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On steepness of 3-jet non-degenerate functions

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Abstract

We consider geometric properties of 3-jet non-degenerate functions in connection with Nekhoroshev theory. In particular, after showing that 3-jet non-degenerate functions are "almost quasi-convex", we prove that they are steep and compute explicitly the steepness indices (which do not exceed 2) and the steepness coefficients.

Keywords Steepness · Steep functions · 3-Jet non-degeneracy · Nekhoroshev's theorem · Hamiltonian systems · Steepness indices · Exponential stability

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Contents

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1 Introduction

In 1977–1979, N.N. Nekhoroshev published a fundamental theorem [17–20] about the "exponential stability" of nearly integrable, real-analytic Hamiltonian systems with Hamiltonian given, in standard action-angle coordinates, by

$$H(I, \varphi) = h(I) + \varepsilon f(I, \varphi), \qquad (I, \varphi) \in U \times \mathbb{T}^n \,, \tag{1}$$

where $U \subseteq \mathbb{R}^n$ is an open region, $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$ is the standard flat n-dimensional torus and ε is a small parameter. The integrable limit $h(I)$ is assumed to satisfy a geometric condition, called by Nekhoroshev "steepness", which can be formulated as follows (compare, also, Definition 2, § 2).

A function $f \in C^1(U)$, with U a bounded region (i.e. open, bounded and connected set) of $\mathbb{R}^n$, is said to be steep in U with *steepness indices* $\delta_1, \ldots, \delta_{n-1} \geq 1$ and (strictly positive) *steepness coefficients* $C_1, \ldots, C_{n-1}, \xi_1, \ldots, \xi_{n-1}$, if its gradient $h'(I)$ satisfies the following estimates: $\inf_{I \in U} \|h'(I)\| > 0$ and, for any $I \in U$, for any k-dimensional linear subspace $V^k \subseteq \mathbb{R}^n$ orthogonal to $h'(I)$ with $1 \leq k \leq n-1$, one has[1]

$$\max_{0 \leq \eta \leq \xi} \min_{u \in V^k : \|u\| = \eta} \|P_{V^k} h'(I+u)\| \geq C_k \xi^{\delta_k} \quad \forall\, \xi \in (0, \xi_k],$$

where P_{V^k} denotes the orthogonal projection over V^k.

Nekhoroshev's original exponential stability statement is, then, the following:

Let H in (1) be real-analytic with h steep. Then, there exist positive constants a, b and ε_0 such that for any $0 \leq \varepsilon < \varepsilon_0$ the solution (I_t, φ_t) of the (standard) Hamilton equations for $H(I, \varphi)$ with initial data (I_0, φ_0) satisfies

$$|I_t - I_0| \leq \varepsilon^b$$

for any time t satisfying

$$|t| \;\leq\; \frac{1}{\varepsilon} \exp\left(\frac{1}{\varepsilon^a}\right).$$

The values of the parameters a, b, ϵ_0 in the original statements of [17–20], as well as in the recent improvement [6,7], depend on the steepness indices and coefficients. Precisely a, b depend only on the values of the steepness indices and the number of the degrees of freedom, while ϵ_0 depends also on the values of the steepness coefficients. In [7], the explicit dependence of a, b, ϵ_0 on the steepness indices and parameters, as well as on the parameters depending on the perturbation f, is given, and the estimate of the stability exponent:

$$a = \frac{1}{2n\delta_1 \cdot \delta_{n-2}}$$

has been conjectured to be optimal.

Nekhoroshev proved in [16,19,20] that steepness is a generic property of C^∞ functions. Later, Niederman [21] proved that for real-analytic h, steepness is equivalent to require that h has no critical points and that its restriction to any affine subspace of dimension $1 \leq k \leq n-1$ admits only isolated critical points. However, neither from Nekhoroshev's genericity techniques nor from Niederman's theorem there follow directly explicit conditions to determine whether a given function is steep or not. Indeed, very little is known about the evaluation of steepness parameters (index and coefficients) for general classes of functions,

[1] For any vector $u \in \mathbb{C}^n$, we denote by $\|u\| := \sqrt{\sum_i |u_i|^2}$ its Hermitian norm.

evaluation which is necessary in order to give explicit exponential estimates for perturbations of a specific steep Hamiltonian.

Essentially, the only general class of steep functions, which is well understood, is that of "quasi-convex" functions. Quasi-convexity is the simplest instance of steepness, and the quasi-convex case has been used for decades to improve the theoretical stability bounds of Nekhoroshev's theorem, especially the stability exponent a. In the quasi-convex case, the proof of the theorem has been significantly simplified (compare [2,3,5]), and furthermore, the stability exponent has been improved up to $a = (2n)^{-1}$ (compare [11,12,24]; see, also, [4] for exponents which are intermediate between $a = (2n)^{-1}$ and $a = (2(n-1))^{-1}$). Such exponents in the convex case have been proved to be nearly optimal [29].

Beyond the quasi-convex case, Nekhoroshev provided other sufficient conditions to recognize if a given C^k function is steep in a neighbourhood of a point I. Such conditions are formulated in terms of the jet of partial derivatives of h (compare [14–20]).

From this point of view, quasi-convex functions are identified as 2-jet non-degenerate functions. Precisely, in [17,18], it is proved that if $\nabla h(I) \neq 0$, and the jet of order 2 of the function h at I is non-degenerate, i.e. if the system:

$$\begin{cases} \displaystyle\sum_{i=1}^{n} \frac{\partial h}{\partial I_i}(I) u_i = 0 \\ \displaystyle\sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial I_i \partial I_j}(I) u_i u_j = 0 \end{cases} \tag{2}$$

has a unique solution $u = (0, \ldots, 0)$ in $\mathbb{R}^n$, then h is steep in a neighbourhood of I with steepness indices $\delta_1 = \cdots = \delta_n = 1$; the steepness coefficients follow from standard convexity estimates, since the restriction to any linear space V^k orthogonal to $\nabla h(I)$ of a quasi-convex function h (or[2] $-h$) is convex (compare, also, Remark (v), Sect. 2).

Therefore, one is left with the problem of computing steepness parameters of functions whose 2-jet is degenerate.

In [17,18], Nekhoroshev pointed out also the steepness of functions h such that $\nabla h(I) \neq 0$ and with jet of order 3 at I is non-degenerate, meaning that the system

$$\begin{cases} \displaystyle\sum_{i=1}^{n} \frac{\partial h}{\partial I_i}(I) u_i = 0 \\ \displaystyle\sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial I_i \partial I_j}(I) u_i u_j = 0 \\ \displaystyle\sum_{i,j,k=1}^{n} \frac{\partial^3 h}{\partial I_i \partial I_j \partial I_k}(I) u_i u_j u_k = 0 \end{cases} \tag{3}$$

has a unique solution $u = (0, \ldots, 0)$ in $\mathbb{R}^n$.

Actually, steepness of 3-jet non-degenerate functions is not proved in [17,18], but rather it is obtained as a consequence of a former result of Nekhoroshev [16,19,20] about the steepness of functions whose jets of order j are outside the closure of the set P_j of jets which satisfy

[2] Steepness is invariant under the change $h \to -h$; hence, convexity and concavity are "equivalent" in this context and one usually refers only to convexity for simplicity.

certain algebraic conditions[3]. From such papers, it follows that the steepness indices of 3-jet non-degenerate functions are bounded from above with functions depending on n, and no computations of the steepness coefficients are provided.

Nevertheless, the 3-jet condition (being the only explicit general steepness condition apart from quasi-convexity) has gained quite a relevance in Nekhoroshev theory. Indeed, it enlarged significantly the range of applications of Nekhoroshev's Theorem, especially to celestial mechanics ([1,10,13,22,23,25,27], see also the review paper [9]). Furthermore, numerical studies revealed the difference of the asymptotic stability between convex functions and 3-jet non-degenerate functions (compare [8,28]).

On the other hand, 2-jet and 3-jet non-degeneracies, presently, appear to be the only algebraic sufficient conditions for steepness which are formulated with equations independent on the number n of the degrees of freedom. For example, there are functions whose 4-jet is non-degenerate, in the sense that the system

$$\begin{cases} \displaystyle\sum_{i=1}^{n} \frac{\partial h}{\partial I_i}(I)u_i = 0 \\ \displaystyle\sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial I_i \partial I_j}(I)u_i u_j = 0 \\ \displaystyle\sum_{i,j,k=1}^{n} \frac{\partial^3 h}{\partial I_i \partial I_j \partial I_k}(I)u_i u_j u_k = 0 \\ \displaystyle\sum_{i,j,k,\ell=1}^{n} \frac{\partial^4 h}{\partial I_i \partial I_j \partial I_k \partial I_\ell}(I)u_i u_j u_k u_\ell = 0 \end{cases}$$

has only the trivial solution $u = (0, \ldots, 0)$, but are not steep (see Example 1, [26])). Algebraic conditions for the steepness of functions of $n = 3$ and $n = 4$ variables which are 3-jet degenerate have been formulated in [26]; no general conditions for the steepness of 3-jet degenerate functions formulated using the 4-jet are known up to now. In fact, the sufficient jet conditions provided by Nekhoroshev in [14–16,19,20] are formulated in terms of the closure C_j of a set whose definition depends on the number n of the degrees of freedom; explicit equations for $C_j(n)$, valid for arbitrary n, are not known.

In this paper, we investigate further 3-jet non-degenerate functions in connection with their steepness properties.

A key property of such functions is a "spectral non-degeneracy" of their Hessian matrix. More precisely, if h is 3-jet non-degenerate and V^k is a linear subspace of $h'(I)^\perp$, then the symmetric operator $P_{V^k} h''(I) : V^k \to V^k$ is strictly definite apart, possibly, from one single direction: in other words, there may be at most one small (or vanishing) eigenvalue; the precise statement is the content of Lemma 1 in Sect. 3. In this sense, one might say that *3-jet non-degenerate functions are "almost quasi-convex"*.

This observation allows to concentrate the study of steepness on *lines* (one-dimensional vector spaces) in $h'(I)^\perp$: this quantitative analysis is the content of Lemma 2 of Sect. 3.

Putting together these two facts, one can finally prove (Sect. 4) the steepness of 3-jet non-degenerate functions and compute explicitly the steepness indices, which do not exceed 2, and the steepness coefficients.

[3] We remark that the result does not follow by simply checking if the 3-jet of a 3-jet non-degenerate function is outside the closure of P_3 but, depending on the value of n, there is suitably large j such that the jet of a 3-jet non-degenerate function is outside the closure of P_j.

2 Main result

We start with some standard notation:

- (**Tensors of derivatives**) Given $G \subseteq \mathbb{R}^n$ open, $p \in \mathbb{N}$ and a C^p function $h : G \to \mathbb{R}$, $D^p h(I) = h^{(p)}(I)$ denotes the symmetric p-tensor at $I \in G$ of the p-derivatives acting on[4] $(u_1, \dots, u_p) \in (\mathbb{R}^n)^p$ as

$$D^p h(I)[u_1, \dots, u_p] := \sum_{1 \le i_1, \dots, i_p \le n} \frac{\partial^p h(I)}{\partial I_{i_1} \cdots \partial I_{i_p}} u_{1i_1} \cdots u_{pi_p} \,.$$

In particular, for $p = 1$, $h^{(1)}$ is (identified with) the gradient

$$h' := \nabla h := \left(\frac{\partial h}{\partial I_1}, \dots, \frac{\partial h}{\partial I_n} \right)$$

and, for $p = 2$, $h^{(2)}$ is (identified with) the Hessian matrix

$$h'' := \left(\frac{\partial^2 h}{\partial I_i \partial I_j} \right)_{i,j=1,\dots,n} .$$

For $n \ge p \ge 2$, $D^p h(I)[u_2, \dots, u_p]$ denotes the vector in $\mathbb{R}^n$ with ith-component given by:

$$e_i \cdot D^p h(I)[u_2, \dots, u_p] = D^p h(I)[e_i, u_2, \dots, u_p] \,,$$

where $\{e_1, \dots, e_n\}$ is the standard orthonormal bases of $\mathbb{R}^n$ ($e_{ij} = \delta_{ij}$) and $u \cdot v = \sum_{i=1}^n u_i v_i$ the standard inner product in $\mathbb{R}^n$.

Analogously, for $n \ge p \ge 3$, $D^p h(I)[u_3, \dots, u_p]$ denotes the $(n \times n)$-matrix with entries given by:

$$D^p h(I)[u_3, \dots, u_p] e_i \cdot e_j = D^p h(I)[e_i, e_j, u_2, \dots, u_p] \,;$$

and so on for higher-order tensors (which, however, we shall not need).
Finally, for $n \ge p \ge k$ we shall also let

$$D^p h(I)[u]^k := D^p h(I)[\underbrace{u, \dots, u}_{k \text{ times}}] \,.$$

- (**Norms**) In $\mathbb{R}^n$, $\|x\| = \sqrt{x \cdot x} = \sqrt{\sum_{i=1}^n x_i^2}$ denotes Euclidean norm.

The norm of tensors of derivatives is the standard "functional norm":

$$\|D^p h(I)\| := \sup_{\substack{u_j : \|u_j\| = 1 \\ j=1,\dots,p}} \left| D^p h(I)[u_1, \dots, u_p] \right|$$

$$\|D^p h\|_D := \sup_D \|D^p h(I)\| \,.$$

4 $u_i = (u_{i1}, \dots, u_{in})$.

From Cauchy–Schwarz inequality, there follows that

$$\|D^p h\|_D \leq \sqrt{\sum_{1\leq i_1,\ldots,i_p\leq n} \sup_{I\in D}\left|\frac{\partial^p h(I)}{\partial I_{i_1}\cdots\partial I_{i_p}}\right|^2} =: M_p \tag{4}$$

- (**Projections**) In what follows, V^k will denote a k-dimensional linear proper subspace of $\mathbb{R}^n$, $1 \leq k \leq n-1$, and $P_{V^k} : \mathbb{R}^n \to V^k$ the orthogonal projection on V^k: if $\{\bar{e}_1, \ldots, \bar{e}_k\}$ is an orthonormal basis of V^k, then

$$P_{V^k} v = \sum_{j=1}^{k} (v \cdot \bar{e}_j)\, \bar{e}_j\ , \qquad \|P_{V^k} v\|^2 = \sum_{j=1}^{k} |v \cdot \bar{e}_j|^2\ . \tag{5}$$

Recall that projections P are symmetric operators with $\|P\| \leq 1$ and such that $P^2 = P$.

Below, linear spaces V^k will be always subspaces of the orthogonal complement of $h'(I)$,

$$h'(I)^{\perp} := \{u \in \mathbb{R}^n \big|\ u \cdot h'(I) = 0\}\ ,$$

with I regular point for h (i.e. $h'(I) \neq 0$).

We, now, recall the general notion of "jet non-degeneracy":

Definition 1 Let $p \in \mathbb{N}$ and $G \subseteq \mathbb{R}^n$ be open. A C^p function $h : G \to \mathbb{R}$ is said to be p**-jet non-degenerate at** $I \in G$ if

$$D^k h(I)[u]^k = 0\ , \quad \forall\, 1 \leq k \leq p \qquad \Longrightarrow \qquad u = 0\ . \tag{6}$$

The function h is said to be p**-jet non-degenerate on** $D \subseteq G$ if h is p-jet non-degenerate at every $I \in D$.

Remarks (i) A 1-jet non-degenerate function at I is simply a function regular at I, i.e. such that $h'(I) \neq 0$.

A 2-jet non-degenerate function with nonvanishing gradient is, by definition, a **quasi-convex** function (at I); in other words, a quasi-convex function is a function h which is strictly convex (or concave) on $h'(I)^{\perp}$, I being a regular point for h.

(ii) From (6), it follows immediately that if h is 2-jet non-degenerate with nonvanishing gradient (quasi-convex) on a compact set $D \subseteq G$ then,

$$M_2 \overset{(4)}{\geq} \|D^2 h\|_D \geq \min_{I\in D} \min_{\substack{u\in h'(I)^{\perp}\\ \|u\|=1}} \left|h^{(2)}(I)[u]^2\right| =: \beta > 0\ .$$

Analogously, if h is 3-jet non-degenerate at I with nonvanishing gradient, then

$$M_3 \overset{(4)}{\geq} \|D^3 h\|_D \geq \min_{\substack{u\in h'(I)^{\perp}\\ \|u\|=1}} \max\left\{\left|h^{(2)}(I)[u]^2\right|,\ \left|h^{(3)}(I)[u]^3\right|\right\} := \beta(I) > 0\ , \tag{7}$$

and, if h is 3-jet non-degenerate on a compact set $D \subseteq G$, then[5]

$$M_3 \geq \min_{\substack{I \in D, \|u\|=1 \\ u \in h'(I)^\perp}} \max\left\{\left|h^{(2)}(I)[u]^2\right| \,,\ \left|h^{(3)}(I)[u]^3\right|\right\} := \beta > 0 \,, \tag{8}$$

(obviously, $\beta(I) \geq \beta$).

(iii) For every $v \in \mathbb{R}^n$ and for every $u \in V^k$ with $\|u\| = 1$, one has

$$\|P_{V^k} v\| \geq |P_{V^k} v \cdot u| = |v \cdot P_{V^k} u| = |v \cdot u| \,. \tag{9}$$

Applying these inequalities with $v = h''(I)u$, one sees that if h is 2-jet non-degenerate on D it follows that, $\forall\, I \in D$, $\forall\, V^k \subseteq h'(I)^\perp$,

$$\|P_{V^k} h''(I)u\| \geq \beta \,, \qquad \forall\, u \in V^k \,,\ \|u\| = 1 \,. \tag{10}$$

Analogously, applying (9) with $v = h^{(2)}(I)u$ and $v = h^{(3)}(I)[u]^2$ one sees that if h is 3-jet non-degenerate on D it follows that, $\forall\, I \in D$, $\forall\, V^k \subseteq h'(I)^\perp$,

$$\max\left\{\|P_{V^k} h''(I)u\|, \|P_{V^k} h^{(3)}(I)[u]^2\|\right\} \geq \beta \,, \qquad \forall\, u \in V^k \,,\ \|u\| = 1 \,. \tag{11}$$

Notice also that the eigenvalues of $P_{V^k} h''(I)$ have absolute value bounded by M_2: indeed, if $P_{V^k} h''(I)\bar{e} = \lambda \bar{e}$ with $\|\bar{e}\| = 1$, then

$$|\lambda| = \|\lambda \bar{e}\| = \|P_{V^k} h''(I)\bar{e}\| \leq M_2 \,. \tag{12}$$

Let us now turn to the definition of steepness as originally given by N.N. Nekhoroshev:

Definition 2 (Nekhoroshev [17,18]) Let $n \geq 2$ be an integer and $G \subseteq \mathbb{R}^n$ an open set. A C^1 function $h : G \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be **steep at a point** $I \in G$ if I is a regular point for h (i.e. $h'(I) \neq 0$) and, for each $1 \leq k \leq n-1$, there exist positive constants C_k, ξ_k, δ_k such that the inequality

$$\max_{0 \leq \eta \leq \xi} \min_{\substack{u \in V^k \\ \|u\|=1}} \left\|P_{V^k} h'(I + \eta u)\right\| \geq C_k \xi^{\delta_k} \,,\ \forall\, 0 < \xi \leq \xi_k \tag{13}$$

holds for every linear subspace $V^k \subseteq h'(I)^\perp$.

The numbers C_k and ξ_k are called **steepness coefficients**, while δ_k is called **the steepness index** of order k.

A C^1 function h is said to be **steep** on $D \subseteq G$ if there exist positive constants C_k, ξ_k, δ_k such that, for every $I \in D$, h is steep at I with coefficients C_k, ξ_k and indices δ_k.

Let us make a few more remarks.

(iv) Definition 2 is well posed since the function (defined for η small enough)

$$\eta \to F_{V^k}(\eta) := \min_{\substack{u \in V^k \\ \|u\|=1}} \left\|P_{V^k} h'(I + \eta u)\right\| \tag{14}$$

is upper semi-continuous, and hence, it achieves maximum on compact sets (note, however, that $F_{V^k}(0) = 0$).

[5] Notice that, if $S^{n-1} := \{u \in \mathbb{R}^n \,|\, \|u\| = 1\}$, the function $F : (I, u) \in D \times S^{n-1} \to F(I, u) := \max\left\{\left|h^{(2)}(I)[u]^2\right| \,,\ \left|h^{(3)}(I)[u]^3\right|\right\}$ is continuous on the compact set $\{(I, u) \in D \times S^{n-1} \,|\, h'(I) \cdot u = 0\}$ and therefore attains a minimum β on such a set: such a minimum is strictly positive since when $h'(I) \cdot u = 0$, by (3), $F(I, u) > 0$.

(v) *Quasi-convex functions are the steepest functions*: they are steep with lowest possible indices, namely $\delta_k = 1$ for all k.

Let us recall the elementary argument: by (10) and by Taylor's formula, for all $u \in V^k \cap G$ with $\|u\| = 1$, V^k linear subspace of $h'(I)^\perp$, and for small enough $\xi > 0$, one has:

$$\begin{aligned}\|P_{V^k} h'(I + \xi u)\| &= \|P_{V^k} h'(I) + \xi P_{V^k} D^2 h(I) u + o(\xi)\| \\ &= \|\xi P_{V^k} h''(I) u + o(\xi)\| \\ &\geq \xi\big(\|P_{V^k} h''(I) u\| - o(1)\big) \overset{(10)}{\geq} \frac{\beta}{2}\,\xi\ ,\end{aligned}$$

and steepness at I follows with $\delta_k = 1$ for all k (and $C_k = \beta/2$). The argument extends uniformly on compact sets.
Notice also that this proves a stronger property than steepness since it has been enough to consider only $\eta = \xi$ in (13) (rather than all $0 < \eta \leq \xi$).

We are ready to formulate the main result:

Theorem *Let D be a compact subset of an open set $G \subseteq \mathbb{R}^n$ such that $B_r(I) \subseteq G$ for all $I \in D$. Let $h : G \to \mathbb{R}$ be a C^4 function and assume that $h' \neq 0$ on D and that h is 3-jet non-degenerate on D. Let β as in (8), M_p as in (4) and define*

$$M := \max\{M_2, M_3, M_4\}\ ,\quad \gamma := \frac{\beta}{M}\ ,\quad \theta := \frac{1}{8(3 + 2\sqrt{2})}$$

and, for $1 \leq k \leq n - 1$,

$$C_k = C := \theta\,\beta\ ,\qquad \xi_k := \min\Big\{r, \frac{\theta}{2k^2}\,\gamma^3\Big\}\ . \tag{15}$$

Then, h is steep on D with coefficients C_k, ξ_k and indices $\delta_k \leq 2$.

Remarks (vi) From the definitions given it follows immediately that

$$\gamma \leq 1\ ,\qquad \xi_k \leq \frac{1}{16(3 + 2\sqrt{2})}\ .$$

(vii) We shall first prove steepness for the third-order truncation of the Taylor expansion of h and then extend it to the full function: this is not surprising, as the main hypothesis regards the 3-jet of h, which may be identified with the third-order Taylor polynomial of h. For this purpose, let us denote $\bar{h}$ the third-order Taylor polynomial of h at[6] $I \in D$:

$$\bar{h}(I') := \sum_{j=0}^{3} \frac{1}{j!} D^j h(I)[I' - I]^j$$

so that by Taylor's formula with integral remainder it is

$$h'(I + u) = \bar{h}'(I + u) + R(u; I)\ , \tag{16}$$

$$R(u; I) := \frac{1}{2}\int_0^1 (1 - t)^2 h^{(4)}(I + tu)[u]^3 dt\ ,$$

$$\|R(u; I)\| \leq \frac{M_4}{6}\|u\|^3\ ,\quad \forall\, I \in D\ ,\ \|u\| \leq r\ . \tag{17}$$

[6] $D^0 h := h$.

We shall also define the truncated "Nekhoroshev's function", for given $I \in D$ and $V^k \subseteq h(I)^\perp$,

$$\bar{F}_{V^k}(\eta) := \min_{\substack{u\in V^k \\ \|u\|=1}} \left\| P_{V^k} \bar{h}'(I+\eta u) \right\| \ . \tag{18}$$

Thus, from definitions (14), (18) and (16), one has

$$F_{V^k}(\eta) \geq \bar{F}_{V^k}(\eta) - \frac{M_4}{6}\eta^3 \ . \tag{19}$$

(viii) Some of the steepness indices δ_k of 3-jet non-degenerate functions can be equal to 1; this happens (trivially) for quasi-convex functions where $\delta_k = 1$ for all k. Also, $\delta_k = 1$ for some k if the restriction of the Hessian matrix of h to any k-dimensional linear space orthogonal to $h'(I)$ is non-degenerate [1,8].

3 Two lemmas

In this section, we show two properties of 3-jet non-degenerate functions: the first is a simple but crucial *spectral non-degeneracy property*, namely that the restriction of the Hessian of a 3-jet non-degenerate function on a linear space orthogonal to its gradient has at most *one* "small" eigenvalue: *3-jet non-degenerate functions are "almost quasi-convex"*.

The second property is the direct, explicit check of *steepness of 3-jet non-degenerate functions on lines* (one-dimensional linear subspaces).

These two properties together lead to a simple proof of steepness (given in Sect. 4).

Lemma 1 (Almost quasi-convexity) *If h is 3-jet non-degenerate at I, and $V^k \subseteq h'(I)^\perp$ with $k \geq 2$, then the spectrum of $P_{V^k} h''(I) : V^k \to V^k$ has at most one eigenvalue in absolute value strictly smaller than $\beta(I)$, where $\beta(I)$ is defined in* (7).

Proof Assume, by contradiction, that the conclusion is false. Then, there is an orthonormal basis of eigenvectors $\{\bar{e}_1, \ldots, \bar{e}_k\} \subseteq V^k$ of $P_{V^k} h''(I)$ with corresponding eigenvalues λ_k so that $|\lambda_1| \leq \cdots \leq |\lambda_k|$ and $|\lambda_1| \leq |\lambda_2| < \beta(I)$. For $t \in [0, 2\pi]$, consider the unitary vectors in V^k given by $u_t := (\cos t)\bar{e}_1 + (\sin t)\bar{e}_2$. Then,

$$|h''(I)u_t \cdot u_t| = |P_{V^k} h''(I)u_t \cdot u_t| = |\lambda_1 \cos^2 t + \lambda_2 \sin^2 t| \leq |\lambda_2| < \beta(I)$$

but this implies, by 3-jet non-degeneracy and the definition of $\beta(I)$, that $|h^{(3)}[u_t]^3| \geq \beta(I)$ for any t, and this is not possible since the real continuous function $t \in [0, 2\pi] \to h^{(3)}[u_t]^3$ changes sign[7] and hence must have a zero. □

Lemma 2 (Steepness on lines) *Under the assumptions of the Theorem, let $\theta_0 := 4\theta$*

$$\kappa := \theta_0 \beta \ , \qquad c := \sqrt{2\kappa/M} \ . \tag{20}$$

For every $I \in D$, $u \in h'(I)^\perp$ with $\|u\| = 1$ and $0 < \xi \leq \gamma$, there exists $\eta = \eta_{u,\xi}$ such that[8]

$$c\xi \leq \eta_{u,\xi} \leq \xi \ , \tag{21}$$

and

$$\bar{F}_{V_u^1}(\eta_{u,\xi}) = \min_{\sigma=\pm 1} |\bar{h}'(I + \sigma\eta_{u,\xi} u) \cdot u| \geq \kappa\xi^2 \ , \tag{22}$$

[7] For example, $h^{(3)}[u_0]^3 = -h^{(3)}[u_\pi]^3$.

[8] Notice that $2\kappa/M = \beta/(M(3+2\sqrt{2})) < 1$, so that $c < 1$.

where we have denoted by V_u^1 the 1-dimensional space generated by u:

$$V_u^1 := \{tu \big| t \in \mathbb{R}\} .$$

Proof The equality in (22) follows from representation (5) with $k = 1$ and observing that $\{u' \in V_u^1 : \|u'\| = 1\} = \{\pm u\}$.

Since $\bar{h}$ is the third-order Taylor polynomial of h at I, for $\sigma = \pm 1$, it is:

$$\bar{F}_{V_u^1}(\eta) = |P_{V_u^1}\bar{h}(I + \sigma\eta u)| = \left|\sigma\eta P_{V_u^1} h''(I)u + \frac{\eta^2}{2} P_{V_u^1} h^{(3)}[u]^2\right|$$

so that

$$\bar{F}_{V_u^1}(\eta) \geq \left|a\eta - \frac{b}{2}\eta^2\right| \tag{23}$$

having set $a := \|P_{V_u^1} h''(I)u\|$ and $b := \|P_{V_u^1} h^{(3)}[u]^2\|$. Note that, by (11), (4) and the definition of M, it is

$$\beta \leq \max\{a, b\} \leq M . \tag{24}$$

We consider various cases.

$$a \geq \beta . \tag{a}$$

Taking $\eta_{u,\xi} = \xi$:

$$\bar{F}_{V_u^1}(\eta_{u,\xi}) \overset{(23)}{\geq} a\xi - \frac{b}{2}\xi^2 \overset{(a),(24)}{\geq} \beta\xi - \frac{M}{2}\xi^2 \geq \frac{\beta}{2}\xi > \kappa\xi^2$$

where in the last two inequalities we used, respectively,

$$\xi \leq \gamma := \beta/M , \qquad \frac{\beta}{2\kappa} \overset{(20)}{=} (3 + 2\sqrt{2}) > 1 .$$

Next case is:

$$a = 0 \tag{b}$$

In view of (24), this implies $b \geq \beta$. Then, taking $\eta_{u,\xi} = \xi$:

$$\bar{F}_{V_u^1}(\eta_{u,\xi}) = \frac{\xi^2}{2} b \geq \frac{\xi^2}{2}\beta \overset{(20)}{=} (3 + 2\sqrt{2})\kappa\xi^2 > \kappa\xi^2 .$$

We are left with the case:

$$0 < a < \beta . \tag{c}$$

Note that, again because of (24),

$$0 < a < \beta \overset{(24)}{\leq} b \leq M . \tag{25}$$

Let $\alpha := \sqrt{2\kappa/b}$ and observe that

$$c = \sqrt{\frac{2\kappa}{M}} \leq \alpha \leq \sqrt{\frac{2\kappa}{\beta}} \overset{(20)}{=} \frac{1}{\sqrt{1 + \sqrt{2}}}. \tag{26}$$

We then have three subcases:

$$\alpha\xi > \frac{a}{b} \tag{c$_1$}$$

In this case, we choose

$$\eta_{u,\xi} := \frac{a + \sqrt{a^2 + 2\kappa b \xi^2}}{b} = \frac{a}{b} + \sqrt{\left(\frac{a}{b}\right)^2 + \alpha^2\xi^2} \tag{27}$$

so that

$$\left|a\eta_{u,\xi} - \frac{b}{2}\eta_{u,\xi}^2\right| = b\eta_{u,\xi}\left|\frac{a}{b} - \frac{\eta_{u,\xi}}{2}\right| = \frac{b}{2}\alpha^2\xi^2 = \kappa\xi^2 ,$$

showing, by (23), that (22) is satisfied. Furthermore, by the hypothesis $a/b < \alpha\xi^2$, (26) and the definition of κ in (20), one finds

$$c\xi \overset{(26)}{\leq} \alpha\xi \overset{(27)}{\leq} \eta_{u,\xi} \overset{(c_1)}{<} \alpha\xi(1+\sqrt{2}) \overset{(26)}{\leq} \xi ,$$

proving (21).

Next, we consider the case

$$\xi \geq \frac{a}{b} \geq \alpha\xi \tag{c$_1$}$$

and choose $\eta_{u,\xi} := a/b$. We then find

$$\left|a\eta_{u,\xi} - \frac{b}{2}\eta_{u,\xi}^2\right| = \frac{b}{2}\left(\frac{a}{b}\right)^2 \geq \frac{b}{2}\alpha^2\xi^2 = \kappa\xi^2 ,$$

showing, again by (23), that (22) is satisfied. Inequalities (21) follow immediately by the hypothesis and the fact that $c \leq \alpha$.

The final case is

$$\frac{a}{b} > \xi . \tag{c$_3$}$$

We choose again $\eta_{u,\xi} = \xi$, so that:

$$\left|a\xi - \frac{b}{2}\xi^2\right| \geq b\left(\frac{a}{b}\xi - \frac{\xi^2}{2}\right) > \frac{b}{2}\,\xi^2 \overset{(25)}{\geq} \frac{\beta}{2}\xi^2 \overset{(26)}{>} \kappa\xi^2 .$$

□

4 Proof of the Theorem

First we prove steepness for the third-order polynomial truncation of h.

For $k = 1$, steepness for the third-order polynomial truncation of h follows from Lemma 2. We therefore assume $2 \leq k \leq n-1$, fix $I \in D$, fix V^k a linear space of dimension k in $h(I)^\perp$ and let, as above, $\bar{h}$ denote the third-order Taylor polynomial of h at I. Let $\{\bar{e}_1, \ldots, \bar{e}_k\} \subseteq V^k$ be an orthonormal basis of eigenvectors of $P_{V^k}h''(I)$ with corresponding eigenvalues λ_k so that $|\lambda_1| \leq \cdots \leq |\lambda_k|$. Then, by Lemma 1 and (12) one has

$$\beta \leq |\lambda_j| , \ \forall j \geq 2 ; \qquad |\lambda_j| \leq M_2 \leq M , \forall j . \tag{28}$$

Fix $0 < \xi \leq \xi_k$ and a unit vector $u \in V^k$ and define

$$\bar{\eta} := \eta_{\bar{e}_1,\xi} \tag{29}$$

with $\eta_{\bar{e}_1,\xi}$ as in Lemma 2. Recall that, by (21), it is

$$c\xi \le \bar{\eta} \le \xi\ . \tag{30}$$

We claim that

$$\bar{F}_{V^k}(\bar{\eta}) = \min_{\substack{u\in V^k\\ \|u\|=1}} \left\| P_{V^k}\bar{h}'(I+\bar{\eta}u)\right\| \ge \frac{\kappa}{2}\xi^2\ , \qquad \forall 0<\xi\le\xi_k\ . \tag{31}$$

Estimate (31) says that steepness on V^k can be controlled in terms of steepness along the line in V^k corresponding to the "degenerate" eigenvector $\bar{e}_1$ of $P_{V^k}h''(I)$, where degeneracy means here that $|\lambda_1|$ may be smaller in absolute value than β (and even vanish).
To prove the claim, we let

$$u = \sum_{j=1}^k x_j\bar{e}_j\ , \qquad \sum_{j=1}^k x_j^2 = 1\ ,$$

be the expansion of u in the orthonormal basis $\{\bar{e}_j\}$ and fix

$$\nu := k\,\frac{M}{\beta} = \frac{k}{\gamma}\ . \tag{32}$$

Notice that

$$2\le k\le \nu\ , \qquad \xi_k \overset{(15)}{\le} \frac{2\theta}{3k}\,\frac{1}{\nu} < \frac{1}{\nu}\ . \tag{33}$$

We distinguish two cases: first, assume that:

$$\sum_{j=2}^k x_j^2 \ge \nu^2\bar{\eta}^2\ . \tag{A}$$

In this case, recalling (28), we have

$$\sum_{j=2}^k |\lambda_j||x_j| \ge \sqrt{\sum_{j=2}^k |\lambda_j|^2|x_j|^2} \ge \beta\sqrt{\sum_{j=2}^k x_j^2} \overset{(A)}{\ge} \beta\nu\bar{\eta}\ . \tag{34}$$

Then, observe that, for all $1\le j\le k$,

$$\begin{aligned}\|P_{V^k}\bar{h}'(I+\bar{\eta}u)\| &\ge \|P_{V^k}\bar{h}'(I+\bar{\eta}u)\cdot\bar{e}_j\| \ge \bar{\eta}\|P_{V^k}h^{(2)}(I)u\cdot\bar{e}_j\| - \frac{\bar{\eta}^2}{2}M\\ &= \bar{\eta}|\lambda_j||x_j| - \frac{\bar{\eta}^2}{2}M\ ,\end{aligned}$$

so that, summing over $2\le j\le k$, one gets

$$\begin{aligned}\|P_{V^k}\bar{h}'(I+\bar{\eta}u)\| &\ge \frac{\bar{\eta}}{k}\sum_{2=1}^k |\lambda_j||x_j| - \frac{\bar{\eta}^2}{2}M\\ &\overset{(34)}{\ge} \bar{\eta}^2\Big(\frac{\beta\nu}{k} - \frac{M}{2}\Big) \overset{(32)}{=} \bar{\eta}^2\frac{M}{2}\\ &\overset{(30)}{\ge} c^2\frac{M}{2}\xi^2 = \kappa\xi^2 > \frac{\kappa}{2}\xi^2\ ,\end{aligned}$$

proving the claim (31) in case (A).

Assume now that

$$\sum_{j=2}^{k} x_j^2 < \nu^2 \bar\eta^2 \,. \tag{B}$$

Notice that by (33) $\nu\bar\eta \le \nu\xi_k < 1$ so that $\sum_{j=2}^{k} x_j^2 < 1$ and, hence, $x_1 \neq 0$.

If $\sum_{j=2}^{k} x_j^2 = 0$, i.e. $x_1 = \pm 1$, the claim follows directly from Lemma 2 in view of the choice of $\bar\eta$ in (29).

Therefore, we assume $0 < |x_1| < 1$. Assumption (B) implies that $|x_1|$ is close to 1:

$$1 - |x_1| < 1 - x_1^2 = \sum_{j=2}^{k} x_j^2 \overset{\text{(B)}}{<} \nu^2 \bar\eta^2 \,. \tag{35}$$

Let $\sigma = \operatorname{sign}(x_1)$ so that

$$x_1 - \sigma = \sigma(|x_1| - 1|) \,. \tag{36}$$

Then, recalling (29), by Lemma 2, one has

$$|\bar h'(I + \sigma\bar\eta\bar e_1) \cdot \bar e_1| \ge \kappa\xi^2 \,. \tag{37}$$

We want to approximate $\bar h'(I + \bar\eta u) \cdot \bar e_1$ with $\bar h'(I + \sigma\bar\eta\bar e_1) \cdot \bar e_1$. We do it in two steps.

First, expanding u in the eigen-base $\{\bar e_j\}$ and cancelling out the equal terms, we find:

$$\begin{aligned}
&\bar h'(I + \bar\eta u) \cdot \bar e_1 - \bar h'(I + \bar\eta x_1 \bar e_1) \cdot \bar e_1 \\
&= \bar\eta h''(I) u \cdot \bar e_1 - \bar\eta x_1 h''(I) \bar e_1 \cdot \bar e_1 + \frac{\bar\eta^2}{2} h^{(3)}(I)[\bar e_1, u, u] - \frac{\bar\eta^2}{2} x_1^2 h^{(3)}(I)[\bar e_1]^3 \\
&= \frac{\bar\eta^2}{2} h^{(3)}(I)[\bar e_1, u, u] - \frac{\bar\eta^2}{2} x_1^2 h^{(3)}(I)[\bar e_1]^3 \\
&= \frac{\bar\eta^2}{2} \sum_{(i,j)\neq(1,1)} x_i x_j h^{(3)}(I)[\bar e_1, \bar e_i, \bar e_j] \\
&= \bar\eta^2 x_1 \sum_{j=2}^{k} x_j h^{(3)}(I)[\bar e_1, \bar e_1, \bar e_j] + \frac{\bar\eta^2}{2} \sum_{i,j=2}^{k} x_i x_j h^{(3)}(I)[\bar e_1, \bar e_i, \bar e_j].
\end{aligned}$$

Thus, by Cauchy–Schwarz inequality, (B), (4), (32) and (33),

$$\begin{aligned}
|\bar h'(I + \bar\eta u) \cdot \bar e_1 - \bar h'(I + \bar\eta x_1 \bar e_1) \cdot \bar e_1| &\le \sqrt{k-1}\,\bar\eta^3 \nu M + \frac{k-1}{2} \bar\eta^4 \nu^2 M \\
&\le \xi^2 \cdot \xi_k \Big(\sqrt{k-1} \frac{k}{\gamma^2} + \frac{k-1}{2} \xi_k \frac{k^2}{\gamma^2} \Big) \beta \\
&\le \xi^2 \Big(\xi_k \, 2 \frac{k^2}{\gamma^2} \Big) \\
&\le \xi^2 \theta \beta = \frac{\kappa}{4} \xi^2 \,.
\end{aligned} \tag{38}$$

Next, by (36), (35), (28) and (33), we find:

$$
\begin{aligned}
&|\bar h'(I+\bar\eta x_1\bar e_1)\cdot\bar e_1-\bar h'(I+\bar\eta\sigma\bar e_1)\cdot\bar e_1|\\
&=\Big|\bar\eta\lambda_1\sigma(1-|x_1|)+\frac{\bar\eta^2}{2}(x_1^2-1)h^{(3)}(I)[\bar e_1]^3\Big|\\
&\le\bar\eta^3\nu^2\Big(M+\frac{\xi_k}{2}M\Big)\\
&\le\xi^2\cdot\xi_k\frac{k^2}{\gamma^2}\Big(\frac{\beta}{\gamma}+\frac{\xi_k}{2}\frac{\beta}{\gamma}\Big)\\
&\le\xi^2\theta\beta=\frac{\kappa}{4}\xi^2\,.
\end{aligned}
\tag{39}
$$

Thus, for $\xi\le\xi_k$, by (38) and (39), one gets

$$
\begin{aligned}
\|P_{V^k}\bar h'(I+\bar\eta u)\|&\ge|\bar h'(I+\bar\eta u)\cdot\bar e_1|\\
&\ge|\bar h'(I+\sigma\bar\eta\bar e_1)\cdot\bar e_1|-|\bar h'(I+\bar\eta x_1\bar e_1)\cdot\bar e_1-\bar h'(I+\bar\eta\sigma\bar e_1)\cdot\bar e_1|\\
&\qquad-|\bar h'(I+\bar\eta u)\cdot\bar e_1-\bar h'(I+\bar\eta x_1\bar e_1)\cdot\bar e_1|\\
&\ge\frac{\kappa}{2}\xi^2\,.
\end{aligned}
\tag{40}
$$

proving claim (31) also in case (B).

Finally, from (19), (40) and the definition of ξ_k, Theorem follows. □

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