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Kac–Moody Lie Algebras *see* Solitons and Kac–Moody Lie Algebras

KAM Theory and Celestial Mechanics

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Introduction

Kolmogorov–Arnol'd–Moser (KAM) theory deals with the construction of quasiperiodic trajectories in nearly integrable Hamiltonian systems and it was motivated by classical problems in celestial mechanics such as the n-body problem. Notwithstanding the formidable bulk of results, ideas and techniques produced by the founders of the modern theory of dynamical systems, most notably by H Poincaré and G D Birkhoff, the fundamental question about the persistence under small perturbations of invariant tori of an integrable Hamiltonian system remained completely open until 1954. In that year, A N Kolmogorov stated what is now usually referred to as the KAM theorem (in the real-analytic setting) and gave a precise outline of its proof, presenting a strikingly new and powerful method to overcome the so-called small-divisor problem (resonances in Hamiltonian dynamics produce, in the perturbation series, divisors which may become arbitrarily small, making convergence argument extremely delicate). Subsequently, KAM theory has been extended and applied to a large variety of different problems, including infinite-dimensional dynamical systems and partial differential equations with Hamiltonian structure. However, establishing the existence of quasiperiodic motions in the n-body problem turned out to be a longer story, which only very recently has reached a satisfactory level; the point being that the n-body problems present strong degeneracies, which violate the main hypotheses of the KAM theorem.

This article gives an account of the ideas and results concerning the construction of quasiperiodic solutions in the planetary n-body problem. The synopsis of the article is the following.

The next section gives the analytical description of the planetary $(1+n)$-body problem.

In the subsection "Kolmogorov's theorem and the RPC3BP (1954)," original version of the KAM theorem is recalled, giving an outline of its proof and showing its implications for the simplest many-body case, namely, the restricted, planar, and circular three-body problem.

In the section "Arnol'd's theorem," the existence of a positive measure set of initial data in phase space giving rise to quasiperiodic motions near coplanar and nearly circular unperturbed Keplerian trajectories is presented. The rest of the section is devoted to the proof of Arnol'd's theorem following the historical developments: Arnol'd's proof (1963a) for the planar three-body case is presented, the extension to the spatial three-body case due to Laskar and Robutel (1995) is discussed, and Herman's proof – in the form given by Féjoz in 2004 – of the general spatial $(1+n)$-case is presented.

In the section "Lower dimensional tori," a brief discussion of the construction of lower-dimensional elliptic tori bifurcating from the Keplerian unperturbed motions is given (these results have been established in the early 2000s).

Finally, the problem of taking into account real astronomical parameter values is considered and a recent result on an application of (computer-assisted) KAM techniques to the solar subsystem formed by Sun, Jupiter, and the asteroid Victoria is briefly mentioned.

The Planetary $(1+n)$-Body Problem

The evolution of $(1+n)$-body systems (assimilated to point masses) interacting only through gravitational attraction is governed by Newton's equations.

If $u^{(i)} \in \mathbb{R}^3$ denotes the position of the ith body in a given reference frame and if m_i denotes its mass, then Newton's equations read

$$\frac{d^2 u^{(i)}}{dt^2} = -\sum_{\substack{0\le j\le n \\ j\ne i}} m_j \frac{u^{(i)} - u^{(j)}}{|u^{(i)} - u^{(j)}|^3}, \quad i = 0, 1, \dots, n \quad [1]$$

Here the gravitational constant is taken to be equal to 1 (which amounts to rescale the time t). Equations [1] are equivalent to the standard Hamilton's equations corresponding to the Hamiltonian function

$$\mathcal{H}_{\text{New}} := \sum_{i=0}^{n} \frac{|U^{(i)}|^2}{2m_i} - \sum_{0\le i<j\le n} \frac{m_i m_j}{|u^{(i)} - u^{(j)}|} \quad [2]$$

where $(U^{(i)}, u^{(i)})$ are standard symplectic variables and the phase space is the "collisionless domain" $\widehat{\mathcal{M}} := \{U^{(i)}, u^{(i)} \in \mathbb{R}^3 \colon u^{(i)} \ne u^{(j)}, 0 \le i \ne j \le n\}$; the symplectic form is the standard one: $\sum_i dU^{(i)} \wedge du^{(i)} := \sum_{i,k} dU_k^{(i)} \wedge du_k^{(i)}$; $|\cdot|$ denotes the standard Euclidean norm. Introducing the symplectic coordinate change $(U, u) = \phi_{\text{hel}}(R, r)$,

$$\phi_{\text{hel}} : \begin{cases} u^{(0)} = r^{(0)}, \quad u^{(i)} = r^{(0)} + r^{(i)} \quad (i = 1, \dots, n) \\ U^{(0)} = R^{(0)} - \sum_{i=1}^{n} R^{(i)}, \quad U^{(i)} = R^{(i)} \\ (i = 1, \dots, n) \end{cases} \quad [3]$$

one sees that the Hamiltonian $\mathcal{H}_{\text{hel}} := \mathcal{H}_{\text{New}} \circ \phi_{\text{hel}}$ does not depend upon $r^{(0)}$ (recall that a local diffeomorphism is called symplectic if it preserves the symplectic form). This means that $R^{(0)}$ ($\equiv$ total linear momentum) is a global integral of motion. Without loss of generality, one can restrict attention to the invariant manifold $\mathcal{M}_0 := \{R^{(0)} = 0\}$ (invariance of eqn [1] by changes of inertial reference frames).

In the "planetary" case, one assumes that one of the bodies, say $i = 0$ (the Sun), has mass much larger than that of the other bodies (this accounts for the index "hel," which stands for "heliocentric"). To make the perturbative character of the problem transparent, one may introduce the following rescalings. Let

$$m_i = \varepsilon \bar{m}_i, \quad X^{(i)} = \frac{R^{(i)}}{\varepsilon m_0^{5/3}}, \quad x^{(i)} = \frac{r^{(i)}}{m_0^{2/3}} \\ (i = 1, \dots, n) \quad [4]$$

and rescale time by a factor $\varepsilon m_0^{7/3}$ (which amounts to dividing the new Hamiltonian by such a factor); then, the flow of the Hamiltonian $\mathcal{H}_{\text{hel}}$ on $\mathcal{M}_0$ is equivalent to the flow of the Hamiltonian

$$\mathcal{H}_{\text{plt}} := \sum_{i=1}^{n} \left(\frac{|X^{(i)}|^2}{2\mu_i} - \frac{\mu_i M_i}{|x^{(i)}|} \right) + \varepsilon \sum_{1\le i<j\le n} \left(X^{(i)} \cdot X^{(j)} - \frac{\bar{m}_i \bar{m}_j / m_0^2}{|x^{(i)} - x^{(j)}|} \right) \quad [5]$$

on the phase space $\mathcal{M} := \{X^{(i)}, x^{(i)} \in \mathbb{R}^3 \colon 1 \le i \le n$ and $0 \ne x^{(i)} \ne x^{(j)}\}$ with respect to the standard symplectic form $\sum_{i=1}^{n} dX^{(i)} \wedge dx^{(i)}$; the mass parameters are defined as

$$M_i := 1 + \varepsilon \frac{\bar{m}_i}{m_0}, \qquad \mu_i := \frac{\bar{m}_i}{m_0 + \varepsilon \bar{m}_i} = \frac{\bar{m}_i}{m_0} \frac{1}{M_i} \quad [6]$$

The following observations can be made:

1. The Hamiltonian
$$\mathcal{H}_{\text{plt}}^{(0)} := \sum_{i=1}^{n} \left(\frac{|X^{(i)}|^2}{2\mu_i} - \frac{\mu_i M_i}{|x^{(i)}|} \right)$$
is integrable and represents the sum of n two-body systems formed by the Sun and the ith planet (disregarding the interaction with the other planets).
2. The transformation ϕ_{hel} in eqn [3] preserves the total angular momentum $\widehat{C} := \sum_{i=0}^{n} U^{(i)} \times u^{(i)}$, which is a vector-valued integral for $\mathcal{H}_{\text{New}}$. Thus, the three components, C_k, of $C := \sum_{i=1}^{n} X^{(i)} \times x^{(i)}$ (which is proportional to $\widehat{C}$ and is termed the "total angular momentum"), are integrals for $\mathcal{H}_{\text{plt}}$. The integrals C_k do not commute: if $\{\cdot,\cdot\}$ denotes the standard Poisson bracket, then $\{C_1, C_2\} = C_3$ (and, cyclically, $\{C_2, C_3\} = C_1, \{C_3, C_1\} = C_2$). Nevertheless, one can form two (independent) commuting integrals, for example, $|C|^2$ and C_3. This shows that the (spatial) $(1+n)$-body problem has $(3n-2)$ degrees of freedom.
3. An important special case is the planar $(1+n)$-body problem. In such a case, one assumes that all the "single" angular momenta $C^{(i)} := X^{(i)} \times x^{(i)}$ are parallel. In this case, the motion takes place on a fixed plane orthogonal to C and (up to a rotation of the reference frame) one can take, as symplectic variables, $X^{(i)}, x^{(i)} \in \mathbb{R}^2$. The Hamiltonian $\mathcal{H}_{\text{pln}}$ governing the dynamics of the planar $(1+n)$-body problem is, then, given on the right-hand side of eqn [5] with $X^{(i)}, x^{(i)} \in \mathbb{R}^2$. Notice that the planar $(1+n)$-body problem has $2n$ degrees of freedom.
4. For a deeper understanding of the perturbation theory of the planetary many-body problem, it is necessary to find "good" sets of symplectic coordinates, which the founders of celestial

mechanics (most notably, Jacobi, Delaunay, and Poincaré) have done. In particular, Delaunay introduced an analytic set of symplectic "action-angle" variables. Recall the Delaunay variables for the two-body "reduced Hamiltonian"

$$\mathcal{H}_{\text{Kep}} = \frac{|X|^2}{2\mu} - \frac{\mu M}{|x|}$$

Let $\{k_1, k_2, k_3\}$ be a standard orthonormal basis in the x-configuration space; let the angular momentum $C = X \times x$ be nonparallel to k_3 and let the energy $E = \mathcal{H}_{\text{Kep}} < 0$. In such a case, $x(t)$ describes an ellipse lying in the plane orthogonal to C, with focus in the origin and fixed symmetry axes. Let a be the semimajor axis of the ellipse spanned by x; $\imath$ (the inclination) be the angle between k_3 and C; $G = |C|$; $\Theta = G \cos \imath = C \cdot k_3$; $L = m\sqrt{Ma}$; ℓ be the mean anomaly of x ($:= 2\pi$ times the normalized area spanned by x measured from the perihelion P, which is the point of the ellipse closest to the origin); θ be the angle between k_1 and $N := k_3 \times C$ (:= oriented "node"); and g be the argument of the perihelion (:= the angle between N and (O, P)). Then (letting $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$)

$$\begin{aligned} &(L, G, \Theta) \in \{L > 0\} \times \{G > \Theta > 0\} \\ &(\ell, g, \theta) \in \mathbb{T}^3 \end{aligned} \qquad [7]$$

are conjugate symplectic coordinates and if ϕ_{Del} is the corresponding symplectic map, then $\mathcal{H}_{\text{Kep}} \circ \phi_{\text{Del}} = -(\mu^3 M^2)/(2L^2)$.

Note that the Delaunay variables become singular when C is vertical (the node is no more defined) and in the circular limit (the perihelion is not unique). In these cases different variables have to be used.

5. Let $(X^{(i)}, x^{(i)}) = \phi_{\text{Del}}((L_i, G_i, \Theta_i), (\ell_i, g_i, \theta_i))$. Then $\mathcal{H}_{\text{plt}}$ expressed in the Delaunay variables $\{(L_i, G_i, \Theta_i), (\ell_i, g_i, \theta_i): 1 \leq i \leq n\}$ becomes

$$\mathcal{H}_{\text{Del}} = \mathcal{H}^{(0)}_{\text{Del}} + \varepsilon \mathcal{H}^{(1)}_{\text{Del}}, \quad \mathcal{H}^{(0)}_{\text{Del}} := -\sum_{i=1}^{n} \frac{\mu_i^3 M_i^2}{2L_i^2} \qquad [8]$$

Note that the number of action variables on which the integrable Hamiltonian $\mathcal{H}^{(0)}_{\text{Del}}$ depends is strictly less than the number of degrees of freedom. This "proper degeneracy," as we shall see in next sections, brings in an essential difficulty one has to face in the perturbative approach to the many-body problem. In fact, this feature of the many-body problem is common to several other problems of celestial mechanics.

Maximal KAM Tori

Kolmogorov's Theorem and the RPC3BP (1954)

Kolmogorov's invariant tori theorem deals with the persistence, in nearly integrable Hamiltonian systems, of Lagrangian (maximal) tori, which, in general, foliate the integrable limit. Kolmogorov (1954) stated his theorem and gave a precise outline of the proof. Let us briefly recall this milestone of the modern theory of dynamical systems.

Let $\mathcal{M} := B^d \times \mathbb{T}^d$ (B^d being a d-dimensional ball in $\mathbb{R}^d$ centered at the origin) be endowed with the standard symplectic form $dy \wedge dx := \sum dy_i \wedge dx_i$ ($y \in B^d$, $x \in \mathbb{T}^d$). A Hamiltonian function N on $\mathcal{M}$ having a Lagrangian invariant d-torus of energy E on which the N-flow is conjugated to the linear dense translation $x \to x + \omega t$, $\omega \in \mathbb{R}^d \backslash \mathbb{Q}^d$ can be put in the form

$$\begin{aligned} &N := E + \omega \cdot y + Q(y, x) \\ &\partial_y^\alpha Q(0, x) = 0, \quad \forall \alpha \in \mathbb{N}^d, \quad |\alpha| \leq 1 \end{aligned} \qquad [9]$$

(as usual, $|\alpha| = \alpha_1 + \cdots + \alpha_d$, $\omega \cdot y := \sum_{i=1}^d \omega_i y_i$, and $\partial_y^\alpha = \partial_{y_1}^{\alpha_1} \cdots \partial_{y_d}^{\alpha_d}$); in such a case, the Hamiltonian N is said to be in Kolmogorov normal form. The vector ω is called the "frequency vector" of the invariant torus $\{y = 0\} \times \mathbb{T}^d$. The Hamiltonian N is said to be nondegenerate if

$$\det \langle \partial_y^2 Q(0, \cdot) \rangle \neq 0 \qquad [10]$$

where the brackets denote average over $\mathbb{T}^d$ and ∂_y^2 the Hessian with respect to the y-variables.

We recall that a vector $\omega \in \mathbb{R}^d$ is said to be "Diophantine" if there exist $\kappa > 0$ and $\tau \geq d - 1$ such that

$$|\omega \cdot k| \geq \frac{\kappa}{|k|^\tau}, \quad \forall k \in \mathbb{Z}^d \backslash \{0\} \qquad [11]$$

The set $\mathcal{D}^d$ of all Diophantine vectors in $\mathbb{R}^d$ is a set of full Lebesgue measure. We also recall that Hamiltonian trajectory is called quasiperiodic with (rationally independent) frequency $\omega \in \mathbb{R}^d$ if it is conjugate to the linear translation $\theta \in \mathbb{T}^d \to \theta + \omega t \in \mathbb{T}^d$.

Theorem (Kolmogorov 1954) *Consider a one-parameter family of real-analytic Hamiltonian functions $H_\varepsilon := N + \varepsilon P$ where N is in Kolmogorov normal form (as in* eqn [9]*) and $\varepsilon \in \mathbb{R}$. Assume that ω is Diophantine and that N is nondegenerate. Then, there exists $\varepsilon_0 > 0$ and for any $|\varepsilon| \leq \varepsilon_0$, a real-analytic symplectic transformation $\phi_\varepsilon : \mathcal{M} \to \mathcal{M}$ putting H_ε in Kolmogorov normal form, $H_\varepsilon \circ \phi_\varepsilon = N_\varepsilon$, with $N_\varepsilon := E_\varepsilon + \omega \cdot y' + Q_\varepsilon(y', x')$. Furthermore, $|E_\varepsilon - E|$, $\|\phi_\varepsilon - \text{id}\|_{C^2}$, and $\|Q_\varepsilon - Q\|_{C^2}$ are small with ε.*

In other words, the Lagrangian unperturbed torus $\mathcal{T}_0 := \{y=0\} \times \mathbb{T}^d$ persists under small perturbation and is smoothly deformed into the H_ε-invariant torus $\mathcal{T}_\varepsilon := \phi_\varepsilon(\{y'=0\} \times \mathbb{T}^d)$; the dynamics on such torus, for all $|\varepsilon| \le \varepsilon_0$, consists of dense quasiperiodic trajectories. Note that the H_ε-flow on $\mathcal{T}_\varepsilon$ is analytically conjugated by ϕ_ε to the translation $x' \to x' + \omega t$ with the same frequency vector of N, while the energy of $\mathcal{T}_\varepsilon$, namely E_ε, is in general different from the energy E of $\mathcal{T}_0$.

Kolmogorov's proof is based on an iterative (Newton) scheme. The map ϕ_ε is obtained as $\lim_{k\to\infty} \phi^{(1)} \circ \cdots \circ \phi^{(k)}$, where the $\phi^{(j)}$'s are (ε-dependent) symplectic transformations of $\mathcal{M}$ successively closer to the identity. It is enough to describe the construction of $\phi^{(1)}$; $\phi^{(2)}$ is then obtained by replacing H_ε with $H_\varepsilon \circ \phi^{(1)}$, and so on. The map $\phi^{(1)}$ is ε-close to the identity and it is generated by $g(y',x) := y' \cdot x + \varepsilon(b \cdot x + s(x) + y' \cdot a(x))$, where s and a are (resp. scalar- and vector-valued) real-analytic functions on $\mathbb{T}^d$ with zero average and $b \in \mathbb{R}^d$; this means that the symplectic map $\phi^{(1)} : (y', x') \to (y, x)$ is implicitly given by the relations $y = \partial_x g$ and $x' = \partial_{y'} g$. It is easy to see that there exists a unique g of the above form such that for a suitable $\varepsilon_0 > 0$,

$$H_\varepsilon \circ \phi^{(1)} = E_1 + \omega \cdot y' + Q_1(y', x') + \varepsilon^2 P_1 \quad \forall\ |\varepsilon| \le \varepsilon_0 \qquad [12]$$

with $\partial_y^\alpha Q_1(0, x') = 0$, for any $\alpha \in \mathbb{N}^d$ and $|\alpha| \le 1$; here, E_1, Q_1, and P_1 depend on ε and, for a suitable $c_1 > 0$ and for $|\varepsilon| \le \varepsilon_0$, $|E - E_1| \le c_1|\varepsilon|$, $\|Q - Q_1\|_{C^2} \le c_1|\varepsilon|$, and $\|P_1\|_{C^2} \le c_1$.

Notice that the symplectic transformation $\phi^{(1)}$ is actually the composition of two "elementary" transformations: $\phi^{(1)} = \phi_1^{(1)} \circ \phi_2^{(1)}$ where $\phi_2^{(1)}: (y', x') \to (\eta, \xi)$ is the symplectic lift of the $\mathbb{T}^d$-diffeomorphism given by $x = \xi + \varepsilon a(\xi)$ (i.e., $\phi_2^{(1)}$ is the symplectic map generated by $y' \cdot \xi + \varepsilon y' \cdot a(\xi)$), while $\phi_1^{(1)} : (\eta, \xi) \to (y, x)$ is the angle-dependent action translation generated by $\eta \cdot x + \varepsilon(b \cdot x + s(x))$; $\phi_2^{(1)}$ acts in the "angle direction" and straightens out the flow up to order $O(\varepsilon^2)$, while $\phi_1^{(1)}$ acts in the "action direction" and is needed to keep the frequency of the torus fixed.

Since $H_\varepsilon \circ \phi_1 =: N_1 + \varepsilon^2 P_1$ is again a perturbation of a nondegenerate Kolmogorov normal form (with same frequency vector ω), one can repeat the construction by obtaining a new Hamiltonian of the form $N_2 + \varepsilon^4 P_2$. Iterating, after k steps, one gets a Hamiltonian $N_k + \varepsilon^{2^k} P_k$. Carrying out the (straightforward but lengthy) estimates, one can check that $\|P_k\|_{C^2} \le c_k \le c^{2^k}$, for a suitable constant $c > 1$ independent of k (the fast growth of the constant c_k is due to the presence of the small divisors appearing in the explicit construction of the symplectic transformations $\phi^{(j)}$). Thus, it is clear that taking ε_0 small enough the iterative procedure converges (superexponentially fast) yielding the thesis of the above theorem.

6. While the statement of the invariant tori theorem and the outline of the proof are very clearly explained in Kolmogorov (1954), Kolmogorov did not fill out the details nor gave any estimates. Some years later, Arnol'd (1963a) published a detailed proof, which, however, did not follow Kolmogorov's idea. In the same year, J K Moser published his invariant curve theorem (for area-preserving twist diffeomorphisms of the annulus) in smooth setting. The bulk of techniques and theorems stemmed out from these works is normally referred to as KAM theory; for reviews, see Arnol'd (1988) or Bost (1984–85). A very complete version of the "KAM theorem" both in the real-analytic and in the smooth case (with optimal smoothness assumptions) is given in Salamon (2004); the proof of the real-analytic part is based on Kolmogorov's scheme. The KAM theory of M Herman, used in his approach to the planetary problem, is based on the abstract functional theoretical approach of R Hamilton (which, in turn, is a development of Nash–Moser implicit function theorem; see Bost (1984–85) for references); it is interesting, however, to note that the heart of Herman's KAM method is based on the above-mentioned Kolmogorov's transformation $\phi^{(1)}$ (compare Féjoz (2002)).
7. In the nearly integrable case, one considers a one-parameter family of Hamiltonians $H_0(I) + \varepsilon H_1(I, x)$ with $(I, x) \in \mathcal{M} := U \times \mathbb{T}^d$ standard symplectic action-angle variables, U being an open subset of $\mathbb{R}^d$. When $\varepsilon = 0$, the phase space $\mathcal{M}$ is foliated by H_0-invariant tori $\{I_0\} \times \mathbb{T}^d$, on which the flow is given by $x \to x + \partial_y H_0(I_0)t$. If I_0 is such that $\omega := \partial_y H_0(I_0)$ is Diophantine and if $\det \partial_y^2 H_0(I_0) \ne 0$, then from Kolmogorov's theorem it follows that the torus $\{I_0\} \times \mathbb{T}^d$ persists under perturbation. In fact, introduce the symplectic variables $(y,\ x)$ with $y = I - I_0$ and let $N(y) := H_0(I_0 + y)$, which by Taylor's formula can be written as $H_0(I_0) + \omega \cdot y + Q(y)$ with $Q(y)$ quadratic in y and $\partial_y^2 Q(0) = \partial_y^2 H_0(I_0)$ invertible. One can then apply Kolmogorov's theorem with $P_1(y, x) := H_1(I_0 + y, x)$.

   Notice that Kolmogorov's nondegeneracy condition $\det \partial_y^2 H_0(I_0) \ne 0$ simply means that the frequency map

$$I \in B^d \subset U \to \omega(I) := \partial_y H_0(I) \qquad [13]$$

is a local diffeomorphism (B^d being a ball around I_0).

8. The symplectic structure implies that if n denotes the number of degrees of freedom (i.e., half of the dimension of the phase space) and d is the number of independent frequencies of a quasi-periodic motion, then $d \leq n$; if $d = n$, the quasi-periodic motion is called maximal. Kolmogorov's theorem gives sufficient conditions in order to get maximal quasiperiodic solutions. In fact, Kolmogorov's nondegeneracy condition is an open condition and the set of Diophantine vectors is a set of full Lebesgue measure. Thus, in general, Kolmogorov's theorem yields a positive invariant measure set spanned by maximal quasiperiodic trajectories.

As mentioned above, the planetary many-body models are properly degenerate and violate Kolmogorov's nondegeneracy conditions and, hence, Kolmogorov's theorem – clearly motivated by celestial mechanics – cannot be applied.

There is, however, an important case to which a slight variation of Kolmogorov's theorem can be applied (Kolmogorov did not mention this in 1954). The case referred to here is the simplest nontrivial three-body problem, namely, the restricted, planar, and circular three-body problem (RPC3BP for short). This model, largely investigated by Poincaré, deals with an asteroid of "zero mass" moving on the plane containing the trajectory of two unperturbed major bodies (say, Sun and Jupiter) revolving on a Keplerian circle. The mathematical model for the restricted three-body problem is obtained by taking $n = 2$ and setting $m_2 = 0$ in eqn [1]: the equations for the two major bodies ($i = 0, 1$) decouple from the equation for the asteroid ($i = 2$) and form an integrable two-body system; the problem then consists in studying the evolution of the asteroid $u^{(2)}(t)$ in the given gravitational field of the primaries. In the circular and planar cases, the motion of the two primaries is assumed to be circular and the motion of the asteroid is assumed to take place on the plane containing the motion of the two primaries; in fact (to avoid collisions), one considers either inner or outer (with respect to the circle described by the relative motion of the primaries) asteroid motions. To describe the Hamiltonian $\mathcal{H}_{\mathrm{rcp}}$ governing the motion of the RCP3BP problem, introduce planar Delaunay variables $((L, G), (\ell, \hat{g}))$ for the asteroid (better, for the reduced heliocentric Sun–asteroid system). Such variables, which are closely related to the above (spatial) Delaunay variables, have the following physical interpretation: G is proportional to the absolute value of the angular momentum of the asteroid, L is proportional to the square root of the semimajor axis of the instantaneous Sun–asteroid ellipse, ℓ is the mean anomaly of the asteroid, while $\hat{g}$ the argument of the perihelion. Then, in suitably normalized units, the Hamiltonian governing the RPC3BP is given by

$$\mathcal{H}_{\mathrm{rcp}}(L, G, \ell, g; \varepsilon) := -\frac{1}{2L^2} - G + \varepsilon \mathcal{H}_1(L, G, \ell, g; \varepsilon) \quad [14]$$

where $g := \hat{g} - \tau, \tau \in \mathbb{T}$ being the longitude of Jupiter; the variables $((L, G), (\ell, g))$ are symplectic coordinates (with respect to the standard symplectic form); the normalizations have been chosen so that the relative motion of the primary bodies is 2π periodic and their distance is 1; the parameter ε is (essentially) the ratio between the masses of the primaries; the perturbation $\mathcal{H}_1$ is the function $x^{(2)} \cdot x^{(1)} - 1/|x^{(2)} - x^{(1)}|$ expressed in the above variables, $x^{(2)}$ being the heliocentric coordinate of the asteroid and $x^{(1)}$ that of the planet (Jupiter): such a function is real-analytic on $\{0 < G < L\} \times \mathbb{T}^2$ and for small ε (for complete details, see, e.g., Celletti and Chierchia (2003)).

The integrable limit

$$\mathcal{H}^{(0)}_{\mathrm{rcp}} := \mathcal{H}_{\mathrm{rcp}}|_{\varepsilon = 0} = -1/(2L^2) - G$$

has vanishing Hessian and, hence, violates Kolmogorov's nondegeneracy condition (as described in item (7) above). However, there is another nondegeneracy condition which leads to a simple variation of Kolmogorov's theorem, as explained briefly below.

Kolmogorov's nondegeneracy condition $\det^2_y H_0(I_0) \neq 0$ allows one to fix d-parameters, namely, the d-components of the (Diophantine) frequency vector $\omega = \partial_y H_0(I_0)$. Instead of fixing such parameters, one may fix the energy $E = H_0(I_0)$ together with the direction $\{s\omega : s \in \mathbb{R}\}$ of the frequency vector: for example, in a neighborhood where $\omega_d \neq 0$, one can fix E and ω_i/ω_d for $1 \leq i \leq d-1$. Notice also that if ω is Diophantine, then so is $s\omega$ for any $s \neq 0$ (with same τ and rescaled κ). Now, it is easy to check that the map $I \in H_0^{-1}(E) \to (\omega_1/\omega_d, \ldots, \omega_{d-1}/\omega_d)$ is (at fixed energy E) a local diffeomorphism if and only if the $(d+1) \times (d+1)$ matrix

$$\begin{pmatrix} \partial^2_y H_0 & \partial_y H_0 \\ \partial_y H_0 & 0 \end{pmatrix}$$

evaluated at I_0 is invertible (here the vector $\partial_y H_0$ in the upper right corner has to be interpreted as a column while the vector $\partial_y H_0$ in the lower left corner has to be interpreted as a row). Such "iso-energetic nondegeneracy" condition, rephrased in terms of Kolmogorov's normal forms, becomes

$$\det\begin{pmatrix} \langle \partial_y^2 Q(0,\cdot)\rangle & \omega \\ \omega & 0 \end{pmatrix} \neq 0 \qquad [15]$$

Kolmogorov's theorem can be easily adapted to the fixed energy case. Assuming that ω is Diophantine and that N is isoenergetically nondegenerate, the same conclusion as in Kolmogorov's theorem holds with $N_\varepsilon := E + \omega_\varepsilon \cdot y' + Q_\varepsilon(y', x')$, where $\omega_\varepsilon = \alpha_\varepsilon \omega$ and $|\alpha_\varepsilon - 1|$ is small with ε.

In the RCP3BP case, the isoenergetic nondegeneracy is met, since

$$\det\begin{pmatrix} \partial^2_{(L,G)}\mathcal{H}^{(0)}_{\text{rcp}} & \partial_{(L,G)}\mathcal{H}^{(0)}_{\text{rcp}} \\ \partial_{(L,G)}\mathcal{H}^{(0)}_{\text{rcp}} & 0 \end{pmatrix} = \frac{3}{L^4}$$

Therefore, one can conclude that on each negative energy level, the RCP3BP admits a positive measure set of phase points, whose time evolution lies on two-dimensional invariant tori (on which the flow is analytically conjugate to linear translation by a Diophantine vector), provided the mass ratio of the primary bodies is small enough; such persistent tori are a slight deformation of the unperturbed "Keplerian" tori corresponding to the asteroid and the Sun revolving on a Keplerian ellipse on the plane where the Sun and the major planet describe a circular orbit.

In fact, one can say more. The phase space for the RCP3BP is four dimensional, the energy levels are three dimensional, and Kolmogorov's invariant tori are two dimensional. Thus, a Kolmogorov torus separates the energy level, on which it lies, into two invariant components, and two Kolmogorov's tori form the boundary of a compact invariant region so that any motion starting in such region will never leave it. Thus, the RCP3BP is "totally stable": in a neighborhood of any phase point of negative energy, if the mass ratio of the primary bodies is small enough, the asteroid stays forever on a nearly Keplerian ellipse with nearly fixed orbital elements L and G.

Arnol'd's Theorem

Consider again the planetary $(1+n)$-body problem governed by the Hamiltonian $\mathcal{H}_{\text{plt}}$ in eqn [5]. In the integrable approximation, governed by the Hamiltonian $\mathcal{H}^{(0)}_{\text{plt}}$, the n planets describe Keplerian ellipses focused on the Sun. Arnol'd (1963b) has stated the following theorem.

Theorem (Arnol'd 1963b) *Let $\varepsilon > 0$ be small enough. Then, there exists a bounded, $\mathcal{H}_{\text{plt}}$-invariant set $\mathcal{F}(\varepsilon) \subset \mathcal{M}$ of positive Lebesgue measure corresponding to planetary motions with bounded relative distances; $\mathcal{F}(0)$ corresponds to Keplerian ellipses with small eccentricities and small relative inclinations.*

This theorem represents a major achievement in celestial mechanics solving more than tri-ćentennial mathematical problem. Arnol'd (1963b) gave a complete proof of this result only in the planar three-body case and gave some indications of how to extend his approach to the general situation. However, to give a full proof of Arnol'd's theorem in the general case turned out to be more than a technical problem and new ideas were needed: the complete proof (due, essentially, to M Herman) has been given only in 2004.

In the following subsections, we briefly review the history and the ideas related to the proof of Arnol'd's theorem. As for credits: the proof of Arnol'd's theorem in the planar 3BP case is due to Arnol'd himself (Arnol'd 1963b); the spatial 3BP case is due to Laskar and Robutel (1995) and Robutel (1995); the general case is due to Herman (1998) and Féjoz (2004). The exposition we have given does not always follow the original references.

The planar three-body problem Recall the Hamiltonian $\mathcal{H}_{\text{pln}}$ of the planar $(1+n)$-body problem given in item (3) of the section "The planetary $(1+n)$-body problem." A convenient set of symplectic variables for nearly circular motions are the "planar Poincaré variables." To describe such variables, consider a single, planar two-body system with Hamiltonian

$$\frac{|X|^2}{2\mu} - \frac{\mu M}{|x|}, \quad X \in \mathbb{R}^2, \quad 0 \neq x \in \mathbb{R}^2 \qquad [16]$$
(with respect to $dX \wedge dx$)

and introduce – as done before formula [14] for $\mathcal{H}^{(0)}_{\text{rcp}}$ – planar Delaunay variables $((L,G),(\ell,g))$ (here, $g = \hat{g}$ = argument of the perihelion). To remove the singularity of the Delaunay variables near zero eccentricities, Poincaré introduced variables $((\Lambda,\eta),(\lambda,\xi))$ defined by the following formulas:

$$\begin{aligned} &\Lambda = L, \quad H = L - G \\ &\lambda = \ell + g, \quad h = -g \\ &\sqrt{2H}\cos h = \eta \\ &\sqrt{2H}\sin h = \xi \end{aligned} \qquad [17]$$

As Poincaré showed, such variables are symplectic and analytic in a neighborhood of $(0,\infty) \times \mathbb{T} \times \{0,0\}$; notice that the symplectic map $((\Lambda,\eta),(\lambda,\xi)) \to (X,x)$ depends on the parameters μ, M, and ε. In Poincaré variables, the two-body Hamiltonian in eqn [16] becomes $-\kappa/(2\Lambda^2)$, with $\kappa := (\mu/m_0)^3/M$. Now, re-insert the index i, let $\phi_i : ((\Lambda_i, \eta_i), (\lambda_i, \xi_i)) \to (X^{(i)}, x^{(i)})$ and $\phi(\Lambda, \eta, \lambda, \xi) = (\phi_1(\Lambda_1, \eta_1, \lambda_1, \xi_1), \ldots, \phi_n(\Lambda_n, \eta_n, \lambda_n, \xi_n))$. Then, the Hamiltonian for the planar $(1+n)$-body problem takes the form

$$\begin{aligned} \mathcal{H}_{\mathrm{pln}} \circ \phi &= \mathcal{H}_0(\Lambda) + \varepsilon \mathcal{H}_1(\Lambda, \lambda, \eta, \xi) \\ \mathcal{H}_0 &:= -\frac{1}{2}\sum_{i=1}^{n} \frac{\kappa_i}{\Lambda_i^2}; \quad \kappa_i := \left(\frac{\mu_i}{m_0}\right)^3 \frac{1}{M_i} \\ \mathcal{H}_1 &:= \mathcal{H}_1^{\mathrm{compl}} + \mathcal{H}_1^{\mathrm{princ}} \end{aligned} \qquad [18]$$

where the so-called "complementary part" $\mathcal{H}_1^{\mathrm{compl}}$ and the "principal part" $\mathcal{H}_1^{\mathrm{princ}}$ of the perturbation are, respectively, the functions

$$\sum_{1 \le i < j \le n} X^{(i)} \cdot X^{(j)} \quad \text{and} \quad \sum_{1 \le i < j \le n} \frac{\mu_i \mu_j}{m_0^2} \frac{1}{|x^{(i)} - x^{(j)}|} \qquad [19]$$

expressed in Poincaré variables.

The scheme of proof of Arnol'd's theorem in the planar, three-body case (one star, $n=2$ planets) is as follows. The Hamiltonian is given by eqn [13] with $n=2$; the phase space is eight dimensional (four degrees of freedom). This system, as mentioned several times, is properly degenerate and Kolmogorov's theorem cannot be applied directly; furthermore, a full (four-dimensional) set of action variables needs to be identified.

A first observation is that, in the planetary model, there are "fast variables" (the λ_i's describing the revolutions of the planets) and "secular variables" (the η_i's and ξ_i's describing the variations of position and shape of the instantaneous Keplerian ellipses). By averaging theory (see, e.g., Arnol'd (1998)), one can "neglect," in nonresonant regions, the fast-angle dependence up to high order in ε obtaining an effective Hamiltonian, which, up to $O(\varepsilon^2)$, is given by the "secular" Hamiltonian

$$\begin{aligned} \mathcal{H}_{\mathrm{sec}} &:= \mathcal{H}_0(\Lambda) + \varepsilon \bar{\mathcal{H}}_1(\Lambda, \eta, \xi) \\ \bar{\mathcal{H}}_1(\Lambda, \eta, \xi) &:= \int \mathcal{H}_1 \frac{d\lambda}{(2\pi)^2} \end{aligned} \qquad [20]$$

"Nonresonant region" means, here, an open Λ-set where $\partial_\Lambda \mathcal{H}_0 \cdot k \neq 0$ for $k \in \mathbb{Z}^2, |k_1| + |k_2| \le K$ and for a suitable $K \ge 1$.

In order to analyze the secular Hamiltonian, we shall beriefly consider $\bar{\mathcal{H}}_1$ as a function of the symplectic variables η and ξ, regarding the "slow actions" Λ_i as parameters.

For symmetry reasons, $\bar{\mathcal{H}}_1$ is even in (η, ξ) and the point $(\eta, \xi) = (0, 0)$ is an elliptic equilibrium for $\bar{\mathcal{H}}_1$: the eigenvalues of the matrix $S\partial^2_{(\eta,\xi)}\bar{\mathcal{H}}_1(\Lambda, 0, 0)$, S being the standard symplectic matrix, are purely imaginary numbers $\{\pm i\Omega_1, \pm i\Omega_2\}$. The real numbers $\{\Omega_i\}$ are symplectic invariants of the secular Hamiltonian and are usually called first (or linear) Birkhoff invariants. In a neighborhood of an elliptic equilibrium, one can use Birkhoff's normal form theory (see, e.g., Siegel (1971)): if the linear invariants (Ω_1, Ω_2) are nonresonant up to order r (i.e., if $\Omega \cdot k := \Omega_1 k_1 + \Omega_2 k_2 \neq 0$ for any $k \in \mathbb{Z}^2$ such that $|k_1| + |k_2| \le r$), then one can find a symplectic transformation ϕ_{Bir} so that

$$\bar{\mathcal{H}}_1 \circ \phi_{\mathrm{Bir}} = F(J_1, J_2; \Lambda) + o_r, \quad J_i = \frac{\eta_i^2 + \xi_i^2}{2} \qquad [21]$$

where F is a polynomial of degree $[r/2]$ of the form $\Omega_1 J_1 + \Omega_2 J_2 + (1/2)\mathcal{M}J \cdot J + \cdots$, $\mathcal{M} = \mathcal{M}(\Lambda)$ being a (2×2) matrix (and $o_r/|J|^{r/2} \to 0$ as $|J| \to 0$). Arnol'd, using computations performed by Le Verrier, checked the nonresonance condition up to order $r=6$ in the asymptotic regime $a_1/a_2 \to 0$ (where a_i denote the semimajor axes of approximate Keplerian ellipses of the two planets); these computations represent one of the most delicate parts of the paper.

Thus, combining averaging theory and Birkhoff normal form theory, one can construct a symplectic change of variables defined on an open subset of the phase space (avoiding some linear resonances) $(\Lambda, \lambda, \eta, \xi) \to (\Lambda', \lambda', J, \varphi)$, where $\eta_j + i\xi_j = \sqrt{2J_j}\exp(i\varphi_j)$, casting the three-body Hamiltonian into the form

$$\begin{aligned} &\mathcal{H}_0(\Lambda') + \varepsilon\big(\Omega(\Lambda') \cdot J + \tfrac{1}{2}\mathcal{M}(\Lambda')J \cdot J\big) \\ &\quad + \varepsilon^2 \mathcal{F}_1(\Lambda', J) + \varepsilon^p \mathcal{F}_2(\Lambda', \lambda', J, \varphi) \\ &:= \widetilde{\mathcal{H}}_0(\Lambda', J; \varepsilon) + \varepsilon^p \mathcal{F}_2(\Lambda', \lambda', J, \varphi) \end{aligned} \qquad [22]$$

for a suitable prefixed order $p \ge 3$; notice that the nonresonance condition needed to apply averaging theory is not particularly hard to check since it involves the unperturbed and completely explicit Kepler Hamiltonian $\mathcal{H}_0$. The idea is now to consider $\varepsilon^p \mathcal{F}_2$ as a perturbation of the completely integrable Hamiltonian $\widetilde{\mathcal{H}}_0$ and to apply Kolmogorov's theorem. Finally, one can check the Kolmogorov's nondegeneracy condition, which since

$$\det \partial^2_{(\Lambda', J)} \widetilde{\mathcal{H}}_0(\Lambda', J'; \varepsilon) = \varepsilon^2\big((\det \mathcal{H}_0'') \det \mathcal{M} + O(\varepsilon)\big)$$

amounts to check the invertibility of the matrix $\mathcal{M}$. Such a condition is also checked in Arnol'd (1963b) with the aid of Le Verrier's tables and in the asymptotic regime $a_1/a_2 \to 0$.

The spatial three-body problem In order to extend the previous argument to the spatial case, Arnol'd suggested connecting the planar and spatial case through a limiting procedure. Such strategy presents analytical problems (the symplectic variables for the spatial case become singular in the planar limit), which have not been overcome. However, the particular structure of the three-body case allows one to derive a four-degree-of-freedom Hamiltonian, to which the proof of the planar case can be easily adapted. The procedure described below is based on the classical Jacobi's reduction of the nodes.

First, we inroduce a convenient set of symplectic variables. Let, for $i=1,2,((L_i,G_i,\Theta_i),(\ell_i,g_i,\theta_i))$ denote the Delaunay variables introduced in items (5) and (6) above: these are the Delaunay variables associated to the two-body system, Sun–ith planet. Then, as Poincaré showed, the variables $((\Lambda_i^*,\lambda_i^*),(\eta_i^*,\xi_i^*),(\Theta_i,\theta_i))$, where

$$\begin{aligned} \Lambda_i^* &= L_i \\ \lambda_i^* &= \ell_i + g_i \\ \eta_i^* &= \sqrt{2(L_i - G_i)}\cos g_i \\ \xi_i^* &= -\sqrt{2(L_i - G_i)}\sin g_i \end{aligned} \qquad [23]$$

are symplectic and analytic near circular, non-coplanar motions; for a detailed discussion of these and other sets of interesting classical variables, see, for example, Biasco *et al.* (2003) and references therein; the asterisk is introduced to avoid confusion with a closely related but different set of Poincaré variables (see below). Let us denote by

$$\mathcal{H}_{3\text{bp}} := \mathcal{H}^{(0)}(\Lambda^*) + \varepsilon\mathcal{H}^{(1)}(\Lambda^*,\lambda^*,\eta^*,\xi^*,\Theta,\theta)$$

the Hamiltonian equation [8] (with $n=2$) expressed in terms of the symplectic variables $((\Lambda^*,\lambda^*),(\eta^*,\xi^*),(\Theta,\theta)),\ \Lambda^*=(\Lambda_1^*,\Lambda_2^*)$, etc. Recalling the physical meaning of the Delaunay variables, one realizes that $\Theta_1+\Theta_2$ is the vertical component, $C_3=C\cdot k_3$, of the total argument $C=C^{(1)}+C^{(2)}$, where $C^{(i)}$ denotes the angular momentum of the ith planet with respect to the origin of an inertial heliocentric frame $\{k_1,k_2,k_3\}$. This suggests that the symplectic variables can be introduced:

$$(\Lambda^*,\lambda^*,\eta^*,\xi^*,\Psi,\psi) = \phi(\Lambda^*,\lambda^*,\eta^*,\xi^*,\Theta,\theta)$$

with $(\Psi_1,\Psi_2,\psi_1,\psi_2):=(\Theta_1,\Theta_1+\Theta_2,\theta_1-\theta_2,\theta_2)$.

Let

$$\mathcal{H}_{3\text{bp}}^* := \mathcal{H}_{3\text{bp}} \circ \phi^{-1}$$

denote the Hamiltonian of the spatial three-body problem in these symplectic variables. Since the Poisson bracket of $\Psi_2=\Theta_1+\Theta_2$ and $\mathcal{H}_{3\text{bp}}^*$ vanishes (C_3 being an integral for the $\mathcal{H}_{3\text{bp}}$-flow), the conjugate angle ψ_2 is cyclic for $\mathcal{H}_{3\text{bp}}^*$, that is,

$$\mathcal{H}_{3\text{bp}}^* = \mathcal{H}_{3\text{bp}}^*(\Lambda^*,\lambda^*,\eta^*,\xi^*,\Psi_1,\Psi_2,\psi_1)$$

Now (because the total angular momentum C is preserved), one may restrict attention to the ten-dimensional invariant (and symplectic) submanifold $\mathcal{M}_{\text{ver}}$ defined by fixing the total angular momentum to be vertical. Such submanifold is easily described in terms of Delaunay variables; in fact, $C\cdot k_1=0=C\cdot k_2$ is equivalent to

$$\theta_1-\theta_2=\pi \quad\text{and}\quad G_1^2-\Theta_1^2=G_2^2-\Theta_2^2 \qquad [24]$$

Thus, $\mathcal{M}_{\text{ver}}^*:=\phi(\mathcal{M}_{\text{ver}})$ is given by

$$\mathcal{M}_{\text{ver}}^* = \left\{\psi_1=\pi,\ \Psi_1=\widehat{\Psi}_1(\Lambda^*,\eta^*,\xi^*;\Psi_2)\right\}$$

with

$$\widehat{\Psi}_1 := \frac{\Psi_2}{2} + \frac{(\Lambda_1^*-H_1^*)^2-(\Lambda_2^*-H_2^*)^2}{2\Psi_2}$$
$$H_i^* := \frac{\eta_i^{*2}+\xi_i^{*2}}{2}$$

Since $\mathcal{M}_{\text{ver}}^*$ is invariant for the flow ϕ_*^t of $\mathcal{H}_{3\text{bp}}^*$, $\psi_1(t)\equiv\pi$ and $\dot\psi_1\equiv 0$ for motions starting on $\mathcal{M}_{\text{ver}}^*$, which implies that $(\partial_{\Psi_1}\mathcal{H}_{3\text{bp}}^*)|_{\mathcal{M}_{\text{ver}}^*}=0$. This fact allows one to introduce, for fixed values of the vertical angular momentum $\Psi_2=c\neq 0$, the following reduced Hamiltonian

$$\begin{aligned} &\mathcal{H}_{\text{red}}^c(\Lambda^*,\lambda^*,\eta^*,\xi^*) \\ &:= \mathcal{H}_{3\text{bp}}^*(\Lambda^*,\lambda^*,\eta^*,\xi^*,\widehat{\Psi}_1(\Lambda^*,\eta^*,\xi^*;c),c,\pi) \end{aligned}$$

on the eight-dimensional phase space $\mathcal{M}_{\text{red}}:=\{\Lambda_i^*>0,\ \lambda\in\mathbb{T}^2,(\eta^*,\xi^*)\in B^4\}$ endowed with the standard symplectic form $\mathrm{d}\Lambda^*\wedge\mathrm{d}\lambda^*+\mathrm{d}\eta^*\wedge\mathrm{d}\xi^*$ (B^4 being a ball around the origin in $\mathbb{R}^4$). In fact, the (standard) Hamilton's equations for $\mathcal{H}_{\text{red}}^c$ are immediately recognized to be a subsystem of the full (standard) Hamilton's equations for $\mathcal{H}_{3\text{bp}}$ when the initial data are restricted on $\mathcal{M}_{\text{ver}}^*$ and the constant value of Ψ_2 is chosen to be c. More precisely, if the Hamiltonian flow of $\mathcal{H}_{\text{red}}^c$ on $\mathcal{M}_{\text{red}}$ is denoted by ϕ_c^t, then

$$\begin{aligned} &\phi_*^t\left(z^*,\widehat{\Psi}_1(\Lambda^*,\eta^*,\xi^*;c),c,\pi,\psi_2\right) \\ &= \left(\phi_c^t(z^*),\widehat{\Psi}_1(t),c,\pi,\psi_2(t)\right) \end{aligned} \qquad [25]$$

where we have used the shorthand notations: $z^*=(\Lambda^*,\lambda^*,\eta^*,\xi^*)\in\mathcal{M}_{\text{red}}$; $\widehat{\Psi}_1(t)=\widehat{\Psi}_1\circ\phi_c^t(z^*)$; $\psi_2(t)=\psi_2+\int_0^t\partial_{\Psi_2}\mathcal{H}_{3\text{bp}}^*(\phi_c^s(z^*),\widehat{\Psi}_1(s),c,\pi)\mathrm{d}s$. At this point, the scheme used for the planar case may be easily adapted to the present situation. The nondegeneracy conditions have been checked in Robutel (1995) where indications, based on a computer program, have been given for the validity of the theorem in a wider set of initial data.

Notice that the dimension of the reduced phase space of the spatial case is 8, which is also the dimension of the phase space of the planar case.

Therefore, also the Lagrangian tori obtained with this procedure have the same dimension of the tori obtained in the planar case (i.e., four).

The general case Consider the general case following the strategy of M Herman as presented by Féjoz (2004), to which the reader is referred for complete proofs and further references.

The symplectic variables used in Féjoz (2004), to cope with the spatial planetary $(1+n)$-body problem (Sun and n planets), are closely related to the variables defined in eqn [23]. For $1 \le i \le n$, let $((L_i, G_i, \Theta_i), (\ell_i, g_i, \theta_i))$ denote the Delaunay variables associated with the two-body system, Sun–ith planet. Then (as shown by Poincaré) the variables $((\Lambda_i, \lambda_i), (\eta_i, \xi_i), (p_i, q_i))$, where $\Lambda_i = L_i, \lambda_i = \ell_i + g_i + \theta_i$, and

$$\begin{aligned} \eta_i &= \sqrt{2(L_i - G_i)}\cos(g_i + \theta_i) \\ \xi_i &= -\sqrt{2(L_i - G_i)}\sin(g_i + \theta_i) \\ p_i &= \sqrt{2(G_i - \Theta_i)}\cos\theta_i \\ q_i &= -\sqrt{2(G_i - \Theta_i)}\sin\theta_i \end{aligned} \qquad [26]$$

are symplectic and analytic near circular, noncoplanar motions (see, e.g., Biasco *et al.* (2003)). Let

$$\mathcal{H}_{\rm nbp} := \mathcal{H}^{(0)}(\Lambda) + \varepsilon\mathcal{H}^{(1)}(\Lambda, \lambda, \eta, \xi, p, q) \qquad [27]$$

denote the Hamiltonian (eqn [8]) expressed in terms of the Poincaré symplectic variables $((\Lambda, \lambda), (\eta, \xi), (p, q))$, $\Lambda = (\Lambda_1, \ldots, \Lambda_n)$, etc.

As the number of the planets increases, the degeneracies become stronger and stronger. Furthermore, a clean reduction, such as the reduction of the nodes, is no more available if $n > 2$. To overcome these problems Herman proposed a new approach, which is described below.

Instead of Kolmogorov's nondegeneracy assumption – which says that the frequency map [13] $I \to \omega(I)$ is a local diffeomorphism – one may consider weaker nondegeneracy conditions. In particular, in Féjoz (2004), one considers nonplanar frequency maps. A smooth curve $u \in A \to \omega(u) \in \mathbb{R}^d$, where A is an open nonempty interval, is called "nonplanar" at $u_0 \in A$ if all the u-derivatives up to order $(d-1)$ at $u_0, \omega(u_0), \omega'(u_0), \ldots, \omega^{(d-1)}(u_0)$ are linearly independent in $\mathbb{R}^d$; a smooth map $u \in A \subset \mathbb{R}^p \to \omega(u) \in \mathbb{R}^d, p \le d$, is called nonplanar at $u_0 \in A$ if there exists a smooth curve $\varphi : \hat{A} \to A$ such that $\omega \circ \varphi$ is nonplanar at $t_0 \in \hat{A}$ with $\varphi(t_0) = u_0$. A S Pyartli has proved (see, e.g., Féjoz (2004)) that if the map $u \in A \subset \mathbb{R}^p \to \omega(u) \in \mathbb{R}^d$ is nonplanar at u_0, then there exists a neighborhood $B \subset A$ of u_0 and a subset $C \subset B$ of full Lebesgue measure (i.e., $\text{meas}(C) = \text{meas}(B)$) such that $\omega(u)$ is Diophantine for any $u \in C$. The nonplanarity condition is weaker than Kolmogorov's nondegeneracy conditions; for example, the map

$$\begin{aligned} \omega(I) &:= \partial_I\left(\frac{I_1^4}{4} + I_1^2 I_2 + I_1 I_3 + I_4\right) \\ &= (I_1^3 + 2I_1 I_2 + I_3, I_1^2, I_1, 1) \end{aligned}$$

violates both Kolmogorov's nondegeneracy and the isoenergetic nondegeneracy conditions but is nonplanar at any point of the form $(I_1, 0, 0, 0)$, since $\omega(I_1, 0, 0, 0) = (I_1^3, I_1^2, I_1, 1)$ is a nonplanar curve (at any point).

As in the three-body case, the frequency map is that associated with the averaged secular Hamiltonian

$$\begin{aligned} \mathcal{H}_{\rm sec} &:= \mathcal{H}^{(0)}(\Lambda) + \varepsilon\bar{\mathcal{H}}^{(1)} \\ \bar{\mathcal{H}}^{(1)}(\Lambda, \eta, \xi, p, q) &:= \int \mathcal{H}^{(1)} \frac{d\lambda}{(2\pi)^n} \end{aligned} \qquad [28]$$

which has an elliptic equilibrium at $\eta = \xi = p = q = 0$ (as above, Λ is regarded as a parameter). It is a remarkably well-known fact that the quadratic part of $\bar{\mathcal{H}}^{(1)}$ does not contain "mixed terms," namely,

$$\begin{aligned} \bar{\mathcal{H}}^{(1)} = \bar{\mathcal{H}}_0^{(1)} &+ \varepsilon(\mathcal{Q}_{\rm pln}\,\eta\cdot\eta + \mathcal{Q}_{\rm pln}\,\xi\cdot\xi + \mathcal{Q}_{\rm spt}\,p\cdot p \\ &+ \mathcal{Q}_{\rm spt}\,q\cdot q + O_4) \end{aligned} \qquad [29]$$

where the function $\bar{\mathcal{H}}_0^{(1)}$ and the symmetric matrices $\mathcal{Q}_{\rm pln}$ and $\mathcal{Q}_{\rm spt}$ depend upon Λ while O_4 denotes terms of order 4 in (η, ξ, p, q). The eigenvalues of the matrices $\mathcal{Q}_{\rm pln}$ and $\mathcal{Q}_{\rm spt}$ are the first Birkhoff invariants of $\bar{\mathcal{H}}^{(1)}$ (with respect to the symplectic variables (η, ξ, p, q)). Let $\sigma_1, \ldots, \sigma_n$ and $\varsigma_1, \ldots, \varsigma_n$ denote, respectively, the eigenvalues of $\mathcal{Q}_{\rm pln}$ and $\mathcal{Q}_{\rm spt}$; then the frequency map for the $(1+n)$-body problem will be defined as (recall eqn [18])

$$\Lambda \to (\hat{\omega}, \varepsilon\Omega) \qquad [30]$$

with

$$\begin{aligned} \hat{\omega} &:= \left(\frac{\kappa_1}{\Lambda_1^3}, \ldots, \frac{\kappa_n}{\Lambda_n^3}\right) \\ \Omega &:= (\sigma, \varsigma) := ((\sigma_1, \ldots, \sigma_n), (\varsigma_1, \ldots, \varsigma_n)) \end{aligned} \qquad [31]$$

Herman pointed out, however, that the frequencies σ and ς satisfy two independent linear relations, namely (up to renumbering the indices),

$$\varsigma_n = 0, \qquad \sum_{i=1}^{n}(\sigma_i + \varsigma_i) = 0 \qquad [32]$$

which clearly prevents the frequency map to be nonplanar; the second relation in eqn [32] is usually called "Herman resonance" (while the first relation is a well-known consequence of rotation invariance).

The degeneracy due to rotation invariance may be easily taken care of by considering (as in the three-body case) the $(6n-2)$-dimensional invariant symplectic manifold $\mathcal{M}_{\text{ver}}$, defined by taking the total angular momentum C to be vertical, that is, $C \cdot k_1 = 0 = C \cdot k_2$. But, when $n > 2$, Jacobi's reduction of the nodes is no more available and to get rid of the second degeneracy (Herman's resonance), the authors bring in a nice trick, originally due – once more! – to Poincaré. In place of considering $\mathcal{H}_{\text{nbp}}$ restricted on $\mathcal{M}_{\text{ver}}$, Féjoz considers the modified Hamiltonian

$$\mathcal{H}^{\delta}_{\text{nbp}} := \mathcal{H}_{\text{nbp}} + \delta C_3^2, \quad C_3 := C \cdot k_3 = |C| \qquad [33]$$

where $\delta \in \mathbb{R}$ is an extra artificial parameter. By an analyticity argument, it is then possible to prove that the (rescaled) frequency map

$$(\Lambda, \delta) \to (\hat{\omega}, \sigma_1, \ldots, \sigma_n, \varsigma_1, \ldots, \varsigma_{n-1}) \in \mathbb{R}^{3n-1}$$

is nonplanar on an open dense set of full measure and this is enough to find a positive measure set of Lagrangian maximal $(3n-1)$-dimensional invariant tori for $\mathcal{H}^{\delta}_{\text{nbp}}$; but, since $\mathcal{H}^{\delta}_{\text{nbp}}$ and $\mathcal{H}_{\text{nbp}}$ commute, a classical Lagrangian intersection argument allows one to conclude that such tori are invariant also for $\mathcal{H}_{\text{nbp}}$ yielding the complete proof of Arnol'd's theorem in the general case. Notice that this argument yields $(3n-1)$-dimensional tori, which in the three-body case means five dimensional. Instead, the tori found in the section "The spatial three-body problem" are four dimensional. The point is that in the reduced phase space, the motion of the nodeline – denoted as $\psi_2(t)$ in eqn [25] – does not appear.

We conclude this discussion by mentioning that the KAM theory used in Féjoz (2004) is a modern and elegant function-theoretic reformulation of the classical theory and is based on a C^∞ local inversion theorem (F Sergeraert and R Hamilton) on "tame" Frechet spaces (which, in turn, is related to the Nash–Moser implicit function theorem; see Bost (1984–85)).

Lower Dimensional Tori

The maximal tori for the many-body problems described above are found near the elliptic equilibria given by the decoupled Keplerian motions. It is natural to ask what happens of such elliptic equilibria when the interaction among planets is taken into account. Even though no complete answer has yet been given to such a question, it appears that, in general, the Keplerian elliptic equilibria "bifurcate" into elliptic n-dimensional tori. This section presents a short and nontechnical account of the existing results on the matter (the general theory of lower-dimensional tori is, mainly, due to J K Moser and S M Graff for the hyperbolic case and V K Melnikov, H Eliasson, and S B Kuksin for the technically more difficult elliptic case; for references, see, e.g., Chierchia *et al.* (2004)).

The normal form of a Hamiltonian admitting an n-dimensional elliptic invariant torus $\mathcal{T}$ of energy E, proper frequencies $\hat{\omega} \in \mathbb{R}^n$, and "normal frequencies" $\Omega \in \mathbb{R}^p$ in a $2d$-dimensional phase space with $d = n + p$ is given by

$$N := E + \hat{\omega} \cdot y + \sum_{j=1}^{p} \Omega_j \frac{\eta_j^2 + \xi_j^2}{2} \qquad [34]$$

Here the symplectic form is given by $dy \wedge dx + d\eta \wedge d\xi$, $y \in \mathbb{R}^n$, $x \in \mathbb{T}^n$, $(\eta, \xi) \in \mathbb{R}^{2p}$; $\mathcal{T}$ is then given by $\mathcal{T} := \{y = 0\} \times \{\eta = \xi = 0\}$. Under suitable assumptions, a set of such tori persists under the effect of a small enough perturbation $P(y, x, \eta, \xi)$. Clearly, the union of the persistent tori (if $n < d$) forms a set of zero measure in phase space; however, in general, n-parameter families persist.

In the many-body case considered in this article, the proper frequencies are the Keplerian frequencies given by the map $\Lambda \to \hat{\omega}(\Lambda)$ (eqn [31]), which is a local diffeomorphism of $\mathbb{R}^n$. The normal frequencies Ω, instead, are proportional to ε and are the first Birkhoff invariants around the elliptic equilibria as discussed above. Under these circumstances, the main nondegeneracy hypothesis needed to establish the persistence of the Keplerian n-dimensional elliptic tori boils down to the so-called Melnilkov condition:

$$\Omega_j \neq 0 \neq \Omega_i - \Omega_j, \quad \forall j \neq i \qquad [35]$$

Such condition has been checked for the planar three-body case in Féjoz (2002), for the spatial three-body case in Biasco *et al.* (2003) and for the planar n-body case in Biasco *et al.* (2004). The general spatial case is still open: in fact, while it is possible to establish lower-dimensional elliptic tori for the modified Hamiltonian $\mathcal{H}^{\delta}_{\text{nbp}}$ in [33], it is not clear how to conclude the existence of elliptic tori for the actual Hamiltonian $\mathcal{H}_{\text{nbp}}$ since the argument used above works only for Lagrangian (maximal) tori; on the other hand, the direct asymptotics techniques used in Biasco *et al.* (2003) do not extend easily to the general spatial case.

Clearly, the lower-dimensional tori described in this section are not the only ones that arise in n-body dynamics. For more lower-dimensional tori in the planar three-body case, see Féjoz (2002).

Physical Applications

The above results show that, in principle, there may exist "stable planetary systems" exhibiting quasiperiodic motions around coplanar, circular Keplerian trajectories – in the Newtonian many-body approximation – provided the masses of the planets are much smaller than the mass of the central star.

A quite different question is: in the Newtonian many-body approximation, is the solar system or, more in generally, a solar subsystem stable?

Clearly, even a precise mathematical reformulation of such a question might be difficult. However, it might be desirable to develop a mathematical theory for important physical models, taking into account observed parameter values.

As a very preliminary step in this direction, consider one of the results of Celletti and Chierchia (see Celletti and Chierchia (2003), and references therein).

In Celletti and Chierchia (2003), the (isolated) subsystem formed by the Sun, Jupiter, and asteroid Victoria (one of the main objects in the Asteroidal belt) is considered. Such a system is modeled by an order-10 Fourier truncation of the RPC3BP, whose Hamiltonian has been described in the section "Kolmogorov's theorem and the RPC3BP (1954)." The Sun–Jupiter motion is therefore approximated by a circular one, the asteroid Victoria is considered massless, and the motions of the three bodies are assumed to be coplanar; the remaining orbital parameters (Jupiter/Sun mass ratio, which is approximately 1/1000; eccentricity and semimajor axis of the osculating Sun–Victoria ellipse; and "energy" of the system) are taken to be the actually observed values. For such a system, it is proved that there exists an invariant region, on the observed fixed energy level, bounded by two maximal two-dimensional Kolmogorov tori, trapping the observed orbital parameters of the osculating Sun–Victoria ellipse.

As mentioned above, the proof of this result is computer assisted: a long series of algebraic computations and estimates is performed on computers, keeping a rigorous track of the numerical errors introduced by the machines.

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See also: Averaging Methods; Diagrammatic Techniques in Perturbation Theory; Gravitational *N*-Body Problem (Classical); Hamiltonian Systems: Stability and Instability Theory; Hamilton–Jacobi Equations and Dynamical Systems: Variational Aspects; Korteweg–de Vries Equation and Other Modulation Equations; Stability Problems in Celestial Mechanics; Stability Theory and KAM.

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