Absolutely continuous spectra of quasiperiodic Schrödinger operators

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Several aspects of the general and constructive spectral theory of quasiperiodic Schrödinger operators in one dimension are discussed. An explicit formula for the absolutely continuous (a.c.) spectral densities that yields an immediate proof of the fact that the Kolmogorov–Arnold–Moser (KAM) spectrum constructed by Dinaburg, Sinai, and Rüssmann [Funkt. Anal. Prilozen. **9**, 8 (1975); Ann. Acad. Sci. **357**, 90 (1980)] is a subset of the a.c. spectrum is provided. Some quasiperiodicity properties of the Deift–Simon a.c. eigenfunctions are proved, namely, that the normalized phase of such eigenfunctions is a quasiperiodic distribution. In the constructive part the Dinaburg–Sinai–Rüssmann theory is extended to quasiperiodic perturbations of periodic Schrödinger operators using a KAM Hamiltonian formalism based on a new treatment of perturbations of harmonic oscillators. Particular attention is devoted to the dependence upon the eigenvalue parameter and a complete control of KAM objects is achieved using the notion of Whitney smoothness.

I. INTRODUCTION

Let L_θ be a quasiperiodic Schrödinger operator in one dimension,[1–6]

$$L_\theta \equiv L(v_\theta) \equiv -\frac{d^2}{dx^2} + v_\theta(x),$$
$$v_\theta(x) \equiv V(T_x\theta), \quad T_x\theta \equiv \theta + \omega x,$$

where $x \in \mathbb{R}$, $\theta \in \mathbb{T}^d \equiv \mathbb{R}^d/2\pi\mathbb{Z}^d$, $\omega \in \mathbb{R}^d$ is a rationally independent vector and V is a real function defined on $\mathbb{T}^d$. In this paper we discuss, from two points of view, the absolutely continuous (a.c.) spectrum of L_θ. First, continuing the analysis in Refs. 2, 3, and 7, we study some general problems such as characterization almost everywhere (with respect to Lebesgue measure on $\mathbb{R}$ and/or Haar measure on $\mathbb{T}^d$) of the a.c. eigenfunctions and of spectral densities. Then we turn to the explicit construction of many (in the sense of Lebesgue measure) quasiperiodic a.c. eigenfunctions for a special class of potentials v. This second part should be regarded as a refinement of the theory in Refs. 8 and 9.

Our results in the general part are described by the following three theorems. Before describing them let us recall a few definitions. The spectral class measure $d\mu^\theta$ is given by one of the following mutually equivalent measures:

$$d\mu^\theta = \sum_1^\infty a_n \, d\mu^\theta_{\phi_n}, \quad a_n > 0, \quad \phi_n \in C_0^\infty,$$

where $\Sigma a_n < \infty$, $\{\phi_n\}$ is an L^2-dense set of C^∞ functions with compact support and $d\mu^\theta_{\phi_n}$ denotes the standard spectral measure of L_θ based upon ϕ_n. Now let $d\mu^\theta_{\text{a.c.}}$ be the a.c. part of $d\mu^\theta$ in the Jordan–Lebesgue decomposition. The essential support S of $d\mu^\theta_{\text{a.c.}}$ is uniquely determined (modulo sets of zero Lebesgue measure) by the requirement that if $A \subset S$ is also a support for $d\mu^\theta_{\text{a.c.}}$ then $\text{meas}(S - A) = 0$. Finally, let $f_\pm(x,\theta,E)$ be the solution of

$$L_\theta f = Ef \tag{1.1}$$

with $\text{Im}\, E \neq 0$, $f \in L^2(\mathbb{R}_\pm)$, $\mathbb{R}_+ \equiv (0,\infty)$, $\mathbb{R}_- \equiv (-\infty,0)$. Denoting by d/dE the Radon–Nicodym derivative with respect to Lebesgue measure, by $[g,h]$ the Wronskian $gh' - g'h \equiv g(dh/dx) - (dg/dx)h$ and by (ϕ,f) the L^2 product $\int_{\mathbb{R}} \phi(x)\bar f(x)dx$, we have the following theorem.

Theorem 1.1: For any $\phi \in C_0^\infty$, for a.e. $(\theta,E) \in \mathbb{T}^d \times S$,

$$\frac{d\mu^\theta_{\phi,\text{a.c.}}}{dE} = \frac{1}{2\pi i} \frac{|(\phi,f)|^2 + |(\phi,\bar f)|^2}{[f,\bar f]},$$

where $f(x,\theta,E) \equiv \lim_{\epsilon \downarrow 0} f_+(x,\theta,E+i\epsilon)$ and $i[f,\bar f] > 0$.

Throughout this paper a fundamental role is played by Bloch waves (or Floquet solutions). These are eigensolutions of the form $\psi = e^{i\beta x}\chi$ with $\beta \in \mathbb{R}$ and χ a quasiperiodic function with basic frequencies ω.

Theorem 1.2: Let $I \subset \mathbb{R}$ be a set of positive Lebesgue measure and assume that for a.e. (θ,E) in $\mathbb{T}^d \times I$ there exists a Bloch wave ψ. Then $I \subset S$, $[\psi,\bar\psi] \neq 0$ and, for any $\phi \in C_0^\infty$,

$$\frac{d\mu^\theta_{\phi,\text{a.c.}}}{dE} = \frac{1}{2\pi} \frac{|(\phi,\psi)|^2 + |(\phi,\bar\psi)|^2}{|[\psi,\bar\psi]|},$$
$$(\theta,E) \text{ a.e. in } \mathbb{T}^d \times I.$$

In Ref. 3 Deift and Simon showed that, for a.e. (θ,E) in $\mathbb{T}^d \times S$ there exist eigensolutions $g = e^{i[\alpha x + \beta(x)]}r(x)$, with $[g,\bar g] = -2i$ and α being the Johnson–Moser rotation number,[7,10] such that r is an L^2 quasiperiodic function, i.e., $r(x,\theta,E) = R(T_x\theta)$ with $R(\cdot,E) \in L^2(\mathbb{T}^d)$. However, no quasiperiodicity properties were proved for the phase β. Now assume that ω satisfies a Diophantine condition like

$$|\omega\cdot\nu| \equiv \left|\sum_{i=1}^d \omega_i\nu_i\right| > \frac{1}{c|\nu|^\tau},$$
$$|\nu| \equiv \sum|\nu_i| \quad (\text{any } \nu \in \mathbb{Z}^d - 0, \text{ some } c,\tau > 0), \tag{1.2}$$

and denote by

$$\mathbf{0} \equiv \left\{\Phi \in C^\infty(\mathbb{T}^d) \colon \int \Phi = 0\right\}.$$

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Then β is an (ω) quasiperiodic distribution on $\mathbf{0}$ in the sense of the following theorem.

Theorem 1.3: There exists a distribution B on $C^\infty(\mathbb{T}^d)$ such that for any $\Phi\in\mathbf{0}$,

$$\langle B,\Phi\rangle = \lim_{x\uparrow\infty}\frac{1}{x}\int_0^x \beta(y,\theta,E)\overline{\Phi}(T_y\theta)dy,$$
$$(\theta,E) \text{ a.e. in } \mathbb{T}^d\times S. \qquad (1.3)$$

Remark 1.4: Equation (1.3) determines B uniquely on $\mathbf{0}$. Taking $\Phi = e^{i\nu\cdot\theta}$, (1.3) shows that all the quasiperiodic Fourier coefficients (with $\nu\neq 0$) of β are well defined and are equal to $\hat{B}_\nu e^{i\nu\cdot\theta}$, $\hat{B}_\nu\equiv\langle B,e^{i\nu\cdot\theta}\rangle$.

Remark 1.5: Theorems 1.1 and 1.2 can be trivially extended to the case of almost periodic Schrödinger operators. Theorem 1.3 is false if ω fails to satisfy any Diophantine condition, i.e., if ω is a "Liouville vector"; compare Ref. 1.

The problem of characterizing the a.c. spectrum in terms of genuine Bloch waves remains open but we will see that it is closely related to the analysis of regularity properties of a nonlinear partial differential equation (PDE) on $\mathbb{T}^d$, namely,

$$D_\omega^2\, F = \frac{1}{F^3} + (V-E)F, \quad F(\theta)>0 \quad \text{for a.e. } \theta, \qquad (1.4)$$

where

$$D_\omega \equiv \sum_{i=1}^d \omega_i\frac{\partial}{\partial\theta_i}.$$

Equation (1.4) will be shown to be satisfied, for a.e. E in S, by $R(\cdot,E)$ in the sense of distributions.

We pass now to the constructive part of the theory. The operators that we shall consider are of the form $L^{(\epsilon)}\equiv L(v+\epsilon w)$, $v+\epsilon w\equiv V(\omega_1 x)+\epsilon W(\omega_2 x,...,\omega_d x)$, with V,W real analytic on, respectively, $\mathbb{T},\mathbb{T}^{d-1}$ and ϵ a positive number. The vector ω is assumed to satisfy a generalized Diophantine condition

$$|\omega\cdot\nu|\geqslant 1/c\Omega(|\nu|), \quad \nu\in\mathbb{Z}^d-0,\ c>0 \quad \text{(fixed)}, \qquad (1.5)$$

where $\Omega(r)\geqslant r^{d-1}$ is a monotone function growing not too fast as $r\uparrow\infty$ (see Ref. 9). Then, employing a Kolmogorov–Arnold–Moser (KAM) technique,[11–13] we will construct, for small $c\epsilon/\kappa$, a subset $\mathbf{E}^{(\epsilon)}$ of $\sigma(L^{(0)})\cap\sigma(L^{(\epsilon)})$ and for each $E\in E^{(\epsilon)}$ a Bloch wave $e^{i\alpha x}\chi(\omega x)$ with (α,ω) rationally independent and $\chi(\theta)$ analytic on $\mathbb{T}^d$. The parameter κ is a function of E asymptotic to $\sqrt{E}$ and, for some $a,b>0$ and for any $E_0>0$, the set $\mathbf{E}^{(\epsilon)}$ satisfies

$$\text{meas}\{(\sigma(L^{(0)})-\mathbf{E}^{(\epsilon)})\cap[E_0,\infty)\}$$
$$\leqslant\frac{a}{c}\Big(\sum_{|\nu|>bE_0}\frac{|\nu|\log\log\Omega(|\nu|)}{\Omega(|\nu|)}\Big). \qquad (1.6)$$

The connection with the general part is then given by Theorem 1.2 which yields immediately $\mathbf{E}^{(\epsilon)}\subset\sigma_{\text{a.c.}}(L^{(\epsilon)})$.

Before constructing such Bloch waves we will explain that the existence of quasiperiodic eigenfunctions corresponds to quasiperiodic Hamiltonian flows on $(d+1)$-dimensional tori; see, also, Refs. 14 and 15. In general, to any operator $L_{\theta_0}(u)$, $u(x)\equiv U(T_x\theta_0)$, we can associate the $(d+1)$-dimensional Hamiltonian

$$H_U(p,B,q,\theta;E)\equiv p^2/2+\omega\cdot B+(q^2/2)[E-U(\theta)],$$

where $(p,B)\in\mathbb{R}^{d+1}$ denote the generalized momenta and $(q,\theta)\in\mathbb{R}\times\mathbb{T}^d$ denote the generalized coordinates. It is readily checked that the evolution equation for q [with initial data $q(0),p(0)\equiv q'(0)$, $\theta(0)\equiv\theta_0$] is nothing but the eigenvalue equation $L_{\theta_0}q=Eq$. What we will see is that, for $E\in\mathbf{E}^{(\epsilon)}$, $H_{V+\epsilon W}$ is canonically conjugate to a system of harmonic oscillators with Hamiltonian $\alpha A_0+\omega_1 A_1+\cdots+\omega_d A_d$ in action-angle variables $(A,\theta)\in\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{T}^{d+1}$. This fact, from one side, clarifies the use of KAM techniques in the theory of quasiperiodic Schrödinger operators and, on the other side, gives a rather natural interpretation of spectral quantities such as a.c. eigenfunctions and the rotation number in terms of Hamiltonian objects. Actually we believe that the Hamiltonian H_U is integrable whenever $E\in\sigma_{\text{a.c.}}(L(u))$.

In our treatment of these matters we refine some aspects of the Dinaburg–Sinai–Rüssmann theory. For example, we will see that KAM objects, such as rotation number and Bloch waves constructed on $\mathbf{E}^{(\epsilon)}$ are C^∞ functions of $E\in\mathbf{E}^{(\epsilon)}$ in the sense of Whitney.[16–18] Exploiting this fact it will be easy to give a self-contained and complete description of the KAM spectrum $\mathbf{E}^{(\epsilon)}$ that was still missing in the literature.

Since the basic KAM techniques are by now well known (see, e.g., Ref. 19), most of the proofs in this second part will be outlined without going into detail.

The content of the rest of the paper is the following: Sec. II, proof of Theorem 1.1; Sec. III, Bloch waves; Sec. IV, weak Bloch waves; Sec. V, periodic Schrödinger operators as harmonic oscillators; Sec. VI, quasiperiodic perturbations; Sec. VII, KAM Bloch waves; Sec. VIII, Whitney smoothness; Sec. IX, structure of KAM spectra; Appendix A: on a new condition in analytic KAM; Appendix B: Moser–Deift–Simon inequality on KAM spectra.

II. PROOF OF THEOREM 1.1

We need the following facts:

(a) $$\frac{d\mu_{\phi,\text{a.c.}}}{dE}=\lim_{\epsilon\downarrow 0}\text{Im}(R_{E+i\epsilon}\phi,\phi),$$
$$R_E\equiv(L-E)^{-1},$$

for any $\phi\in C_0^\infty$ and a.e. E in S.

(b) $$R_E(x,y)\equiv g(x,y;E)$$
$$=f_+(x)f_-(y)/[f_+,f_-],\quad \text{Im}\,E\neq 0,$$

for $x\geqslant y$ and symmetrically for $x<y$ ($f_\pm$ are the eigenfunctions introduced in Sec. I).

(c) $$f_\pm(x,\theta,E)=\text{const}(f_1(x,\theta,E)$$
$$\pm h_\pm(\theta,E)f_2(x,\theta E)),$$

where f_1, f_2 solve (1.1) with $f_1(0)=f_2'(0)=1, f_1'(0)=f_2(0)=0$ and $h_\pm$ are, for every θ, the Herglotz functions defined by $\lim_{x\to\pm\infty}\mp f_1/f_2$. We recall that a function h is Herglotz if it maps holomorphically the open upper half plane $\mathbb{C}_+$ into itself. We will denote the boundary value of h, existing a.e. on $\mathbb{R}$, by the same symbol. For more information see, e.g., Ref. 20.

(d) For a.e. E in $\mathbb{R}$,

$$-\lim_{\epsilon\downarrow 0} \operatorname{Re} \int_{\mathbb{T}^d} h_+(\theta,E+i\epsilon)d\theta$$
$$\equiv \gamma(E) = \gamma_+(E,\theta)$$
$$\equiv (\text{highest}) \text{ Lyapunov exponent for } L_\theta - E$$

and

$$S = \{E: \gamma(E) = 0\} \subset \{E: h_+ = -\bar{h}_-, \operatorname{Im} h_+ > 0\}.$$

Equation (a) is a simple consequence of Stone's formula (see, e.g., Ref. 21). Equations (b) and (c) are the main results of Weyl's limit-point theory (see Ref. 22). Equation (d) is proved in Ref. 2.

Notice that, for a.e. $E, f_\pm(x,\theta,E+i\epsilon)$ converge, as $\epsilon\downarrow 0$, uniformly on compact x sets. Also, for a.e. E in S, (d) shows that $f_+ = \bar{f}_-$ with $[f_+,f_-] = -2i \operatorname{Im} h_+$. These observations together with (a) and the evaluation

$$\operatorname{Re} \iint_{x>y} \psi(x)\bar{\psi}(y)\phi(x)\bar{\phi}(y)dx\,dy = \frac{1}{2}|(\bar{\psi},\phi)|^2,$$

valid for any $\psi\in C(\mathbb{R})$ and $\phi\in C_0^\infty(\mathbb{R})$, make Theorem 1.1 plain.

III. BLOCH WAVES

In this section we prove some elementary properties of (genuine≡smooth) Bloch waves and Theorem 1.2.

Lemma 3.1: (i) If $\psi(x) = e^{i\beta x}\chi(\omega x)$ is a Bloch wave for $L_0 - E$ then $\psi(x,\theta) \equiv e^{i\beta x}\chi(T_x\theta)$ is a Bloch wave for $L_\theta - E$.

(ii) Let I be as in Theorem 1.2. Then ψ can be written (a.e. on I) in the form $e^{i\alpha x}\chi(T_x\theta)$ with (α,ω) rationally independent.

(iii) If $\psi = e^{i\alpha x}\chi(\omega x)$ is a Bloch wave (α,ω) rationally independent, then $[\psi,\bar{\psi}]\neq 0$ and $\min_{\mathbb{T}^d}|\psi| > 0$.

Proof: Since ψ solves (1.1) with $\theta = 0$, χ satisfies

$$D_\omega^2\chi + 2i\beta D_\omega\chi + (E - \beta^2 - V)\chi = 0 \tag{3.1}$$

at $\theta = \omega x$. But because $\{\theta = \omega x: x\in\mathbb{R}\}$ is dense in $\mathbb{T}^d$, (3.1) holds identically on $\mathbb{T}^d$. In particular, it holds at $\theta + \omega x$ and (i) is proved.

Property (ii) follows easily from (i) and the fact that $E\in\mathbb{R}\to\alpha(E)\in\bar{\mathbb{R}}_+$ is an increasing function, constant only on spectral gaps where it takes value in $\{\omega\cdot\nu/2, \nu\in\mathbb{Z}^d\}$; see Ref. 7.

If $[\psi,\bar{\psi}] = 0$ we would have $\chi = ae^{-2i\alpha x}\bar{\chi}$, for some $a\in\mathbb{C}$. But two quasiperiodic functions cannot be equal unless they have the same basic frequencies; see, e.g., Ref. 23. Thus $[\psi,\bar{\psi}]\neq 0$. If $|\psi|$ were not bounded away from 0, there would exist $x_n\uparrow\infty$ for which $\psi(\omega x_n)\to 0$, but this would imply $[\psi,\bar{\psi}] = 0$, a contradiction. □

Proof of Theorem 1.2: From the above lemma $[\psi,\bar{\psi}]\neq 0$ (a.e.) on I. Thus the Lyapunov number vanishes a.e. on I and $I\subset S$ by Kotani's results [see (d), Sec. II]. Now fix E (a.e.) in I and let g be the Deift–Simon function described in Sec. I. Then, for a.e. θ and all x,

$$g(x,\theta) = a\psi(x,\theta) + b\bar{\psi}(x,\theta)$$

for some complex numbers a,b depending on θ. Taking absolute values one obtains

$$R^2(T_x\theta) + c|\chi(T_x\theta)|^2$$
$$= de^{2i\alpha x}\chi^2(T_x\theta) + \bar{d}e^{-2i\alpha x}\bar{\chi}^2(T_x\theta),$$

where $c = -(|a|^2 + |b|^2)$ and $d = 2a\bar{b}$. Now take $y>0$ and $\nu\in\mathbb{Z}^d$. Multiply the above equation by $(1/y)\exp[-2i(t+\alpha x) - iT_x\theta\cdot\nu]$ and integrate it from 0 to y with respect to x. Since R^2 and $|\chi|^2$ belong to $L^1(\mathbb{T}^d)$ we can use the ergodic theorem to let $y\uparrow\infty$ and conclude

$$0 = de^{-2it}(\hat{\chi}^2)_\nu, \text{ for all } \nu, \text{ a.e. } (t,\theta)\in\mathbb{T}^{d+1},$$

where $(\hat{\ })_\nu$ denote Fourier coefficients. This shows that $d = 0$, i.e., either $a = 0$ or $b = 0$. Theorem 1.2 follows now from Theorem 1.1. □

IV. WEAK BLOCH WAVES

Here we discuss the a.c. Deift–Simon eigenfunctions g on S and prove Theorem 1.3. Henceforth we will often omit the sentence (E,θ) a.e. in $S\times\mathbb{T}^d$.

Since $[g,\bar{g}] = -2i$, $r(x)$ never vanishes and the normalized phase β is a well-defined function from $\mathbb{R}\to\mathbb{R}$. The Schrödinger equation for g implies

$$r'' = 1/r^3 + (v_\theta - E)r, \tag{4.1}$$
$$\beta' = 1/r^2 - \alpha, \tag{4.2}$$

with initial data $r(0) = R(\theta)$, $r'(0) = D_\omega R(\theta)$, and $\beta(0) = 0 \pmod{2\pi}$. [The initial value for β is explained by the identification $g(x,\theta,E) = R(\theta)f(x,\theta,E)$, cf. Ref. 3]. Deift and Simon in Ref. 3, extending to S a formula by Johnson and Moser, proved

$$\lim_{x\uparrow\infty}\frac{1}{x}\int_0^x\frac{1}{r^2} = \int_{\mathbb{T}^d}\frac{1}{R^2} = \alpha. \tag{4.3}$$

This, together with the Schrödinger equation for g, yields easily the finiteness of

$$\overline{\lim}\,\frac{1}{x}\int_0^x|g'|^2 \text{ and } \overline{\lim}\,\frac{|r'| + r + r^{-1}}{x}. \tag{4.4}$$

Our next goal is to show that (4.1) and (4.3) imply $R^{-3}\in L^1$ and that R is a distributional solution of (1.4). Let $0\leqslant\Phi\in C^\infty(\mathbb{T}^d)$ and write $\phi(x) = \phi(x,\theta)\equiv\Phi(T_x\theta)$. Then by the ergodic theorem, (4.4) and (4.1),

$$\int_{\mathbb{T}^d}RD_\omega^2\phi = \lim_{x\uparrow\infty}\frac{1}{x}\int_0^x R(T_y\theta)(D_\omega^2\Phi)(T_y\theta)dy$$
$$= \lim\frac{1}{x}\int_0^x r\phi''$$
$$= \lim\frac{1}{x}\left\{[r\phi']_0^x - [r'\phi]_0^x + \int_0^x r''\phi\right\}$$
$$= \lim\frac{1}{x}\int_0^x\left(\frac{1}{r^3} + (v_\theta - E)r\right)\phi. \tag{4.5}$$

Another application of the ergodic theorem to positive random variables shows

$$\lim \frac{1}{x}\int_0^x \frac{\phi}{r^3}=\int_{\mathbb{T}^d}\frac{\Phi}{R^3}.$$

But then from (4.5) we conclude, *a fortiori*, that

$$\int_{\mathbb{T}^d}\frac{\Phi}{R^3}=\int_{\mathbb{T}^d}(E-V)R\Phi+\int_{\mathbb{T}^d}RD_\omega^2\,\Phi<\infty. \qquad (4.6)$$

In particular, by taking $\Phi\equiv 1$ we get

$$\int_{\mathbb{T}^d}\frac{1}{R^3}=\int_{\mathbb{T}^d}(E-V)R. \qquad (4.7)$$

Now we can repeat the computation in (4.5) with an arbitrary $\Phi\in C^\infty(\mathbb{T}^d)$ and get back (4.6). This is the same as saying the R is a weak solution of (1.4).

Next we turn to the proof of Theorem 1.3. Since ω satisfies (1.2),

$$B(\theta)\equiv\sum_{\nu\neq 0}\frac{1}{i\omega\cdot\nu}\left(\widehat{\frac{1}{R^2}}\right)_\nu e^{i\nu\cdot\theta}$$

is seen to be a distribution on $C^\infty(\mathbb{T}^d)$. In fact, if $t>\tau+d/2$,

$$\begin{aligned}\sum_{\nu\neq 0}|\widehat{B}_\nu|^2(1+\nu\cdot\nu)^t &\equiv\sum_{\nu\neq 0}\frac{1}{|\omega\cdot\nu|^2}\left|\left(\widehat{\frac{1}{R^2}}\right)_\nu\right|^2\frac{1}{(1+\nu\cdot\nu)^t}\\ &<c^2\left(\int\frac{1}{R^2}\right)^2\sum\frac{|\nu|^{2m}}{(1+\nu\cdot\nu)^t}<\infty\end{aligned}$$

shows that $B\in H_{-t}(\mathbb{T}^d)$. Now denote by D_ω^{-1} the linear operator

$$D_\omega^{-1}:\Phi\in\mathbf{0}\to D_\omega^{-1}\Phi\equiv\sum_{\nu\neq 0}\frac{\widehat{\Phi}_\nu}{i\omega\cdot\nu}e^{i\nu\cdot\theta}\in\mathbf{0}.$$

Then by (4.3), the ergodic theorem, and (4.2) we have

$$\begin{aligned}\langle B,\Phi\rangle &=\langle B,D_\omega D_\omega^{-1}\Phi\rangle\\ &=-\langle D_\omega B,D_\omega^{-1}\Phi\rangle\\ &=-\langle 1/R^2-\alpha,D_\omega^{-1}\Phi\rangle\\ &\equiv-\int\left(\frac{1}{R^2}-\alpha\right)\overline{D_\omega^{-1}\Phi}\\ &=-\lim\frac{1}{x}\int_0^x\left(\frac{1}{R^2}-\alpha\right)\overline{D_\omega^{-1}\Phi}(T_y\theta)dy\\ &=-\lim\frac{1}{x}\int_0^x\beta'\,\overline{D_\omega^{-1}\Phi}(T_y\theta)dy\\ &=-\lim\frac{1}{x}\left[\beta\,\overline{D_\omega^{-1}\Phi}(T_y\theta)\right]_0^x\\ &\quad+\lim\frac{1}{x}\int_0^x\beta\overline{\Phi}(T_y\theta)\\ &=\lim\frac{1}{x}\int_0^x\beta\overline{\Phi}(T_y\theta),\end{aligned}$$

in which the last equality holds because α is the rotation number of g so that $\lim(1/x)\,\beta(x)=0$. □

To connect the existence of smooth Bloch waves with regularity properties for (1.4), assume that V is of class $C^\infty(\mathbb{T}^d)$ and that R is a smooth solution of (1.4). Then, by the equation, $\min R>0$ and $1/R^2-\alpha$ belongs to $C^\infty(\mathbb{T}^d)$. Thus also B, as defined above, is a smooth function and we can identify $\beta(x,\theta)$ with the quasiperiodic solution of (4.2) $B(T_x\theta)-B(\theta)$. Unfortunately, regularity properties for such nonlinear equations on tori are difficult to obtain by general PDE methods. (See, however, Ref. 24.)

V. PERIODIC SCHRÖDINGER OPERATORS AS HARMONIC OSCILLATORS

From now on we will be concerned with the constructive part of the theory. In this section we look at periodic Schrödinger operators $L^{(0)}$ from the Hamiltonian point of view described in the Introduction. We show briefly that for each E in the interior $\overset{0}{\sigma}$ of the spectrum σ of $L^{(0)}$, the Hamiltonian H_V of Sec. I is conjugated to $\alpha_0A_0+\omega_1A_1$, $(A_0,A_1)\in\mathbb{R}_+\times\mathbb{R}$, $\alpha_0\equiv$ rotation number for $L^{(0)}-E$. (We learned about the integrability of H_V in Ref. 15.) For more details on this and the following sections see Ref. 25.

From Floquet theory[26] one knows that, for each $E\in\overset{0}{\sigma}$, there exist two independent Bloch waves $f_0,\bar{f}_0$ of the form

$$\begin{aligned}f_0(x)&\equiv e^{i\alpha_0x}\chi_0(\omega_1x)\\ &\equiv f_1(x)+\frac{e^{i\alpha_0(2\pi/\omega_1)}-f_1(2\pi/\omega_1)}{f_2(2\pi/\omega_1)}f_2(x),\\ &\qquad\chi_0\in C(\mathbb{T}),\end{aligned}$$

with

$$\kappa\equiv\frac{i}{2}[f_0,\bar{f}_0]=\operatorname{Im}f_0'(0)=\frac{\sin(\alpha_0(2\pi/\omega_1))}{f_2(2\pi/\omega_1)}>0.$$

Now define

$$Q(\theta_0,\theta_1)\equiv\operatorname{Re}F_0(\theta_0,\theta_1),\quad P(\theta_0,\theta_1)\equiv\operatorname{Re}DF_0(\theta_0,\theta_1),$$

where

$$F_0(\theta_0,\theta_1)\equiv e^{i(\theta_0-(\alpha_0/\omega_1)\theta_1)}f_0(\theta_1/\omega_1),\quad(\theta_0,\theta_1)\in\mathbb{T}^2,$$

$$D\equiv\alpha_0\frac{\partial}{\partial\theta_0}+\omega_1\frac{\partial}{\partial\theta_1}.$$

One recognizes easily that $x\to F_0(\theta_0+\alpha_0x,\theta_1+\omega_1x)$ is an eigensolution for $L(v_{\theta_1})$ and that

$$\frac{d}{dx}F_0(\theta_0+\alpha_0x,\theta_1+\omega_1x)=DF_0(\theta_0+\alpha_0x,\theta_1+\omega_1x).$$

Moreover, from

$$\frac{\partial Q}{\partial\theta_0}P-Q\frac{\partial P}{\partial\theta_0}=\frac{i}{2}[f,\bar{f}]=\frac{i}{2}[f_0,\bar{f}_0]=\kappa>0,$$

it follows readily that the map

$$\begin{aligned}(r,B,\theta_0,\theta_1)\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{T}^2&\to(p,B,q,\theta_1)\\ &\equiv(rP(\theta_0,\theta_1),B,rQ(\theta_0,\theta_1),\theta_1)\end{aligned}$$

is a diffeomorphism onto the phase space of H_V, i.e., $\mathbb{R}^3\times\mathbb{T}-(0,\mathbb{R},0,\mathbb{T})$. Now we can construct a diffeomorphism

$$\mathbf{C}:(p,B,q,\theta_1)\to(A_0,A_1,\theta_0,\theta_1)\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{T}^2$$

by setting $A_0=(r^2/2)\kappa$,

$$A_1=B+\frac{r^2}{2}\left(\frac{\partial Q}{\partial\theta_1}P-Q\frac{\partial P}{\partial\theta_1}\right).$$

Straightforward computations will show first that $dp\wedge dq + dB\wedge d\theta_1 = dA_0\wedge d\theta_0 + dA_1\wedge d\theta_1$, so that $\mathbf{C}$ is canonical, and then

$$(p^2/2) + \omega_1 B + (q^2/2)(E - V) = \alpha_0 A_0 + \alpha_1 A_1,$$

confirming what we claimed above.

Remark 5.1: Even for V merely continuous, α_0, f_1, f_2, and κ are real analytic functions of E.[26] Furthermore, α_0 maps $\sigma(L^{(0)})$ onto $[0,\infty)$, $d\alpha_0/dE>0$ on $\overset{0}{\sigma}$ and, setting $\partial\overset{0}{\sigma} \equiv \{E_0^0 < E_1^0 < \cdots\}$, $a_k^0 \equiv \alpha_0(E_k^0) = h\omega_1/2$ for some integer h. Now denoting by $e_0(a)$, $a\in R_+$, $a\neq a_k^0$, the inverse function of α_0, one can easily show that, if $\rho<\omega_1/4$, e_0 admits a holomorphic extension to

$$D(\rho;\mathbf{A}_0) \equiv \bigcup_{a_0\in\mathbf{A}_0} \{a\in\mathbb{C}: |a-a_0|<\rho\},$$

with

$$\mathbf{A}_0 \equiv \bigcup_{k=0}^{\infty} [a_k^0+\rho, a_{k+1}^0-\rho].$$

This will be of later use.

VI. QUASIPERIODIC PERTURBATIONS

Now let $\epsilon>0$. Under the canonical transformation

$$(p,B_1,\ldots,B_d,q,\theta_1,\ldots,\theta_d)\in\mathbb{R}^{d+2}\times\mathbb{T}^d - (0,\mathbb{R}^d,0,\mathbb{T}^d)\rightarrow(A,\theta)$$
$$\equiv(A_0,A_1,\ldots,A_d,\theta_0,\theta_1,\ldots,\theta_d)\in\mathbf{M}\equiv\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{T}^{d+1},$$
$$(A_0,A_1,\theta_0,\theta_1)\equiv\mathbf{C}(p,B,q,\theta_1),\quad A_i=B_i,\quad i\geqslant 2$$

($\mathbf{C}$ as in Sec. V), the Hamiltonian $H_{V+\epsilon W}(\cdot;E)$, $E\in\overset{0}{\sigma}(L^{(0)})$, takes the form

$$H_\epsilon(A,\theta;E)\equiv\omega^{(0)}\cdot A + \epsilon A_0 F(\theta)$$

with

$$\omega^{(0)}\equiv(\alpha_0,\omega), F(\theta)\equiv -[Q^2(\theta_0,\theta_1)/\kappa]W(\theta_2,\ldots,\theta_d).$$

In this section we describe an iterative scheme that will allow us to integrate H_ϵ for special values of the parameter E and small ϵ. Henceforth, it will be more convenient to consider H_ϵ parametrized by the rotation number $\alpha_0\equiv a$ rather than by the eigenvalue E. It will be only later that we shall express our result directly in terms of eigenvalues. We start by considering the jth order analog of H_ϵ. Let $j\geqslant 0$ and, for $(A,\theta)\in\mathbf{M}$, let

$$H^{(j)}(A,\theta;a,\epsilon)\equiv\omega^{(j)}(a;\epsilon)\cdot A + \epsilon^{2^j}A_0F^{(j)}(\theta;a,\epsilon),$$
$$\omega^{(j)}\equiv(\omega_0^{(j)},\omega_1,\ldots,\omega_d).$$

Assume that $\omega_0^{(j)}$ and $F^{(j)}$, as functions of a, are holomorphic in

$$D_j\equiv D(\rho_j;\mathbf{A}^{(j)})\equiv\bigcup_{a_0\in A^{(j)}}\{a\in\mathbb{C}: |a-a_0|\leqslant\rho_j\}$$

for some $\mathbf{A}^{(j)}\subset\mathbb{R}$. Also, as a function of $\theta\in\mathbb{R}^{d+1}$, $F^{(j)}$ is required to have holomorphic extension to

$$S_j\equiv S^{d+1}(\xi_j)\equiv\{\theta\in\mathbb{C}^{d+1}: |\mathrm{Im}\,\theta_i|\leqslant\xi_j\},\quad \xi_j>0,$$

with

$$\|F^{(j)}\|_{\xi_j,\rho_j}\equiv\sup_{(\theta,a)\in S_j\times D_j}|F^{(j)}|\leqslant M_j$$

independently of ϵ. Notice that because of the analyticity assumptions on V and W, $H_\epsilon(A,\theta;e_0(a))$, $\mathbf{A}_0$ as in Remark 5.1, satisfies the above hypothesis, thus we can set $H^{(0)}\equiv H_\epsilon$. Now, let $\delta_j<\xi_j/2$ and let us define the main recursive objects

$$\zeta(s)\equiv 1+\sum_{\nu\in\mathbb{Z}^{d+1}-0}|\nu|\Omega(|\nu|)e^{-s|\nu|},\ s>0,$$
$$N_j\equiv 2^{j+1}\delta_j^{-1}\log\epsilon^{-1},$$
$$F_R^{(j)}(\theta)\equiv\sum_{|\nu|>N_j}\hat{F}_\nu^{(j)}e^{i\nu\cdot\theta},$$
$$\rho_{j+1}\equiv\min\left\{\left(2cN_j\Omega(N_j)\sup_{a\in D_j}\left|\frac{d\omega_0^{(j)}}{da}\right|\right)^{-1},\frac{\rho_j}{2}\right\},$$
$$\xi_{j+1}\equiv\xi_j-2\delta_j,$$
$$A^{(j+1)}\equiv\{a\in\mathbf{A}^{(j)}: |\omega^{(j)}(a)\cdot\nu|\geqslant 1/c\Omega(|\nu|),$$
$$\nu\in\mathbb{Z}^{d+1}-0,\ |\nu|\leqslant N_j\},$$
$$D_{j+1}\equiv D(\rho_{j+1};\mathbf{A}^{(j+1)}),$$
$$S_{j+1}\equiv S^{d+1}(\xi_{j+1}).$$

Lemma 6.1 (Inductive Lemma): If $a\in\mathbf{A}^{(j+1)}$ and ϵ is small enough, i.e.,

$$K_1\zeta(\delta_j)\delta_j^{-1}cM_j\epsilon^{2^j}\leqslant 1, \tag{6.1}$$

where K_1 is a universal constant, then the function

$$(A',\theta)\in\mathbf{M}\rightarrow A'\cdot\theta+\epsilon^{2^j}A_0\Phi_j(\theta;a,\epsilon),$$
$$\Phi_j\equiv\sum_{0<|\nu|\leqslant N_j}\frac{\hat{F}_\nu^{(j)}}{-i\omega^{(j)}\cdot\nu}e^{i\nu\cdot\theta}$$

is the generating function of a surjective canonical transformation, $(A,\theta)\rightarrow(A',\theta')=(A'(A,\theta),\theta'(\theta))$, that conjugates $H^{(j)}(A,\theta)$ to

$$H^{(j+1)}(A',\theta';a,\epsilon)$$
$$\equiv H^{(j)}(A(A',\theta'),\theta(\theta'))$$
$$=\omega^{(j+1)}\cdot A'+\epsilon^{2^{j+1}}A_0'F^{(j+1)}(\theta';a,\epsilon),$$

where

$$\omega^{(j+1)}\equiv(\omega_0^{(j)}+\epsilon^{2^j}\hat{F}_0^{(j)},\omega),$$
$$F^{(j+1)}(\theta'(\theta))\equiv\frac{\partial\Phi_j}{\partial\theta_0}(\theta)F^{(j)}(\theta)+\frac{F_R^{(j)}(\theta)}{\epsilon^{2^j}}.$$

Furthermore, $a\in\mathbf{A}^{(j+1)}\rightarrow\omega_0^{(j)}(a)$ and $(\theta,a)\in R^{d+1}\times\mathbf{A}^{(j+1)}\rightarrow F^{(j+1)}(\theta;a)$ have holomorphic extensions to, respectively, D_{j+1} and $S_{j+1}\times D_{j+1}$ with

$$\|F^{(j+1)}\|_{\xi_{j+1},\rho_{j+1}}\leqslant K_2\zeta(\delta_j)\delta_j^{-(d+1)}c\,M_j^2\equiv M_{j+1}, \tag{6.2}$$

in which K_2 is a second universal constant.

Applying this Lemma infinitely many times one can integrate H_ϵ for $a\in\mathbf{A}^{(\infty)}\equiv\bigcap_{j=0}^{\infty}\mathbf{A}^{(j)}$.

Theorem 6.2: Let $\{\delta_j\}$ be such that $\Sigma_{j=0}^{\infty}\delta_j<\xi/2$, let $a\in\mathbf{A}^{(\infty)}$ and let ϵ verify

$$(K_1/K_2)\epsilon\tau\leqslant 1 \tag{6.3}$$

with

$$\tau\equiv K_2\psi cM_0,\quad \psi\equiv\prod_{j=0}^{\infty}[\zeta(\delta_j)\delta_j^{-(d+1)}]^{1/2^j}.$$

Then the Hamiltonian $H^{(0)}$ is conjugate to

$$H^{(\infty)} \equiv \omega^{(\infty)} \cdot A,$$

where $\omega^{(\infty)} \equiv (\omega_0^{(\infty)}, \omega)$ satisfies

$$c|\omega_0^{(\infty)} - a| \leqslant \frac{1}{K_2} \sum_{j=0}^{\infty} (\epsilon\tau)^{2^j}, \quad |\omega^{(\infty)} \cdot \nu| \geqslant \frac{1}{c\Omega(|\nu|)}, \quad \nu \in \mathbf{Z}^{d+1} - 0. \tag{6.4}$$

The (surjective) canonical transformation conjugating $H^{(0)}$ to $H^{(\infty)}$ has the form

$$(A', \theta') \in \mathbf{M} \to (A, \theta) \equiv (S(\theta')A', \theta'_0 + \epsilon\Delta(\theta'_0), \theta'_1, \ldots, \theta'_d) \in \mathbf{M} \tag{6.5}$$

with S a $(d+1)\times(d+1)$ matrix of the form

$$\begin{bmatrix} 1+\epsilon s_0 & 0 & 0 & \cdots & 0 \\ \epsilon s_1 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ \epsilon s_d & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Moreover, the vector $s \equiv (s_0, \ldots, s_d)$ and Δ have holomorphic extensions to $S^{d+1}(\xi_\infty)$, $\xi_\infty \equiv \xi - 2\sum_{j=0}^{\infty} \delta_j$, and

$$\max\{\|s\|_{\xi_\infty}, \|\Delta\|_{\xi_\infty}\} \leqslant (K_1/K_2)\tau.$$

Remark 6.3: Examples of $\{\delta_j\}$ and Ω such that $\psi < \infty$ are displayed in Appendix A.

Remark 6.4: Perturbations of the Hamiltonian of the form $h(A,\theta) \equiv \bar{\omega} \cdot A$ were investigated in Refs. 27 and 28 using Moser's idea of "modified systems."[29]

Remark 6.5: An easy corollary of Theorem 6.2 is that all the eigensolutions of $L^{(\epsilon)}$ for $E \in \widetilde{\mathbf{E}}^{(\epsilon)} \equiv \alpha_0^{-1}(\mathbf{A}^{(\infty)})$ are quasiperiodic with basic frequencies $(\omega_0^{(\infty)}, \omega)$. Also, since all the transformations involved in the process are close to the identity it is easy to see that $\omega_0^{(\infty)}$ coincides with the rotation number α.

Remark 6.6: From an elementary asymptotic analysis ($E \gg 1$) of the periodic case, one realizes that $\|Q^2\|/K \sim 1/\sqrt{E}$ so that $M_0 \sim \|W\|/\sqrt{E}$.

VII. KAM BLOCH WAVES

Even though we already obtained a complete description of the quasiperiodic eigenfunctions of $L^{(\epsilon)}$ for $E \in \widetilde{\mathbf{E}}^{(\epsilon)}$, it is not immediate from the above analysis that such eigenfunctions are of the form $e^{i\alpha x}\chi(\omega x)$. Since this representation is crucial in the application of Theorem 1.2, we proceed now with a direct construction of Bloch waves for values of E in a set $\mathbf{E}^{(\epsilon)} \subset \sigma(L^{(0)})$, which *a priori* need not be identical to $\widetilde{\mathbf{E}}^{(\epsilon)}$.

The eigenvalue equation $L^{(\epsilon)} f = Ef$ is equivalent to the first-order system

$$y' = \begin{bmatrix} 0 & 1 \\ V(\theta_1) - E & 0 \end{bmatrix} y + \epsilon W(\theta_2, \ldots, \theta_d) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} y, \quad \theta' = \omega, \tag{7.1}$$

with $y = \binom{f}{f'}$. A fundamental matrix for (7.1) at $\epsilon = 0$ is

$$Y = \begin{bmatrix} f_0 & \bar{f}_0 \\ f'_0 & \bar{f}'_0 \end{bmatrix} \quad (f_0 \text{ as in Sec. V}).$$

By setting

$$Y = Te^{Cx}, \quad T \equiv \begin{bmatrix} \chi_0(\omega_1 x) & \bar{\chi}_0 \\ i\alpha_0\chi_0 + \omega_1\chi'_0 & -i\alpha_0\bar{\chi}_0 + \omega_1\bar{\chi}'_0 \end{bmatrix}, \quad C \equiv \begin{bmatrix} i\alpha_0 & 0 \\ 0 & -i\alpha_0 \end{bmatrix},$$

the system (7.1) becomes, under the change of variable $y = Tz$,

$$z' = Cz + \epsilon Pz, \quad \theta' = \omega \tag{7.2}$$

with

$$P \equiv \frac{W(\theta_2, \ldots, \theta_d)}{2\kappa} \begin{bmatrix} -i|\chi_0(\theta_1)|^2 & -i\bar{\chi}_0^2(\theta_1) \\ i\chi_0^2 & i|\chi_0|^2 \end{bmatrix}.$$

Notice that $P \in \mathbf{G}_0 \equiv \{G \in \mathbf{G}: \operatorname{tr} \int_{\mathbf{T}^d} G = 0\}$ where $\mathbf{G}$ denotes the ring of matrix-valued functions on $\mathbf{T}^d$ of the form

$$G = \begin{bmatrix} g & h \\ \bar{h} & \bar{g} \end{bmatrix}.$$

Theorem 7.1: If ϵ satisfies the smallness condition

$$(K_1/K_2)\epsilon\tau \leqslant 1, \quad \tau \equiv K_2 \psi c M_0,$$

where K_1 and K_2 are suitable universal constants, then one can construct a set $\mathbf{E}^{(\epsilon)} \subset \sigma(L^{(0)})$ and, for each $E \in \mathbf{E}^{(\epsilon)}$, a change of variables $z = (I + \epsilon U)w$, with $U \in G_0$, which transforms (7.2) into the trivial system

$$w' = \begin{bmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{bmatrix} w, \quad \theta' = \omega.$$

Furthermore, U as a function of $\theta \in \mathbf{T}^d$ admits a holomorphic extension to $S^d(\xi_\infty)$, for a suitable $\xi_\infty > 0$, with $\|U\|_{\xi_\infty} \leqslant (K_1/K_2)\tau$ and α verifies

$$|\alpha - |\omega \cdot \nu/2|| \geqslant \frac{1}{c\Omega(|\nu|)}, \quad \nu \in \mathbf{Z}^d - 0, \quad E \in \mathbf{E}^{(\epsilon)},$$

$$\sup_{E \in \mathbf{E}^{(\epsilon)}} |\alpha - \alpha_0| < \epsilon\tau/c.$$

Remark 7.2: Above we used the same symbols for quantities that are analogous, but not always identical, to the ones appearing in Sec. VI.

The proof of this result is based on a scheme very similar to the one described in Sec. VI: One removes infinitely many times the order of the perturbation of systems like

$$z'_j = \begin{bmatrix} i\alpha_j & 0 \\ 0 & -i\alpha_j \end{bmatrix} z_j + \epsilon^{2^j} P^{(j)} z_j, \quad \theta' = \omega \quad (P^{(j)} \in \mathbf{G}_0),$$

by the aid of a change of variable $(I + \epsilon^{2^j} U_j(\theta)) z_{j+1} = z_j$. The set $\mathbf{E}^{(\epsilon)}$ will be given by $\alpha_0^{-1}(\mathbf{A}^{(\infty)})$ where $\mathbf{A}^{(\infty)} \equiv \cap \mathbf{A}^{(j)}$, where as in Sec. V, $\mathbf{A}^{(0)}$ is the positive half-line minus suitable intervals of length 2ρ and

$$\mathbf{A}^{(j+1)} \equiv \{a \in \mathbf{A}^{(j)}: |\alpha_j(a) - \omega \cdot \nu/2| \geqslant 1/c\Omega(|\nu|), \ \nu \in \mathbf{Z}^d, \ 0 < |\nu| \leqslant N_j\}$$

with N_j denoting the jth cutoff in the Fourier expansion of $P^{(j)}$.

VIII. WHITNEY SMOOTHNESS

In this section we study the E dependence of the KAM limits. Following Ref. 16 we say that a function $f: \mathbf{A} \subset \mathbf{R} \to \mathbf{R}$ belongs to $C_W^n(\mathbf{A})$ if there exist, on $\mathbf{A}$, functions f_k, $0 \leqslant k \leqslant n$,

$f_0 \equiv f$, with the following property: For each $x_0 \in A$ and $\epsilon > 0$ there is a $\delta > 0$ s.t. if $x, x' \in \{y \in \mathbf{A}: |y - x_0| < \delta\}$ then

$$\left| f_k(x) - \sum_{h=0}^{n-k} \frac{f_{h+k}(x')}{h!}(x - x')^h \right| \leqslant \epsilon |x - x'|^{n-k}. \quad (8.1)$$

At interior points this definition coincides with the standard one but the next lemma shows how nontrivial $C^n_W(\mathbf{A})$ functions can arise.

Lemma 8.1: Let $\mathbf{A} \subset \mathbb{R}$, $r_j \downarrow 0$, and $\{g_j\}$ be a sequence of holomorphic functions on $D(r_j, \mathbf{A})$ which are real on $\mathbf{A}$. If

$$\sum \|g_j\|_{r_j} r_j^{-n} < \infty,$$

then $g \equiv \Sigma g_j$ belongs to $C^n_W(\mathbf{A})$.

Proof: Since for any $k \leqslant n$

$$\sup_{\mathbf{A}} \left| \frac{d^k g}{dx^k} \right| \leqslant \sum \left\| \frac{d^k g_j}{dx^k} \right\|_{r_j/2} \leqslant 2^k \sum \|g_j\|_{r_j} r_j^{-k} < \infty,$$

we can define

$$\frac{d^k g}{dx^k} \equiv \sum \frac{d^k g_j}{dx^k}$$

on A. To check that the $d^k g/dx^k$ are the Whitney derivatives of g, let x, $x' \in \mathbf{A}$, let $s = s(|x - x'|)$ be such that $r_{s+1} \leqslant |x - x'| < r_s$, and consider the splitting $g = g^{[s]} + \tilde{g}^{[s]}$ with $g^{[s]} \equiv \Sigma^s g_j$. The lemma follows now from $g^{[s]} \in C^\infty \times (D(r_s; \mathbf{A}) \cap \mathbb{R})$, the inequality

$$\sup_{\mathbf{A}} \left| \frac{d^k \tilde{g}^{[s]}}{dx^k} \right| \leqslant 2^n \sum_{j=s+1}^{\infty} \|g_j\|_{r_j} r_j^{-n}$$

and from $\lim_{|x-x'| \downarrow 0} s(|x - x'|) = \infty$. □

The KAM limits of Secs. VI and VII are exactly of the above kind. For example,

$$\omega_0^{(\infty)} = a + \sum_0^\infty \epsilon^{2^j} \hat{F}_0^{(j)}(a),$$

with $\hat{F}_0^{(j)}$ holomorphic on $D(\rho_j; \mathbf{A}^{(\infty)})$ and one has the following theorem.

Theorem 8.2: If Ω and $\{\delta_j\}$ are such that

$$\prod_{j=0}^{\infty} \zeta(\delta_j)^{1/2^j} < \infty, \quad (8.2)$$

$$\lim_{j \uparrow \infty} \frac{\log \Omega(2^j \delta_j^{-1})}{2^j} = 0, \quad (8.3)$$

and

$$2\left[\epsilon M_0 \rho_0^{-1} + (\epsilon\tau)^2 \sum_{j=0}^{\infty} (\epsilon\tau)^{2^j} N_j \Omega(N_j)\right] < 1, \quad (8.4)$$

then $\omega_0^{(\infty)} \in C^\infty_W(\mathbf{A}^{(\infty)})$.

The proof follows easily after noticing that (8.4) yields

$$\sup_{D(\rho_{j+1}; \mathbf{A}^{(j-1)})} \left| \frac{d\omega_0^{(j)}}{da} - 1 \right| < \frac{2}{3}, \quad (8.5)$$

so that

$$\rho_j^{-1} < 4N_{j-1} \Omega(N_{j-1}) c. \quad (8.6)$$

For more details see Ref. 25, Sec. 2.6.

Remark 8.3: Whitney smoothness is obviously preserved under composition with smooth functions. Thus $\omega_0^{(\infty)}(\alpha_0(E))$ $(= \alpha(E))$ belongs to $C^\infty_W(\tilde{\mathbf{E}}^{(\epsilon)})$, $\tilde{\mathbf{E}}^{(\epsilon)} \equiv \alpha_0^{-1}(\mathbf{A}^{(\infty)})$.

Remark 8.4: While a condition analogous to (8.2) appears in the (analytic) KAM literature,[8] condition (8.3) is new. This condition is necessary in order to be able to meet the smallness condition (8.4) and, as we shall see, to give a complete description of $\mathbf{A}^{(\infty)}$. We also point out that (8.2) and (8.3) are independent (see Appendix A).

IX. STRUCTURE OF KAM SPECTRA

The main theorem in Ref. 16 is that any function $g \in C^n_W(A)$, $\mathbf{A}$ closed, can be extended to a $C^n(\mathbb{R})$ function which is real analytic on $\mathbb{R} - \mathbf{A}$; a simple corollary of this and of the maximum principle imply

$$\sup_R \left| \frac{d^k g}{dx^k} \right| \leqslant \max_{\mathbf{A}} \left| \frac{d^k g}{dx^k} \right|, \quad k \leqslant n.$$

Here we show how to use the above facts in order to give a precise description of the KAM spectrum $\mathbf{E}^{(\epsilon)}$.

Denote by $\mathbf{R}$ the "resonant" set of $a = \alpha_0(E)$ for which we cannot apply the KAM scheme,

$$\mathbf{R} \equiv \mathbf{A}^{(0)} - \mathbf{A}^{(\infty)} \equiv \bigcup_{j=0}^{\infty} \bigcup_{\substack{\nu \in \mathbb{Z}^d \\ 0 < |\nu| \leqslant N_j}} \mathbf{R}_\nu^{(j)},$$

where for $0 < |\nu| \leqslant N_j$,

$$\mathbf{R}_\nu^{(j)} \equiv \{a \in \mathbf{A}^{(j)}: |\alpha_j(a) - \omega \cdot \nu/2| < 1/c\Omega(|\nu|)\}.$$

A condition analogous to (8.4) implies easily that the Whitney extension of the α_j's satisfy

$$\sup_R \left| \frac{d\alpha_j}{da} - 1 \right| \leqslant \frac{2}{3}.$$

Thus defining

$$a_{j,\nu} \equiv \alpha_j^{-1}(\omega \cdot \nu/2), \quad r_\nu \equiv 3/c\Omega(|\nu|),$$

we see that

$$\mathbf{R}_\nu^{(j)} \subset I_\nu^{(j)} \equiv \{a \in \mathbf{A}^{(0)}: |a - a_{j,\nu}| < r_\nu\}.$$

This completes the description of $\mathbf{A}^{(\infty)}$ and hence, via the smooth map α_0^{-1}, of $\mathbf{E}^{(\epsilon)}$.

Finally it is not difficult to show that[25]

$$\bigcup_{j=0}^{\infty} I_\nu^{(j)} \subset \{|a - a_\nu| < r'_\nu\}, \quad r'_\nu \equiv \frac{7 + \log\log 3\Omega(|\nu|)}{c\Omega(|\nu|)}$$

and $a_\nu \sim \omega \cdot \nu/2$. These facts together with the asymptotic evaluation $\alpha_0(E) \sim \sqrt{E}$ yields (1.4).

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APPENDIX A: ON A NEW CONDITION IN ANALYTIC KAM

Here we show that conditions (8.2) and (8.3) are independent as announced in Remark 8.4. To do this we give two examples.

(1) Let $\Omega(r) = r^m$ for some m. Then

$$(8.2) \Leftrightarrow \sum \frac{1}{2^j} \log \delta_j^{-1} < \infty,$$

$$(8.3) \Leftrightarrow (1/2^j) \log \delta_j^{-1} \to 0 (j \uparrow \infty).$$

(2) Let

$$\Omega(r) = \begin{cases} \exp(r/\log^\sigma r), & r \geqslant e^\sigma, \\ \Omega(e^\sigma), & 1 \leqslant r \leqslant e^\sigma, \end{cases} \quad \sigma > 1;$$

$$(8.2) \Leftrightarrow \delta_j^{-1}/j^\sigma \text{ is bounded;}$$

$$(8.3) \Leftrightarrow \delta_j^{-1}/j^\sigma \to 0 (j \uparrow \infty).$$

In the first example (8.2) is stronger than (8.3) but in the second one the opposite occurs.

Notice that since $\Omega(r) \geqslant r^{d-1}$ (8.2) implies easily the finiteness of ψ.

APPENDIX B: MOSER–DEIFT–SIMON INEQUALITY ON KAM SPECTRA

Deift–Simon,[3] extending an idea of Moser,[30] showed that, for general, almost periodic potentials,

$$\lim_{\epsilon \downarrow 0} \frac{\alpha^2(E+\epsilon) - \alpha^2(E-\epsilon)}{2\epsilon} \geqslant 1, \quad E \text{ a.e. in } S. \tag{B1}$$

Here we want to discuss briefly the constructive version of (B1) for $L^{(\epsilon)}$, namely, we sketch the proof of

$$\frac{d}{dE} \alpha^2 \geqslant 1, \quad E \in \widetilde{\mathbf{E}}^{(\epsilon)}, \quad \frac{d}{dE} \equiv \text{Whitney derivative.} \tag{B2}$$

Without loss of generality we can assume that $\widehat{W}_0 = 0$ and, to simplify the Hamiltonian formalism, we consider $V \equiv 0$ in which case $H_{\epsilon W} = \sqrt{E} A_1 + \omega_2 A_2 + \cdots + \omega_d A_d - (\epsilon/\sqrt{E}) A_1 \sin^2\theta_1 W(\theta_2,...,\theta_d)$. Then we have

$$\alpha(E) = \sqrt{E} + \epsilon \hat{F}_0^{(0)} + \epsilon^2 \hat{F}_0^{(1)} + O(\epsilon^4)$$

with $F^{(0)} \equiv -(\sin^2\theta_1/\sqrt{E}) W(\theta_2,...,\theta_d)$ and $F^{(1)}$ as in the inductive Lemma 6.1. Here $\widehat{W}_0 = 0$ implies $\widehat{F}_0 = 0$. Now, setting $\widetilde{\omega} \equiv (\omega_2,...,\omega_d)$, a computation shows that

$$\hat{F}_0^{(1)} = \int \frac{\partial \Phi_0}{\partial \theta_1} F^{(0)} + O(\epsilon)$$

$$\equiv \int \left(\sum_{0 < |\nu| < N_0} \frac{\widehat{F}_\nu^{(0)}}{-i\omega^{(0)} \cdot \nu} \right) F^{(0)} + O(\epsilon)$$

$$= \frac{1}{2} \sum_{\substack{\mu \in \mathbf{Z}^{d-1} \\ |\mu| < N_0}} \frac{|\widehat{W}_\mu|^2}{(\mu \cdot \widetilde{\omega})^2 - 4E} + O(\epsilon),$$

so that

$$\frac{d\alpha^2}{dE} = 1 + \epsilon^2 \frac{\sqrt{E}}{2} \sum_{|\mu| < N_0} \frac{|\widehat{W}_\mu|^2}{[(\mu \cdot \widetilde{\omega})^2 - 4E]^2} + \frac{1}{\sqrt{E}} O(\epsilon^3).$$

The smallness of the parameter $\epsilon\tau$ confirms (B2).

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