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A note on the construction of Hamiltonian trajectories along heteroclinic chains

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Abstract. We provide a short, simple proof of the existence of Hamiltonian trajectories arbitrarily close to a given chain of heteroclinic orbits connecting "codimension-one, KAM, whiskered tori".

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1 Introduction

"Arnold diffusion", i.e., order-one drift of action variables in general nearly-integrable Hamiltonian systems (with more than two degrees of freedom), takes place near long chains of heteroclinic orbits connecting lower-dimensional, invariant, whiskered tori. For a general theory of Arnold diffusion for perturbations of "a-priori unstable" nearly-integrable Hamiltonian systems, see [2] (roughly speaking, "a-priori unstable" nearly-integrable Hamiltonian systems are perturbations of integrable Hamiltonian systems which possess a one-dimensional separatrix).

This note is devoted to give a short (albeit complete) proof of the existence of drifting (or "shadowing") orbits along a *given (transverse) heteroclinic chain of co-dimension-one invariant whiskered KAM tori*. Here, "codimension-one" means $(l-1)$-dimensional, if l is the number of degrees of freedom (with $l > 2$); "whiskered" means (as in [1]) that the $(l-1)$-dimensional tori possess two asymptotic l-dimensional invariant manifolds (phase points on such manifolds evolve approaching or leaving at an exponential rate the associated tori); "KAM" means that the tori together with their whiskers are constructed by a Kolmogorov-Arnold-Moser technique (see [2], § 5 and also [5]). Such KAM technique, in particular, yields a very strong *normal form*, which describes *exactly* the motion of a $(l+1)$-dimensional neighbourhood of the torus: this normal form is at the basis of the construction of the shadowing orbits presented here (as well as in § 8 of [2]). Finally "heteroclinic chain" stands for an ordered set of orbits belonging simultaneously to the departing whisker of one torus and to the approaching whisker of the successive torus in the chain; the word "transverse" means that the approaching and departing whiskers intersect transversally (often the "approaching/departing whiskers" are also called "stable/ unstable whiskers").

The existence problem for chains of whiskered tori is not addressed in this note: we simply mention that the existence of such chains is established, in the context of a-priori unstable nearly-integrable Hamiltonian systems (under suitable "regularity" assumptions), in [2], where, in particular, the "gap bridging problem" has been overcome for the first time. The "gap bridging problem" is the problem of connecting whiskered tori which are separated by the gaps appearing in KAM constructions; the "gap bridging mechanism" introduced in [2] is based on a quantitative comparison between the size of the gaps and the size of the "homoclinic splittings" (a measure of the transversality of the intersection of the approaching and departing whiskers of a same whiskered torus) in certain region of phase space having suitable "non-resonance properties" (see Lemma 3 of § 7 and § 8 in [2]). Here we simply *assume* to have a heteroclinic chain (see item (i) in § 2 below), while the main conclusions of the KAM analysis worked out in § 5 of [2] are summarized in item (ii) and (iii) of § 2 below. Such items may be regarded as a set of three axioms, which are nontrivially verified in cases considered in [2]: items (ii) and (iii) are proven in § 5 of [2] under rather general assumptions, which include, possibly time-dependent, a-priori unstable systems (but may be also applied to a-priori *stable* systems as discussed in § 11 of [2] in the paragraph containing formula (11.4)); item (i) is proven for "general" a-priori unstable systems (perturbed by a trigonometric polynomial in the angle-variables), see § 7 and § 8 up to the paragraph containing formula (8.9) of [2]. With this approach we hope to provide a conceptually clear distinction between the (much more difficult) problem of constructing KAM heteroclinic chains and the problem of constructing shadowing orbits along them.

The proof presented here follows the scheme given in [2] and corrects a minor error in § 8 of [2] (see [3]). In [2] certain parameters (measuring the expansion rates in the "hyperbolic variables" and in the "quasi-periodic variables") are taken to be different and this, in general, might not be possible (or needs at least a justification). If one takes such parameters *equal*, the proof in § 8 of [2] goes through word-by-word. Such a proof is rather intricate mainly because it attempts to find "reasonable bounds" on diffusion times, i.e., on the times needed by shadowing orbits to go from one end of the chain to the other end. In fact, the proof presented here would lead to time estimates that are even worse than the exponential estimates claimed in [2] (here "exponential" means "exponential in an inverse power of the perturbation parameter" and "even worse" refers to a *chain* of exponentials).

The proof discussed here presents also the following two differences with that of [2]: 1) the construction of the shadowing orbits here is based on special curves parameterized by the "hyperbolic variables" and this has the advantage of stressing the role played by the hyperbolic variables in the Hamiltonian context and of making (possibly) more clear a comparison with standard (but technically quite different) tools of hyperbolic dynamics such as, for example, the so-called "lambda lemma"; 2) the hypotheses made here are slightly more general than what is used in § 8 of [2] (in particular, the whiskers here are *not* assumed to be "graphs over the angles" as it happens in the applications discussed in [2]; compare, also, Lemma 2 below).

In view of the considerable interest devoted to Arnold diffusion, we feel it worthwhile to produce a short, self-contained and—we believe—conceptually clear proof of the existence of orbits shadowing a given chain of whiskered tori.

Finally, here we do *not* address at all the interesting problem of finding "good bounds" on diffusion times also because (probably due to the full constructiveness of our arguments) the time estimates, which would follow from the below analysis, would be quite unsatisfactory. In this respect, we solely mention that the method of [2] and of the present paper can be improved so as to give much more reasonable estimates: see [4], where a complete theory (i.e. construction of heteroclinic chains, construction of shadowing orbits and explicit estimates on diffusion times) is worked out in the case of "isochronous systems". "Isochronous systems" are systems having fixed quasi-periodic frequencies and may be also viewed as quasi-periodic-in-time perturbations of a pendulum (we remark, however, that isochronous systems are "gapless systems", a fact, that not only makes possible to avoid completely the above mentioned "gap bridging problem" but also renders shadowing of transition chains easier).

Next section contains precise definitions and a complete proof of the construction of shadowing orbits. An easy technical consequence (Lemma 1) of a suitable transversality assumptions is included for completeness in the appendix[1].

2 Construction of shadowing orbits

Let ϕ^t denote the flow at time t generated by a (real-analytic) Hamiltonian H on a fixed energy level $\mathscr{S}_E \equiv \{H = E\}$ contained in the phase space $V \times \mathbb{T}^l$ endowed with standard canonical "action-angle variables"; here V is some bounded domain in $\mathbb{R}^l$ and $\mathbb{T}^l$ denotes the standard flat l-dimensional torus. We *assume that, in $\mathscr{S}_E$, there exist a "transverse, heteroclinic chain of whiskered, codimension-one KAM tori"*, i.e., there exist $(l-1)$-dimensional (different) tori $\mathscr{T}_1, \ldots, \mathscr{T}_N$ such that each torus $\mathscr{T}_i$ is invariant (for ϕ^t) and is included into two invariant l-dimensional manifolds ("whiskers") W_i^s and[2] W_i^u on which the motions are asymptotic to it and, furthermore:

(i) For $1 \le i \le N-1$, W_i^u intersects W_{i+1}^s transversally in $\mathscr{S}_E$ at a point[3] z_i.

(ii) The dynamics in a neighbourhood U_i of the torus $\mathscr{T}_i$ is described by standard "KAM normal" coordinates[4] $(\vec{A}', \vec{\psi}, p, q) \in \mathbb{R}^{l-1} \times \mathbb{T}^{l-1} \times \mathbb{R} \times \mathbb{R}$: there exists

[1] In the rest of this note we do quite a large use of footnotes: such footnotes contains no essential arguments, may be skipped and are intended as side comments or reminders.

[2] The superscripts s and u stand for "stable" and "unstable".

[3] We recall that if S is a differentiable manifold and if M and N are two submanifolds of S one says that M intersects transversally N in S at a point $p \in M \cap N$ if $T_pM + T_pN = T_pS$ ("T_p" denoting, here, "tangent space at p of").

[4] The notations we use here follow quite closely those of [2]: in [2] the canonical variables on the phase space $V \times \mathbb{T}^l$ are denoted $(\vec{A}, I, \vec{\alpha}, \varphi)$ and the "KAM normal" coordinates $(\vec{A}', \vec{\psi}, p,$

$r, \hat{r} > 0$, a canonical change of variables $\mathscr{C}$, and (real-analytic) functions of one variable $\vec{A}'_i, \vec{\omega}_i, g_i$, with $|\vec{A}'_i(J) - \vec{A}'_i(0)| < r$ for every $|J| < \hat{r}^2$, satisfying the following properties:

$$\mathscr{C} : \hat{U}_i \equiv \{|\vec{A}' - \vec{A}'_i(0)| < r, \vec{\psi} \in \mathbb{T}^{l-1}, |p| < \hat{r}, |q| < \hat{r}\} \to U_i$$

$$H \circ \mathscr{C}(\vec{A}'_i(pq), \vec{\psi}, p, q) \equiv E, \quad \forall \vec{\psi} \in \mathbb{T}^{l-1}, |p| < \hat{r}, |q| < \hat{r}, \tag{2.1}$$

$$\partial_p(H \circ \mathscr{C})(\vec{A}'_i(0), \vec{\psi}, 0, \hat{r}/2) \neq 0, \quad \forall \vec{\psi} \in \mathbb{T}^{l-1},$$

and, for p, q, t satisfying $|pe^{-g_i(pq)t}| < \hat{r}, |qe^{g_i(pq)t}| < \hat{r}$,

$$\phi^t \circ \mathscr{C}(\vec{A}'_i(pq), \vec{\psi}, p, q)) = \mathscr{C}(\vec{A}'_i(pq), \vec{\psi} + \vec{\omega}_i(pq)t, pe^{-g_i(pq)t}, qe^{g_i(pq)t}). \tag{2.2}$$

The dependence of $\vec{\omega}_i$ upon the variable $J = pq$ is scalar: $\vec{\omega}_i(J) = \tau_i(J)\vec{\omega}_i^0$ and the functions τ_i and g_i are uniformly bounded and bounded away from zero; the constant vectors $\vec{\omega}_i^0$ are rationally independent[5].

(iii) The torus $\mathscr{T}_i$ in the above KAM coordinates is given by

$$\mathscr{T}_i = \mathscr{C}(\{(\vec{A}'_i(0), \vec{\psi}, 0, 0) : \vec{\psi} \in \mathbb{T}^{l-1}\}),$$

while, if we denote by

$$W^s_{i,\mathrm{loc}} = \mathscr{C}(\{(\vec{A}'_i(0), \vec{\psi}, p, 0) : \vec{\psi} \in \mathbb{T}^{l-1}, |p| < \hat{r}\}),$$

$$W^u_{i,\mathrm{loc}} = \mathscr{C}(\{(\vec{A}'_i(0), \vec{\psi}, 0, q) : \vec{\psi} \in \mathbb{T}^{l-1}, |q| < \hat{r}\}),$$

the "local stable/unstable whiskers", then

$$W^s_i = \bigcup_{t \le 0} \phi^t(W^s_{i,\mathrm{loc}}), \quad W^u_i = \bigcup_{t \ge 0} \phi^t(W^u_{i,\mathrm{loc}}).$$

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q); as in [2], the vector symbol $\vec{}$ will be attached to quantities in $\mathbb{R}^{l-1}$ or in $\mathbb{T}^{l-1}$; ϕ^t and $\mathscr{C}$ are denoted in [2], respectively, S_t and $\mathscr{C}_\infty$ (see, in particular, [2], page 33, fifth paragraph: $\mathscr{C}$ is the transformation generated by $\tilde{\phi}_\infty$); $\vec{\omega}_i$ and g_i in (2.2) correspond, respectively, to $(1 + \gamma(J, \sigma_i, \mu))\vec{\omega}_{\sigma_i}$ and $g_{\sigma_i}(1 + \gamma'(pq, \sigma_i, \mu))$ in [2].

[5] In fact the rotation vectors arising in the KAM construction of [2] satisfy standard diophantine conditions. The condition in the third line of (2.1) is a nondegeneracy condition for points on the unstable whisker; obviously (reversing time) the analogous condition for the stable whisker would do as well. We also remark that there is nothing special about the section $\{q = \hat{r}/2\}$ and the nondegeneracy condition might be replaced, e.g., by requiring that $\partial_p(H \circ \mathscr{C})(\vec{A}'_i(0), \vec{\psi}, 0, r') \neq 0, \forall \vec{\psi} \in \mathbb{T}^{l-1}$ for some $0 < r' < \hat{r}$.

As mentioned in the introduction items (ii) and (iii) are the main content of the KAM linearization proved, in the context of a-priori unstable nearly-integrable Hamiltonian systems, in § 5 of [2] (for the correspondence between symbols used here and those used in [2] we refer to the footnote 4). Also, assumption (i) may be checked, under general conditions, in the context of a-priori unstable nearly-integrable Hamiltonian systems (see § 6 ÷ § 8 of [2]).

Note that the set of points $\{(\vec{A}'_i(pq), \vec{\psi}, p, q)\}$ form an $(l+1)$-dimensional manifold in the $(2l-1)$-dimensional energy manifold (if $l = 3$ this is a 4-dimensional manifold in the 5 dimensional energy manifold). Hence the above item *does not* describe all the motions near the torus, but only a small subset of them (which, however, are in a way "perfectly" described).

We now consider an l-dimensional connected submanifold Δ_i^s of the stable whisker W_{i+1}^s lying in U_i and containing a point w_i in the transverse intersection between W_i^u and W_{i+1}^s. Such a submanifold may be parameterized, in terms of normal coordinates, as:

$$\Delta_i^s \equiv \mathscr{C}(\{(\vec{A}_i^*(u), \vec{\psi}_i^*(u), p_i^*(u), q_i^*(u)), u \in U \subset \mathbb{R}^l\}),$$

where U is some l-ball of parameters and the starred letters denote smooth functions. Without loss of generality, we can also write

$$w_i = \mathscr{C}(\hat{w}_i) \quad \text{with} \quad \hat{w}_i = (\vec{A}'_i(0), \vec{\psi}_i, 0, \hat{r}/2) = (\vec{A}_i^*(0), \vec{\psi}_i^*(0), p_i^*(0), q_i^*(0)),$$

(to get the q-value equal, for all i, to $\hat{r}/2$, one can move w_i with the flow). We now build a curve in Δ_i^s, passing through w_i, parameterized by the hyperbolic variable p and with q-coordinate fixed. More precisely we have the following

Lemma 1. *With the above assumptions and notations, there exist a suitable* $\tilde{r} > 0$ *and a suitable (smooth) function* $\vec{\psi}_i(p)$ *such that* $\vec{\psi}_i(0) = \vec{\psi}_i$ *and the curve* Γ_i *defined by* $\Gamma_i \equiv \{\Gamma_i(p) \equiv (\vec{A}'_i(p\hat{r}/2), \vec{\psi}_i(p), p, \hat{r}/2), |p| < \tilde{r}\}$ *satisfies*

$$\mathscr{C}(\Gamma_i) \subseteq \Delta_i^s.$$

The proof of this statement is a simple consequence of the above hypotheses: the transversality assumption implies that the map $u \to \alpha(u) \equiv (\vec{A}_i^*(u), q_i^*(u))$ is invertible near $u = 0$ and, if one defines $v(p) \equiv \alpha^{-1}(\vec{A}'_i(p\hat{r}/2), \hat{r}/2)$, by energy conservations it is $p_i^*(v(p)) = p$ so that the lemma follows by choosing $\vec{\psi}_i(p) \equiv \vec{\psi}_i^*(v(p))$. For completeness we include the details in appendix.

The "local" construction described in the next Proposition will lead at once to the existence of shadowing orbits (and hence to Arnold diffusion whenever the heteroclinic chain is long enough).

Proposition 1. *Assume* (i), (ii) *and* (iii) *above and denote, for* $1 \le i \le N-1$, *by* Δ_i^s *a connected l-submanifold of* W_{i+1}^s *contained in* U_i *and intersecting transversally* $W_{i,\mathrm{loc}}^u$ *in*

$\mathscr{S}_E$ at $w_i \in U_i$. Given a neighbourhood B_{i-1} of some[6] *$\xi_{i-1} \in \Delta^s_{i-1} \cap (W^u_{i-1,\mathrm{loc}})^c$ one can find $\xi_i \in \Delta^s_i \cap (W^u_{i,\mathrm{loc}})^c$, a neighbourhood B_i of ξ_i and a time $T_i > 0$ such that $\phi^{-T_i} B_i \subset B_{i-1}$.*

Proof. Since $\xi_{i-1} \in \Delta^s_{i-1} \subset W^s_i$, ξ_{i-1} will evolve, in a suitable time $T^*_i > 0$, into a point $\xi'_i \in U_i$ having normal coordinates given by $(\vec{A}'_i(0), \vec{\chi}'_i, \hat{r}/2, 0)$ for a suitable $\vec{\chi}'_i \in \mathbb{T}^{l-1}$. Set $B'_i \equiv \phi^{T^*_i}(B_{i-1})$. We have to show that in $\Delta^s_i \cap (W^u_{i,\mathrm{loc}})^c$ one can find a point ξ_i and a time $t^*_i > 0$ such that $\phi^{-t^*_i}(\xi_i) \in B'_i$ (from this the existence of B_i with the desired properties follows at once). The point ξ_i will be selected on the curve Γ_i lying in Δ^s_i constructed in the above Lemma 1. Let $\hat{B}_i$ be a small sphere of radius $s > 0$ (in normal coordinates) centered at $\mathscr{C}^{-1}(\xi'_i)$ and such that $\mathscr{C}(\hat{B}_i) \subset B'_i$. By the construction of Γ_i and (2.2) one has, for p and t satisfying $|pe^{-g_i(p\hat{r}/2)t}| < \hat{r}$ and $\hat{r}e^{g_i(p\hat{r}/2)t}/2 < \hat{r}$,

$$\phi^t \circ \mathscr{C}(\Gamma_i(p)) = \mathscr{C}(\vec{A}'_i(p\hat{r}/2), \vec{\psi}_i(p) + \vec{\omega}_i(p\hat{r}/2)t, pe^{-g_i(p\hat{r}/2)t}, \hat{r}e^{g_i(p\hat{r}/2)t}/2). \tag{2.3}$$

Define, for $p > 0$, $t_i(p)$ by setting $pe^{g_i(p\hat{r}/2)t_i(p)} = \hat{r}/2$ and notice that[7] $\tau_i(p\hat{r}/2)t_i(p) \to \infty$ as $p \to 0$. Since the flow $t \to \vec{\omega}^0_i t$ is dense on $\mathbb{T}^{l-1}$, for any $r_0 > 0$ we can find $0 < p^*_i < r_0$ such that $\vec{\psi}_i - \vec{\omega}_i(p^*_i\hat{r}/2)t_i(p^*_i) \equiv \vec{\psi}_i - \vec{\omega}^0_i\tau_i(p^*_i\hat{r}/2)t_i(p^*_i)$ is arbitrarily close to $\vec{\chi}'_i$. Now, choose $0 < r_0 < \tilde{r}$ be such that $|\vec{A}'_i(p\hat{r}/2) - \vec{A}'_i(0)| < s$ and $|\vec{\psi}_i(p) - \vec{\psi}_i| < s/2$ for $|p| < r_0$ and let $t_0 > 0$ be such that $e^{-g_i(p\hat{r}/2)t}\hat{r}/2 < s$ for all $t > t_0$ and $|p| < r_0$. Then one can find $0 < p^*_i < r_0$ so that $t_i(p^*_i) > t_0$ and $|\vec{\psi}_i - \vec{\omega}_i(p^*_i\hat{r}/2)t_i(p^*_i) - \vec{\chi}'_i| < s/2$, which, in view of the above choices, implies that $\phi^{-t_i(p^*_i)} \circ \mathscr{C}(\Gamma_i(p^*_i)) \in \mathscr{C}(\hat{B}_i) \subset B'_i$. □

We remark that in this proof it is essential that $\vec{\omega}_i(p\hat{r}/2) = \tau_i(pr/2)\vec{\omega}^0_i$ is *parallel to a fixed rationally independent vector* $\vec{\omega}^0_i$ as stated in the last lines of item (ii) at the beginning of the section: a less careful normal form could give a rotation vector which changes with p also its direction taking, in particular, resonant values for a dense set of values of p in the interval of variation of p and the above argument would not work any more.

Since ϕ^{-t} is a diffeomorphism (and therefore preserves transversality), one can take as Δ^s_{i-1} the image under ϕ^{-T} (for a suitable $T > 0$) of a connected component of W^s_i in a neighbourhood of z_{i-1} (compare assumption (i)). An iteration of Proposition 1 leads then immediately to the following

Corollary. *With the same assumptions made in Proposition 1, there exist a positive T^* and a positive measure set of initial data in an arbitrary neighbourhood of $\mathscr{T}_1$ whose ϕ^{T^*}-image is contained into an arbitrary neighbourhood of $\mathscr{T}_N$.*

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[6] The superscript ^{c} denotes complementary set.

[7] Recall the definitions of and the assumptions on g_i and τ_i given in (ii).

Often, in applications, the whiskers appear naturally as *graphs* over the angles. In such a case transversality takes a simple form:

Lemma 2. *Let $\Delta_i^s \subset U_i$ be a submanifold of W_{i+1}^s which may be expressed as a graph over $(\vec{\psi}, q)$, i.e., there exist functions $\vec{A}_i(\vec{\psi}, q)$ and $p_i(\vec{\psi}, q)$ defined (for suitable $d > 0, \vec{\psi}_i \in \mathbb{T}^{l-1}, 0 < |q_i| < \hat{r} - d$) for $|\vec{\psi} - \vec{\psi}_i| < d$ and $|q - q_i| < d$ such that*

$$(2.4)\quad \begin{aligned} \Delta_i^s &= \mathscr{C}(\{(\vec{A}_i(\vec{\psi}, q), \vec{\psi}, p_i(\vec{\psi}, q), q) : |\vec{\psi} - \vec{\psi}_i| < d, |q - q_i| < d\}), \\ \vec{A}_i(\vec{\psi}_i, q_i) &= \vec{A}_i'(0), \quad p_i(\vec{\psi}_i, q_i) = 0. \end{aligned}$$

Then Δ_i^s intersects transversally $W_{i,\mathrm{loc}}^u$ in $\mathscr{S}_E$ at a point $w_i \equiv \mathscr{C}(\vec{A}_i(\vec{\psi}_i, q_i), \vec{\psi}_i, p_i(\vec{\psi}_i, q_i), q_i)$ if and only if

$$(2.5)\quad \det \frac{\partial \vec{A}_i}{\partial \vec{\psi}}(\vec{\psi}_i, q_i) \neq 0.$$

The proof of this statement is elementary (going along the lines of the proof of Lemma 1 given in the appendix) and is omitted.

Formula (2.5) gives a simple criterion that can be rather easily tested by making use of the theories of the homoclinic splitting.

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A Proof of Lemma 1

From the transversality (in the energy level) between Δ_i^s and W_i^u at w_i we have

$$(\mathrm{A}.1)\quad 2l - 1 = \operatorname{rank} \begin{pmatrix} \partial_u \vec{A}_i^*(0) & 0 & 0 \\ \partial_u \vec{\psi}_i^*(0) & I & 0 \\ \partial_u p_i^*(0) & 0 & 0 \\ \partial_u q_i^*(0) & 0 & 1 \end{pmatrix} = \operatorname{rank} \begin{pmatrix} \partial_u \vec{A}_i^*(0) & 0 \\ \partial_u p_i^*(0) & 0 \\ \partial_u q_i^*(0) & 1 \end{pmatrix} + l - 1$$

where I denotes here the $(l-1) \times (l-1)$ identity matrix. By energy conservation ($\Delta_i^s \subset \mathscr{S}_E$ and second line of (2.1))

$$(\mathrm{A}.2)\quad H \circ \mathscr{C}(\vec{A}_i^*(u), \vec{\psi}_i^*(u), p_i^*(u), q_i^*(u)) = E = H \circ \mathscr{C}(\vec{A}_i(0), \vec{\psi}, 0, q)$$

so that, differentiating with respect to u the first equality and with respect to $\vec{\psi}$ and q the second one, one obtains

(A.3) $\partial_u p_i^*(0) = \vec{a} \cdot \partial_u \vec{A}_i^*(0)$

with[8] $\vec{a} \equiv -\partial_{\vec{A}'}(H \circ \mathscr{C})(\hat{w}_i)/\partial_p(H \circ \mathscr{C})(\hat{w}_i)$.

Furthermore, since $w_i \in W_i^u \cap W_{i+1}^s$ the trajectory $w(t) \equiv \phi^t(w_i)$ is a *heteroclinic* orbit belonging to $W_i^u \cap W_{i+1}^s$ for all times. In particular (since $w(t) \in W_i^u$) $w(t) = \mathscr{C}(\vec{A}_i'(0), \vec{\psi}_i^h(t), 0, q_i^h(t))$ for suitable functions $\vec{\psi}_i^h, q_i^h(t)$ while (since $w(t)$ lies, at least for small $|t|$, in $\Delta_i^s \subset W_{i+1}^s$) there exists a (smooth) function $u(t)$ such that $u(0) = 0$ and $w(t) = \mathscr{C} \circ (\vec{A}_i^*(u(t)), \vec{\psi}_i^*(u(t)), p_i^*(u(t)), q_i^*(u(t)))$. Thus $\vec{A}_i^*(u(t)) = \vec{A}_i'(0)$, $p_i^*(u(t)) = 0$ and $q_i^*(u(t)) = q_i^h(t)$ (for $|t|$ small enough). Differentiating with respect to t the relations $\vec{A}_i^*(u(t)) = \vec{A}_i'(0)$ and $q_i^*(u(t)) = q_i^h(t)$ one sees that

$$\text{(A.4)} \qquad \begin{pmatrix} \partial_u \vec{A}_i^*(0) \\ \partial_u q_i^*(0) \end{pmatrix} \bar{a} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

with $\bar{a} \equiv \dot{u}(0)/\partial_p(H \circ \mathscr{C})(\hat{w}_i)$. So, from (A.1), (A.3), (A.4) it follows that

$$l = \operatorname{rank} \begin{pmatrix} \partial_u \vec{A}_i^*(0) & 0 \\ \partial_u p_i^*(0) & 0 \\ \partial_u q_i^*(0) & 1 \end{pmatrix} = \operatorname{rank} \begin{pmatrix} \partial_u \vec{A}_i^*(0) & 0 \\ \partial_u q_i^*(0) & 1 \end{pmatrix} = \operatorname{rank} \begin{pmatrix} \partial_u \vec{A}_i^*(0) \\ \partial_u q_i^*(0) \end{pmatrix}$$

thus the matrix $\partial_u(\vec{A}_i^*, q_i^*)\big|_{u=0}$ is nonsingular. By the Inverse Function Theorem, there exists a function $v(p)$ such that $v(0) = 0$ and, for small p (say $|p| < \tilde{r}$),

$$\text{(A.5)} \qquad \vec{A}_i^*(v(p)) = \vec{A}_i'(p\hat{r}/2), \quad q_i^*(v(p)) = \hat{r}/2.$$

Now, define $\vec{\psi}_i(p) \equiv \vec{\psi}_i^*(v(p))$ and observe that, from (A.2) evaluated at $u = v(p)$ and from the second line in (2.1) evaluated at $q = \hat{r}/2, \vec{\psi} = \vec{\psi}_i^*(v(p))$, there follows

$$\begin{aligned} &H \circ \mathscr{C}(\vec{A}_i^*(v(p)), \vec{\psi}_i^*(v(p)), p_i^*(v(p)), q_i^*(v(p))) \\ &\quad = H \circ \mathscr{C}(\vec{A}_i'(p\hat{r}/2), \vec{\psi}_i^*(v(p)), p_i^*(v(p)), \hat{r}/2)) \\ &\quad = E = H \circ \mathscr{C}(\vec{A}_i'(p\hat{r}/2), \vec{\psi}_i^*(v(p)), p, \hat{r}/2), \end{aligned}$$

which implies[9] that $p_i^*(v(p)) = p$.

So

$$\Delta_i^s \supseteq \mathscr{C}\{(\vec{A}_i^*(v(p)), \vec{\psi}_i^*(v(p)), p_i^*(v(p)), q_i^*(v(p))), |p| < \tilde{r}\} + \mathscr{C}(\Gamma_i). \qquad \square$$

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[8] Recall that, since $\hat{w}_i = (\vec{A}_i'(0), \vec{\psi}_i, 0, \hat{r}/2)$, (ii) implies that $\partial_p(H \circ \mathscr{C})(\hat{w}_i) \neq 0$.

[9] Use (A.5) and recall again that $\partial_p(H \circ \mathscr{C})(\hat{w}_i) \neq 0$.

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