ON RIGOROUS STABILITY RESULTS FOR LOW-DIMENSIONAL KAM SURFACES

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The stability of invariant (KAM) surfaces for nonintegrable dynamical systems with few degrees of freedom, as a nonlinearity parameter is increased, is considered. A rigorous method, which allows one to construct explicitly such surfaces, is discussed. A byproduct of this method allows one to give lower bounds on breakdown thresholds and applications to the standard map and to a two wave hamiltonian system yield results that agree within 60% with the numerical expectations.

1. Introduction

Transition to stochastic regimes, in hamiltonian mechanics with few degrees of freedom, seems to be intimately related to the disappearance, as a non-linearity parameter is increased, of KAM tori with highly irrational rotation numbers [1].

In this Letter we consider the "paradigm hamiltonian" of Escande [1]

$$H(y,x,t;\epsilon)\equiv \tfrac{1}{2}y^2+\epsilon f(x,t)$$
$$\equiv \tfrac{1}{2}y^2+\epsilon[\cos x+\cos(x-t)] \qquad (1)$$

and illustrate the main ideas which are needed in order to give a rigorous proof of the stability of the torus with rotation number $\omega=(\sqrt{5}-1)/2$ for complex values of ϵ with $|\epsilon|\leqslant 0.015$. The experimental value at which this torus is expected to disappear is about 0.0276 [2,3] [#1].

The complete mathematical proof of this stability result [4] is quite technical and involves computer-assisted estimations, but its basic scheme is rather simple and will be described here.

This scheme, which is quite general and can be applied to higher dimensional hamiltonian systems as well as (directly) to monotone twist diffeomorphisms of the plane, is based on two fundamental steps. One is an implementation of a recent Newton method [5], which allows one to establish the existence of KAM surfaces by solving directly a "KAM-torus equation" (see eq. (2) below and compare also with ref. [6]) rather than by exploiting repeated use of symplectic transformations as in classical KAM theory [7,8]. The second step consists in finding a good initial guess for the Newton method and is based on smoothness properties in the non-linearity parameter ϵ. Roughly speaking, under analyticity assumptions, KAM surfaces depend analytically on ϵ and from the KAM-torus equation one can compute recursively a few terms of the ϵ-power-series expansion of the KAM surface. Such truncated series will be used as initial guess. (For related power series methods, see ref. [9].)

As mentioned above our method can be applied *tout court* to twist maps and, here, we just mention that an application to the Chirikov–Greene standard map yields existence of the golden-mean curve for $|\epsilon|\leqslant 0.65$ [4], while the experimental value obtained by Greene [8] predicts a critical threshold of 0.97.

Finally, numerical extrapolations of our method give results in complete agreement with the experimental ones.

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[#1] Falcolini [3] applying Greene's residue criterion [10] to a "leap-frog map with large integrator step" obtained as Poincaré section, obtains a critical value of 0.02758. The value indicated by Escande [2], based on the renormalization theory of ref. [1], is 0.0276.

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2. Stability theorem

We recall that a KAM torus for (1) with rotation number ω is an invariant two-dimensional torus that can be described parametrically $\{(x,t)=(\theta+u(\theta,t;\epsilon),t\colon(\theta,t)\in T^2\}$ where T^2 denotes the standard torus $R^2/(2\pi Z)^2$ and u is a periodic function of (θ,t) satisfying $1+u_\theta\neq 0$ and such that in the (θ,t)-coordinates the H-flow is simply given by $(\theta_0,t_0)\to(\theta_0+\omega t, t_0+t)$. From this and Hamilton equations, it follows that u satisfies the following partial differential equation on T^2 ("KAM-torus equation")

$$D^2u+\epsilon\frac{\partial f}{\partial x}(\theta+u,t)=0,\quad D\equiv\omega\frac{\partial}{\partial\theta}+\frac{\partial}{\partial t}. \tag{2}$$

Thus, to prove the existence tori is equivalent to solve (2). We summarize our resu!t in the following

Theorem. Let $\omega=\sqrt{5}-1)/2$ and let ϵ be a complex number with $|\epsilon|\leqslant 0.015$. Then eq. (2) has a unique solution with mean value (over T^2) zero, which is a real analytic function of θ, t and ϵ. Such a solution can be written in the form

$$u\equiv\sum_{l=1}^{l_0}u^{(l)}(\theta,t)\epsilon^l+R_{l_0}(\theta,t;\epsilon),$$

where the $u^{(l)}$ are odd trygonometric polynomials and, for $l_0=24$,

$$\max_{\substack{(\theta,t)\in T^2\\|\epsilon|\leqslant 0.015}}|R_{24}(\theta,t;\epsilon)|\leqslant 6.85\times10^{-5}.$$

It is worth mentioning that the $u^{(l)}$ can be computed "explicitly"; for example, the first two terms are simply given by

$$u^{(1)}=-\left(\frac{1}{\omega^2}\sin\theta+\frac{1}{(\omega-1)^2}\sin(\theta-t)\right),$$

$$u^{(2)}=\frac{1}{2}\left[\frac{1}{4\omega^4}\sin 2\theta+\left(\frac{1}{\omega^2}+\frac{1}{(\omega-1)^2}\right)\frac{1}{(2\omega-1)^2}\sin(2\theta-t)+\left(\frac{1}{\omega^2}-\frac{1}{(\omega-1)^2}\right)\sin t+\frac{\sin[2(\theta-t)]}{4(\omega-1)^4}\right].$$

3. Power series expansions

As it turns, $u(\theta,t;\epsilon)$ is an analytic function of ϵ near $\epsilon=0$. Thus, we set

$$u=\sum_{l=1}^{\infty}u^{(l)}(\theta,t)\epsilon^l$$

and, expanding (2) in a series, we get

$$D^2u^{(1)}=\sin\theta+\sin(\theta-t),$$

$$D^2u^{(l)}=\sum_{k=1}^{l-1}\frac{1}{k!}\left(\frac{\partial^k}{\partial\theta^k}[\sin\theta+\sin(\theta-t)]\right)\times\sum_{\substack{l_1+...+l_k=l-1\\l_j\geqslant 1}}u^{(l_1)}...u^{(l_k)},\quad l\geqslant 2.$$

This system of equations has a unique solution $(u^{(1)}, u^{(2)},...)$ with $\int u^{(l)}=0$. In Fourier series,

$$u^{(l)}=\sum_{(n,m)\in Z^2}\hat{u}^{(l)}_{(n,m)}e^{i(n\theta+mt)},$$

such a solution is given by

$$\hat{u}^{(1)}_{(n,m)}=\frac{1}{(\omega n+m)^2}in\zeta_{(n,m)},$$

$$\hat{u}^{(l)}_{(n,m)}=\frac{1}{(\omega n+m)^2}\times\sum_{k=1}^{l-1}\sum_{\substack{l_1+...+l_k=l-1\\l_j\geqslant 1}}\sum_{\substack{\nu_0+...+\nu_k=(n,m)\\\nu_j\equiv(\nu_{j1},\nu_{j2})\in Z^2}}\frac{(i\nu_{01})^{k+1}}{k!}\times\zeta_{\nu_0}\hat{u}^{(l_1)}_{\nu_1}...\hat{u}^{(l_k)}_{\nu_k},$$

where $\zeta_{(n,m)}=\frac{1}{2}$ if $(n,m)=\pm(1,0),\pm(1,-1)$ and 0 otherwise. For $l=1,2$ one will easily recover the functions of the preceding section.

It will be readily realized that the number of Fourier coefficients of $u^{(l)}$ grows quite rapidly with l. To handle them and to obtain a rigorous evaluation of them we used a computer performing the so-called interval-arithmetic [11].

As mentioned in the introduction we took

$$v_0=\sum_{l=1}^{24}u^{(l)}\epsilon^l$$

as initial guess for the Newton iteration, which we proceed now to describe.

4. Newton iteration

Let v_j be some approximate solution of (2), namely, let

$$D^2 v_j + \epsilon f_x(\theta + v_j, t) = e_j , \qquad (3)_j$$

with e_j small (in suitable sup-norms). We look for a new function $v_{j+1} = v_j + w_j$ satisfying $(3)_{j+1}$ with $w_j \simeq O(e_j)$ and $e_{j+1} \simeq O(e_j)^2$. Recalling that a KAM torus satisfies $1 + v_\theta \neq 0$, we assume that

$$1 + \partial v_j / \partial\theta \neq 0 , \qquad (4)_j$$

so that we can define $z = z(\theta, t; \epsilon)$ as the unique solution with mean value zero of

$$D[(1 + \partial v_j/\partial\theta)^2 Dz] = -(1 + \partial v_j/\partial\theta) e_j .$$

Notice that this equation can be solved because the right-hand side has mean value zero as it follows from $(3)_j$. Then, one sets $w_j \equiv (1 + \partial v_j/\partial\theta) z$, defines e_{j+1} as the left-hand side of $(3)_{j+1}$ and checks that w_j and e_{j+1} satisfy the above smallness requirements. Finally, observe that, if v_j and e_j are analytic in ϵ in some disk D, so are v_{j+1} and e_{j+1}, provided $(4)_j$ holds uniformly in D. This last requirement gives a restriction on the size of the radius of D.

5. Convergence

Equipping this iteration with a "close-to-optimal" set of estimates, we obtain an algorithm that, given v_0 and some initial guess on the ϵ-analyticity radius ρ, yields a sequence of new functions v_j and e_j with relative bounds. The v_j will converge to the solution of (2), if one can show that

$$1 + \partial v_j/\partial\theta \neq 0 \quad \forall j; \quad \lim_{j\uparrow\infty} e_j = 0 . \qquad (5)$$

Because of the fast rate of convergence of Newton procedures, it is quite easy to check if (5) holds. We did this by making use of a "KAM condition" that, if satisfied by v_{j_0} and e_{j_0} for some j_0, yields (5). Such a condition, which is nothing else than a completely explicit version of standard KAM theorems in the style of, e.g., ref. [12], is necessary in order to establish a rigorous result and, by its nature, cannot be close-to-optimal; nevertheless its effect can be highly mitigated by the step-by-step application of the Newton iteration as indicated above.

In our computations ($\rho_0 = 0.015$, $l_0 = 24$) we checked the KAM condition for $j_0 = 10$, that is, after applying 10 times the Newton steps described above.

6. Conclusion

The above analysis, together with numerical extrapolations based on it, seems to suggest that, at least in the case considered here, the radius of convergence in ϵ of the parametric representation of KAM tori might actually coincide with their stability threshold.

This phenomenon, if further confirmed, might be relevant in understanding the breakdown of analytic KAM tori even in higher dimensions, at least in the case of systems that are a direct generalization of (1).

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