

ON THE MEASURE OF KAM TORI IN TWO DEGREES OF FREEDOM

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ABSTRACT. A conjecture of Arnold, Kozlov and Neishtadt on the exponentially small measure of the “non-torus” set in analytic systems with two degrees of freedom is discussed.

1. Introduction and main result. In this paper we consider real-analytic, nearly-integrable mechanical systems with two-degrees of freedom, namely, Hamiltonian systems governed by a Hamiltonian, in action-angle variables, of the form

$$H_\varepsilon(y, x) := \frac{1}{2}|y|^2 + \varepsilon f(x) := \frac{y_1^2 + y_2^2}{2} + \varepsilon f(x_1, x_2), \quad (1)$$

with

$$y = (y_1, y_2) \in \mathbb{R}^2, \quad x = (x_1, x_2) \in \mathbb{T}^2 := \mathbb{R}^2 / (2\pi\mathbb{Z})^2, \quad f : \mathbb{T}^2 \rightarrow \mathbb{R}$$

real-analytic, ε a small non negative parameter. The phase space $\mathbb{R}^2 \times \mathbb{T}^2$ is endowed with the standard symplectic form $dy_1 \wedge dx_1 + dy_2 \wedge dx_2$ so that the Hamiltonian flow induced by H_ε ,

$$\phi_{H_\varepsilon}^t : (y_0, x_0) \in \mathbb{R}^2 \times \mathbb{T}^2 \mapsto (y(t), x(t)) := \phi_{H_\varepsilon}^t(y_0, x_0) \in \mathbb{R}^2 \times \mathbb{T}^2,$$

is the solution of standard Hamiltonian equations

$$\begin{cases} \dot{y} = -\partial_x H_\varepsilon = -\varepsilon f_x \\ \dot{x} = \partial_y H_\varepsilon = y + \varepsilon f_y \end{cases}, \quad (y(0), x(0)) = (y_0, x_0).$$

Such equation are equivalent to the Lagrangian Newtonian equations on \mathbb{T}^2 with potential f , i.e.¹,

$$\ddot{x} = -\varepsilon f_x(x), \quad \begin{cases} x(0) = x_0 \\ \dot{x}(0) = y_0 \end{cases}.$$

For $\varepsilon = 0$, the system is integrable, the action variables y_1 and y_2 are integrals of the motions, and all trajectories are simply given by $y(t) = y_0$ and $x(t) = x_0 + \omega t$ where the frequency ω coincides with the constant value y_0 . In particular the 2-tori $\{y_0\} \times \mathbb{T}^2$ are all left invariant by the Hamiltonian flow and whenever the ratio of the frequencies is an irrational number, such tori are spanned densely by any orbit.

As well known, according to classical KAM theory “most” integrable tori $\{y_0\} \times \mathbb{T}^2$ persist for small ε undergoing a small deformation and fill any bounded region of the phase space up to a set of measure at most $\sqrt{\varepsilon}$ (as $\varepsilon \rightarrow 0$); these tori – which

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¹As standard, dot denotes the derivative with respect to “time” t and $\partial_y = (\partial_{y_1}, \partial_{y_2})$ and $\partial_x = (\partial_{x_1}, \partial_{x_2})$ denote the gradients with respect to the variables y and x .

are sometimes called *primary tori* – are Lagrangian graphs over \mathbb{T}^2 and the motion is analytically conjugated to a translation by a Diophantine frequency² ω on \mathbb{T}^2 ; (see, [2] for general information).

This bound on the measure of the complement of primary tori is sharp as it follows immediately by considering the trivial example

$$H_\varepsilon = \frac{y_1^2 + y_2^2}{2} + \varepsilon \cos x_1, \quad (2)$$

which governs the mechanics of a simple pendulum with small gravity coupled with a free rotator. Indeed, this is an integrable system having different topologies for $\varepsilon = 0$ and $\varepsilon > 0$, and for $\varepsilon > 0$ the measure of primary tori in any region $\{|y_i| \leq R\} \times \mathbb{T}^2$ with $\sqrt{\varepsilon} < R/2$, is given by $(4\pi R)^2(1 - \frac{4}{\pi R}\sqrt{\varepsilon})$.

Of course, if one takes into account *all* invariant tori, i.e., primary and *secondary* tori (namely, the invariant tori that arise by effect of the perturbation and that in this trivial example correspond to the (y_1, x_1) -librational orbits of the pendulum with initial data inside the separatrix $\{\frac{1}{2}y_1^2 + \varepsilon \cos x_1 = \varepsilon\}$), one has that the phase space of this integrable system is filled by invariant Lagrangian tori, up to a set of measure zero.

For general systems one does not expect to have a full set of invariant tori (see, also, [12]), however, Arnold, Kozlov and Neishtadt, in Remark 6.17 of [2], write:

It is natural to expect that in a generic (analytic) system with two degrees of freedom and with frequencies that do not vanish simultaneously the total measure of the “non-torus” set corresponding to all the resonances is exponentially small. However, this has not been proved.

Indeed, we can prove the following result.

For $s > 0$, denote

$$\mathbb{T}_s^2 := \{x = (x_1, x_2) \in \mathbb{C}^2 \mid |\operatorname{Im} x_j| < s\} / (2\pi\mathbb{Z}^2), \quad (3)$$

and let \mathbb{B}_s^2 be the Banach space of *real-analytic* functions on \mathbb{T}_s^2 having zero average and finite ℓ^∞ -Fourier norm³:

$$\mathbb{B}_s^2 := \left\{ f = \sum_{\substack{k \in \mathbb{Z}^2 \\ k \neq 0}} f_k e^{ik \cdot x} \mid \|f\|_s := \sup_{\substack{k \in \mathbb{Z}^2 \\ k \neq 0}} |f_k| e^{|k|_1 s} < \infty \right\}. \quad (4)$$

Theorem A. *Let $s > 0$. There exists a set $\mathcal{P}_s \subseteq \mathbb{B}_s^2$, containing an open and dense set, such that the following holds.*

Fix $0 < r < R$, let $D := \{y \in \mathbb{R}^2 \mid r \leq |y| \leq R\}$ and consider the mechanical Hamiltonian system with phase space $D \times \mathbb{T}^2$ and Hamiltonian H_ε as in (1) with potential f belonging to \mathcal{P}_s . Then, there exists $\varepsilon_0, a > 0$ small enough such that, whenever $0 < \varepsilon < \varepsilon_0$, the Liouville measure of the complementary of $\phi_{H_\varepsilon}^t$ -invariant tori in the phase region D is smaller than $R^2 \exp(-\operatorname{const}/\varepsilon^a)$.

²“Diophantine” means that there exists $\alpha, \tau > 0$ such that $|\omega \cdot k| = |\omega_1 k_1 + \omega_2 k_2| \geq \alpha/|k|^\tau$ for any non vanishing integer vector k .

³In this paper $x \cdot y$ denotes the inner product $x_1 y_1 + x_2 y_2$, $|x|$ the Euclidean norm $\sqrt{x_1^2 + x_2^2}$ and $|x|_1$ the 1-norm $|x_1| + |x_2|$; f_k denotes the Fourier coefficient of order k , i.e., $(2\pi)^{-2} \int_{\mathbb{T}^2} f(x) e^{-ik \cdot x} dx$.

Remark 1.1. (i) Notice that in the mechanical case the frequencies $\omega_i := \partial_{y_i} H_0 = y_i$ vanish simultaneously only at $y = 0$: this accounts for the annular shape of the action domain D considered in the above theorem.

(ii) The exponent a is computed in [7], where a detailed proof of the Theorem A will appear.

(iii) The exponential smallness of the “non torus set” (i.e., of the complementary of $\phi_{H_\varepsilon}^t$ -invariant tori) in two degrees of freedom is due to the fact that, in regions where the frequencies do not vanish simultaneously (the origin, in the mechanical case) *there do not appear double resonances* (compare Lemma 3.1 below).

(iv) In three or more degrees of freedom, multiple resonances instead are unavoidable and the exponential bound is in general no more valid. What one can prove is the following

Theorem ([3, 4].) *Consider a real-analytic nearly-integrable mechanical system with potential f , namely, a Hamiltonian system with real-analytic Hamiltonian*

$$H_\varepsilon(y, x) = \frac{1}{2} \sum_{i=1}^n y_i^2 + \varepsilon f(x) ,$$

$(y, x) \in \mathbb{R}^n \times \mathbb{T}^n$ being standard action-angle variables. For “general non-degenerate” potentials f ’s there exists $\varepsilon_0, a > 0$ such that, if $0 < \varepsilon < \varepsilon_0$, then the Liouville measure of the complementary of $\phi_{H_\varepsilon}^t$ -invariant tori is smaller than $\varepsilon |\log \varepsilon|^a$.

The class of “general non-degenerate” potentials is the natural extension to higher dimension of the class \mathcal{P}_s defined in Sect. 2 below. Also this theorem is in agreement (up to the logarithmic correction) with a conjecture by Arnold, Kozlov and Neishtadt⁴.

In the rest of the paper, we shall define \mathcal{P}_s and sketch the proof of Theorem A.

2. The generic set \mathcal{P}_s . Fix once and for all $s > 0$.

In this section we define the generic set of potentials \mathcal{P}_s .

Denote by \mathcal{G}_1^2 the “generators” of one-dimensional maximal lattices in \mathbb{Z}^2 , i.e.,

$$\mathcal{G}_1^2 := \{k = (k_1, k_2) \in \mathbb{Z}^2 : k_1 > 0 \text{ and } \gcd(k_1, k_2) = 1\} \cup \{(0, 1)\} . \tag{5}$$

Then, the list of one-dimensional maximal lattices in \mathbb{Z}^2 is given by the sets $\mathbb{Z}k$ with $k \in \mathcal{G}_1^2$ (explaining the name given to \mathcal{G}_1^2).

Given a function $f \in \mathbb{B}_s^2$ and given $k \in \mathcal{G}_1^2$, we can project f , in Fourier space, on the lattice generated by $k \in \mathcal{G}_1^2$ obtaining a function of the “angle” $k_1 x_1 + k_2 x_2$, as follows

$$\sum_{j \in \mathbb{Z}} f_{jk} e^{ijk \cdot x} =: F^k(k \cdot x)$$

where $\theta \rightarrow F^k(\theta)$ is a real-analytic function on \mathbb{T} defined by

$$F^k(\theta) = \sum_{\substack{j \in \mathbb{Z} \\ j \neq 0}} f_{jk} e^{ij\theta} . \tag{6}$$

One can, then, decompose (in a unique way) the potential f as sum of “one dimensional” functions of the angles $x \cdot k$, as $k \in \mathcal{G}_1^2$:

$$f(x) = \sum_{\substack{k \in \mathbb{Z}^2 \\ k \neq 0}} f_k e^{ik \cdot x} = \sum_{k \in \mathcal{G}_1^2} F^k(x \cdot k) . \tag{7}$$

⁴ [2, Remark 6.18, p. 285]: “It is natural to expect that in a generic system with three or more degrees of freedom the measure of the “non-torus” set has order ε ”.

The functions F^k will play a fundamental role in the forthcoming analysis.

Definition 2.1. Let $0 < \delta \leq 1$ and let

$$K_o(\delta) := c \max \left\{ 1, \frac{1}{s}, \frac{1}{s} \log \frac{1}{s\delta} \right\}, \tag{8}$$

where $c > 1$ is a suitable universal constant. Denote by $\mathcal{P}_s(\delta)$ the set of functions in \mathbb{B}_s^2 such that, for all $k \in \mathcal{G}_1^2$ with $|k|_1 > K_o(\delta)$, one has:

(P1) $|f_k| \geq \delta |k|_1^{-2} e^{-|k|_1 s}$,

while, for all $k \in \mathcal{G}_1^2$ with $|k|_1 \leq K_o(\delta)$, one has:

(P2) $\min_{\theta \in \mathbb{T}} (|\partial_\theta F^k(\theta)| + |\partial_\theta^2 F^k(\theta)|) > 0$;

(P3) $F^k(\theta_1) \neq F^k(\theta_2), \forall 0 \leq \theta_1 < \theta_2 < 2\pi$ such that $\partial_\theta F^k(\theta_1) = \partial_\theta F^k(\theta_2) = 0$.

Then, $\mathcal{P}_s := \bigcup_{\delta > 0} \mathcal{P}_s(\delta)$.

Remark 2.1. (i) It is easy to produce functions in $\mathcal{P}_s(\delta)$. Consider, for example, the function

$$f(x) := 2\delta \sum_{k \in \mathcal{G}_1^2} |k|_1^{-2} e^{-|k|_1 s} \cos(k \cdot x). \tag{9}$$

Such function has Fourier coefficients

$$f_k = \begin{cases} \delta |k|_1^{-2} e^{-|k|_1 s}, & \text{if } \pm k \in \mathcal{G}_1^2 \\ 0, & \text{otherwise} \end{cases}$$

and Fourier projections

$$F^k(\theta) = \delta |k|_1^{-2} e^{-|k|_1 s} \cos \theta.$$

As it is plain, $f \in \mathcal{P}_s(\delta)$.

(ii) The functions in \mathcal{P}_s are general in several ways.

For example, from Proposition 3.1 of [6], it follows easily that:

- (a) \mathcal{P}_s contains an open and dense set in \mathbb{B}_s^2 .
- (b) \mathcal{P}_s is a prevalent set⁵.
- (c) The (weighted) Fourier map

$$j : f \in \mathbb{B}_s^2 \rightarrow \{f_k e^{|k|_1 s}\}_{k \in \mathcal{G}_1^2} \in \ell^\infty(\mathcal{G}_1^2)$$

yields a natural isomorphisms between functions in \mathbb{B}_s^2 and bounded sequences of complex numbers supported on \mathcal{G}_1^2 .

Denote by \mathbf{B}_1 the closed ball of radius one in \mathbb{B}_s^2 and by \mathcal{B} the Borellians in \mathbf{B}_1 .

On \mathbf{B}_1 one can introduce a natural (product) *probability measure*, as follows. Consider, first, the probability measure given by the normalised Lebesgue-product measure on the unit closed ball of $\ell^\infty(\mathcal{G}_1^2)$, namely, the unique probability measure μ on the Borellians of $\{z \in \ell^\infty(\mathcal{G}_1^2) \mid |z|_\infty \leq 1\}$ such that, given Lebesgue measurable sets A_k in the unit complex disk $D_1 := \{w \in \mathbb{C} : |w| \leq 1\}$ with $A_k \neq D_1$ only for finitely many k , one has

$$\mu \left(\prod_{k \in \mathcal{G}_1^2} A_k \right) = \prod_{\{k \in \mathcal{G}_1^2 : A_k \neq D_1\}} \frac{1}{\pi} \text{meas}(A_k)$$

⁵Recall that a Borel set P of a Banach space X is called *prevalent* if there exists a compactly supported probability measure ν on the Borellians of X such that $\nu(x + P) = 1$ for all $x \in X$; compare, e.g., [10], [11].

where “meas” denotes the Lebesgue measure on the unit complex disk D_1 .

Then, *the isometry j induces a probability measure μ_s on the Borellians \mathcal{B} and one has that*

$$\mathcal{P}_s \cap \mathbf{B}_1 \in \mathcal{B}, \quad \text{and} \quad \mu_s(\mathcal{P}_s \cap \mathbf{B}_1) = 1.$$

- (d) Assumption **(P3)** is made in order to simplify (the quite technical and intricate) proofs but it is possible to obtain the main result also without such assumption.

Assumption **(P2)** was used in [13] (see also [14]).

3. Sketch of the proof of Theorem A. Let $f \in \mathcal{P}_s$ (Definition 2.1), i.e., $f \in \mathcal{P}_s(\delta)$ for some $\delta > 0$, which will henceforth be fixed.

In what follows, we denote by c various (possibly different) constants, which may depend upon s, δ, r and R .

3.1. Small divisors and geometry of resonances. Let $\alpha > 0$ and $K \in \mathbb{N}$: α will measure the small divisors appearing and K will be a Fourier cut-off. Later on these parameters will be suitably chosen as functions of ε (see (11) below). In terms of these two parameters we shall describe the geometry of resonances.

Define

- $D^0 := \{y \in D \mid |y \cdot k| \geq \alpha, \quad \forall k \in \mathcal{G}_1^2, \quad |k|_1 \leq K\}$;
- $D^{1,k} := \{y \in D \mid |y \cdot k| < \alpha\}$, for $k \in \mathcal{G}_1^2$;
- $D^1 := \bigcup_{k \in \mathcal{G}_1^2, |k|_1 \leq K} D^{1,k}$;
- For $k \in \mathbb{R}^2 \setminus \{0\}$, denote by $\pi_k : \mathbb{R}^2 \rightarrow \langle k \rangle := \{tk \mid t \in \mathbb{R}\}$ the orthogonal projection onto the 1-dimensional vector space containing k , i.e.,

$$\pi_k y := \frac{y \cdot k}{|k|^2} k,$$

and by π_k^\perp the orthogonal projection onto $\langle k \rangle^\perp$, the vector space orthogonal to k . Notice, that since we are in two space dimensions, $\langle k \rangle^\perp$ is the one-dimensional vector space containing $(k_2, -k_1)$, so that:

$$\pi_k^\perp y := y - \frac{y \cdot k}{|k|^2} k = \frac{y_1 k_2 - y_2 k_1}{|k|^2} (k_2, -k_1). \tag{10}$$

Remark 3.1. (i) Recall that for the model at hand, frequencies $\omega = \partial_y H_0$ and actions y coincide.

(ii) In the language of [16], D^0 is a (α, K) -completely non resonant set, while $D^{1,k}$ is an α -neighbourhood of an exact resonance $y \cdot k = 0$ with $k \in \mathcal{G}_1^2$ and $|k|_1 \leq K$; compare also Appendix A.1.

(iii) Obviously, by the definitions given, it follows immediately that

$$D = D^0 \cup D^1.$$

(iv) For general “geometry of resonances” in the context of nearly-integrable Hamiltonian systems, see, e.g., [15], [16] and, more recently, [9]. For a geometry of resonances specific for two-frequencies systems, see [8], [1] and [13].

Lemma 3.1. *Let $\alpha \leq r/32K$, $k \in \mathcal{G}_1^2$ with $|k| \leq K$. Let, also, $\ell \in \mathbb{Z}^2 \setminus k\mathbb{Z}$ with $|\ell| \leq 8K$. Then,*

$$|y \cdot \ell| \geq \frac{r}{4|k|}, \quad \forall y \in D^{1,k}.$$

Proof. By (10) and the definition of $D^{1,k}$,

$$|\pi_k^\perp y| \geq |y| - \frac{|y \cdot k|}{|k|} > r - \frac{\alpha}{|k|} \geq \frac{r}{2},$$

and, observing that $k_2 \ell_1 - k_1 \ell_2 \in \mathbb{Z} \setminus \{0\}$ (since $\ell \notin k\mathbb{Z}$),

$$|\pi_k^\perp \ell| = \frac{|k_2 \ell_1 - k_1 \ell_2|}{|k|} \geq \frac{1}{|k|}.$$

Thus, (using again that $\langle k \rangle^\perp$ is one-dimensional),

$$\begin{aligned} |y \cdot \ell| &= |\pi_k^\perp y \cdot \pi_k^\perp \ell + \pi_k y \cdot \ell| \geq |\pi_k^\perp y \cdot \pi_k^\perp \ell| - |\pi_k y \cdot \ell| \\ &\geq |\pi_k^\perp y| |\pi_k^\perp \ell| - |\pi_k y| |\ell| \geq \frac{r}{2|k|} - \alpha \frac{|\ell|}{|k|} \geq \frac{r}{4|k|}. \quad \blacksquare \end{aligned}$$

From now on we fix:

$$\alpha := \frac{r}{32K}, \quad K := \varepsilon^{-a}, \tag{11}$$

where $0 < a < 1/6$ will be chosen later small enough.

3.2. Averaging and normal forms. In this section we construct suitable normal forms in the sets D^0 and $D^{1,k}$. The main tool is Proposition 4.1 of [6], which, for convenience of the reader, is reported in Appendix A.1.

To describe the normal forms, we need to introduce proper norms.

Given a domain $D \subset \mathbb{R}^2$ and $r > 0$, we denote by D_r the complex neighbourhood

$$D_r := \{y \in \mathbb{C}^2 \mid |y - y_0| < r, \text{ for some } y_0 \in D\};$$

for a real-analytic function $f : \mathbb{T}_s^n \rightarrow \mathbb{C}$ or $f : D_r \times \mathbb{T}_s^n \rightarrow \mathbb{C}$, we let, respectively,

$$\|f\|_s = \sup_{j \in \mathbb{Z}^n} |f_j| e^{|j|_1 s}, \quad \|f\|_{r,s} = \sup_{j \in \mathbb{Z}^n} \sup_{y \in D_r} |f_j(y)| e^{|j|_1 s}, \tag{12}$$

where $f_j, f_j(y)$ denote Fourier coefficients.

For a given sublattice $\Lambda \subseteq \mathbb{Z}^2$, we denote by p_Λ the Fourier-projection on Λ :

$$p_\Lambda f := \sum_{k \in \Lambda} f_k e^{ik \cdot x}.$$

3.2.1. Normal form on the non-resonant set D^0 . Set

$$r_0 := \alpha/2K.$$

then

$$|y \cdot k| \geq \alpha/2, \quad \forall y \in D_{r_0}^0, \quad \forall 0 < |k| \leq K.$$

From Proposition A.1 it follows that, for ε small enough, there exists a symplectic change of variables

$$\phi_0 : D_{r_0/2}^0 \times \mathbb{T}_{s(1-2/K)}^2 \rightarrow D_{r_0}^0 \times \mathbb{T}_s^2, \tag{13}$$

such that⁶

$$H_\varepsilon \circ \Psi_0 = \frac{|y|^2}{2} + \varepsilon g^\circ(y) + \varepsilon f^\circ(y, x), \quad \langle f^\circ \rangle = 0, \tag{14}$$

where $\langle \cdot \rangle = p_{\{0\}}(\cdot)$ denotes the average with respect to the angles x , and:

$$\sup_{D_{r_0/2}^0} |g^\circ - \langle f \rangle| \leq c \frac{\varepsilon K^2}{\alpha^2}, \quad \|f^\circ\|_{r_0/2, s(1-2/K)/2} \leq e^{-Ks/3}. \tag{15}$$

⁶ f° corresponds to f_{**} in Proposition A.1.

3.2.2. *Normal forms on simply-non-resonant sets* $D^{1,k}$. Fix $k \in \mathcal{G}_{1,K}^n$ and let

$$r_k := \frac{r}{32|k|K} \tag{16}$$

then

$$y \in D_{r_k}^{1,k}, \quad \ell \in \mathbb{Z}^2 \setminus k\mathbb{Z}, \quad |\ell| \leq 8K \implies |y \cdot \ell| \geq \frac{r}{4|k|}.$$

By Proposition A.1, with (α, K) replaced by $(\frac{r}{4|k|}, 8K)$, we see that, for ε small enough, there exists a symplectic change of variables

$$\Psi_k : D_{r_k/2}^{1,k} \times \mathbb{T}_{s_*}^n \rightarrow D_{r_k}^{1,k} \times \mathbb{T}_s^n, \quad s_* := s(1 - 1/K) \tag{17}$$

such that⁷

$$H_\varepsilon \circ \Psi_k =: \frac{|y|^2}{2} + \varepsilon G_0^k(y) + \varepsilon G^k(y, k \cdot x) + \varepsilon f^k(y, x) \tag{18}$$

where

$$\langle G^k(y, \cdot) \rangle = 0, \quad p_{k\mathbb{Z}} f^k = 0, \tag{19}$$

and⁸

$$\sup_{D_{r_k/2}^{1,k}} |G_0^k(y)|, \quad \|G^k - F^k\|_{r_k/2, s_*|k|_1} \leq c\varepsilon|k|_1^2 K^2, \quad \|f^k\|_{r_k/2, s_*/2} \leq 2e^{-(4K-1)s}. \tag{20}$$

Remark 3.2. The function $G^k(y, \theta)$ will be called the **effective potential** since, disregarding the small remainder f^k , it governs the (integrable) Hamiltonian evolution at simple resonances.

3.3. **Exponential density of primary tori in $D^0 \times \mathbb{T}^2$.** In this brief section we show how the exponential density of primary tori in the region $D^0 \times \mathbb{T}^2$ is an immediate consequence of the KAM Theorem, if one chooses suitably the parameter K as a function of ε .

Indeed, we can apply the KAM Theorem A.1 to the Hamiltonian in (14) with $h(y) = |y|^2/2 + \varepsilon g^o(y)$: in this case $h_{yy} = \mathbb{I} + O(\varepsilon)$ and the perturbation εf^o has norm bounded by (see (15)) $\varepsilon e^{-Ks/3}$. Therefore, recalling (11), where we chose $K = 1/\varepsilon^a$, one sees that the KAM condition (45) is met for ε small enough and that, by (47), the relative measure of Diophantine primary tori in $D^0 \times \mathbb{T}^2$ is at least

$$1 - \exp\left(-\frac{s}{6\varepsilon^a}\right). \tag{21}$$

3.4. **The typical effective potential at simple resonances.** In the neighbourhoods $D^{1,k}$ of simple resonances, after the averaging of § 3.2.2, the strategy is to put the integrable Hamiltonian⁹

$$h = \frac{|y|^2}{2} + \varepsilon G_0^k(y) + \varepsilon G^k(y, k \cdot x)$$

into action-angle variables, to check Kolmogorov’s non-degeneracy and then to apply the KAM Theorem A.1.

To do this one has, first, to understand the topological structure associated to the effective potentials G^k for $|k|_1 \leq K$.

⁷ f^k corresponds to f_{**} in Proposition A.1.

⁸ Beware that F^k and G^k are functions of one angle variable, while f^k depends on two angle variables.

⁹ Integrable, since it depends only on the angle $Q = k \cdot x \in \mathbb{T}^1$.

Remark 3.3. (i) In the case¹⁰ $|k|_1 \leq K_o(\delta)$, the topology of the phase space of the effective integrable Hamiltonian can be quite arbitrary, as long as it is non-degenerate, in the sense that the critical points of $\theta \mapsto G^k(y, \theta)$ are non-degenerate and at different energy levels (compare **(P2)**, **(P3)** above).

(ii) On the other hand, for $|k|_1 > K_o(\delta)$, all effective potentials G^k have the same topological features of a pendulum, as we shall briefly describe.

We stress that, while the case in (i) concerns a fixed (i.e., ε -independent) number of modes, the case $K_o(\delta) < |k|_1 \leq K$ concerns a number of modes, which goes to infinity when ε goes to zero. It is therefore essential to have *uniform control* of the case $K_o(\delta) < |k|_1 \leq K$.

(iii) From now on, to simplify the exposition, we shall consider only the case of simple resonances with $K_o(\delta) < |k|_1 \leq K$.

The case $0 < |k|_1 \leq K_o(\delta)$, is similar but more complicated and we omit the details in the present sketch of proof.

Thus, from now on, we fix $k \in \mathcal{G}_1^2$ with $K_o(\delta) < |k|_1 \leq K$.

3.4.1. *Uniform pendulum-like structure of the effective Hamiltonian ($|k|_1 > K_o(\delta)$).* Because of the fast decay of Fourier modes due to analyticity, F^k (recall (6)) has the form:

$$F^k(\theta) = (f_k e^{i\theta} + f_{-k} e^{-i\theta}) + O(e^{-2|k|_1 s}) = 2|f_k| \cos(\theta + \theta_k) + O(e^{-2|k|_1 s}),$$

for a suitable $\theta_k \in [0, 2\pi)$. Recalling **(P1)**, we can factor $|f_k|$, getting

$$F^k(\theta) = 2|f_k| \left(\cos(\theta + \theta_k) + O(|k|_1^2 e^{-|k|_1 s}) \right).$$

In fact, these identities hold in a strong norm (e.g., in $\|\cdot\|_b$ with $b > 1$; compare (12)).

Then, by (20) and **(P1)**, one has¹¹:

$$\frac{1}{|f_k|} \|G^k - F^k\|_{r_k/2, 2} \leq c \frac{1}{|f_k|} e^{-s|k|_1} \|G^k - F^k\|_{r_k/2, s_*|k|_1} \leq c |k|_1^4 K^2 \varepsilon.$$

Hence, recalling (18), and using again **(P1)**, one gets

$$H_\varepsilon \circ \Psi_k =: \frac{|y|^2}{2} + \varepsilon G_0^k(y) + 2|f_k| \varepsilon \left(\cos(k \cdot x + \theta^{(k)}) + \mathbf{G}^k(y, k \cdot x) + \mathbf{f}^k(y, x) \right) \quad (22)$$

with

$$\|\mathbf{G}^k\|_{r_k/2, 2} \leq c K^6 \varepsilon =: \eta, \quad \|\mathbf{f}^k\|_{r_k/2, s_*/2} \leq e^{-5Ks/2}. \quad (23)$$

Recalling that in (11) we assumed $a < 1/6$, we get

$$\eta = O(\varepsilon^{1-6a}) \ll 1. \quad (24)$$

3.4.2. *Rescaling.* For the upcoming analysis it is convenient to make the rescaling¹²

$$y \rightarrow \lambda y, \quad \text{where } \lambda := \sqrt{2|f_k| \varepsilon} \quad (25)$$

followed by a time-rescaling obtained by dividing the Hamiltonian by $\lambda^2 = 2|f_k| \varepsilon$, so as to obtain the Hamiltonian

$$\mathbf{H}_k := h_k(y) + \left(\cos(k \cdot x + \theta^{(k)}) + \mathbf{G}^k(\lambda y, k \cdot x) + \mathbf{f}^k(\lambda y, x) \right), \quad (26)$$

¹⁰Recall (8) for the definition of $K_o(\delta)$.

¹¹Notice that: if $0 < s' < s$ and $\langle f \rangle = 0$, then $\|f\|_{s'} \leq e^{s'-s} \|f\|_s$.

¹²In the following, for ease of notation, we shall sometimes drop the dependence on k , which has been fixed.

where

$$h_k(y) := \frac{|y|^2}{2} + \frac{1}{2|f_k|} G_0^k(\lambda y) . \tag{27}$$

3.4.3. *The fast angle* $Q_2 = k \cdot x$. By Bezout’s Lemma we can find $\bar{k} = (\bar{k}_1, \bar{k}_2) \in \mathbb{Z}^2$ with $|\bar{k}|_\infty \leq |k|_\infty$ such that

$$\bar{k}_1 k_1 - \bar{k}_2 k_2 = 1 .$$

Let

$$A := \begin{pmatrix} \bar{k}_1 & \bar{k}_2 \\ k_1 & k_2 \end{pmatrix} .$$

Applying the canonical transformation

$$\Psi_A : (P, Q) \mapsto (y, x) , \quad y := A^T P , \quad x := A^{-1} Q$$

and noting that $k \cdot x = Q_2$ we get

$$\mathbf{H}_k \circ \Psi_A = h_k(A^T P) + (\cos(Q_2 + \theta^{(k)}) + \mathbf{G}^k(\lambda A^T P, Q_2) + \mathbf{f}^k(\lambda A^T P, A^{-1} Q)) . \tag{28}$$

The aim of this transformation is that, now, the effective potential

$$\cos(Q_2 + \theta^{(k)}) + \mathbf{G}^k(\lambda A^T P, Q_2)$$

depends only on one angle, i.e. Q_2 .

Remark 3.4. The norms of A and A^{-1} is proportional to $|k|_\infty$, and therefore the angle analyticity domain becomes $\mathbb{T}_{s/cK}^2$.

3.4.4. *Decoupling the kinetic energy.* However, this has the unpleasant cost that the main part of the quadratic part in P (the “kinetic energy”) $\frac{1}{2}|A^T P|^2$ is no longer diagonal. In order to diagonalise it one can consider the symplectic map

$$\Psi_U : (p, q) \mapsto (P, Q) , \quad P := U p , \quad Q := (U^{-1})^T q , \tag{29}$$

where

$$U := \begin{pmatrix} 1 & 0 \\ -\bar{k} \cdot k |k|^{-2} & 1 \end{pmatrix} ,$$

Indeed, using such a map, since $A^T U = [\pi_k^\perp \bar{k}, k]$, one finds

$$\frac{1}{2}|A^T U p|^2 = \frac{1}{2}|\pi_k^\perp \bar{k}|^2 p_1^2 + \frac{1}{2}|k|^2 p_2^2 .$$

However, Ψ_U does not yield a diffeomorphism on \mathbb{T}^2 as, in general, $\frac{\bar{k} \cdot k}{|k|^2} \in \mathbb{Q}$ is not integer and, therefore,

$$Q_1 = q_1 + \frac{\bar{k} \cdot k}{|k|^2} q_2$$

is not well defined for $q_2 \in \mathbb{T}^1$. Nevertheless, applying Ψ_U to the “effective Hamiltonian”

$$h_k(A^T P) + (\cos(q_2 + \theta^{(k)}) + \mathbf{G}^k(\lambda A^T P, q_2))$$

we get

$$\frac{1}{2}|\pi_k^\perp \bar{k}|^2 p_1^2 + \frac{1}{2}|k|^2 p_2^2 + W(p) + (\cos(q_2 + \theta^{(k)}) + V(p, q_2)) , \tag{30}$$

where

$$W(p) := \frac{1}{2|f_k|} G_0^k(\lambda A^T U p) , \quad V(p, q_2) := \mathbf{G}^k(\lambda A^T U p, q_2)$$

satisfy

$$\sup_{\mathbf{D}_{r_k}^k} \|\partial_p^2 W\| \leq \eta , \quad \|V\|_{r_k, 2} \leq \eta , \tag{31}$$

with¹³:

$$\mathbf{D}^k := \frac{1}{\lambda} U^{-1} (A^{-1})^T D^{1,k}, \quad \mathbf{r}_k := \frac{r_k}{4\lambda |k|} \geq 1 \quad (32)$$

for ε small enough (recall (11), (16) and (25)).

3.4.5. *Action-angle variables.* Since the “effective Hamiltonian” in (30) does not depend on the angle q_1 , the action p_1 is an integral of motion and plays the role of a parameter. Then, disregarding the dynamically irrelevant term $\frac{1}{2} |\pi_k^{\perp} h|^2 p_1^2$, we study the “pendulum-like Hamiltonian”

$$H_{\text{pend}}(p_2, q_2; p_1) := \frac{1}{2} |k|^2 p_2^2 + W(p) + \left(\cos(q_2 + \theta^{(k)}) + V(p, q_2) \right).$$

H_{pend} is a one dimensional Hamiltonian depending on the parameter p_1 and, therefore, it is integrable introducing suitable action angle variable.

The separatrix divides the phase of H_{pend} into three (p_1 -dependent) open regions: \mathcal{D}_+ , above the separatrix, \mathcal{D}_- , below the separatrix, and \mathcal{D}_0 , inside the separatrix (excluding the elliptic equilibrium), which will contain the (projection of) the secondary tori, i.e., those Lagrangian tori, which are not graphs over the angles.

Next, we construct, in each region, action-angle variables (p_2, q_2) through p_1 -dependent symplectic transformations

$$p_2 = p_2^\sigma(I_2, \varphi_2; p_1), \quad q_2 = q_2^\sigma(I_2, \varphi_2; p_1), \quad (33)$$

with $\sigma = +, -$ or 0 , such that, in the new variable H_{pend} reads

$$H_{\text{pend}}(p_2^\sigma, q_2^\sigma; p_1) =: E^\sigma(p_1, I_2) \quad (34)$$

(which is integrable). Note that the maps in (33) can be easily completed into symplectic transformations

$$\Psi_{\text{aa}}^\sigma : (I_1, I_2, \varphi_1, \varphi_2) \mapsto (p_1, p_2, q_1, q_2)$$

fixing $p_1 = I_1$.

It is important to remark that, even though Ψ_U (defined in (29)) is not well defined on the angles, the composition

$$\Psi_U \circ \Psi_{\text{aa}}^\pm \circ \Psi_U^{-1}$$

is instead well defined.

On the other hand, in the region \mathcal{D}_0 , in view of the different topology, it is actually enough to consider the symplectic transformation

$$\Psi_U \circ \Psi_{\text{aa}}^0,$$

which is well defined.

In the variables (I, φ) the Hamiltonian takes the form

$$\mathbf{h}_k(I) + \mathbf{f}_k(I, \varphi),$$

where

$$\mathbf{h}_k(I) := \frac{1}{2} |\pi_k^{\perp} \bar{k}|^2 I_1^2 + E^\sigma(I), \quad \mathbf{f}_k = O(\exp(-5Ks/2)). \quad (35)$$

¹³The fact that we can choose $r_k/4\lambda|k|$ as new analyticity radius follows by (25) and estimating the operatorial norms of the matrices A and U as $\|A\| \leq 2|k|$ and $\|U\| \leq 2$.

3.4.6. *Kolmogorov’s non-degeneracy.* In order to apply the KAM Theorem A.1 to such Hamiltonian, we need to show that \mathbf{h}_k twists, namely, that the determinant of its Hessian is bounded away from zero.

Remark 3.5. Notice that, recalling (31), for $\eta = 0$, $E^\sigma(I)|_{\eta=0}$ reduces to the pendulum (in action variables) and the twist can be checked by direct computations. As far as one stays away from the separatrix, one can still check the twist perturbatively. However, we need estimates in regions which are exponentially (in $1/\varepsilon$) close to the separatrix and this regime is no longer perturbative, as we are going to explain.

Indeed, denoting by z the distance in energy from the separatrix, it can be shown that, asymptotically as $\eta, z \rightarrow 0$, one has (up to multiplicative $|\log z|^b$ -corrections)

$$\begin{aligned} \det \partial_I^2 \mathbf{h}_k &\cong \det \begin{pmatrix} |\pi_k^\perp \bar{k}|^2 + O(\eta/z) & O(\eta/z) \\ O(\eta/z) & c_0/z \end{pmatrix} \\ &= \frac{c_1}{z} + \frac{O(\eta)}{z^2} \end{aligned} \tag{36}$$

with $c_1 = |\pi_k^\perp \bar{k}|^2 c_0 \neq 0$, and, since z can be much smaller than η , we see that the evaluation in (36) turns into a singular perturbation problem, and hence cannot be handled by usual perturbation techniques.

To overcome this problem, we consider the inverse of the function $I_2 \mapsto E = E^\sigma(I_1, I_2)$, parameterised by I_1 : let us call it $I_2^\sigma(z; I_1)$, where $z := E - E_0$, $E_0 = E_0(I_1)$ being the energy of the separatrix.

Now, one can prove that

$$I_2^\sigma(z; I_1) = \phi^\sigma(z; I_1) + \chi^\sigma(z; I_1) z \log z,$$

with ϕ^σ and χ^σ analytic in z near the origin.

By using analyticity arguments, we can then show that:

For any $\theta > 0$ small enough, up to a region θ -bounded away from separatrices and of measure of order θ^{c_1} for some $0 < c_1 < 1$, the following estimates hold uniformly in $|k| \leq K$:

$$\|\partial_I^2 \mathbf{h}_k\| \leq \frac{1}{\theta}, \quad |\det \partial_I^2 \mathbf{h}_k| \geq \theta. \tag{37}$$

3.5. Exponential density of primary and secondary tori in $D^1 \times \mathbb{T}^2$. In the region where (37) holds, we can apply the KAM Theorem A.1 with $d = \theta$, $M = 1/\theta$, $\mu = \theta^3$, $\varepsilon_0 = O(\exp(-5Ks/2))$, recall (35), $\text{diam } D \leq K/c\lambda$, $r = \theta/cK$ and¹⁴ $s = 1/cK$. Then $\epsilon \leq e^{-5Ks/2}/\theta^2$ and the KAM condition in (45) is satisfied choosing

$$\theta = \exp(-c_2/\varepsilon^a), \tag{38}$$

for a suitable c_2 small enough and ε small enough. Since C in (48) is bounded, for ε small enough, by $1/\lambda^2 \theta^{22} \leq e^{Ks}/\theta^{23}$ (recall (25)), then the measure of the complement of invariant tori is bounded, recalling (47), by

$$C \exp(-5Ks/4) \leq \theta^{-24} \exp(-5Ks/4) \leq \exp(-Ks/4) \leq \theta,$$

for ε small enough. In conclusion, the measure of the complement of invariant tori is bounded by 2θ . Recalling (32) we have that in the starting domain $D^{1,k} \times \mathbb{T}^2$ the

¹⁴Recall Remark 3.4.

measure of the complement of invariant tori is bounded by

$$\frac{1}{c} K^2 \lambda^2 \theta.$$

Then, in the whole region $D^1 \times \mathbb{T}^2$ the measure of the complement of invariant tori is bounded by

$$\frac{1}{c} K^4 \lambda^2 \theta \leq \frac{1}{c'} K^4 \varepsilon \theta \leq \theta$$

for ε small enough.

This last estimate, recalling the definition (38), together with the estimates of § 3.3, concludes the proof of Theorem A.

Appendix A. Normal forms and KAM.

A.1. A normal form lemma. The following normal form lemma is proven in [6, Proposition 4.1]. Before stating it we need some definitions.

- For functions $f : D_r \times \mathbb{T}_s^n \rightarrow \mathbb{C}$ we set

$$\|f\|_{D,r,s} = \|f\|_{r,s} := \sup_{y \in D_r} \sum_{k \in \mathbb{Z}^n} |f_k(y)| e^{|k|_1 s}. \tag{39}$$

The norms $\|\cdot\|_{r,s}$ and $|\cdot|_{r,s}$ are not equivalent, however the following relation holds

$$\begin{aligned} \|f\|_{r,s} \leq \|f\|_{r,s} &\leq (\coth^n(\sigma/2) - 1) \|f\|_{r,s+\sigma} \\ &\leq (2n/\sigma)^n \|f\|_{r,s+\sigma}. \end{aligned} \tag{40}$$

- Given an integrable Hamiltonian $h(y)$, positive numbers α, K and a lattice $\Lambda \subset \mathbb{Z}^n$, a (real or complex) domain U is (α, K) non-resonant modulo Λ (with respect to h) if

$$|h'(y) \cdot k| \geq \alpha, \quad \forall y \in U, \forall k \in \mathbb{Z}^n \setminus \Lambda, |k|_1 \leq K. \tag{41}$$

- Given $f(y, x) = \sum_{k \in \mathbb{Z}^n} f_k(y) e^{ik \cdot x}$ and a sublattice Λ of \mathbb{Z}^n , we denote by p_Λ the projection on the Fourier coefficients in Λ , namely

$$p_\Lambda f := \sum_{k \in \Lambda} f_k(y) e^{ik \cdot x}.$$

and by p_Λ^\perp its “orthogonal” operator (projection on the Fourier modes in $\mathbb{Z}^n \setminus \Lambda$):

$$p_\Lambda^\perp f := \sum_{k \notin \Lambda} f_k(y) e^{ik \cdot x}.$$

Proposition A.1 ([6]). *Let $r, s, \alpha > 0$, $K \in \mathbb{N}$, $K \geq 2$, $D \subseteq \mathbb{R}^n$, and let Λ be a lattice of \mathbb{Z}^n . Let*

$$H(y, x) = h(y) + f(y, x)$$

be real-analytic on $D_r \times \mathbb{T}_s^n$ with $\|f\|_{r,s} < \infty$. Assume that D_r is (α, K) -non-resonant modulo Λ and that

$$\vartheta_\star := \frac{2^{11} K^2}{\alpha r s} \|f\|_{r,s} < 1.$$

Then, there exists a real-analytic symplectic change of variables

$$\Psi : (y', x') \in D_{r_\star} \times \mathbb{T}_{s_\star}^n \mapsto (y, x) \in D_r \times \mathbb{T}_s^n \quad \text{with} \quad r_\star := r/2, \quad s_\star := s(1 - 1/K)$$

satisfying

$$|y - y'|_1 \leq \frac{\vartheta_\star}{2^7 K} r, \quad \max_{1 \leq i \leq n} |x_i - x'_i| \leq \frac{\vartheta_\star}{16 K^2} s,$$

and such that

$$H \circ \Psi = h + f^b + f_\star, \quad f^b := p_\Lambda f + T_K^\perp p_\Lambda^\perp f \tag{42}$$

with

$$|f_\star|_{r_\star, s_\star} \leq \frac{1}{K} \vartheta_\star |f|_{r, s}, \quad |T_K p_\Lambda^\perp f_\star|_{r_\star, s_\star} \leq (\vartheta_\star/8)^K \frac{8}{eK} |f|_{r, s}.$$

Moreover, re-writing (42) as

$$H \circ \Psi = h + g + f_{\star\star} \quad \text{where} \quad p_\Lambda g = g, \quad p_\Lambda f_{\star\star} = 0,$$

one has

$$|g - p_\Lambda f|_{r_\star, s_\star} \leq \frac{1}{K} \vartheta_\star |f|_{r, s}, \quad |f_{\star\star}|_{r_\star, s/2} \leq 2e^{-(K-2)\bar{s}} |f|_{r, s},$$

where

$$\bar{s} := \min \left\{ \frac{s}{2}, \log \frac{8}{\vartheta_\star} \right\}.$$

Remark A.1. The main point of Proposition A.1 concerns the analyticity domain in the angular variables of the renormalised Hamiltonian, which is close to optimal. Indeed, the Fourier coefficients of the new Hamiltonian are shown to decay at the exact same exponential rate as the Fourier coefficients of the original Hamiltonian, at least up to order K , and this fact plays a crucial role in our analysis.

A.2. A KAM theorem.

Theorem A.1. Let $r, s > 0$, $n \geq 2$, $D \subseteq \mathbb{R}^n$ be a bounded set and $H(y, x) = h(y) + f(y, x)$ be a real-analytic Hamiltonian on $D_r \times \mathbb{T}_s^n$, such that

$$M := \sup_D |h_{pp}| < +\infty, \quad d := \inf_D |\det h_{pp}| > 0, \quad \varepsilon_0 := \sup_{D_r \times \mathbb{T}_s^n} |f| < +\infty. \tag{43}$$

Let also

$$\mu := \frac{d}{M^n}, \tag{44}$$

and fix $\tau > n - 1$.

Then, there exists positive constants $c < 1$ depending only on n and τ such that, if

$$\epsilon := \frac{\varepsilon_0}{Mr^2} \leq c \mu^8 s^{4\tau+8}, \tag{45}$$

then the following holds. Define

$$\alpha := \frac{Mr}{\mu s^{3\tau+6}} \sqrt{\epsilon}, \quad \hat{r} := \mu^2 r, \quad r_\epsilon := \frac{1}{c} \frac{\sqrt{\epsilon} r}{\mu}. \tag{46}$$

Then, there exists a positive measure set $\mathcal{T}_\alpha \subseteq D_{\hat{r}} \times \mathbb{T}^n$ formed by “primary” Kolmogorov’s tori; more precisely, for any point $(p, q) \in \mathcal{T}_\alpha$, $\phi_H^t(p, q)$ covers densely an H -invariant, analytic, Lagrangian torus, with H -flow analytically conjugated to a linear flow with (α, τ) -Diophantine frequencies $\omega = h_p(p_0)$, for a suitable $p_0 \in D$; each of such tori is a graph over \mathbb{T}^n r_ϵ -close to the unperturbed trivial graph $\{(p, \theta) = (p_0, \theta) \mid \theta \in \mathbb{T}^n\}$.

Finally, the Lebesgue outer measure of $(D \times \mathbb{T}^n) \setminus \mathcal{T}_\alpha$ is bounded by:

$$\text{meas}((D \times \mathbb{T}^n) \setminus \mathcal{T}_\alpha) \leq C \sqrt{\epsilon} \tag{47}$$

with

$$C := (\max\{\mu^2 r, \text{diam } D\})^n \cdot \frac{1}{c \mu^{n+5} s^{3\tau+6}}. \tag{48}$$

Remark A.2. (i) Theorem A.1 is an immediate consequence of Theorem 1 in [5] (actually, it is just a slightly simplified version of it).

(ii) Notice that $\mu \leq 1$: in fact, since the eigenvalues of h_{pp} are bounded in absolute value by $\|h_{pp}\| \leq M$, one has that $d \leq \sup_D |\det h_{pp}| \leq M^n$.

(iii) The main point of Theorem A.1 is to have a quantitative smallness condition with explicit dependence on the domain D : This is important for our application, since domains (after rescalings and changes of variables) may become very large.

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