and does not depend upon the particular variables used.

We close this brief introduction by mentioning that the above analysis is the starting point to prove analytically (using, also, Nekhoroshev techniques) the *exponential stability* of the angular momentum (and hence its 'adiabatic invariance') of a planet in a neighborhood of a spin/orbit resonance (Biasco and Chierchia, to appear); the peculiarity of the case 1:1 and 2:1 makes, however, the proofs in such two cases much more involved.

2. The D'Alembert Hamiltonian Planetary Model

In this section, we revisit briefly the Hamiltonian version of the planetary D'Alembert model as presented, for example, in (Chierchia and Gallavotti, 1998).

Consider an oblate planet $\mathcal{P}$ of mass $m_{\mathcal{P}}$ modeled by a rotational ellipsoid slightly flattened along the symmetry axis ('north–south axis'); assume that its center of mass revolves on a *Keplerian orbit* (of small eccentricity) around a *fixed star* of mass $m_{\mathcal{S}}$ occupying one of the foci of the ellipse.* The problem is then to study the *relative motion* of the planet and, in particular, the motion (and stability) of its *angular momentum*.

We start by writing down the Lagrangian of this system. Let (i, j, k) be an orthogonal fixed basis with 'origin' on the star so that: i is the unit versor pointing towards the aphelion (or, equivalently, pointing towards the other focus of the ellipse); j is the versor in the ecliptic plane (i.e. the plane containing the Keplerian ellipse) orthogonal to i oriented according to the motion of the center of mass of the planet (i.e. if $x_{\mathcal{P}}(t)$ denotes the position at time t of the center of mass of $\mathcal{P}$ and if $x_{\mathcal{P}}$ passes at time t_0 at the aphelion, then** $j \cdot \dot{x}_{\mathcal{P}}(t_0) > 0$); $k = i \times j$ is the unit normal to ecliptic plane.*** Let (i_1, i_2, i_3) be a co-moving frame with 'origin' in the center of mass of the planet so that: i_1 and i_2 determine the 'equatorial plane' and i_3 points towards the 'north pole'‡; clearly i_1 and i_2 may be interchanged; for simplicity, *we shall assume that* i_3 *is never orthogonal nor parallel to the ecliptic plane* and that the basis (i_1, i_2, i_3) is such that

$$0 < i_3 \cdot k < 1. \tag{1}$$

*In other words, we assume that the motion of the star is not influenced by the form of the planet.

**$a \cdot b$ denotes the standard inner product in $\mathbf{R}^n$ (here $n = 3$); and $\dot{a}$ denotes the time derivative of a.

***Here, '$\times$' denotes the standard 'vector' (or 'external') skew-symmetric product in $\mathbf{R}^3$. Informally, an observer 'standing' on the ecliptic in the position identified by k would see the center of mass of $\mathcal{P}$ revolve 'counter-clockwise'.

‡Recall that we are assuming the the planet is a rotational ellipsoid; thus the 'equatorial plane' is the plane identified by the maximal circle of the ellipsoid and the 'north-south' axis is the line orthogonal to the equatorial plane.

Let, $\theta := (\theta_1, \theta_2, \theta_3)$ denote the Euler angles of the planet, namely, if n denotes a unit vector identifying the *equatorial node* on the ecliptic (i.e. the line obtained as intersection between the ecliptic plane and the equatorial plane), then

$$\theta_1 = \text{angle } (i, n), \qquad \theta_2 = \text{angle } (i_3, k), \qquad \theta_3 = \text{angle}(n, i_1). \tag{2}$$

Then, if $I_1 = I_2$ and I_3 denotes the inertia moments of the planet, γ denotes the gravitational constant, $x_{\mathcal{P}}(t)$ denotes, as above, the position at time t of the center of mass of $\mathcal{P}$ and $\mathcal{P}(t)$ denotes the space region occupied at time t by the planet, then *the Lagrangian describing the above model is given by*

$$\begin{aligned}\mathcal{L} \;=\; & \frac{1}{2} I_3(\dot\theta_1 \cos\theta_2 + \dot\theta_3)^2 + \frac{1}{2} I_1(\dot\theta_2^2 + \dot\theta_1^2 \sin^2\theta_2) + \\ & + \gamma\, \frac{m_{\mathcal{P}} m_{\mathcal{S}}}{Vol \mathcal{P}} \int_{\mathcal{P}(t)} \frac{\mathrm{d}x}{|x_{\mathcal{P}}(t) + x|}. \end{aligned} \tag{3}$$

Thanks to a well known result by Andoyer and Deprit (see, e.g. Gallavotti, 1983; Arnold, 1988), *the Legendre transform of $\mathcal{L}$ is equivalent, in suitable physical units, to the following Hamiltonian function*⋆

$$\begin{aligned} H_{\varepsilon,\mu}(J, \psi) \;:=\; & \frac{(\bar J_1 + J_1)^2}{2} + \bar\omega(J_3 - J_2) + \\ & + \varepsilon F_0(J_1, J_2, \psi_1, \psi_2) + \varepsilon\mu F_1(J_1, J_2, \psi_1, \psi_2, \psi_3; \mu), \end{aligned} \tag{4}$$

where:

(a) $\bar J_1$ is constant parameter, which may be interpreted as a 'reference datum' in a neighborhood of which the system will be studied;

(b) ε and μ are two *small* non-negative parameters measuring, respectively, the flatness of the planet and the eccentricity of the Keplerian orbit described by the center of mass of the planet;

(c) $(J, \psi) := (J_1, J_2, J_3, \psi_1, \psi_2, \psi_3) \in A \times \mathbf{T}^3$ are standard symplectic coordinates⋆⋆; the domain $A \subset \mathbf{R}^3$ is given by

$$A := \{|J_1| < d\ , \quad |J_2 - \bar J_2| < d\ , \quad J_3 \in \mathbf{R}\}, \tag{5}$$

where d is a suitable fixed (and small) positive number while $\bar J_2$ is fixed 'reference datum' (verifying, together with $\bar J_1$, certain assumptions spelled out below);

(d) $2\pi/\bar\omega$ is the period of the Keplerian motion ('year of the planet');

(e) the function F_0 is a trigonometric polynomial given by

$$F_0 = \sum_{\substack{j \in \mathbf{Z} \\ |j| \leqslant 2}} c_j \cos(j\psi_1) + d_j \cos(j\psi_1 + 2\psi_2),$$

where c_j and d_j are functions of $(\bar J_1 + J_1, J_2)$ listed in the following item;

⋆See (Chierchia and Gallavotti, 1998).

⋆⋆The symbol $\mathbf{T}^n$ denotes the standard n-dimensional flat torus $\mathbf{R}^n/(2\pi\mathbf{Z}^n)$.

(f) let

$$\kappa_1 := \kappa_1(J_1) := \frac{L}{\bar{J}_1 + J_1}, \qquad \kappa_2 := \kappa_2(J_1, J_2) := \frac{J_2}{\bar{J}_1 + J_1},$$
$$\nu_1 := \nu_1(J_1) := \sqrt{1 - \kappa_1^2}, \qquad \nu_2 := \nu_2(J_1, J_2) := \sqrt{1 - \kappa_2^2};$$

where L is a real parameter; the parameters $\bar{J}_i$, L and the constant d are assumed to satisfy

$$L + d < \bar{J}_1, \qquad |\bar{J}_2| + 2d < \bar{J}_1; \tag{6}$$

in this way $0 < \kappa_i < 1$ (and the ν_i's are well defined on the domain A). Then, the functions c_j and d_j are defined by

$$\begin{aligned}
c_0(J_1, J_2) &:= \tfrac{1}{4}(2\kappa_1^2\nu_2^2 + \nu_1^2(1 + \kappa_2^2)),\\
d_0(J_1, J_2) &:= -\frac{\nu_2^2}{4}(2\kappa_1^2 - \nu_1^2),\\
c_{\pm 1}(J_1, J_2) &:= \frac{\kappa_1\kappa_2\nu_1\nu_2}{2},\\
d_{\pm 1}(J_1, J_2) &:= \mp\frac{(1 \pm \kappa_2)\kappa_1\nu_1\nu_2}{2},\\
c_{\pm 2}(J_1, J_2) &:= -\frac{\nu_1^2\nu_2^2}{8},\\
d_{\pm 2}(J_1, J_2) &:= -\frac{\nu_1^2(1 \pm \kappa_2)^2}{8}.
\end{aligned} \tag{7}$$

(g) the function F_1 is a convergent series in μ of trigonometric polynomials (with increasing degrees); for example $F_1|_{\mu=0} := F_1^0$ is given by

$$\begin{aligned}
F_1^0 = \sum_{\substack{j\in\mathbf{Z}\\ |j|\leqslant 2}} &(-3)c_j \cos(j\psi_1 + \psi_3) + \\
&+ \frac{d_j}{2}\{\cos(j\psi_1 + 2\psi_2 + \psi_3) - 7\cos(j\psi_1 + 2\psi_2 - \psi_3)\}.
\end{aligned}$$

Remark 1. (i) Since J_3 appears only linearly with coefficient $\bar{\omega}$, the angle ψ_3 corresponds to time t and $H_{\varepsilon,\mu}$ is actually a two-degrees-of-freedom Hamiltonian depending explicitly on time in a periodic way (with period $2\pi/\bar{\omega}$).

(ii) The *physical interpretation* of the action–variables J_1, J_2, the parameter L and the angles ψ_i, which are closely related to (but do not coincide with) the Andoyer canonical variables, is the following. *In suitable physical units*, the variable $\bar{J}_1 + J_1$ corresponds to the absolute value of the angular momentum of the planet; the variable J_2 corresponds to the value of the projection of the angular momentum of the planet onto the direction k orthogonal to the *ecliptic* plane and L corresponds to the value of the projection of the angular momentum of the planet in the direction i_3 of the polar axis of the planet (and, because of the symmetry of the planet, is a constant of the motion). In formulae, if K_P denotes the angular momentum of the planet, then:

$$\bar{J}_1 + J_1 = |K_P|, \qquad J_2 = K_P \cdot k, \qquad L = K_P \cdot i_3 = \text{ const.} \tag{8}$$

To describe the angles ψ_i let us introduce two more relevant 'nodes': let m be a versor in the direction of the line of intersection ('node') of the ecliptic plane with the 'angular momentum plane' (i.e. the plane orthogonal to the angular momentum of the planet); let, also, n_0 be a versor in the direction of the line of intersection ('node') of the equatorial plane with the angular momentum plane. Then: ψ_3 is the so-called 'mean anomaly' and is proportional to time, as seen above; ψ_1 is the angle between the nodes m and n_0; ψ_2 is the *difference* between the angle between m and i and ψ_3. In formulae:

$$\begin{aligned} \psi_3 &= \text{ const. } + \bar{\omega}t, \qquad \psi_1 = \text{ angle } (m, n_0), \\ \psi_2 &= \text{ angle } (m, i) - \psi_3. \end{aligned} \tag{9}$$

(iii) Under our assumptions (i.e. that $0 < d \ll 1$), the *over the angles of* $H_{\varepsilon,0}$ is given by

$$\frac{(\bar{J}_1 + J_1)^2}{2} + \bar{\omega}(J_3 - J_2) + \varepsilon\frac{1}{4}\left\{(2 - \bar{\nu}_1^2) - (2 - 3\bar{\nu}_1^2)\frac{J_2^2}{\bar{J}_1^2} + O(d)\right\}, \tag{10}$$

where $\bar{\nu}_1 := \nu_1(0) = \sqrt{1 - (L/\bar{J}_1)^2}$. The number $\bar{\nu}_1$ is the so-called *Euler nutation constant*. By (ii) we see that $\bar{\nu}_1 \ll 1$ corresponds to rotations of the planet with spin axis nearly parallel to the polar axis (a case common, e.g., in the solar system). In such a case the *average over the angles of* $H_{\varepsilon,0}$ *is not a convex function of the action variables* (J_1, J_2). This lack of convexity is quite a common feature in celestial mechanics and is exhibited, for example, also in three-body problems.

We are interested in studying the above system in a neighborhood of a day/year (or 'spin/orbit') resonance. Since the daily rotation is measured by the angle ψ_1 and since in the unperturbed situation ($\varepsilon = 0$ and $J_1 = 0$) $\psi_1 = \psi_1^0 + \bar{J}_1 t$, we see that *an approximate day/year resonance corresponds to take the 'reference datum'* $\bar{J}_1$ *(which, in our units, coincides with the daily frequency) in a rational relation with the year frequency* $\bar{\omega}$, that is, $\bar{J}_1 = \frac{p}{q}\bar{\omega}$ with p and q co-prime positive integers; we shall speak in such a case of a '$p : q$ spin/orbit-resonance'.

Setting

$$\bar{J}_1 := \frac{p}{q}\bar{\omega}, \qquad \omega := \frac{\bar{\omega}}{q}, \tag{11}$$

we see that *the dynamics near a* $p : q$ *spin/orbit resonance is described by the Hamiltonian*

$$\begin{aligned} H_{\varepsilon,\mu}(J,\psi) \;:=\; & \frac{J_1^2}{2} + \omega(pJ_1 - qJ_2 + qJ_3) + \\ & + \varepsilon F_0(J_1, J_2, \psi_1, \psi_2) + \\ & + \varepsilon\mu F_1(J_1, J_2, \psi_1, \psi_2, \psi_3; \mu), \end{aligned} \tag{12}$$

(where we have omitted the constant term $\bar{J}_1^2/2$).

Finally, to make the analysis perturbative, we shall take as action–variable domain an ε-dependent subset of A:

(h) the domain of definition A introduced in item (c) above will, from here on, be replaced by its subset

$$A_\varepsilon := \{|J_1| < d\varepsilon^\ell \,, \quad |J_2 - \bar{J}_2| < d \,, \quad J_3 \in \mathbf{R}\}, \tag{13}$$

where $0 \leqslant \ell < 1$, $(0 < \varepsilon < 1)$.

The Hamiltonian $H_{\varepsilon,\mu}$ in (12) *will be called the 'resonant D'Alembert Hamiltonian' and, in the rest of this paper, we shall consider only the resonant D'Alembert Hamiltonian defined on the domain $A_\varepsilon \times \mathbf{T}^3$.*

3. Linear Analysis and the Effective Hamiltonian

The appearance of the linear combination $(pJ_1 - qJ_2 + qJ_3)$ in the D'Alembert Hamiltonian $H_{\varepsilon,\mu}$ suggests to look for a linear symplectic (ε-independent) change of variables casting $H_{\varepsilon,\mu}$ in a simpler and more informative form. Calling

$$\Phi_L\colon (I,\varphi) \to (J,\psi) = \Phi_L(I,\varphi) \tag{14}$$

such a linear change of variables, it is quite natural to set*

$$I_3 := pJ_1 - qJ_2 + qJ_3. \tag{15}$$

Besides (15) we shall also require the following condition, which needs a little explanation (given below):

$$\int_0^{2\pi} F_0 \circ \Phi_L(I,\varphi) \frac{\mathrm{d}\varphi_3}{2\pi} \;=\; \text{function depending on } I \text{ and at most on one angle.} \tag{16}$$

The idea beyond these conditions is the following. The unperturbed frequencies of the transformed Hamiltonian (i.e. $\nabla_I H_{0,0} \circ \Phi_L$) are given by**

$$J_1(I)\, \nabla_I J_1(I) + (0,0,\omega) = (0,0,\omega) + O(\varepsilon^\ell).$$

*Clearly, the choice of the index 3 is arbitrary.

**Recall that in our domain A_ε, (13), J_1 has been taken of order ε^ℓ. The precise quantitative analysis will be described in the next section, where we will also assume that $\mu \leqslant \varepsilon^c$ for some $c > 0$.

This implies that φ_1 and φ_2 are 'slow' angles, while φ_3 is a 'fast' angle so that φ_3 'averages out' (see next section for a precise mathematical statement) leaving an 'effective Hamiltonian' given by

$$\begin{aligned} H_{\text{eff}} &:= \int_0^{2\pi} H_{\varepsilon,0} \circ \Phi_L(I,\varphi) \frac{\mathrm{d}\varphi_3}{2\pi} \\ &= \frac{J_1(I)^2}{2} + \omega I_3 + \varepsilon \int_0^{2\pi} F_0 \circ \Phi_L(I,\varphi) \frac{\mathrm{d}\varphi_3}{2\pi}, \end{aligned} \tag{17}$$

which, in view of (16), is a one-degree-of-freedom Hamiltonian (and hence integrable): H_{eff} depends, possibly, on all the actions I_i but, because of (16), *on at most one angle* (φ_1 or φ_2). In the case H_{eff} depends explicitly on one angle, say φ_1, then the actions I_2 and I_3 are just parameters for the dynamics generated by H_{eff}.

The rest of this section is devoted to find linear symplectic diffeomorphisms, Φ_L, of $A \times \mathbf{T}^3$ satisfying (16) and the upshot will be that *if $p : q$ is different from* 1:1 *or* 2:1, *then H_{eff} depends only on the action variables while in the other cases H_{eff} depends explicitly on one angle also*: in the first case the phase portrait of the integrable system associated to H_{eff} is entirely foliated by (homotopically non trivial) invariant curves while in the latter case there are also hyperbolic equilibria, separatrices and curves with different topology (exactly as in the phase portrait of the standard pendulum).

The linear symplectic diffeomorphism Φ_L has a generating function given, up to an arbitrary (and meaningless) plus or minus sign, by*

$$\begin{aligned} &S(J,\varphi) := MJ \cdot \varphi\,, \quad \text{with} \quad M \in SL(3,\mathbf{Z}), \\ &J = M^{-1} I, \qquad \psi = M^T \varphi. \end{aligned} \tag{18}$$

The relation (15) means that M has the form

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ p & -q & q \end{pmatrix}, \tag{19}$$

with integers $a, \ldots, f$ to be determined. Thus, by (18) and (19), we have that

$$\begin{aligned} \psi_1 &= a\varphi_1 + d\varphi_2 + p\varphi_3, \qquad \psi_2 = b\varphi_1 + e\varphi_2 - q\varphi_3, \\ \psi_3 &= c\varphi_1 + f\varphi_2 + q\varphi_3. \end{aligned} \tag{20}$$

* $SL(3, \mathbf{Z})$ denotes the group of real (3×3) matrices with integer entries and determinant one; the superscript T denotes matrix transposition.

By (e) above and (20), we find

$$\begin{aligned}
&\int_0^{2\pi} F_0(J_1, J_2, \psi(\varphi)) \frac{\mathrm{d}\varphi_3}{2\pi} \\
&\quad = \int_0^{2\pi} \Bigg[\sum_{|j| \leqslant 2} c_j \cos j(a\varphi_1 + d\varphi_2 + p\varphi_3) + \\
&\qquad + d_j \cos((aj + 2b)\varphi_1 + (dj + 2e)\varphi_2 + (pj - 2q)\varphi_3) \Bigg] \frac{\mathrm{d}\varphi_3}{2\pi} \\
&\quad = c_0 + \sum_{\substack{|j| \leqslant 2 \\ pj = 2q}} d_j \cos((aj + 2b)\varphi_1 + (dj + 2e)\varphi_2). \qquad (21)
\end{aligned}$$

If (p, q) *is different from* $(1, 1)$ *and* $(2, 1)$ *we see that there are no integers* j *with* $|j| \leqslant 2$ *such that* $pj = 2q$*, so that, in this case, the sum in the last line of* (21) *is absent and we have that* H_{eff} *depends only on the action variables and is given by*

$$H_{\text{eff}}(J(I), \psi(\varphi)) = \frac{J_1(I)^2}{2} + \omega I_3 + \varepsilon c_0(J_1(I), J_2(I)). \qquad (22)$$

Next we show that *when* (p, q) *is equal to* $(1, 1)$ *or* $(2, 1)$*, then* H_{eff} *cannot be as in* (22) *and it must depend explicitly on one angle* (φ_1 or φ_2).

Let us consider first the case $(p, q) = (1, 1)$. In this case, $pj = 2q$ means $j = 2$ and (21) implies that

$$\int_0^{2\pi} F_0(J_1, J_2, \psi(\varphi)) \frac{\mathrm{d}\varphi_3}{2\pi} = c_0 + d_2 \cos(2(a + b)\varphi_1 + 2(d + e)\varphi_2). \qquad (23)$$

Thus, H_{eff} independent on angles means

$$a + b = 0 = d + e,$$

a relation which makes the first two columns of the matrix M proportional (one is the opposite of the other) and this implies that the determinant of M would vanish. The case $(p, q) = (2, 1)$ is similar: $pj = 2q$ means $j = 1$ and (21) implies that

$$\int_0^{2\pi} F_0(J_1, J_2, \psi(\varphi)) \frac{\mathrm{d}\varphi_3}{2\pi} = c_0 + d_1 \cos((a + 2b)\varphi_1 + (d + 2e)\varphi_2). \qquad (24)$$

Thus, H_{eff} independent on angles means

$$a + 2b = 0 = d + 2e,$$

a relation which, as above, makes the first two columns of the matrix M be one the opposite of the other, implying, again, the vanishing of the determinant of M.

Remark 2. In what follows we shall make particular (and ‘convenient’) choices for the matrix M (and hence for the symplectic transformation Φ_L), but one should bear in mind that in doing this there is quite a bit of freedom but that the physical relevant quantities (such as $H_{\rm eff}$) are essentially intrinsic.

In the case $p = 1, 2$ and $q = 1$, by the above analysis, we see that (16) is satisfied *provided*

$$\text{either} \quad a + pb = 0 \quad \text{or} \quad d + pe = 0.$$

We then take $d = 0 = e$ and*

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ p & -1 & 1, \end{pmatrix}, \qquad (p = 1, 2), \tag{25}$$

leading to the linear symplectic transformation

$$\Phi_L\colon (I, \varphi) \to \begin{cases} J = (I_1, pI_1 + I_2 - I_3, I_2), \\ \psi = (\varphi_1 + p\varphi_3, -\varphi_3, \varphi_2 + \varphi_3), \end{cases} \quad (p = 1, 2). \tag{26}$$

In the new coordinates the Hamiltonian becomes

$$\begin{aligned} H_{\varepsilon,\mu} \circ \Phi_L &= \frac{I_1^2}{2} + \omega I_3 + \varepsilon F_0(I_1, pI_1 + I_2 - I_3, \varphi_1 + p\varphi_3, -\varphi_3) + \\ &\quad + \varepsilon\mu F_1(I_1, pI_1 + I_2 - I_3, \varphi_1 + p\varphi_3, -\varphi_3, \varphi_2 + \varphi_3; \mu) \\ &:= H_{00}(I_1, I_3) + \varepsilon G_0(I, \varphi) + \varepsilon\mu G_1(I, \varphi; \mu); \end{aligned} \tag{27}$$

and the averaged resonant D’Alembert Hamiltonian is (recall (23) and (24))

$$\begin{aligned} H_{\rm eff}(I, \varphi_1; \varepsilon) &:= \int_0^{2\pi} H_{\varepsilon,0} \circ \Phi_L(I, \varphi) \frac{d\varphi_3}{2\pi} \\ &= \frac{I_1^2}{2} + \omega I_3 + \\ &\quad + \varepsilon\{c_0(I_1, pI_1 + I_2 - I_3) + \\ &\quad + d_{j_p}(I_1, pI_1 + I_2 - I_3)\cos(j_p\varphi_1)\} \\ &:= H_{00}(I_1, I_3) + \varepsilon H_{01}(I, \varphi_1), \end{aligned} \tag{28}$$

where $j_1 := 2$ and $j_2 := 1$.

Let us turn to the case in which (p, q) is different from $(1, 1)$ and $(2, 1)$. In this case, as discussed above, (16) is always satisfied and $H_{\rm eff}$ does not depend on angles. To make a particular choice, let a and b be integers such that

$$aq + bp = 1. \tag{29}$$

*In (Chierchia and Gallavotti, 1998), where it is studied the resonant D’Alembert Hamiltonian when $(p, q) = (2, 1)$, it is taken $\begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ p & -1 & 1 \end{pmatrix}$.

In view of an elementary algebraic identity,* such (infinitely many) integers always exist and we shall fix the ones that minimize the sum $|a| + |b|$. Then, we define

$$M = \begin{pmatrix} a & b & -b \\ 0 & 1 & 0 \\ p & -q & q \end{pmatrix}, \qquad (p, q) \neq (1, 1), (2, 1), \tag{30}$$

leading to the linear symplectic transformation

$$\Phi_L\colon (I, \varphi) \to \begin{cases} J = (qI_1 + bI_3, I_2, -pI_1 + I_2 + aI_3) \\ \psi = (q\varphi_1 - p\varphi_3, \varphi_2 + \varphi_3, b\varphi_1 + a\varphi_3). \end{cases} \tag{31}$$

In the new coordinates the resonant D'Alembert Hamiltonian becomes

$$\begin{aligned} H_{\varepsilon,\mu} \circ \Phi_L &= \frac{(qI_1 + bI_3)^2}{2} + \omega I_3 + \\ &\quad + \varepsilon F_0(qI_1 + bI_3, I_2, q\varphi_1 - p\varphi_3, \varphi_2 + \varphi_3) + \\ &\quad + \varepsilon\mu F_1(qI_1 + bI_3, I_2, q\varphi_1 - p\varphi_3, \varphi_2 + \varphi_3, b\varphi_1 + a\varphi_3; \mu) \\ &:= H_{00}(I_1, I_3) + \varepsilon G_0(I, \varphi) + \varepsilon\mu G_1(I, \varphi; \mu), \end{aligned} \tag{32}$$

and, in this case, the averaged resonant D'Alembert Hamiltonian is simply

$$\begin{aligned} H_{\text{eff}}(I; \varepsilon) &:= \int_0^{2\pi} H_{\varepsilon,0} \circ \Phi_L(I, \varphi) \frac{\mathrm{d}\varphi_3}{2\pi} \\ &= \frac{(qI_1 + bI_3)^2}{2} + \omega I_3 + \varepsilon c_0(qI_1 + bI_3, I_2) \\ &:= H_{00}(I_1, I_3) + \varepsilon H_{01}(I). \end{aligned} \tag{33}$$

Remark 3. Notice that we are using a unified notation for different objects (such as G_i or H_{00} or H_{01}), which, in fact, depend explicitly on the resonance (p, q).

4. Fast Averaging

In this section, we shall average out the φ_3 dependence in the (transformed) D'Alembert Hamiltonian (27) or (32), showing that *the D'Alembert model in a neighborhood of a spin/orbit resonance is equivalent, up to an exponentially small term, to a two-degrees-of-freedom Hamiltonian system; furthermore, such reduced system is a properly–degenerate Hamiltonian system,*** *whose 'intermediate part' corresponds to* H_{01} *in* (28) *or* (33).

The mathematical technical tool we shall use in order to carry over the averaging is a 'normal form lemma' taken from Nekhoroshev theory. The following

*The so-called 'Bezout identity,' which is an immediate consequence of the Euclidean algorithm.

**That is, when $\varepsilon = 0$ the Hamiltonian depends only on one action variable.

formulation, apart from minor technical points, is taken from Pöschel (1993, p. 192).

In the following we shall use the following notations: if $A \subset \mathbf{R}^d$ and $r > 0$, we denote by A_r the subset of points in $\mathbf{C}^d$ at distance less than r from A; $\mathbf{T}^d_s$ denotes the complex set $\{z \in \mathbf{C}^d\colon\ |\mathrm{Im} z_j| < s \text{ for all } j\}$ (thought of as a complex neighborhood of $\mathbf{T}^d$). If $f(I,\varphi)$ is a real analytic function on $A_r \times \mathbf{T}^d_s$ we let $\|f\|_{r,s}$ denote the following norm*

$$\|f\|_{r,s} := \sum_{k\in\mathbf{Z}^d} \sup_{I\in A_r} |f_k(I)| e^{|k|s}, \tag{34}$$

$f_k(I)$ being the Fourier coefficients of the periodic function $\varphi \to f(I,\varphi)$.

LEMMA 1. *Let n_1 and n_2 be two non-negative integers such that $n_1 + n_2 = 3$ and let $D \subset \mathbf{R}^{n_1}$ and $D' \subset \mathbf{R}^{n_2}$. Consider a Hamiltonian $H(I,\varphi) := h(I) + f(I,\varphi)$ real-analytic on $W_{r_1,r_2,s} := (D_{r_1} \times D'_{r_2}) \times \mathbf{T}^3_s$ for some $r_2 \geqslant r_1 > 0$ and $s > 0$. Assume that there exist $K \geqslant 6/s$ and $\alpha > 0$ such that*

$$\begin{aligned} &|\omega(I)\cdot k| \geqslant \alpha, \qquad \forall\, k \in \mathbf{Z}^3,\\ &\quad |k| \leqslant K, \qquad k_3 \neq 0, \qquad \forall\, I \in D_{r_1} \times D'_{r_2}, \end{aligned} \tag{35}$$

where $\omega(I) := \nabla h(I)$. Assume also that

$$\|f\|_{r_1,r_2,s} \leqslant \frac{\alpha r_1}{2^8 K}. \tag{36}$$

Then, there exists a real-analytic symplectic transformation

$$\Phi\colon (\hat I, \hat\varphi) \in W_{r_1/2, r_2/2, s/6} \to (I,\varphi) = \Phi(\hat I, \hat\varphi) \in W_{r_1,r_2,s}$$

such that

$$H \circ \Phi(\hat I, \hat\varphi) = h(\hat I) + g(\hat I, \hat\varphi_1, \hat\varphi_2) + f_*(\hat I, \hat\varphi) \tag{37}$$

with

$$\begin{aligned} &\Big\|g - \frac{1}{2\pi}\int_0^{2\pi} f_K(\hat I, \hat\varphi)\,\mathrm{d}\hat\varphi_3\Big\|_{r_1/2,r_2/2,s/6}\\ &\quad \leqslant \frac{2^{11}}{\alpha r_1 s}\,(\|f\|_{r_1,r_2,s})^2 \leqslant \frac{1}{4}\|f\|_{r_1,r_2,s},\\ &\|f_*\|_{r_1/2,r_2/2,s/6} \leqslant \|f\|_{r_1,r_2,s}\ \exp\left(-\frac{Ks}{6}\right),\\ &\|\Phi(\hat I,\hat\varphi) - (\hat I,\hat\varphi)\|_{r_1/2,r_2/2,s/6} \leqslant \hat c\ \|f\|_{r_1,r_2,s}, \end{aligned} \tag{38}$$

where $f_K(\hat I,\hat\varphi) := \sum_{|k|\leqslant K} \hat f_k(\hat I)\exp(ik\cdot\hat\varphi)$ and $\hat c > 0$ is a suitable constant.

*The specific choice of norm will play no role in the sequel; obviously if f is a real-analytic function on $\mathbf{T}^d_s$, $\|f\|_s$ stands for $\sum_{k\in\mathbf{Z}^d} |f_k| e^{|k|s}$, f_k being the Fourier coefficients of f, while, if f is a real-analytic function on A_r, then $\|f\|_r = \sup_{I\in A_r} |f(I)|$.

Remark 4. In the formulation presented here (and used in forthcoming papers by the authors), one allows for *different analyticity radii*. In fact, in the above D'Alembert problem, as well as other problems arising in celestial mechanics, there appears quite naturally different scales in the action space and it is convenient, for a quantitative analysis, not to mix such different scales. The modifications of the proof in (Pöschel, 1993) in order to get the more general statement in Lemma 1 are routine (notice, in particular, that in the 'smallness condition' (36) there appears the *smallest* radius). One more technical comment: in (Pöschel, 1993) there appears the condition $r \leqslant \alpha/(\text{const. } K)$; such a condition is needed to control the small divisor bounds on complex domains. Since in the above formulation it is assumed that the small divisor bounds are valid *directly* on complex domains, such a condition is not needed.

We shall apply the Lemma to the (transformed) resonant D'Alembert Hamiltonians $H_{\varepsilon,\mu} \circ \Phi_L$ (27) and (32).

We first observe that under the map Φ_L, the domain A_ε (see (13)) gets transformed into

$$A'_\varepsilon := \{I \in \mathbf{R}^3 \colon |I_1| < d\varepsilon^\ell, \quad |(pI_1 + I_2 - I_3) - \bar{J}_2| < d, \quad I_2 \in \mathbf{R}\}, \tag{39}$$

when $(p, q) = (1, 1)$ or $(p, q) = (1, 2)$, while, when $(p, q) \neq (1, 1)$ and $(p, q) \neq (1, 2)$, A_ε gets transformed into

$$A'_\varepsilon := \{I \in \mathbf{R}^3 \colon \; |qI_1 + bI_3| < d\varepsilon^\ell, \quad |I_2 - \bar{J}_2| < d\}. \tag{40}$$

Let $0 < \ell < 1$ and ε small; let $\bar{J}_2$ and L be so that (6) (with $\bar{J}_1 = \bar{\omega}p/q = p\omega$) is (abundantly) verified. Then the sets D and D' in the above lemma can be chosen as follows. When $(p, q) = (1, 1)$ or $(p, q) = (1, 2)$, we let

$$\begin{aligned} &n_1 = 1, \qquad n_2 = 2, \qquad D := (-d\varepsilon^\ell, d\varepsilon^\ell), \\ &D' := \{(I_2, I_3) \in \mathbf{R}^2 \colon \; |I_2 - I_3 - \bar{J}_2| < 2d\}; \end{aligned} \tag{41}$$

while, when $(p, q) \neq (1, 1)$ and $(p, q) \neq (1, 2)$, we let

$$\begin{aligned} &n_1 = 2, \qquad n_2 = 1, \qquad D := \{(I_1, I_3) \in \mathbf{R}^2 \colon \; |q_1 + bI_3| < d\varepsilon^\ell\}, \\ &D' := (\bar{J}_2 - d, \bar{J}_2 + d). \end{aligned} \tag{42}$$

With such choices one has

$$A'_\varepsilon \subset D \times D'. \tag{43}$$

If we choose also

$$r_1 := \frac{d}{10}\,\varepsilon^\ell, \qquad r_2 := \frac{d}{10}, \tag{44}$$

we see that the functions ν_i and κ_i (and hence the functions c_i, d_i, G_i) are analytic and bounded, for a suitable $s > 0$ (depending on the analyticity domain of F_1), in the domain $(D_{r_1} \times D'_{r_2}) \times \mathbf{T}^3_s$. We can now apply Lemma 1 to the Hamiltonian $H_{\varepsilon,\mu} \circ \Phi_L(I,\varphi) = H(I,\varphi) = h(I) + f(I,\varphi)$ with*

$$h(I) := H_{00}(I_1, I_3), \qquad f(I,\varphi) := \varepsilon G_0(I_1, I_2, \varphi) + \varepsilon\mu G_1(I_1, I_2, \varphi; \mu).$$

Under the above positions, for $0 \leqslant \mu \leqslant 1$, we have that

$$\|f\|_{r_1,r_2,s} \leqslant \text{const. } \varepsilon.$$

Thus, letting $\alpha = \omega/2$ and $K := \omega/(4\varepsilon^\ell)$, we see that (36) is satisfied for any $\ell < 1/2$, provided $\varepsilon > 0$ is small enough. Thus, by Lemma 1 and (27) or (32), we find that $H_{\varepsilon,\mu} \circ \Phi_L \circ \Phi(\hat{I},\hat{\varphi})$ has the form**

$$\begin{aligned} &H_{\text{eff}}(\hat{I}_1, \hat{I}_2, \hat{\varphi}_1; \varepsilon) + \tilde{g}(\hat{I}, \hat{\varphi}_1, \hat{\varphi}_2; \varepsilon, \mu) + f_*(\hat{I}_1, \hat{I}_2, \hat{\varphi}; \varepsilon, \mu) \\ &\quad := H_{00}(\hat{I}_1, \hat{I}_3) + \varepsilon H_{01}(\hat{I}, \hat{\varphi}_1) + \\ &\qquad + \tilde{g}(\hat{I}, \hat{\varphi}_1, \hat{\varphi}_2; \varepsilon, \mu) + f_*(\hat{I}_1, \hat{I}_2, \hat{\varphi}; \varepsilon, \mu), \end{aligned} \tag{45}$$

where (if g is as in the Lemma) $\tilde{g} := g - \varepsilon H_{01}$. The function f_* is exponentially small,

$$\|f_*\|_{r_1/2, r_2/2, s/6} \leqslant \|f\|_{r_1,r_2,s} \exp\left(-\frac{Ks}{6}\right) \leqslant \text{const. } \varepsilon \exp\left(-\frac{\omega s}{24}\frac{1}{\varepsilon^\ell}\right), \tag{46}$$

and, in view of (38), the definition of $\tilde{g}$ and (27) or (32), the function $\tilde{g}$ satisfies the bound

$$\|\tilde{g}\|_{r_1/2, r_2/2, s/6} \leqslant \text{const. } (\varepsilon^{2-\ell} + \varepsilon\mu). \tag{47}$$

Thus, *assuming* $|\mu| \leqslant \varepsilon^c$ *with* $c > 0$ *and* $0 < \ell < 1/2$, in the above region of phase space, the resonant D'Alembert Hamiltonian is described, *up to the exponentially small term in* (46), by the Hamiltonian***

$$\begin{aligned} H_D(\hat{I}_1, \hat{I}_2, \hat{\varphi}_1, \hat{\varphi}_2; \varepsilon, \mu) \; := \; & H_{00}(\hat{I}_1) + \varepsilon H_{01}(\hat{I}_1, \hat{I}_2, \hat{\varphi}_1) + \\ & + \varepsilon^a G(\hat{I}_1, \hat{I}_2, \hat{\varphi}_1, \hat{\varphi}_2; \varepsilon, \mu), \end{aligned} \tag{48}$$

where

$$\begin{aligned} a \; &:= \; \min\{\, 2 - \ell, \; 1 + c \,\} > 1, \\ G \; &:= \; \frac{\tilde{g}}{\varepsilon^a}, \qquad \|G\|_{r_1/2, r_2/2, s/6} \leqslant \text{const.} \end{aligned} \tag{49}$$

*See, respectively (27) and (32).

**Here we are using a unified notation, but bear in mind that all functions (G_0, G_1, H_{eff}, H_{00}, and H_{01}, as well as the sets A'_ε) depend on p and q and differ qualitatively if (p,q) is equal to (1, 1) or (2, 1) or are equal to $(p,q) \neq (1,1), (2,1)$; in particular if $(p,q) \neq (1,1)$ and (2, 1) then $H_{01}(\hat{I},\hat{\varphi}) = H_{01}(\hat{I})$.

***If we disregard f_* then $\hat{I}_3$ becomes a dumb parameter, which we drop. Once again, if $(p,q) \neq$ (1, 1) and (2, 1) then $H_{01}(\hat{I},\hat{\varphi}) = H_{01}(\hat{I})$.

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