Z angew Math Phys 47 (1996)
(ZAMP)

0044-2275/96/020210-11 \$ 1.50 + 0.20

A note on quasi-periodic solutions of some elliptic systems

By L. Chierchia, Dip di Matematica, Terza Università di Roma, via Segre 2, 00146 Roma, Italy (Internet: luigi@matrm3.mat.uniroma3.it), (Fax: (39-6) 55176263) and C. Falcolini, Dip di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy (Internet: falcolini@mat.utovrm.it) (Fax: (39-6) 7259 4699)

1. Introduction

1) In this paper we are concerned with the construction of *quasi-periodic* solutions of systems of nonlinear partial differential equations, for the unknown vector function $u: y \in \mathbb{R}^M \to u(y) \in \mathbb{R}^N$, of the type:

$$\Delta u = \varepsilon f_x(u, y) \qquad i.e.\ \Delta u_i = \varepsilon \frac{\partial f}{\partial x_i}(u, y), \qquad i = 1, \ldots, N, \tag{1.1}$$

where $(x, y) \in \mathbb{R}^{N+M}$, $\Delta = \Sigma_{j=1}^M (\partial^2/\partial y_j^2)$, $f = f(x, y)$ is a smooth (later real-analytic) function *periodic* in each of its $N + M$ variables and ε is a parameter; "quasi-periodic" means that there exist an $N \times M$ matrix, Ω, and a (smooth) periodic function, $U: (\theta, \psi) \in \mathbb{T}^{N+M} \equiv \mathbb{R}^{N+M}/2\pi\mathbb{Z}^{N+M} \to U(\theta, \psi) \in \mathbb{R}^N$, such that

$$u(y) = \Omega y + U(\Omega y, y), \qquad \forall y \in \mathbb{R}^M. \tag{1.2}$$

(To be precise we should replace $(\Omega y, y)$ by $\pi_0(\Omega y, y)$ as argument of U, π_0 being the projection of $\mathbb{R}^{N+M}$ onto $\mathbb{T}^{N+M}$: we shall however omit such projection). System (1.1) corresponds (formally) to the Euler equations of the variational problem associated to the functional

$$\int F(\nabla u, u, y)\, dy \equiv \int \left\{ \frac{1}{2} \sum_{i=1}^N |\nabla u_i|^2 + \varepsilon f(u, y) \right\} dy. \tag{1.3}$$

The interest for quasi-periodic solutions of (1.1) is motivated by the following two examples.

(i) $M = 1$: (1.1) are then the Euler-Lagrange equations,

$$\frac{d}{dt} F_p(\nabla u, u, t) = F_x(\nabla u, u, t),$$

associated to the "time-dependent" Lagrangian $F(p, x, t)$ having the particular form

$$F = \frac{1}{2}|p|^2 + \varepsilon f(x, t), \qquad (p, x) \in \mathbb{R}^N \times \mathbb{T}^N, \qquad t \equiv y \in \mathbb{T},$$

bars denoting Euclidean norm in $\mathbb{R}^N$. Alternatively, (1.1) can also be interpreted as the Hamiltonian equations generated by the non-autonomous Hamiltonian $H(p, x, t) \equiv \frac{1}{2}|p|^2 - \varepsilon f(x, t)$ (with respect to the standard symplectic form $dp \wedge dx : \dot{x} = H_p,\ \dot{p} = -H_x,\ x = x(t) = u(t)$). In this case $\Omega \equiv \omega$ is an N-vector and if such a vector is rationally independent with 1 (*i.e.* $\omega \cdot n + q \equiv \Sigma_{j=1}^N \omega_j n_j + q = 0$ for some $n \in \mathbb{Z}^N$, $q \in \mathbb{Z} \Rightarrow n = 0 = q$), then quasi-periodic solutions correspond to invariant $(N+1)$-dimensional tori embedded (if $\det[I + \partial_\theta U] \neq 0$) in the phase space $\mathbb{R}^N \times \mathbb{T}^{N+1}$. In parametric equations such tori are given by $(\theta, t) \in \mathbb{T}^{N+1} \to (\theta + U(\theta, t), t)$ on which the Hamiltonian flow becomes $(\theta_0, t_0) \to (\theta_0 + \omega t, t_0 + t)$. The existence (under suitable hypothesis on ω, F and ε) of quasi-periodic solutions is the main object of the so called *KAM* (Kolmogorov-Arnold-Moser) *theory* (see [1] and references therein, and [4] for some recent developments).

(ii) $N = 1$: In this case (1.1) is a single elliptic equation corresponding to the Euler equation of the scalar variational problem (1.3) (with $N = 1$). The existence of quasi-periodic solutions (satisfying $1 + \partial_\theta U > 0$) is equivalent to establish *minimal foliations* for (1.3), *i.e.* foliations of codimension 1 on $\mathbb{T}^{M+1}$ whose leaves are minimal for the variational problem (1.3). In 1988, J. Moser, continuing the analysis started in [12], proved a *stability theorem* ("ε small") (for smooth F satisfying a "Legendre condition" and for suitable $\Omega \in \mathbb{R}^M$) for such minimal foliations ([14], see also [13], [15]).

2) In this note we construct quasi-periodic solutions of (1.1) adapting to the PDE case a "direct method à la Siegel", which has been used, for the first time, in the Hamiltonian (ODE) case by H. Eliasson [7] and which has been recently revived (always in the ODE case: see [5], [8], [9]). The "direct method" is based onto two clearly distinct parts: an *algebraic* (and short) part and a *quantitative* (rather lengthy and technical) one. It is interesting to note that to generalize the proof to the PDE case one needs to discuss *only* the *algebraic part* as the quantitative estimates follows almost at once from the analogous estimates of the ODE case. We also notice that in the case $N = 1$ we obtain a new proof of a particular case of Moser's theorem [12].

3) Let us briefly describe the precise results discussed in this note. We assume a "Diophantine" condition on Ω, which generalize the similar conditions needed in (i) and (ii). We denote by A the $M \times (N + M)$ matrix (Ω^T, I_M) ($I_M \equiv M \times M$-identity matrix) and assume that

$$|Al| \geq \frac{1}{\gamma |l|^\tau}, \qquad \forall l \in \mathbb{Z}^{N+M} \backslash \{0\}, \qquad A \equiv (\Omega^T, I_M). \tag{1.4}$$

For example such a condition is verified if one of the column of Ω is a Diophantine vector *i.e.* denoting $\Omega = [\omega_1, \ldots, \omega_M]$, ($\omega_j \in \mathbb{R}^N$ to be interpreted as column vectors: $\Omega_{ij} = \omega_{j,i} \equiv i^{\text{th}}$ component of the vector ω_j) if, for some $1 \le j \le M$, there exist $\gamma, \tau > 0$ such that[1]

$$|\omega_j \cdot n + q| \ge \frac{1}{\gamma |n|^\tau}, \qquad \forall n \in \mathbb{Z}^N \backslash \{0\}, \qquad q \in \mathbb{Z}. \tag{1.5}$$

It is well known (see, *e.g.*, [1]) that, given $\tau > N$, the set of vectors $\omega_j \in \mathbb{R}^N$ satisfying (1.5) for some $\gamma > 0$, has full Lebesgue measure. In particular, (1.4) implies that the map $y \in \mathbb{R}^M \to (\Omega y, y)$ is dense on $\mathbb{T}^{N+M}$. Thus, it is immediate to check that *u in* (1.2) *solves* (1.1) *if and only if $U(\theta, \psi)$ solves the degenerate system on $\mathbb{T}^{N+M}$ given by*

$$LU = \varepsilon f_x(\theta + U, \psi), \qquad L \equiv \sum_{j=1}^{M} (\omega_j \cdot \partial_\theta + \partial_{\psi_j})^2, \tag{1.6}$$

(where, as above, dot denotes inner product: $\omega_j \cdot \partial_\theta = \Sigma_{i=1}^N \omega_{j,i}\, \partial_{\theta_i}$). Notice that, in Fourier space, L acts on periodic functions $g(\theta, \psi)$ as a multiplication operator transforming the Fourier coefficient g_l, $l \in \mathbb{Z}^{N+M}$, into $(-|Al|^2 g_l)$.

Theorem 1.1. Let f in (1.6) be real-analytic on $\mathbb{T}^{N+M}$ and let Ω satisfy (1.4). Then there exists a unique solution $U = U(\theta, \psi; \varepsilon)$ of (1.6) jointly real-analytic in (θ, ψ) and ε (for ε in a complex neighborhood of $\varepsilon = 0$), and such that

$$\langle U \rangle \equiv \int_{\mathbb{T}^{N+M}} U(\theta, \psi; \varepsilon) \frac{d\theta\, d\psi}{(2\pi)^{N+M}} = 0. \tag{1.7}$$

Condition (1.7) is just a normalization condition necessary to have uniqueness: in fact, if U is a solution of (1.6) then so is $(\theta, \psi) \to \theta_0 + U(\theta_0 + \theta, \psi_0 + \psi)$ for any prefixed $(\theta_0, \psi_0) \in \mathbb{T}^{N+M}$. An immediate corollary of this theorem and of the above mentioned abundance of matrices Ω satisfying (1.4) is the existence of uncountably many quasi-periodic solutions u of (1.1) related to U by (1.2).

To be more quantitative, we introduce, on the space of analytic functions $g: \mathbb{T}^r \to \mathbb{C}$, the norm $\|g\|_\sigma \equiv \Sigma_{n \in \mathbb{Z}^r} |g_n|\, e^{|n|\sigma}$; here g_n denote Fourier coefficients and σ is a prefixed, positive parameter. Obviously, g is real-analytic on $\mathbb{T}^r$ if and only if there exists a $\sigma > 0$ such that $\|g\|_\sigma < \infty$. If $g \equiv (g_1, \ldots, g_s)$ is a vector-valued analytic function, we set $\|g\|_\sigma \equiv \sup_j \|g_j\|_\sigma$.

[1] $a \cdot b = \Sigma_{i=1}^N a_i b_i$, $|a| = (\Sigma_{i=1}^N |a_i|^2)^{1/2}$.

We will then show that there exist constants $\beta = \beta(\tau)$ *and* $c = c(\tau, \gamma)$ (*recall* (1.4)) *such that if* $\|f\|_{\bar\sigma} < \infty$ *then the solution* U *described in Theorem* 1.1 *is real-analytic in the domain*

$$\mathscr{D}_{\sigma,\varepsilon_0} \equiv \{\theta \in \mathbb{C}^N\colon |\mathrm{Im}\,\theta_i| \le \sigma\} \times \{\psi \in \mathbb{C}^M\colon |\mathrm{Im}\,\psi_i| \le \sigma\} \\ \times \{\varepsilon \in \mathbb{C}\colon |\varepsilon| \le \varepsilon_0\}$$

for any $0 < \sigma < \bar\sigma$ *and for*

$$\varepsilon_0 \equiv \frac{(\bar\sigma - \sigma)^\beta}{c\|f\|_{\bar\sigma}}. \tag{1.8}$$

Moreover, for any complex ε *with* $|\varepsilon| < \varepsilon_0$, U *satisfies the bound*:

$$\|U(\cdot\,;\varepsilon)\|_\sigma \le \frac{\bar\sigma - \sigma}{2}\frac{|\varepsilon|}{\varepsilon_0 - |\varepsilon|}. \tag{1.9}$$

Remark 1.1. ("*No epsilon case*") Obviously, a similar result holds for the systems ("no epsilon"):

$$\Delta u = f_x(u, y), \qquad LU = f_x(\theta + U, \psi), \tag{1.10}$$

provided $\|f\|_{\bar\sigma}$ is small enough *i.e.* $\|f\|_{\bar\sigma} \le (\bar\sigma - \sigma)^\beta/(2c)$ (just insert the *auxiliary parameter* ε in the r.h.s. of (1.10): the smallness condition on f guarantees that $\varepsilon = 1$ is well inside the disk of ε-analyticity and the solution $U(\theta, \psi)$ of (1.10) is obtained by taking $U(\theta, \psi) \equiv U(\theta, \psi; 1)$, which satisfies $\|U\|_\sigma \le (\bar\sigma - \sigma)/2$).

Remark 1.2. ("*Autonomous case*") In the case $f = f(x)$ does not depend upon y, the definition of quasi-periodic solutions has to be adapted in the obvious way: U in (1.2) will depend only on θ and the operator L becomes simply $\Sigma_{j=1}^M (\omega_j \cdot \partial_\theta)^2$ (and the matrix A in (1.4) has to be replaced by Ω^T). Then, Theorem 1.1 still holds together with the bounds (1.9), (1.8). Furthermore, in the autonomous case, the problem with no epsilon (1.10) is transformed into the original problem (1.6) by a simple rescaling of the frequency matrix: $\Omega \to \Omega/\sqrt{\varepsilon}$. Thus, since Diophantine vectors form a set of positive measure, we conclude immediately from Theorem 1.1 that *there exist uncountably many quasi-periodic solutions of* $\Delta u = f_x(u)$ *for any analytic function* f, *without assuming* $\|f\|_{\bar\sigma}$ *to be small* (compare [13], [6]).

Remark 1.3. It is probably possible to extend Theorem 1.1 so as to cover a broader class of elliptic systems corresponding to more general Lagrangian functions (satisfying a suitable non-degeneracy condition). On the other hand, our techniques do not seem apt to cover the interesting case in which f is assumed to be (sufficiently) differentiable.

4) The proof of Theorem 1.1 is "elementary and direct": the existence and uniqueness of *formal solution* of (1.6), $U \sim \Sigma_{k \geq 1} \varepsilon^k U^{(k)}(\theta, \psi)$, with $U^{(k)}$ real-analytic on $\mathbb{T}^{N+M}$, is first explicitly established and then it is shown that the $U^{(k)}$'s satisfy the bounds

$$\| U^{(k)} \|_\sigma \leq \frac{1}{2} (\bar{\sigma} - \sigma) \varepsilon_0^{-k}, \tag{1.11}$$

which leads at once to Theorem 1.1 and to (1.9). In particular *no use is made* of fast converging schemes as it is typical in KAM theory.

The main (and well known) difficulty in such strategy is that $U^{(k)}(\theta, \psi)$ (the k^{th} term of the formal solution) can be written as a sum of many "elementary contributions" of the form

$$q \left(\prod_{j=1}^{k} f_{l_j} \right) \varrho^{-2}, \tag{1.12}$$

where q is an integer, f_{l_j} are Fourier coefficients and ϱ^{-2} is a product of so-called *small divisors i.e.* $\varrho = \Pi_{j \in J} |Al_j|$ with J a finite set made up of at least k elements: the "divisors" Al_j may get arbitrarily small when $|l_j| \to \infty$ and the repetition of particularly small divisors generates elementary contributions of size as large as $\sim k!$. Therefore, to get a bound like (1.11) one has to show that all the large elementary contributions *compensate*.

Here, we shall not carry out the explicit evaluation of the constants β and c in (1.8), which could be easily done by mimicking the similar estimates performed in full details in [5] leading to (the certainly not optimal values of)

$$\beta = 12\tau + 6, \qquad c = \gamma^2 2^{30\tau + 16}(12\tau + 6)!$$

5) We close this introduction sketching the main steps on which the proof of Theorem 1.1 relies.

(1) One establishes, for (1.6), existence and uniqueness of formal quasi-periodic solutions $U \sim \Sigma_{k \geq 1} U^{(k)} \varepsilon^k$ by a simple recursive argument similar to that used in [6]. For this step it is crucial that the equations (1.1) are "conservative".

(2) The formal solution is shown to admit a very explicit representation in terms of elementary graph theory or, more precisely, in terms of labeled rooted trees[2].

(3) The first estimates on products of small divisors ϱ are proved. These estimates are based on a simple generalization of the original argument given by C. L. Siegel [16] in order to linearize germs of analytic functions of the form $f(z) = e^{i\omega} z + z^2 g(z)$, $z \in \mathbb{C}$, with ω a

[2] In this paper we shall need no more than the basic definitions from graph theory, which may be found in the first chapters of most books on the subject (such as, *e.g.*, [3]) or in Appendix A of [5].

real number satisfying a Diophantine condition. These first estimates cannot be generalized in a straightforward way to the case considered in the present paper (not even for the case $N = M = 1$). What happens is that in products (1.12) the *same small divisor may appear several times* as a consequence of vanishing combinations of Fourier indices ("*resonances*"): such repetitions may lead to contributions (1.12) of size $k!$ (see [5], Appendix B).

(4) One produces a partition of the terms forming the tree-decomposition of $U^{(k)}$ in well behaving families *i.e.* in families whose elements behave as if there were no resonances. This step is algebraic in character and constitutes the heart of the proof.

(5) Using steps 3 and 4, one can bound the sum of the contributions in each of the families of the above partition by a constant to the k^{th} power using Cauchy's classical method of majorants (see, *e.g.*, [11] Chapter 3).

6) We emphasize that, basically, the only difference from the proof of the ODE case is step 4, which is explained here in full details; the other parts are essentially contained in [5] and here we simply point out the needed (notational) adjustments.

2. Proof of Theorem 1.1

Step 1: Existence and uniqueness of formal solutions

Given a (convergent) ε-power series, $g = \Sigma_{k \geq 0} g_k \varepsilon^k$, we denote by $[\cdot]_k$ the "coefficient operator", which associates to g its k^{th} coefficient $[g]_k = g_k$. A formal solution with real-analytic coefficients of (1.6) is a sequence of real-analytic functions $\{U^{(k)}\}_{k \geq 0}$, $U^{(k)}$: $(\theta, \psi) \in \mathbb{T}^{N+M} \to U^{(k)}(\theta, \psi) \in \mathbb{R}^N$, satisfying

$$LU^{(0)} = 0, \qquad LU^{(k)} = \left[f_x\left(\theta + \sum_{h=0}^{k-1} \varepsilon^h U^{(h)}, \psi \right) \right]_{k-1}, \qquad (k \geq 1). \tag{2.1}$$

Obviously, if the series $\Sigma_{k \geq 0} \varepsilon^k U^{(k)}$ is convergent (*i.e.* $\exists \sigma > 0$ such that the numerical series $\Sigma_{k \geq 0} \varepsilon^k \| U^{(k)} \|_\sigma$ has positive radius of convergence in the complex ε-plane), then $U \equiv \Sigma_{k \geq 0} \varepsilon^k U^{(k)}$ satisfies (1.6) if and only if (2.1) holds for all k. It is easy to prove, by induction, that given a real-analytic f and a real matrix Ω satisfying (1.4), there exists a unique formal solution satisfying (2.1) and the *normalization condition* $\langle U^{(k)} \rangle = 0$: see, for example, the detailed proof given (in the ODE case) in Appendix B of [5]. The unique (normalized) solution is given by

$$U^{(0)} = 0, \qquad U^{(k)} = - \sum_{\substack{l \neq 0 \\ l = (n,m) \in \mathbb{Z}^{N+M}}} \frac{F_l^{(k)}}{|Al|^2} e^{i(n \cdot \theta + m \cdot \psi)}, \qquad (k \geq 1) \tag{2.2}$$

($F^{(h)}$ being recursively defined in terms of $U^{(1)}, \ldots, U^{(h-1)}$).

Step 2: Tree-representation of the formal solution

Given a rooted tree T with root in $r \in V \equiv V(T)$ ($\equiv$ {vertices of T}), we introduce on V a *partial ordering* by setting $u \geq v$ if the path with endpoints r and v contains u; $u > v$ means $u \geq v$ and $u \neq v$. Given a rooted tree T and a function $\alpha\colon v \in V \to \alpha_v \in \mathbb{Z}^{N+M}$, we denote by $\delta_v(T;\alpha)$ (or by $\delta_v(T)$ or simply δ_v) the $\mathbb{R}^M$-vector

$$\delta_v = A \sum_{\substack{v' \in V \\ v' \leq v}} \alpha_{v'}.$$

As in [5], we select the root r of T by adding to the set of edges of the tree T an *extra* edge ηr where η is a symbol (not a vertex) (notice that in this way for rooted trees one has $\# E = \# V$ where E denotes, as customary, the set of edges of T). Finally, we denote by $\mathscr{T}^k$ the set of labeled rooted trees of order k. The method of proof of Proposition 3.1 in [5] yields the following:

Lemma 2.1. For $1 \leq j \leq N$, $l \in \mathbb{Z}^{N+M} \backslash \{0\}$, let $U_{lj}^{(k)} \equiv U_{jl}^{(k)}$ denote the j^{th} component of the Fourier coefficient of index l of the real-analytic function $U^{(k)}(\theta, \psi)$ in (2.2). Then

$$U_{jl}^{(k)} = \frac{1}{k!} \sum_{T \in \mathscr{T}^k} \mathscr{U}_{jl}(T), \qquad \text{with:}$$

$$\mathscr{U}_{jl}(T) \equiv (-i) \sum_{\substack{\alpha \in \mathscr{A}(T) \text{s.t.} \\ \alpha(T) = l, n_\eta \equiv e_j}} \prod_{v \in V} f_{\alpha_v} \prod_{vv' \in E} n_v \cdot n_{v'} \prod_{v \in V} |\delta_v|^{-2}, \tag{2.3}$$

where $\mathscr{A}(T)$ is the set of functions from the vertices of T into $\mathbb{Z}^{N+M} \backslash \{0\}$ such that $\Sigma_{v' \leq v} \alpha_{v'} \neq 0$ $\forall v \in T$, n_v denotes, as above, the first N components of the vector α_v, $e_j \in \mathbb{Z}^N$ denotes the versor in the j^{th} direction and $\alpha(T)$ is short for $\Sigma_{v \in T} \alpha_v$.

Step 3: Small divisors bounds (non-critical case)

As already mentioned in the introduction, in the sum at the r.h.s. of (2.3) there are terms that, for k large, behave like $k!$ (an explicit identification and evalaution of such terms is given, *e.g.*, in [5], Appendix B). This phenomenon is due to (rather obsessive) repetitions of certain small divisors δ_v. Repetitions of small divisors are "dangerous" whenever they occur on related vertices *i.e.* when happens that $\delta_u = \delta_w$ for some $u > w$. In such a case, if we let $R(u, w)$ denote the subtree with vertices $V(R(u, w)) = \{v \leq u\} \backslash \{v \leq w\}$, it is $\alpha(R(u, w)) = 0$. A subtree R is called a *resonance* (or a resonant subtree) if: (i) $\deg R \equiv$ number of edges connecting R with[3] $T \backslash R$ is two; (ii) R is null *i.e.* $\alpha(R) = 0$; (iii) R cannot be discon-

[3] $V(T \backslash R) = V(T) \backslash V(R)$, $E(T \backslash R) = E(T) \backslash \{uv \in E(T)\colon v \in V(R)\}$.

nected, by removal of one edge, in two null subtrees. Actually, as the following Lemma shows, only certain ("critical") resonances are really dangerous. Fix $\lambda = 1/5$ (definitions and estimates given below will depend upon a free parameter $\lambda \le 1/4$, which, for definiteness, we fix, once and for all, to be $1/5$) and call a *λ-resonance* a resonant subtree R, for which, setting[4]

$$l \equiv \sum_{v < R} \alpha_v, \qquad \delta_{\min}(R) \equiv \min_{\substack{v_1, v \in R\\ v_1 \neq v}} |\delta_v(R_{v_1})|,$$

one has $|Al| \le \lambda \delta_{\min}(R)$. Then the proof of Lemma 5.2 in [5] yields at once the following:

Lemma 2.2. Let T be a rooted tree of order k, $\alpha \in \mathscr{A}(T)$. Assume that there are no λ-resonances. Then there exist constants $c_1 > 1$ and $\beta_1 > 1$ such that

$$\prod_{v \in T} |\delta_v|^{-1} \le c_1^k \prod_{v \in T} |\alpha_v|^{\beta_1}. \tag{2.4}$$

Step 4: Small divisors compensations

Let us now prove, in full details, the crucial algebraic property (analogous to Lemma 5.3 in [5]). A resonance produces a repetition of divisors: Let R be a tree, let $w \in R$, let z be an extra vertex (*i.e.* $z \notin V(R)$) and consider the rooted tree R_u^w with root in u defined by $V(R_u^w) \equiv V(R) \cup \{z\}$, $E(R_u^w) \equiv E(R_u) \cup \{wz\}$. Fix an α function, $\alpha\colon V(R_u^w) \to \mathbb{Z}^{N+M} \backslash \{0\}$ with $\alpha_z \equiv l \neq 0$ and $\alpha(R) = 0$, so that R is a resonance for R_u^w. In the product $\Pi_{v \in R_u^w} |\delta_v(R_u^w)|^{-2}$, the divisor $\delta \equiv \delta_z(R_u^w) \equiv Al$ appears at least twice, as $\delta_u(R_u^w) = \delta$. We now produce a combination of trees similar to R_u^w, so that the above repetition disappears. A precise statement is as follows. For $v \in V(R_u^w)$ we denote, as usual, $\alpha_v = (n_v, m_v)$. Fix $1 \le i, j \le N$, and consider the meromorphic function of $t \in \mathbb{C}$ given by

$$\varphi(t) \equiv \sum_{u, w \in R} n_{ui} n_{wj} \prod_{\substack{v \in P(u,w)\\ v \neq u}} |\delta_v(R_u) + t\delta|^{-2} \prod_{v \in R \backslash P(u,w)} |\delta_v(R_u)|^{-2}, \qquad \delta = Al,$$

where $P(u, w)$ is the path joining u and w (if $u = w$ the first product is missing, while if R is a path the second product is missing). Note that

$$|\delta|^{-4} \varphi(1) = \sum_{u, w \in R} n_{ui} n_{wj} \prod_{v \in R_u^w} |\delta_v(R_u^w)|^{-2}.$$

[4] $v < R$ means $v \in V(T) \backslash V(R)$ and $v < u$ for some $u \in R$, and recall that $\delta_v(R_u) = A \sum_{\substack{v' < v\\ v' \in R}} \alpha_{v'}$ where the order $\le$ is that of the tree R rooted at u.

But, in fact, *one of the* $|\delta|^{-2}$ *is fictitious*:

$$\varphi(0) = \frac{d\varphi}{dt}(0) = 0. \tag{2.5}$$

To prove (2.5), recall that $\alpha(R) = 0$ and define, for any u, w in R

$$\mu(u) \equiv \prod_{\substack{v \in R \\ v \neq u}} |\delta_v(R_u)|, \qquad \nu(w) \equiv \sum_{u \in R, u \neq w} n_{ui} \sum_{\substack{v \in P(u,w) \\ v \neq u}} \frac{\delta \cdot \delta_v(R_u)}{|\delta_v(R_u)|^2}.$$

We claim that *the functions* $\mu(u)$ *and* $\nu(w)$ *are independent of, respectively,* u *and* w*:* $\mu(u) \equiv \mu$, $\nu(w) \equiv \nu$. The independence of $\mu(u)$ from u comes immediately from the identities

$$\delta_v(R_u) = \delta_v(R_w), \qquad \forall v \notin P(u, w), \tag{2.6}$$

(where as usual $P(u, w)$ denotes the path joining u and w) while, since $\alpha(R) = 0$,

$$\prod_{\substack{v \in P(u,w) \\ v \neq u}} |\delta_v(R_u)| = \prod_{\substack{v \in P(u,w) \\ v \neq w}} |-\delta_v(R_w)|, \qquad \forall u \neq w. \tag{2.7}$$

Now, for any u and w with $u \neq w$ (2.6) and (2.7) imply

$$\begin{aligned}
\mu(u) &= \prod_{\substack{v \in P(u,w) \\ v \neq u}} |\delta_v(R_u)| \prod_{v \notin P(u,w)} |\delta_v(R_u)| \\
&= \prod_{\substack{v \in P(u,w) \\ v \neq w}} |\delta_v(R_w)| \prod_{v \notin P(u,w)} |\delta_v(R_w)| = \mu(w),
\end{aligned}$$

which proves the independency of μ from points in R. Next, observe that to prove the independence of ν from points in R it is enough to check that $\nu(w) = \nu(w')$ for *adjacent points* w and w'. Thus, let w, w' be adjacent points in R. Then

$$\begin{aligned}
\nu(w) - \nu(w') &= \sum_{\substack{u \in R \\ u \neq w,w'}} n_{ui} \left[\sum_{\substack{v \in P(u,w) \\ v \neq u}} \frac{\delta \cdot \delta_v(R_u)}{|\delta_v(R_u)|^2} - \sum_{\substack{v \in P(u,w') \\ v \neq u}} \frac{\delta \cdot \delta_v(R_u)}{|\delta_v(R_u)|^2} \right] \\
&\quad + \chi_R(w') n_{w'i} \frac{\delta \cdot \delta_w(R_{w'})}{|\delta_w(R_{w'})|^2} - \chi_R(w) n_{wi} \frac{\delta \cdot \delta_{w'}(R_w)}{|\delta_{w'}(R_w)|^2} \\
&= \sum_{\substack{u \in R \\ u \neq w,w'}} n_{ui} \frac{\delta \cdot \delta_w(R_{w'})}{|\delta_w(R_{w'})|^2} + \chi_R(w') n_{w'i} \frac{\delta \cdot \delta_w(R_{w'})}{|\delta_w(R_{w'})|^2} \\
&\quad + \chi_R(w) n_{wi} \frac{\delta \cdot \delta_w(R_{w'})}{|\delta_w(R_{w'})|^2} \\
&= \left(\sum_{u \in R} n_{ui} \right) \frac{\delta \cdot \delta_w(R_{w'})}{|\delta_w(R_{w'})|^2} = 0,
\end{aligned}$$

where χ_R denotes characteristic function of the set R and, in the second equality, we used $\delta_{w'}(R_w) = -\delta_w(R_{w'})$. This finishes the proof of the claim. Equalities (2.5) are now an immediate consequence of $\alpha(R) = 0$:

$$\varphi(0) = \sum_{u,w \in R} n_{ui} n_{wj} \prod_{\substack{v \in R \\ v \neq u}} |\delta_v(R_u)|^{-2} = \mu^{-2} \sum_{u,w \in R} n_{ui} n_{wj} = 0$$

and

$$\begin{aligned}\frac{d\varphi}{dt}(0) &= -2 \sum_{\substack{u,w \in R \\ u \neq w}} n_{ui} n_{wj} \prod_{\substack{v \in R \\ v \neq u}} |\delta_v(R_u)|^{-2} \sum_{\substack{v \in P(u,w) \\ v \neq u}} \frac{\delta \cdot \delta_v(R_u)}{|\delta_v(R_u)|^2} \\ &= -2\mu^{-2} \sum_{w \in R} n_{wj} \left(\sum_{\substack{u \in R \\ u \neq w}} n_{ui} \sum_{\substack{v \in P(u,w) \\ v \neq u}} \frac{\delta \cdot \delta_v(R_u)}{|\delta_v(R_u)|^2} \right) \\ &= -2\mu^{-2} v \sum_{w \in R} n_{wj} = 0.\end{aligned}$$

Step 5: Small divisors bounds (general case)

To any tree $T \in \mathscr{T}^k$ and any $\alpha \in \mathscr{A}(T)$ one can associate a family of trees $\mathscr{F}(T)$, called "the complete family of T" and $\mathscr{T}^k$ can be *partitioned* into such complete families. Essentially, a complete family is constructed by associating to critical resonances of T the trees which yield the compensations described in step (4) *i.e.* the trees obtained by "rotating" in all possible ways the two edges connecting a critical resonance R with $T \backslash R$. The precise construction of complete families can be found in §4 of [5].

The following estimate is then an easy consequence of Lemma 5.4 in [5].

Lemma 2.3. Let T be a rooted tree of order k, $\alpha \in \mathscr{A}(T)$ and let $\mathscr{F} = \mathscr{F}(T)$ be the associated complete family. Then there exist constants $c_2 > c_1^2$, $\beta_2 > 2\beta_1$ such that

$$\left| \sum_{T' \in \mathscr{F}} \prod_{vv' \in E(T')} n_v \cdot n_{v'} \prod_V \delta_v^{-2} \right| \leq \prod_{v \in V} |\alpha_v|^{\deg_{\mathscr{F}} v} c_2^{|V|} \prod_{v \in V} |\alpha_v|^{\beta_2}. \tag{2.8}$$

where $\deg_{\mathscr{F}} v = \max_{T' \in \mathscr{F}} \deg_{T'} v$.

The proof of Theorem 1.1 now follows using, *e.g.*, the well known method of majorants (see, *e.g.*, [11] §3 and its similar application in [5]). □

References

[1] Encyclopaedia of Math. Sciences, Vol. 3, *Dynamical Systems*, V. I. Arnold (ed.), Springer-Verlag, Berlin 1988.

[2] Bruno, A. D., *Convergence of transformations of differential equations to normal form*, Dokl. Akad. Nauk SSR *165*, 987–989 (1965); *Analytic form of differential equations*, Trans. Moscow Math. Soc. *25*, 131–288 (1971) and *26*, 199–239 (1972).

[3] Bollobas, B., *Graph Theory*, Springer-Verlag, Berlin 1979.
[4] Chierchia, L., *Birth and death of invariant tori for Hamiltonian systems*, in *The Geometry of Hamiltonian Systems*, T. Ratiu (ed.), Springer-Verlag, Berlin 1991.
[5] Chierchia, L. and Falcolini, C., *A direct proof of a theorem by Kolmogorov in Hamiltonian systems*, Ann. Sc. Norm. Super. Pisa, Cl. Sci., Ser. IV, *XXI*, Fasc. *4*, 541–593 (1994).
[6] Chierchia, L. and Zehnder, E., *Asymptotic expansions of quasiperiodic solutions*. Ann. Sc. Norm. Super. Pisa, Cl. Sci., Ser. IV, *XVI*, Fasc. *2*, 245–258 (1989).
[7] Eliasson, L. H., *Absolutely convergent series expansions for quasi periodic motions*, Reports Department of Math., Univ of Stockholm, Sweden, No. 2, 1–31 (1988); *Generalization of an estimate of small divisors by Siegel*, in *Analysis, et cetera*, P. H. Rabinowitz and E. Zehnder (eds.), Academic Press, New York 1990.
[8] Gallavotti, G., *Twistless KAM tori, quasi flat homoclinic intersections, and other cancellations in the perturbation series of certain completely integrable hamiltonian systems. A review*. Rev. on Math. Phys. *6*, 343–411 (1994).
[9] Gallavotti, G. and Gentile, G., *Majorant series convergence for twistless KAM tori*. Ergod. Th. & Dynam. Sys. *15*, 857–869 (1995).
[10] Goulden, I. P. and Jackson, D. M., *Combinatorial Enumeration*, Wiley Interscience Series in Discrete Math. 1983.
[11] John, F., *Partial Differential Equations*, Springer-Verlag, Berlin 1982.
[12] Moser, J., *Minimal solutions of variational problems*. Ann. Inst. Henri Poincaré (Anal. Nonlinéaire) *3*, 229–262 (1986).
[13] Moser, J., *Minimal foliations on a torus. Four lectures at CIME Conf. on Topics in Calculus of Variations*. M. Giaquinta (ed.), Springer Lect. Notes in Maths. *1365*, (1987).
[14] Moser, J., *A stability theorem for minimal foliations on a torus*. Ergod. Th. & Dynam. Sys. *8**, 251–281 (1988).
[15] Moser, J., *Quasi-periodic solutions of nonlinear elliptic partial differential equations*. Bol. Soc. Brasil. Mat. (N.S.) *20*, 29–45 (1989).
[16] Siegel, C. L., *Iterations of analytic functions*, Annals of Math. *43*, No. 4, 607–612 (1942).

Abstract

We extend a recent method of proof of a theorem by Kolmogorov on the conservation of quasi-periodic motion in Hamiltonian systems so as to prove existence of (uncountably many) real-analytic quasi-periodic solutions for elliptic systems $\Delta u = \varepsilon f_x(u, y)$, where $u: y \in \mathbb{R}^M \to u(y) \in \mathbb{R}^N$, $f = f(x, y)$ is a real-analytic periodic function and ε is a small parameter. Kolmogorov's theorem is obtained (in a special case) when $M = 1$ while the case $N = 1$ is (a special case of) a theorem by J. Moser on minimal foliations of codimension 1 on a torus $\mathbb{T}^{M+1}$. In the autonomous case, $f = f(x)$, the above result holds for any ε.

(Received: December 5, 1994)