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Low-order resonances in weakly dissipative spin–orbit models ☆

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ABSTRACT

Second-order differential equations with small nonlinearity and weak dissipation, such as the spin–orbit model of celestial mechanics, are considered. Explicit conditions for the coexistence of periodic orbits and estimates on the measure of the basins of attraction of stable periodic orbits are discussed.

Contents

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1. Introduction and results

In this paper we consider second-order differential equations of the following type

$$\ddot{x} + \bar{\eta}(\dot{x} - \bar{\nu}) + \bar{\varepsilon} f_x(x, t) = 0, \tag{1}$$

where f is a smooth 2π-periodic both in $x \in \mathbb{R}$ and in $t \in \mathbb{R}$; $x = x(t)$ and dot denotes time derivative; $\bar{\eta}$, $\bar{\nu}$ and $\bar{\varepsilon}$ are nonnegative parameters.

This equation models, for $\bar{\eta}$ and $\bar{\varepsilon}$ small, a *nearly-integrable, weakly-dissipative system*: for $\bar{\eta} = 0$ the equation is conservative, being the Euler–Lagrange equation of the nearly-integrable Lagrangian

$$\mathcal{L}_\varepsilon(\dot{x}, x, t) := \frac{1}{2}\dot{x}^2 - \bar{\varepsilon} f(x, t),$$

and the associated dynamical system exhibits the reach dynamics of nearly-integrable Lagrangian systems (periodic orbits of any period, KAM tori, Aubry–Mather sets, etc.; see [2] for general information); on the other hand, for $\bar{\varepsilon} = 0$ and $\bar{\eta} > 0$ the equation is dissipative and all solutions are of the form

$$x(t) = x_0 + \bar{\nu}t + \frac{1 - \exp(-\bar{\eta}t)}{\bar{\eta}}(v_0 - \bar{\nu}),$$

and tend exponentially fast to the global attractor $\{\dot{x} = \bar{\nu}\}$.

Given two coprime positive integers p and q, it is particularly interesting to study the existence, stability and basins of attraction of "periodic orbits of type (p, q)" (or "(p, q)-periodic orbits"), i.e., solutions $t \in \mathbb{R} \to x_{pq}(t) = x(t) \in \mathbb{R}$ of (1) satisfying

$$x(t + T) = x(t) + 2\pi p, \quad \forall t, \tag{2}$$

with $T := 2\pi q$.

Remark 1.1. (i) Since the "potential" f in (1) is 2π-periodic in x and t, the (extended) phase space for Eq. (1) may be taken to be

$$\mathcal{M} := \big\{\big((x, t), y\big) \in \mathbb{T}^2 \times \mathbb{R}\big\},$$

where $\mathbb{T}^2$ is the standard 2-torus $\mathbb{R}^2/(2\pi\mathbb{Z}^2)$, so that an orbit satisfying (2) describes, projected on $\mathcal{M}$, a periodic trajectory with period $T = 2\pi q$ and winding (or rotation) number

$$\omega = \lim_{s\to\infty} \frac{1}{s} x(t + s) = \frac{p}{q}.$$

(ii) A solution of type (p,q) can be written as

$$x_{pq}(t) = \xi + \frac{p}{q}t + u\left(\frac{t}{q}\right), \quad \langle u \rangle = 0, \tag{3}$$

where $\xi \in \mathbb{R}$, u is 2π-periodic and[1]

$$\langle u \rangle := \frac{1}{2\pi}\int_0^{2\pi} u(s)\,ds = 0.$$

The main motivation for studying Eq. (1) and its (p,q)-periodic orbits, comes from celestial mechanics. In fact, Eq. (1) describes, in a suitable simplified model, the rotations of a satellite whose center of mass revolves on a fixed Keplerian orbit of eccentricity $\mathrm{e} \in [0,1)$ and is subject to the gravitational attraction of a major body sitting on one of the foci of the ellipse.

For such a model, (p,q)-periodic orbits correspond to $p:q$ *spin–orbit resonances*, i.e., to periodic motions where the satellite turns on its spin axis exactly p times while doing q revolutions around its star/planet.[2]

The simplifying physical assumptions and the meaning of the quantities appearing in Eq. (1) are the following (for precise definitions we refer to Section 2.5 and to Appendix A; for general information on spin–orbit resonances, see, e.g., [5,6,8,10–12,15] and [7]):

- the satellite is modeled by a nonsymmetric ellipsoid subject to the gravitational attraction of a pointmass star sitting on a focus of the Keplerian ellipse ("restricted model");
- we assume that the satellite has fixed vertical spin axis coinciding with the shortest physical axis ("no obliquity");
- the dissipation is modeled by a linear dependence upon the angular velocity (and it is meant to reflect the internal non-rigidity of the planet taking into account a time lag introduced by tides);
- the parameters $\bar\eta$ and $\bar\nu$ are real-analytic functions of e while $\bar\varepsilon$ is a measure of the oblateness of the satellite (Section 2.5); in typical examples in the Solar system (such as Moon–Earth or Mercury–Sun) $\bar\varepsilon \sim 10^{-4}$ and $\bar\eta \sim 10^{-8}$;
- the "Newtonian potential" f has the Fourier representation

$$f(x,t) = f(x,t;\mathrm{e}) = \sum_{\substack{j\in\mathbb{Z}\\ j\neq 0}} \alpha_j \cos(2x - jt), \tag{4}$$

  where α_j are suitable real-analytic (nontrivial) functions of e; see Section 2.5 and Appendix A for the analytic description.

We can now state our main results.

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[1] Indeed, from (2) it follows that $\tilde x(t) := x_{pq}(t) - \frac{p}{q}t$ is $2\pi q$-periodic, and ξ is easily recognized as the $\lim_{t\to\infty}\frac{1}{t}\int_0^t \tilde x(s)\,ds$.

[2] In the Solar system there are, including our Moon, 23 moon's observed in a 1 : 1 spin–orbit resonance and one planet (Mercury) observed in 3 : 2 spin–orbit resonance.

Theorem 1.2. *Let p and q be positive coprime integers with $q = 1, 2$ or 4 and fix $0 < \kappa < 1$. Then, there exist $\bar\varepsilon_0 > 0$ and $\bar\eta_0 > 0$ such that for any $0 < \bar\varepsilon \leqslant \bar\varepsilon_0$ and $0 \leqslant \bar\eta \leqslant \bar\eta_0$, the spin–orbit problem modeled by Eqs.* (1), (4) *has periodic solutions x_{pq} of type (p, q), provided*

$$\left|\bar\nu - \frac{p}{q}\right| < \begin{cases} 2\kappa \frac{\bar\varepsilon}{\bar\eta}|\alpha_{2p}| & \text{if } q = 1, \\ 2\kappa \frac{\bar\varepsilon}{\bar\eta}|\alpha_p| & \text{if } q = 2, \\ 16\kappa \frac{\bar\varepsilon^2}{\bar\eta}\left|\sum_{j\in\mathbb{Z},\, j\neq 0,p} \frac{\alpha_{p-j}\alpha_j}{(p-2j)^2}\right| & \text{if } q = 4. \end{cases} \tag{5}$$

Furthermore, representing the solution x_{pq} as in (3), *one has that u depends smoothly on $\bar\varepsilon$ and $|u| \leqslant c|\bar\varepsilon|$ for a suitable ($\bar\varepsilon$-independent) constant $c > 0$.*

The second result deals with the basins of attraction of stable (p, q)-orbits for low q.

Theorem 1.3. *Let x_{pq} be a (p, q)-periodic orbit of* (1) *as in Theorem* 1.2 *with $q = 1, 2$. Assume that*

$$\theta_0 := \frac{1}{2\pi}\int_0^{2\pi} f_{xx}(\xi + pt, qt)\, dt > 0. \tag{6}$$

There exist $0 < \bar\varepsilon_* \leqslant \bar\varepsilon_0$ and $0 < \bar\eta_* \leqslant \bar\eta_0$ and constants $0 < \bar c_1 \leqslant \bar c_2$ such that if $0 < \bar\varepsilon \leqslant \bar\varepsilon_*$, $0 \leqslant \bar\eta \leqslant \bar\eta_*$ and

$$\bar\eta^2 < \bar\varepsilon \min\{\theta_0, 1\}, \tag{7}$$

then any solution $x(t)$ with initial conditions sufficiently close to the initial conditions of x_{pq} tends exponentially fast to $x_{pq}(t)$; more precisely, if $x(t)$ is a solution of (1) *with*

$$\sqrt{\bar\varepsilon}\left|x(0) - x_{pq}(0)\right| + \left|\dot x(0) - \dot x_{pq}(0)\right| \leqslant \bar c_1 \bar\eta \tag{8}$$

then

$$\sqrt{\bar\varepsilon}\left|x(t) - x_{pq}(t)\right| + \left|\dot x(t) - \dot x_{pq}(t)\right| \leqslant \bar c_2 \bar\eta e^{-\bar\eta t/2}, \quad \forall t \geqslant 0. \tag{9}$$

Let us make a few comments.

(i) *Coexistence of low-period spin–orbit resonances.*[3]
Condition (5) yields a quantitative relation between the various quantities entering (1) and the integers (p, q) characterizing the periodic orbit. As it is evident, condition (5) may be satisfied by several couples (p, q) provided the dissipation $\bar\eta$ is smaller than the nonlinearity $\bar\varepsilon$. For example, considering the astronomical parameters of the Mercury–Sun spin–orbit model (i.e., $\mathrm{e} \simeq 0.2056$, $\bar\varepsilon = 10^{-4}$ and $\bar\eta \simeq \bar\varepsilon^2$) one can show that (p, q)-periodic orbits with $(p, q) = (1, 1), (5, 4), (3, 2), (2, 1), (5, 2)$ and $(3, 1)$ coexist; compare Appendix B. Furthermore, from Theorem 1.3 it follows that, for $q = 1, 2$, there are stable periodic orbits with a basin of attraction bounded below by $\bar\eta/\bar\varepsilon$.

(ii) *Quasi-periodic attractors.*[4]
It can be shown [6] that (1) (with f real-analytic) admits, *uniformly in* $\bar\eta \in [0, 1]$, also *quasi-periodic solutions* (attractors) $x(t) = \omega t + U(\omega t, t)$, with $U(\theta, t)$ analytic on $\mathbb{T}^2$ and ω Diophantine,

[3] For model problems exhibiting many coexisting periodic attractors with low periods, see [9].

[4] For general information on dissipative quasi-periodic attractors (nonuniform in the dissipation parameter) and their bifurcation analysis, see [3].

provided $\bar\varepsilon$ is small enough and the driving frequencies $\bar\nu$ is "finely tuned" with ω, i.e., it satisfies the compatibility condition $\bar\nu = \omega(1 + \langle U_\theta^2\rangle)$. At contrast with the coexistence of (p, q)-periodic orbits (and with the conservative case), such quasi-periodic attractors, when they exist, are unique.

(iii) *On the Mercury–Sun model.*
In [7] the spin–orbit problem (1), (4) has been numerically investigated with the scope of finding stable resonances together with their basins of attraction. For several astronomically parameter values, the occurrence of periodic and quasi-periodic attractors has been studied by a Monte Carlo method on the initial conditions; the percentage of initial data, which evolve towards an attractor has been computed and interpreted as a "basin-of-attraction measure," providing, in particular, a possible dynamical-system interpretation of the observed capture in the 3 : 2 spin–orbit resonance of Mercury. Some of the results in [7] are reported in Appendix B and compared with the theoretical predictions based upon Theorem 1.2 showing an excellent agreement.

(iv) *On the proof of Theorem* 1.2.
The proof of Theorem 1.2 (performed at the end of Section 2) is based upon a Lyapunov–Schmidt decomposition[5] of the functional equation satisfied by u in (3). Such functional equation can be written in the form

$$Lu = \Phi_\xi(u),$$

where L is the linear operator $\partial_t^2 + \bar\eta\partial_t$ and Φ_ξ is a nonlinear operator parameterized by the phase $\xi \in [0, 2\pi]$ (compare Section 2.1). The operator L has (on suitable function spaces) kernel formed by the constants and range formed by periodic functions with zero average; splitting accordingly the function spaces, one is led to consider the "range equation" and the "kernel (or bifurcation) equation." The range equation is easily solved, for $\bar\varepsilon$ small, by standard contraction mapping arguments, while the bifurcation equation reduces to an equation on $\mathbb{R}$ for the unknown parameter ξ. Actually, in the spin–orbit problem, the most delicate step consists in solving the bifurcation equation, which, in general is degenerate and requires higher-order analysis. In this paper, for simplicity and for the relevance in the Mercury–Sun model, we discuss only the $q = 1, 2$ case (corresponding to the nondegenerate case) and the $q = 4$ case (the first degenerate case).

(v) *On the proof of Theorem* 1.3.
The idea of the proof of Theorem 1.3 (performed in Section 3) is the following. Given a stable (p, q)-periodic orbit $x_{pq}(t) = \xi + pt/q + u(t/q)$, for which condition (6) holds, one studies solutions $x(t) = x_{pq}(t) + w(t)$ with $|w(0)| + |\dot w(0)|$ small. Setting $z(t) := e^{-\alpha t} w(t)$, with $\alpha = \bar\eta q/2$, one sees that z satisfies

$$\mathcal{L}z = \bar\varepsilon e^{\alpha t} Q\left(e^{-\alpha t} z\right),$$

where $\mathcal{L}$ is a Hill's operator of the form $\partial_t^2 + V_{\bar\varepsilon,\bar\eta}(t)$ with $V_{\bar\varepsilon,\bar\eta}$ 2π-periodic (and depending on x_{pq}), while Q is a *quadratic* operator (i.e. $|Q(v)| \leqslant \text{const} \cdot |v|^2$). By standard Floquet theory one can show that $\mathcal{L}$ is invertible, provided (7) holds and $\bar\varepsilon > 0$ is small. However, $V_{\bar\varepsilon,\bar\eta}(t) \to 0$ when $\bar\varepsilon \to 0$ and the Hill's operator $\mathcal{L}$ degenerates. So we have to perform a further analysis to estimate the blow-up of $\mathcal{L}^{-1}$ for $\bar\varepsilon > 0$ close to zero (see Remark 3.3 and Lemma 3.4). Finally simple a priori bounds show that (8) implies (9).

(vi) *Developments.*
Let us indicate a few possible developments and open problems.

- Extending the approach presented here, give lower bounds on the basins of attraction of the quasi-periodic attractors found in [6]. At this respect, let us remark that, numerically, the basins of attraction of quasi-periodic solutions appear to be much larger than those of periodic solutions (compare [7]).

[5] For a systematic usage of the Lyapunov–Schmidt decomposition in functional differential equations, see [1] and references therein; for an earlier related approach to the study of periodic solutions in the Kepler problem, see [14].

- Provide explicit conditions for the existence of (p,q)-periodic orbits for *any* q, showing, in particular, that one recovers, for $\bar\eta \to 0$, all the (p,q)-periodic solutions of the conservative case.
- Discuss lower bounds on the basins of attraction for any q.
- Prove (or disprove) that for $q = 1, 2$, the basin of attraction of a (p,q)-periodic orbit is actually "large."
- Discuss more general models (nonrestricted, nonplanar, obliquity, more general dissipations,...).

2. Periodic orbits and spin–orbit resonances

2.1. Functional equation

Let p and q be positive coprime integers, then $x_{pq}(t) = \xi + pt/q + u(t/q)$ is a (p,q)-periodic orbit of (1) if and only if u satisfies

$$u''(t/q) + \bar\eta q\big(u'(t/q) + p - \bar\nu q\big) + \bar\varepsilon q^2 f_x\big(\xi + pt/q + u(t/q), t\big) = 0. \tag{10}$$

Setting

$$\eta := q\bar\eta, \qquad \nu := q\bar\nu - p, \qquad \varepsilon := q^2\bar\varepsilon, \tag{11}$$

and replacing t with qt, (10) becomes

$$u''(t) + \eta\big(u'(t) - \nu\big) + \varepsilon f_x\big(\xi + pt + u(t), qt\big) = 0. \tag{12}$$

We can rewrite Eq. (12) as follows. Let

$$\begin{cases} Lu = L_\eta u := u'' + \eta u', \\ \big[\Phi_\xi(u)\big](t) := \big[\Phi_\xi(u; \eta\nu, \varepsilon, p, q)\big](t) := \eta\nu - \varepsilon f_x\big(\xi + pt + u(t), qt\big). \end{cases} \tag{13}$$

Then, (12) is equivalent to

$$Lu = \Phi_\xi(u). \tag{14}$$

Remark 2.1. (i) In Eq. (14), $\eta, \nu, \varepsilon, p, q$ are parameters, while the unknowns are the (2π-periodic with zero average) function u and the "phase" ξ.

(ii) The nonlinear operator Φ_ξ is 2π-periodic in ξ, therefore, from now on we shall consider $\xi \in [0, 2\pi]$.

(iii) The kernel and the range of the linear operator L are, roughly speaking, the constants and the zero average functions, respectively.

2.2. Green operator

The linear operator L defined in (13) is invertible on the space of periodic functions with zero average; here we describe its inverse operator $G = L^{-1}$.

Let $C^k_{\rm per}$ be the Banach space of $C^k(\mathbb{R})$ functions 2π-periodic endowed with the C^k-norm,[6] let $C^k_{{\rm per},0}$ be the closed subspace of $C^k_{\rm per}$ formed by functions with vanishing average over $[0, 2\pi]$, and denote

$$\mathbb{B} := C^0_{{\rm per},0}.$$

[6] $\|v\|_{C^k} := \sum_{0 \leqslant j \leqslant k} \sup|D^j v|$.

Fix $\eta_0 > 0$. Let $\eta \in [0, \eta_0]$ and, for $g \in \mathbb{B}$, define the linear "Green operator" $G = G_\eta$ by[7]

$$(G_\eta g)(t) := x_\eta + y_\eta \frac{1-e^{-\eta t}}{\eta} + \int_0^t e^{-\eta\tau} \int_0^\tau e^{\eta s} g(s)\, ds\, d\tau \tag{15}$$

$$= x_\eta + y_\eta \frac{1-e^{-\eta t}}{\eta} + \int_0^t \frac{1-e^{\eta(s-t)}}{\eta} g(s)\, ds, \tag{16}$$

with

$$x_\eta := -y_\eta \frac{e^{-2\pi\eta} - 1 + 2\pi\eta}{2\pi\eta^2} - \frac{1}{2\pi\eta^2} \int_0^{2\pi} \big(e^{\eta(s-2\pi)} - 1 - \eta(s-2\pi)\big) g(s)\, ds, \tag{17}$$

$$y_\eta := \frac{1}{e^{-2\pi\eta} - 1} \int_0^{2\pi} \big(1 - e^{\eta(s-2\pi)}\big) g(s)\, ds = \frac{1}{e^{2\pi\eta} - 1} \int_0^{2\pi} e^{\eta s} g(s)\, ds. \tag{18}$$

The operator G is a bounded linear isomorphism with inverse L; more precisely:

Lemma 2.2. *G_η maps $\mathbb{B} = C^0_{\mathrm{per},0}$ onto $C^2_{\mathrm{per},0}$; for any $g \in C^0_{\mathrm{per},0}$, $u := G_\eta g$ satisfies $L_\eta u = u'' + \eta u' = g$. Furthermore, there exists a constant $\kappa_0 = \kappa_0(\eta_0) > 0$ such that $\|u\|_{C^2} \leqslant \kappa_0 \|g\|_{C^0}$, i.e., $\|G_\eta\| := \|G\|_{L(\mathbb{B}, C^2_{\mathrm{per},0})} \leqslant \kappa_0$, for all $\eta \leqslant \eta_0$.*

The proof is elementary and is left to the reader.[8]

Remark 2.3. (i) The Green operator G has a very simple expression in Fourier series:

$$G_\eta \Big[\sum_{n\neq 0} g_n e^{\mathrm{i}nt}\Big] := \sum_{n\neq 0} \frac{g_n}{n^2 - \mathrm{i}\eta n} e^{\mathrm{i}nt}. \tag{19}$$

(ii) Solutions of (14) are recognized as fixed points of the operator $G_\eta \circ \Phi_\xi$:

$$u = G_\eta \circ \Phi_\xi(u), \tag{20}$$

where ξ appear as a parameter.

2.3. Lyapunov–Schmidt decomposition

To solve Eq. (20), we shall perform a *Lyapunov–Schmidt decomposition* (compare Remark 2.1(iii)).
For p and q (positive coprime integers) and $\xi \in [0, 2\pi]$, let

$$\hat{\Phi}_\xi := \hat{\Phi}_\xi(\cdot; p, q) : C^0_{\mathrm{per}} \to \mathbb{B} = C^0_{\mathrm{per},0}, \tag{21}$$

where

[7] The formulas for $\eta = 0$ have to be intended as the limit for $\eta \to 0$.

[8] The vanishing of the average of g together with the definition of y_η guarantees that $G_\eta g \in C^2$ and is periodic; the definition of x_η implies that $\langle G_\eta g\rangle = 0$.

$$\begin{aligned}\big[\hat{\Phi}_\xi(v)\big](t) &:= \frac{1}{\varepsilon}\big[\Phi_\xi(v) - \big\langle \Phi_\xi(v)\big\rangle\big]\\ &= -f_x\big(\xi + pt + v(t), qt\big) + \phi_v(\xi),\end{aligned} \tag{22}$$

with

$$\phi_v(\xi) = \phi_v(\xi;\varepsilon,p,q) := \frac{1}{2\pi}\int_0^{2\pi} f_x\big(\xi + pt + v(t;\xi), qt\big)\,dt. \tag{23}$$

Then, Eq. (20) can be splitted into a "range equation"

$$u = \varepsilon G_\eta \circ \hat{\Phi}_\xi(u) \tag{24}$$

(where $u = u(\cdot;\xi)$) and a "kernel (or bifurcation) equation"

$$\phi_u(\xi) = \frac{\eta\nu}{\varepsilon} \quad \Longleftrightarrow \quad \big\langle \Phi_\xi\big(u(\cdot;\xi)\big)\big\rangle = 0. \tag{25}$$

Remark 2.4. (i) If $(u,\xi) \in \mathbb{B} \times [0,2\pi]$ solves (24), (25), then, by Lemma 2.2, $x_{pq}(t)$ defined in (3) solves (1).

(ii) $\forall \xi \in [0,2\pi]$, $\hat{\Phi}_\xi \in C^1(\mathbb{B},\mathbb{B})$; moreover, $\forall (u,\xi) \in \mathbb{B} \times [0,2\pi]$,

$$\big\|\hat{\Phi}_\xi(u)\big\|_{C^0} \leqslant 2\sup_{\mathbb{T}^2}|f_x| \leqslant 2\|f\|_{C^2}, \qquad \|D_u\hat{\Phi}_\xi\|_{\mathcal{L}(\mathbb{B},\mathbb{B})} \leqslant 2\sup_{\mathbb{T}^2}|f_{xx}| \leqslant 2\|f\|_{C^2}. \tag{26}$$

The usual way to proceed to solve (24), (25) is the following:

1. for any $\xi \in [0,2\pi]$, find $u = u(\cdot;\xi) = u(\cdot;\xi,\varepsilon)$ solving (24);
2. insert $u = u(\cdot,\xi)$ into the kernel equation (25) and determine $\xi \in [0,2\pi]$ so that (25) holds.

2.4. Perturbation theory

2.4.1. The range equation

For ε small the range equation is easily solved by standard contraction arguments.

Fix $\xi \in [0,2\pi]$, $0 < \kappa_1 \leqslant 1$ and let

$$\begin{cases} \mathbb{B}_{\kappa_1} := \big\{v \in \mathbb{B}\colon\ \|v\|_{C^0} \leqslant \kappa_1\big\},\\ \varphi : v \in \mathbb{B}_{\kappa_1} \to \varphi(v) := \varepsilon G_\eta\big(\hat{\Phi}_\xi(v)\big). \end{cases} \tag{27}$$

As above $\eta_0 > 0$ is fixed and $0 \leqslant \eta \leqslant \eta_0$.

Proposition 2.5. *Let*

$$\kappa_2 := 2\kappa_0\|f\|_{C^2}, \qquad \varepsilon_0 := \frac{\kappa_1}{2\kappa_2}. \tag{28}$$

Then, for $|\varepsilon| \leqslant \varepsilon_0$, the map φ in (27) maps $\mathbb{B}_{\kappa_1}$ into itself and is a contraction with Lipschitz constant $|\varepsilon|\kappa_2 \leqslant \varepsilon_0\kappa_2 \leqslant 1/2$:

$$\big\|\varphi(v) - \varphi(w)\big\|_{C^0} \leqslant |\varepsilon|\kappa_2\|v - w\|_{C^0}, \quad \forall v, w \in \mathbb{B}_{\kappa_1}. \tag{29}$$

Therefore, for every $\xi \in [0, 2\pi]$, there exists a unique $u := u(\cdot;\xi) \in \mathbb{B}_{\kappa_1}$ such that $\varphi(u) = u$. Furthermore, u has the following representation:

$$u = \sum_{k=1}^{\infty} \tilde{u}_k, \quad \textit{with } \|\tilde{u}_k\|_{C^0} \leqslant \big(|\varepsilon|\kappa_2\big)^k; \tag{30}$$

in particular, for any $|\varepsilon| \leqslant \varepsilon_0$, one has

$$\|u\|_{C^0} \leqslant \sum_{k=1}^{\infty} \|\tilde{u}_k\|_{C^0} \leqslant \frac{|\varepsilon|\kappa_2}{1 - |\varepsilon|\kappa_2} \leqslant 2\kappa_2|\varepsilon|. \tag{31}$$

Proof. From Lemma 2.2 and (26) there follows at once that φ maps $\mathbb{B}_{\kappa_1}$ into itself and contracts as in (29). Thus, from the contraction principle it follows that

$$v_j := \varphi^j(0) := \underbrace{\varphi \circ \cdots \circ \varphi}_{j \text{ times}}(0) \tag{32}$$

converges uniformly to the unique fixed point u. Furthermore, $\|v_j - v_{j-1}\|_{C^0} \leqslant (|\varepsilon|\kappa_2)^j$, so that, setting $\tilde{u}_k := v_k - v_{k-1}$, one obtains (30), which, in turn, implies immediately (31). □

Remark 2.6. (i) The solution u depends on the various parameters in a regular way: for example, if f is C^∞ then also u is C^∞ in all its variables $t, \varepsilon, \xi, \eta$.

(ii) If f is real-analytic (as it is the case in the spin–orbit problem) also the fixed point $u(\cdot;\xi)$ is real-analytic and from Proposition 2.5 and Cauchy integral formula for analytic function, it follows that $u = \sum_{k\geqslant 1} \varepsilon^k u_k(t;\xi)$ (with u_k independent of ε) and that u_k satisfies $\|u_k\| \leqslant 2\kappa_2\varepsilon_0^{-k}$, provided the above sup-norms are taken on suitable complex domains.

2.4.2. The bifurcation equation

From now on f will be assumed to be sufficiently smooth so that, by Proposition 2.5 and Remark 2.6, functions depend smoothly on ε; in particular, the solution u of the range equation (24) has the form

$$u(t) = u(t;\xi,\varepsilon) = \varepsilon u_1(t;\xi) + \varepsilon^2 u_2(t;\xi) + \cdots, \tag{33}$$

with

$$u_1 := G_\eta \hat{\Phi}_\xi(0) = -G_\eta\left(f_x(\xi + pt, qt) - \frac{1}{2\pi}\int_0^{2\pi} f_x(\xi + pt, qt)\,dt\right). \tag{34}$$

Therefore, $\phi_u(\xi;\varepsilon)$ in (23) can be written as

$$\begin{aligned} \phi(\xi;\varepsilon) := \phi_u(\xi;\varepsilon) &= \phi^{(0)}(\xi) + \varepsilon\tilde{\phi}^{(1)}(\xi,\varepsilon) \\ &= \phi^{(0)}(\xi) + \varepsilon\phi^{(1)}(\xi) + \varepsilon^2\tilde{\phi}^{(2)}(\xi,\varepsilon), \end{aligned} \tag{35}$$

with

$$\phi^{(0)}(\xi) = \phi^{(0)}(\xi; p, q) := \frac{1}{2\pi}\int_0^{2\pi} f_x(\xi + pt, qt)\,dt, \tag{36}$$

$$\phi^{(1)}(\xi) = \phi^{(1)}(\xi; p, q) := \frac{1}{2\pi}\int_0^{2\pi} f_{xx}(\xi + pt, qt)u_1\,dt \tag{37}$$

and

$$\sup_{\substack{|\varepsilon|\leqslant\varepsilon_0\\ \xi\in[0,2\pi]}} \left|\tilde\phi^{(i)}\right| \leqslant M_i \tag{38}$$

for suitable $M_i > 0$. We note that $\phi^{(0)}(\xi)$ has zero average, being the derivative of a periodic function; therefore if $\phi^{(0)}(\xi)$ is constant, it must be identically zero.

The following alternative holds:

(i) $\xi \to \phi^{(0)}(\xi)$ is not identically zero ("nondegenerate case"), i.e.,

$$\phi^{(0)}_- := \min_{\xi\in[0,2\pi]} \phi^{(0)}(\xi) < \max_{\xi\in[0,2\pi]} \phi^{(0)}(\xi) =: \phi^{(0)}_+. \tag{39}$$

(ii) $\xi \to \phi^{(0)}(\xi)$ is identically zero ("degenerate case").

(i) In the nondegenerate case, fix

$$0 < \delta < \left(\phi^{(0)}_+ - \phi^{(0)}_-\right)/2.$$

Then, if

$$|\varepsilon| \leqslant \varepsilon_1 := \min\{\varepsilon_0, \delta/M_1\} \tag{40}$$

one has that the range of ϕ contains the interval $[\phi^{(0)}_- + \delta, \phi^{(0)}_+ - \delta]$. Thus, (by continuity of ϕ) for all

$$\frac{\eta\nu}{\varepsilon} \in \left[\phi^{(0)}_- + \delta, \phi^{(0)}_+ - \delta\right] \tag{41}$$

there exists $\xi \in [0, 2\pi]$ solving the bifurcation equation.

(ii) In the degenerate case further assumptions, in general, are needed. For example, assume that $\phi^{(1)}(\xi)$ is not identically constant:

$$\phi^{(1)}_- := \min_{\xi\in[0,2\pi]} \phi^{(1)}(\xi) < \max_{\xi\in[0,2\pi]} \phi^{(1)}(\xi) =: \phi^{(1)}_+. \tag{42}$$

As above, fix

$$0 < \delta < \left(\phi^{(1)}_+ - \phi^{(1)}_-\right)/2,$$

and let

$$|\varepsilon| \leqslant \varepsilon_2 := \min\{\varepsilon_0, \delta/M_2\}. \tag{43}$$

Then, one has that the range of ϕ contains the interval $[\varepsilon(\phi_-^{(1)}+\delta), \varepsilon(\phi_+^{(1)}-\delta)]$ so that for all

$$\frac{\eta\nu}{\varepsilon} \in \left[\varepsilon\left(\phi_-^{(1)}+\delta\right), \varepsilon\left(\phi_+^{(1)}-\delta\right)\right] \tag{44}$$

there exists $\xi \in [0, 2\pi]$ solving the bifurcation equation.

We have proven the following

Proposition 2.7. *Let ε_0 and u (for $|\varepsilon| \leqslant \varepsilon_0$) be as in Proposition* 2.5; *write $\phi(\xi;\varepsilon) = \phi_u(\xi;\varepsilon,p,q)$ (defined in* (23)) *in the form* (35)–(38).

(i) *Assume* (39) *and let $|\varepsilon| \leqslant \varepsilon_1$ with ε_1 as in* (40). *Then, there exists $\xi \in [0, 2\pi]$ solving the bifurcation equation* (25) *provided* (41) *holds.*

(ii) *Assume $\xi \to \phi^{(0)}(\xi)$ is identically zero and that* (42) *holds; let $|\varepsilon| \leqslant \varepsilon_2$ with ε_2 as in* (43). *Then, there exists $\xi \in [0, 2\pi]$ solving the bifurcation equation* (25) *provided* (44) *holds.*

In either case, for the above ε, ξ, η and ν, the couple (u, ξ) solves (24)–(25), *so that $x_{pq}(t)$ defined in* (3) *solves* (1).

Explicit examples of nondegenerate and degenerate situations will be discussed in the next section.

2.5. The bifurcation equation for the spin–orbit model

As briefly mentioned in Section 1, the (planar) dissipative spin–orbit system is governed by Eq. (1) with parameters $\bar\eta$, $\bar\nu$, $\bar\varepsilon$ and the potential f defined as follows:

$$\begin{cases} \bar\eta = K\Omega_{\mathrm{e}}, \quad \bar\nu = \bar\nu_{\mathrm{e}}, \quad \bar\varepsilon = \dfrac{3}{2}\dfrac{B-A}{C}, \\ f = f(x,t;\mathrm{e}) := -\dfrac{1}{2\rho_{\mathrm{e}}(t)^3}\cos\bigl(2x - 2\mathrm{f}_{\mathrm{e}}(t)\bigr), \end{cases} \tag{45}$$

where

- $K \geqslant 0$ is a physical constant depending on the internal (non-rigid) structure of the satellite;
- $\Omega_{\mathrm{e}} > 0$, $N_{\mathrm{e}} > 0$ and $\bar\nu_{\mathrm{e}} > 1$ are known functions of the eccentricity $\mathrm{e} \in [0, 1)$ and are given by

$$\Omega_{\mathrm{e}} := \left(1 + 3\mathrm{e}^2 + \frac{3}{8}\mathrm{e}^4\right)\frac{1}{(1-\mathrm{e}^2)^{9/2}}, \tag{46}$$

$$N_{\mathrm{e}} := \left(1 + \frac{15}{2}\mathrm{e}^2 + \frac{45}{8}\mathrm{e}^4 + \frac{5}{16}\mathrm{e}^6\right)\frac{1}{(1-\mathrm{e}^2)^6},$$

$$\bar\nu_{\mathrm{e}} := \frac{N_{\mathrm{e}}}{\Omega_{\mathrm{e}}} = 1 + 6\mathrm{e}^2 + \frac{3}{8}\mathrm{e}^4 + O\left(\mathrm{e}^6\right); \tag{47}$$

- $0 < A < B < C$ are the principal moments of inertia of the satellite;
- $\rho_{\mathrm{e}}(t)$ and $\mathrm{f}_{\mathrm{e}}(t)$ are, respectively, the (normalized) orbital radius and the true anomaly of the Keplerian motion, which (with suitable normalizations) are 2π-periodic function of time t. To describe ρ_{e} and f_{e}, let $\mathrm{u} = \mathrm{u}_{\mathrm{e}}(t)$ ("eccentric anomaly") be the 2π-periodic function obtained by inverting

$$t = \mathrm{u} - \mathrm{e}\sin\mathrm{u}, \quad \text{("Kepler's equation")}; \tag{48}$$

then

$$\rho_{\rm e}(t) = 1 - {\rm e}\cos {\rm u_e}(t),$$
$$f_{\rm e}(t) = 2\arctan\left(\sqrt{\frac{1+{\rm e}}{1-{\rm e}}}\tan\frac{{\rm u_e}(t)}{2}\right). \tag{49}$$

In fact, the potential f has a very particular Fourier expansion: it can be shown that (4) holds with

$$\alpha_j = \alpha_j({\rm e}) := \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos(2\tilde{f}_{\rm e} - ju + {\rm e}j\sin u)}{\tilde\rho_{\rm e}^2}\,du; \tag{50}$$

for details see Appendix A.

To discuss the bifurcation equation for the spin–orbit problem, one needs to study the derivatives of the spin–orbit potential evaluated along the unperturbed periodic orbit $t \to (\xi + pt, qt)$. We collect a few elementary facts in the following

Lemma 2.8. *For $k \in \mathbb{N}$ let*

$$f^{(k)}(\xi,t) = f^{(k)}(\xi,t;p,q,{\rm e}) := \frac{\partial^k f}{\partial x^k}(\xi + pt, qt) =: \sum_{n\in\mathbb{Z}} f_n^{(k)}(\xi)e^{{\rm i}nt}; \tag{51}$$

let, also,

$$m_j = m_j(p,q) := 2p - jq, \qquad J_0 := \{j \in \mathbb{Z}:\ j \neq 0,\ m_j = 0\}, \tag{52}$$

and denote by $\mathcal{F}_q$ the set of Fourier modes[9] *of $f^{(k)}$. Then*:

(i) $f^{(2h)} = (-4)^h f^{(0)}$, $f^{(2h+1)} = (-4)^h f^{(1)}$.

(ii) $\begin{cases} m_j \neq m_k,\ \forall j \neq k; \\ m_j = -m_k \ \Leftrightarrow\ q = 1, 2 \text{ or } 4,\ k = \frac{4p}{q} - j. \end{cases}$

(iii) $\langle f^{(1)}(\xi,\cdot)\rangle = -2\sin(2\xi)\sum_{j\in J_0}\alpha_j$.

(iv) $\mathcal{F}_q = \{m_j:\ j \in \mathbb{Z}\}$ *if* $q = 1, 2$ *or* 4; $\mathcal{F}_q = \{\pm m_j:\ j \in \mathbb{Z}\setminus\{0\}\}$ *if* $q \neq 1, 2, 4$.

(v) *If $q = 4$, then $m_j = -m_{p-j}$ and*

$$f^{(k)}(\xi,t) = \sum_{j\in\mathbb{Z}} c_j^{(k)} e^{{\rm i}m_j t} \tag{53}$$

with

$$c_j^{(0)} := \begin{cases} \frac12\alpha_p e^{-{\rm i}2\xi}, \\ \frac12\alpha_p e^{{\rm i}2\xi}, \\ \frac12(\alpha_j e^{{\rm i}2\xi} + \alpha_{p-j}e^{-{\rm i}2\xi}), \end{cases} \qquad c_j^{(1)} := \begin{cases} -{\rm i}\alpha_p e^{-{\rm i}2\xi} & \text{if } j = 0, \\ {\rm i}\alpha_p e^{{\rm i}2\xi} & \text{if } j = p, \\ {\rm i}(\alpha_j e^{{\rm i}2\xi} - \alpha_{p-j}e^{-{\rm i}2\xi}) & \text{if } j \neq 0, p. \end{cases} \tag{54}$$

Proof. (i) and (ii) follow immediately from the definitions in (4) and (52).

[9] I.e., the set of integers $n \in \mathbb{Z}$ such that $f_n^{(k)} \neq 0$.

(iii) From (51) and (4) one has that

$$\begin{cases} f^{(0)}(\xi,t) = \dfrac{1}{2}\displaystyle\sum_{j\neq 0}\bigl(\alpha_j e^{\mathrm{i}2\xi} e^{\mathrm{i}m_j t} + \alpha_j e^{-\mathrm{i}2\xi} e^{-\mathrm{i}m_j t}\bigr), \\ f^{(1)}(\xi,t) = \mathrm{i}\displaystyle\sum_{j\neq 0}\bigl(\alpha_j e^{\mathrm{i}2\xi} e^{\mathrm{i}m_j t} - \alpha_j e^{-\mathrm{i}2\xi} e^{-\mathrm{i}m_j t}\bigr) \end{cases} \tag{55}$$

so that (iii) follows immediately after integrating $f^{(1)}$ with respect to t.

(iv) follows from (ii) and (55).

(v) From (ii) one has that, since $q=4$, $m_{j-p} = -m_j$. Hence, from Eq. (55) there follows[10]

$$\begin{aligned} f^{(0)}(\xi,t) &= \frac{1}{2}\sum_{j\neq 0}\bigl(\alpha_j e^{\mathrm{i}2\xi} e^{\mathrm{i}m_j t} + \alpha_j e^{-\mathrm{i}2\xi} e^{-\mathrm{i}m_j t}\bigr) \\ &= \frac{1}{2}\sum_{j\neq 0}\alpha_j e^{\mathrm{i}2\xi} e^{\mathrm{i}m_j t} + \frac{1}{2}\sum_{j\neq p}\alpha_{p-j} e^{-\mathrm{i}2\xi} e^{\mathrm{i}m_j t} \\ &= \sum_{j\in\mathbb{Z}} c_j^{(0)} e^{\mathrm{i}m_j t}, \end{aligned}$$

and analogously for $f^{(1)}$. □

If $q=1$ or $q=2$ the spin–orbit problem is nondegenerate in the sense of Section 2.4.2. In fact:

Proposition 2.9. *The spin–orbit model is nondegenerate if and only if $q=1$ or $q=2$. Indeed, denoting $\phi^{(0)}(\xi) = \phi^{(0)}(\xi; p, q)$, one has*:

$$\phi^{(0)}(\xi; p, q) = \begin{cases} -2\alpha_{2p}\sin(2\xi) & \text{if } q=1, \\ -2\alpha_p \sin(2\xi) & \text{if } q=2, \\ 0, & \forall q \geqslant 3. \end{cases} \tag{56}$$

Proof. By definition, $\phi^{(0)}(\xi; p, q) = \langle f^{(1)}(\xi,\cdot)\rangle$; thus, the claim follows from (iii) of Lemma 2.8 by noticing that $m_j = 0$ if and only if $j = (2p)/q$ so that (since p and q are coprime)

$$J_0 = \begin{cases} \{2p\} & \text{if } q=1, \\ \{p\} & \text{if } q=2, \\ \emptyset & \text{if } q \geqslant 3, \end{cases} \tag{57}$$

from which (56) follows at once. □

Next result tells us when $\phi^{(1)}$ is nonconstant.

Proposition 2.10. *Let $q \geqslant 3$. Then $\xi \to \phi^{(1)}(\xi) = \phi^{(1)}(\xi; p, q)$ is nonconstant if and only if $q=4$, in which case*

$$\phi^{(1)}(\xi; p, 4) = \eta a_0 - 4\sin(4\xi) \sum_{\substack{j\in\mathbb{Z} \\ j\neq 0,p}} \frac{\alpha_{p-j}\alpha_j}{4(p-2j)^2 + \eta^2}, \tag{58}$$

[10] In the second equality change the summation index $j \to j-p$ and use that $m_{j-p} = -m_j$; for the third equality consider separately the cases $j=0$, $j=p$ and $j \neq 0, p$.

where

$$a_0 := \frac{2}{p}\frac{\alpha_p^2}{4p^2+\eta^2} - \sum_{j\neq 0,p} \frac{\alpha_j^2-\alpha_{p-j}^2}{(p-2j)(4(p-2j)^2+\eta^2)}. \tag{59}$$

Proof. Since $q \geqslant 3$, by (57), $\langle f^{(1)}(\xi,\cdot)\rangle = 0$. Hence by (37), (33), (i) of Lemma 2.8 and (19) one has that

$$\phi^{(1)}(\xi) = \langle f^{(2)} u_1\rangle = 4\langle f^{(0)} G_\eta f^{(1)}\rangle = 4\sum_{0\neq n\in\mathbb{Z}} \frac{f_{-n}^{(0)} f_n^{(1)}}{n^2 - \mathrm{i}\eta n}. \tag{60}$$

Now, if $q \neq 4$, by (iv) of Lemma 2.8, $\mathcal{F}_q = \{\pm m_j\colon\ j\neq 0\}$ and (55) represents the Fourier expansion of $f^{(k)}$. Thus,

$$\phi^{(1)}(\xi) = 4\sum_{j\neq 0}\left(\frac{f_{-m_j}^{(0)} f_{m_j}^{(1)}}{m_j^2 - \mathrm{i}\eta m_j} + \frac{f_{m_j}^{(0)} f_{-m_j}^{(1)}}{m_j^2 + \mathrm{i}\eta m_j}\right) = -4\eta\sum_{j\neq 0}\frac{\alpha_j^2}{m_j^3+\eta^2 m_j}, \tag{61}$$

proving that $\xi \to \phi^{(1)}(\xi)$ is constant for $4 \neq q \geqslant 3$.

Fix now $q=4$. Then, by (60), point (v) of Lemma 2.8, one finds[11]

$$\begin{aligned}\phi^{(1)}(\xi) &= 4\sum_{j,k\in\mathbb{Z}} \frac{c_k^{(0)} c_j^{(1)}}{m_j^2 - \mathrm{i}\eta m_j}\langle e^{\mathrm{i}(m_k+m_j)t}\rangle \\ &= 4\sum_j \frac{c_{p-j}^{(0)} c_j^{(1)}}{m_j^2 - \mathrm{i}\eta m_j} \\ &= 4\sum_j \operatorname{Re}\left(\frac{c_{p-j}^{(0)} c_j^{(1)}}{m_j^2 - \mathrm{i}\eta m_j}\right).\end{aligned}$$

Then, considering separately the cases $j=0$, $j=p$ and $j\neq 0, p$, one finds easily (58) and (59). □

Thanks to Propositions 2.7, 2.9 and 2.10 (recall (11)), we are now ready for the

Proof of Theorem 1.2. Define

$$\beta_{pq} := \begin{cases} -2\alpha_{2p} & \text{if } q=1,\\ -2\alpha_p & \text{if } q=2,\\ \sum_{j\in\mathbb{Z},\ j\neq 0,p} \frac{\alpha_{p-j}\alpha_j}{(p-2j)^2} & \text{if } q=4.\end{cases} \tag{62}$$

Then, for $q=1$ and $q=2$, the thesis follows at once by the definition of β_{pq}, (56) and Proposition 2.7, if one chooses $\delta < (1-\kappa)|\beta_{pq}|$ in (41).

In the case $q=4$, define

$$\tilde\beta := \phi^{(1)}(\xi; p, 4) - \beta_{p4}$$

[11] Recall, in particular (54) and notice also that $\overline{c_j^{(k)}} = c_{p-j}^{(k)}$.

and observe that, by (58), there exists $c > 0$ such that $|\tilde{\beta}| \leqslant c|\bar{\eta}|$ for $|\bar{\eta}| < \bar{\eta}_0$. Then, again the thesis follows from Proposition 2.7, by choosing, in (44), $\delta < (1-\kappa)|\beta_{pq}|/2$ and letting $\bar{\eta}_0$ be so small that $c\bar{\eta}_0 < \delta$. $\square$

In [7] the above dissipative spin–orbit problem is investigated from the numerical point of view, with the purpose of detecting stable periodic orbits ("resonances") together with their basins of attraction; a numerical evaluation of the normalized size of the basins of attraction, for different values of the dissipation parameter, is then related to mechanism of "capture in resonance" in our Solar system. The above Theorem 1.2 gives a theoretical prediction for periodic orbits to coexist (according to whether condition (5) is satisfied or not): in Appendix B we compare the above theoretical predictions with the experimental numerical results of [7], finding a rather striking agreement.

3. Local basins of attractions

In the previous section we have found $2\pi q$-periodic solutions $x_{pq}(t)$ of (1) of the form (3) with u 2π-periodic, $\langle u\rangle = 0$ and $|u| \leqslant c|\varepsilon|$ for $|\varepsilon| \leqslant \varepsilon_0$ (compare also Theorem 1.2).

In this section we discuss the *basin of attraction* of stable periodic orbit, giving an estimate on the size of initial data which approach (exponentially fast) the periodic orbit.

For simplicity, we consider only the nondegenerate case and *assume that ξ satisfies* (6), i.e.,

$$\theta_0 = \partial_\xi \phi^{(0)}(\xi; p, q)t > 0$$

(so that the periodic orbit is *elliptic*). In the spin–orbit case, by Proposition 2.9,

$$\theta_0 = \begin{cases} -4\alpha_{2p}\cos(2\xi) & \text{if } q = 1, \\ -4\alpha_p \cos(2\xi) & \text{if } q = 2, \end{cases}$$

and nondegenerate minima are in $\xi = 0, \pi$ if $\alpha_j > 0$ or in $\xi = \pi/2, 3\pi/2$ if $\alpha_j < 0$.

Therefore, we look at the behavior of solutions

$$x(t) = x_{pq}(t) + \tilde{w}(t) \tag{63}$$

of (1) with

$$\left|x(0) - x_{pq}(0)\right| + q\left|\dot{x}(0) - \dot{x}_{pq}(0)\right| = \left|\tilde{w}(0)\right| + q\left|\dot{\tilde{w}}(0)\right|$$

small. Clearly, $\tilde{w}(t)$ exists $\forall t \in \mathbb{R}$ and satisfies the differential equation

$$\ddot{\tilde{w}} + \bar{\eta}\dot{\tilde{w}} + \bar{\varepsilon} f_x(x + \tilde{w}, t) - \bar{\varepsilon} f_x(x, t) = 0.$$

Setting

$$w(t) := \tilde{w}(qt)$$

and replacing, as above, t with qt, we see that w satisfies

$$w'' + \eta w' + \varepsilon f_x(\bar{x} + w, qt) - \varepsilon f_x(\bar{x}, qt) = 0,$$

with η, ε defined in (11) and $\bar{x}$ defined to be

$$\bar{x}(t) := \xi + pt + u(t).$$

To clarify the forthcoming analysis, let us give the following definitions

$$\begin{cases} \big(Q(w)\big)(t) := f_x\big(\bar{x}(t) + w(t), qt\big) - f_x\big(\bar{x}(t), qt\big) - f_{xx}\big(\bar{x}(t), qt\big) w(t), \\ \theta = \theta(\varepsilon) := \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} f_{xx}\big(\bar{x}(t), qt\big)\, dt \overset{\varepsilon \to 0}{\longrightarrow} \theta_0, \\ \gamma(t) := f_{xx}\big(\bar{x}(t), qt\big) - \theta, \\ \alpha := \eta/2, \\ z(t) := e^{\alpha t} w(t), \\ \mathcal{L}z := z'' + \big((\varepsilon\theta - \alpha^2) + \varepsilon\gamma(t)\big) z. \end{cases} \tag{64}$$

From such definitions there follows immediately that w and z satisfy the following differential equations

$$\begin{cases} w'' + \eta w' + \varepsilon\big(\theta + \gamma(t)\big) w + \varepsilon\big(Q(w)\big)(t) = 0, \\ \mathcal{L}z = -\varepsilon e^{\alpha t} Q\left(e^{-\alpha t} z\right). \end{cases} \tag{65}$$

Remark 3.1. (i) Q is a "quadratic operator," i.e., there exists a constant c_1 (depending only on f) such that

$$\big|\big(Q(w)\big)(t)\big| \leqslant c_1 \big|w(t)\big|^2, \quad \forall t \in \mathbb{R}. \tag{66}$$

(ii) The function γ is 2π-periodic and $\langle\gamma\rangle = 0$.

(iii) The homogeneous equation associated to the nonlinear equation for z in (65) is

$$\mathcal{L}z = 0, \tag{67}$$

which (in view of the preceding point (ii)) is a *Hill's equation.*

Now, if $\mathfrak{c}(t)$ and $\mathfrak{s}(t)$ denote the ("fundamental") solutions of (67) with initial data

$$\mathfrak{c}(0) = 1 = \mathfrak{s}'(0), \qquad \mathfrak{c}'(0) = 0 = \mathfrak{s}(0), \tag{68}$$

as well known from classical Floquet theory, *if the solutions $\rho = \rho_\pm$ of the characteristic equation*

$$\rho^2 - \big[\mathfrak{c}(2\pi) + \mathfrak{s}'(2\pi)\big]\rho + 1 = 0 \tag{69}$$

are distinct, then Eq. (67) *has two independent solutions of the form*

$$z_\pm(t) = e^{\pm \mathrm{i}\lambda t} P_\pm(t), \tag{70}$$

where λ is such that $e^{\pm \mathrm{i}\lambda} = \rho_\pm$ and $P_\pm$ are 2π-periodic functions; see, e.g., *Floquet's theorem in* [13, p. 4].

(iv) The inverse of the operator $\mathcal{L}$ is given by

$$\big(\mathcal{G}[h]\big)(t) := \mathfrak{s}(t) \int_0^t \mathfrak{c}(\tau) h(\tau)\, dt - \mathfrak{c}(t) \int_0^t \mathfrak{s}(\tau) h(\tau)\, dt, \tag{71}$$

i.e., $x = \mathcal{G}[h]$ is the unique solution of $\mathcal{L}x = h$ with initial data $x(0) = 0 = x'(0)$. Thus, the solution of the equation $\mathcal{L}z = h$, $z(0) = z_0$, $z'(0) = v_0$ is given by $z = z_{\mathrm{hom}} + \mathcal{G}[h]$, where z_{hom} is the solution of the homogeneous equation $\mathcal{L}z_{\mathrm{hom}} = 0$ and initial data z_0, v_0.

Indeed, in the case of the Hill's operator (65), $\rho_+ \neq \rho_-$, provided certain smallness conditions are satisfied. Let, in fact, suppose that η *is so small that*

$$\frac{\eta^2}{4} =: \alpha^2 < \frac{1}{4}\varepsilon\theta_0 \tag{72}$$

(which holds by (7)) and ε *is so small that*

$$\frac{1}{2}\theta_0 \leqslant \theta = \theta(\varepsilon) \leqslant 2\theta_0, \tag{73}$$

and let

$$2\varepsilon\theta_0 \geqslant \varepsilon\theta \geqslant \omega^2 := \varepsilon\theta - \alpha^2 \geqslant \frac{1}{2}\varepsilon\theta \geqslant \frac{1}{4}\varepsilon\theta_0, \qquad \omega \approx \sqrt{\varepsilon}. \tag{74}$$

Let also

$$g(t) := \varepsilon\omega^{-2}\gamma(t), \tag{75}$$

so that g is 2π-periodic, $\langle g\rangle = 0$, $|g(t)| \leqslant 4|\gamma(t)|/\theta_0$ $\forall t \in \mathbb{R}$, and rewrite the operator $\mathcal{L}$ in (64) as

$$\mathcal{L}z = z'' + \omega^2\big(1 + g(t)\big)z. \tag{76}$$

Then, the following result holds:

Lemma 3.2. *For $\omega > 0$ small enough there exists $0 < \delta = 1 - 2\pi^2\omega^2 - O(\omega^4) < 1$ such that the solutions of the characteristic equation* (69) *associated to* (67) *are given by $\rho_\pm = \delta \pm \mathrm{i}\sqrt{1-\delta^2}$ and, hence, are distinct. Thus λ in* (70) *is real, $\lambda = 2\pi\omega + O(\omega^3)$ and all solutions of* (67) *are bounded together with their derivatives. Finally δ and λ smoothly depend on ω^2 and ω respectively.*

Proof. Let us first rewrite (67) as a system: define

$$T := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B(t) := \begin{pmatrix} 0 & 0 \\ -1 - g(t) & 0 \end{pmatrix}, \qquad A(t) := T + \omega^2 B(t).$$

Then, (67) is equivalent to the system

$$\mathfrak{u}' = A\mathfrak{u}, \quad \text{with } \mathfrak{u} := \begin{pmatrix} z \\ z' \end{pmatrix}. \tag{77}$$

The fundamental solution of[12] (77) is given by

$$U = \begin{pmatrix} \mathfrak{c} & \mathfrak{s} \\ \mathfrak{c}' & \mathfrak{s}' \end{pmatrix}$$

with $\mathfrak{c}$ and $\mathfrak{s}$ solutions of (67) with initial data as in (68). Now expand $U = U(t;\omega^2)$ (with $t \in [0, 2\pi]$) in power of ω^2:

$$U\big(t;\omega^2\big) = U_0(t) + U_1(t)\omega^2 + O\big(\omega^4\big).$$

[12] I.e., the two-by-two matrix $U(t)$ satisfying Cauchy problem $U' = AU$, $U(0) = \mathrm{Id}$ = identity matrix.

Then, U_0 solves

$$U_0' = TU_0, \qquad U_0(0) = \mathrm{Id},$$

i.e.,

$$U_0(t) = e^{Tt} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

and U_1 solves

$$U_1' = TU_1 + BU_0, \qquad U_1(0) = 0,$$

i.e. (by the formula of "variation of constants")

$$U_1(t) = e^{Tt}\int_0^t e^{-Ts}B(s)e^{Ts}\,ds = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\int_0^t \begin{pmatrix} s+sg(s) & s^2+s^2g(s) \\ -1-g(s) & -s-sg(s) \end{pmatrix} ds.$$

Thus, since $\langle g\rangle = 0$,

$$U_1(2\pi) = \begin{pmatrix} 1 & 2\pi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ -2\pi & -a \end{pmatrix} = \begin{pmatrix} a-4\pi^2 & b-2\pi a \\ -2\pi & -a \end{pmatrix}$$

for suitable $a, b \in \mathbb{R}$. In conclusion,

$$\begin{aligned} \mathfrak{c}(2\pi) + \mathfrak{s}'(2\pi) &= \operatorname{Tr} U(2\pi) \\ &= \operatorname{Tr} U_0(2\pi) + \omega^2 \operatorname{Tr} U_1(2\pi) + O\big(\omega^4\big) \\ &= 2 - 4\pi^2\omega^2 + O\big(\omega^4\big) =: 2\delta, \end{aligned}$$

so that, if ω is small enough, then $0 < \delta < 1$ and the solutions of Eq. (69) are given by $\rho_\pm = \delta \pm \mathrm{i}\sqrt{1-\delta^2}$. Finally λ in (70) satisfies $\cos\lambda = \delta$, $\sin\lambda = \sqrt{1-\delta^2}$, then it is real and has the asymptotic behavior stated in the thesis. $\square$

Remark 3.3. When $\varepsilon = 0$ (so that, by (72), $\eta = \alpha = 0$) the Hill's operator $\mathcal{L}$ degenerates becoming ∂_{tt}. The solutions of the characteristic equation (69) coincide $\rho_+ = \rho_- = 1$ and the corresponding solutions of the homogeneous equation $\partial_{tt} z = 0$ *are not more bounded for* $t \geqslant 0$.

The following lemma, whose proof is postponed in Section 3.1, describes the behavior of the fundamental solutions of the homogeneous equation (67) when ε approaches zero. We will use it in (84) to estimate the inverse $\mathcal{G}$ of $\mathcal{L}$.

Lemma 3.4. *There exists $c_2 \geqslant 1$ such that, for ε small enough, the fundamental solutions $\mathfrak{c}$ and $\mathfrak{s}$ of* (67)–(68) *satisfy*

$$\big|\mathfrak{c}(t)\big|, \sqrt{\varepsilon}\big|\mathfrak{s}(t)\big|, \big|\mathfrak{c}'(t)\big|/\sqrt{\varepsilon}, \big|\mathfrak{s}'(t)\big| \leqslant c_2, \quad \forall t \geqslant 0. \tag{78}$$

Let, now, $z(t)$ be solution of (65) and let

$$a := z(0), \qquad b := z'(0), \qquad z_{\text{hom}}(t) := a\mathfrak{c}(t) + b\mathfrak{s}(t), \qquad v := z - z_{\text{hom}}. \tag{79}$$

Then, by point (iv) of Remark 3.1, since $\mathcal{L}z = \mathcal{L}v$, v is seen to satisfy[13]

$$v = -\varepsilon\mathcal{G}\big[e^{\alpha t} Q\left(e^{-\alpha t} z_{\text{hom}} + e^{-\alpha t} v\right)\big]. \tag{80}$$

The main technical estimate is contained in the following lemma, which states that if the initial data of $z(t)$ are small enough, then v and its derivatives are small for all times.

Lemma 3.5. *Let $c_3 := 1/(32c_1{c_2}^3)$ and $c_4 := 1/(16c_1{c_2}^2)$. If $\sqrt{\varepsilon}|a|, |b| \leqslant c_3\alpha$ then*

$$\sqrt{\varepsilon}\big|v(t)\big| + \big|v'(t)\big| < c_4\alpha, \quad \forall t \geqslant 0. \tag{81}$$

Proof. By contradiction, let us suppose that there exists $\bar{t} > 0$ such that (81) holds for every $0 \leqslant t < \bar{t}$ but

$$\sqrt{\varepsilon}\big|v(\bar{t})\big| + \big|v'(t)\big| = c_4\alpha. \tag{82}$$

By hypothesis (78) and (79)

$$\big|z_{\text{hom}}(t)\big| \leqslant c_4\alpha/\sqrt{\varepsilon}, \quad \forall t \geqslant 0.$$

By (66)

$$\big|Q\left(e^{-\alpha t} z_{\text{hom}}(t) + e^{-\alpha t} v(t)\right)\big| \leqslant c_1 e^{-2\alpha t}\big(\big|z_{\text{hom}}(t)\big| + \big|v(t)\big|\big)^2 \leqslant 4c_1{c_4}^2 e^{-2\alpha t}\frac{\alpha^2}{\varepsilon}, \quad \forall 0 \leqslant t \leqslant \bar{t}. \tag{83}$$

By (78), (71) and (83) one has

$$\sqrt{\varepsilon}\big|\big(\mathcal{G}[h]\big)(t)\big|, \left|\frac{d}{dt}\big(\mathcal{G}[h]\big)(t)\right| \leqslant 2{c_2}^2\int_0^t \big|h(\tau)\big|\,d\tau, \quad \forall t \geqslant 0. \tag{84}$$

By (82), (80) and (84)

$$\begin{aligned}
c_4\alpha &= \sqrt{\varepsilon}\big|v(\bar{t})\big| + \big|v'(\bar{t})\big| \\
&\leqslant 4{c_2}^2\varepsilon\int_0^{\bar{t}} e^{\alpha\tau}\big|Q\left(e^{-\alpha\tau} z_{\text{hom}}(\tau) + e^{-\alpha\tau} v(\tau)\right)\big|\,d\tau \\
&\leqslant 16c_1{c_2}^2{c_4}^2\alpha^2\int_0^{\bar{t}} e^{-\alpha\tau}\,d\tau \\
&= c_4\alpha^2\int_0^{\bar{t}} e^{-\alpha\tau}\,d\tau \\
&< c_4\alpha,
\end{aligned}$$

which is a contradiction. □

[13] Note that, from (79), $v(0) = v'(0) = 0$.

We can proceed with the

Proof of Theorem 1.3. By (78), (79) and (81) we have that if

$$\sqrt{\varepsilon}\big|z(0)\big|, \big|z'(0)\big| \leqslant c_3\alpha$$

then

$$\sqrt{\varepsilon}\big|z_{\mathrm{hom}}(t)\big| + \big|z'_{\mathrm{hom}}(t)\big| \leqslant 2\alpha c_4, \quad \forall t \geqslant 0,$$

so that, by Lemma 3.5,

$$\sqrt{\varepsilon}\big|z(t)\big| + \big|z'(t)\big| \leqslant 3c_4\alpha, \quad \forall t \geqslant 0.$$

Next, by (64) we get $z(0) = w(0)$ and $z'(0) = \alpha w(0) + w'(0)$, so that, assuming

$$\sqrt{\varepsilon}\big|w(0)\big|, \big|w'(0)\big| \leqslant c_3\alpha/2$$

one gets

$$\sqrt{\varepsilon}\big|z(0)\big|, \big|z'(0)\big| \leqslant c_3\alpha,$$

provided $\alpha \leqslant \sqrt{\varepsilon}/2$, namely (recall (11)) $\bar{\eta} \leqslant \sqrt{\varepsilon}/q = \sqrt{\bar{\varepsilon}}$, which holds by (7). Since, by (64)

$$\sqrt{\varepsilon}\big|w(t)\big| + \big|w'(t)\big| \leqslant 2e^{-\alpha t}\big(\sqrt{\varepsilon}\big|z(t)\big| + \big|z'(t)\big|\big) \leqslant 6c_4\alpha e^{-\alpha t}, \tag{85}$$

and since $w(t/q) = \tilde{w}(t)$ and $q\dot{\tilde{w}}(t) = w'(t/q)$ we see that (85) (with t replaced by t/q) implies (9) finishing the proof, choosing $\bar{c}_1 = c_3/2$ and $\bar{c}_2 = 6c_4$. □

3.1. *Proof of Lemma 3.4*

Note that $\zeta_\pm(t) = e^{\pm \mathrm{i}\lambda t}\mathcal{P}_\pm(t)$ solves (67) if and only if $\mathcal{P}_\pm$ solves

$$\ddot{\mathcal{P}}_\pm \pm \mathrm{i}2\lambda\dot{\mathcal{P}}_\pm + \lambda^2 G\mathcal{P}_\pm = 0, \tag{Eq$_\pm$}$$

where $G(t;\lambda) := \lambda^{-2}\omega^2(1+g) - 1$ smoothly depends on the parameter λ; note that here we are thinking ω as a smooth function of λ, inverting the expression $\lambda = 2\pi\omega + O(\omega^3)$ found in Lemma 3.2. Note that by (74)

$$\lambda \approx \omega \approx \sqrt{\varepsilon}. \tag{86}$$

Lemma 3.6. *Let* $0 < |\lambda| < 1/2$. *If* $\mathcal{P}_+$ *satisfies* (Eq$_+$) *with Dirichlet boundary condition*

$$\mathcal{P}_+(0) = \mathcal{P}_+(2\pi) = 1, \tag{87}$$

then $\zeta_+(t) := e^{\mathrm{i}\lambda t}\mathcal{P}_+(t)$ *and*[14] $\zeta_-(t) := e^{-\mathrm{i}\lambda t}\overline{\mathcal{P}_+(t)}$ *are two independent solutions of* (67).

[14] We denote by $\bar{\zeta}$ the complex conjugated of $\zeta \in \mathbb{C}$.

Proof. Note that $\mathcal{P}_+$ and $\overline{\mathcal{P}_+}$ satisfy (Eq_+) and (Eq_-) respectively, and, therefore, $\zeta_\pm$ satisfy (67). Now we prove that $\zeta_\pm$ are independent. Indeed, if by contradiction $\zeta_- \equiv c\zeta_+$ for some $c \in \mathbb{C}$, then $\overline{\mathcal{P}_+(t)} = ce^{\mathrm{i}2\lambda t}\mathcal{P}_+(t)$, $\forall t \in [0, 2\pi]$. Then for $t = 0$ we get $1 = \overline{\mathcal{P}_+(0)} = c$ and, for $t = 2\pi$, $1 = \overline{\mathcal{P}_+(2\pi)} = e^{\mathrm{i}4\pi\lambda}$, namely $2\lambda \in \mathbb{Z}$, which is impossible since $0 < |\lambda| < 1/2$. □

Lemma 3.7. *Let $0 < |\lambda| < 1/2$. Let $z_\pm(t) = e^{\pm \mathrm{i}\lambda t} P_\pm(t)$, with $P_\pm$ 2π-periodic, be two independent solutions of (67). If $\mathcal{P}_+ : [0, 2\pi] \to \mathbb{C}$ solves (Eq_+) and (87), then $\mathcal{P}_+$ and $\mathcal{P}_- := \overline{\mathcal{P}_+}$ can be extended on $\mathbb{R}$ to two 2π-periodic solutions of ($\mathrm{Eq}_\pm$).*

Proof. Let $\zeta_\pm(t) := e^{\pm \mathrm{i}\lambda t}\mathcal{P}_\pm(t)$. Since $\zeta_\pm(t)$ solve (67) for $t \in [0, 2\pi]$, there exist $c_\pm, d_\pm \in \mathbb{C}$ such that

$$\mathcal{P}_+(t) = c_+ P_+(t) + c_- e^{-\mathrm{i}2\lambda t} P_-(t), \qquad \mathcal{P}_-(t) = d_+ e^{\mathrm{i}2\lambda t} P_+(t) + d_- P_-(t). \tag{88}$$

Evaluating (88) for $t = 0, 2\pi$ and since $P_\pm(0) = P_\pm(2\pi)$, we get

$$C_+ + C_- = 1, \qquad D_+ + D_- = 1, \qquad C_+ + C_- e^{-\mathrm{i}4\pi\lambda} = 1, \qquad D_+ e^{\mathrm{i}4\pi\lambda} + D_- = 1, \tag{89}$$

where $C_\pm := c_\pm P_\pm(0)$ and $D_\pm := d_\pm P_\pm(0)$. By (89) we have

$$C_-\big(e^{-\mathrm{i}4\pi\lambda} - 1\big) = 0, \qquad D_+\big(e^{\mathrm{i}4\pi\lambda} - 1\big) = 0.$$

Since $0 < |\lambda| < 1/2$, $e^{\pm \mathrm{i}4\pi\lambda} - 1 \neq 0$ and

$$C_- = c_- P_-(0) = 0 = d_+ P_+(0) = D_+.$$

Since $z_\pm$ are independent solutions, $P_+(0)$ and $P_-(0)$ cannot be *both zero* and, therefore, $c_- d_+ = 0$. If, e.g., $c_- = 0$ by (88) we get $\mathcal{P}_+ = c_+ P_+$; so that $\mathcal{P}_+$ and $\mathcal{P}_- = \overline{\mathcal{P}_+}$ can be extended on $\mathbb{R}$ to two 2π-periodic solutions of ($\mathrm{Eq}_\pm$). Analogously if $d_+ = 0$. □

A solution $\mathcal{P}_+ : [0, 2\pi] \to \mathbb{C}$ of (Eq_+) and (87) can be constructed, for example, as a fixed point of $\mathcal{P}_+ = \Psi(\mathcal{P}_+)$ where

$$\big[\Psi(\mathcal{P}_+; \lambda)\big](t) := 1 + \lambda v_0 t - \mathrm{i}2\lambda \int_0^t (\mathcal{P}_+ - 1) - \lambda^2 \int_0^t \int_0^s G\mathcal{P}_+ \, d\xi \, d\sigma,$$

$$v_0(\mathcal{P}_+; \lambda) := \frac{\mathrm{i}}{\pi} \int_0^{2\pi} (\mathcal{P}_+ - 1) + \frac{\lambda}{2\pi} \int_0^{2\pi} \int_0^s G\mathcal{P}_+ \, d\xi \, d\sigma.$$

The map $\Psi(\mathcal{P}_+; \lambda)$ smoothly depends on λ and will be a contraction for λ small enough (e.g. in the space $C^1([0, 2\pi])$). Then also the fixed point $\mathcal{P}_+(t) = \mathcal{P}_+(t; \lambda)$ smoothly depends on λ and it is simple to see that

$$\mathcal{P}_+(t; \lambda) = 1 + O\big(\lambda^2\big), \qquad \dot{\mathcal{P}}_+(t; \lambda) = O\big(\lambda^2\big). \tag{90}$$

By Lemma 3.2 λ is real. For λ small enough, by Floquet Theory (recall point (iii) of Remark 3.1) and Lemma 3.7 we can extend $\mathcal{P}_+$ and $\mathcal{P}_- := \overline{\mathcal{P}_+}$ on $\mathbb{R}$ to two 2π-periodic solutions of ($\mathrm{Eq}_\pm$). By Lemma 3.6 $\zeta_+(t) := e^{\mathrm{i}\lambda t}\mathcal{P}_+(t)$ and $\zeta_-(t) := e^{-\mathrm{i}\lambda t}\mathcal{P}_-(t)$ are two independent solutions of (67).

The fundamental solutions $\mathfrak{c}, \mathfrak{s}$ of (67) defined in (68) write

$$\mathfrak{c}(t) = \mathfrak{c}_+ \zeta_+(t) + \mathfrak{c}_- \zeta_-(t), \qquad \mathfrak{s}(t) = \mathfrak{s}_+ \zeta_+(t) + \mathfrak{s}_- \zeta_-(t), \tag{91}$$

for suitable $\mathfrak{c}_\pm, \mathfrak{s}_\pm \in \mathbb{C}$. By (68) and (90) we get

$$\mathfrak{c}_+ = \frac{1}{2} + O(\lambda), \qquad \mathfrak{c}_- = \frac{1}{2} + O(\lambda), \qquad \mathfrak{s}_+ = \frac{1}{\mathrm{i}2\lambda} + O(1), \qquad \mathfrak{s}_+ = -\frac{1}{\mathrm{i}2\lambda} + O(1)$$

and by (91)

$$\mathfrak{c}(t;\lambda) = \cos(\lambda t) + O(\lambda), \qquad \mathfrak{s}(t;\lambda) = \frac{\sin(\lambda t)}{\lambda} + O(1). \tag{92}$$

Then the estimates in (78) follow by (86).

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Appendix A. The Newtonian spin–orbit potential

In this appendix we discuss some classical facts about the spin–orbit potential $f(x,t;\mathrm{e})$ defined in (45)–(49), which are not so easy to find in the literature; see, however, A. Cayley [4].

(i) The first step in the definition of the potential f is the inversion of Kepler's equation (48). From standard contraction mapping arguments, it follows immediately, that, *there exists a unique smooth function*

$$(t,\mathrm{e}) \in \mathbb{R} \times [0,1) \to U_\mathrm{e}(t) \in \mathbb{R}$$

such that, for any $\mathrm{e} \in [0,1)$, $t \to t + \mathrm{e}U_\mathrm{e}(t)$ *is the inverse function of the function* $u \to u - \mathrm{e}\sin u$, *i.e.,* U_e *verifies*

$$U_\mathrm{e}(t) = \sin\big(t + \mathrm{e}U_\mathrm{e}(t)\big), \quad \forall (t,\mathrm{e}) \in \mathbb{R} \times [0,1).$$

Furthermore, U_e *is* 2π*-periodic and odd in*[15] t.
In fact, U_e is real-analytic in (t,e), and, for t real, the smallest radius of convergence in e (which depends on t) is $r^* = 0.6627434\ldots$; compare [16, §284].

(ii) From point (i), it follows immediately that the true anomaly $\mathrm{f}_\mathrm{e}(t)$ defined in (49) has the form

$$\mathrm{f}_\mathrm{e}(t) = t + \mathrm{e}\check{\mathrm{f}}_\mathrm{e}(t)$$

with $t \to \check{\mathrm{f}}_\mathrm{e}(t)$ 2π-periodic and odd in t (and with the same regularity properties of U_e).

(iii) From the above symmetry properties it follows that

$$G(t) = G_\mathrm{e}(t) := \frac{\exp(2\mathrm{i}\mathrm{f}_\mathrm{e}(t))}{\rho_\mathrm{e}(t)^3} = \sum_{j\in\mathbb{Z}} G_j \exp(\mathrm{i}jt), \quad \text{with } G_j = G_j(\mathrm{e}) \in \mathbb{R}.$$

Proof. By (i) and (ii) above, $t \to \mathrm{f}_\mathrm{e}(t)$ is odd while $t \to \rho_\mathrm{e}(t) = 1 - \mathrm{e}\cos \mathrm{u}_\mathrm{e}(t)$ is even. Thus, for $t \in \mathbb{R}$, $G(-t) = \overline{G(t)}$ so that, by Fourier expansion,

[15] This claim follows immediately since the map $v \to \sin(t + \mathrm{e}v)$ is a contraction with Lipschitz constant e from $X := \{v : \mathbb{R} \to \mathbb{R}$ continuous, 2π-periodic and odd$\}$ into itself endowed with the sup-norm.

$$\sum_{j\in\mathbb{Z}} G_j \exp(-\mathrm{i} jt) = G(-t) = \overline{G(t)} = \sum_{j\in\mathbb{Z}} \overline{G_j} \exp(-\mathrm{i} jt),$$

showing that $G_j = \overline{G_j}$. □

(iv) $f(x,t;\mathrm{e}) = \sum_{j\in\mathbb{Z}} \alpha_j \cos(2x - jt)$ with $\alpha_j = G_j \in \mathbb{R}$.

Proof. From the definition of f in (45)–(49), the definition of g in the preceding point and the oddness of $\mathrm{f_e}(t)$, it follows that

$$\begin{aligned} f(x,t) &= \operatorname{Re} \frac{\exp(\mathrm{i}2(x - \mathrm{f_e}(t)))}{\rho_\mathrm{e}(t)^3} = \operatorname{Re} \frac{\exp(\mathrm{i}2x)\exp(\mathrm{f_e}(-t))}{\rho_\mathrm{e}(t)^3} \\ &= \operatorname{Re}\big(\exp(\mathrm{i}2x) G(-t)\big) \\ &= \operatorname{Re}\Big(\exp(\mathrm{i}2x) \sum_{j\in\mathbb{Z}} G_j \exp(-\mathrm{i} jt)\Big) \\ &= \operatorname{Re}\Big(\sum_{j\in\mathbb{Z}} G_j \exp\big(\mathrm{i}(2x - jt)\big)\Big) \\ &= \sum_{j\in\mathbb{Z}} G_j \cos(2x - jt), \end{aligned}$$

proving (iv). □

(v) $\alpha_j = \alpha_j(\mathrm{e}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos(2\tilde{\mathrm{f}}_\mathrm{e} - ju + \mathrm{e} j \sin u)}{\tilde{\rho}_\mathrm{e}^2}\, du$ where

$$\begin{cases} \tilde{\mathrm{f}}_\mathrm{e} = \tilde{\mathrm{f}}_\mathrm{e}(\mathrm{u}) := 2\arctan\Big(c_\mathrm{e} \tan\dfrac{\mathrm{u}}{2}\Big), \quad c_\mathrm{e} := \sqrt{\dfrac{1+\mathrm{e}}{1-\mathrm{e}}}, \\ \tilde{\rho}_\mathrm{e} = \tilde{\rho}_\mathrm{e}(\mathrm{u}) := 1 - \mathrm{e}\cos\mathrm{u}. \end{cases}$$

Proof. By the parity properties of $\mathrm{f_e}(t)$ and $\rho_\mathrm{e}(t)$, $t \to \rho_\mathrm{e}^{-3} \sin(2\mathrm{f_e} - jt)$ is odd, hence

$$\begin{aligned} \alpha_j = G_j &:= \frac{1}{2\pi}\int_0^{2\pi} G(t)\exp(-\mathrm{i} jt)\, dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{\cos(2\mathrm{f_e}(t) - jt)}{\rho_\mathrm{e}^3(t)} + \mathrm{i}\, \frac{\sin(2\mathrm{f_e}(t) - jt)}{\rho_\mathrm{e}^3(t)}\right) dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos(2\mathrm{f_e}(t) - jt)}{\rho_\mathrm{e}^3(t)}\, dt. \end{aligned} \tag{A.1}$$

From Kepler's equation (48), it follows that

$$\mathrm{u}_\mathrm{e}'(t) = \frac{1}{\rho_\mathrm{e}(t)}$$

so that, changing variable of integration from t to $u = \mathrm{u_e}$ in (A.1) one gets[16] (v). □

[16] By (48), $t = u - \mathrm{e}\sin u$.

(vi) $\alpha_0(\mathrm{e}) = 0$.

Proof. From elementary trigonometry and the definition of c_{e}, it follows that

$$\cos(2\tilde{\mathrm{f}}_{\mathrm{e}}) = \frac{3\mathrm{e}^2 - 4\mathrm{e}\cos u + (2-\mathrm{e}^2)\cos(2u)}{2(1-\mathrm{e}\cos u)^2};$$

thus, changing variable of integration setting $t = \tan(u/2)$, after some algebra, one gets

$$\alpha_0 = \frac{1}{2\pi}\int_0^{2\pi} \cos(2\tilde{\mathrm{f}}_{\mathrm{e}})\,du = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{3\mathrm{e}^2 - 4\mathrm{e}\cos u + (2-\mathrm{e}^2)\cos(2u)}{2(1-\mathrm{e}\cos u)^2}\,du$$

$$= \frac{(1+a_{\mathrm{e}}^2)^2}{2}\int_{\infty}^{\infty}(1+t^2)\frac{a_{\mathrm{e}}^4 - 6a_{\mathrm{e}}^2 t^2 + t^4}{(a_{\mathrm{e}}^2 + t^2)^4}\,dt,$$

where $a_{\mathrm{e}} := 1/c_{\mathrm{e}} = \sqrt{(1-\mathrm{e})/(1+\mathrm{e})} \in (0,1]$.
By residues, this latter integral vanishes. $\square$

(vii) Points (iv), (v) and (vi) prove Eqs. (4)–(50).

Appendix B. Comparison with the numerical results of [7]

In [7] the spin–orbit problem described in Section 2.5 has been numerically investigated with the scope of finding stable periodic orbits (resonances) together with their basins of attraction. In synthesis, for several astronomically relevant parameter values, the occurrence of periodic and quasi-periodic attractors has been studied by a Monte Carlo method on the initial conditions; the percentage of initial data, which evolve towards an attractor has been computed and interpreted as a "basin-of-attraction measure," providing, in particular, a possible dynamical-system interpretation of the observed capture in the 3 : 2 spin–orbit resonance of Mercury, which is the only planet or satellite of the Solar system observed in such a state.

In particular, in [7], for the value of eccentricity $\mathrm{e} \sim 0.2056$ (which corresponds to observed eccentricity of the orbit of Mercury around the Sun), for[17] $\bar{\varepsilon} = 10^{-3}$ and for various value of the dissipation[18] K ranging from 10^{-3} up to $K = 5 \cdot 10^{-6}$, the results reported in Table 1 have been obtained.

Now, by Theorem 1.2, in order to have a periodic orbit of type p/q condition (5) has to be satisfied. Since κ is an arbitrary number smaller than one, for the purpose of the following (approximate) computation, we shall take $\kappa = 1$. Thus, we define

$$T(p,q,K) := \begin{cases} \frac{\bar{\varepsilon}}{\bar{\eta}}|\beta_{pq}| - |\bar{\nu}(\mathrm{e}) - \frac{p}{q}| & \text{if } q = 1, 2, \\ \frac{\bar{\varepsilon}^2}{\bar{\eta}}16|\beta_{pq}| - |\bar{\nu}(\mathrm{e}) - \frac{p}{q}| & \text{if } q = 4, \end{cases} \tag{B.1}$$

so that $T(p,q,K) > 0$ means that conditions (5) (with $\kappa = 1$) is satisfied while $T(p,q,K) < 0$ means that conditions (5) is not satisfied.

A (straightforward) numerical evaluation of $T(p,q,K)$ yields the results reported in Table 2.

[17] Actually, the physical values fro the Sun–Mercury spin–orbit model are: $\mathrm{e} = 0.2056$, $\bar{\varepsilon} \sim 10^{-4}$ and $K \sim 10^{-8}$ but such values would lead to computations beyond the actual possibilities of computers.

[18] Recall that dissipation constant K is related to $\bar{\eta}$ by the equation $\bar{\eta} = K\Omega_{\mathrm{e}}$; compare (45).

Table 1
Numerical results from [7]: on the top of the columns (starting from the second column) the type p/q of the $2\pi q$-periodic orbit is indicated and, below such value, the percentage of initial data corresponding to orbits approaching the p/q-orbit is reported; each row corresponds to the dissipation value reported on the first column and the "complementary percentage"[a] correspond to a seemingly quasi-periodic attractor with "irrational" frequency $1.256\ldots$.

K	1/1	5/4	3/2	2/1	5/2	3/1
10^{-3}	2%	–	5.7%	–	–	–
$5\cdot 10^{-4}$	3.9%	1%	7.6%	–	–	–
10^{-4}	4.4%	6%	10.9%	1.8%	–	–
$5\cdot 10^{-5}$	4.4%	7.7%	11.6%	3%	0.6%	–
10^{-5}	4.7%	8.4%	12.6%	2.9%	1.1%	0.5%
$5\cdot 10^{-6}$	4.7%	6.8%	13.3%	2.7%	0.6%	0.3%

[a] I.e., 100 minus the sum of values in each row (i.e., 92.3% in the first row, 87.5% in the second, etc.).

Table 2
Numerical evaluation of $T(p,q,K)$ defined in (B.1). In the first row (starting from the second column) it is indicated the couple (p,q) and below each couple, the value $T(p,q,K)$ is reported corresponding to the value of the dissipation constant K reported in the first column.

K	$(1,1)$	$(5,4)$	$(3,2)$	$(2,1)$	$(5,2)$	$(3,1)$
10^{-3}	1.05	0.0058	0.7	−0.27	−1.0	−1.67
$5\cdot 10^{-4}$	2.35	0.017	1.65	0.19	−0.84	−1.59
10^{-4}	12.81	0.11	9.24	3.92	0.75	−1.01
$5\cdot 10^{-5}$	25.88	0.22	18.74	8.60	2.75	−0.28
10^{-5}	130.46	1.16	94.69	45.99	18.76	5.55
$5\cdot 10^{-6}$	261.17	2.33	189.62	92.72	38.77	12.86

Thus, one sees that there is an almost complete agreement between the numerical experiments of [7] reported in Table 1 and the (numerical) evaluation of condition (5) (corresponding to a positive value of T if (5) is satisfied and to a negative value of T otherwise); the only exception are given by $(p,q,K)=(5,4,10^{-3})$, $(p,q,K)=(2,1,5\cdot 10^{-4})$ and $(p,q,K)=(3,1,5\cdot 10^{-5})$ where the periodic orbits are "found" theoretically but not by numerical experiments (notice also in these three cases the values of T is smaller than 1).

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