ON THE CONVERGENCE OF FORMAL SERIES CONTAINING SMALL DIVISORS

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1. Introduction

Poincaré–Lindstedt series for the (formal) computation of quasi–periodic solutions (in the context of real–analytic, nearly–integrable Hamiltonian dynamical systems) with *fixed frequencies* have been extensively studied, for over a century, from both the theoretical and applicative point of view. For applications, the Poincaré–Lindstedt series provide a simple practical tool to explicitly compute the first few orders of perturbation theory; the problem of convergence has been instead much more controversial (famous is Poincaré dubious statement in his *Méthodes nouvelles de la Méchanique Céleste*). The matter was settled *indirectly* in the sixties thanks to KAM (Kolmogorov, Arnold, Moser) theory. "Indirectly" means that the convergence is obtained as a *byproduct of estimates uniform in the smallness parameter* rather than *directly* looking at the formal series and trying to check convergence by studying the rate of growth of coefficients (as in the classical Siegel's approach to the small divisor problem arising in linearization of germs of complex analytic functions). As it is well known, the main problem with the "direct approach" is that the k^{th} coefficient of the formal power series, if expanded in sums of monomials composed by Fourier coefficients of the Hamiltonian and of "small divisors" (appearing in the denominators of the monomials as linear integer combinations of the basic frequencies), contains, in general, monomials which diverge as $k!$. Hence, a "direct proof" has necessarily to deal with *compensations*, i.e., with the problem of grouping together all the "diverging" terms showing that they sum up to much smaller contributions, which can be bounded by a constant to the k^{th} power. Direct proofs (in Hamiltonian setting) were given by H. Eliasson in 1988, by Gallavotti, Gentile and Mastropietro and, independently, by the authors in 1993 (for bibliography and more technical discussions see [1, 2, 3] and references therein).

It is conceivable that direct techniques may solve small divisor problems out of the reach of the more general KAM techniques. As a possible instance we formulate, in §2, a small divisor problem (for which KAM [4] yields rather weak results) in a language particularly suitable to study compensations.

In §3 we prove the convergence of Poincaré–Lindstedt series for "isoenergetic maximal quasi–periodic solutions": even though it is certainly possible to obtain such a result (which we couldn't find explicitly stated in the literature) by using the direct (but rather

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involved) compensation techniques, we shall discuss a (short) proof based on a KAM result for Hamiltonians depending analytically on some parameters.

2. Tree Expansion of Lower Dimensional Elliptic Tori

Let us consider the following model problem. Let $\mathcal{M}$ be the phase space $\mathbb{R}^{N+M} \times \mathbb{T}^{N+M}$ endowed with the standard symplectic form $\sum_{j=1}^{N} dy_j \wedge dx_j + \sum_{j=1}^{M} dp_j \wedge dq_j$ where $(y,x) \in \mathbb{R}^N \times \mathbb{T}^N$, $(p,q) \in \mathbb{R}^M \times \mathbb{T}^M$. Let H be the Hamiltonian[1]

$$H \equiv \frac{1}{2}|y|^2 + \frac{1}{2}|p|^2 + \varepsilon f_0(q) + \mu f(x,q) \ . \tag{1}$$

As it is well known, if $\mu = \varepsilon^2$, (1) reflects essentially the structure of a Hamiltonian $\frac{1}{2}(|y|^2 + |p|^2) + \varepsilon F(x,q)$ near a "multiple resonance" of the form $(y,p) = (\omega, 0)$ with $\omega \in \mathbb{R}^N$ some rationally independent vector[2]. We are interested in finding *quasi-periodic solutions* with basic frequencies $\omega \in \mathbb{R}^N$ ("non maximal quasi-periodic solutions"), i.e. solutions of the form $x(t) \equiv \omega t + X(\omega t)$, $q(t) \equiv Q(\omega t)$ with X and Q smooth functions over $\mathbb{T}^N$. Inserting such an expression in the Hamilton equations[3] and using the rational independence of ω, one sees that the function $\theta \in \mathbb{T}^N \to (X(\theta), Q(\theta)) \in \mathbb{R}^{N+M}$ satisfies the system

$$D^2 X = -\mu \partial_x f(\theta + X, Q) \ , \qquad D^2 Q = -\varepsilon \partial_q f_0(Q) - \mu \partial_q f(\theta + X, Q) \ ,$$

where $D \equiv \omega \cdot \partial_\theta \equiv \sum_{i=1}^{N} \omega_i \partial_{\theta_i}$.

Let now $q_0 \in \mathbb{T}^M$ be a *critical point of* f_0. The problem is: *Fix ε (say) positive (and small) and study the μ-analytic continuation of the unperturbed ($\mu = 0$) motion $(\omega, 0, x, q_0) \to (\omega, 0, x + \omega t, q_0)$* (proving possibly that the radius of μ-analyticity is greater than ε^2).

This problem is completely understood in the case that ω is Diophantine[4] and the Hessian matrix $A_0 \equiv \partial_q^2 f(q_0)$ is *negative definite* ("partially hyperbolic" case): in such a situation one can show, by the method mentioned in §1, that the associated formal (μ) power series[5] have a radius of convergence that is greater than ε^2; see [3, §6] (for related methods and results see [2] and references therein). If the matrix A_0 is instead *positive definite* ("partially elliptic" case) the question is much more subtle and has not been settled yet (see however [4]).

Here, under suitable assumptions on ω and the matrix A_0, we shall write down explicitly the tree expansion of the formal solution leaving however open the question of its convergence (which we plan to discuss in a future paper).

Assumption *Let us assume that $\omega \in \mathbb{R}^N$ and that the (positive) eigenvalues λ_i ($i = 1, \ldots, M$) of $A \equiv \varepsilon A_0 \equiv \varepsilon \partial_q^2 f_0(q_0)$ are such that there exist positive constants γ and τ for which*

$$|\omega \cdot n \pm \sqrt{\lambda_i}| \geq \frac{\gamma}{|n|^\tau} \ , \qquad \forall\, n \in \mathbb{Z}^N \backslash \{0\} \ , \qquad \forall\, i = 0, \ldots, M, \qquad \lambda_0 \equiv 0 \ . \tag{2}$$

[1] If w is a vector in $\mathbb{R}^n$ we denote by $|w|^2 = w \cdot w = \sum_{j=1}^{n} w_j^2$.

[2] In fact, f_0 would correspond to the average over the "fast angles" x of F. Of course, in general f would also depend on the action variables (y,p) (besides depending on ε). For simplicity we shall consider here only the case (1).

[3] $\ddot{x} = -\mu \partial_x f(x,q)$, $\quad \ddot{q} = -\varepsilon \partial_q f_0(q) - \mu \partial_q f(x,q)$.

[4] That is, $|\omega \cdot n| \geq \gamma |n|^{-\tau}$ for some $\gamma, \tau > 0$ and all $n \in \mathbb{Z}^N$ different from 0.

[5] I.e. $X \sim \sum_{j \geq 1} \mu^j X^j(\theta)$, $Q \sim q_0 + \sum_{j \geq 1} \mu^j Q^j(\theta)$.

By standard arguments, it is easy to check that if $\tau > N-1$, the set of ω's in $\mathbb{R}^N$ satisfying (2) for some γ has full Lebesgue measure.

Following [3, §6], we set

$$Z^{(1)} \equiv X \ , \quad Z^{(2)} \equiv Q \ , \quad \partial^{(1)} \equiv \partial_x \ , \quad \partial^{(2)} \equiv \partial_q \ , \quad F_0 \equiv \varepsilon f_0(q) \ , \quad F_1 \equiv f(x,q) \ .$$

Let us also denote by $[\cdot]_k$ the operator acting on (formal) power series, $a = \sum a_j \mu^j$, by associating to a its $k^{\rm th}$ coefficients a_k: $[a]_k \equiv a_k$. One checks immediately that if $Z^{(\rho)} \sim \sum Z^{(\rho)k}\mu^k$, then the recursive equations for $Z^{(\rho)k}$ $(\rho = 1,2)$ are

$$-\Big(D^2 + (\rho-1)A\Big) Z^{(\rho)k} = \sum_{\chi=0,1} [\partial^{(\rho)} F_\chi]_{k-\chi}^{(k-1)} \ , \qquad \rho = 1,2 \ , \tag{3}$$

where the suffix $^{(k-1)}$ means that the argument of the function within square brackets is, for $k \geq 2$, the polynomial in μ of degree $(k-1)$ given by

$$x = \theta + \sum_{h=1}^{k-1} \mu^h Z^{(1)h} \ , \qquad q = q_0 + \sum_{h=1}^{k-1} \mu^h Z^{(2)h} \ ,$$

and, for $k = 1$, is $(x,q) = (\theta, q_0)$. To construct the formal solution, i.e., to solve recursively (3) we first observe that if $Z^{(\rho)h}$ solve (3) (with h in place of k) for $0 \leq h \leq k-1$ ($Z^{(1)0} \equiv 0$, $Z^{(2)0} \equiv q_0$) then *the right hand side of* (3) *has, for* $\rho = 1$, *vanishing mean value over* $\mathbb{T}^N$: this is a non trivial fact[6] reflecting the symplectic structure of the Hamiltonian equations. Thus, we can solve (3) for $\rho = 1$ by inverting the operator D^2 (an operation that leaves free the average of $Z^{(1)k}$, which we shall normalize to 0) and for $\rho = 2$ by inverting the operator $D^2 + A$. Notice that by the above Assumption one has

$$\left\| \Big((\omega\cdot n)^2 + \ell A\Big)^{-1} \right\| \leq \frac{1}{\min\limits_{0\leq i\leq M} |(\omega\cdot n)^2 - \lambda_i|} \leq \gamma^{-2}|n|^{2\tau} \ , \qquad \text{for } \ell = 0,1 \ .$$

Taking Fourier coefficients of (3) we get

$$Z_n^{(\rho)k} = \sum_{\substack{\sigma\in\{0,2\}^* \\ \chi\in\{0,1\}}} (\omega\cdot n)^{-\sigma} \left\{[D_n^{(\rho)} F_\chi]_{k-\chi}^{(k-1)}\right\}_n \ ;$$

where $\sigma \in \{0,2\}^*$ means that $\sigma + \rho \in \{2,3\}$ and that $n = 0 \implies \sigma = 0$, and the vector valued operator $D_n^{(\rho)}$ is defined by:

$$D_n^{(\rho)} \equiv \Big((\omega\cdot n)^2 - A\Big)^{1-\rho}\, \partial^{(\rho)} \ .$$

Notice that the components $D_{nj}^{(\rho)}$ are N if $\rho = 1$ and M if $\rho = 2$; we therefore let $N_1 \equiv N$, $N_2 \equiv M$ so that $j \in \{1,\ldots,N_\rho\}$. The *tree expansion* of such solution takes the form (compare with [3, formula (4.11)])

$$Z_{nj_0}^{(\rho_0)k} = \frac{1}{k!} \sum_{T_r\in\mathcal{T}_k} \sum_{\substack{\alpha:V\to\alpha_v\in\mathbb{Z}^N \\ \Sigma_v\alpha_v = n}} \sum_{\substack{\beta:V\to\beta_v\in B \\ \rho_r=\rho_0}} \sum_{\substack{j:V\to j_v\in\{1,\ldots,N_{\rho_v}\} \\ j_r = j_0}} \prod_{v\in T_r} \{\Lambda_v F_{\chi_v}\}_{\alpha_v} \prod_{v\in T_r} \gamma_v \ , \tag{4}$$

[6] Essentially known to Poincaré and Lindstedt (at least in the case $M = 0$).

where: 1) $Z_{nj_0}^{(\rho_0)k}$ denotes the j_0^{th} component of the n–Fourier coefficient of the function $Z^{(\rho_0)k}$; 2) $\mathcal{T}_k$ is the set of all *labeled, weighted, rooted trees* with k vertices introduced in [3][7]; 3) the index set B is defined as: $B \equiv \{\beta = (\sigma, \rho)\colon \sigma \in \{0,2\},\ \rho \in \{1,2\}$ with the constraints $\sigma + \rho \in \{2,3\}$ and $\delta_v = 0 \implies \sigma = 0\}$ with[8] $\delta_v \equiv \sum_{v' \le v} \alpha_{v'}$; 4) the operator Λ_v and the positive number γ_v are defined respectively as[9]

$$\Lambda_v \equiv \Lambda_v(T_r, \alpha, \beta, j) \equiv D^{(\rho_v)}_{\delta_v j_v} \prod_{v' \in \mathcal{N}_v} \partial^{(\rho_{v'})}_{j_{v'}} \ , \quad \gamma_v \equiv \gamma_v(T_r, \alpha, \beta) \equiv \begin{cases} (\omega \cdot \delta_v)^{-\sigma_v} & \text{if } \delta_v \ne 0, \\ 1 & \text{if } \delta_v = 0. \end{cases}$$

The proof of formula (4) can be easily checked by mimicking the deduction of formula (4.9) (with specifications as in §6) in [3] (where the lower dimensional partially hyperbolic case is treated in detail). If one could prove *compensations* in the sense mentioned in §1 (see [3]) for (4) one would be able to conclude the *convergence of the formal power series* obtaining a new result[10].

3. ε–Analyticity of Isoenergetic Tori

Using the standard (analytic) implicit function theorem and a KAM result for Hamiltonian systems depending analytically on several parameters[11] we shall prove here the following "KAM isoenergetic" result. Let H be a real–analytic, nearly–integrable Hamiltonian of the form $H \equiv h(y) + \varepsilon f(y,x)$, $(y,x) \in V \times \mathbb{T}^N$, V being, say, a sphere centered in $y_0 \in \mathbb{R}^N$.

Theorem 3.1 *Assume that $\omega \equiv h_y(y_0)$ is a Diophantine vector, that* $\det h_{yy}(y_0) \ne 0$ *and that*[12] *$\omega \cdot (h_{yy}(y_0))^{-1}\omega \ne 0$. Let $E \equiv h(y_0)$. Then there exist unique functions $u(\theta;\varepsilon)$, $v(\theta;\varepsilon)$, $\alpha(\varepsilon)$ depending analytically on $\theta \in \mathbb{T}^N$ and ε (near 0) satisfying: $u(\theta;0) = 0$, $v(\theta;0) = y_0$, $\int_{\mathbb{T}^N} u = 0$ and, letting $\omega_\alpha \equiv (1+\alpha)\omega$ and $D_{\omega_\alpha} \equiv \omega_\alpha \cdot \partial_\theta$,*

$$\omega_\alpha + D_{\omega_\alpha} u = H_y(v, \theta + u; \varepsilon) \ , \quad D_{\omega_\alpha} v = -H_x(v, \theta + u; \varepsilon) \ , \quad H(v, \theta + u; \varepsilon) \equiv E \ .$$

Proof Let $\widetilde{H} \equiv h(\frac{y}{1+\alpha}) + \varepsilon f(\frac{y}{1+\alpha}, x)$ with α scalar near $\alpha_0 \equiv 0$. Then the Proposition in footnote 11 holds. If $\bar{v} \equiv \frac{\tilde{v}}{1+\alpha}$ and $\bar{u} \equiv \tilde{u}$, then the Proposition implies that $\bar{u}$ and $\bar{v}$

[7] Given a rooted (unlabeled) tree T_r (rooted at r) one calls a function of the vertices of T_r, $\chi : v \in V(T_r) \to \chi_v \in \{0,1\}$, a *weight function*. A *weighted rooted tree* is a couple (T_r, χ) with T_r a rooted tree and χ a weight function. We now denote $\widetilde{\mathcal{T}}_k$ the set of weighted rooted trees satisfying (i) $\deg v \le 2 \implies \chi_v = 1$, (ii) $\sum_{v \in V(T_r)} \chi_v = k$. Finally, the class $\mathcal{T}_k$ of *labeled, weighted rooted trees* is obtained from $\widetilde{\mathcal{T}}_k$ by labelling with k different labels the k vertices with weight 1.

[8] Rooted trees can be naturally equipped with a partial order: $v' \le v$ if the path joining the root r of T_r with v' contains v.

[9] The set $\mathcal{N}_v$ denotes the vertices smaller than v and adjacent to it.

[10] The lower dimensional partially elliptic tori found in [4] are not even continuous in μ and they are proved to exist only for values of μ in a Cantor set.

[11] **Proposition** *Let $\widetilde{H} = h(y;\alpha) + \varepsilon f(y,x;\alpha)$ with h and f depending (analytically) on y near $y_0 \in \mathbb{R}^N$, on $x \in \mathbb{T}^N$ and on α near $\alpha_0 \in \mathbb{R}^n$ (w.r.t. the standard form $dy \wedge dx$). Let $\omega = h_y(y_0;\alpha_0)$ be a Diophantine vector and assume that $\det h_{yy}(y_0;\alpha_0) \ne 0$. Then there exist (unique) functions $\tilde{u}$ and $\tilde{v}$, depending analytically on $\theta \in \mathbb{T}^N$, ε and α in neighbourhoods of, respectively, 0 and α_0, satisfying: $\tilde{u}(\theta;0,\alpha) = 0$, $\tilde{v}(\theta;0,\alpha_0) = y_0$, $\int_{\mathbb{T}^N} \tilde{u} = 0$ and $\omega + D\tilde{u} = \widetilde{H}_y(\tilde{v}, \theta + \tilde{u}; \varepsilon, \alpha)$, $D\tilde{v} = -\widetilde{H}_x(\tilde{v}, \theta + \tilde{u}; \varepsilon, \alpha)$ $(D = \omega \cdot \partial_\theta)$.* [A proof may be found in A. Celletti, L. Chierchia (1997) On the Stability of Realistic Three Body Problems, *Commun. Math. Phys.*, **186**, 413-449].

[12] Notice that (since $T = h_{yy}(y_0)$ is invertible) the condition $\omega \cdot T^{-1}\omega \ne 0$ is equivalent to the standard isoenergetic non–degeneracy condition, i.e., $\det \begin{pmatrix} T & \omega \\ \omega & 0 \end{pmatrix} \ne 0$.

are analytic in $\theta \in \mathbb{T}^N$ and in (ε, α) near $(0,0)$ and verify $\omega_\alpha + D_{\omega_\alpha}\bar{u} = H_y(\bar{v}, \theta + \bar{u}; \varepsilon)$, $D_{\omega_\alpha}\bar{v} = -H_x(\bar{v}, \theta + \bar{u}; \varepsilon)$. To fix the energy we want to determine α as a function of ε using the (analytic) implicit function theorem. Let $F(\varepsilon, \alpha) \equiv H(\bar{v}(0; \varepsilon, \alpha), \bar{u}(0; \varepsilon, \alpha); \varepsilon) - E$. Then $F(0,0) = h(y_0) - E = 0$ and it remains to check that $F_\alpha(0,0) \neq 0$. But, $F_\alpha(0,0) = h_y(y_0)\bar{v}_\alpha(0;0,0) = \omega A$ where A is the matrix $\bar{v}_\alpha(0;0,0)$. To compute A, observe that evaluating at $\varepsilon = 0$ the equations satisfied by $\tilde{u}$ and $\tilde{v}$, one finds that $h_y(\bar{v}(0;0,\alpha)) = \omega_\alpha$. Differentiating this relation one obtains $A = h_{yy}(y_0)^{-1}\omega$. Whence $F_\alpha(0,0) = \omega \cdot h_{yy}(y_0)^{-1}\omega$, which is different from 0 by hypothesis. From the implicit function theorem it follows that there exists an analytic function $\alpha(\varepsilon)$ such that $H(\bar{v}(0; \varepsilon, \alpha(\varepsilon)), \bar{u}(0; \varepsilon, \alpha(\varepsilon); \varepsilon) - E \equiv 0$ and the conclusion of the Theorem holds if $u(\theta; \varepsilon) \equiv \bar{u}(\theta; \varepsilon, \alpha(\varepsilon))$, $v(\theta; \varepsilon) \equiv \bar{v}(\theta; \varepsilon, \alpha(\varepsilon))$. ∎

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