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The planetary N-body problem: symplectic foliation, reductions and invariant tori

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Abstract The $6n$-dimensional phase space of the planetary $(1+n)$-body problem (after the classical reduction of the total linear momentum) is shown to be foliated by symplectic leaves of dimension $(6n-2)$ invariant for the planetary Hamiltonian $\mathcal{H}$. Such foliation is described by means of a new global set of Darboux coordinates related to a symplectic (partial) reduction of rotations. On each symplectic leaf $\mathcal{H}$ has the same form and it is shown to preserve classical symmetries. Further sets of Darboux coordinates may be introduced on the symplectic leaves so as to achieve a complete (total) reduction of rotations. Next, by explicit computations, it is shown that, in the reduced settings, certain degeneracies are removed. In particular, full torsion is checked both in the partially and totally reduced settings. As a consequence, a new direct proof of Arnold's theorem (Arnold in Russ. Math. Surv. 18(6):85–191, 1963) on the stability of planetary system (both in the partially and in the totally reduced setting) is easily deduced, producing Diophantine Lagrangian invariant tori of dimension $(3n-1)$ and $(3n-2)$. Finally, elliptic lower dimensional tori bifurcating from the secular equilibrium are easily obtained.

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1 Introduction

A major breakthrough in the mathematical treatment of the three-body problem was the "reduction of the nodes" introduced by Jacobi in 1842 [19]. Jacobi's reduction allows to lower the number of differential equations which describe the dynamics, frees the system from extra integrals of motion (related to invariance by rotation) and, in general, clarifies the structure of the phase space.[1] Applications of the reduction of the nodes in the general theory of the three-body problem are countless.

[1] For a symplectic version of Jacobi's reduction of the nodes, see, e.g., [6, §4.4].

Jacobi's reduction was fully generalized to the general spatial N-body problem only in 1983 by A. Deprit [12], but, strangely enough, Deprit's reduction did not have a similar mathematical success.[2] This fact might be partly due to the pessimistic attitude of Deprit himself, who at p. 194 of [12], writes: "Whether the new phase variables (34) and (35) are practical in the General Theory of Perturbations is an open question. At least for planetary theories, the answer is likely to be negative [...]".

In this paper we show how, starting from Deprit's reduction of the nodes, one can give a new description of the analytic structure of the phase space of the general N-body system, which not only might have some theoretical interest in itself, but can be useful in practical computations. For example, we will explicitly compute the second order Birkhoff invariants of the (reduced) secular planetary[3] perturbation showing, in particular, full torsion of the secular planetary system. This fact leads immediately to a new "direct" proof of a celebrated theorem by V.I. Arnold on the existence of a positive measure set of phase points evolving in relatively bounded motions in the planetary $(1+n)$-body problem for small values of the planet masses.

Before describing our results, let us briefly discuss the history of Arnold's theorem, which illustrates quite well the fundamental rôle of having "proper" analytic symplectic variables for general N-body systems. In 1963 V.I. Arnold [2] claimed that, in the general spatial planetary $(1+n)$-body problem, there is a positive Liouville measure set of phase space points whose evolution lies on invariant $(3n-1)$-dimensional Lagrangian Diophantine tori, solving a more than centennial problem. Arnold gave a complete proof for the planar three-body case,[4] giving some indications on how to generalize his approach. Very roughly, Arnold's scheme of proof consists in: averaging the planetary Hamiltonian over the mean anomalies ("fast angles"); put the averaged system (the "secular system") in Birkhoff normal form up to order four[5]; introduce polar symplectic variables for the secular system and check Kolmogorov's non-degeneracy, i.e., the non-degeneracy of the matrix of the second order Birkhoff invariants ("torsion"), so as to apply a KAM theorem for properly-degenerate systems.[6]

[2] At the present date, the MathSciNet database (MR0682462) shows only one citation of Deprit's paper.

[3] In the planetary $(1+n)$-body problem one considers one star and n planets—all regarded as point masses—interacting only through gravity.

[4] In this case the degrees of freedom are four and the tori are 4-dimensional.

[5] Actually, Arnold requires the Birkhoff normal form up to order six, but this is not necessary, compare [10].

[6] We recall that a properly-degenerate Hamiltonian system is a system for which the unperturbed Hamiltonian does not depend on all the action-variables. In [2, Chap. 4], Arnold worked out a general KAM theory for properly–degenerate Hamiltonian systems for which the aver-

For many years it was believed that the problem was completely settled by Arnold, also in view of authoritative endorsements, compare, e.g., Siegel-Moser's *Lectures on Celestial Mechanics* [27, p. 277]. However, M. Herman in the 1990's realized that there were serious hindrances preventing the extension of Arnold's approach to the general case. Incidentally, by using Jacobi's reduction of the nodes, in 1995, Robutel [25] extended Arnold's proof to the spatial three-body case.

The first problem is related to the presence of *secular resonances*, i.e., resonances among the first order Birkhoff invariants Ω_i of the secular perturbation. In particular, Herman discovered a strange resonance that seemed not to have been fully noticed before, namely that the sum of Ω_i's vanishes.[7] These resonances prevent standard applications of Birkhoff normal form theory[8] and, what is worst for Herman's approach (which we will shortly recall), they imply that the frequency map lies on a plane. A second problem in trying to pursue Arnold's strategy is the allegedly lack of torsion of the secular perturbation. Herman, using Poincaré variables,[9] tried to compute the torsion in the three-body case (in a suitable asymptotics) but did not succeed and computed, instead, the Birkhoff invariants for a modified system.[10] Indeed, it is a fact that the torsion computed in Poincaré variables are degenerate at all orders [9].

To overcome such problems, Herman (unpublished), and later Féjoz in 2004 [15, 16], introduced two ideas: 1) use a weaker KAM theory based on non-degeneracy conditions involving only the first order Birkhoff invariants Ω_i, i.e., one requires that the map $a \to \Omega(a)$ is *non planar* (a being the vector of the n semimajor axes and "non planar" means that Ω has not to be contained in any hyperplane); 2) use a trick by Poincarè, consisting in modifying the Hamiltonian by adding a commuting Hamiltonian, so as to remove the degeneracy: by a Lagrangian intersection theory argument, commuting Hamiltonians have the same maximal transitive invariant tori, so that the KAM tori constructed for the modified Hamiltonian are indeed invariant tori also for the original system. Incidentally, we mention that Herman's KAM theory in [15] yields smooth tori also if the Hamiltonian system is analytic; for a real-analytic version of Herman's KAM theory, see [11].

aged perturbation admits an elliptic equilibrium. Such theory has been revisited and strengthened in [10]. We also recall that in properly-degenerate systems there appear two perturbative parameters: one (say μ) measures the size of the perturbation and a second one (say ϵ) measures the distance from the reference elliptic equilibrium of the averaged perturbation.

[7] This resonance is now known as "Herman resonance"; for more information, compare [1], [15] and Sect. 7.1 below.

[8] For generalities on Birkhoff normal forms and Birkhoff invariants, see [18].

[9] See, e.g., [15, §6,1] and references therein.

[10] Compare [17] and in particular the remark at the end of p. 24 where Herman says: "J'ignore si $\det T_2(a)$ est identiquement nulle!", where T_2 is the (4×4)-matrix of the second order Birkhoff invariants for the three-body case and a the ratio of the semimajor axes.

Let us now describe the results obtained in this paper.

First, we recall that the phase space of the planetary $(1+n)$-body problem, after the classical Poincarè reduction of the linear momentum, is $6n$-dimensional and is endowed with standard (heliocentric) symplectic variables. On such phase space we introduce a new set of Darboux coordinates, which we call *Regularized Planetary Symplectic (RPS) variables*: such variables are analytic in a neighborhood of the secular elliptic equilibrium, corresponding to co-planar and co-circular motions, of the secular Hamiltonian. The RPS variables are obtained, first, by considering an action-angle version of Deprit's variables and then performing a Poincarè regularization to remove the singularity due to the vanishing of the eccentricities and mutual inclinations.

The RPS variables $(\Lambda, \lambda, z) := (\Lambda, \lambda, \eta, \xi, p, q) \in \mathbb{R}^n \times \mathbb{T}^n \times B^{4n}$, $\mathbb{T}^n$ being the standard flat n-dimensional torus and B^{4n} denoting a ball around the origin in $\mathbb{R}^{4n}$, share with the spatial Poincaré variables several nice features related to symmetries. In fact, the averaged perturbation, expressed in RPS variables, is even in the secular variables z and commutes with rotations; the d'Alembert relations in the quadratic forms describing the linearized secular system are retained. But unlike Poincaré variables, RPS variables may be used to perform a partial and total symplectic reduction of the $(1+n)$-body problem.[11] In particular, the partial reduction leads to a remarkable *symplectic foliation* of the phase space into invariant symplectic submanifold ("symplectic leaves") with a prescribed orientation of the total angular momentum.

The *partial symplectic reduction* ($3n-1$ degrees of freedom) may be described as follows. The region $\mathcal{M}^{6n}$ of the phase space corresponding to bounded motions in the integrable limit is foliated by $(6n-2)$-dimensional symplectic leaves $\mathcal{M}^{6n-2}_{p_n^*, q_n^*}$, which are invariant for the planetary flow. In fact, two conjugated variables p_n and q_n, depending only upon the total angular momentum, are both cyclic for the planetary Hamiltonian $\mathcal{H}$, and the invariant leaves $\mathcal{M}^{6n-2}_{p_n^*, q_n^*}$ are sections obtained by fixing $p_n = p_n^*$ and $q_n = q_n^*$. The induced symplectic form on any leaf is simply $d\Lambda \wedge d\lambda + d\eta \wedge d\xi + d\bar{p} \wedge d\bar{q}$ where $\bar{p}$ and $\bar{q}$ denote the first $(n-1)$ components of p and q. Furthermore, the restriction of $\mathcal{H}$ to each symplectic leaf is the same and is given by $h_{\rm Kep}(\Lambda) + \mu f(\Lambda, \lambda, \eta, \xi, \bar{p}, \bar{q})$, where $h_{\rm Kep}$ coincide with the classical expression of n-decoupled two-body systems in Delaunay variables.

We then study the secular Hamiltonian $f_{\rm av}$ (i.e., the average over the λ's of f) on $\mathcal{M}^{6n-2}_{p_n^*, q_n^*}$. The first order Birkhoff invariants of $f_{\rm av}$ satisfy identically only Herman's resonance, while the well known "rotational resonance" (one of the first order Birkhoff invariant vanishing identically) present in the unreduced setting, disappears.

[11] For *formal series* reductions based on Poincaré variables, see [21].

Now, it is a general fact (discussed in Sect. 7.2 below), that, for rotational invariant Hamiltonian systems, the construction of the Birkhoff normal form is simpler: indeed the only dangerous resonances (leading to zero divisors) are those non-vanishing integer vectors k for which $\sum k_i = 0$. Since $f_{\rm av}$ is rotation invariant, one sees immediately that Herman's resonance does not affect the construction of Birkhoff normal form, which is explicitly carried out up to order four in the limit of well separated semimajor axes. In particular, we show that the matrix formed by the second order Birkhoff invariant is non-singular, proving full torsion of the secular Hamiltonian.[12]

The *total symplectic reduction* ($3n-2$ degrees of freedom) may be described as follows. Since the planetary Hamiltonian is the same on any symplectic leaf $\mathcal{M}^{6n-2}_{p_n^*,q_n^*}$, without loss of generality, one can consider only the "vertical leaf" $\mathcal{M}^{6n-2}_0 = \mathcal{M}^{6n-2}_{0,0}$. In an open region avoiding certain "conic singularities" (compare (9.7)), we introduce a new set of Darboux coordinates, which includes the symplectic couple $(G, g) \in \mathbb{R}_+ \times \mathbb{T}$, where G denotes the Euclidean length of the total angular momentum. Since G is an integral, the periodic variable g is cyclic and the motion is described by a $(3n-2)$-degrees of freedom system $(\hat{\mathcal{M}}^{6n-4}_{G}, \hat{\mathcal{H}}_{G})$ where the phase space $\hat{\mathcal{M}}^{6n-4}_{G}$ is endowed with the standard symplectic form $d\Lambda \wedge d\hat{\lambda} + d\hat{\eta} \wedge d\hat{\xi} + d\hat{p} \wedge d\hat{q}$, with $(\hat{p}, \hat{q}) \in \mathbb{R}^{2(n-2)}$. Symmetries are now broken and in particular the secular origin is no longer an equilibrium for the averaged perturbation. However, (essentially through the standard Implicit Function Theorem) one can perform a symplectic re-centering of the variables and still compute the Birkhoff normal form up to order four and check again full torsion.

In particular, we see that in the partially reduced setting there still is an independent commuting integral (namely, G), while in the totally reduced system the are no other integrals related to invariance by rotations. Similarly, in the partially reduced setting, Herman's resonance is the only resonance among the first order Birkhoff invariants, while in the totally reduced setting no secular resonances exist.

As an application of the above theory, we consider the persistence of quasi-periodic motions for small values of the planetary masses.

We show how to construct the Birkhoff normal form up to order four in both the partially and totally reduced settings and then check the non-vanishing of the torsion in all symplectic leaves. This fact allow to resume Arnold's direct approach and give a complete proof of Arnold's theorem. Furthermore, having Kolmogorov's non-degeneracy allows for explicit (and sharp) measure estimates: in the partially reduced case, if ϵ denotes the distance from the elliptic equilibrium, the Kolmogorov's set (i.e., the union of

[12]On the other hand, as mentioned above, the torsion evaluated in the unreduced $(6n)$-dimensional phase space is identically zero.

$(3n-1)$-dimensional Diophantine tori) fill asymptotically an open set of measure ϵ^{4n-2} with a density of order $1-\sqrt{\epsilon}$; compare (11.2) below.

In the fully reduced setting one obtains a positive measure set of invariant Lagrangian KAM tori in $\hat{\mathcal{M}}^{6n-4}_{G}$ with $(3n-2)$ Diophantine frequencies; indeed, the estimate on the measure of the Kolmogorov's set in $\hat{\mathcal{M}}^{6n-4}_{G}$ improves, being proportional to ϵ^{4n-4}. Lifting such tori in $\mathcal{M}^{6n-2}$ amounts to a trivial rotation, leading to $(3n-1)$-dimensional tori; however such tori may now be Diophantine, Liouvillean or resonant according to the value of the quasi-periodic average of $\partial_G \hat{\mathcal{H}}_G$ (integrated along the motion). Incidentally, we notice that, in the $6n$-dimensional "ambient" phase space all the invariant tori constructed are resonant since the RPS variables p_n and q_n do not move.

We finally turn to study the bifurcation of the linear elliptic equilibrium corresponding to Keplerian motions in the fast variables. We prove that such linear equilibrium bifurcates, in the fully non-linear setting, into Cantor families of n-dimensional elliptic Diophantine tori; for more comments and references on this topic, see Sect. 11.2 below.

The organization of the paper is reflected by the table of contents reported above. As the reader might have noticed the paper contains also a lengthy appendix (Appendix B), where the averaged secular Hamiltonian is expanded in RPS variables up to order 4. This computation is clearly central for the computation of the Birkhoff normal forms and for checking the full torsion of the planetary system, which is the main hypothesis for constructing KAM tori, both in partially and totally reduced settings. On the other hand, once the symmetries are established and the form of the expansion is therefore derived (Sect. 6), these computations are in a sense "straightforward" and this is the reason why they are relegated to the appendix. Another reason to spell out these computations is to convince the reader that they can be done by pencil and paper.

2 The planetary Hamiltonian in Cartesian variables

Let us consider the planetary $(1+n)$-body system, i.e., a system of $n+1$ bodies (point masses) with masses m_0 (corresponding to "the star" or the Sun) and μm_i (corresponding to n "planets"), subject only to the mutual gravitation attraction.[13] By translation invariance of the Newton equations governing this system, we can restrict our attention to the invariant, symplectic submanifold of vanishing total linear momentum and zero center of mass. On such manifold, following Poincaré, one can introduce symplectic heliocentric coordinates, eliminating the coordinates of the Sun and lowering the number of

[13] Eventually, μ will be taken $0<\mu\ll 1$.

degrees of freedom by 3 units. In such coordinates the planetary system is governed by the Hamiltonian

$$\begin{aligned}\mathcal{H}_{\rm plt} &= \sum_{1\le i\le n}\left(\frac{|y^{(i)}|^2}{2M_i} - \frac{M_i\bar m_i}{|x^{(i)}|}\right) + \mu \sum_{1\le i<j\le n}\left(\frac{y^{(i)}\cdot y^{(j)}}{m_0} - \frac{m_i m_j}{|x^{(i)}-x^{(j)}|}\right)\\ &=: h_{\rm plt} + \mu\, f_{\rm plt} \end{aligned}\tag{2.1}$$

where $x^{(i)} = (x_1^{(i)}, x_2^{(i)}, x_3^{(i)}) \in \mathbb{R}^3$, $y^{(i)} = (y_1^{(i)}, y_2^{(i)}, y_3^{(i)}) \in \mathbb{R}^3$ are standard symplectic conjugate variables, $x\cdot y = \sum_{1\le i\le 3} x_i y_i$ and $|x| := (x\cdot x)^{1/2}$ denote, respectively, the standard inner product in $\mathbb{R}^3$ and the Euclidean norm;

$$M_i := \frac{m_0 m_i}{m_0+\mu m_i}, \qquad \bar m_i := m_0 + \mu m_i;$$

the phase space is the collisionless domain

$$\begin{aligned}\mathcal{P}^{6n} := \big\{(y,x) &= \big((y^{(1)},\dots,y^{(n)}), (x^{(1)},\dots,x^{(n)})\big)\\ &\in \mathbb{R}^{3n}\times\mathbb{R}^{3n} :\ 0\ne x^{(i)} \ne x^{(j)}\ \forall i\ne j\big\},\end{aligned}\tag{2.2}$$

endowed with the standard symplectic form

$$\sum_{1\le i\le n} dy^{(i)}\wedge dx^{(i)} = \sum_{1\le i\le n}\sum_{1\le j\le 3} dy_j^{(i)}\wedge dx_j^{(i)}.$$

Physically, the coordinates $x^{(i)}$ represent the difference between the position of the $i^{\rm th}$ planet and the position of the Sun, while $y^{(i)}$ are the associated symplectic momenta rescaled by μ; the position and velocity of the Sun is recovered by recalling that the center of mass and the total linear momentum are zero. For details, see, e.g., [15, §6.1] and references therein.[14]

The Hamiltonian $h_{\rm plt}$, as well known, describes the integrable limit given by n decoupled two-body systems formed by the Sun and the $i^{\rm th}$ planet.

In this way symplectic reduction of the total linear momentum has been achieved, but the three components $\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3$ of the *total angular momentum*

$$\mathrm{C} := \sum_{1\le i\le n} x^{(i)}\times y^{(i)}$$

are still integrals for the Hamiltonian (2.1). Such integrals are independent but do not commute, as the well known cyclic Poisson commutation rules

[14]The parameters μ, M_j, $\bar m_j$ here correspond to, respectively, ϵ, μ_j, M_j in [15].

hold. Nevertheless one can form, out of them, two independent commuting integrals, for example:

$$\mathrm{C}_3 = \sum_{1\le i\le n} \big(x_1^{(i)} y_2^{(i)} - x_2^{(i)} y_1^{(i)}\big) \quad \text{and} \quad G := |\mathrm{C}| = \Big| \sum_{1\le i\le n} x^{(i)} \times y^{(i)} \Big|. \tag{2.3}$$

The conservation of C_3 and G along the $\mathcal{H}_{\rm plt}$-trajectories is equivalent to the invariance of $\mathcal{H}_{\rm plt}$ under rotations around the $k^{(3)}$-axis[15] and around the C-axis, and will be at the basis of the further symplectic reductions described in Sects. 5 and 9 below.

3 Deprit's action-angle variables

Following Deprit [12], we introduce, in an open subset of $\mathcal{P}^{6n}$ avoiding co-circular and co-planar phase points, a remarkable set of action-angle variables. In the next two sections, after regularizing such variables (allowing again for co-circular and co-planar points), we will achieve the symplectic reduction of the inclination of the total angular momentum C and find a global symplectic chart for the reduced symplectic submanifolds of dimension $(6n-2)$, which foliate the phase space.

Consider the flow $(y^{(i)}, x^{(i)}) \to \phi^t_{h_i}(y^{(i)}, x^{(i)})$ generated by the *two-body problem* Hamiltonian

$$h_i(y^{(i)}, x^{(i)}) := \frac{|y^{(i)}|^2}{2M_i} - \frac{M_i \bar{m}_i}{|x^{(i)}|} \quad \text{with } y^{(i)} \in \mathbb{R}^3,\ x^{(i)} \in \mathbb{R}^3 \setminus \{0\}. \tag{3.1}$$

As well known, initial data $(y^{(i)}, x^{(i)})$ with strictly negative energies $h_i(y^{(i)}, x^{(i)}) = E_i < 0$ give rise to bounded motions, evolving on Keplerian ellipses $\mathfrak{E}_i = \mathfrak{E}_i(y^{(i)}, x^{(i)})$, having one focus in the origin. Let $a_i = a_i(y^{(i)}, x^{(i)})$, $e_i = e_i(y^{(i)}, x^{(i)})$ denote, respectively, the semimajor axis and the eccentricity of $\mathfrak{E}_i(y^{(i)}, x^{(i)})$,

$$\mathrm{C}^{(i)} := x^{(i)} \times y^{(i)}$$

the angular momentum of the $i^{\rm th}$ the planet, orthogonal to the plane of $\mathfrak{E}_i$ and, finally,

$$S^{(i)} := \sum_{j=1}^{i} \mathrm{C}^{(j)}$$

[15] $(k^{(1)}, k^{(2)}, k^{(3)})$ denotes the standard orthonormal basis in $\mathbb{R}^3$, i.e., $k^{(i)}_j = \delta_{ij}$, where δ_{ij} denotes the usual Kronecker delta.

the "partial" angular momentum of the first i planets, for[16] $i = 2, \dots, n$.

We consider the following phase spaces.

(D) $\mathcal{D}^{6n}$:= the set of phase points[17] $(y, x) \in \mathcal{P}^{6n}$ such that: $E_i < 0$; $e_i < 1$; $S^{(i)}$ is not anti-parallel[18] to $\mathrm{C}^{(i)}$ for $2 \le i \le n$; C is not anti-parallel to $k^{(3)}$.

(D$_*$) $\mathcal{D}^{6n}_*$:= the subset of $\mathcal{D}^{6n}$ where the following inequalities also hold:

$$\mathcal{D}^{6n}_* \subset \mathcal{D}^{6n}: \quad \begin{cases} e_i(y^{(i)}, x^{(i)}) > 0, \\ S^{(i)} \times \mathrm{C}^{(i)} \neq 0, \qquad \forall 2 \le i \le n. \\ k^{(3)} \times \mathrm{C} \neq 0, \end{cases} \tag{3.2}$$

Deprit action-angle variables[19] $((L, \Gamma, \Psi), (\ell, \gamma, \psi))$ are defined through a symplectic map

$$\phi_* : \ (y, x) \in \mathcal{D}^{6n}_* \to \big((L, \Gamma, \Psi), (\ell, \gamma, \psi)\big) \in (0, +\infty)^{3n-1} \times \mathbb{R} \times \mathbb{T}^{3n} \tag{3.3}$$

which we now proceed to describe.

(D$_1$) *The variables* $L = (L_1, \dots, L_n)$, $\ell = (\ell_1, \dots, \ell_n)$ *and* $\Gamma = (\Gamma_1, \dots, \Gamma_n)$ *are standard planar Delaunay variables,*[20] namely:

- $L_i := M_i\sqrt{\bar{m}_i a_i}$, where a_i is *semimajor axis* of $\mathfrak{E}_i$;
- the angle conjugated to L_i is the *mean anomaly* ℓ_i of $x^{(i)}$, i.e., the area spanned by the orbital radius starting from the i^{th} perihelion P_i (:= the point of $\mathfrak{E}_i$ at minimum distance from a focus) and ending at $x^{(i)}$, normalized to 2π;
- the action variable $\Gamma_i = |\mathrm{C}^{(i)}|$ is the Euclidean length of the i^{th} angular momentum[21] $\mathrm{C}^{(i)}$.

(D$_2$) For $1 \le i \le n-1$, $\Psi_i := |S^{(i+1)}|$, while $\Psi_n := \mathrm{C}_3$ is the vertical component of the total angular momentum as in (2.3). Notice that $\Psi_{n-1} = |S^{(n)}| = |\mathrm{C}^{(1)} + \dots + \mathrm{C}^{(n)}| = |\mathrm{C}| =: G$.

To describe the remaining Deprit angles γ_i and ψ_i, we introduce the following notations. Given three non vanishing vectors u, v and w of $\mathbb{R}^3$, with w

[16] In [12], Deprit uses the different (but equivalent) convention $S^{(i)} = \sum_{j=i+1}^{n} \mathrm{C}^{(j)}$.

[17] Taking the phase points in the collisionless phase space $\mathcal{P}^{6n}$ (2.2) is actually only needed in considering the planetary Hamiltonian, but could be disregarded in the general symplectic-geometry arguments.

[18] We say that u and v in $\mathbb{R}^3$ are anti-parallel if $u \times v = 0$ and $u \cdot v < 0$.

[19] Actually, the variables in [12] are slightly different from the variables in (3.3), compare Remark 3.1-(i) below.

[20] For an analytical description of planar Delaunay variables see, e.g., [7, §3.2].

[21] Recall also the classical relation $\Gamma_i := |\mathrm{C}^{(i)}| = L_i\sqrt{1 - e_i^2}$.

orthogonal to u and v, we denote as $\alpha_w(u, v)$ the positively oriented angle between u and v in the plane π_w orthogonal to[22] w. We also denote by $\mathcal{R}_w(g)$ the positive (3×3)-matrix rotation by an angle g around the axis oriented as[23] w.

In view of (3.2), on $\mathcal{D}_*^{6n}$ the following "nodes" are well defined.[24]

$$\bar{\nu} := k^{(3)} \times \mathrm{C}, \qquad \nu_i := \begin{cases} \mathrm{C}^{(1)} \times \mathrm{C}^{(2)} & i = 1, \\ S^{(i)} \times \mathrm{C}^{(i)} & 2 \le i \le n. \end{cases} \tag{3.4}$$

(D3) $\gamma_i := \alpha_{\mathrm{C}^{(i)}}(\nu_i, P_i)$, i.e., γ_i is (in the plane orthogonal $\mathrm{C}^{(i)}$) the "argument of the perihelion with respect to the node the ν_i". Thus, γ_i differs form the Delaunay's angle g_i, which is defined as the argument of the perihelion with respect to the node $\bar{\nu}_i := k^{(3)} \times \mathrm{C}^{(i)}$, simply by a shift:

$$\gamma_i = \alpha_{\mathrm{C}^{(i)}}(\nu_i, P_i) = \alpha_{\mathrm{C}^{(i)}}(\nu_i, \bar{\nu}_i) + \alpha_{\mathrm{C}^{(i)}}(\bar{\nu}_i, P_i) = \alpha_{\mathrm{C}^{(i)}}(\nu_i, \bar{\nu}_i) + \mathrm{g}_i,$$
$$\bar{\nu}_i := k^{(3)} \times \mathrm{C}^{(i)}. \tag{3.5}$$

(D4) $\psi_{n-1} := \alpha_{\mathrm{C}}(\bar{\nu}, \nu_n)$ and[25] $\psi_n := \alpha_{k^{(3)}}(k^{(1)}, \bar{\nu})$.

(D5) When $n > 2$, the angles $\psi_1, \dots, \psi_{n-2}$ are defined as[26] $\psi_i := \alpha_{S^{(i+1)}}(\nu_{i+2}, \nu_{i+1})$, $1 \le i \le n-2$.

Remark 3.1 (i) In Appendix A.1 it is given the analytic expression of the map ϕ_*^{-1}. Here, we just remark that the variables $(L, \Gamma, \ell, \gamma)$ are "horizontal variables": each quadruple $(L_i, \Gamma_i, \ell_i, \gamma_i)$ describes the position of the i^{th} planet on its orbital plane, while the variables (Ψ, ψ) play the rôle of "vertical variables", since they are related to the orientations of the different orbital planes.

In the original paper by Deprit we find a different set of horizontal variables, namely, the set

$$(R, \Phi, r, \phi) = \big((R_1, \dots, R_n), (\Phi_1, \dots, \Phi_n), (r_1, \dots, r_n), (\phi_1, \dots, \phi_n)\big), \tag{3.6}$$

[22] For example, if $u = k^{(1)}$, $v = k^{(2)}$ and $w = k^{(3)}$, then $\alpha_{k^{(3)}}(k^{(1)}, k^{(2)}) = \pi/2$.

[23] For example, $\mathcal{R}_{k^{(3)}}(\pi/2)k^{(1)} = k^{(2)}$.

[24] I.e., do not vanish. Notice that (for later convenience) the node ν_1 is defined as $\nu_1 = \nu_2$ (and that assumption (3.2) with $i = 2$ implies $\nu_1 \ne 0$). In [12] ν_1 is chosen with the opposite sign, possibly to recover, for $n = 2$, the so-called "Jacobi's opposition of the nodes" (namely, the relation $\mathrm{C} \times \mathrm{C}^{(1)} = -\mathrm{C} \times \mathrm{C}^{(2)}$). Notice also that $\bar{\nu}$ is orthogonal to $k^{(3)}$ and ν_i is orthogonal to $\mathrm{C}^{(i)}$ and $S^{(i)}$.

[25] Such angles will be often denoted, respectively, by g and ζ.

[26] Note that $\nu_{i+2} := (S^{(i+2)} \times \mathrm{C}^{(i+2)}) = (S^{(i+1)} \times \mathrm{C}^{(i+2)})$ so that also $\nu_{i+2} \in \pi_{S^{(i+1)}}$.

where (r_i, ϕ_i) are the usual polar coordinates of $x^{(i)}$ on $\pi_{C^{(i)}}$ with respect to ν_i (as polar axis); (R_i, Φ_i) are their respective conjugated actions. In [12] Deprit proved that the variables $(R, \Phi, \Psi, r, \phi, \psi)$ are symplectic. Since the map $(R_i, \Phi_i, r_i, \phi_i) \to (L_i, \Gamma_i, \ell_i, \gamma_i)$ is well known to be symplectic,[27] it follows that the variables $\mathrm{D}1$–$\mathrm{D}5$ are symplectic[28]; for a different (inductive) proof of the symplecticity of the variables $\mathrm{D}1$–$\mathrm{D}5$, see [8].

(ii) A remarkable property of Deprit's variables is the presence, among the action variables Ψ, of the two commuting $\mathcal{H}_{\rm plt}$-integrals $G = \Psi_{n-1}$ and $\mathrm{C}_3 = \Psi_n$. The variables $\Psi_1, \dots, \Psi_{n-2}$ complete Ψ_{n-1}, Ψ_n so as to obtain a commuting set of action variables $\Psi = (\Psi_1, \dots, \Psi_n)$.

(iii) Besides G and C_3, also the angle ζ (the longitude of the node among the planes $\pi_{k^{(3)}}$ and π_{C}) is an integral of motion: giving G, C_3 and ζ corresponds to giving the three components of the angular momentum C. This fact will be crucial in the forthcoming reduction.

(iv) As in Delaunay variables, the integrable part $h_{\rm plt}$ of the planetary Hamiltonian (2.1) takes the well known "Keplerian form":

$$h_{\rm Kep}(L) = -\sum_{1\le i\le n} \frac{M_i^3 \bar m_i^2}{2L_i^2}. \tag{3.7}$$

4 Regularized Planetary Symplectic (RPS) variables (Λ, λ, z)

The action-angle Deprit variables defined in Sect. 3 become singular when some of the inequalities in (3.2) fails. We describe now a regularization procedure which allows for $e_i = 0$ and for $\mathrm{C}^{(j+1)}$ parallel to $S^{(j)}$, and C parallel to $k^{(3)}$, including, in particular, the case of co-circular and co-planar motions.[29] This procedure is analogous to the Poincaré regularization of the Delaunay variables. The new symplectic variables (Λ, λ, z) with

$$\begin{aligned}
&\Lambda = (\Lambda_1, \dots, \Lambda_n), \qquad \lambda = (\lambda_1, \dots, \lambda_n), \qquad z = (\eta, \xi, p, q)\\
&\eta = (\eta_1, \dots, \eta_n), \qquad \xi = (\xi_1, \dots, \xi_n),\\
&p = (p_1, \dots, p_n), \qquad q = (q_1, \dots, q_n)
\end{aligned}$$

[27] Compare, e.g., §3.2 of [7].

[28] Note that the variables (Ψ, ψ) correspond, in Deprit's notation [12] (up to an unessential reordering of $\mathrm{C}^{(1)}, \dots, \mathrm{C}^{(n)}$), to: $\Psi_i = \Theta^*_{n-1-i}$ for $1 \le i \le n-1$; $\Psi_n = N^*_0$; $\psi_1 = \theta^*_{n-2} + \pi$, $\psi_i = \theta^*_{n-1-i}$ for $2 \le i \le n-2$; $\psi_{n-1} = \theta^*_0 + \pi$; $\psi_n = \nu^*_0$. Notice also that the names in [12] of the variables (Φ, ϕ) in (3.6) are (Θ, θ).

[29] Actually, the regularization procedure depends on whether one wants to allow parallel or antiparallel angular momenta and C parallel or antiparallel to $k^{(3)}$. For definiteness, we shall describe only the case corresponding to parallel angular momenta, and C parallel to $k^{(3)}$ (the other cases being analogous).

will be called *Regularized Planetary Symplectic* (RPS) variables and are defined as follows.

Let $(L, \Gamma, \Psi, \ell, \gamma, \psi) = \phi_*(y, x)$; let $\psi_0 := 0$, $\Psi_0 := \Gamma_1$ and $\Gamma_{n+1} := 0$. Then, for $1 \le i \le n$, we define:

$$\Lambda = L, \qquad \lambda_i = \ell_i + \gamma_i + \psi_{i-1}^n, \quad \text{where } \psi_i^n = \sum_{i \le j \le n} \psi_j,$$

$$\begin{cases} \eta_i = \sqrt{2(\Lambda_i - \Gamma_i)} \cos(\gamma_i + \psi_{i-1}^n) \\ \xi_i = -\sqrt{2(\Lambda_i - \Gamma_i)} \sin(\gamma_i + \psi_{i-1}^n) \end{cases}$$
$$\begin{cases} p_i = \sqrt{2(\Gamma_{i+1} + \Psi_{i-1} - \Psi_i)} \cos \psi_i^n \\ q_i = -\sqrt{2(\Gamma_{i+1} + \Psi_{i-1} - \Psi_i)} \sin \psi_i^n \end{cases} \tag{4.1}$$

Remark 4.1 Writing out (p_n, q_n) explicitly one finds

$$\begin{cases} p_n = \sqrt{2(\Psi_{n-1} - \Psi_n)} \cos \psi_n = \sqrt{2(G - \mathrm{C}_3)} \cos \zeta \\ q_n = -\sqrt{2(\Psi_{n-1} - \Psi_n)} \sin \psi_n = -\sqrt{2(G - \mathrm{C}_3)} \sin \zeta \end{cases} \tag{4.2}$$

showing that the conjugated variables p_n and q_n are both integrals for $\mathcal{H}_{\mathrm{plt}}$. This fact is at the basis of the partial symplectic reduction described in next section.

The map from the Cartesian heliocentric variables (y, x) to the RPS variables (Λ, λ, z) will be denoted by $\phi : (y, x) \in \mathcal{D}_*^{6n} \to (\Lambda, \lambda, z)$. Its inverse map ϕ^{-1} can be described as follows (see Appendix A.2 for full details):

$$\phi^{-1}: \quad \begin{cases} y^{(i)} = \mathfrak{R}_i(\Lambda, z) y_{\mathrm{pl}}^{(i)}(\Lambda_i, \lambda_i, \eta_i, \xi_i) \\ x^{(i)} = \mathfrak{R}_i(\Lambda, z) x_{\mathrm{pl}}^{(i)}(\Lambda_i, \lambda_i, \eta_i, \xi_i) \end{cases} \tag{4.3}$$

where:

- $(\Lambda_i, \lambda_i, \eta_i, \xi_i) \to (y_{\mathrm{pl}}^{(i)}, x_{\mathrm{pl}}^{(i)})$ denotes the planar Poincaré map, i.e., the map which sends a point $(\Lambda_i, \lambda_i, \eta_i, \xi_i) \in (0, +\infty) \times \mathbb{T} \times \{\frac{\eta_i^2 + \xi_i^2}{2\Lambda_i} < 1\}$ to the point $(y_{\mathrm{pl}}^{(i)}, x_{\mathrm{pl}}^{(i)}) \in \mathbb{R}^2 \times \mathbb{R}^2 \subset \mathbb{R}^3 \times \mathbb{R}^3$ recovering the Cartesian coordinates on the "instantaneous orbital plane" (i.e. the $\mathfrak{E}_i$ plane). The planar Poincaré map is explicitly given by[30]

$$x_{\mathrm{pl}}^{(i)} = \big(\mathrm{x}_1^{(i)}, \mathrm{x}_2^{(i)}, 0\big), \qquad y_{\mathrm{pl}}^{(i)} = \beta_i \, \partial_{\lambda_i} x_{\mathrm{pl}}^{(i)} \tag{4.4}$$

[30]Compare, [3, Lemma 2.1].

where

$$\begin{cases} \mathrm{x}_1^{(i)} := \frac{1}{\bar m_i}\big(\frac{\Lambda_i}{M_i}\big)^2\big(\cos u_i - \frac{\xi_i}{2\Lambda_i}\,(\eta_i\sin u_i+\xi_i\cos u_i) - \frac{\eta_i}{\sqrt{\Lambda_i}}\sqrt{1-\frac{{\eta_i}^2+{\xi_i}^2}{4\Lambda_i}}\big) \\ \mathrm{x}_2^{(i)} := \frac{1}{\bar m_i}\big(\frac{\Lambda_i}{M_i}\big)^2\big(\sin u_i - \frac{\eta_i}{2\Lambda_i}\,(\eta_i\sin u_i+\xi_i\cos u_i) \\ \qquad\qquad + \frac{\xi_i}{\sqrt{\Lambda_i}}\sqrt{1-\frac{{\eta_i}^2+{\xi_i}^2}{4\Lambda_i}}\big) \\ \beta_i := \frac{\bar m_i^2 M_i^4}{\Lambda_i^3} \end{cases} \tag{4.5}$$

and $u_i = u_i(\Lambda_i, \lambda_i, \eta_i, \xi_i) = \lambda_i + O(|(\eta_i, \xi_i)|)$ is the unique solution of the *(regularized) Kepler equation*

$$u_i - \frac{1}{\sqrt{\Lambda_i}}\sqrt{1-\frac{{\eta_i}^2+{\xi_i}^2}{4\Lambda_i}}\,(\eta_i\sin u_i+\xi_i\cos u_i) = \lambda_i; \tag{4.6}$$

- for $1 \le i \le n$, $\mathfrak{R}_i$ are products of matrices:

$$\mathfrak{R}_i = \mathcal{R}_n^*\mathcal{R}_{n-1}^*\cdots\mathcal{R}_i^*\mathcal{R}_i \tag{4.7}$$

where $\mathcal{R}_1 = \mathrm{id}$ and $\mathcal{R}_i, \mathcal{R}_j^*$ are 3×3 unitary matrices depending on (Λ, z) but with $\mathcal{R}_i, \mathcal{R}_j^*$, for $1 \le i \le n$ and $2 \le j \le n-1$, independent of the cyclic couple (p_n, q_n). The matrices $\mathcal{R}_i$, $\mathcal{R}_j^*$ have the form

$$\begin{aligned} \mathcal{R}_i^* &= \begin{pmatrix} 1-q_i^2\mathfrak{c}_i^* & -p_iq_i\mathfrak{c}_i^* & -q_i\mathfrak{s}_i^* \\ -p_iq_i\mathfrak{c}_i^* & 1-p_i^2\mathfrak{c}_i^* & -p_i\mathfrak{s}_i^* \\ q_i\mathfrak{s}_i^* & p_i\mathfrak{s}_i^* & 1-(p_i^2+q_i^2)\mathfrak{c}_i^* \end{pmatrix}, \quad 1 \le i \le n \\ \mathcal{R}_i &= \begin{pmatrix} 1-q_{i-1}^2\mathfrak{c}_i & -p_{i-1}q_{i-1}\mathfrak{c}_i & -q_{i-1}\mathfrak{s}_i \\ -p_{i-1}q_{i-1}\mathfrak{c}_i & 1-p_{i-1}^2\mathfrak{c}_i & -p_{i-1}\mathfrak{s}_i \\ q_{i-1}\mathfrak{s}_i & p_{i-1}\mathfrak{s}_i & 1-(p_{i-1}^2+q_{i-1}^2)\mathfrak{c}_i \end{pmatrix}, \quad 2 \le i \le n \end{aligned} \tag{4.8}$$

where $\mathfrak{c}_i$, $\mathfrak{s}_i$, $\mathfrak{c}_i^*$, $\mathfrak{s}_i^*$, explicitly computed in Appendix A (compare (A.22), (A.23) and (A.27)), are analytic functions of $\rho_i = \frac{\eta_i^2+\xi_i^2}{2}$, $r_j = \frac{p_j^2+q_j^2}{2}$ for $1 \le i \le n$ and $1 \le j \le n-1$.

Remark 4.2 (i) By (4.8), when $p = 0 = q$, one has $\mathcal{R}_i = \mathrm{id} = \mathcal{R}_i^*$ for all i's. In this case, by (4.3) and (4.7), the remaining variables $(\Lambda, \lambda, \eta, \xi)$ are seen to coincide with the planar Poincaré variables.

(ii) By (4.1), $(\eta_i, \xi_i) = 0$ corresponds to $e_i = 0$ and $(p_j, q_j) = 0$ to $S^{(j)}$ parallel to $\mathrm{C}^{(j+1)}$ $(j \ne n)$. By (4.2), $(p_n, q_n) = 0$ corresponds to $G = \mathrm{C}_3$, i.e., to C parallel to $k^{(3)}$.

(iii) From (4.3)–(4.8) it follows at once that *the map* $\phi : (y,x) \in \mathcal{D}_*^{6n} \to (\Lambda, \lambda, z) \in \phi(\mathcal{D}_*^{6n})$ *defined in* (4.1) *can be extended to a real-analytic diffeomorphism*

$$\phi : (y,x) \in \mathcal{D}^{6n} \to \mathcal{M}^{6n} := \phi(\mathcal{D}^{6n}) \tag{4.9}$$

where $\mathcal{D}^{6n}$ *is the set defined in Sect.* 3–(D).

(iv) The variables (Λ, λ, z) are symplectic. In fact, let, on $\mathcal{D}_*^{6n}$, (ρ_i, φ_i), (r_i, χ_i) denote the symplectic polar coordinates associated to[31] (η_i, ξ_i), (p_i, q_i), and let $\rho = (\rho_1, \dots, \rho_n)$, $r = (r_1, \dots, r_n)$, $\varphi = (\varphi_1, \dots, \varphi_n)$, $\chi = (\chi_1, \dots, \chi_n)$. In terms of Λ, ρ, r, λ, φ, χ, (4.1) become

$$\begin{pmatrix} \Lambda \\ \rho \\ r \end{pmatrix} = \mathrm{M} \begin{pmatrix} L \\ \Gamma \\ \Psi \end{pmatrix}, \qquad \begin{pmatrix} \lambda \\ \varphi \\ \chi \end{pmatrix} = \hat{\mathrm{M}} \begin{pmatrix} \ell \\ \gamma \\ \psi \end{pmatrix}$$

where M and $\hat{\mathrm{M}}$ denote the matrices of order $3n$ uniquely defined by the equations ($1 \le i \le n-1$)

$$\begin{cases} \Lambda_i = L_i \\ \rho_i = L_i - \Gamma_i \\ r_i = \Gamma_{i+1} + \Psi_{i-1} - \Psi_i \end{cases} \qquad \begin{cases} \lambda_i = \ell_i + \gamma_i + \psi_{i-1}^n \\ \varphi_i = -\gamma_i - \psi_{i-1}^n \\ \chi_i = -\psi_i^n \end{cases}$$

One easily recognizes that the matrices M and $\hat{\mathrm{M}}$ are related by $\hat{\mathrm{M}} = (\mathrm{M}^{\mathrm{t}})^{-1}$, where $(\cdot)^{\mathrm{t}}$ denotes matrix transposition. This relation implies that ϕ is symplectic on $\mathcal{D}_*^{6n}$, hence, by regularity, on $\mathcal{D}^{6n}$.

The properties in (iii) and (iv) explain the name given to the variables (Λ, λ, z).

(v) The following relations are immediately checked

$$\sum_{1 \le i \le n} \rho_i = \sum_{1 \le i \le n} \frac{\eta_i^2 + \xi_i^2}{2} = \sum_{1 \le i \le n} \Lambda_i - \sum_{1 \le i \le n} \Gamma_i \tag{4.10}$$

$$\sum_{1 \le j \le n-1} r_j = \sum_{1 \le j \le n-1} \frac{p_j^2 + q_j^2}{2} = \sum_{1 \le j \le n} \Gamma_j - \Psi_{n-1} = \sum_{1 \le j \le n} \Gamma_j - G \tag{4.11}$$

$$r_n = \frac{p_n^2 + q_n^2}{2} = \Psi_{n-1} - \Psi_n = G - \mathrm{C}_3. \tag{4.12}$$

[31] I.e., in complex notation, $\eta_j + \mathrm{i}\xi_j = \sqrt{2\rho_j} e^{\mathrm{i}\varphi_j}$ and $p_j + \mathrm{i}q_j = \sqrt{2r_j} e^{\mathrm{i}\chi_j}$, where $\mathrm{i} := \sqrt{-1}$.

5 Partial symplectic reduction of the planetary system

We now go back to the planetary many-body problem showing, in particular, that its phase space $\mathcal{D}^{6n}$ (defined in Sect. 3-(D)) is foliated by $\mathcal{H}_{\rm plt}$-invariant symplectic submanifolds of dimension $(6n-2)$ with a natural "global" symplectic structure.

Let $\mathcal{H}(\Lambda,\lambda,z) := \mathcal{H}_{\rm plt}\circ\phi^{-1} = h_{\rm Kep}(\Lambda)+\mu f$ denote the planetary Hamiltonian expressed in RPS variables with phase space given by $\mathcal{M}^{6n}$ as in (4.9), endowed with the standard symplectic form $d\Lambda\wedge d\lambda + d\eta\wedge d\xi + dp\wedge dq$.

As mentioned in Remark 4.1, the variables p_n, q_n in (4.2) are both integrals and cyclic for $\mathcal{H}$. This means that *the perturbation function f does not depend upon (p_n, q_n)*, i.e., $f = f(\Lambda,\lambda,\bar z)$ with

$$\bar z := (\eta,\xi,\bar p,\bar q) := \big((\eta_1,\ldots,\eta_n),(\xi_1,\ldots,\xi_n),(p_1,\ldots,p_{n-1}),\\ (q_1,\ldots,q_{n-1})\big).$$

The upshot is that *the phase space $\mathcal{M}^{6n}$ is foliated by symplectic $\mathcal{H}$-invariant submanifolds*

$$\mathcal{M}^{6n-2}_{p_n^*,q_n^*} := \{(\Lambda,\lambda,z)\in\mathcal{M}^{6n}:\ p_n = p_n^*,\ q_n = q_n^*\}, \tag{5.1}$$

p_n^* and q_n^* being fixed constants. A global symplectic chart is given simply by the first $(6n-2)$ variables $(\Lambda,\lambda,\bar z)$, the restriction of the symplectic form on each "symplectic leaf" $\mathcal{M}^{6n-2}_{p_n^*,q_n^*}$ being

$$d\Lambda\wedge d\lambda + d\eta\wedge d\xi + d\bar p\wedge d\bar q.$$

Finally, *the restriction of the planetary Hamiltonian to each leaf $\mathcal{M}^{6n-2}_{p_n^*,q_n^*}$ is the same*, as the perturbation f is independent from the couple (p_n,q_n): we shall keep denoting $\mathcal{H}$ the partially reduced Hamiltonian and f the planetary perturbation:

$$\mathcal{H}(\Lambda,\lambda,\bar z) = \mathcal{H}_{\rm plt}\circ\phi^{-1} = h_{\rm Kep}(\Lambda) + \mu f(\Lambda,\lambda,\bar z). \tag{5.2}$$

Remark 5.1 (i) In view of this last observation, without loss of generality, we can restrict our attention to the symplectic leaf $\mathcal{M}^{6n-2}_0 := \mathcal{M}^{6n-2}_{0,0}$ with $p_n^* = 0 = q_n^*$, which corresponds to the "vertical submanifold" $\{\mathrm{C}_1 = 0 = \mathrm{C}_2\}$ where the total angular momentum is oriented in the $k^{(3)}$-direction.

(ii) When the "vertical variables" p and q are set to be zero, that is, when the secular set z is taken $z = z_{\rm pl} = (\eta,\xi,0,0)$, the map ϕ^{-1} reduces to the planar Poincaré map.

(iii) The expression "partial reduction" refers to the fact that, on each leaf $\mathcal{M}^{6n-2}_{p_n^*,q_n^*}$, $\mathcal{H}$ admits another independent integral, namely, the restriction of

$G = \Psi_{n-1} = |\mathrm{C}|$ onto $\mathcal{M}^{6n-2}_{p_n^*,q_n^*}$. From (4.10) and (4.11) there follows

$$G = \sum_{1\le i\le n} \Lambda_i - \frac{1}{2}|\bar{z}|^2. \tag{5.3}$$

We also note that the expression of C_3 on $\mathcal{M}^{6n}$ is given[32] by

$$\mathrm{C}_3 = G - \frac{p_n^2+q_n^2}{2} = \sum_{1\le i\le n} \Lambda_i - \frac{1}{2}|z|^2. \tag{5.4}$$

Notice that the formula for C_3 on $\mathcal{M}^{6n}$ coincides with the well known one in spatial Poincaré variables.

(iv) The conservation of G along $\mathcal{H}$-trajectories induces into the averaged perturbation[33] f_{av} some symmetries (discussed in detail in Sect. 6 below), which imply, in particular, that f_{av} is an even function of $\bar{z}$ and that its quadratic part splits into the sum of two separated terms:[34]

$$\mathrm{Q_h}(\Lambda)\cdot\frac{\eta^2+\xi^2}{2} + \bar{\mathrm{Q}}_{\mathrm{v}}(\Lambda)\cdot\frac{\bar{p}^2+\bar{q}^2}{2} \tag{5.5}$$

where the "horizontal part" $\mathrm{Q_h}$ is a quadratic form of order n coinciding with that of the planar problem, while the "vertical part" $\bar{\mathrm{Q}}_{\mathrm{v}}$ is a quadratic form of order $n-1$. As we will see below, $\mathrm{Q_h}$ and $\bar{\mathrm{Q}}_{\mathrm{v}}$ are non-degenerate,[35] hence, the point $\bar{z} = (\eta, \xi, \bar{p}, \bar{q}) = 0$, corresponding to zero eccentricities and zero relative inclinations, *is a non-degenerate elliptic equilibrium for the secular Hamiltonian* f_{av}.

(v) In Poincaré variables[36] a splitting similar to (5.5) holds with *the same horizontal quadratic form* and with the vertical quadratic form replaced by a quadratic form $\mathcal{Q}_v$ of order n. One can show that the eigenvalues of the

[32] Recall (4.12) and observe that, clearly, $\mathrm{C}_3 = G$ on $\mathcal{M}^{6n-2}_0$.

[33] Here and in what follows, the index "av" denotes the average over the fast angles conjugated to Λ; the function f_{av} is usually called "secular Hamiltonian".

[34] For exact notation, see footnote 39 below. The symmetries expressed by (5.5) is sometimes called "d'Alembert relations".

[35] Actually, they are, respectively, negative and positive definite. The positive-definiteness of $\mathrm{Q_h}$ has been proved in [15, PROPOSITION 73]. From the analytical expression of $\bar{\mathrm{Q}}_{\mathrm{v}}$ there follows that, since $C_1 < 0$, its eigenvalues $\bar{\varsigma}_i$'s are (maybe not strictly) positive. From their asymptotic evaluation (compare the proof of Proposition 7.2 below) there follows that they are actually strictly positive on an open set $\mathcal{A}$ of semimajor axes described below (see (7.2)). By standard arguments of complex analysis there follows that $\bar{\mathrm{Q}}_{\mathrm{v}}$ is strictly positive on an open dense set.

[36] For the analytic definition of spatial Poincaré variables, see [15] or [3]. Notice that secular "vertical" variables of the Poincaré set (i.e., the p's and the q's) are in $\mathbb{R}^n$.

quadratic form $\bar{\mathrm{Q}}_{\mathrm{v}}$ into (5.5) coincide with the non identically vanishing eigenvalues of $\mathcal{Q}_v$ (compare [9]). As shown in [15] and Proposition 7.1 below, the trace $\mathrm{tr}\,(\mathrm{Q_h}+\mathrm{Q_v}) = \mathrm{tr}\,(\mathrm{Q_h}+\mathcal{Q}_{\mathrm{v}}) = 0$, where $\mathrm{Q_v}$ is the $(n\times n)$ matrix $\mathrm{Q_v} := \left(\begin{smallmatrix}\bar{\mathrm{Q}}_{\mathrm{v}} & 0\\ 0 & 0\end{smallmatrix}\right)$; this relation is known as "Herman's resonance".

6 Symmetries of the partially reduced secular Hamiltonian

The planetary perturbation function f in (5.2) enjoys several symmetry properties, which, in particular, simplify the Taylor expansion of the secular Hamiltonian[37] $f_{\mathrm{av}}(\Lambda, z) = (2\pi)^{-n}\int_{\mathbb{T}^n} f\,d\lambda$.

In view of Remark 5.1-(i), we will consider only the symplectic "vertical leaf" $\mathcal{M}_0^{6n-2}$.

The angular momentum integral G Poisson commutes with the unperturbed Keplerian Hamiltonian h_{Kep}, whence G commutes with f. This fact implies that f is invariant under the Hamiltonian flow $g \to \mathcal{R}^g$ at time g generated by G, i.e.:

$$f\big(\mathcal{R}^g(\Lambda,\lambda,\bar{z})\big) = f(\Lambda,\lambda,\bar{z}) \quad \text{for any } g\in\mathbb{T},\ (\Lambda,\lambda,\bar{z})\in\mathcal{M}_0^{6n-2}. \tag{6.1}$$

The action of $\mathcal{R}^g$ corresponds, in Cartesian variables (y,x), to a positive rotation of all the $y^{(i)}$'s and the $x^{(i)}$'s by an angle g around the C-axis, which coincide with the $k^{(3)}$-axis on $\mathcal{M}_0^{6n-2}$. By the expression (5.3) of G in RPS variables, such flow is given by

$$\mathcal{R}^g:\quad \Lambda' = \Lambda,\quad \lambda_i' = \lambda_i + g,\quad \bar{z}' = \mathcal{S}^g\,\bar{z} \tag{6.2}$$

where $\mathcal{S}^g$ acts as synchronous rotation in the symplectic (η_i,ξ_i) and (p_i,q_i)-planes:

$$\begin{aligned}\mathcal{S}^g:\quad &\begin{pmatrix}\eta_i'\\ \xi_i'\end{pmatrix} = \mathcal{R}(-g)\begin{pmatrix}\eta_i\\ \xi_i\end{pmatrix},\quad \begin{pmatrix}p_j'\\ q_j'\end{pmatrix} = \mathcal{R}(-g)\begin{pmatrix}p_j\\ q_j\end{pmatrix},\\ &1\le i\le n,\ 1\le j\le n-1\end{aligned} \tag{6.3}$$

$\mathcal{R}(g)$ being the plane rotation by g

$$\mathcal{R}(g) := \begin{pmatrix}\cos g & -\sin g\\ \sin g & \cos g\end{pmatrix}. \tag{6.4}$$

[37] By (4.4) and (4.3), the λ-average of the perturbation (2.1) expressed in RPS variables (Λ,λ,z) reduces to the averaging the newtonian potential $V(\Lambda,\lambda,z) := \sum_{1\le i<j\le n}\big(-\frac{m_i m_j}{|x^{(i)}-x^{(j)}|}\big)$.

As one checks immediately, the planetary perturbation $f_{\rm plt}$ (2.1) is invariant also under the following transformations $(x^{(i)}, y^{(i)}) \to (x'^{(i)}, y'^{(i)})$:

$$
\begin{aligned}
&\mathcal{R}_1^-: && x'^{(i)} = \big(-x_1^{(i)},\ x_2^{(i)},\ x_3^{(i)}\big), && y'^{(i)} = \big(y_1^{(i)},\ -y_2^{(i)},\ -y_3^{(i)}\big)\\
&\mathcal{R}_2^-: && x'^{(i)} = \big(x_1^{(i)},\ -x_2^{(i)},\ x_3^{(i)}\big), && y'^{(i)} = \big(-y_1^{(i)},\ y_2^{(i)},\ -y_3^{(i)}\big)\\
&\mathcal{R}_3^-: && x'^{(i)} = \big(x_1^{(i)},\ x_2^{(i)},\ -x_3^{(i)}\big), && y'^{(i)} = \big(y_1^{(i)},\ y_2^{(i)},\ -y_3^{(i)}\big)\\
&\mathcal{R}_{1\leftrightarrow 2}: && x'^{(i)} = \big(x_2^{(i)},\ x_1^{(i)},\ x_3^{(i)}\big), && y'^{(i)} = \big(-y_2^{(i)},\ -y_1^{(i)},\ -y_3^{(i)}\big).
\end{aligned}
\tag{6.5}
$$

Such transformations correspond, respectively, to the reflections with respect to the planes $\{x_1 = 0\}$, $\{x_2 = 0\}$, $\{x_3 = 0\}$ and $\{x_1 = x_2\}$. Notice that the invariance by $\mathcal{R}_2^-$ and $\mathcal{R}_{1\leftrightarrow 2}$ is implied by the invariance by $\mathcal{R}_1^-$ and $\mathcal{R}^g$.

The actions of the reflections (6.5) in the variables (Λ, λ, z) are found using (4.3)–(4.8) and their expressions are given by

$$
\begin{aligned}
&\mathcal{R}_1^-: && \Lambda_i' = \Lambda_i, && \lambda_i' = \pi - \lambda_i, && \bar z' = (-\eta, \xi, \bar p, -\bar q) := \mathcal{S}_{14}^- \bar z\\
&\mathcal{R}_2^-: && \Lambda_i' = \Lambda_i, && \lambda_i' = -\lambda_i, && \bar z' = (\eta, -\xi, -\bar p, \bar q) := \mathcal{S}_{23}^- \bar z\\
&\mathcal{R}_3^-: && \Lambda_i' = \Lambda_i, && \lambda_i' = \lambda_i, && \bar z' = (\eta, \xi, -\bar p, -\bar q) := \mathcal{S}_{34}^- \bar z\\
&\mathcal{R}_{1\leftrightarrow 2}: && \Lambda_i' = \Lambda_i, && \lambda_i' = \tfrac{\pi}{2} - \lambda_i, && \bar z' = (\xi, \eta, \bar q, \bar p) := \mathcal{S}_{1\leftrightarrow 2} \bar z.
\end{aligned}
\tag{6.6}
$$

Let us check, for instance, the expression of $\mathcal{R}_1^-$, the others being similar. From Kepler's equation (4.6), it follows that $u_i' := u_i(\Lambda_i, \pi - \lambda_i, -\eta_i, \xi_i) = \pi - u_i(\Lambda_i, \lambda_i, \eta_i, \xi_i) := \pi - u_i$, hence, $\cos u_i' = -\cos u_i$ and $\sin u_i' = \sin u_i$. By (4.5), if $x^{(i)}(\Lambda_i, \lambda_i, \eta_i, \xi_i) = (\mathrm{x}_1^{(i)}, \mathrm{x}_2^{(i)}, 0)$, then

$$
(\mathrm{x}_1'^{(i)}, \mathrm{x}_2'^{(i)}, 0) := x^{(i)}(\Lambda_i, \pi - \lambda_i, -\eta_i, \xi_i) = (-\mathrm{x}_1^{(i)}, \mathrm{x}_2^{(i)}, 0).
$$

If one changes simultaneously $(\bar p, \bar q) \to (\bar p, -\bar q)$, then, by (4.3) and (4.8), one obtains that, if

$$
\begin{aligned}
x^{(i)}(\Lambda, \lambda, \eta, \xi, \bar p, \bar q) = \big(&x_1^{(i)}(\Lambda, \lambda, \eta, \xi, \bar p, \bar q), x_2^{(i)}(\Lambda, \lambda, \eta, \xi, \bar p, \bar q),\\
&x_3^{(i)}(\Lambda, \lambda, \eta, \xi, \bar p, \bar q)\big) := (x_1^{(i)}, x_2^{(i)}, x_3^{(i)})
\end{aligned}
$$

then[38]

$$x'^{(i)} := x^{(i)}(\Lambda, \pi - \lambda, -\eta, \xi, \bar{p}, -\bar{q}) = (-x_1^{(i)}, x_2^{(i)}, x_3^{(i)}). \tag{6.7}$$

This is the first identity into the first line of (6.5). The second one is then obtained taking the derivative of (6.7) with respect to λ_i, multiplied by β_i.

Remark 6.1 (i) The fast angles λ's are averaged out when considering the secular Hamiltonian f_{av}.

(ii) The invariance of f under the reflections (6.6) yields the following symmetries for f_{av}:

$$f_{\text{av}}(\Lambda, \bar{z}) = f_{\text{av}}(\Lambda, \mathcal{S}^g \bar{z}) = f_{\text{av}}(\Lambda, \mathcal{S}\bar{z}) \quad \text{for any } \mathcal{S} \in \{\mathcal{S}_{14}^-, \mathcal{S}_{23}^-, \mathcal{S}_{34}^-, \mathcal{S}_{1\leftrightarrow 2}\}. \tag{6.8}$$

(iii) By the symmetries $\mathcal{S}_{14}^-$, $\mathcal{S}_{23}^-$, and $\mathcal{S}_{34}^-$, it follows that f_{av} is an even function of the variables $(\eta, \bar{q})$, $(\xi, \bar{p})$ and $(\bar{p}, \bar{q})$. Thus, the parity holds also in the variables (η, ξ) and $(\eta, \bar{p})$. By the $\mathcal{S}_{1\leftrightarrow 2}$-invariance, f_{av} does not change when (η, ξ) is changed to (ξ, η) and, simultaneously, $(\bar{p}, \bar{q})$ to $(\bar{q}, \bar{p})$.

We are now ready to discuss the form of the *Taylor expansion of the secular Hamiltonian* f_{av} around the elliptic equilibrium $\bar{z} = 0$ up to the fourth order. By parity, in second order term, only the monomials[39] η^2, ξ^2 will appear and, in the fourth order term, only the monomials η^4, $\eta^2\xi^2$, ξ^4, $\bar{p}^4$, $\bar{p}^2\bar{q}^2$, $\bar{q}^4$, $\eta^2\bar{q}^2$, $\xi^2\bar{p}^2$, $\xi^2\bar{q}^2$, $\xi^2\bar{p}^2$, $\eta\xi\bar{p}\bar{q}$. By the symmetry $\mathcal{S}_{1\leftrightarrow 2}$, the tensors in front of each monomial of the couples (η^2, ξ^2), $(\bar{p}^2, \bar{q}^2)$, (η^4, ξ^4), $(\bar{p}^4, \bar{q}^4)$, $(\eta^2\bar{q}^2, \xi^2\bar{p}^2)$, $(\eta^2\bar{p}^2, \xi^2\bar{q}^2)$ will be pairwise equal. Let us denote $\mathrm{Q_h} \cdot \frac{\eta^2}{2}$, $\bar{\mathrm{Q}}_{\mathrm{v}} \cdot \frac{\bar{p}^2}{2}$ the quadratic forms associated to the monomials η^2, $\bar{p}^2$ and

$$\mathrm{F_h} \cdot \eta^2\xi^2, \qquad \mathrm{F_v} \cdot \bar{p}^2\bar{q}^2, \qquad \mathrm{F_{hv}} \cdot \eta^2\bar{q}^2, \qquad \mathrm{F}'_{\mathrm{hv}} \cdot \eta^2\bar{p}^2$$

the quartic forms associated to, respectively, $\eta^2\xi^2$, $\bar{p}^2\bar{q}^2$, $\eta^2\bar{q}^2$, $\eta^2\bar{p}^2$. By the invariance for the transformation $\eta_i \to (\eta_i - \xi_i)/\sqrt{2}$, $\xi_i \to (\eta_i + \xi_i)/\sqrt{2}$,

[38] $\pi - \lambda$ denotes, for short, the vector of components $(\pi - \lambda_1, \pi - \lambda_2, \dots, \pi - \lambda_n)$. Notice that $\mathcal{R}_n^* = \text{id}$ since we have assumed $p_n = 0 = q_n$ (compare (4.8)). This relation easily implies (6.7).

[39] For tensors, we use the same notation as in [15]. Thus, if $a = (a_{i,j})_{i\in\{1,\dots,n\}^{r_1}, j\in\{1,\dots,n-1\}^{r_2}}$ is a tensor with $r = r_1 + r_2$ indices and k_i is an integer between 0 and r_i, then, $a \cdot \eta^{k_1}\xi^{r_1-k_1}\bar{p}^{k_2}\bar{q}^{r_2-k_2}$ denotes

$$\sum_{i\in\{1,\dots,n\}^{r_1}, j\in\{1,\dots,n-1\}^{r_2}} a_{i,j}\, \eta_{i_1}\cdots\eta_{i_{k_1}}\xi_{i_{k_1+1}}\cdots\xi_{i_{r_1}}\, p_{j_1}\cdots p_{j_{k_2}} q_{j_{k_2+1}}\cdots q_{j_{r_2}}.$$

In the Taylor expansion are present only those monomials $\eta^a\xi^b p^c q^d$ which are left unvaried by the symmetries $\mathcal{S}_{14}^-$, $\mathcal{S}_{34}^-$, $\mathcal{S}_{23}^-$, $\mathcal{S}_{1\leftrightarrow 2}$ in (6.6) (and by their compositions).

$\bar{p}_i \to (\bar{p}_i - \bar{q}_i)/\sqrt{2}$, $\bar{q}_i \to (\bar{p}_i + \bar{q}_i)/\sqrt{2}$ (which corresponds to a rotation by $\pi/4$ around the C-axis: compare (6.3)), the quartic tensors associated to monomials η^4, $\bar{p}^4$ coincide, respectively, with one half the quartic tensors $\mathrm{F_h}$, $\mathrm{F_v}$ associated to $\eta^2\xi^2$, $\bar{p}^2\bar{q}^2$. By the same reason, the tensor associated to $\eta\xi\,\bar{p}\bar{q}$ coincides with one half the difference $\mathrm{F}'_{\mathrm{hv}} - \mathrm{F_{hv}}$, provided (as it is always possible) the entries $(\mathrm{F_{hv}})_{ijkl}$, $(\mathrm{F}'_{\mathrm{hv}})_{ijkl}$ of such tensors are chosen so as to satisfy $(\mathrm{F}'_{\mathrm{hv}})_{ijkl} = (\mathrm{F}'_{\mathrm{hv}})_{jilk}$, $(\mathrm{F_{hv}})_{ijkl} = (\mathrm{F_{hv}})_{jilk}$.

By the previous considerations, the expansion of f_{av} has the form

$$\begin{aligned} f_{\mathrm{av}}(\Lambda, \bar{z}) = C_0(\Lambda) + \mathrm{Q_h}(\Lambda)\cdot\frac{\eta^2+\xi^2}{2} + \bar{\mathrm{Q}}_{\mathrm{v}}(\Lambda)\cdot\frac{\bar{p}^2+\bar{q}^2}{2} \\ + \frac{1}{2}\mathrm{F}(\Lambda)\cdot\bar{z}^4 + \mathcal{P}(\Lambda,\bar{z}) \end{aligned} \tag{6.9}$$

where $\mathcal{P}$ has a zero of order 6 (due to the parity of f_{av} as a function of $\bar{z}$) in $\bar{z}=0$ and $\frac{1}{2}\mathrm{F}(\Lambda)\cdot\bar{z}^4$ denotes the fourth order term

$$\begin{aligned} \frac{1}{2}\mathrm{F}(\Lambda)\cdot\bar{z}^4 := \mathrm{F_h}(\Lambda)\cdot\frac{\eta^4+\xi^4+2\eta^2\xi^2}{2} + \mathrm{F_v}(\Lambda)\cdot\frac{\bar{p}^4+\bar{q}^4+2\bar{p}^2\bar{q}^2}{2} \\ + \mathrm{F_{hv}}\cdot\frac{\eta^2\bar{q}^2+\xi^2\bar{p}^2-2\eta\xi\,\bar{p}\bar{q}}{2} \\ + \mathrm{F}'_{\mathrm{hv}}\cdot\frac{\eta^2\bar{p}^2+\xi^2\bar{q}^2+2\eta\xi\,\bar{p}\bar{q}}{2}. \end{aligned} \tag{6.10}$$

Remark 6.2 The invariance by $\mathcal{R}_3^-$ actually implies that the whole perturbation (and not only its average) is even in $(\bar{p}, \bar{q})$.

The explicit values of the tensors appearing in (6.9), (6.10) will be given in Appendix B.

7 Birkhoff normal form of the partially reduced secular Hamiltonian

In this section we shall discuss non-resonance properties of the first order Birkhoff invariants of the partially reduced secular planetary Hamiltonian and its full torsion (i.e., the non-vanishing of the determinant of the second order Birkhoff invariants). In particular, we will show that: (i) the first order Birkhoff invariants do not verify, at any order, any other resonance besides Herman's resonance (which, actually, as follows from Sect. 7.2 below, does not affect the construction of Birkhoff normal forms); (ii) the matrix of the second order Birkhoff invariants is not singular ("full torsion"). Furthermore, we will construct explicitly the Birkhoff normal form up to order four for any n, in the well-spaced regime. The secular planetary Hamiltonian is rotation invariant[40] and this simplifies the structure of the "dangerous resonances" in constructing Birkhoff normal forms: this is a general fact that will be explained in Sect. 7.2 below.

The computations in this section are based upon the Taylor expansion up to order four of the secular planetary perturbation, as given in Appendix B.

7.1 First order Birkhoff invariants

Proposition 7.1 (Herman's resonance)

$$\mathrm{tr}\,(\mathrm{Q_h}+\mathrm{Q_v}) = \sum_{i=1}^{n}\sigma_i + \sum_{i=1}^{n-1}\varsigma_i = 0, \qquad \mathrm{Q_v} := \begin{pmatrix} \bar{\mathrm{Q}}_{\mathrm{v}} & 0 \\ 0 & 0 \end{pmatrix}, \tag{7.1}$$

where $(\sigma,\bar\varsigma)\in\mathbb{R}^n\times\mathbb{R}^{n-1}$ *are the eigenvalues of the two quadratic forms* $\mathrm{Q_h}$ *and* $\bar{\mathrm{Q}}_{\mathrm{v}}$ *in* (6.9).

Proof By (B.1) and (B.3), the diagonal entries of the matrices $\mathrm{Q_h}$ and $\bar{\mathrm{Q}}_{\mathrm{v}}$ are given by $(\mathrm{Q_h})_{ii} = \sum_{1\le k\ne i\le n} m_i m_k \frac{C_1(a_i,a_k)}{\Lambda_i}$ and $(\bar{\mathrm{Q}}_{\mathrm{v}}(\Lambda))_{ii} = -\sum_{1\le j<k\le n} m_j m_k C_1(a_j,a_k)(\mathcal{L}_{ji}-\mathcal{L}_{ki})^2$. Using (B.4), one readily finds

$$(\mathcal{L}_{ji}-\mathcal{L}_{ki})^2 = \begin{cases} \frac{1}{\Lambda_{i+1}}+\frac{1}{\mathfrak{L}_i}, & k=i+1, \qquad 1\le j\le i \\ \frac{1}{\Lambda_{i+1}}-\frac{1}{\mathfrak{L}_{i+1}}, & i+1<k\le n, \qquad j=i+1 \\ \frac{1}{\mathfrak{L}_i}-\frac{1}{\mathfrak{L}_{i+1}}, & i+1<k\le n, \qquad 1\le j<i+1 \\ & (\text{when } j\le k-2) \\ 0, & \text{otherwise.} \end{cases}$$

This implies that $\sum_{1\le i\le n-1}(\mathcal{L}_{ji}-\mathcal{L}_{ki})^2 = (\frac{1}{\Lambda_j}+\frac{1}{\Lambda_k})$ and hence

$$\begin{aligned} \sum_{i=1}^{n-1}\varsigma_i = \mathrm{tr}\,\bar{\mathrm{Q}}_{\mathrm{v}} &= \sum_{1\le i\le n-1}(\bar{\mathrm{Q}}_{\mathrm{v}}(\Lambda))_{ii} \\ &= -\sum_{1\le i\le n-1}\sum_{1\le j<k\le n} m_j m_k C_1(a_j,a_k)(\mathcal{L}_{ji}-\mathcal{L}_{ki})^2 \\ &= -\sum_{1\le j<k\le n} m_j m_k C_1(a_j,a_k)\sum_{1\le i\le n-1}(\mathcal{L}_{ji}-\mathcal{L}_{ki})^2 \end{aligned}$$

[40] I.e., it is invariant under $\mathcal{S}^g$; compare (6.8) and (6.3).

$$= - \sum_{1 \le j < k \le n} m_j m_k C_1(a_j, a_k) \left(\frac{1}{\Lambda_j} + \frac{1}{\Lambda_k} \right) = - \operatorname{tr} \mathrm{Q_h} = - \sum_{i=1}^{n} \sigma_i . \qquad \square$$

We shall now prove that $\sigma(\Lambda)$ and $\bar{\varsigma}(\Lambda)$ do not satisfy any other resonance, up to a prefixed order, on a suitable open set of Λ's. More precisely, we consider the following subset of $\{a_1 < \cdots < a_n\}$

$$\mathcal{A} := \left\{ \Lambda : \ \underline{a}_j < a_j < \overline{a}_j \text{ for any } 1 \le j \le n \right\} \tag{7.2}$$

where $\underline{a}_1, \dots, \underline{a}_n, \overline{a}_1, \dots, \overline{a}_n$, are positive numbers verifying $\underline{a}_j < \overline{a}_j < \underline{a}_{j+1}$ for any $1 \le j \le n$, $\overline{a}_{n+1} := \infty$ ("well-spaced regime").

Proposition 7.2 *For any $n \ge 2$, $s \in \mathbb{N}$, there exist $\underline{a}_j, \overline{a}_j$ and d such that*

$$|(\sigma, \bar{\varsigma}) \cdot k| \ge d > 0 \quad \textit{for any } \Lambda \in \mathcal{A},\ k \in \mathbb{Z}^{2n-1} \colon 0 < |k| \le s \\ \textit{with } k_i \ne k_j \textit{ for some } i \ne j. \tag{7.3}$$

In particular, the first order Birkhoff invariants $(\sigma, \bar{\varsigma}) \in \mathbb{R}^n \times \mathbb{R}^{n-1}$ of the partially reduced planetary system verify, identically, only Herman's resonance (7.1).

To prove Proposition 7.2, we need the following simple[41]

Lemma 7.1 *Let $M^* \in \mathrm{Mat}(m \times m)$, $M_* \in \mathrm{Mat}(r \times r)$, $M^\# \in \mathrm{Mat}(m \times r)$, $M_\# \in \mathrm{Mat}(r \times m)$. Let*

$$M_\delta := \begin{pmatrix} M^* + \mathrm{O}(\delta) & \delta M^\# \\ \delta M_\# & \delta^{1+t} M_* + \mathrm{O}(\delta^2) \end{pmatrix}, \qquad 0 \le t < 1, \tag{7.4}$$

where δ is a small parameter. Then,

- (i) *if $\lambda^* \ne 0$ is a simple eigenvalue of M^*, then, for $|\delta|$ small enough, M_δ has an eigenvalue of the form $\lambda^*_\delta = \lambda^* + \mathrm{O}(\delta)$;*
- (ii) *if λ_* is a simple eigenvalue of M_* and* $\det M^* \ne 0$, *then, for $|\delta|$ small enough, M_δ has an eigenvalue of the form $\lambda^\delta_* = \delta^{1+t}\lambda_* + \mathrm{O}(\delta^2)$;*
- (iii) *if $r = 1$ and* $(\mathrm{U}^*)^{\mathrm{t}} M^* \mathrm{U}^* = \mathrm{diag}\,[\lambda^*_1, \dots, \lambda^*_m]$, *with* $\mathrm{U}^* \in \mathrm{SO}(m)$ *and* $0 \ne \lambda^*_i \ne \lambda^*_j$, *then there exists* $\mathrm{U}_\delta = \mathrm{diag}\,[\mathrm{U}^*, 1] + \mathrm{O}(\delta)$ *such that* $(\mathrm{U}_\delta)^{\mathrm{t}} M_\delta \mathrm{U}_\delta$ *is diagonal.*

Proof Statement (i) follows applying the IFT (Implicit Function Theorem) to the function $\mathcal{F}_1(\lambda, \delta) := \det(\mathcal{M}_\delta - \lambda\, \mathrm{id}_{m+r})$, noticing that $\mathcal{F}_1(\lambda, 0) =$

[41] Compare also Lemma B.2 of [4].

$(-\lambda)^r \det(M^* - \lambda\,\mathrm{id}_m)$. For statement (ii), apply the IFT to

$$\mathcal{F}_2(\lambda,\delta) = \det\begin{pmatrix} M^* - \delta^{1+t}\lambda\,\mathrm{id}_m & \delta^{1-t}M^{\#} \\ M_{\#} & M_* - \lambda\,\mathrm{id}_r \end{pmatrix}$$

noticing that $\det(\mathcal{M}_\delta - \delta^{1+t}\lambda\,\mathrm{id}_{m+r}) = \delta^{(1+t)r}\mathcal{F}_2(\lambda,\delta)$. Finally, for statement (iii), apply the IFT the $m+1$ functions $(w,\delta) \to (\mathcal{F}^{(i)}, 1-|w|^2)$, with $1 \le i \le m+1$, where $\mathcal{F}^{(i)}(w,\delta) = (\mathcal{M}_\delta^{(i)} - \lambda_\delta^{(i)}\,\mathrm{id}_m)w$, $\lambda_\delta^{(1)},\ldots,\lambda_\delta^{(m+1)}$ denote the eigenvalues of M_δ which are obtained by continuation of $\lambda_1^*,\ldots,\lambda_m^*$ and $\lambda := M_*$; $\mathcal{M}_\delta^{(i)}$ the matrix which is obtained by $\mathcal{M}_\delta$ dropping its i^{th} row and column. □

Proof of Proposition 7.2 Fix $s \ge 1$. We shall prove by induction that, for any $n \ge 2$, the eigenvalues $\sigma_2,\ldots,\sigma_n$, $\varsigma_1,\ldots,\varsigma_{n-1}$ do not satisfy any non trivial resonance up to order s in $\mathcal{A}$ and thus (7.3) follows. For $n=2$, the assertion follows by the direct computation:

$$\begin{aligned} \sigma_2 &= -\frac{3}{4}m_1 m_2 \frac{a_1}{a_2^2\Lambda_2}\left(\frac{a_1}{a_2} + \mathrm{O}\left(\frac{a_1}{a_2}\right)^2\right), \\ \varsigma &= +\frac{3}{4}m_1 m_2 \frac{a_1}{a_2^2}\left(\frac{1}{\Lambda_1} + \frac{1}{\Lambda_2}\right)\left(\frac{a_1}{a_2} + \mathrm{O}\left(\frac{a_1}{a_2}\right)^3\right). \end{aligned} \tag{7.5}$$

By (7.5), the functions $a_2 \to |k_2\sigma_2 + \kappa\varsigma|$ with $\underline{a}_1 < a_1 < \overline{a}_1$ and $(k_2,\kappa) \ne 0$, have a positive infimum on a suitable neighborhood of $a_2 = +\infty$.

Assume now that, when $n-1 \ge 2$, $\hat\sigma = (\hat\sigma_1,\ldots,\hat\sigma_{n-1})$, $\hat\varsigma = (\hat\varsigma_1,\ldots,\hat\varsigma_{n-2})$, $\hat{\mathcal{A}}$ replace n, σ, $\bar\varsigma$, $\mathcal{A}$, where $\hat\sigma = (\hat\sigma_1,\ldots,\hat\sigma_{n-1})$, $\hat\varsigma = (\hat\varsigma_1,\ldots,\hat\varsigma_{n-2})$ denote the eigenvalues of the matrices $\hat{\mathrm{Q}}_h$, $\hat{\mathrm{Q}}_v$ at rank $n-1$ and $\hat{\mathcal{A}}$ is a suitable set of the form (7.2), with $n-1$ replacing n, then, $\hat\sigma_2,\ldots,\hat\sigma_{n-1}$, $\hat\varsigma_1,\ldots,\hat\varsigma_{n-2}$ do not satisfy any non trivial linear combination on $\hat{\mathcal{A}}$. Then, (7.3) holds with $n-1$, $\hat{\mathcal{A}}$, $\hat\sigma$, $\hat\varsigma$. In particular, on $\hat{\mathcal{A}}$,

$$0 \ne \hat\sigma_i \ne \hat\sigma_j \quad \text{and} \quad 0 \ne \hat\varsigma_h \ne \hat\varsigma_k, \quad \forall 1 \le i < j \le n-1,\ \forall 1 \le h < k \le n-2. \tag{7.6}$$

As it follows from formulae in (B.1)–(B.4), at rank n, the matrices Q_h and $\bar{\mathrm{Q}}_\mathrm{v}$ are given by

$$\begin{aligned} \mathrm{Q}_\mathrm{h} &= \begin{pmatrix} \hat{\mathrm{Q}}_\mathrm{h} + \mathrm{O}(\delta) & \mathrm{O}(\delta) \\ \mathrm{O}(\delta) & \alpha\delta(1+\mathrm{O}(\delta^{2/3})) \end{pmatrix}, \\ \bar{\mathrm{Q}}_\mathrm{v} &= \begin{pmatrix} \hat{\mathrm{Q}}_v + \mathrm{O}(\delta) & \mathrm{O}(\delta) \\ \mathrm{O}(\delta) & \beta\delta(1+\mathrm{O}(\delta^{2/3})) \end{pmatrix} \end{aligned} \tag{7.7}$$

where[42]

$$\delta := a_n^{-3}, \qquad \alpha = -\frac{3m_n}{4\Lambda_n}\sum_{1\le i<n} m_i a_i^2,$$

$$\beta = +\frac{3}{4}\left(\frac{1}{\mathfrak{L}_{n-1}}+\frac{1}{\Lambda_n}\right)m_n \sum_{1\le i<n} m_i a_i^2$$

and $\mathfrak{L}_{n-1} = \sum_{1\le i\le n-1}\Lambda_i$. By Lemma 7.1, $\mathrm{Q_h}$ has $n-1$ eigenvalues $(\sigma_1,\ldots,\sigma_{n-1})$ which go to $\hat\sigma = (\hat\sigma_1,\ldots,\hat\sigma_{n-1})$ as $\delta\to 0$ and a non vanishing eigenvalue $\sigma_n = \alpha\delta + \mathrm{O}(\delta^{5/3})$; similarly, $\bar{\mathrm{Q}}_\mathrm{v}$ has $n-2$ eigenvalues $(\varsigma_1,\ldots,\varsigma_{n-2})$ which go to $\hat\varsigma = (\hat\varsigma_1,\ldots,\hat\varsigma_{n-2})$ and a non vanishing eigenvalue $\varsigma_{n-1} = \beta\delta + \mathrm{O}(\delta^{5/3})$. The assertion at rank n for the frequencies $\sigma_2,\ldots,\sigma_n, \varsigma_1,\ldots,\varsigma_{n-1}$ then follows. □

Remark 7.1 Actually, as mentioned in Remark 5.1-(v) above, the first order Birkhoff invariants of the reduced secular Hamiltonian coincide with the first $2n-1$ invariants of the (non-reduced) secular Hamiltonian computed in spatial Poincaré variables.[43] Therefore, the non-resonances shown above could have been indirectly deduced by comparison with the non-reduced setting. However, the details would not have been shorter (and can be found in [9]).

7.2 Birkhoff normal forms for rotation invariant functions

For rotation invariant Hamiltonians (i.e., invariant under $\mathcal{S}^g$) the construction of Birkhoff normal forms simplifies. As a consequence we shall see that Herman's resonance (7.1) does not play any rôle in the construction of Birkhoff normal forms (at any order) for the planetary system in RPS variables.

Let us consider a general setting, namely, let $\mathcal{B}$ an open, bounded, connected set of $\mathbb{R}^n$; let $\mathcal{D} := \mathcal{B}\times\mathbb{T}^n\times B_r^{2m}$, endowed with the standard symplectic form $dI\wedge d\varphi + du\wedge dv$ and consider a (φ-independent) real-analytic function f: $(I,\varphi,w)\in\mathcal{D}\to \mathrm{f}(I,w)\in\mathbb{R}$ of the form

$$\mathrm{f}(I,w) = \mathrm{f}_0(I) + \Omega\cdot\mathrm{r} + \mathcal{P}(I,w)$$

[42] From (B.1)–(B.4) it follows that $(\mathrm{Q_h})_{nn} = -\sum_{1\le i\le n-1}\frac{m_i m_n}{2\Lambda_n}\frac{a_i}{a_n^2} b_{3/2,1}(a_i/a_n)$ and (B.4), that $(\bar{\mathrm{Q}}_\mathrm{v})_{n-1,n-1} = \sum_{1\le i\le n-1}\frac{m_i m_n}{2}(\frac{1}{\mathfrak{L}_{n-1}}+\frac{1}{\Lambda_n})\frac{a_i}{a_n^2} b_{3/2,1}(a_i/a_n)$. In fact that, by (B.4), the differences $\mathcal{L}_{j,n-1}-\mathcal{L}_{k,n-1}$ vanish unless $k=n$, in which case they take the value $-\sqrt{\frac{1}{\mathfrak{L}_{n-1}}+\frac{1}{\Lambda_n}}$ for any $1\le j\le n-1$.

[43] In such set-up there appear a "vertical" invariant $\varsigma_n = 0$, which in the reduced setting is absent.

$$\text{where} \begin{cases} w = (u_1, \ldots, u_m, v_1, \ldots, v_m) \in \mathbb{R}^{2m} \\ \mathrm{r} = (\mathrm{r}_1, \ldots, \mathrm{r}_m), \quad \mathrm{r}_j = \frac{u_j^2+v_j^2}{2} \end{cases} \tag{7.8}$$

with $\mathcal{P}(I, w) = \mathrm{o}(|w|^2)$. The components Ω_j of Ω are called the first order Birkhoff invariants.

Denote by $\mathcal{R}^g$ the symplectic rotation of $\mathcal{D}$ into itself

$$\mathcal{R}^g : (I, \varphi, w) \to (I', \varphi', w') \quad \text{with } I_i' = I_i,\ \varphi_i' = \varphi_i + g,\ w' = \mathcal{S}^g(w), \tag{7.9}$$

$\mathcal{S}^g$ being as in (6.3) (i.e. rotates simultaneously by $\mathcal{R}(-g)$ in the symplectic planes (u_i, v_i)). We say that *a function $F(I, w)$ is rotation-invariant if $F(I, \mathcal{S}^g w) = F(I, w)$ for all angles g*. We then have:

Proposition 7.3 *Assume that* f *is rotation-invariant and that the first order Birkhoff invariants Ω_j verify, for some $a > 0$ and integer s,*

$$|\Omega(I) \cdot k| \geq a > 0, \quad \forall k \in \mathbb{Z}^m\colon\ \sum_{i=1}^m k_i = 0,\ 0 < |k|_1 := \sum_{i=1}^m |k_j| \leq 2s,\ I \in \mathcal{B}. \tag{7.10}$$

Then, there exists $0 < \breve{r} \leq r$ and a symplectic transformation $\breve{\phi} : (I, \breve{\varphi}, \breve{w}) \in \breve{\mathcal{D}} := \mathcal{B} \times \mathbb{T}^n \times B^{2m}_{\breve{r}} \to (I, \varphi, w) \in \mathcal{D}$ which puts f *into Birkhoff normal form up to the order*[44] $2s$. *Furthermore, $\breve{\phi}$ leaves the I-variables unvaried, acts as a $\breve{\varphi}$-independent shift on $\breve{\varphi}$, is $\breve{\varphi}$-independent on the remaining variables and is such that*

$$\breve{\phi} \circ \mathcal{R}^g = \mathcal{R}^g \circ \breve{\phi}. \tag{7.11}$$

Proof As customary, we pass to complex symplectic variables[45]

$$(t, t^*) = \big((t_1, \ldots, t_m), (t_1^*, \ldots, t_m^*)\big)\colon \quad \begin{cases} t_j = \frac{u_j - \mathrm{i} v_j}{\sqrt{2}} \\ t_j^* = \frac{u_j + \mathrm{i} v_j}{\sqrt{2}\,\mathrm{i}} \end{cases} \tag{7.12}$$

[44] I.e., such that $\mathrm{f} \circ \breve{\phi} = \mathrm{f}_0 + \Omega \cdot \breve{r} + \sum_{2 \leq h \leq s} P_h(\breve{r}; I) + o(|\breve{w}|^{2s})$, where P_h are homogeneous polynomials in $\breve{r}_j = |\breve{w}_j|^2/2 := (\breve{u}_j^2 + \breve{v}_j^2)/2$ of degree h.

[45] As above, i denotes the imaginary unit $\sqrt{-1}$.

and write f in such variables[46]

$$\mathrm{f}(I,t,t^*) = \sum_{0\le k<+\infty} \sum_{|\alpha|_1+|\alpha^*|_1=k} c_{\alpha,\alpha^*}(I) \prod_{1\le i\le m} t_i^{\alpha_i} t_i^{*\alpha_i^*}. \tag{7.13}$$

In terms of the coordinates t, t^* the rotations $\mathcal{R}^g$ in (7.9) becomes

$$\mathcal{R}^g\colon I_i' = I_i, \quad \varphi_i' = \varphi_i + g, \quad 1 \le i \le n; \qquad (t', t'^*) = \mathcal{S}^g(t,t^*) \tag{7.14}$$

where $\mathcal{S}^g : (t,t^*) \to (t', t'^*)$ with $t_j' := t_j e^{\mathrm{i}g}$ and $t_j'^* := t_j^* e^{-\mathrm{i}g}$. Thus, one sees immediately that f *being rotation invariance is equivalent to have*

$$c_{\alpha,\alpha^*}(I) = 0, \quad \forall |\alpha|_1 \ne |\alpha^*|_1. \tag{7.15}$$

In particular, *a rotation invariant function is even in* (t,t^*). Thus, the first non-vanishing (and not in normal form) polynomial of the Taylor expansion of f with respect to the w-variables has degree 4, i.e.,

$$\begin{aligned} \mathrm{f}(I,t,t^*) &= \mathrm{f}_0 + \sum_{1\le j\le m} \Omega_j |t_j|^2 + \mathcal{P}_4(I,t,t^*) + \mathrm{O}(|t|^6), \\ \mathcal{P}_4 &= \sum_{|\alpha|_1+|\alpha^*|_1=4} c^{(4)}_{\alpha,\alpha^*} \prod_{1\le j\le m} t_j^{\alpha_j} t_j^{*\alpha_j^*}; \end{aligned} \tag{7.16}$$

f_0, Ω_j and $c^{(4)}_{\alpha,\alpha^*}$ depend on I and we denote $|t_j|^2 := \mathrm{i} t_j t_j^*$. Because of rotation invariance, $c^{(4)}_{\alpha,\alpha^*} = 0$ for $|\alpha|_1 \ne |\alpha^*|_1$. From Birkhoff normal form theory (see, e.g., [18]) one knows that the symplectic transformation putting (7.16) into Birkhoff normal form of order 2 can be obtained as the time-1 flow ϕ_2 generated by the Hamiltonian function

$$\begin{aligned} K_4 &= \sum_{\alpha\ne\alpha^*} \frac{c^{(4)}_{\alpha,\alpha^*}}{\mathrm{i}\Omega\cdot(\alpha-\alpha^*)} \prod_{1\le j\le n} t_j^{\alpha_j} t_j^{*\alpha_j^*} \\ &= \sum_{\substack{|\alpha|_1=|\alpha^*|_1 \\ \alpha\ne\alpha^*}} \frac{c^{(4)}_{\alpha,\alpha^*}}{\mathrm{i}\Omega\cdot(\alpha-\alpha^*)} \prod_{1\le j\le n} t_j^{\alpha_j} t_j^{*\alpha_j^*}. \end{aligned} \tag{7.17}$$

Notice that, since $\sum_{1\le i\le m}(\alpha_i - \alpha_i^*) = |\alpha|_1 - |\alpha^*|_1$, $\sum_{1\le i\le m}(\alpha_i - \alpha_i^*) = 0$ in (7.17). Hence, in view of (7.10), we have that $|\Omega(I)\cdot(\alpha-\alpha^*)| \ge a > 0$ so that K_4 is well-defined and analytic in a neighborhood of $t = 0 = t^*$. Clearly, K_4

[46]With abuse of notation, we denote with the same symbol the function f in complex symplectic coordinates.

is rotation invariant (and φ-independent). Let $(I, \varphi, t, t^*) \to \Phi^\theta_{_{K_4}}(I, \varphi, t, t^*)$ the Hamiltonian flow (which leaves I unvaried) generated by K_4 at time θ. Since the transformations $\mathcal{R}^g$ in (7.9) are symplectic and K_4 is invariant by $\mathcal{R}^g$, the flow $\Phi^\theta_{_{K_4}}$ commutes with $\mathcal{R}^g$:

$$\Phi^\theta_{_{K_4}} \circ \mathcal{R}^g(I, \varphi, t, t^*) = \mathcal{R}^g \circ \Phi^\theta_{_{K_4}}(I, \varphi, t, t^*).$$

Taking $\theta = 1$, we find that the transformation $\phi_2 := \Phi^1_{_{K_4}}$ commutes with $\mathcal{R}^g$. Furthermore, ϕ_2 puts f in Birkhoff normal form up to the order 4, acting on the φ-variables as a φ-independent shift and as the identity on the I-variables and being φ-independent on the remaining variables. This implies that the function $\mathrm{f}_2 := \mathrm{f} \circ \phi_2$ is φ-independent, too. To check that f_2 is again rotation invariant, we denote $(I, (t, t^*)) \to (t_{\phi_2}, t^*_{\phi_2})$ the projection of ϕ_2 on the (t, t^*)-variables and $\phi_g := \mathcal{R}^g \circ \phi_2 = \phi_2 \circ \mathcal{R}^g$. Then, the projection of ϕ_g the (t, t^*)-variables is given by $\mathcal{S}^g(t_{\phi_2}, t^*_{\phi_2})$, thus

$$\begin{aligned}
\mathrm{f}_2\big(I, \mathcal{S}^g(t, t^*)\big) &= \mathrm{f} \circ \phi_2\big(I, \mathcal{S}^g(t, t^*)\big) = \mathrm{f} \circ \phi_2 \circ \mathcal{R}^g\big(I, (t, t^*)\big) \\
&= \mathrm{f} \circ \phi_g\big(I, (t, t^*)\big) = \mathrm{f}\big(I, \mathcal{S}^g(t_{\phi_2}, t^*_{\phi_2})\big) \\
&= \mathrm{f}\big(I, (t_{\phi_2}, t^*_{\phi_2})\big) = \mathrm{f} \circ \phi_2\big(I, (t, t^*)\big) \\
&= \mathrm{f}_2\big(I, (t, t^*)\big).
\end{aligned}$$

The argument can now be iterated, with $\mathrm{f}_2 = \mathrm{f} \circ \phi_2$ replacing f. After $s - 1$ steps, we have the thesis of Proposition 7.3. □

Remark 7.2 Birkhoff normal form up to the order $2s$ for the secular Hamiltonian is achieved in $s - 1$ steps, rather than $2s - 2$, because of parity.

7.3 Second order Birkhoff invariants

From now on we shall consider, in phase space, a neighborhood of the secular origin (which corresponds to co-planar and co-circular motions) with well-spaced semimajor axes. In such neighborhood we will construct the Birkhoff normal form. Therefore, we let[47]

$$\tilde{\mathcal{M}}^{6n-2} := \mathcal{A} \times \mathbb{T}^n \times B^{2(2n-1)}_{\epsilon_0} \subset \mathcal{M}^{6n-2}_0 \tag{7.18}$$

where $\mathcal{A}$ is as in (7.2) and ϵ_0 is some positive number.

[47] Actually such neighborhood can be lifted to any leaf $\mathcal{M}^{6n-2}_{p^*, q^*}$.

Now, let $\mathrm{U_h} = \mathrm{U_h}(\Lambda) \in \mathrm{SO}(n)$ a matrix which diagonalizes $\mathrm{Q_h}(\Lambda)$ and $\bar{\mathrm{U}}_\mathrm{v} = \bar{\mathrm{U}}_\mathrm{v}(\Lambda) \in \mathrm{SO}(n-1)$ a matrix which diagonalizes $\bar{\mathrm{Q}}_\mathrm{v}(\Lambda)$:

$$\mathrm{U_h^t Q_h U_h} = \mathrm{diag}[\sigma_1, \dots, \sigma_n], \qquad (\bar{\mathrm{U}}_\mathrm{v})^\mathrm{t} \bar{\mathrm{Q}}_\mathrm{v} \bar{\mathrm{U}}_\mathrm{v} = \mathrm{diag}[\varsigma_1, \dots, \varsigma_{n-1}]. \tag{7.19}$$

Put $\bar{\mathfrak{U}} := \mathrm{diag}\,[\mathrm{U_h}, \mathrm{U_h}, \bar{\mathrm{U}}_\mathrm{v}, \bar{\mathrm{U}}_\mathrm{v}]$ and consider the unitary transformation of $B^{2(2n-1)}_{\epsilon_0}$ into itself $\bar{z} = \bar{\mathfrak{U}}(\Lambda)\,\tilde{z} = \bar{\mathfrak{U}}(\Lambda)\,(\tilde{\eta}, \tilde{\xi}, \tilde{p}, \tilde{q})$, namely, the transformation

$$\eta = \mathrm{U_h}\tilde{\eta}, \qquad \xi = \mathrm{U_h}\tilde{\xi}, \qquad \bar{p} = \bar{\mathrm{U}}_\mathrm{v}\tilde{p}, \qquad \bar{q} = \bar{\mathrm{U}}_\mathrm{v}\tilde{q}. \tag{7.20}$$

In the coordinates $\tilde{z}$, the quadratic part of f_av in (6.9) is in diagonal form

$$\sum_{1\le i\le n} \sigma_i(\Lambda)\frac{\tilde{\eta}_i^2 + \tilde{\xi}_i^2}{2} + \sum_{1\le i\le n-1} \varsigma_i(\Lambda)\frac{\tilde{p}_i^2 + \tilde{q}_i^2}{2}, \tag{7.21}$$

with the first order Birkhoff invariants $(\sigma, \bar{\varsigma})$ satisfying in the set $\mathcal{A}$ Herman's resonance and only that.

The change of coordinates (7.20) is lifted to a symplectic transformation $\tilde{\phi}$ of the domain $\tilde{\mathcal{M}}^{6n-2}$ into itself which, leaving the Λ's unvaried, acts as

$$\tilde{\phi}\colon \quad (\Lambda, \tilde{\lambda}, \tilde{z}) \to (\Lambda, \tilde{\lambda} - \lambda_1(\Lambda, \tilde{z}), \bar{\mathfrak{U}}(\Lambda)\,\tilde{z}) \tag{7.22}$$

where $\lambda_1(\Lambda, \tilde{z})$ is a suitable shift making $\tilde{\phi}$ symplectic.[48]

Let us denote

$$\begin{aligned} &\tilde{\mathcal{H}}(\Lambda, \lambda, \tilde{z}) := \mathcal{H} \circ \tilde{\phi}(\Lambda, \tilde{\lambda}, \tilde{z}) = h_\mathrm{Kep}(\Lambda) + \mu \tilde{f}(\Lambda, \tilde{\lambda}, \tilde{z}), \\ &\quad (\Lambda, \tilde{\lambda}, \tilde{z}) \in \tilde{\mathcal{M}}^{6n-2}. \end{aligned} \tag{7.23}$$

It is easy to see that the transformation $\tilde{\phi}$ in (7.22) preserves G. Indeed, since it acts as the identity on Λ and as a unitary transformation $\bar{z} = \bar{\mathfrak{U}}(\Lambda)\tilde{z}$ on $\tilde{z}$, we have

$$\begin{aligned} G \circ \tilde{\phi}(\Lambda, \tilde{\lambda}, \tilde{z}) = G(\Lambda, \bar{\mathfrak{U}}(\Lambda)\tilde{z}) &= \sum_{1\le i\le n} \Lambda_i - \frac{1}{2}|\bar{\mathfrak{U}}(\Lambda)\tilde{z}|^2 \\ &= \sum_{1\le i\le n} \Lambda_i - \frac{1}{2}|\tilde{z}|^2 = G(\Lambda, \tilde{z}). \end{aligned} \tag{7.24}$$

[48] The generating function of $\tilde{\phi}$ is $S(\tilde{\Lambda}, \tilde{\eta}, \tilde{p}, \lambda, \xi, \bar{q}) := \tilde{\Lambda}\cdot\lambda + \xi \cdot \mathrm{U_h}(\tilde{\Lambda})\tilde{\eta} + \bar{q}\cdot \bar{\mathrm{U}}_\mathrm{v}(\tilde{\Lambda})\tilde{p}$ so that $\tilde{\lambda}_1(\Lambda, \tilde{z}) = \xi \cdot \partial_\Lambda \mathrm{U_h}(\Lambda)\tilde{\eta} + \bar{q}\cdot\partial_\Lambda \bar{\mathrm{U}}_\mathrm{v}(\Lambda)\tilde{p}$ evaluated at $\xi = \mathrm{U_h}(\Lambda)\tilde{\xi}$, $\bar{p} = \bar{\mathrm{U}}_\mathrm{v}(\Lambda)\tilde{p}$.

This fact implies that $\tilde\phi$ commutes with the G-flow $\mathcal{R}^g$ (6.2) and hence, since the Hamiltonian (5.2) is $\mathcal{R}^g$-invariant, also the Hamiltonian (7.23) is $\mathcal{R}^g$-invariant. Since $\tilde\phi$ commutes also with the reflections $\mathcal{R}_1^-$, $\mathcal{R}_3^-$, in (6.6) and $\mathcal{H}$ is invariant by them, we have also that $\tilde{\mathcal{H}}$ is invariant by $\mathcal{R}_1^-$, $\mathcal{R}_3^-$ (and hence also by $\mathcal{R}_2^-$ and $\mathcal{R}_{1\leftrightarrow 2}$).

Thus, the averaged perturbation $\tilde f_{\rm av}$ admits the same symmetries as $f_{\rm av}$, namely, for any $(\Lambda, \tilde z)\in \mathcal{A}\times B_{\epsilon_0}^{2(2n-1)}$ one has

$$\tilde f_{\rm av}(\Lambda,\tilde z)=\tilde f_{\rm av}(\Lambda,\mathcal{S}^g\tilde z)=\tilde f_{\rm av}(\Lambda,\mathcal{S}\tilde z),\quad \forall\mathcal{S}\in\{\mathcal{S}_{14}^-,\mathcal{S}_{23}^-,\mathcal{S}_{34}^-,\mathcal{S}_{1\leftrightarrow 2}\} \tag{7.25}$$

where $\mathcal{S}_{14}^-$, $\mathcal{S}_{23}^-$, $\mathcal{S}_{34}^-$, $\mathcal{S}_{1\leftrightarrow 2}$ are defined in (6.6). As discussed in Sect. 6 above, such symmetries imply that $\tilde f_{\rm av}$ has the form[49]

$$\begin{aligned}\tilde f_{\rm av}(\Lambda,\tilde z):=(f\circ\tilde\phi)_{\rm av}(\Lambda,\tilde z)=f_{\rm av}\circ\tilde\phi(\Lambda,\tilde z)\\ =C_0(\Lambda)+\sum_{1\le i\le n}\sigma_i(\Lambda)\frac{\tilde\eta_i^2+\tilde\xi_i^2}{2}\\ +\sum_{1\le i\le n-1}\varsigma_i(\Lambda)\frac{\tilde p_i^2+\tilde q_i^2}{2}\\ +\frac{1}{2}\tilde{\rm F}(\Lambda)\cdot\tilde z^4+\tilde{\mathcal{P}}(\Lambda,\tilde z)\end{aligned} \tag{7.26}$$

where $\tilde{\mathcal{P}}(\Lambda,\tilde z)$ is of order $|\tilde z|^6$ and (compare (6.9) and (6.10))

$$\begin{aligned}\frac{1}{2}\tilde{\rm F}(\Lambda)\cdot\tilde z^4:=\tilde{\rm F}_{\rm h}(\Lambda)\cdot\frac{\tilde\eta^4+\tilde\xi^4+2\tilde\eta^2\tilde\xi^2}{2}+\tilde{\rm F}_{\rm v}(\Lambda)\cdot\frac{\tilde p^4+\tilde q^4+2\tilde p^2\tilde q^2}{2}\\ +\tilde{\rm F}_{\rm hv}\cdot\frac{\tilde\eta^2\tilde q^2+\tilde\xi^2\tilde p^2-2\tilde\eta\tilde\xi\tilde p\tilde q}{2}+\tilde{\rm F}'_{\rm hv}\cdot\frac{\tilde\eta^2\tilde p^2+\tilde\xi^2\tilde q^2+2\tilde\eta\tilde\xi\tilde p\tilde q}{2}.\end{aligned}$$

The new quartic tensors $\tilde{\rm F}_{\rm h}$, $\tilde{\rm F}_{\rm v}$, $\tilde{\rm F}_{\rm hv}$, $\tilde{\rm F}'_{\rm hv}$ are easily identified observing that

$$\tilde{\rm F}(\Lambda)\cdot\tilde z^4={\rm F}(\Lambda)\cdot(\bar{\mathfrak{U}}(\Lambda)\tilde z)^4 \tag{7.27}$$

with ${\rm F}(\Lambda)$ as in (6.10).

[49]Since $\tilde\phi$ acts as a $\tilde\lambda$-independent shift on the $\tilde\lambda$ variables and is $\tilde\lambda$-independent on the remaining variables, then, the actions of averaging and applying $\tilde\phi$ can be interchanged, i.e., $(f\circ\tilde\phi)_{\rm av}=f_{\rm av}\circ\tilde\phi$, whence, (7.26).

Expression (7.26) is the starting point for the construction of the normal form up to fourth order for the averaged perturbation of the planetary problem: we make use of Proposition 7.3 with

$$m = 2n-1, \qquad \mathcal{B} = \mathcal{A}, \qquad r = \epsilon_0, \qquad s = 2, \quad \text{and} \quad f = \tilde{f}_{\rm av}.$$

Herman's resonance, indeed, does not violate assumption (7.10) (at any order $2s$ of the Birkhoff transformation), since it holds with $k = (1, \dots, 1)$, hence, $\sum_{1 \le i \le 2n-1} k_i = 2n-1 > 0$.

Let

$$\begin{aligned} &\epsilon_0 > \epsilon_1 := \breve{r} > 0, \\ &\breve{\phi} : (\Lambda, \breve{\lambda}, \breve{z}) \in \breve{\mathcal{M}}^{6n-2} := \mathcal{A} \times \mathbb{T}^n \times B^{2(2n-1)}_{\epsilon_1} \to (\Lambda, \tilde{\lambda}, \tilde{z}) \in \tilde{\mathcal{M}}^{6n-2} \end{aligned} \tag{7.28}$$

be the Birkhoff transformation given by Proposition 7.3. Denote

$$\breve{\mathcal{H}} := \tilde{\mathcal{H}} \circ \breve{\phi}(\Lambda, \breve{\lambda}, \breve{z}) = h_{\rm kep}(\Lambda) + \mu \breve{f}(\Lambda, \breve{\lambda}, \breve{z}) \tag{7.29}$$

the normalized Hamiltonian, with[50]

$$\begin{aligned} \breve{f}_{\rm av}(\Lambda, \breve{z}) &:= (\tilde{f} \circ \breve{\phi})_{\rm av}(\Lambda, \breve{z}) = \tilde{f}_{\rm av} \circ \breve{\phi}(\Lambda, \breve{z}) \\ &= C_0(\Lambda) + \Omega \cdot \breve{\mathrm{R}} + \frac{1}{2} \tau \breve{\mathrm{R}} \cdot \breve{\mathrm{R}} + \breve{\mathcal{P}}(\Lambda, \breve{z}) \end{aligned} \tag{7.30}$$

where $\breve{z} := (\breve{\eta}, \breve{\xi}, \breve{p}, \breve{q})$, $\breve{\mathcal{P}}(\Lambda, \breve{z})$ is of order $|\breve{z}|^6$ (compare (7.26)), $\breve{\mathrm{R}} = (\breve{\rho}, \breve{r})$, $\breve{\rho} = (\breve{\rho}_1, \dots, \breve{\rho}_n)$, $\breve{r} = (\breve{r}_1, \dots, \breve{r}_{n-1})$, $\breve{\rho}_i := \frac{\breve{\eta}_i^2 + \breve{\xi}_i^2}{2}$, $\breve{r}_i = \frac{\breve{p}_i^2 + \breve{q}_i^2}{2}$, and, finally, $\Omega = (\sigma, \varsigma)$ (compare Proposition 7.2).

Remark 7.3 (i) Notice that, since $\tilde{f}$ is invariant by $\mathcal{R}^g$, by (7.11) the perturbation $\breve{f}$ in (7.29) is again invariant by $\mathcal{R}^g$:

$$\begin{aligned} \breve{f} \circ \mathcal{R}^g(\Lambda, \breve{\lambda}, \breve{z}) = \tilde{f} \circ \breve{\phi} \circ \mathcal{R}^g(\Lambda, \breve{\lambda}, \breve{z}) &= \tilde{f} \circ \mathcal{R}^g \circ \breve{\phi}(\Lambda, \breve{\lambda}, \breve{z}) \\ &= \tilde{f} \circ \breve{\phi}(\Lambda, \breve{\lambda}, \breve{z}) = \breve{f}(\Lambda, \breve{\lambda}, \breve{z}). \end{aligned}$$

(ii) Since $\tilde{f}_{\rm av}$ is even in $\tilde{z}$, to explicitly evaluate the second order Birkhoff invariants, we can use the procedure described in the proof of Proposition 7.3, with $\mathrm{f} = \tilde{f}_{\rm av}$ and $m = 2n-1$. In the three-body case ($n = 2$) one can compute the Birkhoff invariants explicitly (Sect. 8), while for arbitrary n, we shall evaluate the Birkhoff invariants in the asymptotic limit of well-spaced semimajor axes, i.e., on the set $\mathcal{A}$ in (7.2) (Sect. 8.2).

[50] Again, $(\tilde{f} \circ \breve{\phi})_{\rm av}(\Lambda, \breve{z}) = \tilde{f}_{\rm av} \circ \breve{\phi}(\Lambda, \breve{z})$, since $\breve{\phi}$ acts as a $\breve{\lambda}$-independent shift on the $\breve{\lambda}$-variables and is $\breve{\lambda}$-independent on the remaining variables.

(iii) As remarked in [15], since the Birkhoff invariants are real-analytic functions of the semimajor axes, they define complex holomorphic functions. Now, since in Sects. 8–8.2 we will see that the determinant of the second order Birkhoff invariants (which is also an holomorphic function of the semimajor axes) does not vanish on the set $\mathcal{A}$, by complex function theory,[51] it then follows that the determinant does not vanish on an open dense set.

8 Full torsion of the partially reduced planetary system

In this section we shall check the full torsion of the partially reduced planetary $(1+n)$-body problem, i.e., we shall show that the matrix of the second order Birkhoff invariants τ computed above is non singular. In the case $n=2$ we will evaluate exactly τ, while for the general case we shall deduce a simple inductive formula in the asymptotics of well-spaced semimajor axes.

8.1 Full torsion in the three-body case

In the case $n=2$, we can compute explicitly τ, re-obtaining, in the planar limit, the computation performed by Arnold in [2].

In this case, we have two couples of horizontal variables (η_1, ξ_1), (η_2, ξ_2) and one couple (p, q) of vertical variables. Writing down the matrices $\mathrm{Q_h}$ and $\bar{\mathrm{Q}}_\mathrm{v}$ (of order 2, 1 respectively), the quartic tensors $\mathrm{F_h}$, $\mathrm{F_v}$, $\mathrm{F_{hv}}$, $\mathrm{F'_{hv}}$ appearing in the expansion (6.9) (compare Appendix B, definitions in (B.1)–(B.17)), computing the matrix $\mathrm{U_h}$ which diagonalizes $\mathrm{Q_h}$ (the matrix $\bar{\mathrm{U}}_\mathrm{v}=1$ being trivial) as in (7.19) and computing finally the normal form of the polynomial (7.27) with $\bar{\mathfrak{U}} = \mathrm{diag}\,[\mathrm{U_h}, \mathrm{U_h}, 1, 1]$ as described in proof of Proposition 7.3, one finds the following expressions of the second order Birkhoff invariants:

$$
\begin{aligned}
\tau_{11} &= \frac{4m_1m_2}{(1+d^2)^2}\Bigg[\frac{\mathrm{r}_1(a_1,a_2)}{\Lambda_1^2} + \frac{d^4\mathrm{r}_1(a_2,a_1)}{\Lambda_2^2} + \frac{2d^2\mathrm{r}_2(a_1,a_2)}{\Lambda_1\Lambda_2} \\
&\quad - \frac{2d\mathrm{r}_3(a_1,a_2)}{\Lambda_1\sqrt{\Lambda_1\Lambda_2}} - \frac{2d^3\mathrm{r}_3(a_2,a_1)}{\Lambda_2\sqrt{\Lambda_1\Lambda_2}} + \frac{d^2\mathrm{r}_4(a_1,a_2)}{\Lambda_1\Lambda_2}\Bigg] \\
\tau_{12} &= \frac{4m_1m_2}{(1+d^2)^2}\Bigg[\frac{(1-d^2)^2\mathrm{r}_2(a_1,a_2)}{\Lambda_1\Lambda_2} + \frac{2d^2\mathrm{r}_1(a_1,a_2)}{\Lambda_1^2} + \frac{2d^2\mathrm{r}_1(a_2,a_1)}{\Lambda_2^2} \\
&\quad + \frac{2d(1-d^2)\mathrm{r}_3(a_1,a_2)}{\Lambda_1\sqrt{\Lambda_1\Lambda_2}} - \frac{2d(1-d^2)\mathrm{r}_3(a_2,a_1)}{\Lambda_2\sqrt{\Lambda_1\Lambda_2}} \\
&\quad + \frac{(1-6d^2+d^4)\mathrm{r}_4(a_1,a_2)}{4\Lambda_1\Lambda_2}\Bigg]
\end{aligned}
\tag{8.1}
$$

[51] Compare, also, with the argument used in Proposition 74, p. 1573 of [15].

$$\tau_{22} = \frac{4m_1m_2}{(1+d^2)^2}\left[\frac{\mathrm{r}_1(a_2,a_1)}{\Lambda_2^2} + \frac{d^4\mathrm{r}_1(a_1,a_2)}{\Lambda_1^2} + \frac{2d^2\mathrm{r}_2(a_1,a_2)}{\Lambda_1\Lambda_2} - \frac{2d\mathrm{r}_3(a_2,a_1)}{\Lambda_2\sqrt{\Lambda_1\Lambda_2}} - \frac{2d^3\mathrm{r}_3(a_1,a_2)}{\Lambda_1\sqrt{\Lambda_1\Lambda_2}} + \frac{d^2\mathrm{r}_4(a_1,a_2)}{\Lambda_1\Lambda_2}\right]$$

$$\begin{aligned}
\tau_{13} = \frac{m_1m_2}{(1+d^2)^2}\Bigg[&\frac{1}{\Lambda_1}\left(\frac{1}{\Lambda_1}+\frac{1}{\Lambda_2}\right)(\mathrm{s}_1(a_1,a_2)+\mathrm{s}_1^*(a_1,a_2))\\
&-\left(\frac{1}{\Lambda_1^2}+\frac{d^2}{\Lambda_2^2}\right)C_1(a_1,a_2)\\
&-\frac{d}{\sqrt{\Lambda_1\Lambda_2}}\left(\frac{1}{\Lambda_1}+\frac{1}{\Lambda_2}\right)(\mathrm{s}_2(a_1,a_2)+\mathrm{s}_2^*(a_1,a_2))\\
&+\frac{d^2}{\Lambda_2}\left(\frac{1}{\Lambda_1}+\frac{1}{\Lambda_2}\right)(\mathrm{s}_1(a_2,a_1)+\mathrm{s}_1^*(a_2,a_1))\Bigg]\\
\tau_{23} = \frac{m_1m_2}{(1+d^2)^2}\Bigg[&\frac{1}{\Lambda_2}\left(\frac{1}{\Lambda_1}+\frac{1}{\Lambda_2}\right)(\mathrm{s}_1(a_2,a_1)+\mathrm{s}_1^*(a_2,a_1))\\
&-\left(\frac{1}{\Lambda_2^2}+\frac{d^2}{\Lambda_1^2}\right)C_1(a_1,a_2)\\
&+\frac{d}{\sqrt{\Lambda_1\Lambda_2}}\left(\frac{1}{\Lambda_1}+\frac{1}{\Lambda_2}\right)(\mathrm{s}_2(a_1,a_2)+\mathrm{s}_2^*(a_1,a_2))\\
&+\frac{d^2}{\Lambda_1}\left(\frac{1}{\Lambda_1}+\frac{1}{\Lambda_2}\right)(\mathrm{s}_1(a_1,a_2)+\mathrm{s}_1^*(a_1,a_2))\Bigg]\\
\tau_{33} = 4m_1m_2\Bigg[&\mathrm{r}_1^*(a_1,a_2)\left(\frac{1}{\Lambda_1}+\frac{1}{\Lambda_2}\right)^2 + \frac{C_1(a_1,a_2)}{4\Lambda_1\Lambda_2}\Bigg]
\end{aligned}$$

where

$$d := \frac{|a-b|}{2c}\left(\sqrt{1+\frac{4c^2}{(a-b)^2}}-1\right) \quad \text{with } a := \frac{C_1(a_1,a_2)}{\Lambda_1},\ b := \frac{C_1(a_1,a_2)}{\Lambda_2},\quad c := \frac{C_2(a_1,a_2)}{\sqrt{\Lambda_1,\Lambda_2}}$$

and C_1, C_2, $\mathrm{r}_1,\dots,\mathrm{r}_4$, r_1^*, r_2^*, s_1, s_2, s_1^*, s_2^* are defined in terms of the Laplace coefficients $\beta_k^{(r)}(\alpha) := b_{r/2,k}(\alpha)$ in Appendix B (compare (B.2), (B.6), and (B.8)).

The expression (8.1) generalizes the normal form of the planar three-body problem (the only known), computed in[52] [2, Arnold, 1963]. And in fact, if we consider Arnold's case ($n = 2$ and $p = 0 = q$) and use the following asymptotics (where we regard $a_1 = \mathrm{O}(1)$ and $1/a_2$ small)

$$\begin{aligned}
\mathrm{r}_1(a_1, a_2) &= \frac{3}{16a_2}\left(\frac{a_1^2}{a_2^2} + \mathrm{O}\left(\frac{a_1^4}{a_2^4}\right)\right),\\
\mathrm{r}_1(a_2, a_1) &= -\frac{3}{4a_2}\left(\frac{a_1^2}{a_2^2} + \mathrm{O}\left(\frac{a_1^4}{a_2^4}\right)\right)\\
\mathrm{r}_2(a_1, a_2) &= \mathrm{r}_2(a_2, a_1) = -\frac{9}{16a_2}\left(\frac{a_1^2}{a_2^2} + \mathrm{O}\left(\frac{a_1^4}{a_2^4}\right)\right),\\
\mathrm{r}_3(a_1, a_2) &= \mathrm{O}\left(\frac{a_1^3}{a_2^4}\right) = \mathrm{r}_3(a_2, a_1)\\
\mathrm{r}_4(a_1, a_2) &= \mathrm{O}\left(\frac{a_1^4}{a_2^5}\right), \qquad d = \mathrm{O}(a_2^{-5/4})
\end{aligned} \tag{8.2}$$

then, asymptotically, the matrix τ of the coefficients of (7.30), which is reduced to "horizontal" submatrix of (8.1) $\tau_{\mathrm{pl}} = \left(\begin{smallmatrix}\tau_{11} & \tau_{12}\\ \tau_{12} & \tau_{22}\end{smallmatrix}\right)$ takes the value

$$\tau_{\mathrm{pl}} = m_1 m_2 \frac{a_1^2}{a_2^3}\begin{pmatrix} \frac{3}{4\Lambda_1^2} & -\frac{9}{4\Lambda_1\Lambda_2}\\ -\frac{9}{4\Lambda_1\Lambda_2} & -\frac{3}{\Lambda_2^2}\end{pmatrix}(1 + \mathrm{O}(a_2^{-5/4})). \tag{8.3}$$

We, then, recover

$$\det \tau_{\mathrm{pl}} = -\frac{117}{16\Lambda_1^2\Lambda_2^2}\left(m_1 m_2 \frac{a_1^2}{a_2^3}\right)^2 (1 + \mathrm{O}(a_2^{-5/4})). \tag{8.4}$$

This is Arnold's check[53] of non-degeneracy (or "*full torsion*") of the frequency-map of the planar three-body problem.

The first non-trivial case after that considered by Arnold, is the (spatial) three-body problem. As Arnold, we consider well-spaced semimajor axes. Then, using the expansions

$$\mathrm{r}_1^*(a_1, a_2) = -\frac{3}{16a_2}\left(\frac{a_1^2}{a_2^2} + \mathrm{O}\left(\frac{a_1^4}{a_2^4}\right)\right), \qquad \mathrm{s}_1(a_1, a_2) = \frac{3}{a_2}\left(\frac{a_1^2}{a_2^2} + \mathrm{O}\left(\frac{a_1^4}{a_2^4}\right)\right)$$

[52] Recall that when $p = 0 = q$, the RPS map coincides with the planar Poincaré map.

[53] Compare (8.4) with [2, p. 138, (3.4.31)]. Notice that, in [2], the second order Birkhoff invariants are defined as one half the τ_{ij}'s and that, in [2], a_2^4 should be corrected into a_2^7.

$$
\begin{aligned}
&\mathrm{s}_1(a_2, a_1) = \frac{9}{8a_2}\left(\frac{a_1^2}{a_2^2} + \mathrm{O}\left(\frac{a_1^4}{a_2^4}\right)\right),\\
&\mathrm{s}_1^*(a_1, a_2) = -\frac{3}{4a_2}\left(\frac{a_1^2}{a_2^2} + \mathrm{O}\left(\frac{a_1^4}{a_2^4}\right)\right)\\
&\mathrm{s}_1^*(a_2, a_1) = \frac{9}{8a_2}\left(\frac{a_1^2}{a_2^2} + \mathrm{O}\left(\frac{a_1^4}{a_2^4}\right)\right), \qquad \mathrm{s}_2(a_1, a_2),\ \mathrm{s}_2^*(a_1, a_2) = \mathrm{O}\left(\frac{a_1^3}{a_2^4}\right)\\
&C_1(a_1, a_2) = -\frac{3}{4a_2}\left(\frac{a_1^2}{a_2^2} + \mathrm{O}\left(\frac{a_1^4}{a_2^4}\right)\right),
\end{aligned} \tag{8.5}
$$

we find, from (8.1), that the planar matrix (8.3) is completed, in the spatial case, to

$$
\tau = m_1 m_2 \frac{a_1^2}{a_2^3}\begin{pmatrix} \frac{3}{4\Lambda_1^2} & -\frac{9}{4\Lambda_1\Lambda_2} & \frac{3}{\Lambda_1^2} \\ -\frac{9}{4\Lambda_1\Lambda_2} & -\frac{3}{\Lambda_2^2} & \frac{9}{4\Lambda_1\Lambda_2} \\ \frac{3}{\Lambda_1^2} & \frac{9}{4\Lambda_1\Lambda_2} & -\frac{3}{4\Lambda_1^2} \end{pmatrix}(1 + \mathrm{O}(a_2^{-5/4})). \tag{8.6}
$$

For small $1/a_2$, this matrix is non singular:

$$
\begin{aligned}
\det \tau &= -\frac{27}{16\Lambda_1^4\Lambda_2^2}\left(m_1 m_2 \frac{a_1^2}{a_2^3}\right)^3 (1 + \mathrm{O}(a_2^{-5/4}))\\
&= -\frac{27}{16}\frac{m_2}{m_1 m_0^3}\frac{a_1^4}{a_2^7}(1 + \mathrm{O}(\mu) + \mathrm{O}(a_2^{-5/4}))
\end{aligned} \tag{8.7}
$$

having used $\Lambda_i^2 = m_i^2 m_0 a_i + \mathrm{O}(\mu)$.

8.2 Asymptotic full torsion in the general case

The asymptotic evaluation (8.7) can be extended to the general case:

Proposition 8.1 (Full torsion of the planetary frequency map) *For $n \geq 2$ and $0 < \delta_\star < 1$ there exist*[54] $\bar{\mu} > 0$, $0 < \underline{a}_1 < \overline{a}_1 < \cdots < \underline{a}_n < \overline{a}_n$ *such that, on the set $\mathcal{A}$ defined in* (7.2) *and for* $0 < \mu < \bar{\mu}$, *the matrix* $\tau = (\tau_{ij})$ *is non-singular*: $\det \tau = d_n(1 + \delta_n)$, *where* $|\delta_n| < \delta_\star$ *with*

[54] $\bar{\mu}$ is taken small only to simplify (8.8), but a similar evaluation hold with $\bar{\mu} = 1$.

$$d_n = (-1)^{n-1}\frac{3}{5}\left(\frac{45}{16m_0^2}\right)^{n-1}\frac{m_2}{m_1m_0}a_1\left(\frac{a_1}{a_n}\right)^3\prod_{2\le k\le n}\left(\frac{1}{a_k}\right)^4. \tag{8.8}$$

Proof The proof is by induction on n. The case $n=2$ has been proved by explicit computation, compare (8.7).

Assume now that, when, in the statement of Proposition 8.1, $n-1\ge 2$, $\hat{\mathcal{A}}$ and $\hat\tau$ replace n, $\mathcal{A}$, τ, then,

$$\det\hat\tau = d_{n-1}(1+\mathrm{O}(\mu)+\mathrm{O}(1/\sqrt{a_2})) \tag{8.9}$$

and let us prove this equality at rank n. Proceeding as in the proof of Proposition 7.2 and using Lemma 7.1-(iii), one sees that, at rank n, matrices which diagonalize $\mathrm{Q_h}$, $\bar{\mathrm{Q}}_\mathrm{v}$ may be taken of the form

$$\mathrm{U_h} = \mathrm{U_h^{(0)}} + \mathrm{O}(\delta), \qquad \bar{\mathrm{U}}_\mathrm{v} = \bar{\mathrm{U}}_\mathrm{v}^{(0)} + \mathrm{O}(\delta) \tag{8.10}$$

where $\mathrm{U_h^{(0)}} = \mathrm{diag}\,[\hat{\mathrm{U}}_\mathrm{h}, 1]$, $\bar{\mathrm{U}}_\mathrm{v}^{(0)} = \mathrm{diag}\,[\hat{\mathrm{U}}_\mathrm{v}, 1]$ and $\hat{\mathrm{U}}_\mathrm{h}$, $\hat{\mathrm{U}}_\mathrm{v}$ denote matrices which diagonalize the second oder matrices $\hat{\mathrm{Q}}_\mathrm{h}$, $\hat{\mathrm{Q}}_\mathrm{v}$ at the previous rank.

Using (B.5)–(B.17), one also sees that the polynomial of degree 4 in (6.9) has the form

$$\begin{aligned}\mathrm{F}(\Lambda)\cdot\bar z^4 = (\hat{\mathrm{F}}(\hat\Lambda)+\mathrm{F}_0(\Lambda))(z^{(n-1)})^4 + \sum_{1\le i\le 3}\mathrm{F}_i(\Lambda)\cdot(z^{(n-1)})^{4-i}\mathrm{z}^i\\ + \mathrm{F}_4(\Lambda)\cdot\mathrm{z}^4,\end{aligned} \tag{8.11}$$

where F_i are $\mathrm{O}(\delta)$, $\hat\Lambda$, $z^{(n-1)}$, z denote the sets of variables

$$\begin{aligned}&\hat\Lambda = (\Lambda_1,\dots,\Lambda_{n-1}),\\ &z^{(n-1)} = (\eta_1,\dots,\eta_{n-1},\xi_1,\dots,\xi_{n-1},p_1,\dots,p_{n-2},q_1,\dots,q_{n-2})\\ &\mathrm{z} := (\eta_n,\xi_n,p_{n-1},q_{n-1})\end{aligned}$$

and $\hat{\mathrm{F}}\cdot(z^{(n-1)})^4$ is the fourth order polynomial in (6.9) associated to rank $n-1$. Later, we will need the explicit expression of the polynomial $\mathrm{F}_4\cdot\mathrm{z}^4$ in (8.11), which, as we will shortly see, is given by

$$\begin{aligned}\mathrm{F}_4\cdot\mathrm{z}^4 = \bar{\mathrm{F}}_\mathrm{h}(\Lambda)(\eta_n^2+\xi_n^2)^2 + \bar{\mathrm{F}}_\mathrm{v}(\Lambda)(p_{n-1}^2+q_{n-1}^2)^2\\ + \bar{\mathrm{F}}_\mathrm{hv}(\eta_n q_{n-1}-\xi_n p_{n-1})^2 + \bar{\mathrm{F}}'_\mathrm{hv}(\eta_n p_{n-1}+\xi_n q_{n-1})^2,\end{aligned} \tag{8.12}$$

where

$$\begin{aligned}
\bar{\mathrm{F}}_{\mathrm{h}}(\Lambda) &= \sum_{1\le j<n} m_j m_n \frac{\mathrm{r}_1(a_n, a_j)}{\Lambda_n^2} \\
\bar{\mathrm{F}}_{\mathrm{v}}(\Lambda) &= \sum_{1\le j<n} m_j m_n \left[\mathrm{r}_1^*(a_j, a_n)\left(\frac{1}{\mathfrak{L}_{n-1}} + \frac{1}{\Lambda_n}\right)^2 + \frac{C_1(a_j, a_n)}{4\mathfrak{L}_{n-1}\Lambda_n}\right] \\
\bar{\mathrm{F}}_{\mathrm{hv}} &= \sum_{1\le j<n} m_j m_n \left[\left(\frac{1}{\mathfrak{L}_{n-1}} + \frac{1}{\Lambda_n}\right)\frac{\mathrm{s}_1(a_n, a_j)}{\Lambda_n} - \frac{C_1(a_j, a_n)}{2\Lambda_n^2}\right] \\
\bar{\mathrm{F}}'_{\mathrm{hv}} &= \sum_{1\le j<n} m_j m_n \left[\left(\frac{1}{\mathfrak{L}_{n-1}} + \frac{1}{\Lambda_n}\right)\frac{\mathrm{s}_1^*(a_n, a_j)}{\Lambda_n} - \frac{C_1(a_j, a_n)}{2\Lambda_n^2}\right].
\end{aligned} \tag{8.13}$$

In fact, the first line in (8.13) is found using (B.5). Next, by (B.7), the action $\mathrm{F}_{\mathrm{v}1}$, $\mathrm{F}_{\mathrm{hv}1}$, $\mathrm{F}'_{\mathrm{hv}1}$ on z^4, is found selecting in the formula of $\mathrm{F}_{\mathrm{v}1}$ only the indices $j = k = n-1$ and, in the formula of F_{hv}, $\mathrm{F}'_{\mathrm{hv}}$, the indices $j = n$ and $h = n-1$. Using, as in the proof of Proposition 7.2, that $\mathcal{L}_{i,n-1} - \mathcal{L}_{j,n-1} = -\sqrt{\frac{1}{\Lambda_n} + \frac{1}{\mathfrak{L}_{n-1}}}\delta_{j,n}$, one finds the contribution to $\bar{\mathrm{F}}_{\mathrm{v}}$, $\bar{\mathrm{F}}_{\mathrm{hv}}$ and $\bar{\mathrm{F}}'_{\mathrm{hv}}$ proportional to, respectively, r_1^*, s_1 and s_1^* in (8.13). Finally, the actions of the tensors $\mathrm{F}_{\mathrm{v}2}$, $\mathrm{F}_{\mathrm{hv}2}$, $\mathrm{F}'_{\mathrm{hv}2}$ on z^4 are found observing that the matrices T_{i+1}, T_i^*, $\bar{\mathrm{T}}_i$ defined in (B.9)–(B.17) contain only (η_j, ξ_j)-variables with indices $j \le i+1$ and (p_j, q_j)-variables with indices $j \le i$. Hence, the monomials with literal part z^4 in the polynomial (B.17) are only those appearing in $\sum_{1\le i<n} m_i m_n C_1(a_i, a_n)\,\mathrm{tr}\,\bar{\mathrm{T}}_{n-1}$. Since the quartic part of the trace of $\bar{\mathrm{T}}_{n-1}$ is $-\bar{Q}_{n-1}(p_{n-1}^2 + q_{n-1}^2)$, with $\bar{Q}_{n-1} = -\frac{\mathfrak{L}_{n-1}\Lambda_n(p_{n-1}^2+q_{n-1}^2)-2(\mathfrak{L}_{n-1})^2(\eta_n^2+\xi_n^2)}{4\mathfrak{L}_{n-1}^2\Lambda_n^2}$ (up to monomials which do not depend on z: compare (B.9)), it follows that the sum of the actions of $\mathrm{F}_{\mathrm{v}2}$, $\mathrm{F}_{\mathrm{hv}2}$, $\mathrm{F}'_{\mathrm{hv}2}$ on z^4 is identified with the polynomial

$$\sum_{1\le i<n} m_i m_n C_1(a_i, a_n)(p_{n-1}^2 + q_{n-1}^2) \\ \times \frac{\mathfrak{L}_{n-1}\Lambda_n(p_{n-1}^2 + q_{n-1}^2) - 2\mathfrak{L}_{n-1}^2(\eta_n^2 + \xi_n^2)}{4\mathfrak{L}_{n-1}^2\Lambda_n^2}$$

and hence (8.13) follow.

By (8.10) and (8.11), letting, for short, $\bar{\mathfrak{U}}^{(0)} := \mathrm{diag}\,[\mathrm{U}_{\mathrm{h}}^{(0)}, \mathrm{U}_{\mathrm{h}}^{(0)}, \bar{\mathrm{U}}_{\mathrm{v}}^{(0)}, \bar{\mathrm{U}}_{\mathrm{v}}^{(0)}]$ and $\hat{\mathfrak{U}} := \mathrm{diag}\,[\hat{\mathrm{U}}_{\mathrm{h}}, \hat{\mathrm{U}}_{\mathrm{h}}, \hat{\mathrm{U}}_{\mathrm{v}}, \hat{\mathrm{U}}_{\mathrm{v}}]$, one readily finds that the polynomial (7.27) is given by

$$\begin{aligned}\tilde{\mathrm{F}}(\Lambda)\cdot\tilde{z}^4 &= \mathrm{F}(\Lambda)\cdot(\bar{\mathfrak{U}}\tilde{z})^4 = \mathrm{F}(\Lambda)\cdot((\bar{\mathfrak{U}}^{(0)}+\mathrm{O}(\delta))\tilde{z})^4\\ &= (\hat{\mathrm{F}}(\hat{\Lambda})+\tilde{\mathrm{F}}_0(\Lambda))(\hat{\mathfrak{U}}\tilde{z}^{(n-1)})^4 + \sum_{1\le i\le 3}\tilde{\mathrm{F}}_i(\Lambda)\cdot(\tilde{z}^{(n-1)})^{4-i}\tilde{\mathrm{z}}^i\\ &\quad + (\mathrm{F}_4(\Lambda)+\mathrm{O}(\delta^2))\cdot\tilde{\mathrm{z}}^4,\end{aligned} \tag{8.14}$$

where $\tilde{\mathrm{F}}_i$ are $\mathrm{O}(\delta)$ and

$$\begin{aligned}&\tilde{z}^{(n-1)} := (\tilde{\eta}_1,\ldots,\tilde{\eta}_{n-1},\tilde{\xi}_1,\ldots,\tilde{\xi}_{n-1},\tilde{p}_1,\ldots,\tilde{p}_{n-2},\tilde{q}_1,\ldots,\tilde{q}_{n-2}),\\ &\tilde{\mathrm{z}} := (\tilde{\eta}_n,\tilde{\xi}_n,\tilde{p}_{n-1},\tilde{q}_{n-1}).\end{aligned}$$

Then, computing the second order Birkhoff invariants $\tau=(\tau_{ij})$ from (8.14), one finds

$$\tau = \begin{pmatrix}\hat{\tau}+\mathrm{O}(\delta) & \mathrm{O}(\delta)\\ \mathrm{O}(\delta) & \bar{\tau}+\mathrm{O}(\delta^2)\end{pmatrix} \tag{8.15}$$

where $\hat{\tau}$ is the second order Birkhoff invariants matrix at rank $n-1$ and $\bar{\tau}$ is the matrix of the Birkhoff invariants of order 2 associated to the polynomial in (8.12), i.e.,

$$\begin{aligned}\bar{\tau}_{11} &= 4\bar{\mathrm{F}}_{\mathrm{h}} = \frac{4m_n}{\Lambda_n^2}\sum_{1\le j<n} m_j \mathrm{r}_1(a_n,a_j)\\ \bar{\tau}_{12} = \bar{\tau}_{21} &= \bar{\mathrm{F}}_{\mathrm{hv}}+\bar{\mathrm{F}}'_{\mathrm{hv}} = m_n\sum_{1\le j<n} m_j\Bigg[\left(\frac{1}{\mathfrak{L}_{n-1}}+\frac{1}{\Lambda_n}\right)\\ &\qquad\times\frac{\mathrm{s}_1(a_n,a_j)+\mathrm{s}_1^*(a_n,a_j)}{\Lambda_n}-\frac{C_1(a_j,a_n)}{\Lambda_n^2}\Bigg]\\ \bar{\tau}_{22} &= 4\bar{\mathrm{F}}_{\mathrm{v}} = 4m_n\sum_{1\le j<n} m_j\Bigg[\mathrm{r}_1^*(a_j,a_n)\left(\frac{1}{\mathfrak{L}_{n-1}}+\frac{1}{\Lambda_n}\right)^2\\ &\qquad+\frac{C_1(a_j,a_n)}{4\mathfrak{L}_{n-1}\Lambda_n}\Bigg].\end{aligned}$$

Using now $\mathfrak{L}_{n-1}=\Lambda_{n-1}(1+\mathrm{O}(\Lambda_{n-2}/\Lambda_{n-1}))$, $\Lambda_i^2=m_i^2 m_0 a_i(1+\mathrm{O}(\mu))$ and the asymptotics expressions in (8.2) and (8.5), one finds

$$\begin{aligned}\bar{\tau}_{11} &= -3\frac{m_{n-1}a_{n-1}^2}{m_0 m_n a_n^4}(1+\mathrm{O}(1/\Lambda_{n-1})+\mathrm{O}(\mu)),\\ \bar{\tau}_{12} &= \frac{9}{4}\frac{a_{n-1}^2}{m_0\sqrt{a_{n-1}a_n}\,a_n^3}(1+\mathrm{O}(1/\Lambda_{n-1})+\mathrm{O}(\mu))\end{aligned} \tag{8.16}$$

$$\bar{\tau}_{22} = -\frac{3}{4}\frac{m_n a_{n-1}}{m_0 m_{n-1} a_n^3}(1 + \mathrm{O}(1/\Lambda_{n-1}) + \mathrm{O}(\mu)).$$

Finally in view of (8.15),

$$\det \tau = \det \hat{\tau} \det \bar{\tau}(1 + \mathrm{O}(\delta)) = -\frac{45}{16 m_0^2}\frac{a_{n-1}^3}{a_n^7} d_{n-1}(1 + \mathrm{O}(1/\sqrt{a_2}) + \mathrm{O}(\mu))$$

relation which immediately implies (8.8) at step n, because of the inductive assumption (8.9). □

9 The totally reduced planetary system

We turn now to the total reduction of the planetary system. Indeed, the partially reduced Hamiltonian (in Birkhoff normalized variables (7.28)–(7.30)) $\breve{\mathcal{H}} = h_{\mathrm{kep}} + \mu \breve{f}$ *still commutes with the length of the total angular momentum* $G = |\mathrm{C}|$, or, equivalently, is invariant under the $\mathcal{R}^g$ action; compare Remark 7.3. This allows to reduce completely the total angular momentum by introducing symplectic variables, which include an action-angle couple (G, g), with g cyclic for the reduced planetary Hamiltonian. However, to achieve the total reduction one has, in general, to exclude certain conical singularities, compare (9.7) below.[55]

Recall (7.28) and let, at first,

$$\breve{\mathcal{M}}_*^{6n-2} := \mathcal{A} \times \mathbb{T}^n \times B_* \tag{9.1}$$

where B_* denotes the open set

$$B_* := B_{\epsilon_1}^{2(2n-1)} \cap \big\{\breve{z}\colon (\breve{\eta}_i, \breve{\xi}_i) \neq 0,\ (\breve{p}_j, \breve{q}_j) \neq 0$$
$$\text{for any } 1 \le i \le n,\ 1 \le j \le n-1\big\}.$$

For $\breve{z} \in B_*$, let

$$\begin{cases} \breve{\rho}_i := \frac{\breve{\eta}_i^2 + \breve{\xi}_i^2}{2} \\ \breve{\varphi}_i := \arg(\breve{\xi}_i, \breve{q}_i) \end{cases}, \qquad \begin{cases} \breve{r}_j := \frac{\breve{p}_j^2 + \breve{q}_j^2}{2} \\ \breve{\chi}_j := \arg(\breve{p}_j, \breve{q}_j) \end{cases} \tag{9.2}$$

[55] These singularities include the secular origin (i.e., co-circular and co-planar phase points); but, say, all co-circular and all-but-one co-planar are allowed; compare Remark 9.1-(iii) below. In the case $n = 2$, these singularities are removable; see [9].

be the symplectic polar coordinates associated to $(\breve\eta_i, \breve\xi_i)$, $(\breve p_j, \breve q_j)$. The integral G becomes then a linear function of Λ, $\breve\rho_1, \ldots, \breve\rho_n, \breve r_1, \ldots, \breve r_{n-1}$,

$$G = \sum_{1\le i\le n} \Lambda_i - \sum_{1\le i\le n} \breve\rho_i - \sum_{1\le j\le n-1} \breve r_j. \tag{9.3}$$

Denote $\hat r := (\breve r_1, \ldots \breve r_{n-2})$, $\hat\chi := (\breve\chi_1, \ldots \breve\chi_{n-2})$, $1_k := (1, \ldots, 1) \in \mathbb{R}^k$ and observe that

$$\begin{aligned}\Lambda \cdot d\breve\lambda + \breve\eta \cdot d\breve\xi + \breve p \cdot d\breve q &= \Lambda \cdot d\breve\lambda + \breve\rho \cdot d\breve\varphi + \breve r \cdot d\breve\chi \\ &= \Lambda \cdot d\breve\lambda + \breve\rho \cdot d\breve\varphi + \hat r \cdot d\hat\chi + \breve r_{n-1} \cdot d\breve\chi_{n-1} \\ &= \Lambda \cdot d(\breve\lambda + \breve\chi_{n-1} 1_n) + \breve\rho \cdot d(\breve\varphi - \breve\chi_{n-1} 1_n) \\ &\quad + \hat r \cdot d(\hat\chi - \breve\chi_{n-1} 1_{n-2}) + G d(-\breve\chi_{n-1}).\end{aligned} \tag{9.4}$$

Define, now, the symplectic map $\hat\phi^{-1} : (\Lambda, \breve\lambda, \breve z) \to (\Lambda, G, \hat\lambda, g, \hat z)$ by letting, for $1 \le i \le n$ and $1 \le j \le n-2$:

$$\begin{aligned}&\hat\lambda = (\hat\lambda_1, \ldots \hat\lambda_n) \in \mathbb{T}^n, \qquad (G, g) \in \mathbb{R}_+ \times \mathbb{T}, \qquad \hat z = (\hat\eta, \hat\xi, \hat p, \hat q) \\ &\quad \text{with } \hat\eta = (\hat\eta_1, \ldots, \hat\eta_n),\ \hat\xi = (\hat\xi_1, \ldots, \hat\xi_n) \in \mathbb{R}^n;\ \hat p = (\hat p_1, \ldots, \hat p_{n-2}), \\ &\quad\quad \hat q = (\hat q_1, \ldots, \hat q_{n-2}) \in \mathbb{R}^{n-2} \\ &\begin{cases}\Lambda_i \\ \hat\lambda_i := \breve\lambda_i + \breve\chi_{n-1}\end{cases} \qquad \begin{cases} G = G(\Lambda, \breve z) \\ g := -\breve\chi_{n-1} = -\arg(\breve p_{n-1}, \breve q_{n-1})\end{cases} \\ &\begin{cases}\hat\eta_i = \sqrt{2\breve\rho_i}\cos(\breve\varphi_i - \breve\chi_{n-1}) \\ \hat\xi_i = \sqrt{2\breve\rho_i}\sin(\breve\varphi_i - \breve\chi_{n-1})\end{cases} \qquad \begin{cases}\hat p_j = \sqrt{2\breve r_j}\cos(\breve\chi_j - \breve\chi_{n-1}) \\ \hat q_j = \sqrt{2\breve r_j}\sin(\breve\chi_j - \breve\chi_{n-1})\end{cases}\end{aligned} \tag{9.5}$$

Conclusions are drawn in the following

Remark 9.1 (i) In view of (9.4) *the map* $\hat\phi^{-1} : \breve{\mathcal{M}}^{6n-2}_* \to \hat{\mathcal{M}}^{6n-2}_* := \hat\phi^{-1}(\breve{\mathcal{M}}^{6n-2}_*)$ *is symplectic*.

(ii) Inverting formulae (9.5) we find that the symplectic map $\hat\phi$ is explicitly given by

$$\begin{aligned}&\begin{pmatrix}\breve\eta_i \\ \breve\xi_i\end{pmatrix} = \mathcal{R}(-g)\begin{pmatrix}\hat\eta_i \\ \hat\xi_i\end{pmatrix}, \qquad \breve\lambda_i = \hat\lambda_i + g, \quad 1 \le i \le n \\ &\begin{pmatrix}\breve p_j \\ \breve q_j\end{pmatrix} = \mathcal{R}(-g)\begin{pmatrix}\hat p_j \\ \hat q_j\end{pmatrix}, \qquad \begin{pmatrix}\breve p_{n-1} \\ \breve q_{n-1}\end{pmatrix} = \mathcal{R}(-g)\begin{pmatrix}\sqrt{\varrho^2 - |\hat z|^2} \\ 0\end{pmatrix}, \\ &\quad 1 \le j \le n-2\end{aligned} \tag{9.6}$$

where $\varrho^2 := 2(\sum_{1\le i\le n}\Lambda_i - G)$ and $\mathcal{R}(g)$ is the matrix (6.4). Thus, $\hat\phi$ *can be extended to a real-analytic map* $\hat\phi : \hat{\mathcal{M}}^{6n-2} \to \breve{\mathcal{M}}^{6n-2}$ *with*[56]:

$$\hat{\mathcal{M}}^{6n-2} := \Bigg\{ \Lambda \in \mathcal{A},\ G \in \mathbb{R}_+,\ |\hat z| < \varrho(\Lambda, G) \\ := \sqrt{2\Big(\sum_{1\le i\le n}\Lambda_i - G\Big)} < \epsilon_1,\ \hat\lambda \in \mathbb{T}^n,\ g \in \mathbb{T}\Bigg\}.$$

(iii) In particular, points on the hyperplanes $(\hat\eta_i, \hat\xi_i) = 0$ or $(\hat p_j, \hat q_j) = 0$, corresponding to co-circular or co-planar motions for $n - 1$ planets (and one possibly co-circular with the others but not co-planar) are regular points for $\hat\phi$.

(iv) Consider the Hamiltonian $\hat{\mathcal{H}} := \breve{\mathcal{H}} \circ \hat\phi$ on the phase space $\hat{\mathcal{M}}^{6n-2}$. As already mentioned, the invariance of $\breve{\mathcal{H}}$ by $\mathcal{R}^g$ implies that *the variable g is cyclic for* $\breve{\mathcal{H}} \circ \hat\phi$. We then set

$$\hat{\mathcal{H}}_{\rm G}(\Lambda, \hat\lambda, \hat z) := \breve{\mathcal{H}} \circ \hat\phi(\Lambda, \hat\lambda, \hat z; G) = h_{\rm Kep}(\Lambda) + \mu \hat f_{\rm G}(\Lambda, \hat\lambda, \hat z). \tag{9.7}$$

For fixed values of the parameter $G \in \mathbb{R}_+$, *the "totally reduced planetary Hamiltonian"* $\hat{\mathcal{H}}_{\rm G}$ *governs the motions of the planetary system on the* $(6n-4)$*-dimensional phase space*

$$\hat{\mathcal{M}}_{\rm G}^{6n-4} := \hat{\mathcal{A}}_{\rm G} \times \mathbb{T}^n \times B^{2(2n-2)}_{\varrho(\Lambda,G)}\,, \qquad \hat{\mathcal{A}}_{\rm G} := \{\Lambda \in \mathcal{A} :\ 0 < \varrho(\Lambda, G) < \epsilon_1\}, \tag{9.8}$$

endowed with the standard symplectic form $d\Lambda \wedge d\hat\lambda + d\hat\eta \wedge d\hat\xi + d\hat p \wedge d\hat q$.

(v) *The full motion on* $\hat{\mathcal{M}}^{6n-2}$ *is then simply obtained by integrating the cyclic variable* g, *which amounts to a rotation around the direction of C with instant speed given by* $\partial_G \hat{\mathcal{H}}_{\rm G}$.

10 Birkhoff normal form and full torsion of the totally reduced secular Hamiltonian

In this section, by means of the standard IFT, we cast the totally reduced secular Hamiltonian $\hat f_{\rm G,av}$ in Birkhoff normal form up to order four on $\hat{\mathcal{M}}_{\rm G}^{6n-4}$. In this context, at contrast with the partially reduced case, there are no secular resonances (see Remark 10.1 below); on the other hand, as in the partially reduced case, the torsion matrix is nonsingular.

[56] Recall (9.1) and notice that by (9.2) and (9.3) $\sum_{1\le i\le n}\Lambda_i - G = |\breve z|^2/2$. Notice also that $\hat\phi$ is well defined for any $\Lambda \in \mathbb{R}^n_+$.

10.1 Fourth order Birkhoff normal form of the totally reduced planetary Hamiltonian

We first consider the "ambient phase space" $\hat{\mathcal{M}}^{6n-2}$. Using the expression of $\hat\phi$ in (9.6) and the independence of $\hat f_{\rm G}$ on g, we have

$$\hat f_{\rm G} = \breve f \circ \hat\phi = (\breve f \circ \hat\phi)|_{g=0} = \breve f \circ \hat\phi_0 \tag{10.1}$$

where $\hat\phi_0 := \hat\phi_0^{(\varrho)} := \hat\phi|_{g=0}$ has the simple expression

$$\hat\phi_0^{(\varrho)}:\quad \breve\lambda = \hat\lambda, \quad \begin{cases} \breve\eta = \hat\eta \\ \breve\xi = \hat\xi \end{cases} \quad \begin{cases} \breve p_i = \hat p_i \\ \breve q_i = \hat q_i \end{cases} \quad \begin{cases} \breve p_{n-1} = \sqrt{\varrho^2 - |\hat z|^2} \\ \breve q_{n-1} = 0 \end{cases}$$
$$1 \le i \le n-2. \tag{10.2}$$

Taking the average of (10.1) with respect to $\hat\lambda$ and using (7.30) one readily finds the following expression

$$\hat f_{\rm G,av}(\Lambda, \hat z) = \breve f_{\rm av} \circ \hat\phi_0^{(\varrho)} = \hat C_0(\Lambda, \varrho) + \hat\Omega(\Lambda, \varrho) \cdot \hat R + \frac{1}{2}\hat\tau(\Lambda)\hat{\rm R}\cdot\hat{\rm R} + \hat{\mathcal{F}}(\Lambda, \hat z, \varrho) \tag{10.3}$$

where

$$\hat R := (\hat\rho, \hat r), \quad \hat\rho_i := \frac{\hat\eta_i^2 + \hat\xi_i^2}{2}, \quad \hat r_j := \frac{\hat p_j^2 + \hat q_j^2}{2} \quad 1 \le i \le n,\ 1 \le j \le n-2;$$

the constants $\hat C(\Lambda, \varrho)$ and $\hat\Omega = (\hat\Omega_1, \dots, \hat\Omega_{2n-2})$ are defined as:

$$\hat C_0(\Lambda, \varrho) := C_0(\Lambda) + \frac{\varrho^2}{2}\varsigma_{n-1}(\Lambda) + \frac{\varrho^4}{4}\tau_{mm}(\Lambda), \quad m := 2n-1 \tag{10.4}$$

$$\hat\Omega_i(\Lambda, \varrho) = \Omega_i^0(\Lambda) + \varrho^2(\tau_{im}(\Lambda) - \tau_{mm}(\Lambda)), \quad 1 \le i \le 2n-2 \tag{10.5}$$

with $\Omega^0 := (\Omega_1^0, \dots, \Omega_{2n-2}^0)$ given by

$$\Omega_i^0(\Lambda) := \begin{cases} \sigma_i^0(\Lambda) := \sigma_i(\Lambda) - \varsigma_{n-1}(\Lambda), & 1 \le i \le n \\ \varsigma_{i-n}^0(\Lambda) := \varsigma_{i-n}(\Lambda) - \varsigma_{n-1}(\Lambda), & n+1 \le i \le 2n-2; \end{cases} \tag{10.6}$$

and the matrix $\hat\tau = (\hat\tau_{ij})$ by

$$\hat\tau_{ij}(\Lambda) := \tau_{ij}(\Lambda) - \tau_{im}(\Lambda) - \tau_{jm}(\Lambda) + \tau_{mm}(\Lambda), \quad 1 \le i,\ j \le 2n-2; \tag{10.7}$$

finally,

$$\hat{\mathcal{F}}(\Lambda, \hat z, \varrho) := \breve{\mathcal{P}} \circ \hat\phi_0^{(\varrho)}(\Lambda, \hat z). \tag{10.8}$$

In general, for $n \geq 3$, the function $\hat{\mathcal{F}}(\Lambda, \hat{z}, \varrho)$ has non vanishing derivatives in $\hat{z} = 0$. But the existence of a suitable elliptic equilibrium for $\hat{f}_{\rm G}(\Lambda, \hat{z})$ can be established by a simple IFT argument and the relative normal form around such equilibrium next restored by classical symplectic diagonalization and Birkhoff normal form.

Remark 10.1 The functions (10.6) do not exhibit any resonance of order 4 or lower in the range of well separated semimajor axes. In fact, the existence of a non-trivial linear relation among the Ω_i^0's, would imply a linear relation among $\sigma_1, \ldots, \sigma_n, \varsigma_1, \ldots, \varsigma_{n-1}$ different from Herman's resonance (since the coefficient of ς_{n-1} should be opposite to the sum of the coefficients of $\sigma_1, \ldots, \sigma_n, \varsigma_1, \ldots, \varsigma_{n-2}$), which is in contradiction with Proposition 7.2.

Let

$$
\begin{aligned}
& I := (\Lambda, G), \\
& \mathcal{B}_* = \Big\{ I = (\Lambda, G) \in \mathcal{A} \times \mathbb{R}_+ \colon \sum_{1 \leq i \leq n} \Lambda_i - G > 0 \Big\} \subset \mathbb{R}_+^n \times \mathbb{R}_+ \\
& \hat{\varphi} := (\hat{\lambda}, \hat{g}) \in \mathbb{T}^{n+1}, \qquad \hat{u} := (\hat{\eta}, \hat{p}), \qquad \hat{v} := (\hat{\xi}, \hat{q}), \qquad \hat{z} := (\hat{u}, \hat{v}).
\end{aligned}
$$

Proposition 10.1 *There exists $\epsilon_2 \in (0, \epsilon_1)$ such that, on the domain*

$$
\check{\mathcal{M}}^{6n-2} := \Big\{ I \in \mathcal{B}_*, \ \check{\varphi} \in \mathbb{T}^{n+1}, \ 0 < \varrho(I) < \epsilon_2, \ |\check{z}| < \frac{\varrho(I)}{4} \Big\}, \tag{10.9}
$$

one can find a real-analytic and symplectic transformation with respect to $dI \wedge d\varphi + d\hat{u} \wedge d\hat{v}$,

$$
\check{\phi} \colon \quad (I, \check{\varphi}, \check{z}) \in \check{\mathcal{M}}^{6n-2} \to (I, \hat{\varphi}, \hat{z}) \in \hat{\mathcal{M}}^{6n-2},
$$

with the following properties. The map $\check{\phi}$ leaves the I-variables unvaried, maps $\check{\varphi} \to \varphi = \check{\varphi} + \check{\varphi}_1(I, \check{z})$, and is $\check{\varphi}$-independent on the remaining variables; furthermore, it puts $\hat{\mathcal{H}}_{\rm G}$ into the form

$$
\check{\mathcal{H}}_{\rm G} = \check{\mathcal{H}}_{\rm G} \circ \check{\phi} = h_{\rm Kep}(\Lambda) + \mu \check{f}_{\rm G}(\Lambda, \check{\lambda}, \check{z}) \tag{10.10}
$$

with:

$$
\begin{aligned}
& \check{f}_{{\rm G},\mathrm{av}}(\Lambda, \check{z}) = \check{C}_{\rm G}(\Lambda) + \check{\Omega}_{\rm G}(\Lambda) \cdot \check{\mathrm{R}} + \frac{1}{2} \check{\mathrm{R}} \cdot \check{\tau}_{\rm G}(\Lambda) \check{\mathrm{R}} + \check{\mathcal{P}}_{\rm G}(\Lambda, \check{z}), \\
& \check{z} = (\check{u}, \check{v}) = \big((\check{u}_1, \ldots, \check{u}_{2n-2}), (\check{v}_1, \ldots, \check{v}_{2n-2}) \big), \\
& \check{\mathrm{R}} = (\check{\mathrm{R}}_1, \ldots, \check{\mathrm{R}}_{2n-2}), \qquad \check{\mathrm{R}}_i = \frac{\check{u}_i^2 + \check{v}_i^2}{2};
\end{aligned}
\tag{10.11}
$$

with $\check{\mathcal{P}}_{\rm G}(I, \check{z})$ having a zero of order 5 in $\check{z} = 0$.

The first and second Birkhoff invariants $\check{\Omega}_{\rm G}$, $\check{\tau}_{\rm G}$ *verify*

$$|\check{\Omega}_{\rm G}(\Lambda) - \Omega^0(\Lambda)| \le C\varrho(I)^2, \qquad \|\check{\tau}_{\rm G}(\Lambda) - \hat{\tau}(\Lambda)\| \le C\varrho(I)^2, \tag{10.12}$$

where Ω^0 *and* $\hat{\tau}$ *are as in* (10.6) *and* (10.7). *Finally,* $\check{\Omega}_{\rm G}$ *is non-resonant up to order four.*[57]

Remark 10.2 (i) Clearly, since the Hamiltonian $\hat{\mathcal{H}}_{\rm G}$ does not depend on g, it projects to a Hamiltonian (which we still call $\hat{\mathcal{H}}_{\rm G}$) on the totally reduced phase space

$$\left\{\Lambda \in \mathcal{A},\ \check{\lambda} \in \mathbb{T}^n,\ |\check{z}| < \frac{\varrho(I)}{4} < \frac{\epsilon_2}{4}\right\}, \tag{10.13}$$

endowed with the standard symplectic form $d\Lambda \wedge d\check{\lambda} + d\check{\eta} \wedge d\check{\xi} + d\check{p} \wedge d\check{q}$ ($\check{p}, \check{q} \in \mathbb{R}^{n-2}$ and G is a parameter). Notice that, now, the secular center $\check{z} = 0$ is an elliptic equilibrium for the totally reduced Hamiltonian $\hat{\mathcal{H}}_{\rm G}$.

(ii) For the applications in Sect. 11 below, we will consider, for simplicity, a subset of (10.13) of the form

$$\check{\mathcal{M}}^{6n-4}_{\rm G} := \left\{\Lambda \in\ \check{\mathcal{A}}_{\rm G},\ \check{\lambda} \in \mathbb{T}^n,\ |\check{z}| < \epsilon_3\right\},$$
$$\text{with } \check{\mathcal{A}}_{\rm G} := \{\Lambda \in \mathcal{A} :\ 4\epsilon_3 < \varrho(\Lambda, G) < \epsilon_2\}, \tag{10.14}$$

where ϵ_3 is an arbitrary positive number smaller than $\epsilon_2/4$.

Proof of Proposition 10.1 We divide the proof of into four steps.

Step 1: The new elliptic equilibrium

For $n = 2$, the point $\hat{z}_{\rm e} := 0$ in an equilibrium for $\hat{f}_{\rm G,av}$ for any $I \in \mathcal{B}_*$. When $n \ge 3$, to find an equilibrium $I \to \hat{z}_{\rm e}(I)$, we use the IFT (Proposition 10.2 below).

Fix a number $0 < \theta < 1$. For

$$\check{z} = (\check{u}, \check{v}) = \big((\check{u}_1, \ldots, \check{u}_{2n-2}), (\check{v}_1, \ldots, \check{v}_{2n-2}\big) \in \overline{B}_\theta^{2(2n-2)}, \tag{10.15}$$

let $\check{\phi}^{(\varrho)}$ the map

$$\check{\phi}^{(\varrho)}\colon\quad (\Lambda, \hat{\lambda}, \check{z}) \to \hat{\phi}_0^{(\varrho)}(\Lambda, \hat{\lambda}, \varrho\check{z}) := (\Lambda, \hat{\lambda}, \check{\phi}_{\check{z}}^{(\varrho)}) = (\Lambda, \hat{\lambda}, \varrho\check{\phi}_{\check{z}}^{(1)}),$$

where $\hat{\phi}_0^{(\varrho)}$ is the map (10.2) and hence the $\check{z}$-component of the map $\check{\phi}^{(1)}$ becomes:

$$\check{\phi}_{\check{z}}^{(1)}\colon\quad \check{z} = (\check{u}, \check{v}) \to \check{z} = (\check{u}, \check{v}) : \check{z}_i = \check{z}_i, \quad i \le 2n-2$$

[57] And, in fact, by possibly reducing ϵ_2, at any finite order.

$$\check{u}_{2n-1} = \sqrt{1 - |\check{z}|^2}\,, \quad \check{v}_{2n-1} = 0. \tag{10.16}$$

Let us also put

$$f^{(\varrho)} := \hat{f}_{\text{G,av}}|_{\hat{z}=\varrho\check{z}} = \hat{C}_0(\Lambda, \varrho) + \varrho^2 \Big(\hat{\Omega}(\Lambda, \varrho) \cdot \check{\mathrm{R}} + \frac{\varrho^2}{2} \check{\mathrm{R}} \cdot \hat{\tau}(\Lambda) \check{\mathrm{R}} + \varrho^4 \mathcal{Q}(\Lambda, \check{z}, \varrho) \Big) \quad \text{where } \check{\mathrm{R}}_i := \frac{\check{u}_i^2 + \check{v}_i^2}{2}$$

($1 \le i \le 2n-2$) and, by (10.8) and (10.16), the function $\mathcal{Q}(\Lambda, \check{z}, \alpha)$ is defined by

$$\begin{aligned} \alpha^6 \mathcal{Q}(\Lambda, \check{z}, \alpha) &= \hat{\mathcal{F}}(\Lambda, \alpha\check{z}, \alpha) = \check{\mathcal{P}} \circ \hat{\phi}_0^{(\alpha)}(\Lambda, \alpha\check{z}) \\ &= \check{\mathcal{P}}(\Lambda, \check{\phi}_{\check{z}}^{(\alpha)}(\check{z})) = \check{\mathcal{P}}(\Lambda, \alpha\check{\phi}_{\check{z}}^{(1)}(\check{z})). \end{aligned} \tag{10.17}$$

Since the function $\check{z} \to \check{\mathcal{P}}(\Lambda, \check{z})$ is regular on the closed ball $\overline{B}_{\epsilon_1/2}^{2(2n-1)} \subset B_{\epsilon_1}^{2(2n-1)}$, has a zero of order 6 in 0, $\check{\phi}_{\check{z}}^{(1)}$ is regular on $\overline{B}_{\theta}^{2(2n-2)}$ and $|\check{\phi}_{\check{z}}^{(1)}(\check{z})| = 1$ for any $\check{z} \in \overline{B}_{\theta}^{2(2n-2)}$, by (10.17), for any fixed Λ, the function $(\check{z}, \alpha) \to \mathcal{Q}(\Lambda, \check{z}, \alpha)$ is a regular function of $\check{z}, \alpha$ on the domain

$$|\check{z}| \in \overline{B}_{\theta}^{2(2n-2)} \quad \text{and} \quad 0 \le \alpha \le \frac{\epsilon_1}{2}.$$

We now need a quantitative formulation of the standard IFT[58]:

Proposition 10.2 (Quantitative IFT) *Let $w_0 \in \mathbb{R}^h$, and let A be a compact set of $\mathbb{R}^k$. Let $F:\ (w, \alpha) \in \overline{B}_R(w_0) \times A \to F(w, \alpha) \in \mathbb{R}$ ($\overline{B}_R(w_0)$ denoting the closed ball of radius R and center w_0) be a continuous function with invertible and* C^1 *Jacobian matrix $\partial_w F(w_0, \alpha)$, for any $\alpha \in A$. Denote by $M(\alpha) := (\partial_w F(w_0, \alpha))^{-1}$, by* m *an upper bound on* $\sup_A \|M\|$ ($\|\cdot\|$ *denoting the standard "operator norm" on matrices*). *If*

$$4\mathrm{m}^2 \sup_{\mathrm{A}} |F(w_0, \alpha)| \sup_{\overline{B}_R(w_0) \times A} \|\partial_w^2 F\| \le 1 \tag{10.18}$$

then, there exists a unique continuous function $\alpha \in A \to w(\alpha) \in \overline{B}_\rho(w_0)$ such that $F(w(\alpha), \alpha) \equiv 0$ for any $\alpha \in A$, where $\rho := 2\mathrm{m} \sup_{\mathrm{A}} |F(w_0, \alpha)|$.

[58] The elementary proof can be found in [5, Theorem 1].

We apply Proposition 10.2, with[59] $h = 2n-2$, $k=1$, $A=\{\alpha\}$, $w_0=0$, $R=\theta$ and

$$\begin{aligned} F(\check z,\alpha) &:= \partial_{\check z}(f^{(\alpha)} - \hat C_0(\Lambda,\alpha))\alpha^{-2} \\ &= \partial_{\check z}\left(\hat\Omega(\Lambda,\alpha)\cdot\check{\mathrm R} + \frac{\alpha^2}{2}\check{\mathrm R}\cdot\hat\tau(\Lambda)\check{\mathrm R} + \alpha^4\mathcal Q(\Lambda,\check z,\alpha)\right), \end{aligned}$$

having omitted the dependence on Λ in the notation for F. We have in fact

$$\begin{cases} F(0,\alpha) = \alpha^4\partial_{\check z}\mathcal Q(\Lambda,0,\alpha) \\ \partial_{\check z}F(0,\alpha) = \operatorname{diag}[\hat\Omega,\hat\Omega] + \alpha^4\partial^2_{\check z}\mathcal Q(\Lambda,0,\alpha) \\ \partial^2_{\check z}F(\check z,\alpha) = \alpha^2\partial^3_{\check z}(\frac12\check{\mathrm R}\cdot\hat\tau\check{\mathrm R} + \alpha^2\mathcal Q(\Lambda,\check z,\alpha)) \end{cases}$$

easily implying the existence of three constants[60] c_1, c_2, c_3 (independent on Λ) for which the following bounds hold

$$\begin{cases} |F(0,\alpha)| \le c_1\alpha^4 \\ \|(\partial_{\check z}F(0,\alpha))^{-1}\| \le 2\sup_{\mathcal A}\|(\operatorname{diag}[\hat\Omega,\hat\Omega])^{-1}\| =: \mathrm m \\ \quad \text{when } c_2\alpha^4 \le 1 \\ \sup_{\check z\in\overline B^{2(2n-2)}_\theta}\|\partial^2_{\check z}F(\check z,\alpha)\| \le c_3\alpha^2. \end{cases} \tag{10.19}$$

Notice that the constant m is well defined by the non resonance assumption of the Ω's. Let

$$c_4 := 2\mathrm m c_1, \qquad c_5 := \frac{2c_4}{\theta}, \qquad c_6 := 4\mathrm m^2 c_1 c_3$$

and let $\bar\varrho$ such that the inequalities

$$\max\left\{c_6\alpha^6,\ c_5\alpha^5\right\} \le 1 \tag{10.20}$$

hold for $\alpha \le \bar\varrho$. The assumption (10.18) of Proposition 10.2 is easily met, since, in view of (10.19), (10.20)

$$4\mathrm m^2|F(0,\alpha)|\sup_{\overline B^{2(2n-2)}_\theta}\|\partial^2_{\check z}F\| \le c_6\alpha^6 \le 1.$$

[59]In particular, Proposition 10.2 holds also when the compact A is a set of one point only.

[60]The c_i's can be taken

$$\begin{cases} c_1 := \sup_{\Lambda\in\mathcal A, 0\le\alpha\le\epsilon_1/2}|\partial_{\check z}\mathcal Q(\Lambda,0,\alpha)| \\ c_2 := 2\sup_{\Lambda\in\mathcal A, 0\le\alpha\le\epsilon_1/2}\|(\operatorname{diag}[\hat\Omega,\hat\Omega])^{-1}\partial^2_{\check z}\mathcal Q(\Lambda,0,\alpha)\| \\ c_3 := \sup_{\Lambda\in\mathcal A,\check z\in\overline B^{2n-2}_\theta, 0\le\alpha\le\epsilon_1/2}\|\partial^3_{\check z}(\frac12\mathrm R\cdot\hat\tau\mathrm R + \alpha^2\mathcal Q(\Lambda,\check z,\alpha))\| \end{cases}$$

Then, by Proposition 10.2, a root $\check{z}_{\rm e} = \check{z}_{\rm e}(\Lambda, \alpha)$ of equation $F(\check{z}, \alpha) = 0$ can be found, satisfying

$$|\check{z}_{\rm e}| \leq 2{\rm m}|F(0, \alpha)| \leq c_4\alpha^4 \tag{10.21}$$

for any $\alpha \leq \bar{\varrho}$. Thus, the point $\hat{z}_{\rm e} = \hat{z}_{\rm e}(\Lambda, G) \in \overline{B}^{2(2n-2)}_{c_4\varrho^5} \subset \overline{B}^{2(2n-2)}_{\theta\varrho/2}$ defined as $\hat{z}_{\rm e} := \varrho\check{z}_{\rm e}(\Lambda, \varrho)$ is an equilibrium for $f_{\rm G,av}$ for any (Λ, G), with $\Lambda \in \mathcal{A}$ and G such that $\varrho(\Lambda, G) \leq \epsilon_2$, where $\epsilon_2 \leq \bar{\varrho}$.

Step 2: Symplectic shift of the equilibrium into the origin

The equilibrium point $\hat{z} = \hat{z}_{\rm e}$ can be then shifted into the origin with the change of variables

$$\hat{z} = z^\star + \hat{z}_{\rm e}, \quad \text{i.e.,} \quad \hat{u} = u^\star + \hat{u}_{\rm e}(I), \quad \hat{v} = v^\star + \hat{u}_{\rm e}(I), \tag{10.22}$$

where $z^\star := (u^\star, v^\star)$ is taken varying into the closed ball $\overline{B}^{2(2n-2)}_{\theta\varrho/2}$, so to have

$$|\hat{z}| = |z^\star + \hat{z}_{\rm e}| \leq |z^\star| + |\hat{z}_{\rm e}| \leq \frac{\varrho\theta}{2} + \frac{\varrho\theta}{2} = \varrho\theta < \varrho. \tag{10.23}$$

By construction, the function $f^\star(I, z^\star) := \hat{f}_{\rm G,av}|_{\hat{z}=z^\star+\hat{z}_{\rm e}}$ has vanishing linear part[61] in $z^\star = 0$. Using this fact and (10.3), we can write

$$f^\star(I, z^\star) := \hat{f}_{\rm G,av}|_{\hat{z}=z^\star+\hat{z}_{\rm e}} = C_0^\star(I) + \Omega^\star \cdot {\rm R}^\star + \frac{1}{2}{\rm R}^\star \cdot \hat{\tau}{\rm R}^\star + \mathcal{Q}^\star(I, z^\star), \tag{10.24}$$

where[62]

$$\begin{cases} C_0^\star(I) := \hat{C}_0(I) + \hat{\Omega} \cdot \hat{\rm R}_{\rm e} + \frac{1}{2}\hat{\rm R}_{\rm e} \cdot \hat{\tau}\hat{\rm R}_{\rm e}, \quad \text{where } \hat{\rm R}_{\rm e}^{(i)} := \frac{(\hat{u}_{\rm e}^{(i)})^2 + (\hat{v}_{\rm e}^{(i)})^2}{2} \\ {\rm R}_i^\star := \frac{(u_i^\star)^2 + (v_i^\star)^2}{2} \\ \Omega^\star := \hat{\Omega} + \hat{\tau}\hat{\rm R}_{\rm e} \\ \mathcal{Q}^\star(I, z^\star) := \mathcal{P}^\star(I, z^\star) - \mathcal{L}_{z^\star}(\mathcal{P}^\star(I, z^\star)) \end{cases} \tag{10.25}$$

where

$$\mathcal{P}^\star(I, z^\star) := \hat{\mathcal{F}}(\Lambda, z^\star + \hat{z}_{\rm e}, \varrho) = \breve{\mathcal{P}} \circ \hat{\phi}_0^{(\varrho)}(\Lambda, z^\star + \hat{z}_{\rm e}) \tag{10.26}$$

and $\mathcal{L}_{z^\star}(\mathcal{P}^\star(I, z^\star))$ denotes the linear part of $\mathcal{P}^\star(I, z^\star)$ with respect to $z^\star$ in $z^\star = 0$.

[61] If $z \to g(a, z)$ is analytic on $z = 0$, its linear part in $z = 0$ is $\partial_z g(a, 0) \cdot z$.

[62] If $v \in \mathbb{R}^{2n-2}$, $v^{(i)}$ or v_i denotes its $i^{\rm th}$ component.

The shift (10.22) is next lifted to a symplectic transformation $\phi^\star$ which, leaving the $I's$ unvaried, acts on their respective conjugated angles as[63]

$$\hat\varphi = \varphi^\star + u^\star \cdot \partial_I \hat v_{\rm e}(I) - (v^\star + \hat u_{\rm e}(I))\partial_I \hat u_{\rm e}(I).$$

Step 3: Symplectic diagonalization around the equilibrium

Let $\mathrm{Q}^\star$, $\mathrm{C}^\star$, $\mathrm{F}^\star$ denote, respectively, the coefficients of the expansion

$$\begin{aligned}\mathcal{Q}^\star(I, z^\star) &= \mathcal{P}^\star(I, z^\star) - \mathcal{L}_{z^\star}(\mathcal{P}^\star(I, z^\star))\\ &= \mathcal{Q}_0^\star(I) + z^\star \cdot \mathrm{Q}^\star(I) z^\star + \mathrm{C}^\star(I)\cdot {z^\star}^3 + \frac{1}{2}\mathrm{F}^\star(I)\cdot {z^\star}^4 + \mathcal{Q}_5^\star(I, z^\star),\end{aligned} \tag{10.27}$$

with $\mathcal{Q}_5^\star(I, z^\star) \le C|z^\star|^5$. Using (10.23) and (10.26), one easily finds suitable constants c_7–c_9 such that

$$\sup_{\mathcal{A}} \|\mathrm{Q}^\star(I)\| \le c_7 \varrho^4, \qquad \sup_{\mathcal{A}} \|\mathrm{C}^\star(I)\| \le c_8 \varrho^3, \qquad \sup_{\mathcal{A}} \|\mathrm{F}^\star(I)\| \le c_9 \varrho^2. \tag{10.28}$$

In particular, when ϱ is sufficiently small, the quadratic form

$$\mathrm{Q}^* := \mathrm{diag}\,[\hat\Omega, \hat\Omega] + \mathrm{Q}^\star \tag{10.29}$$

associated to $f^\star(I, z^\star)$ has purely imaginary eigenvalues (hence, the equilibrium $z^\star = 0$ is elliptic). Let them be denoted as $\check\Omega = (\check\Omega_1, \dots, \check\Omega_{m-1})$; notice that in Proposition 10.1 $\check\Omega$ is denoted $\check\Omega_{\rm G}$. Then, $\check\Omega$ satisfy

$$|\check\Omega - \hat\Omega| \le c_{10}\varrho^4. \tag{10.30}$$

Let $\check\Omega \cdot \mathrm{R}^*$, where $\mathrm{R}_i^* = \frac{(u_i^*)^2 + (v_i^*)^2}{2}$, the diagonal form of the quadratic form Q^* as in (10.29) and let $z^* \to \mathcal{L}(I)z^*$, where $z^* = (u^*, v^*)$, the symplectic transformation[64] which transforms Q^* in such diagonal form. By (10.28), $\mathcal{L}$ is ϱ^4-close to the identity and hence we can assume that is well defined on the domain $B^{2(2n-2)}_{\varrho/3}$. Being linear and symplectic, $\mathcal{L}$ can be lifted to a symplectic transformation

$$\phi^* :\ (\Lambda, \lambda^*, z^*) \to (\Lambda, \lambda^\star, z^\star)$$

on the domain

$$\mathcal{D}^*:\quad I \in \mathcal{B}_*, \qquad \varphi \in \mathbb{T}^{n+1}, \qquad 0 < \varrho(I) < \varrho^*, \qquad |z^*| < \frac{\varrho(I)}{3},$$

[63] The generating function of this transformation is $S(I', u', \varphi, v) = I' \cdot \varphi + u' \cdot (v - v_{\rm e}(I') + v \cdot u_{\rm e}(I')$.

[64] With respect to $du^* \wedge dv^*$.

such in a way to leave the I's unvaried and shifting the angles φ^* of a φ^*-independent quantity and is φ^*-independent on the remaining variables. Let

$$f^*_{\rm av} := f^\star_{\rm av} \circ \phi^* = \check{C}_{\rm G} + \check{\Omega} \cdot {\rm R}^* + {\rm C}^* \cdot (z^*)^3 + \frac{1}{2}{\rm R}^* \cdot \hat{\tau} {\rm R}^* + {\rm F}^* \cdot (z^*)^4 + {\rm O}(|z^*|^5),$$

where ${\rm C}^*$ and ${\rm F}^*$ are ϱ^4-close to ${\rm C}^\star$ and ${\rm F}^\star$, respectively.

Step 4: Construction of the Birkhoff normal form

By (10.30) and (10.4), the first invariants $\check{\Omega}$ of $f^*_{\rm av}$ are ϱ^2-close to the functions Ω^0_i into (10.6) which, as already observed, do not satisfy any resonance of order 4 or lower in $\mathcal{A}$. Then, when ϱ is sufficiently small, Birkhoff theory can be applied, proving the thesis of Proposition 10.1.

The final claim follows at once by the non-resonance of the Ω^0_i's (Remark 10.1) and by taking ϵ_2 small enough (compare (10.12)). □

10.2 Full torsion of the totally reduced planetary system

We are now ready to check full torsion in the fully reduced setting.

Corollary 10.1 *Fix $n \geq 2$ and $0 < \delta_\star < 1$. Then, there exist $\bar{\mu} > 0$, $0 < \underline{a}_1 < \overline{a}_1 < \cdots < \underline{a}_n < \overline{a}_n$ such that for any $\mu < \bar{\mu}$ and for any $\Lambda \in \check{\mathcal{A}}_{\rm G}$, where $\check{\mathcal{A}}_{\rm G}$ is the set in (10.14), the matrix $\check{\tau}_{\rm G}$ is non-singular*: $\det \check{\tau}_{\rm G} = \hat{d}_n(1 + \delta_n)$, *with $|\delta_n| < \delta_\star$ and*

$$\hat{d}_n = \frac{m_0 m_n}{m_{n-1}} \frac{a_n^4}{a_{n-1}^2} \tilde{d}_n \tag{10.31}$$

where $\tilde{n}$ is defined as $\frac{17}{3} d_n$, for $n = 2$, and as $\frac{4}{15} d_n$ for $n > 2$, d_n being defined in (8.8).

Proof By the second inequality in (10.12) (eventually, taking ϵ_2 smaller), it suffices to prove that the matrix $\hat{\tau}$ defined (in terms of the unreduced matrix τ) in (10.7) is non-singular. We distinguish two cases.

Case $n = 2$ By the computation of the τ_{ij}'s performed in Sect. 7.3 (compare the exact expression in (8.1), or, better, the asymptotics for small $1/a_2$ in (8.6)), when $1/a_2$ is small enough, one has

$$\hat{\tau} = m_1 m_2 \frac{a_1^2}{a_2^3} \begin{pmatrix} -\frac{6}{\Lambda_1^2} & -\frac{15}{4\Lambda_1^2} \\ -\frac{15}{4\Lambda_1^2} & -\frac{3}{4\Lambda_1^2} \end{pmatrix} (1 + o(1)),$$

easily proving non-singularity:

$$\det \hat{\tau} = -\left(m_1 m_2 \frac{a_1^2}{a_2^3}\right)^2 \frac{153}{16\Lambda_1^4}(1 + o(1)) \neq 0.$$

Case $n \geq 3$ Since $\tau_{i,2n-1} = \tau_{2n-1,i} = \mathrm{O}(\delta)$ for any $1 \leq i \leq 2n-1$, one easily finds that

$$\hat{\tau} = \begin{pmatrix} \check{\tau} + \mathrm{O}(\delta) & \mathrm{O}(\delta) \\ \mathrm{O}(\delta) & \hat{\tau}_{2n-2,2n-2} + \mathrm{O}(\delta^2) \end{pmatrix} \tag{10.32}$$

where $\delta = 1/a_n^3$, $\check{\tau}$ denotes the unreduced matrix relatively to $n-1$ bodies (of order 1 in δ) and

$$\hat{\tau}_{2n-2,2n-2} = \bar{\tau}_{11} - 2\bar{\tau}_{12} + \bar{\tau}_{22} = -\frac{3}{4}\frac{m_n}{\mathfrak{L}_{n-1}^2}\delta \sum_{1 \leq j < n} m_j a_j^2 (1 + o(1)) \tag{10.33}$$

(compare (8.15) and (8.16), (10.7)). Then, by (10.32), (10.33) and the non singularity of $\check{\tau}$ (compare Proposition 8.1, with $n-1$ replacing n), we have

$$\det \hat{\tau} = -\frac{3}{4}\frac{m_n}{\mathfrak{L}_{n-1}^2} \sum_{1 \leq j < n} m_j a_j^2 (\det \check{\tau}) \cdot \delta \cdot (1 + o(1)) \neq 0$$

and (10.31) immediately follows. □

11 Quasi-periodic motions in the planetary problem

In this section we shall see some consequences of the non-degeneracy of the Birkhoff normal forms of the planetary system, providing, in particular:

(i) a new direct proof of Arnold's planetary theorem [2, 15] both in the partially and in totally reduced settings, including explicit measure estimates on the Kolmogorov's set (i.e., the union of Lagrangian KAM tori);
(ii) existence of Cantor families of n-lower dimensional elliptic tori in the partially and totally reduced phase space.

11.1 Lagrangian Diophantine tori in the planetary system

The existence of Lagrangian KAM tori and estimates on the measure of the Kolmogorov's set for the partially reduced and totally reduced planetary models will follow immediately from the following theorem, which is an improvement of Arnold's "Fundamental Theorem" in [2, Sect. 4].

Theorem 11.1 *Let* $\mathcal{P}_\epsilon := V \times \mathbb{T}^{n_1} \times B_\epsilon^{2n_2}$, *where* V *is an open, bounded, connected set of* $\mathbb{R}^{n_1}$ *and* $B_\epsilon^{2n_2}$ *is a* $2n_2$*-dimensional ball of radius* ϵ *centered at the origin. Let* $\epsilon_0 > 0$ *and let* $H(I, \varphi, p, q; \mu) = H_0(I) + \mu P(I, \varphi, p, q; \mu)$ *be a real-analytic Hamiltonian on* $\mathcal{P}_{\epsilon_0}$, *endowed with the standard symplectic form* $dI \wedge d\varphi + dp \wedge dq$. *Assume that* H *verify the following non-degeneracy assumptions*:

(A$_1$) *$I \in V \to \partial_I H_0$ is a diffeomorphism*;
(A$_2$) *$P_{\rm av}(p,q;I) = P_0(I) + \sum_{i=1}^{n_2} \Omega_i(I) r_i + \frac{1}{2}\sum_{i,j=1}^{n_2} \beta_{ij}(I) r_i r_j + o_4$ with $r_i := \frac{p_i^2+q_i^2}{2}$ and $\lim_{(p,q)\to 0} \frac{o_4}{|(p,q)|^4} = 0$*;
(A$_3$) *$|\det \beta(I)| \geq \mathrm{const} > 0$ for all $I \in V$.*

Then, there exist positive numbers $\epsilon_* < \epsilon_0$, C_* and c_* such that, for

$$0 < \epsilon < \epsilon_*, \qquad 0 < \mu < \frac{\epsilon^6}{(\log \epsilon^{-1})^{c_*}}, \tag{11.1}$$

one can find a set $\mathcal{K} \subset \mathcal{P}_\epsilon$ formed by the union of H-invariant n-dimensional Lagrangian tori, on which the H-motion is analytically conjugated to linear Diophantine quasi-periodic motions with frequencies $(\omega_1,\omega_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with $\omega_1 = O(1)$ and $\omega_2 = O(\mu)$. The set $\mathcal{K}$ has positive Liouville-Lebesgue measure and satisfies

$$\mathrm{meas}\,\mathcal{P}_\epsilon > \mathrm{meas}\,\mathcal{K} > (1 - C_*\sqrt{\epsilon})\,\mathrm{meas}\,\mathcal{P}_\epsilon. \tag{11.2}$$

This is Theorem 1.3 in [10], to which we refer for the proof.[65]

- Consider the partially reduced planetary $(1+n)$-body system in normalized Birkhoff coordinates, i.e., consider the Hamiltonian system $(\breve{\mathcal{H}}, \breve{\mathcal{M}}_\epsilon^{6n-2})$ where $\breve{\mathcal{H}}$ is as in (7.29) and $\breve{\mathcal{M}}_\epsilon^{6n-2}$ is defined as in (7.28) with $B_{\epsilon_1}^{2(2n-1)}$ replaced by $B_\epsilon^{2(2n-1)}$ with $\epsilon \leq \epsilon_1$. Consider also the "well separated regime", i.e., let $\mu < \bar{\mu}$ and let the semimajor axes a_i be as in (7.2) with $\underline{a}_j$ and $\overline{a}_j$ as in Proposition 8.1, so that full torsion holds ($\det \tau \neq 0$).

Then, $\breve{\mathcal{M}}_\epsilon^{6n-2}$ has exactly the form of $\mathcal{P}_\epsilon$ with $V = \mathcal{A}$, $n_1 = n$, $n_2 = 2n-1$. Furthermore, by the form of the Keplerian part, by (7.30) and by Proposition 8.1, one sees immediately that the Hamiltonian $\breve{\mathcal{H}}$ satisfies assumptions (A$_1$)–(A$_3$) of Theorem 11.1. Thus, the following result follows at once.

Theorem 11.2 *If $\mu < \bar{\mu}$ and $\epsilon < \epsilon_1$ verify condition (11.1), then each symplectic leaf $\mathcal{M}^{6n-2}_{p_n^*,q_n^*}$ (5.1) contains a positive measure $\mathcal{H}$-invariant Kolmogorov set $\mathcal{K}_{p_n^*,q_n^*}$, which is actually the suspension of the same Kolmogorov's set $\mathcal{K}$, which in normalized Birkhoff symplectic variables $(\Lambda, \breve{\lambda}, \breve{z})$ is $\breve{\mathcal{H}}$-invariant. Furthermore, $\mathcal{K}$ is formed by the union of $(3n-1)$-dimensional Lagrangian, real-analytic tori on which the $\breve{\mathcal{H}}$-motion is analytically conjugated to linear Diophantine quasi-periodic motions with frequencies $(\omega_1,\omega_2) \in \mathbb{R}^n \times \mathbb{R}^{2n-1}$ with $\omega_1 = O(1)$ and $\omega_2 = O(\mu)$. Finally,*

[65] Actually, Theorem 11.1 is a corollary of a more general result holding under much milder conditions than (11.1); compare Theorem 1.2 in [10].

$\mathcal{K}$ satisfies the bound in (11.2) with $\mathcal{P}_\epsilon$ replaced by $\check{\mathcal{M}}^{6n-2}_\epsilon$; in particular $\operatorname{meas}\mathcal{K} \simeq \epsilon^{2(2n-1)}$.

Remark 11.1 (i) In fact, from Remark 7.3-(iii) it follows (giving up the constructive approach) that the same result holds in an open dense set of $\mathcal{M}^{6n}$.

(ii) Since the "secular variables" $\check{z}$ vary in a ball around the origin, which correspond to co-circular and co-planar motions, we recover and strengthen Arnold's results in [2]; compare [2, p. 125 and p. 142]. The approach followed here extends to the general situation Arnold's proof, which was given only for the planar $n = 2$ case.

(iii) The proof in [15] of Arnold's result is rather different from the one presented here. Indeed, the proof in [15] is based on the following: (a) one works in the unreduced phase space endowed with Poincaré spatial symplectic variables; (b) the KAM non-degeneracy condition used in [15] involves only the first order Birkhoff invariants,[66] which requires the frequency-map (formed by $\partial_\Lambda h_{\mathrm{Kep}}$ and by the first order Birkhoff invariants of the secular Hamiltonian) to be non-planar (i.e., not to lie in any hyperplane); (c) in the unreduced phase space, however, two secular resonances are present (Herman resonance and the vanishing of one of the spatial eigenvalues, ς_n); (d) to overcome the problem of the secular resonances, Féjoz, following an idea of Herman (based, in turn, on a Poincaré trick), modifies the system by adding a term proportional to C_3^2, since, by an abstract Lagrangian intersection theory argument, two commuting Hamiltonians have the same transitive tori; (e) to obtain $(3n-1)$-dimensional tori one restricts to the vertical symplectic submanifold[67] $\{C_1 = 0 = C_2\}$.

As side remarks, we point out also that: in the unreduced setting the full torsion is false (i.e., one can show that the determinant of the second order Birkhoff invariants vanishes identically[68]); the KAM theory developed in [15] the invariant tori constructed are C^∞ even if the starting Hamiltonian is analytic[69]; it is not clear what kind of measure estimates one would obtain from the scheme in [15].

- We turn now to the construction of Lagrangian tori in the fully reduced setting and well-spaced regime. Let $\check{\mathcal{M}}^{6n-4}_{\mathrm{G},\epsilon}$ be as in (10.14) with ϵ_3 replaced by a generic $\epsilon \le \epsilon_3$. Then, also in this case, $\check{\mathcal{M}}^{6n-4}_{\mathrm{G},\epsilon}$ has the form $\mathcal{P}_\epsilon$ with $V = \mathcal{A}_{\mathrm{G}}$, $n_1 = n$, $n_2 = 2n-2$ and the totally reduced Hamiltonian $\check{\mathcal{H}}_{\mathrm{G}}$

[66] Such condition is called "Arnold-Pyarti's condition" in [15]; elsewhere is also called "Rüssmann condition".

[67] In [15], however, no explicit symplectic variables are available on the vertical submanifold.

[68] Compare [9].

[69] For a real-analytic version of [15] see [11].

in (10.10)–(10.11), by Proposition 10.1, verifies assumptions (A_1)–(A_3) of Theorem 11.1. Thus, a statement parallel to Theorem 11.2 holds also in the totally reduced case:

Theorem 11.3 *If $\mu < \bar{\mu}$ and $\epsilon < \epsilon_3$ verify condition (11.1), then, $\check{\mathcal{M}}^{6n-4}_{G,\epsilon}$ contains a positive measure Kolmogorov set $\mathcal{K}_G$, which is $\check{\mathcal{H}}_G$-invariant and is formed by the union of $3n-2$-dimensional Lagrangian, real-analytic tori on which the $\check{\mathcal{H}}_G$-motion is analytically conjugated to linear Diophantine quasi-periodic motions with frequencies $(\omega_1, \omega_2) \in \mathbb{R}^n \times \mathbb{R}^{2n-2}$ with $\omega_1 = O(1)$ and $\omega_2 = O(\mu)$; furthermore $\mathcal{K}_G$ satisfies the bound in (11.2) with $\mathcal{P}_\epsilon$ replaced by $\check{\mathcal{M}}^{6n-4}_{G,\epsilon}$. In particular* $\operatorname{meas} \mathcal{K}_G \simeq \epsilon^{2(2n-2)}$.

Remark 11.2 (i) The tori found in this theorem were not mentioned in [2].

(ii) The $(3n-1)$-dimensional invariant tori in the partially reduced phase space obtained by integrating (rotating) the angle g in the Hamilton equation $\dot{g} = \partial_G \hat{\mathcal{H}}_G$, may be resonant according to whether the quasi-periodic average of $\partial_G \hat{\mathcal{H}}_G$ is rationally independent or not with the $(3n-2)$ Diophantine frequencies of the invariant tori belonging to $\mathcal{K}_G$. In general one expects to have all kind of tori (resonant, Liouvillean and Diophantine). Clearly all the $3n$-dimensional tori lifted in the unreduced space will be resonant, since the p_n and q_n variables are, in fact, always constant.

11.2 n-Dimensional elliptic KAM tori in the planetary system

In this final section we discuss briefly elliptic lower dimensional tori in the planetary reduced (and fully reduced) system, generalizing to the spatial case the results in [4].

The Lagrangian tori found in Sect. 11 have fast Keplerian rotations and slow secular quasi-periodic variations around the elliptic linear equilibrium. It is natural to ask whether the linear secular equilibrium bifurcates in the full nonlinear dynamics into lower dimensional elliptic tori of dimension n. This is indeed the case, as we will shortly prove.

Results in this direction were obtained in [14] in the planar three-body case, in [3] for the spatial three-body case and in [4] in the planar $(1+n)$-body case. All these results are based on the application of the lower dimensional KAM theory as developed by Melnikov [22], Kuksin [20], Eliasson [13], Rüssmann [26] and Pöschel [24]. The extension of the previous results to the general spatial case in the partially and fully reduced setting is possible, because of absence of low-order resonances (in the well-spaced region or in an open dense set). We also remark that in this discussion one only needs non degeneracy of the first order Birkhoff invariants ("Melnikov's conditions") and no information is needed on the second order Birkhoff invariants.

The existence of lower dimensional tori will be based upon the following

Theorem 11.4 *Let $\mathcal{P}_\epsilon$ and H be as in Theorem 11.1 with* (A$_2$) *and* (A$_3$) *replaced, respectively, by*

(A$'_2$) $P_{\rm av}(p,q;I) = P_0(I) + \sum_{i=1}^{n_2} \Omega_i(I) r_i + o_2$ *with* $r_i := \frac{p_i^2+q_i^2}{2}$ *and* $\lim_{(p,q)\to 0} \frac{o_2}{|(p,q)|^2} = 0$;

(A$'_3$) $|\Omega_i| \geq {\rm const}$, $|\Omega_i - \Omega_j| \geq {\rm const}$, $\forall i \neq j$, $\forall I \in V$.

Then, if ϵ and μ are small enough, there exists a Cantor set of actions I in V of positive n_1-measure, which parameterizes a family of n_1-dimensional elliptic H-invariant tori on which the H-motion is analytically conjugated to linear quasi-periodic motions with n_1 Diophantine frequencies close to the unperturbed frequencies $\partial_I H_0$.

The proof of Theorem 11.4 is given is Sect. 4 of [4].

Taking $\mathcal{P}_\epsilon = \breve{\mathcal{M}}^{6n-2}$ (as in (7.28) with ϵ_1 replaced by $\epsilon \leq \epsilon_1$) and $\breve{\mathcal{H}} = h_{\rm kep}(\Lambda) + \mu \breve{f}(\Lambda, \breve{\lambda}, \breve{z})$ (as in (7.29)–(7.30)), and observing that assumption (A$'_3$) follows from Proposition 7.2, one obtains the following corollary:

Theorem 11.5 *For ϵ, μ small enough, the well-spaced, partially reduced planetary system $(\breve{\mathcal{H}}, \breve{\mathcal{M}}^{6n-2})$ possesses a Cantor family of n-dimensional elliptic $\breve{\mathcal{H}}$-invariant tori with frequencies close to the unperturbed Keplerian frequencies. Such tori are "surrounded" by the Lagrangian tori constructed in Theorem 11.2.*

An analogous discussion can be done in the totally reduced planetary system, leading to the existence of n-dimensional elliptic tori also in $\breve{\mathcal{M}}^{6n-4}_{\rm G}$ (compare (10.14)).

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Appendix A: Explicit formulae for the RPS map

A.1 Deprit map

In this section we describe the map which relates the action-angle Deprit variables $(\Lambda, \Gamma, \Psi, \lambda, \gamma, \psi)$ defined in Sect. 3 to the standard Cartesian variables $(y, x) \in \mathbb{R}^{3n} \times \mathbb{R}^{3n}$.

To describe properly the rotations occurring in such map, we introduce the following notation. Given two frames $\mathrm{F}=(k^{(1)},k^{(2)},k^{(3)})$ and $\mathrm{G}=(\kappa^{(1)},\kappa^{(2)},\kappa^{(3)})$, we say that G is (i,ψ)-*rotated with respect to* F if

$$\kappa^{(1)}\perp k^{(3)},\qquad \alpha_{\kappa^{(1)}}(k^{(3)},\kappa^{(3)})=i\quad\text{and}\quad \alpha_{k^{(3)}}(k^{(1)},\kappa^{(1)})=\psi\quad i,\psi\in\mathbb{T}. \tag{A.1}$$

So, if (z_1,z_2,z_3), $(\zeta_1,\zeta_2,\zeta_3)$ denote, respectively, the respective sets of Cartesian coordinates of a point P with respect to F, G, i.e., $\mathrm{P}=z_1k^{(1)}+z_2k^{(2)}+z_3k^{(3)}=\zeta_1\kappa^{(1)}+\zeta_2\kappa^{(2)}+\zeta_3\kappa^{(3)}$, *the matrix* R_{FG} *of the change of coordinates from G to F*, i.e., such that

$$\begin{pmatrix}z_1\\ z_2\\ z_3\end{pmatrix}=\mathrm{R}_{\mathrm{FG}}\begin{pmatrix}\zeta_1\\ \zeta_2\\ \zeta_3\end{pmatrix}$$

is given by

$$\mathrm{R}_{\mathrm{FG}}=\mathcal{R}_{31}(\psi,i):=\mathcal{R}_3(\psi)\mathcal{R}_1(i),$$

with $\mathcal{R}_1(i)$, $\mathcal{R}_3(\psi)$ denoting the matrices

$$\mathcal{R}_1(i)=\begin{pmatrix}1&0&0\\0&\cos i&-\sin i\\0&\sin i&\cos i\end{pmatrix},\qquad \mathcal{R}_3(\psi)=\begin{pmatrix}\cos\psi&-\sin\psi&0\\ \sin\psi&\cos\psi&0\\0&0&1\end{pmatrix}. \tag{A.2}$$

Let $\mathrm{F}=(k^{(1)},k^{(2)},k^{(3)})$ a prefixed positively oriented orthonormal frame. With the same notations as in Sect. 3, we introduce the following positively oriented frames.

- The "*orbital frames*" F_j, for $1\le j\le n$, defined by the orthonormal triples $(k^{(1,j)},k^{(2,j)},k^{(3,j)})$, where $k^{(3,j)}$ is in the direction of $\mathrm{C}^{(j)}$ and $k^{(1,j)}$ in the direction of the node ν_j (compare the definition (3.4));
- the frames F^*_j, for $1\le j\le n$, defined as follows. F^*_1 is the orbital frame F_1, defined above. For $2\le j\le n$, the frames F^*_j are defined by the orthonormal triples $(k_*^{(1,j)},k_*^{(2,j)},k_*^{(3,j)})$, where $k_*^{(3,j)}$ is in the direction of $S^{(j)}$ and $k_*^{(1,j)}$ in the direction of ν_{j+1}, with $\nu_{n+1}:=\bar\nu$.

The planes $(k^{(1,j)},k^{(2,j)})$ of the frames $\mathrm{F}_j=(k^{(1,j)},k^{(3,j)},k^{(3,j)})$ contain the osculating ellipses $\mathfrak{E}_j$ (orthogonal to $\mathrm{C}^{(j)}\parallel k^{(3,j)}$). We recall that they are defined as the ellipses with the perihelia P_j forming an angle γ_j with the $k^{(1,j)}=\nu_j$-axis, semimajor axis $a_j=\frac{1}{\bar m_j}(\frac{L_j}{M_j})^2$ and eccentricity $e_j=\sqrt{1-(\frac{\Gamma_j}{L_j})^2}$. We also recall that the symplectic Cartesian coordinates $(\mathrm{y}_{\mathrm{pl}}^{(j)},\mathrm{x}_{\mathrm{pl}}^{(j)})$ with respect to the orbital frame F_j of a point on the osculating ellipse $\mathfrak{E}_j$ are recovered by the so-called Kepler map, analytically defined by equations

$$\mathrm{x}_{\mathrm{pl}}^{(j)} = \mathcal{R}_3(\gamma_j)\mathrm{x}_{\mathrm{orb}}^{(j)}, \qquad \mathrm{y}_{\mathrm{pl}}^{(j)} = \beta_j\,\partial_{\ell_j}\mathrm{x}_{\mathrm{pl}}^{(j)} \tag{A.3}$$

where β_j is as in (4.5) and

$$\mathrm{x}_{\mathrm{orb}}^{(j)} = a_j \begin{pmatrix} \cos u_j - e_j \\ \sqrt{1-e_j^2}\sin u_j \\ 0 \end{pmatrix}, \tag{A.4}$$

u_j being the unique solution of *Kepler's equation* $u_j - e_j \sin u_j = \ell_j$.

By (A.3) and the definition of the frames F_j, the Cartesian coordinates $(y^{(j)}, x^{(j)})$ with respect to the prefixed frame $\mathrm{F} = (k^{(1)}, k^{(2)}, k^{(3)})$ are recovered by the following formulae

$$\begin{cases} y^{(j)} = (y_1^{(j)}, y_2^{(j)}, y_3^{(j)}) = \mathrm{R}_{\mathrm{FF}_j}(\Gamma,\Psi,\psi)\,\mathrm{y}_{\mathrm{pl}}^{(j)}(L_j,\Gamma_j,\ell_j,\gamma_j) \\ \qquad\qquad\qquad\qquad\quad = \beta_j\partial_{\ell_j}x^{(j)} \\ x^{(j)} = (x_1^{(j)}, x_2^{(j)}, x_3^{(j)}) = \mathrm{R}_{\mathrm{FF}_j}(\Gamma,\Psi,\psi)\,\mathrm{x}_{\mathrm{pl}}^{(j)}(L_j,\Gamma_j,\ell_j,\gamma_j) \end{cases} \quad 1 \le j \le n, \tag{A.5}$$

where $\mathrm{R}_{\mathrm{FF}_j}$ denotes the matrix which describes the change of coordinates from F_j to F.

To describe $\mathrm{R}_{\mathrm{FF}_j}$ in terms of the Deprit variables defined in Sect. 3, we consider the following tree of frames

$$\begin{array}{ccccccccccccc} \mathrm{F} \to & \mathrm{F}_n^* & \to & \mathrm{F}_{n-1}^* & \to \cdots \to & \mathrm{F}_j^* & \to \cdots \to & \mathrm{F}_2^* & \to \mathrm{F}_1^* = \mathrm{F}_1 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ & \mathrm{F}_n & & \mathrm{F}_{n-1} & & \mathrm{F}_j & & \mathrm{F}_2 & \end{array} \tag{A.6}$$

and we decompose $\mathrm{R}_{\mathrm{FF}_j}$ as

$$\mathrm{R}_{\mathrm{FF}_j} = \begin{cases} \mathrm{R}_{\mathrm{FF}_n^*}\mathrm{R}_{\mathrm{F}_n^*\mathrm{F}_{n-1}^*}\cdots\mathrm{R}_{\mathrm{F}_{j+1}^*\mathrm{F}_j^*}\mathrm{R}_{\mathrm{F}_2^*\mathrm{F}_1}, & j=1 \\ \mathrm{R}_{\mathrm{FF}_n^*}\mathrm{R}_{\mathrm{F}_n^*\mathrm{F}_{n-1}^*}\cdots\mathrm{R}_{\mathrm{F}_{j+1}^*\mathrm{F}_j^*}\mathrm{R}_{\mathrm{F}_j^*\mathrm{F}_j}, & 2 \le j \le n \end{cases} \tag{A.7}$$

with R_{FG} generically denoting the matrix associated to the change of coordinates from G to F.

We have thus to express the matrices appearing at the right hand side of (A.7) in terms of the Deprit variables. To do it,we define the following "*ab-*

solute" angles i_j, i_j^*, i.e., the angles in $(0, \pi)$ defined by

$$\cos i_j^* = \frac{\Psi_j^2 + \Psi_{j-1}^2 - \Gamma_{j+1}^2}{2\Psi_j \Psi_{j-1}}, \qquad \cos i_n^* = \frac{\mathrm{C}_3}{G}, \quad i_1 := i_1^*,$$
$$\cos i_{j+1} = \frac{\Psi_j^2 + \Gamma_{j+1}^2 - \Psi_{j-1}^2}{2\Psi_j \Gamma_{j+1}} \tag{A.8}$$

where $\Psi_0 := \Gamma_1$ and $1 \le j \le n-1$. Notice that i_n^* has the meaning of the absolute angle between C and $k^{(3)}$ and, considering the triangle with sides $\Psi_{j-2} = |S^{(j-1)}|$, $\Gamma_j = |\mathrm{C}^{(j)}|$ and $\Psi_{j-1} = |S^{(j)}| = |S^{(j-1)} + \mathrm{C}^{(j)}|$, for $1 \le j-1 \le n-1$, i_{j-1}^* is the absolute angle between $S^{(j-1)}$ and $S^{(j)}$ (with $S^{(1)} := \mathrm{C}^{(1)}$), while, for $2 \le j \le n$, i_j is the absolute angle between $\mathrm{C}^{(j)}$ and $S^{(j)}$. Notice also that values 0 or π are never reached because of the assumptions[70] (3.2).

By the above definitions, one has that

- F_n^* is (i_n^*, ζ)-rotated with respect to F and hence, as noticed at the beginning of this section,

$$\mathrm{R}_{\mathrm{FF}_n^*} = \mathcal{R}_{31}(\zeta, i_n^*). \tag{A.9}$$

  In fact, $k_*^{(1,n)} \parallel \nu_{n+1} := \bar{\nu} = k^{(3)} \times \mathrm{C} \perp k^{(3)}$; by the definition of i_n^*, $\alpha_{k_*^{(1,n)}}(k^{(3)}, k_*^{(3,n)}) = \alpha_{\bar{\nu}}(k^{(3)}, \mathrm{C}) = \alpha_{k^{(3)} \times \mathrm{C}}(k^{(3)}, \mathrm{C}) = i_n^*$ and, by definition of ζ, $\alpha_{k^{(3)}}(k^{(1)}, k_*^{(1,n)}) = \alpha_{k^{(3)}}(k^{(1)}, \bar{\nu}) = \zeta$.
- When $n \ge 3$,

$$\mathrm{R}_{\mathrm{F}_j^* \mathrm{F}_{j-1}^*} = \mathcal{R}_{31}(\psi_{j-1}, -i_{j-1}^*) \quad \text{for } 2 \le j-1 \le n-1, \tag{A.10}$$

  since F_{j-1}^* is $(-i_{j-1}^*, \psi_{j-1})$-rotated with respect to F_j^*.

  In fact, one has that $k_*^{(1,j-1)} \| \nu_j = S^{(j)} \times \mathrm{C}^{(j)} \perp S^{(j)} \| k_*^{(3,j)}$; furthermore, $\alpha_{k_*^{(1,j-1)}}(k_*^{(3,j)}, k_*^{(3,j-1)}) = \alpha_{\nu_j}(S^{(j)}, S^{(j-1)}) = \alpha_{S^{(j)} \times \mathrm{C}^{(j)}}(S^{(j)}, S^{(j-1)}) = \alpha_{-S^{(j)} \times S^{(j-1)}}(S^{(j)}, S^{(j-1)}) = -i_{j-1}^*$, by definition of i_{j-1}^*. Finally, by definition of ψ_{j-1}, $\alpha_{k_*^{(3,j)}}(k_*^{(1,j)}, k_*^{(1,j-1)}) = \alpha_{S^{(j)}}(\nu_{j+1}, \nu_j) = \psi_{j-1}$.
- For $2 \le j \le n$, F_j is (i_j, ψ_{j-1})-rotated with respect to F_j^*, hence,

$$\mathrm{R}_{\mathrm{F}_j^* \mathrm{F}_j} = \mathcal{R}_{31}(\psi_{j-1}, i_j), \quad 2 \le j \le n. \tag{A.11}$$

[70] $S^{(j)} \times \mathrm{C}^{(j)} \neq 0$ implies that also $S^{(j-1)} \times S^{(j)} \neq 0$, since $\mathrm{C}^{(j)} = S^{(j)} - S^{(j-1)}$.

In fact, $k^{(1,j)} \parallel \nu_j = S^{(j)} \times \mathrm{C}^{(j)} \perp S^{(i)} \parallel k_*^{(3,j)}$; furthermore, by definition of i_j,

$$\alpha_{k^{(1,j)}}(k_*^{(3,j)}, k^{(3,j)}) = \alpha_{\nu_j}(S^{(j)}, \mathrm{C}^{(j)}) = \alpha_{S^{(j)} \times \mathrm{C}^{(j)}}(S^{(j)}, \mathrm{C}^{(j)}) = i_j$$

and finally, by definition of ψ_{j-1},

$$\alpha_{k_*^{(3,j)}}(k_*^{(1,j)}, k^{(1,j)}) = \alpha_{S^{(j)}}(\nu_{j+1}, \nu_j) = \psi_{j-1}.$$

- For $j = 1$, F_1 is $(-i_1^*, \psi_1)$-rotated with respect to F_2^*, implying

$$\mathrm{R}_{\mathrm{F}_2^*\mathrm{F}_1} = \mathcal{R}_{31}(\psi_1, -i_1^*). \tag{A.12}$$

  In fact, similarly to the previous case, we have that $k^{(1,1)} \perp k_*^{(3,2)}$ and that $\alpha_{k^{(1,1)}}(k_*^{(3,2)}, k^{(3,1)}) = \alpha_{\nu_2}(S^{(2)}, \mathrm{C}^{(1)}) = -\alpha_{\nu_2}(\mathrm{C}^{(1)}, S^{(2)}) = -i_1 = -i_1^*$. Furthermore, since $\nu_1 = \nu_2$, we have that $\alpha_{k_*^{(3,2)}}(k_*^{(1,2)}, k^{(1,1)}) = \alpha_{S^{(2)}}(\nu_3, \nu_1) = \alpha_{S^{(2)}}(\nu_3, \nu_2) = \psi_1$.

Then, in view of (A.7), (A.9), (A.10), (A.11) and (A.12), the description (A.5) of the Deprit map is completed by the following formulae

$$\mathrm{R}_{\mathrm{F}\mathrm{F}_j}(\Gamma, \Psi) = \begin{cases} \mathcal{R}_{31}(\psi_n, i_n^*)\mathcal{R}_{31}(\psi_{n-1}, -i_{n-1}^*) \cdots \\ \quad \mathcal{R}_{31}(\psi_2, -i_2^*)\mathcal{R}_{31}(\psi_1, -i_1^*), \quad j = 1 \\ \mathcal{R}_{31}(\psi_n, i_n^*)\mathcal{R}_{31}(\psi_{n-1}, -i_{n-1}^*) \cdots \\ \quad \mathcal{R}_{31}(\psi_j, -i_j^*)\mathcal{R}_{31}(\psi_{j-1}, i_j), \quad 2 \le j \le n. \end{cases} \tag{A.13}$$

where i_j^*, i_j are as in (A.8).

A.2 RPS map

By (4.1) and (4.2), the angles ζ, $\psi_j^n = \sum_{j \le k \le n} \psi_k$ and γ_j are related to regularized variables by[71]

$$\begin{aligned} &\zeta = \psi_n = -\arg(p_n, q_n), \qquad \psi_j^n = -\arg(p_j, q_j), \\ &\gamma_1 - \arg(p_1, q_1) = -\arg(\eta_1, \xi_1), \\ &\gamma_j - \arg(p_{j-1}, q_{j-1}) = -\arg(\eta_j, \xi_j) \quad 2 \le j \le n. \end{aligned} \tag{A.14}$$

[71] Recall the choice $\psi_0 = 0$. As usual, if $(u, v) \in \mathbb{R}^2 \setminus \{0\}$, $\arg(u, v)$ denotes the unique $t \in \mathbb{T}$ such that $\cos t = \frac{u}{\sqrt{u^2+v^2}}$ and $\sin t = \frac{v}{\sqrt{u^2+v^2}}$. Recall also that assumptions (3.2) imply $(\eta_j, \xi_j) \neq 0$, $(p_j, q_j) \neq 0$ for any $1 \le j \le n$.

From the formula of ψ_j^n, we find the angles $\psi_1, \dots, \psi_{n-2}, \psi_{n-1} = g$:

$$\psi_j = \psi_j^n - \psi_{j+1}^n = \arg(p_{j+1}, q_{j+1}) - \arg(p_j, q_j) \quad \text{for } 1 \le j \le n-1. \tag{A.15}$$

We substitute such expressions of into the matrices $\mathrm{R_{FF_j}}$ defined in (A.13), and we find

$$\begin{aligned} \mathrm{R_{FF_1}} &= \mathfrak{R}_1 \mathcal{R}_3\big(-\arg(p_1, q_1)\big), \\ \mathrm{R_{FF_{\mathit{j}}}} &= \mathfrak{R}_j \mathcal{R}_3\big(-\arg(p_{j-1}, q_{j-1})\big) \quad \text{for } 2 \le j \le n \end{aligned} \tag{A.16}$$

where the matrices $\mathfrak{R}_j$ are the products (4.7), with

$$\mathcal{R}_j^* := \begin{cases} \mathcal{R}_{313}(\arg(p_j, q_j), -i_j^*), & 1 \le j \le n-1 \\ \mathcal{R}_{313}(\arg(p_n, q_n), i_n^*), & j = n \end{cases} \tag{A.17}$$

and

$$\begin{aligned} &\mathcal{R}_j := \mathcal{R}_{313}(\arg(p_{j-1}, q_{j-1}), i_j), \\ &\qquad 2 \le j \le n \text{ with } \mathcal{R}_{313}(\psi, i) := \mathcal{R}_3(-\psi)\mathcal{R}_1(i)\mathcal{R}_3(\psi). \end{aligned} \tag{A.18}$$

Remark A.1 Notice that, if $(\Lambda, \lambda, z) \to \mathcal{R}^g(\Lambda, \lambda, z)$, where $\mathcal{R}^g$ is as in (7.9), then, $\mathfrak{R}_i \to \mathcal{R}(g)\mathfrak{R}_i\mathcal{R}(-g)$ (since the same holds for the matrices $\mathcal{R}_i$, $\mathcal{R}_i^*$, as it follows by their definitions).

Substituting (A.16) into the expression of $y^{(j)}$, $x^{(j)}$ in (A.5), with $\mathrm{x}_{\mathrm{pl}}^{(j)}$ as in (A.3) and using finally the expressions in the second line of (A.14), one easily finds the expression in (4.3), with $\mathfrak{R}_i$ as in (4.7) and $x_{\mathrm{pl}}^{(j)} := \mathcal{R}_3(-\arg(\eta_j, \xi_j))\mathrm{x}_{\mathrm{pl}}^{(j)}$ easily recognized to be the planar Poincaré map defined in (4.4) and (4.5).

To complete the analytical expression of the RPS map (4.3), we have to express the matrices $\mathcal{R}_i^*$, $\mathcal{R}_i$ in (A.17) and (A.18) in terms of the variables (Λ, λ, z).

To this end, we let

$$\begin{aligned} &\rho_i := \frac{\eta_i^2 + \xi_i^2}{2}, \qquad r_i := \frac{p_i^2 + q_i^2}{2}, \qquad \mathfrak{L}_i := \sum_{1 \le j \le i} \Lambda_j \\ &z_0 := (\eta_1, \xi_1), \\ &z_i = (\eta_1, \dots, \eta_{i+1}, \xi_1, \dots, \xi_{i+1}, p_1, \dots, p_i, q_1, \dots, q_i), \quad 1 \le i \le n-1 \end{aligned} \tag{A.19}$$

so that $z_{n-1} = \bar{z}$, $(\bar{z}, (p_n, q_n)) = (z_{n-1}, (p_n, q_n)) = z$. By (4.1), (4.2),

$$\Gamma_i = \Lambda_i - \rho_i, \quad 1 \le i \le n$$

$$\Psi_i = \sum_{j=1}^{i+1} \Lambda_j - \sum_{j=1}^{i+1}(\Lambda_j - \Gamma_j) - \sum_{j=1}^{i}(\Psi_{j-1} + \Gamma_{j+1} - \Psi_j) = \mathfrak{L}_{i+1} - \frac{1}{2}|z_i|^2,$$
$$1 \le i \le n-1$$
$$\mathrm{C}_3 = G - \frac{p_n^2 + q_n^2}{2} = \Psi_{n-1} - \frac{p_n^2 + q_n^2}{2} = \mathfrak{L}_n - \frac{1}{2}|z^2| \tag{A.20}$$

Then, by (A.8), we have

$$1 - \cos i_n^* = \frac{G - \mathrm{C}_3}{G} = \frac{p_n^2 + q_n^2}{2\mathfrak{L}_n - |\bar{z}|^2} = (p_n^2 + q_n^2)\mathfrak{c}_n^* \tag{A.21}$$

where

$$\mathfrak{c}_n^* := \frac{1}{2\mathfrak{L}_n - |\bar{z}|^2} \tag{A.22}$$

Similarly define $\mathfrak{c}_1^*, \ldots, \mathfrak{c}_{n-1}^*, \mathfrak{c}_2, \ldots, \mathfrak{c}_n$ by

$$\begin{aligned} \mathfrak{c}_j^* &:= \frac{2\Lambda_{j+1} - 2\rho_{j+1} - r_j}{(2\mathfrak{L}_{j+1} - |z_j|^2)(2\mathfrak{L}_j - |z_{j-1}|^2)} \quad 1 \le j \le n-1 \\ \mathfrak{c}_j &:= \frac{2\mathfrak{L}_{j-1} - |z_{j-2}|^2 - r_{j-1}}{2(\Lambda_j - \rho_j)(2\mathfrak{L}_j - |z_{j-1}|^2)} \quad 2 \le j \le n \end{aligned} \tag{A.23}$$

such that, by (A.8) and (A.20), for $1 \le j \le n-1$

$$\begin{aligned} 1 - \cos i_j^* &= 1 - \frac{\Psi_j^2 + \Psi_{j-1}^2 - \Gamma_{j+1}^2}{2\Psi_j\Psi_{j-1}} \\ &= \frac{(\Gamma_{j+1} - \Psi_j + \Psi_{j-1})(\Gamma_{j+1} + \Psi_j - \Psi_{j-1})}{2\Psi_j\Psi_{j-1}} \\ &= (p_j^2 + q_j^2)\mathfrak{c}_j^* \quad (|z_0|^2 := 2\rho_1) \end{aligned} \tag{A.24}$$

and, for $2 \le j \le n$

$$\begin{aligned} 1 - \cos i_j &= 1 - \frac{\Psi_{j-1}^2 + \Gamma_j^2 - \Psi_{j-2}^2}{2\Psi_{j-1}\Gamma_j} \\ &= \frac{(\Psi_{j-2} - \Psi_{j-1} + \Gamma_j)(\Psi_{j-2} + \Psi_{j-1} - \Gamma_j)}{2\Psi_{j-1}\Gamma_j} \\ &= (p_{j-1}^2 + q_{j-1}^2)\mathfrak{c}_j. \end{aligned} \tag{A.25}$$

Now, if $\mathcal{R}_1, \mathcal{R}_3$ are as in (A.2) then, the matrix $\mathcal{R}_{313}(\psi, i) = \mathcal{R}_3(-\psi)\mathcal{R}_1(i) \times \mathcal{R}_3(\psi)$ has the expression

$$\mathcal{R}_{313}(\psi, i) = \begin{pmatrix} 1 - \sin^2\psi(1-\cos i) & -\sin\psi\cos\psi(1-\cos i) & -\sin\psi\sin i \\ -\sin\psi\cos\psi(1-\cos i) & 1-\cos^2\psi(1-\cos i) & -\cos\psi\sin i \\ \sin\psi\sin i & \cos\psi\sin i & \cos i \end{pmatrix} \tag{A.26}$$

Then, by (A.21), (A.24), (A.25) and (A.26), we find that the matrices $\mathcal{R}_j^*$, $\mathcal{R}_j$ into (A.17) and (A.18) have the expressions (4.8), with $\mathfrak{c}_j$, $\mathfrak{c}_j^*$ as in (A.22) and (A.23) and

$$\begin{aligned} \mathfrak{s}_j^* &:= \frac{\sin(-i_j^*)}{\sqrt{p_j^2+q_j^2}} = -\sqrt{\mathfrak{c}_j^*\big(2-(p_j^2+q_j^2)\mathfrak{c}_j^*\big)} \quad 1 \le j \le n-1 \\ \mathfrak{s}_n^* &:= \frac{\sin i}{\sqrt{p_n^2+q_n^2}} = \sqrt{\mathfrak{c}_n^*\big(2-(p_n^2+q_n^2)\mathfrak{c}_n\big)} \\ \mathfrak{s}_j &:= \frac{\sin i_j}{\sqrt{p_{j-1}^2+q_{j-1}^2}} = \sqrt{\mathfrak{c}_j\big(2-(p_{j-1}^2+q_{j-1}^2)\mathfrak{c}_j\big)} \quad 2 \le j \le n. \end{aligned} \tag{A.27}$$

Notice that the matrices $\mathcal{R}_j^*$, $\mathcal{R}_j$, hence the matrices $\mathfrak{R}_i$ in (4.7), are regular also when some of (η_j, ξ_j) or (p_j, q_j) vanishes. Since, as it is known, also the planar Poincaré map $(y_{\rm pl}^{(j)}, x_{\rm pl}^{(j)})$ (compare (4.5), (4.6)) is regular, the RPS (4.3) map is so.

Appendix B: Expansion of the secular Hamiltonian up to order four in RPS variables

Here, we will prove that the explicit form of the constant C_0, the quadratic tensors $\rm Q_h$, $\bar{\rm Q}_v$ and the quartic tensors $\rm F_h$, $\rm F_v$, $\rm F_{hv}$, $\rm F'_{hv}$ appearing into the expansion (6.9) and (6.10) are given by the following explicit formulae.

- The constant C_0 is trivial and is given by $C_0(\Lambda) := -\sum_{1\le j<k\le n} \frac{m_j m_k}{a_k} \times b_{1/2,0}(a_j/a_k)$ where $a_k = a_k(\Lambda_k) = \bar{m}_k^{-1}(\Lambda_k/M_k)^2$ is the $k^{\rm th}$ semimajor axis and $b_{h,k}$'s denote the Laplace coefficients.[72]

  Let us, now, denote by $\beta_k^{(r)} := b_{r/2,k}$ and, for two any positive numbers $a \neq b$, $\alpha = \alpha(a, b) := a/b$. Then:

[72]Recall the definition of Laplace coefficients: if $h > 0$, $k \in \mathbb{Z}$, $\alpha \in \mathbb{C}$, with $\alpha \neq \pm 1$, the Laplace coefficient $b_{h,k}(\alpha)$ are the Fourier coefficient of the function $t \to (1+\alpha^2 -$

- *the horizontal quadratic form* $\mathrm{Q_h}$ is[73]

$$\mathrm{Q_h}(\Lambda)\cdot\eta^2 := \sum_{1\le j<k\le n} m_j m_k\Bigg(C_1(a_j,a_k)\left(\frac{\eta_j^2}{\Lambda_j}+\frac{\eta_k^2}{\Lambda_k}\right) + 2C_2(a_j,a_k)\frac{\eta_j\eta_k}{\sqrt{\Lambda_j\Lambda_k}}\Bigg); \tag{B.1}$$

with

$$C_1(a,b) := -\frac{\alpha}{2b}\beta_1^{(3)}(\alpha), \qquad C_2(a,b) := \frac{\alpha}{b}\beta_2^{(3)}(\alpha) \tag{B.2}$$

- *the vertical quadratic form* $\bar{\mathrm{Q}}_\mathrm{v}$ is

$$\bar{\mathrm{Q}}_\mathrm{v}(\Lambda)\cdot\bar p^2 := -\sum_{1\le j<k\le n} m_j m_k C_1(a_j,a_k)\Bigg(\sum_{1\le h\le n-1}(\mathcal{L}_{jh}-\mathcal{L}_{kh})p_h\Bigg)^2 \tag{B.3}$$

where C_1 is as in (B.2) and $\mathcal{L}$ is the $n\times(n-1)$ matrix given by

$$\mathcal{L}_{ij} := \begin{cases} -\sqrt{\frac{\Lambda_{j+1}}{\mathfrak{L}_{j+1}\mathfrak{L}_j}} & \text{for } i=1 \text{ and } 1\le j\le n-1,\ \text{or } 2\le i\le j\le n-1 \\ \sqrt{\frac{\mathfrak{L}_{i-1}}{\mathfrak{L}_i\Lambda_i}} & \text{for } 2\le i\le n \text{ and } j=i-1 \\ 0 & \text{otherwise.}\end{cases} \tag{B.4}$$

where $\mathfrak{L}_j := \sum_{1\le k\le j}\Lambda_k$

- the *horizontal quartic tensor* $\mathrm{F_h}(\Lambda)$ is given by

$$\begin{aligned}\mathrm{F_h}(\Lambda)\cdot\eta^2\xi^2 := \sum_{1\le i<j\le n} m_i m_j\Bigg(&\mathrm{r}_1(a_i,a_j)\frac{\eta_i^2\xi_i^2}{\Lambda_i^2}+\mathrm{r}_1(a_j,a_i)\frac{\eta_j^2\xi_j^2}{\Lambda_j^2}\\ &+\mathrm{r}_2(a_i,a_j)\frac{\eta_i^2\xi_j^2}{\Lambda_i\Lambda_j}+\mathrm{r}_2(a_j,a_i)\frac{\eta_j^2\xi_i^2}{\Lambda_i\Lambda_j}\\ &+\mathrm{r}_3(a_i,a_j)\frac{\eta_i\eta_j\xi_i^2}{\Lambda_i\sqrt{\Lambda_i\Lambda_j}}+\mathrm{r}_3(a_j,a_i)\frac{\eta_i\eta_j\xi_j^2}{\Lambda_j\sqrt{\Lambda_i\Lambda_j}}\end{aligned}$$

$2\alpha\cos t)^{-h}$, i.e., $b_{h,k}(\alpha) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\cos kt}{(1+\alpha^2-2\alpha\cos t)^h}dt$. Recall that we are in a region of phase where $a_1 < a_2 < \cdots < a_n$. Notice that, when $\bar z = (\eta,\xi,\bar p,\bar q) = 0$, the $x^{(i)}$-projection of the RPS map reduces to $x^{(i)} = a_i(\cos\lambda_i, \sin\lambda_i, 0)$, whence the expression of C_0 follows.

[73] Formulae (B.1), (B.2) are as in [15, (36), (37)], where Poincaré variables are used. This is due to the fact that when $\bar p = \bar q = 0$ and $p_n = q_n = 0$, the two sets of variables (Poincaré and RPS) coincide: see Remark 4.2-(i). The "vertical" quadratic form in (10.29), on the other hand, differs from that in [15].

$$+ \mathrm{r}_3(a_i, a_j)\frac{\eta_i^2 \xi_i \xi_j}{\Lambda_i\sqrt{\Lambda_i\Lambda_j}} + \mathrm{r}_3(a_j, a_i)\frac{\eta_j^2 \xi_i \xi_j}{\Lambda_j\sqrt{\Lambda_i\Lambda_j}}$$

$$+ \mathrm{r}_4(a_i, a_j)\frac{\eta_i \eta_j \xi_i \xi_j}{\Lambda_i\Lambda_j}\Bigg) \tag{B.5}$$

where, for $a \neq b$,

$$\begin{aligned}
\mathrm{r}_1(a,b) := -\frac{\alpha}{256b}\,&[(-60\alpha^5 + 4311\alpha^3 - 300\alpha)\,\beta_0^{(9)}(\alpha)\\
&+ 8\cdot(7\alpha^6 - 252\alpha^4 - 222\alpha^2 + 7)\,\beta_1^{(9)}(\alpha)\\
&+ 4\cdot(75\alpha^5 - 503\alpha^3 + 135\alpha)\,\beta_2^{(9)}(\alpha)\\
&+ 24\cdot(23\alpha^4 + 13\alpha^2)\,\beta_3^{(9)} + 37\alpha^4\,\beta_4^{(9)}(\alpha)]
\end{aligned}$$

$$\begin{aligned}
\mathrm{r}_2(a,b) := \frac{3\alpha}{512b}\,&[(84\alpha^5 - 8832\alpha^3 + 84\alpha)\,\beta_0^{(9)}(\alpha)\\
&- 8\cdot(5\alpha^6 - 652\alpha^4 - 652\alpha^2 + 5)\,\beta_1^{(9)}(\alpha)\\
&- 5\cdot(328\alpha^5 - 561\alpha^3 + 328\alpha)\,\beta_2^{(9)}(\alpha)\\
&+ (216\alpha^6 - 1020\alpha^4 - 1020\alpha^2 + 216)\,\beta_3^{(9)}(\alpha)\\
&+ (116\alpha^5 + 200\alpha^3 + 116\alpha)\,\beta_4^{(9)}(\alpha)\\
&- (20\alpha^4 + 20\alpha^2)\,\beta_5^{(9)}(\alpha) + 3\alpha^3\,\beta_6^{(9)}(\alpha)]
\end{aligned}$$

$$\begin{aligned}
\mathrm{r}_3(a,b) := \frac{\alpha}{256b}\,&[(1146\alpha^4 + 1266\alpha^2)\,\beta_0^{(9)}(\alpha)\\
&+ (-744\alpha^5 + 2014\alpha^3 - 864\alpha)\,\beta_1^{(9)}(\alpha)\\
&+ 8\cdot(28\alpha^6 - 321\alpha^4 - 321\alpha^2 + 28)\,\beta_2^{(9)}(\alpha)\\
&+ (552\alpha^5 + 423\alpha^3 + 672\alpha)\,\beta_3^{(9)}(\alpha)\\
&+ 6(29\alpha^4 + 9\alpha^2)\,\beta_4^{(9)}(\alpha) - 5\alpha^3\,\beta_5^{(9)}(\alpha)]
\end{aligned}$$

$$\begin{aligned}
\mathrm{r}_4(a,b) := -\frac{\alpha}{128b}\,&[(-36\alpha^5 - 7956\alpha^3 - 36\alpha)\,\beta_0^{(9)}(\alpha)\\
&+ 8\cdot(\alpha^6 + 828\alpha^4 + 828\alpha^2 + 1)\,\beta_1^{(9)}(\alpha)\\
&+ (-3096\alpha^5 + 1039\alpha^3 - 3096\alpha)\,\beta_2^{(9)}(\alpha)
\end{aligned}$$

$$+ (648\alpha^6 - 1332\alpha^4 - 1332\alpha^2 + 648)\,\beta_3^{(9)}(\alpha)$$
$$+ (348\alpha^5 + 700\alpha^3 + 348\alpha)\,\beta_4^{(9)}(\alpha)$$
$$- 60 \cdot (\alpha^4 + \alpha^2)\,\beta_5^{(9)}(\alpha) + 9\alpha^3\,\beta_6^{(9)}(\alpha)] \tag{B.6}$$

- the *vertical quartic tensor* $\mathrm{F_v}(\Lambda)$ and the *mixed quartic tensors* $\mathrm{F_{hv}}(\Lambda)$, $\mathrm{F'_{hv}}(\Lambda)$ can be splitted as

$$\mathrm{F_v}(\Lambda) = \mathrm{F_{v1}}(\Lambda) + \mathrm{F_{v2}}(\Lambda), \qquad \mathrm{F_{hv}}(\Lambda) = \mathrm{F_{hv1}}(\Lambda) + \mathrm{F_{hv2}}(\Lambda),$$
$$\mathrm{F'_{hv}}(\Lambda) = \mathrm{F'_{hv1}}(\Lambda) + \mathrm{F'_{hv2}}(\Lambda)$$

where

- $\mathrm{F_{v1}}(\Lambda) \cdot \bar p^2 \bar q^2$, $\mathrm{F_{hv1}}(\Lambda) \cdot \eta^2 \bar q^2$ and $\mathrm{F'_{hv1}}(\Lambda) \cdot \eta^2 \bar p^2$ are respectively given by

$$\sum_{1\le i<j\le n} m_i m_j \Bigg(\mathrm{r}_1^*(a_i, a_j) \Bigg(\sum_{1\le h\le n-1} (\mathcal{L}_{ih} - \mathcal{L}_{jh}) p_h \Bigg)^2 \times \Bigg(\sum_{1\le k\le n-1} (\mathcal{L}_{ik} - \mathcal{L}_{jk}) q_k \Bigg)^2 + \mathrm{r}_2^*(a_j, a_k) \Bigg(\sum^{*}_{1\le h<k\le n-1} (\mathcal{L}_{ih} - \mathcal{L}_{jh})(\mathcal{L}_{ik} - \mathcal{L}_{jk})(p_h q_k - p_k q_h) \Bigg)^2 \Bigg) \tag{B.7}$$

$$\sum_{1\le i<j\le n} m_i m_j \Bigg(\mathrm{s}_1(a_i, a_j) \frac{{\eta_i}^2}{\Lambda_i} + \mathrm{s}_2(a_i, a_j) \frac{\eta_i \eta_j}{\sqrt{\Lambda_i \Lambda_j}} + \mathrm{s}_1(a_j, a_i) \frac{{\eta_j}^2}{\Lambda_j} \Bigg) \times \Bigg(\sum_{1\le h\le n-1} (\mathcal{L}_{ih} - \mathcal{L}_{jh}) q_h \Bigg)^2$$

$$\sum_{1\le i<j\le n} m_i m_j \Bigg(\mathrm{s}_1^*(a_i, a_j) \frac{{\eta_i}^2}{\Lambda_i} + \mathrm{s}_2^*(a_i, a_j) \frac{\eta_i \eta_j}{\sqrt{\Lambda_i \Lambda_j}} + \mathrm{s}_1^*(a_j, a_i) \frac{{\eta_j}^2}{\Lambda_j} \Bigg) \times \Bigg(\sum_{1\le h\le n-1} (\mathcal{L}_{ih} - \mathcal{L}_{jh}) p_h \Bigg)^2$$

with

$$\mathrm{r}_1^*(a, b) := -\frac{3}{32} \frac{\alpha^2}{b} (2\beta_0^{(5)}(\alpha) + \beta_2^{(5)}(\alpha))$$

$$
\begin{aligned}
\mathrm{r}_2^*(a,b) &:= -\frac{3}{16}\frac{\alpha^2}{b}(\beta_0^{(5)}(\alpha)+\beta_2^{(5)}(\alpha))\\
\mathrm{s}_1(a,b) &:= \frac{\alpha}{64b}[2\alpha(57\alpha^2+117)\beta_0^{(7)}(\alpha)+(-12\alpha^4-291\alpha^2-12)\beta_1^{(7)}(\alpha)\\
&\quad +2\alpha(15\alpha^2-45)\beta_2^{(7)}(\alpha)+27\alpha^2\beta_3^{(7)}(\alpha)]\\
\mathrm{s}_2(a,b) &:= \frac{\alpha}{64b}[554\alpha^2\beta_0^{(7)}(\alpha)+\alpha(-376-376\alpha^2)\beta_1^{(7)}(\alpha)+2(16\alpha^4\\
&\quad +10\alpha^2+16)\beta_2^{(7)}(\alpha)+56\alpha(\alpha^2+1)\beta_3^{(7)}(\alpha)+2\alpha^2\beta_4^{(7)}(\alpha)]\\
\mathrm{s}_1^*(a,b) &:= \frac{\alpha}{64b}[2\alpha(71\alpha^2+11)\beta_0^{(7)}(\alpha)+(-20\alpha^4+119\alpha^2-20)\beta_1^{(7)}(\alpha)\\
&\quad +2\alpha(-79\alpha^2-19)\beta_2^{(7)}(\alpha)-47\alpha^2\beta_3^{(7)}(\alpha)]\\
\mathrm{s}_2^*(a,b) &:= \frac{\alpha}{64b}[-430\alpha^2\beta_0^{(7)}(\alpha)+\alpha(8+8\alpha^2)\beta_1^{(7)}+2(16\alpha^4+118\alpha^2\\
&\quad +16)\beta_2^{(7)}(\alpha)+56\alpha(\alpha^2+1)\beta_3^{(7)}(\alpha)+2\alpha^2\beta_4^{(7)}(\alpha)].
\end{aligned}
\tag{B.8}
$$

- The definition of $\mathrm{F}_{\mathrm{v}2}(\Lambda)\cdot\bar{p}^2\bar{q}^2$, $\mathrm{F}_{\mathrm{hv}2}(\Lambda)\cdot\eta^2\bar{q}^2$ and $\mathrm{F}'_{\mathrm{hv}2}(\Lambda)\cdot\eta^2\bar{p}^2$ is more involved (reflecting the products (4.7) appearing in the definition of the rotation matrices $\mathfrak{R}_i$).

  Let $c_2,\ldots,c_n,c_1^*,\ldots,c_{n-1}^*,\bar{c}_1,\ldots,\bar{c}_{n-1}$ be defined by

$$
\begin{cases} c_1^* := \mathcal{L}_{11} & i=1\\ c_i := \mathcal{L}_{i,i-1} & 2\le i\le n\end{cases}
\qquad
\begin{cases} c_i^* := \mathcal{L}_{ki} & 2\le i\le n-1,\ 1\le k\le i\\ \bar{c}_i = c_{i+1}-c_i^* = \sqrt{\frac{\mathfrak{L}_{i+1}}{\mathfrak{L}_i\Lambda_{i+1}}} & 1\le i\le n-1\end{cases}
\tag{B.9}
$$

  where $\mathcal{L}_{ij}$ is the matrix (B.4). Notice, for later convenience, that the above definitions imply that the differences $\mathcal{L}_i-\mathcal{L}_j$ of the rows of $\mathcal{L}$ are related to $c_2,\ldots,c_n,c_1^*,\ldots,c_{n-1}^*,\bar{c}_1,\ldots,\bar{c}_{n-1}$ by

$$
\begin{aligned}
\mathcal{L}_i-\mathcal{L}_j &= \begin{cases}(-\alpha_{ij},0_{n-j}), & i=1,\ 2\\ (0_{i-2},-\alpha_{ij},0_{n-j}), & i\ge 3\end{cases}\\
\alpha_{ij} &:= \begin{cases}(-c_1^*,-c_2^*,\ldots,-c_{j-2}^*,\bar{c}_{j-1}), & i=1\\ (-c_i,-c_i^*,\ldots,-c_{j-2}^*,\bar{c}_{j-1}), & i\ge 2\end{cases}
\end{aligned}
\tag{B.10}
$$

  where 0_r denotes the null vector of dimension r.

Define the homogeneous polynomials, of degree 2 in $\bar z$, Q_2, …, Q_n, Q_1^*, …, Q_{n-1}^*, $\bar Q_1$, …, $\bar Q_{n-1}$ by[74]

$$\begin{aligned}
Q_i &:= \frac{2\mathfrak{L}_{i-1}(\mathfrak{L}_{i-1}+2\Lambda_i)\rho_i+\Lambda_i(\mathfrak{L}_{i-1}-\Lambda_i)r_{i-1}-\Lambda_i^2|z_{i-2}|^2}{4\mathfrak{L}_i^2\Lambda_i^2},\\
&\quad 2\le i\le n\\
Q_1^* &:= \frac{2\Lambda_2(2\Lambda_1+\Lambda_2)\rho_1-2\Lambda_1^2\rho_2+\Lambda_1(\Lambda_2-\Lambda_1)r_1}{4\Lambda_1^2\mathfrak{L}_2^2},\\
Q_i^* &:= \frac{-2\mathfrak{L}_i^2\rho_{i+1}+\mathfrak{L}_i(\Lambda_{i+1}-\mathfrak{L}_i)r_i+\Lambda_{i+1}(2\mathfrak{L}_i+\Lambda_{i+1})|z_{i-1}|^2}{4\mathfrak{L}_i^2\mathfrak{L}_{i+1}^2},\\
&\quad 2\le i\le n-1\\
\bar Q_i &:= \frac{\Lambda_{i+1}^2|z_{i-1}|^2+2\mathfrak{L}_i^2\rho_{i+1}-\mathfrak{L}_i\Lambda_{i+1}r_i}{4\mathfrak{L}_i^2\Lambda_{i+1}^2},\quad 1\le i\le n-1
\end{aligned}\tag{B.11}$$

where

$$z_i := (\eta_1,\dots,\eta_{i+1},\xi_1,\dots,\xi_{i+1},p_1,\dots,p_i,q_1,\dots,q_i)\quad \text{for } 1\le i\le n-1.\tag{B.12}$$

Next, for arbitrary numbers Q, $c\neq 0$, r, p, q, denote

$$\begin{aligned}
&\mathrm{T}(c,Q,r,p,q)\\
&:=\begin{pmatrix}
1-q^2(\frac{c^2}{2}+Q) & -pq(\frac{c^2}{2}+Q) & -q\mathcal{S}(c,Q,r)\\
-pq(\frac{c^2}{2}+Q) & 1-p^2(\frac{c^2}{2}+Q) & -p\mathcal{S}(c,Q,r)\\
q\mathcal{S}(c,Q,r) & p\mathcal{S}(c,Q,r) & 1-(p^2+q^2)(\frac{c^2}{2}+Q)
\end{pmatrix};
\end{aligned}\tag{B.13}$$

where

$$\mathcal{S}(c,Q,r) := c+\frac{1}{c}Q-\frac{c^3}{4}r.\tag{B.14}$$

[74]Recall (A.19).

Now, define

$$\begin{aligned} &\mathrm{T}_1 := \mathrm{id}\,, \\ &\mathrm{T}_i := \mathrm{T}(c_i, Q_i, r_{i-1}, p_{i-1}, q_{i-1}) \quad \text{for } 2 \le i \le n \\ &\mathrm{T}^*_i := \mathrm{T}(c^*_i, Q^*_i, r_i, p_i, q_i), \qquad \text{for } 1 \le i \le n-1 \\ &\bar{\mathrm{T}}_i := \mathrm{T}(\bar{c}_i, \bar{Q}_i, r_i, p_i, q_i)) \qquad \text{for } 1 \le i \le n-1; \end{aligned} \tag{B.15}$$

and let, for $1 \le i < j \le n$,

$$\mathrm{T}_{ij} := (\mathrm{T}_i)^{\mathrm{t}} (\mathrm{T}^*_i)^{\mathrm{t}} \cdots (\mathrm{T}^*_{j-2})^{\mathrm{t}} \bar{\mathrm{T}}_{j-1}, \tag{B.16}$$

where T^{t} denotes the transpose of T and the product $(\mathrm{T}^*_i)^{\mathrm{t}} \cdots (\mathrm{T}^*_{j-2})^{\mathrm{t}}$ in front of $\bar{\mathrm{T}}_{j-1}$ is absent when $j = i + 1$.

Let, finally, $\mathfrak{Q}_{ij}$ denote the quartic part (i.e., the homogeneous part of order 4 in z) of the upper left (2×2) submatrix of T_{ij} and define the polynomial

$$\sum_{1 \le i < j \le n} m_i m_j C_1(a_i, a_j) \,\mathrm{tr}\, \mathfrak{Q}_{ij}, \tag{B.17}$$

where $\mathrm{tr}\,\mathfrak{Q}_{ij}$ denotes the trace of $\mathfrak{Q}_{ij}$.

Then, $\mathrm{F}_{\mathrm{v}2}(\Lambda) \cdot \bar{p}^2 \bar{q}^2$, $\mathrm{F}_{\mathrm{hv}2}(\Lambda) \cdot \eta^2 \bar{q}^2$ and $\mathrm{F}'_{\mathrm{hv}2}(\Lambda) \cdot \eta^2 \bar{p}^2$ are identified as the monomials associated respectively to the literal parts $\bar{p}^2 \bar{q}^2$, $\eta^2 \bar{q}^2$, $\eta^2 \bar{p}^2$ of the polynomial (B.17).

Proof of (B.1)–(B.17) Using (4.3), we write the Euclidean distance of $x^{(i)}$ and $x^{(j)}$ as

$$|x^{(i)} - x^{(j)}| = |\mathfrak{R}_i x^{(i)}_{\mathrm{pl}} - \mathfrak{R}_j x^{(j)}_{\mathrm{pl}}| = |x^{(i)}_{\mathrm{pl}} - \mathfrak{R}_{ij} x^{(j)}_{\mathrm{pl}}|, \quad \text{where } \mathfrak{R}_{ij} := \mathfrak{R}^{\mathrm{t}}_i \mathfrak{R}_j,$$

so as to write f_{av} as[75]

$$f_{\mathrm{av}} = -\sum_{1 \le i < j \le n} \frac{m_i m_j}{4\pi^2} \int_{\mathbb{T}^2} \frac{d\lambda_i \, d\lambda_j}{|x^{(i)}_{\mathrm{pl}} - \mathfrak{R}_{ij} x^{(j)}_{\mathrm{pl}}|}. \tag{B.18}$$

Since $x^{(i)}_{\mathrm{pl}}$ and $x^{(j)}_{\mathrm{pl}}$ have vanishing third components, we can substitute, into (B.18), the matrix $\mathfrak{R}_{ij}$ with $\mathfrak{R}^{(2)}_{ij}$, defined as its upper left submatrix of order 2. It is easy to see, using (4.7), (A.17) and (A.18) that $\mathfrak{R}^{(2)}_{ij}$ even in z. We

[75] As we have already observed, also in the variables (Λ, λ, z) the averaged perturbation coincides with the average of the averaged Newtonian potential (B.18).

therefore denote

$$\mathfrak{R}_{ij}^{(2)} = \mathrm{id} + \mathfrak{q}_{ij} + \mathfrak{Q}_{ij} + \mathrm{O}(|z|^6) \tag{B.19}$$

its expansion up to the fourth order, where $\mathfrak{q}_{ij}$ and $\mathfrak{Q}_{ij}$ are respectively matrices of order 2 of homogeneous polynomials in z of degree 2, 4 respectively.

We interrupt for a while the description of the expansion, just to point out how the 2×2 matrices $\mathfrak{q}_{ij}$, $\mathfrak{Q}_{ij}$ may be computed. This is done in the following

Remark B.1 At first, by (4.7), write $\mathfrak{R}_{ij}$ as

$$\mathfrak{R}_{ij} = \mathfrak{R}_i^{\mathrm{t}}\mathfrak{R}_j = (\mathcal{R}_i)^{\mathrm{t}}(\mathcal{R}_i^*)^{\mathrm{t}} \cdots (\mathcal{R}_{j-1}^*)^{\mathrm{t}}\mathcal{R}_j = (\mathcal{R}_i)^{\mathrm{t}}(\mathcal{R}_i^*)^{\mathrm{t}} \cdots (\mathcal{R}_{j-2}^*)^{\mathrm{t}}\bar{\mathcal{R}}_{j-1}$$
$$\text{for } 1 \le i < j \le n \tag{B.20}$$

where $\mathcal{R}_i := \mathrm{id}$,

$$\bar{\mathcal{R}}_{j-1} := (\mathcal{R}_{j-1}^*)^{\mathrm{t}}\mathcal{R}_j \quad \text{for } 1 \le j-1 \le n-1 \tag{B.21}$$

and the products $(\mathcal{R}_i^*)^{\mathrm{t}} \cdots (\mathcal{R}_{j-1}^*)$ do not appear for $j = i+1$. Notice that the matrix $\bar{\mathcal{R}}_{j-1} = (\mathcal{R}_{j-1}^*)^{\mathrm{t}}\mathcal{R}_j$ has the expression (compare (A.17) and (A.18))

$$\begin{aligned}\bar{\mathcal{R}}_{j-1} = (\mathcal{R}_{j-1}^*)^{\mathrm{t}}\mathcal{R}_j &= \mathcal{R}_{313}\big(\arg(p_{j-1}, q_{j-1}), i_{j-1}^* + i_j\big)\\ &= \mathcal{R}_{313}\big(\arg(p_{j-1}, q_{j-1}), \bar{i}_{j-1}\big)\end{aligned} \tag{B.22}$$

where

$$\bar{i}_{j-1} := i_{j-1}^* + i_j \quad \text{for } 1 \le j-1 \le n-1. \tag{B.23}$$

By the definition of i_{j-1}^* and i_j and (B.23), the inclination $\bar{i}_{j-1}$ corresponds to be the outer angle of Ψ_{j-2} and Γ_j in the triangle of Ψ_{j-2}, Γ_j, Ψ_{j-1}. Hence, with the same notations as in Appendix A.2, using (A.20)

$$\begin{aligned}1 - \cos\bar{i}_{j-1} &= 1 - \frac{\Psi_{j-1}^2 - \Psi_{j-2}^2 - \Gamma_j^2}{2\Gamma_j\Psi_{j-2}} = \frac{(\Gamma_j + \Psi_{j-2})^2 - \Psi_{j-1}^2}{2\Gamma_j\Psi_{j-2}}\\ &= \frac{(\Gamma_j + \Psi_{j-2} - \Psi_{j-1})(\Gamma_j + \Psi_{j-2} + \Psi_{j-1})}{2\Gamma_j\Psi_{j-2}}\\ &= (p_{j-1}^2 + q_{j-1}^2)\bar{\mathfrak{c}}_{j-1}\\ \text{where } \bar{\mathfrak{c}}_{j-1} &:= \frac{2\mathfrak{L}_j - |z_{j-2}|^2 - 2\rho_j - r_{j-1}}{(2\Lambda_j - 2\rho_j)(2\mathfrak{L}_{j-1} - |z_{j-2}^2|)} \ (j \ge 2)\end{aligned} \tag{B.24}$$

Then, by (B.22) and (A.26), $\bar{\mathcal{R}}_{j-1}$ has the expression

$$\bar{\mathcal{R}}_{j-1} = \begin{pmatrix} 1-q_{j-1}^2\bar{\mathfrak{c}}_{j-1} & -p_{j-1}q_{j-1}\bar{\mathfrak{c}}_{j-1} & -q_{j-1}\bar{\mathfrak{s}}_{j-1} \\ -p_{j-1}q_{j-1}\bar{\mathfrak{c}}_{j-1} & 1-p_{j-1}^2\bar{\mathfrak{c}}_{j-1} & -p_{j-1}\bar{\mathfrak{s}}_{j-1} \\ q_{j-1}\bar{\mathfrak{s}}_{j-1} & p_{j-1}\bar{\mathfrak{s}}_{j-1} & 1-(p_{j-1}^2+q_{j-1}^2)\bar{\mathfrak{c}}_{j-1} \end{pmatrix} \tag{B.25}$$

where

$$\bar{\mathfrak{s}}_{j-1} := \frac{\sin \bar{i}_{j-1}}{\sqrt{p_{j-1}^2+q_{j-1}^2}} = \sqrt{\bar{\mathfrak{c}}_{j-1}\big(2-(p_{j-1}^2+q_{j-1}^2)\bar{\mathfrak{c}}_{j-1}\big)}. \tag{B.26}$$

(i) To compute the second order term $\mathfrak{q}_{ij}$ of $\mathfrak{R}_{ij}^{(2)}$, where $\mathfrak{R}_{ij}^{(2)}$ is the submatrix of order 2 of the matrix $\mathfrak{R}_{ij}$ in (B.20), we truncate the matrices $\mathcal{R}_i$, $\mathcal{R}_i^*$, $\bar{\mathcal{R}}_i$ to the second order and denote as S_i, S_i^*, $\bar{\mathrm{S}}_i$ the respective truncations. In fact, by (B.20), the matrix $\mathfrak{q}_{ij}$ can be obtained taking the term of degree 2 of the submatrix of order 2 of the products

$$(\mathrm{S}_i)^{\mathrm{t}}(\mathrm{S}_i^*)^{\mathrm{t}}\cdots(\mathrm{S}_{j-2}^*)^{\mathrm{t}}\bar{\mathrm{S}}_{j-1} \quad \text{for } 1\le i<j\le n \ (\mathrm{S}_1 := \mathrm{id}) \tag{B.27}$$

of the respective truncated matrices. By (4.8), (B.25) and the definitions (A.23), (A.27), (B.24) and (B.26) of the functions $\mathfrak{c}_i$, $\mathfrak{c}_i^*$, $\bar{\mathfrak{c}}_i$, $\mathfrak{s}_i$, $\mathfrak{s}_i^*$, $\bar{\mathfrak{s}}_i$, one sees that the expressions of the respective truncated matrices are

$$\mathrm{S}_{i+1} = \mathrm{S}(c_{i+1}, p_i, q_i), \qquad \mathrm{S}_i^* = \mathrm{S}(c_i^*, p_i, q_i), \qquad \bar{\mathrm{S}}_i = \mathrm{S}(\bar{c}_i, p_i, q_i)$$
$$1\le i\le n-1$$

where c_{i+1}, c_i^*, $\bar{c}_i$, are defined in as in (B.9) and $\mathrm{S}(c, p, q)$ denotes

$$\mathrm{S}(c,p,q) := \begin{pmatrix} 1-\frac{1}{2}c^2q^2 & -\frac{1}{2}c^2pq & -cq \\ -\frac{1}{2}c^2pq & 1-\frac{1}{2}c^2p^2 & -cp \\ cq & cp & 1-\frac{1}{2}c^2(p^2+q^2) \end{pmatrix}.$$

It is quite immediate to check that, if

$$\alpha = (\alpha_1,\dots,\alpha_m), \qquad y=(y_1,\dots,y_m), \qquad x=(x_1,\dots,x_m)\in\mathbb{R}^m,$$

then, the second order term of the principal submatrix of order 2 of the product

$$\mathrm{S}(\alpha_1,y_1,x_1)\cdots\mathrm{S}(\alpha_m,y_m,x_m)$$

is given by the matrix

$$\mathfrak{q}(\alpha, y, x) = -\frac{1}{2}\begin{pmatrix} (\alpha\cdot x)^2 & (\alpha\cdot y)(\alpha\cdot x) - \Delta(\alpha, y, x) \\ (\alpha\cdot y)(\alpha\cdot x) + \Delta(\alpha, y, x) & (\alpha\cdot y)^2 \end{pmatrix} \tag{B.28}$$

where $\Delta(\alpha, y, x)$ denotes

$$\Delta(\alpha, y, x) := \sum_{1\le h<k\le m} \alpha_h\alpha_k(y_h x_k - y_k x_h) \quad (\text{when } m \ge 2).$$

Then, taking the quadratic part $\mathfrak{q}_{ij}$ in the products in (B.27), the entries $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ of the matrix $\mathfrak{q}_{ij} = \left(\begin{smallmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij}\end{smallmatrix}\right)$ are found using (B.28) with

$$\begin{aligned} &m = \begin{cases} j-1 & i=1 \\ j-i+1 & i\ge 2 \end{cases} \qquad \alpha = \alpha_{ij} \\ &y = \begin{cases} (p_1, \dots p_{j-1}) & i=1 \\ (p_{i-1}, \dots, p_{j-1}) & i\ge 2 \end{cases} \qquad x = \begin{cases} (q_1, \dots, q_{j-1}) & i=1 \\ (q_{i-1}, \dots, q_{j-1}) & i \ge 2 \end{cases} \end{aligned} \tag{B.29}$$

with α_{ij} as in (B.10). Therefore, one finds

$$\begin{aligned} a_{ij} &= -\frac{1}{2}\Big(\sum_{1\le k\le n-1} (\mathcal{L}_{ik} - \mathcal{L}_{jk}) q_k\Big)^2 \\ b_{ij} &= -\frac{1}{2}\sum_{1\le h,k\le n-1} (\mathcal{L}_{ik} - \mathcal{L}_{jk})(\mathcal{L}_{ih} - \mathcal{L}_{jh}) p_h q_k \\ &\qquad + \frac{1}{2}\sum^{*}_{1\le h<k\le n-1} (\mathcal{L}_{ih} - \mathcal{L}_{jh})(\mathcal{L}_{ik} - \mathcal{L}_{jk})(p_h q_k - p_k q_h) \\ c_{ij} &= -\frac{1}{2}\sum_{1\le k,h\le n-1} (\mathcal{L}_{ik} - \mathcal{L}_{jk})(\mathcal{L}_{ih} - \mathcal{L}_{jh}) p_h q_k \\ &\qquad - \frac{1}{2}\sum^{*}_{1\le h<k\le n-1} (\mathcal{L}_{ih} - \mathcal{L}_{jh})(\mathcal{L}_{ik} - \mathcal{L}_{jk})(p_h q_k - p_k q_h) \\ d_{ij} &= -\frac{1}{2}\Big(\sum_{1\le k\le n-1} (\mathcal{L}_{ik} - \mathcal{L}_{jk}) p_k\Big)^2 \end{aligned} \tag{B.30}$$

where the asterisk in $\sum^*$ means that such sums exist only when $n \ge 3$; we have also used the first relation in (B.10).

(ii) The fourth order matrix $\mathfrak{Q}_{ij}$ has a less explicit expression. To compute it, we have to consider the fourth order truncations of the matrices $\mathcal{R}_i$, $\mathcal{R}_i^*$, $\bar{\mathcal{R}}_i$. By (A.23), (A.27), (B.24), (B.26) one easily sees that the functions $\mathfrak{c}_i$, $\mathfrak{c}_i^*$, $\bar{\mathfrak{c}}_i$, $\mathfrak{s}_i$, $\mathfrak{s}_i^*$, $\bar{\mathfrak{s}}_i$ verify

$$\mathfrak{c}_i = \frac{c_i^2}{2} + Q_i + \mathrm{O}(|\bar{z}|^4), \qquad \mathfrak{s}_i = \mathcal{S}(c_i, Q_i, r_{i-1}) + \mathrm{O}(|\bar{z}|^4)$$

$$\mathfrak{c}_i^* = \frac{(c_i^*)^2}{2} + Q_i^* + \mathrm{O}(|\bar{z}|^4), \qquad \mathfrak{s}_i^* = \mathcal{S}(c_i^*, Q_i^*, r_i) + \mathrm{O}(|\bar{z}|^4)$$

$$\bar{\mathfrak{c}}_i = \frac{\bar{c}_i^2}{2} + \bar{Q}_i + \mathrm{O}(|\bar{z}|^4), \qquad \bar{\mathfrak{s}}_i = \mathcal{S}(\bar{c}_i, \bar{Q}_i, r_i) + \mathrm{O}(|\bar{z}|^4)$$

where c_i, c_i^* $\bar{c}_i$, Q_i, Q_i^*, $\bar{Q}_i$ and $\mathcal{S}(c, Q, r)$ are defined in (B.9), (B.11) and (B.14). Then, since Q_i, Q_i^*, $\bar{Q}_i$, $\mathcal{S}_i$, $\mathcal{S}_i^*$, $\bar{\mathcal{S}}_i$ are $\mathrm{O}(|\bar{z}|^2)$, the fourth order truncations of the matrices $\mathcal{R}_i$, $\mathcal{R}_i^*$, $\bar{\mathcal{R}}_i$ are given by the matrices T_i, T_i^*, $\bar{\mathrm{T}}_i$ into (B.15). Hence,

The matrix $\mathfrak{Q}_{ij}$ into (B.19) *of* $\mathfrak{R}_{ij}^{(2)}$ *is uniquely identified as the principal submatrix of order* 2 *of the fourth order term of the matrix* T_{ij} *into* (B.16).

We proceed with the expansion of the function (B.18). Using (B.19), we write the squared Euclidean distance $D_{ij} := |x_{\mathrm{pl}}^{(i)} - \mathfrak{R}_{ij} x_{\mathrm{pl}}^{(j)}|^2$ as

$$\begin{aligned} D_{ij} = |x_{\mathrm{pl}}^{(i)} - \mathfrak{R}_{ij} x_{\mathrm{pl}}^{(j)}|^2 &= |x_{\mathrm{pl}}^{(i)} - \mathfrak{R}_{ij}^{(2)} x_{\mathrm{pl}}^{(j)}|^2 \\ &= |x_{\mathrm{pl}}^{(i)}|^2 + |x_{\mathrm{pl}}^{(j)}|^2 - 2x_{\mathrm{pl}}^{(i)} \cdot \mathfrak{R}_{ij}^{(2)} x_{\mathrm{pl}}^{(j)} \\ &= |x_{\mathrm{pl}}^{(i)} - x_{\mathrm{pl}}^{(j)}|^2 - 2x_{\mathrm{pl}}^{(i)} \cdot \mathfrak{q}_{ij} x_{\mathrm{pl}}^{(j)} \\ &\quad - 2x_{\mathrm{pl}}^{(i)} \cdot \mathfrak{Q}_{ij} x_{\mathrm{pl}}^{(j)} + \mathrm{O}(|\bar{z}|^6). \end{aligned}$$

We have then the following expansion for the inverse Euclidean distance

$$\begin{aligned} \frac{1}{|x_{\mathrm{pl}}^{(i)} - \mathfrak{R}_{ij} x_{\mathrm{pl}}^{(j)}|} &= \frac{1}{|x_{\mathrm{pl}}^{(i)} - x_{\mathrm{pl}}^{(j)}|} + \frac{x_{\mathrm{pl}}^{(i)} \cdot \mathfrak{q}_{ij} x_{\mathrm{pl}}^{(j)}}{|x_{\mathrm{pl}}^{(i)} - x_{\mathrm{pl}}^{(j)}|^3} + \frac{x_{\mathrm{pl}}^{(i)} \cdot \mathfrak{Q}_{ij} x_{\mathrm{pl}}^{(j)}}{|x_{\mathrm{pl}}^{(i)} - x_{\mathrm{pl}}^{(j)}|^3} \\ &\quad + \frac{3}{2} \frac{(x_{\mathrm{pl}}^{(i)} \cdot \mathfrak{q}_{ij} x_{\mathrm{pl}}^{(j)})^2}{|x_{\mathrm{pl}}^{(i)} - x_{\mathrm{pl}}^{(j)}|^5} + \mathrm{O}(|\bar{z}|^6). \end{aligned} \tag{B.31}$$

Then, multiplying this expression by $-m_i m_i$, taking the average over $\lambda_i, \lambda_j \in \mathbb{T}$ and the sum for $1 \le i < j \le n$, we split $f_{\rm av}$ as

$$f_{\rm av}(\Lambda, \bar z) = f_{\rm h}(\Lambda, (\eta, \xi)) + f^{(1)}_{\rm hv}(\Lambda, \bar z) + f^{(2)}_{\rm hv}(\Lambda, \bar z) + f_{\rm v}\big(\Lambda, \bar z\big) + {\rm O}(|\bar z|^6)$$

where

$$\begin{aligned}
f_{\rm h}\big(\Lambda, (\eta, \xi)\big) &:= -\sum_{1\le i<j\le n} \frac{m_i m_j}{4\pi^2} \int_{\mathbb{T}^2} \frac{d\lambda_i\, d\lambda_j}{|x^{(i)}_{\rm pl} - x^{(j)}_{\rm pl}|} \\
f^{(1)}_{\rm hv}(\Lambda, \bar z) &:= -\sum_{1\le i<j\le n} \frac{m_i m_j}{4\pi^2} \int_{\mathbb{T}^2} \frac{x^{(i)}_{\rm pl} \cdot \mathfrak{q}_{ij} x^{(j)}_{\rm pl}}{|x^{(i)}_{\rm pl} - x^{(j)}_{\rm pl}|^3} d\lambda_i\, d\lambda_j \\
f^{(2)}_{\rm hv}(\Lambda, \bar z) &:= -\sum_{1\le i<j\le n} \frac{m_i m_j}{4\pi^2} \int_{\mathbb{T}^2} \frac{x^{(i)}_{\rm pl} \cdot \mathfrak{Q}_{ij} x^{(j)}_{\rm pl}}{|x^{(i)}_{\rm pl} - x^{(j)}_{\rm pl}|^3} d\lambda_i\, d\lambda_j \\
f_{\rm v}(\Lambda, \bar z) &:= -\sum_{1\le i<j\le n} \frac{m_i m_j}{4\pi^2} \int_{\mathbb{T}^2} \frac{3}{2} \frac{(x^{(i)}_{\rm pl} \cdot \mathfrak{q}_{ij} x^{(j)}_{\rm pl})^2}{|x^{(i)}_{\rm pl} - x^{(j)}_{\rm pl}|^5} d\lambda_i\, d\lambda_j
\end{aligned} \tag{B.32}$$

Remark B.2 Notice that each of the previous functions is rotation invariant, i.e., it is invariant under the transformations $(\Lambda, \lambda, \bar z) \to \mathcal{R}^g(\Lambda, \lambda, \bar z)$ defined in (6.2). Indeed, when $(\Lambda, \lambda, \bar z) \to \mathcal{R}^g(\Lambda, \lambda, \bar z)$, the matrices $\mathfrak{R}_{ij} = \mathfrak{R}^{\rm t}_i \mathfrak{R}_j$ transform as

$$\mathfrak{R}_{ij} \to \mathcal{R}(g)\mathfrak{R}_{ij}\mathcal{R}(-g)$$

since the same holds for the matrices $\mathfrak{R}_i$ (compare Remark A.1) and the planar Poincaré map as $x^{(i)}_{\rm pl} \to \mathcal{R}(g)x^{(i)}_{\rm pl}$. Then, the scalar products $x^{(i)}_{\rm pl} \cdot \mathfrak{R}^{(2)}_{ij} x^{(j)}_{\rm pl} = x^{(i)}_{\rm pl} \cdot \mathfrak{R}_{ij} x^{(j)}_{\rm pl}$ are rotation invariant and hence (since they are so term by term) the scalar products $x^{(i)}_{\rm pl} \cdot \mathfrak{q}_{ij} x^{(j)}_{\rm pl}$ and $x^{(i)}_{\rm pl} \cdot \mathfrak{Q}_{ij} x^{(j)}_{\rm pl}$ are. The rotation invariance of $f_{\rm h}$, $f^{(1)}_{\rm hv}$, $f^{(2)}_{\rm hv}$, $f_{\rm v}$ then follows. Also invariance by the reflections (6.6) can easily be checked. Noticing finally that

- $f_{\rm h}$ depends only on (η, ξ);
- $f^{(1)}_{\rm hv}$ is ${\rm O}(|(\bar p, \bar q)|^2)$ and its quartic part vanishes for $(\eta, \xi) = 0$;
- $f^{(2)}_{\rm hv}$ is ${\rm O}(|\bar z|^4)$ and vanishes for $(\bar p, \bar q) = 0$;
- $f_{\rm v}$ is ${\rm O}(|(\bar p, \bar q)|^4)$;

the following expansions hold

$$f_{\rm h} = C_0(\Lambda) + {\rm Q}_{\rm h} \cdot \frac{\eta^2 + \xi^2}{2} + {\rm F}_{\rm h} \cdot \frac{\eta^4 + \xi^4 + 2\eta^2\xi^2}{2} + {\rm O}(|(\eta, \xi)|^6)$$

$$
\begin{aligned}
f_{\mathrm{hv}}^{(1)} &= \bar{\mathrm{Q}}_{\mathrm{v}} \cdot \frac{\bar{p}^2+\bar{q}^2}{2} + \mathrm{F}_{\mathrm{hv}1} \cdot \frac{\eta^2\bar{q}^2+\xi^2\bar{p}^2-2\eta\xi\,\bar{p}\bar{q}}{2} \\
&\quad + \mathrm{F}'_{\mathrm{hv}1} \cdot \frac{\eta^2\bar{p}^2+\xi^2\bar{q}^2+2\eta\xi\,\bar{p}\bar{q}}{2} + \mathrm{O}(|\bar{z}|^6) \\
f_{\mathrm{hv}}^{(2)} &= \mathrm{F}_{\mathrm{hv}2} \cdot \frac{\eta^2\bar{q}^2+\xi^2\bar{p}^2-2\eta\xi\,\bar{p}\bar{q}}{2} + \mathrm{F}'_{\mathrm{hv}2} \cdot \frac{\eta^2\bar{p}^2+\xi^2\bar{q}^2+2\eta\xi\,\bar{p}\bar{q}}{2} \\
&\quad + \mathrm{F}_{\mathrm{v}2} \cdot \frac{\bar{p}^4+\bar{q}^4+2\bar{p}^2\bar{q}^2}{2} + \mathrm{O}(|\bar{z}|^6) \\
f_{\mathrm{v}} &= \mathrm{F}_{\mathrm{v}1} \cdot \frac{\bar{p}^4+\bar{q}^4+2\bar{p}^2\bar{q}^2}{2} + \mathrm{O}(|\bar{z}|^6)
\end{aligned}
\tag{B.33}
$$

- *The tensor* $\mathrm{F}_{\mathrm{v}1}$

This tensor defines the quartic part of the function f_{v} in (B.32). Since $\mathfrak{q}_{ij}$ is a matrix of degree 2 in $\bar{z}$, to compute $\mathrm{F}_{\mathrm{v}1}$, it suffices to truncate the planar Poincaré map $x_{\mathrm{pl}}^{(i)}$ to its zero order, i.e., to substitute $x_{\mathrm{pl}}^{(i)}$ with

$$
x_{\mathrm{tr}0}^{(i)} = a_i(\cos\lambda_i, \sin\lambda_i) \quad \text{where } a_i = \frac{1}{\bar{m}_i}\left(\frac{\Lambda_i}{M_i}\right)^2. \tag{B.34}
$$

If $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ are the entries of the matrix $\mathfrak{q}_{ij}$ as in (B.30), we have

$$
\begin{aligned}
&\mathrm{F}_{\mathrm{v}1} \cdot \frac{\bar{p}^4+\bar{q}^4+2\bar{p}^2\bar{q}^2}{2} \\
&\quad = -\sum_{1\le i<j\le n} \frac{m_i m_j}{4\pi^2} \int_{\mathbb{T}^2} \frac{3}{2} \\
&\qquad \times \frac{a_i^2 a_j^2 (a_{ij}\cos\lambda_i\cos\lambda_j + b_{ij}\cos\lambda_i\sin\lambda_j + c_{ij}\sin\lambda_i\cos\lambda_j + d_{ij}\sin\lambda_i\sin\lambda_j)^2}{(a_i^2+a_j^2-2a_ia_j\cos(\lambda_i-\lambda_j))^{5/2}}\, d\lambda_i\, d\lambda_j
\end{aligned}
$$

Computing the integral easily gives the quartic form

$$
\begin{aligned}
\mathrm{F}_{\mathrm{v}1} \cdot \frac{\bar{p}^4+\bar{q}^4+2\bar{p}^2\bar{q}^2}{2} = -\sum_{1\le i<j\le n} \frac{3}{16}\frac{a_i^2}{a_j^3}((a_{ij}^2+b_{ij}^2+c_{ij}^2+d_{ij}^2) \\
\times (2\beta_0^{(5)}+\beta_2^{(5)}) + 2(a_{ij}d_{ij}-b_{ij}c_{ij})\beta_2^{(5)})m_i m_j
\end{aligned}
$$

Using the expressions (B.30) and selecting the monomial with literal part $\bar{p}^2\bar{q}^2$, we find the result in (B.7) and (B.8).

- *The tensors* $\mathrm{F}_{\mathrm{hv}2}$ *and* $\mathrm{F}'_{\mathrm{hv}2}$ *and* $\mathrm{F}_{\mathrm{v}2}$

Such tensors define the quartic part of the function $f_{\rm hv}^{(2)}$ into (B.32). Since $\mathfrak{Q}_{ij}$ is of degree 4 in $\bar z$, to compute them we substitute, as before, $x_{\rm pl}^{(i)}$ with its 0-order truncation (B.34). The computation of the integral is immediate and gives the quartic form

$$\begin{aligned}
&\mathrm{F}_{\rm hv2}\cdot\frac{\eta^2\bar q^2+\xi^2\bar p^2-2\eta\xi\,\bar p\bar q}{2}+\mathrm{F}'_{\rm hv2}\cdot\frac{\eta^2\bar p^2+\xi^2\bar q^2+2\eta\xi\,\bar p\bar q}{2}\\
&\qquad+\mathrm{F}_{\rm v2}\cdot\frac{\bar p^4+\bar q^4+2\bar p^2\bar q^2}{2}\\
&\quad=-\sum_{1\le i<j\le n}\frac{m_im_j}{4\pi^2}\int_{\mathbb{T}^2}\frac{x_{\rm tr0}^{(i)}\cdot\mathfrak{Q}_{ij}x_{\rm tr0}^{(j)}}{|x_{\rm tr0}^{(i)}-x_{\rm tr0}^{(j)}|^{3/2}}=-\sum_{1\le i<j\le n}\frac{m_im_j}{4\pi^2}\\
&\qquad\times\int_{\mathbb{T}^2}\frac{a_ia_j(A_{ij}\cos\lambda_i\cos\lambda_j+B_{ij}\cos\lambda_i\sin\lambda_j+C_{ij}\sin\lambda_i\cos\lambda_j+D_{ij}\sin\lambda_i\sin\lambda_j)}{(a_i^2+a_j^2-2a_ia_j\cos(\lambda_i-\lambda_j))^{3/2}}\,d\lambda_i\,d\lambda_j\\
&\quad=-\sum_{1\le i<j\le n}m_im_j\frac{a_i}{2a_j^2}(A_{ij}+D_{ij})\beta_1^{(3)}=\sum_{1\le i<j\le n}m_im_jC_1(a_i,a_j)\,\mathrm{tr}\,\mathfrak{Q}_{ij}
\end{aligned}\tag{B.35}$$

where A_{ij}, B_{ij}, C_{ij}, D_{ij} denote the entries of $\mathfrak{Q}_{ij}=\begin{pmatrix}A_{ij}&B_{ij}\\C_{ij}&D_{ij}\end{pmatrix}$ and $C_1(a_i,a_j)$ is as in (B.2).

- *The tensor* $\mathrm{F}_{\rm h}$

For the computation of the tensor $\mathrm{F}_{\rm h}$, defining the quartic part of $f_{\rm h}$ (see (B.32)), we have to consider the truncation of $x_{\rm pl}^{(i)}=(\mathrm{x}_1^{(i)},\mathrm{x}_2^{(i)},0)$ up to degree 2 in η and ξ separately. Let such truncation be denoted as $x_{\rm tr4}^{(i)}=(\mathrm{x}_{1,\rm tr4}^{(i)},\mathrm{x}_{2,\rm tr4}^{(i)},0)$. Proceeding as in [4], we write the regularized Kepler equation (4.6) in the form

$$v_i=s_i\,\sin v_i+t_i\,\cos v_i,\quad\text{where }v_i:=u_i-\lambda_i\tag{B.36}$$

and

$$\begin{aligned}
s_i&:=\frac{1}{\sqrt{\Lambda_i}}\sqrt{1-\frac{\eta_i^2+\xi_i^2}{4\Lambda_i}}\,(\eta_i\cos\lambda_i-\xi_i\sin\lambda_i)\\
&=\frac{1}{\sqrt{\Lambda_i}}\left(1-\frac{\eta_i^2+\xi_i^2}{8\Lambda_i}\right)(\eta_i\cos\lambda_i-\xi_i\sin\lambda_i)+\mathrm{O}(|(\eta_i,\xi_i)|^5)
\end{aligned}\tag{B.37}$$

$$t_i:=\frac{1}{\sqrt{\Lambda_i}}\sqrt{1-\frac{\eta_i^2+\xi_i^2}{4\Lambda_i}}\,(\eta_i\sin\lambda_i+\xi_i\cos\lambda_i)$$

$$= \frac{1}{\sqrt{\Lambda_i}}\left(1 - \frac{\eta_i^2+\xi_i^2}{8\Lambda_i}\right)(\eta_i \sin\lambda_i + \xi_i\cos\lambda_i) + \mathrm{O}(|(\eta_i,\xi_i)|^5)$$

We then expand, from (B.36), the variable $v_i = u_i - \lambda_i$ in powers of (s_i, t_i)

$$v_i = t_i + s_i\, t_i + s_i^2\, t_i - \frac{1}{2}\, t_i^3 + s_i^3\, t_i - \frac{5}{3}\, s_i\, t_i^3 + \mathrm{O}(|(s_i,t_i)|^5) \tag{B.38}$$

Using (B.37) and (B.38) into (4.5), we find

$$\begin{aligned} \mathrm{x}^{(i)}_{1,\mathrm{tr4}} = \frac{1}{\bar m_i}\left(\frac{\Lambda_i}{M_i}\right)^2 \Bigg(& \cos\lambda_i + \frac{\eta_i}{2\sqrt{\Lambda_i}}(\cos 2\lambda_i - 3) - \frac{\xi_i}{2\sqrt{\Lambda_i}}\sin 2\lambda_i \\ & + \frac{3}{8\Lambda_i}\eta_i^2\,(\cos 3\lambda_i - \cos\lambda_i) - \frac{\eta_i\xi_i}{4\sqrt{\Lambda_i}}(3\,\sin 3\lambda_i + \sin\lambda_i) \\ & - \frac{\xi_i^2}{8\Lambda_i}(3\,\cos 3\lambda_i + 5\,\cos\lambda_i) - \frac{\eta_i^2\xi_i}{16\Lambda_i\sqrt{\Lambda_i}}(16\,\sin 4\lambda_i - 5\,\sin 2\lambda_i) \\ & - \frac{\eta_i\xi_i^2}{16\Lambda_i\sqrt{\Lambda_i}}(16\cos 4\lambda_i + 9\,\cos 2\lambda_i - 3) \\ & - \frac{\eta_i^2\,\xi_i^2}{64\Lambda_i^2}(125\,\cos 5\lambda_i + 9\,\cos 3\lambda_i - 14\,\cos\lambda_i)\Bigg) \end{aligned} \tag{B.39}$$

The expression of $\mathrm{x}^{(i)}_{2,\mathrm{tr4}}$ can be obtained from the right-hand-side of (B.39), letting $(\lambda_i, \eta_i, \xi_i) \to (\frac{\pi}{2} - \lambda_i, \xi_i, \eta_i)$. Using finally such truncated expressions into the definition of f_h in (B.32) will provide, through the computation of the $(\alpha_i, \alpha_j, \beta_i, \beta_j)$-derivatives with respect to $(\eta_i, \eta_j, \xi_i, \xi_j)$ over $\alpha_i!\alpha_j!\beta_i!\beta_j!$, the quartic form $\mathrm{F}_\mathrm{h}\cdot\eta^2\xi^2$ as in (B.5) and (B.6). We omit further details.

- *The tensors* $\bar{\mathrm{Q}}_\mathrm{v}$, $\mathrm{F}_{\mathrm{hv1}}$, $\mathrm{F}'_{\mathrm{hv1}}$

The tensors $\bar{\mathrm{Q}}_\mathrm{v}$, $\mathrm{F}_{\mathrm{hv1}}$, $\mathrm{F}'_{\mathrm{hv1}}$ define the expansion of the function $f^{(1)}_{\mathrm{hv}}$. For their computation (in particular, for the computation of the quartic tensors $\mathrm{F}_{\mathrm{hv1}}$ and $\mathrm{F}'_{\mathrm{hv1}}$), we shall put $\xi = 0$ and then shall select into the expansion of $f^{(1)}_{\mathrm{hv}}|_{\xi=0}$ the monomials with literal part $\bar p^2$, $\eta^2\bar q^2$, $\eta^2\bar p^2$ respectively.

By the previous paragraph, we find the following truncations of the Kepler map up to degree 2 in η and with $\xi = 0$

$$\begin{aligned} \mathrm{x}^{(i)}_{1,\mathrm{tr2}} = \frac{1}{\bar m_i}\left(\frac{\Lambda_i}{M_i}\right)^2 \Bigg(& \cos\lambda_i + \frac{\eta_i}{2\sqrt{\Lambda_i}}(\cos 2\lambda_i - 3) \\ & + \frac{\eta_i^2}{8\Lambda_i}(3\cos 3\lambda_i - 3\cos\lambda_i)\Bigg) \end{aligned} \tag{B.40}$$

$$\mathrm{x}^{(i)}_{2,\mathrm{tr}2} = \frac{1}{\bar{m}_i}\left(\frac{\Lambda_i}{M_i}\right)^2\left(\sin\lambda_i + \frac{\eta_i}{2\sqrt{\Lambda_i}}\sin 2\lambda_i + \frac{\eta_i^2}{8\Lambda_i}(3\sin 3\lambda_i - 5\sin\lambda_i)\right)$$

Denoting $x^{(i)}_{\mathrm{tr}2} := (\mathrm{x}^{(i)}_{1,\mathrm{tr}2}, \mathrm{x}^{(i)}_{2,\mathrm{tr}2})$, by the expression of $f^{(1)}_{\mathrm{hv}}$ in (B.32), since the matrix $\mathfrak{q}_{ij}$ is $\mathrm{O}(|(p,q)|^2)$ and $\mathrm{x}^{(i)}_{1,\mathrm{tr}2}$, $\mathrm{x}^{(i)}_{2,\mathrm{tr}2}$ are, respectively, even, odd in λ_i,

$$\begin{aligned} f^{(1)}_{\mathrm{hv}} &= -\sum_{1\le i<j\le n}\frac{m_i m_j}{4\pi^2}\int_{\mathbb{T}^2}\frac{x^{(i)}_{\mathrm{tr}2}\cdot\mathfrak{q}_{ij}x^{(j)}_{\mathrm{tr}2}}{|x^{(i)}_{\mathrm{tr}2} - x^{(j)}_{\mathrm{tr}2}|^3}d\lambda_i d\lambda_j + \mathrm{O}(|(\bar{p},\bar{q})|^2\eta^4)\\ &= -\sum_{1\le i<j\le n}\frac{m_i m_j}{4\pi^2}\int_{\mathbb{T}^2}\frac{a_{ij}\mathrm{x}^{(i)}_{1,\mathrm{tr}2}\mathrm{x}^{(j)}_{1,\mathrm{tr}2} + d_{ij}\mathrm{x}^{(i)}_{2,\mathrm{tr}2}\mathrm{x}^{(j)}_{2,\mathrm{tr}2}}{|x^{(i)}_{\mathrm{tr}2} - x^{(j)}_{\mathrm{tr}2}|^3}d\lambda_i\, d\lambda_j\\ &\quad + \mathrm{O}(|(\bar{p},\bar{q})|^2\eta^4) \end{aligned} \tag{B.41}$$

where a_{ij}, d_{ij} are the diagonal entries of $\mathfrak{q}_{ij}$ (given in (B.30)). In particular, if we put also $\eta = 0$, so to have $x^{(i)}_{\mathrm{tr}2} = x^{(i)}_{\mathrm{tr}0} = a_i(\cos\lambda_i, \sin\lambda_i)$, by (B.33) and (B.41), we can identify

$$\bar{\mathrm{Q}}_{\mathrm{v}}\cdot\frac{\bar{p}^2+\bar{q}^2}{2} = \sum_{1\le i<j\le n} m_i m_j C_1(a_i,a_j)\,\mathrm{tr}\,\mathfrak{q}_{ij}$$

(the computation being the same as in (B.35)) with $\mathrm{tr}\,\mathfrak{q}_{ij} = a_{ij} + d_{ij}$ the trace of $\mathfrak{q}_{ij}$. Using the expressions for a_{ij}, b_{ij} in (B.30) and selecting the monomial in $\bar{p}^2$, we find the result in (B.3).

If we use the whole expression for $\mathrm{x}^{(i)}_{1,\mathrm{tr}2}$, $\mathrm{x}^{(i)}_{2,\mathrm{tr}2}$ as in (B.40), computing the derivatives of order two with respect to (η_i,η_j), for $(\eta_i,\eta_j) = 0$, we then have the second order in η of $f^{(1)}_{\mathrm{hv}}$, which we identify with $\mathrm{F}_{\mathrm{hv}1}\cdot\eta^2\bar{q}^2 + \mathrm{F}'_{\mathrm{hv}1}\cdot\eta^2\bar{p}^2$. We omit the details of this straightforward computation. □

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