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Stability of Nearly Integrable, Degenerate Hamiltonian Systems with Two Degrees of Freedom

L. Biasco,[1] L. Chierchia,[1] and D. Treschev[2]

[1] Dipartimento di Matematica, Università "Roma Tre," Largo S. L. Murialdo 1, 00146 Roma, Italy
e-mail: {biasco,luigi}@mat.uniroma3.it

[2] Dept. of Mechanics and Mathematics, Moscow State University, Vorob'evy Gory, 119899 Moscow, Russia
e-mail: dtresch@mech.math.msu.su

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Summary. We consider the problem of the stability of action variables in properly degenerate, nearly integrable Hamiltonian systems and prove, in particular, stability results for systems with two degrees of freedom. An application of such results to celestial mechanics is presented.

1. Introduction and Main Results

Consider the n-dimensional Hamiltonian system

$$\dot{y} = -\partial H/\partial x, \quad \dot{x} = \partial H/\partial y, \qquad H = H_0(y) + \varepsilon H_1(x, y; \varepsilon), \tag{1.1}$$
$$y \in D, \quad x \in \mathbf{T}^n = \mathbf{R}^n/(2\pi \mathbf{Z}^n),$$

where $D \subset \mathbf{R}^n$ is an open domain and ε is a small parameter. Variables y and x are called respectively actions and angles. We denote by $y(t) = y(t; y_0, x_0, \varepsilon)$, $x(t) = x(t; y_0, x_0, \varepsilon)$, the Hamiltonian flow of system (1.1) with initial data $y(0) = y_0$, $x(0) = x_0$.

System (1.1) is called nearly integrable since, for $\varepsilon = 0$, it can be simply integrated:

$$\dot{y} = -\partial H_0/\partial x = 0, \quad \dot{x} = \partial H_0/\partial y. \tag{1.2}$$

Then the phase space $\mathcal{M} := D \times \mathbf{T}^n$ is foliated by n-dimensional invariant tori

$$T_{\omega_0} := \{y = y_0,\ x \in \mathbf{T}^n\},$$

on which the angles linearly evolve $x(t) = \omega_0 t + x_0$, where $\omega_0 := \partial H_0/\partial y|_{y=y_0}$, while the actions remain constant: $y(t) = y_0$. We are interested in knowing the behavior of $y(t; y_0, x_0, \varepsilon)$ for $\varepsilon \neq 0$.

The problem of the (in)stability of action variables in nearly integrable Hamiltonian systems consists in (dis)proving that, for any compact set $K \subset D$,

$$\exists\, c(\varepsilon) > 0, \qquad c(\varepsilon) \to 0 \text{ for } \varepsilon \to 0, \quad \text{s.t.} \quad \sup_{t \in \mathbf{R}} |y(t; y_0, x_0, \varepsilon) - y_0| \leq c(\varepsilon), \quad (1.3)$$

for any $(y_0, x_0) \in K$. Property (1.3) is often called *total* stability, in the sense that it holds for *all* times.

1.1. Stability by KAM Theory for Nondegenerate Systems

The celebrated KAM Theorem assures the persistence of the majority[1] of the perturbed invariant tori for ε small, under suitable (general) hypotheses of nondegeneracy on the integrable Hamiltonian H_0.

As a byproduct, the majority of orbits are stable, in the sense that (1.3) holds for any compact set in a "big" set $\mathcal{M}_K$ (meaning that $\text{meas}(\mathcal{M} \backslash \mathcal{M}_K)$ is bounded by a quantity of order $\sqrt{\varepsilon}$).

Denote by D_h the unperturbed energy level in the space of actions:

$$D_h = \{y \in D\colon\ H_0(y) = h\}.$$

Let $\mathbf{P}^{n-1}$ be the $(n-1)$-dimensional projective space and pr: $\mathbf{R}^n \setminus \{0\} \to \mathbf{P}^{n-1}$ the natural projection. Consider the map $\Phi\colon D_h \to \mathbf{P}^{n-1}$, $\Phi(y) = \text{pr} \circ \text{grad} H_0(y)$. Then the system (1.1) is called *isoenergetically nondegenerate* at a point $y \in D_h$ if the map Φ is a local diffeomorphism in the vicinity of y.

According to KAM theory, for small ε, energy levels of an isoenergetically nondegenerate system (1.1) contain a large (in the measure sense) family of invariant n-dimensional tori.

As is well known, for $n = 2$, isoenergetical nondegeneracy prevents the drift of y-variables on trajectories, since the KAM tori are two-dimensional invariant hypersurfaces on three-dimensional energy levels. Hence an isoenergetically nondegenerate system with two degrees of freedom is stable (namely, (1.3) holds).

On the other hand, it is believed that general isoenergetically nondegenerate systems exhibit $O(1)$ drift in actions,[2] whenever $n > 2$. Such drift is known as *Arnold diffusion*, since Arnold showed its existence in a simple ad hoc model in [4] and conjectured[3] its genericity in [3]: "the typical case in higher-dimensional problems is *topological*

[1] Namely, the measure of the complement of the union of the persistent invariant tori does not exceed a quantity of order $\sqrt{\varepsilon}$.

[2] Namely, there exist a constant $c > 0$, an ε-depending family of initial data $(y_0^\varepsilon, x_0^\varepsilon)$, and times T_ε such that

$$|y(T_\varepsilon; y_0^\varepsilon, x_0^\varepsilon, \varepsilon) - y_0^\varepsilon| \geq c.$$

[3] The conjecture is stronger than the simple negation of (1.3) and even of the existence of a drift.

instability: through an arbitrarily small neighborhood of any point there pass phase trajectories along which the actions drift away from their initial value by a quantity of order one."[4]

Genericity of the Arnold diffusion was announced in [18], [19].

Finally we observe that, according to the Nekhoroshev theory (see [20]), the average velocity of such a drift is exponentially small in ε provided H_0 satisfies suitable general "steepness" conditions (see [20] or the recent preprint [21]).

1.2. Isoenergetically and Properly Degenerate Systems with Two Degrees of Freedom

From now on we shall consider nearly integrable Hamiltonian systems with two degrees of freedom.

Suppose that system (1.1) is isoenergetically degenerate on a certain energy level. In this case even for $n = 2$, drift of action variables is possible. Indeed, consider the Nekhoroshev example (see [20]):

$$H = \frac{1}{2}(y_1^2 - y_2^2) - \varepsilon \sin(x_1 + x_2). \tag{1.4}$$

The system is isoenergetically degenerate on the unperturbed energy level D_0. The solution $y_1(t) = y_2(t) = \varepsilon t$, $x_1(t) = \varepsilon t^2/2 = -x_2(t)$, presents $O(1)$ drift of actions along D_0 with velocity of order ε.

In Section 2 below, generalizing the Nekhoroshev example, we will describe a simple "general" method apt to construct drifting orbits for isoenergetically degenerate systems in two degrees of freedom.

Let us point out that the study of degenerate systems is not only a mathematical question but might be of interest also from a physical point of view. In fact, a typical feature in celestial mechanics is that the unperturbed system is *properly degenerate*, i.e., the unperturbed Hamiltonian H_0 in (1.1) does not depend upon all action variables. In such a case the isoenergetical nondegeneracy is obviously *strongly* violated.

Some important examples of properly degenerate models arising in Celestial Mechanics are the three-body problem (see, e.g., [17], [8]); the problem of fast rotations of a symmetric rigid body (see [6], [7], [5]); and the D'Alembert planetary "spin/orbit" model (see, e.g., [14], [12], [13]). The D'Alembert model will be taken up in Section 4 below and will, somehow, be used as a "guideline" for our investigations.

In general, properly degenerate systems are unstable, i.e., (1.3) does not hold. For example, in the system governed by the Hamiltonian

$$H(y_1, y_2, x_1, x_2; \varepsilon) := H_0(y_1) + \varepsilon H_1, \quad \text{where} \quad H_1 := \frac{y_2^2}{2} + \cos x_2,$$

all trajectories with initial positions such that $(y_2, x_2) \neq (0, k\pi)$ violate (1.3). Hence, in order to have stability, one has to make suitable assumptions on the perturbation H_1. Such assumptions arise naturally in the celestial mechanical examples mentioned above.

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[4] See [1, pg. 189] from which the citation is taken.

One requires that the perturbation H_1 is of the form

$$H_1(y_1, y_2, x_1, x_2; \varepsilon) := H_{01}(y_1, y_2, x_1) + O(\varepsilon^c), \quad c > 0. \tag{1.5}$$

Here and in the following, by $O(\varepsilon^c)$, $c > 0$, we mean a function that, divided by ε^c, is smooth (or analytic) in x and y and bounded in all its variables as $\varepsilon \to 0$.

For example, in [11], total stability is proved for the "model problem"

$$H_0 = \frac{y_1^2}{2}, \qquad H_{01} := \pm\frac{y_2^2}{2} + \cos x_1. \tag{1.6}$$

In this paper, in order to prove stability results in two degrees of freedom, we will assume the following stronger assumption on the intermediate term H_{01}: We shall assume that H_{01} is independent on the angle x_1 and consider perturbations of the form

$$H_1 := H_{01}(y_1, y_2) + O(\varepsilon^c), \quad c > 0. \tag{1.7}$$

A first answer on the stability of systems governed by Hamiltonians of the form

$$H(y_1, y_2, x_1, x_2; \varepsilon) := H_0(y_1) + \varepsilon H_{01}(y) + O(\varepsilon^2), \tag{1.8}$$

was given by Arnold (see [3] and compare also [1], Chapter 5, Section 3).

Theorem [3]. *Let H be as in (1.8) and assume that the perturbation removes the degeneracy, in the sense that*

$$\frac{\partial H_0}{\partial y_1} \neq 0, \qquad \frac{\partial^2 H_{01}}{\partial y_2^2} \neq 0. \tag{1.9}$$

Then, for all ε small enough, total stability holds.

However a general function $H_0(y_1)$ will have critical points (where (1.9) is violated) and one may pose the question whether in the vicinity of a critical point y_1^{cr} of $\partial H_0/\partial y_1$ drifting phenomena may occur or total stability holds.

1.3. Main Results: Stability Theorems for Properly Degenerate Systems in Two Degrees of Freedom

Let D be an open bounded set in $\mathbf{R}^2$, let $y \in D$, $x \in \mathbf{T}^2$, $\varepsilon \geq 0$, and consider a system with Hamiltonian

$$H(y, x, \varepsilon) = H_{00}(y_1) + \varepsilon H_{01}(y) + \varepsilon^a H_a(y, x) + O(\varepsilon^{a_1}), \quad 1 < a < a_1. \tag{1.10}$$

We shall assume H to be smooth enough or real-analytic. In the following, by "(total) stability" we mean that there exists a constant $s > 0$ such that (1.3) holds with $c(\varepsilon) := \varepsilon^s$. Notation: from now on, prime denotes derivative with respect to y_2.

Theorem 1. *Suppose that*

(1) *the critical points of* H_{00} *are nondegenerate,*
(2) *for all* y_1^0 *fixed,* $y_2 \mapsto H''_{01}(y_1^0, y_2)$ *is not identically zero on any open subset of* $D \cap (\{y_1^0\} \times \mathbf{R})$.

Then the following condition is sufficient for the stability of system (1.10): For any critical point y_1^{cr} *of* H_{00} *and for any constant* h*, the function* $H_{01}(y_1^{cr}, y_2)$ *is not a quadratic polynomial in* y_2 *of the form*

$$d(y_2 - r)^2 + h, \qquad d\frac{\partial^2 H_{00}}{\partial y_1^2}(y_1^{cr}) < 0, \tag{1.11}$$

for some $d, r \in \mathbf{R}$.

Unfortunately in our astronomical guide-problem (the D'Alemebert spin/orbit problem), $H_{01}(y_1^{cr}, y_2)$ is exactly of the form described in (1.11) and the previous theorem cannot be applied. In such cases, one has to look at "higher order" nondegeneracy conditions such as the ones described in the following:

Theorem 2. *Suppose that* $a < 3/2$ *and hypotheses (1)–(2) of Theorem 2 hold, but, for some critical point* y_1^{cr} *of* H_{00}, $F(y_2) := H_{01}(y_1^{cr}, y_2)$ *is of the type described in (1.11), namely,*

$$F(y_2) \equiv d(y_2 - r)^2 + h \quad \textit{for some } d, r, h \in \mathbf{R}. \tag{1.12}$$

Then stability nevertheless takes place, provided

$$\overline{F}_a(y_2) := \frac{1}{(2\pi)^2}\int_{\mathbf{T}^2} H_a(y_1^{cr}, y_2, x)\, dx \tag{1.13}$$

is not a quadratic polynomial having r *as a root, namely,*

$$\overline{F}_a(y_2) \not\equiv (uy_2 + v)(y_2 - r) \quad \textit{for some } u, v \in \mathbf{R}. \tag{1.14}$$

Let us conclude this introduction by loosely describing the application of the above results to the D'Alembert problem.

The D'Alembert planetary model is a nearly integrable, properly degenerate, time-dependent Hamiltonian system with two (and a half) degrees of freedom describing the positions of a planet, modelled by a nearly spherical ellipsoid, whose center of mass revolves on a Keplerian nearly circular ellipse around a fixed star (see [14], [12], [13] and Section 4 below). In particular one is interested in studying the resulting motions in phase region close to exact "spin/orbit" resonances, i.e., in regions corresponding to unperturbed motions where the day of the planet is commensurable with the year (the period of the Keplerian orbit of the center of mass)

Time is a "fast variable," which can be averaged out up to exponentially small (in the main perturbative parameter)[5] terms. The two-degree-of-freedom (time-independent) resulting Hamiltonian is usually called "the effective Hamiltonian."

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[5] In the D'Alembert problem there are two perturbative parameters, namely, the oblateness ε of the planet and the eccentricity μ of the Keplerian fixed orbit around which is revolving the center of mass of the ellipsoidal planet. Usually one takes $\mu = \varepsilon^c$ for some prefixed $c > 0$. See Section 4 for more information.

Theorem 3. *For all but a finite number of spin/orbit resonances, the effective Hamiltonian of the D'Alembert planetary model is totally stable. Consequently, the action variables of the full three-degree-of-freedom D'Alembert Hamiltonian are stable for an exponentially long time.*

More precise statements (and comparison with known results) are given in Section 4; see, in particular, Theorems 8 and 9.

2. Instability in Degenerate Systems: Resonant Channels

Definition 2.1. *Let H_0 be isoenergetically degenerate at every point of a suitable connected component E_h of D_h. Suppose that* $\mathrm{grad} H_0|_{E_h} \neq 0$. *Then E_h is called a channel.*

Proposition 2.1. *Every channel is a part of a straight line.*

Proof. If $\omega := (\omega_1, \omega_2) := \mathrm{grad} H_0(y)$, the isoenergetic degeneracy implies that $\mathrm{pr}(\omega)$ is constant on E_h. Hence, the direction of $\mathrm{grad} H_0$ is the same on E_h. □

Below, we shall always assume that channels correspond to the energy value $h = 0$.

Corollary 2.1. *Let $E := E_0$ be a channel. Then for some constant vector $\tilde{\omega} \in \mathbf{R}^2$ and some function $g(y)$ smooth in a neighborhood of E,*

$$H_0(y) = \frac{\langle \tilde{\omega}, y \rangle}{g(y)}. \tag{2.1}$$

Definition 2.2. *Let us consider a channel E. If for some integer j_1, j_2, with* $\gcd(\mathrm{j}_1, \mathrm{j}_2) = 1$, *we have* $\mathrm{pr}(\omega_1, \omega_2) = (\mathrm{j}_1, \mathrm{j}_2)$ *on E, we call E resonant.*

Remark 2.1. In a properly degenerate system every channel is resonant.

Let E be a resonant channel of system (1.1). According to Proposition 2.1, there exists $I \subset E$ and suitable $a, b \in D$, $\lambda > 0$, $m_1, m_2 \in \mathbf{Z}^2$ with $\gcd(m_1, m_2) = 1$, such that $a - b = \lambda\mu = \lambda(-m_2, m_1)$ and

$$I = \{y \in \mathbf{R}^2 \colon\ y = as + b(1 - s),\, s \in [0, 1]\}. \tag{2.2}$$

We construct the integer matrix

$$A = \begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \end{pmatrix}, \qquad \det A = 1, \qquad A^{-1} = \begin{pmatrix} n_2 & -m_2 \\ -n_1 & m_1 \end{pmatrix}, \tag{2.3}$$

and define the function

$$\chi(\eta, \xi) = \frac{1}{2\pi} \int_0^{2\pi} H_1\left(b + \mu\eta, A^T \begin{pmatrix} \tau \\ \xi \end{pmatrix}\right) d\tau, \quad \xi \in \mathbf{T}^1,\ \eta \in [0, \lambda].$$

We shall assume that the perturbation H_1 satisfies the following condition:

C1. There exists a constant h_0 and a function ϕ: $[0, \lambda] \to \mathbf{T}^1$ such that

$$\chi(\eta, \xi)|_{\xi=\phi(\eta)} \equiv h_0 \quad \text{and} \quad \left| \frac{\partial \chi}{\partial \xi}\Big|_{\xi=\phi(\eta)} \right| \geq \textit{const.} > 0. \tag{2.4}$$

Theorem 4. *An isoenergetically degenerate system (1.1) admitting a resonant channel is unstable for any perturbations H_1 verifying* **C1**. *That is, for all ε sufficiently small, there exists a solution $(y(t), x(t))$ of system (1.1) such that*

$$|y(0) - b| < c_1\varepsilon, \quad |y(T) - a| < c_2\varepsilon, \quad c_3\varepsilon^{-1} < |T| < c_4\varepsilon^{-1}.$$

Remark 2.2. The time T is positive (respectively, negative) if in Condition **C1** $\frac{\partial \chi}{\partial \xi}|_{\xi=\phi(\eta)} > 0$ (respectively, < 0).

From the previous Theorem and Remark 2.1 we have the following:

Corollary 2.2. *A properly degenerate system (1.1) in two degrees of freedom is unstable for any perturbations H_1 verifying* **C1**.

Consider as a trivial example the system with Hamiltonian (1.4). Then $E = \{y \in \mathbf{R}^2\colon y_1 = y_2 > 0\}$ is a resonant channel. We take I satisfying (2.2) with $a = \alpha\binom{1}{1}$, $b = \beta\binom{1}{1}$, $0 < \beta < \alpha$. Then $\mu = (1, 1)$, $\lambda = \alpha - \beta$. We can put $A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Then

$$\chi(\eta, \xi) = -\frac{1}{2\pi} \int_0^{2\pi} \sin \xi \, d\tau = -\sin \xi,$$

and condition **C1** obviously holds.

Proof of Theorem 4. Without loss of generality, we assume that $b = 0$. Since the channel E is resonant, H_0 satisfies (2.1) with $\tilde{\omega} = (m_1, m_2)$.

Consider the linear symplectic change of variables $(y, x) \to (Y, X)$,

$$Y = Ay, \quad x = A^T X.$$

In the new variables the interval I and the Hamiltonian (1.1) take the form

$$\begin{aligned} I &= \{Y_1 = 0, Y_2 \in [0, \lambda]\}, \\ \mathcal{H}(Y, X; \varepsilon) &= Y_1/G(Y) + \varepsilon H_1(A^{-1}Y, A^T X; \varepsilon), \quad G(Y) = g(A^{-1}Y). \end{aligned}$$

We reduce the order on the energy level

$$\mathcal{H} = \varepsilon h_0, \tag{2.5}$$

where h_0 is a suitable constant such that condition **C1** holds. The solution of equation (2.5) with respect to Y_1 has the form

$$Y_1 = \varepsilon G(0, Y_2)(h_0 - H_1(0, m_1 Y_2, A^T X; 0)) + O(\varepsilon^2), \tag{2.6}$$

where m_1 is defined as above. Now $\tau = X_1$ plays the role of the new time and $\varepsilon \widehat{H} = -Y_1(Y_2, \tau, X_2; \varepsilon)$ of the new Hamiltonian. Since τ is the only fast variable in this system, the dynamics on time interval $\tau \sim \varepsilon^{-1}$ is determined with precision of order ε by the averaged system, i.e., the system with Hamiltonian

$$\varepsilon \widehat{H}^0 = -\varepsilon G(0, Y_2)(h_0 - \chi(Y_2, X_2));$$

see for example, [2]. Condition **C1** implies that the averaged system has a solution that goes along the curve $\{\chi = h_0\}$ with velocity of order ε, and projection of this curve to the line Y_2 covers the interval $[0, \lambda]$. □

3. Proofs of the Stability Theorems

3.1. Technical Lemmata

We now state two classical results. The first one is a normal form theorem, the second one a KAM theorem (such results are derived by Lemma 5.2 and Lemma 5.3 of [9], respectively, while a detailed exposition on normal forms and KAM theory can be found e.g. in [15]). Finally the third result is an ad hoc corollary of the previous two results which will be useful to prove our stability results.

Theorem 5. *Consider a real analytic Hamiltonian* $\mathcal{H} := \mathcal{H}(y, x; t) := h(y) + f(y, x; t)$ *defined for* $t \in \mathbf{T}$, $x \in \mathbf{T}_s := \{x \in \mathrm{C}\ \ s.t.\ \ |\mathrm{Im}\, x| < s\}$ *and* $y \in \mathcal{I} := [y^-, y^+]$, *for* $y^- < y^+$, *with analytic extension on* $\mathcal{I}_r := \{y \in \mathrm{C},\ s.t. \exists\ y^* \in \mathcal{I},\ |y - y^*| < r\}$, *for* $0 < r, s \le 1$. *Let* $\alpha := \min\{1/2,\ \inf_{y \in \mathcal{I}_r} |h'(y)|\}$. *There exists a small constant c (independent on r and* α*) such that, if for some* $K \ge 1$,

$$\beta := \sup_{y \in \mathcal{I}_r,\ x \in \mathbf{T}_s,\ t \in \mathbf{T}} |f(y, x; t)| \le cr\alpha/K,$$

then there exists a time-dependent analytic canonical transformation

$$\Phi\colon\ (\tilde{y}, \tilde{x}; t) \in \mathcal{I}_{r/2} \times \mathbf{T}_{s/6} \times \mathbf{T} \longrightarrow (y, x) \in \mathcal{I}_r \times \mathbf{T}_s,$$

such that, in the new variables $(\tilde{y}, \tilde{x})$, *the new Hamiltonian is*

$$\widetilde{\mathcal{H}} := h(\tilde{y}) + g(\tilde{y}) + \tilde{f}(\tilde{y}, \tilde{x}; t),$$

with

$$\sup_{y \in \mathcal{I}_{r/2}} |g(y)| \le 2\beta, \qquad \sup_{y \in \mathcal{I}_{r/2},\ x \in \mathbf{T}_{s/6},\ t \in \mathbf{T}} |\tilde{f}(y, x; t)| \le \beta \exp(-cK),$$

and

$$|y - \tilde{y}| \le \beta/c.$$

Theorem 6. *Consider a Hamiltonian $\mathcal{H}$ defined as in the previous theorem. Assume that* $\delta := \inf_{y\in\mathcal{I}_r} |h''(y)| > 0$. *Take* $0 < \gamma < 1$ *and define*

$$\begin{aligned} A &:= 1 + \sup_{y\in\mathcal{I}_r}(|h'(y)|^2 + |h''(y)|), \\ F &:= A\gamma^{-2} \sup_{y\in\mathcal{I}_r,\ x\in\mathbf{T}_s,\ t\in\mathbf{T}} |f(y,x;t)|, \\ B &:= 1 + c_1 \frac{\gamma}{Ar|\ln F|^{c_2}}, \\ \hat{F} &:= c_3 B A^2 \delta^{-2} F |\ln F|^{c_4}, \\ \hat{\gamma} &:= c_5 A \delta^{-1}\gamma, \end{aligned}$$

where c_i are suitable constants. Suppose that $\hat{F} \le 1$, then, due to the preservation of KAM tori, the evolution of $y(t)$ remains bounded, namely,

$$|y(t) - y(0)| \le 2\max\{r\hat{F},\ \hat{\gamma}\}. \tag{3.1}$$

Theorem 7. *Consider a real analytic Hamiltonian $\mathcal{H}(y,x,t;\varepsilon)$ defined for y, x, t as above and ε small. Suppose that $\mathcal{H}$ has the form*

$$\mathcal{H} := \varepsilon^{\kappa_1}\left(h(y;\varepsilon) + \varepsilon^{\kappa_2} f(y,x,t;\varepsilon)\right),$$

for some $\kappa_1, \kappa_2 > 0$. Suppose that

$$r \ge \varepsilon^{\sigma_1}, \qquad \inf_{y\in\mathcal{I}_r} |h'(y)| \ge \varepsilon^{\sigma_2}, \qquad \inf_{y\in\mathcal{I}_r} |h''(y)| \ge \varepsilon^{\sigma_3},$$

for $\sigma_1 + \sigma_2 < \kappa_2$, $\sigma_3 > 0$. Then the action y is stable; namely, there exists a $\sigma_4 > 0$ such that

$$|y(t) - y(0)| \le \varepsilon^{\sigma_4}, \quad \forall\, t \in \mathbf{R}.$$

Proof. Since $\sigma_1+\sigma_2 < \kappa_2$, we can apply Theorem 5, obtaining a $O(\varepsilon^{\kappa_1+\kappa_2})$ close-to-the-identity canonical transformation after which the perturbation becomes exponentially small (w.r.t. ε). Then we can apply Theorem 6 with γ small enough such that $\hat{\gamma} = O(\varepsilon^{\kappa_1+\kappa_2})$, noting that condition $\hat{F} \le 1$ is surely verified for ε small enough since the perturbation is exponentially small. Therefore stability follows by (3.1) with any $\sigma_4 < \kappa_1 + \kappa_2$ (and ε small enough). □

3.2. Proof of Theorem 1

Let y^0 be the initial condition for the variable y. We put

$$\omega = \frac{\partial H_{00}}{\partial y_1}(y_1^0).$$

We distinguish two cases: $|\omega| > C\varepsilon^{1/2}$ and $|\omega| \le C\varepsilon^{1/2}$. In the first case we reduce the order on the energy level and, since the obtained system is non degenerate, we can directly apply KAM theory, namely Theorem 7, getting stability. The second case is more complicated. After reducing the order on the energy level, we have that the new time of the obtained $(1+1/2)$-degrees-of-freedom system is a fast angle, so we average with respect to it. To get some "twist" condition for the averaged Hamiltonian, we have to construct action–angle variables that integrate the one-dimensional system obtained neglecting the time-dependent term. After this last change of variables, the twist condition follows from the hypothesis (1.11) of Theorem 1 and, again by KAM theory, we get stability.

Case 1. $|\omega| > C\varepsilon^{1/2}$ with large constant C.
Putting

$$h = H(y^0, x^0, \varepsilon) = H_{00}(y_1^0) + \varepsilon H_{01}(y^0) + O(\varepsilon^a),$$

we reduce the order on the energy level $H = h$. If C is large enough, we can find a unique solution of the equation $H(y, x, \varepsilon) = h$ with respect to y_1, which has the form $y_1 = y_1^0 - \chi(y_2, x, \varepsilon)$, where

$$\chi(y_2, x, \varepsilon) := \omega\left(g(y_2; \varepsilon) + O\left(\frac{\varepsilon^2}{\omega^3} + \frac{\varepsilon^a}{\omega^2}\right)\right). \tag{3.2}$$

Indeed, the equation $H = h$ can be written, dividing by ω^2, in the form

$$-\varphi(\chi) + \frac{\varepsilon}{\omega^2}\tilde{g}(y_2) + O\left(\frac{\varepsilon}{\omega^2}\chi + \frac{\varepsilon^a}{\omega^2}\right) = 0, \tag{3.3}$$

where

$$\begin{aligned} \varphi(\chi) &:= \varphi(\chi; y_1^0) := \omega^{-2}[H_{00}(y_1^0) - H_{00}(y_1^0 - \chi)], \\ \tilde{g}(y_2) &:= H_{01}(y_1^0, y_2) - H_{01}(y^0). \end{aligned}$$

Let us denote by $\xi \to \varphi^{-1}(\xi; y_1^0)$ the inverse function of φ in a neighborhood of zero (such an inverse function exists since $\partial_\chi(0; y_1^0) = \frac{1}{\omega} \neq 0$). Let us consider

$$\psi(\xi) := \psi(\xi; y_1^0) := \frac{1}{\omega}\varphi^{-1}(\xi; y_1^0).$$

We note that ψ can be extended to a smooth function even in $y_1^0 = y_1^{cr}$ (where $\omega = 0$); indeed, it results that $\partial_\xi\psi(0; y_1^{cr}) = 1$, $\partial^2_{\xi^2}\psi(0; y_1^{cr}) = \partial^2 H_{00}(y_1^{cr})$, etc. By (3.3) we get

$$\chi = \omega\,\psi\left(\frac{\varepsilon}{\omega^2}\tilde{g}(y_2) + O\left(\frac{\varepsilon}{\omega^2}\chi + \frac{\varepsilon^a}{\omega^2}\right)\right).$$

Since $\psi(\xi) = \xi + O(\xi^2)$, it results that $\chi = O(\varepsilon/\omega)$. Hence χ is of the form described in (3.2) with

$$g(y_2; \varepsilon) := \psi\left(\frac{\varepsilon}{\omega^2}\tilde{g}(y_2)\right). \tag{3.4}$$

Let us define $\mathrm{g}(y_2; \epsilon) := \psi(\epsilon\tilde{g}(y_2))$ and $\epsilon := \varepsilon/\omega^2$. We have $g(y_2; \varepsilon) = \mathrm{g}(y_2; \epsilon)$ and

$$\begin{aligned} g'(y_2; \varepsilon) &= \mathrm{g}'(y_2; \epsilon) = \epsilon\partial_\xi\psi(\epsilon\tilde{g}(y_2))\tilde{g}'(y_2) \\ &= \epsilon\partial_\xi\psi(\epsilon\tilde{g}(y_2))H'_{01}(y_1^0, y_2), \end{aligned} \tag{3.5}$$

$$\begin{aligned} g''(y_2;\varepsilon) &= \mathrm{g}''(y_2;\epsilon) = \epsilon\left[\epsilon\partial^2_{\xi^2}\psi(\epsilon\tilde{g}(y_2))\left(\tilde{g}'(y_2)\right)^2 + \partial_\xi\psi(\epsilon\tilde{g}(y_2))\tilde{g}''(y_2)\right] \\ &= \epsilon\left[\epsilon\kappa(y_2,\epsilon) + k(y_2)\right], \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} k(y_2) &:= H''_{01}(y_1^0, y_2), \\ \kappa(y_2,\epsilon) &:= \partial^2_{\xi^2}\psi(\epsilon\tilde{g}(y_2))\left(H'_{01}(y_1^0,y_2)\right)^2 + \frac{\partial_\xi\psi(\epsilon\tilde{g}(y_2)) - 1}{\epsilon}H''_{01}(y_1^0,y_2); \end{aligned} \tag{3.7}$$

note that κ is analytic also in $\epsilon = 0$ since $\partial_\xi\psi(0) = 1$.

Then on the energy level $H = h$ the system is determined by the equations

$$\frac{dx_2}{d\tau} = \frac{\partial\chi}{\partial y_2}, \qquad \frac{dy_2}{d\tau} = -\frac{\partial\chi}{\partial x_2}, \quad \tau = x_1, \quad \chi = \chi(y_2,\tau,x_2,\varepsilon). \tag{3.8}$$

The Hamiltonian χ can be regarded as a perturbation of

$$\chi_0(y_2;\varepsilon) := \omega g(y_2;\varepsilon).$$

1A. *First assume that* $|g'(y_2^0;\varepsilon)| \geq \epsilon\varepsilon^\sigma$ *and* $|g''(y_2^0;\varepsilon)| \geq \epsilon\varepsilon^\sigma$ with sufficiently small $\sigma > 0$. Then the system (3.8) is nondegenerate. We can apply Theorem 7 with analyticity radius $r := \text{const}\,\varepsilon^\sigma$, for a suitable const > 0 small, obtaining stability for $y_2(t)$ and, hence, for $y_1(t)$.

1B. Now suppose that $|g'(y_2^0;\varepsilon)|$ or $|g''(y_2^0;\varepsilon)| < \epsilon\varepsilon^\sigma$. Fix $\tilde{\sigma} > 0$ (small enough). If $|y_2(t) - y_2^0| < 2\varepsilon^{\tilde{\sigma}}$ for all $t \in \mathbf{R}$, we get stability. Otherwise there exists $t^* \in \mathbf{R}$ such that $|y_2(t^*) - y_2^0| = 2\varepsilon^{\tilde{\sigma}}$, while $|y_2(t) - y_2^0| < 2\varepsilon^{\tilde{\sigma}}\ \forall |t| < |t^*|$. Let $y_2^* := y_2(t^*)$.

The idea is to use condition (2) of Theorem 1 in order to prove that, even if $|g'(y_2^0;\varepsilon)|$ or $|g''(y_2^0;\varepsilon)|$ can be small, nevertheless $|g'(y_2^*;\varepsilon)|$ and $|g''(y_2^*;\varepsilon)|$ must be sufficiently large (taking $\tilde{\sigma}$ small enough).

We need the following elementary lemma:

Lemma 3.1. *Let $k(y_2)$, $\kappa(y_2,\epsilon)$ be two analytic functions defined on some compact set with k not identically zero. There exist $\hat{\epsilon}, \hat{s}, \hat{c}, \hat{\delta} > 0$ such that, if $|\epsilon| \leq \hat{\epsilon}$ and $0 < \delta \leq \hat{\delta}$, then*

$$|\epsilon\kappa(y_2,\epsilon) + k(y_2)| \leq \delta$$

only for y_2 belonging to a finite set of intervals whose length does not exceed $\hat{c}\delta^{\hat{s}}$.

Due to condition (2) of Theorem 1, recalling (3.5)–(3.7) and applying the previous Lemma, we have, for $\tilde{\sigma}$ small enough,[6] that $|g'(y^*;\varepsilon)|, |g''(y^*;\varepsilon)| \geq \epsilon\varepsilon^\sigma$, and we reduce the situation to the case **1A**, with initial data y^* (noting that $|\partial_{y_1}H_{00}(y_1^*)| \geq C\varepsilon^{1/2}/2$).

At the end of this case 1, we give a simple example in which we can explicitly evaluate the function g in (3.2). Let us set $H_{00}(y_1) := y_1^2/2$. Then $\omega = y_1^0$ and the functions φ, ψ, g take the simple form

$$\varphi(\chi;y_1^0) = \frac{\chi}{y_1^0} - \frac{1}{2}\left(\frac{\chi}{y_1^0}\right)^2, \qquad \psi(\xi;y_1^0) = 1 - \sqrt{1-2\xi},$$

[6] Uniformly in the compact set in which we want to prove stability.

$$g(y_2;\varepsilon) = 1 - \sqrt{1 - 2\varepsilon\tilde{g}(y_2)/\omega^2}.$$

Case 2: $|\omega| < C\varepsilon^{1/2}$.

According to condition (1), we have $y_1^0 = y_1^{cr} + \sqrt{\varepsilon}y_1^*$, where y_1^{cr} is a critical point of H_{00} and $|y_1^*| < C_1$ with sufficiently large constant C_1.

2A. *First, assume that* $|y_1^*| > \varepsilon^\sigma$, *where* $\sigma > 0$ *is again a small constant*. We put

$$w := \frac{\partial^2 H_{00}}{\partial y_1^2}(y_1^{cr}), \qquad \lambda := (y_1^*)^2 w,$$

$$\overline{F}_a(y_2^0) := \frac{1}{(2\pi)^2}\int_{\mathbf{T}^2} H_a(y_1^{cr}, y_2^0, x)\,dx,$$

$$F(y_2^0) := H_{01}(y_1^{cr}, y_2^0), \qquad \mu := \overline{F}_a(y_2^0) - H_a(y_1^{cr}, y_2^0, x^0).$$

We note that $w \neq 0$ since y_1^{cr} is nondegenerate. Below, without loss of generality, we assume that

$$y_1^{cr} = 0, \qquad H_{00}(0) = 0,$$

so that $y_1^0 = \sqrt{\varepsilon}y_1^*$. Consider the new action variables $\widetilde{y}$:

$$y_1 = y_1^0 + \varepsilon^{a-1}\widetilde{y}_1, \qquad y_2 = y_2^0 + \varepsilon^{a-1}\widetilde{y}_2. \tag{3.9}$$

To reduce the order on the energy level $H = \varepsilon h$ with

$$\begin{aligned}\varepsilon h &= H(y^0, x^0, \varepsilon)\\ &= H_{00}(y_1^0) + \varepsilon F(y_2^0) + \varepsilon^a H_a(0, y_2^0, x^0) + O(\varepsilon^{3/2} + \varepsilon^{a_1}),\end{aligned} \tag{3.10}$$

we solve the equation

$$H(y_1^0 + \varepsilon^{a-1}\widetilde{y}_1, y_2^0 + \varepsilon^{a-1}\widetilde{y}_2, x) = \varepsilon h, \tag{3.11}$$

with respect to $\widetilde{y}_1$. Developing H_{00} around the critical point $y_1^{cr} = 0$, we get

$$H_{00}(y_1^0) = \frac{1}{2}w(y_1^0)^2 + O((y_1^0)^3) = \frac{1}{2}\varepsilon w(y_1^*)^2 + O(\varepsilon^{3/2}).$$

Hence

$$\varepsilon h = \frac{1}{2}\varepsilon w(y_1^*)^2 + \varepsilon F(y_2^0) + \varepsilon^a H_a(0, y_2^0, x^0) + O(\varepsilon^{3/2} + \varepsilon^{a_1}),$$

from which we obtain

$$\lambda = 2(h - F(y_2^0) - \varepsilon^{a-1}H_a(0, y_2^0, x^0)) + O(\varepsilon^{1/2} + \varepsilon^{a_1-1}). \tag{3.12}$$

According to the definition of λ (and also using the fact that both w and λ are different from 0), we get $\lambda w > 0$. Multiplying (3.12) by w,

$$\lambda w = 2(h - F(y_2^0))w + O(\varepsilon^{a-1} + \varepsilon^{1/2}) > 0,$$

we finally obtain

$$(h - F(y_2^0))w > 0. \tag{3.13}$$

Notation. From now on, given $0 < \sigma < \alpha$, we denote $f = O_\sigma(\varepsilon^\alpha)$ if $f = O(\varepsilon^{\alpha - n\sigma})$ for a suitable $0 \le n < \alpha/\sigma$. For example, we will write $O_\sigma(\varepsilon)$ instead of $O(\varepsilon^{1-\sigma})$ or $O(\varepsilon^{1-2\sigma})$, provided $\sigma < 1/2$.

Lemma 3.2. *Equation (3.11) has a unique solution of the form*

$$\widetilde{y}_1 = -Y(\widetilde{y}_2, x_1, x_2, \varepsilon), \tag{3.14}$$

where

$$Y := \frac{\sqrt{\varepsilon}}{w y_1^*}\left(\Phi_0 + \Phi_1 \widetilde{y}_2 + \varepsilon^{a-1}\Phi_2 \widetilde{y}_2^2 + \varepsilon^{2a-2}\Phi_3 \widetilde{y}_2^3 + O_\sigma(\varepsilon^{3a-3}\widetilde{y}_2^4)\right).$$

The functions $\Phi_{0,1,2,3}$ are as follows:

$$\Phi_0(x,\varepsilon) = H_a(0, y_2^0, x) - H_a(0, y_2^0, x^0) + O_\sigma(\varepsilon^{1/2} + \varepsilon^{a-1} + \varepsilon^{a_1 - a}), \tag{3.15}$$

$$\begin{aligned}\Phi_1(x,\varepsilon) &= F'(y_2^0) + \frac{\varepsilon^{a-1}\mu}{\lambda}F'(y_2^0) + \varepsilon^{a-1}\left(H_a'(0, y_2^0, x) + \varphi_1(y_2^0, x)\right)\\ &\quad + O_\sigma(\varepsilon^{2a-2} + \varepsilon^{1/2} + \varepsilon^{a_1 - 1}),\end{aligned} \tag{3.16}$$

$$\begin{aligned}\Phi_2(x,\varepsilon) &= \frac{1}{2}F''(y_2^0) + \frac{1}{2\lambda}(F')^2(y_2^0)\\ &\quad + \frac{\varepsilon^{a-1}\mu}{2\lambda}\left(F''(y_2^0) + \frac{3}{\lambda}(F')^2(y_2^0)\right)\\ &\quad + \varepsilon^{a-1}\left(\frac{1}{2}H_a''(0, y_2^0, x) + \frac{1}{\lambda}F'(y_2^0)H_a'(0, y_2^0, x) + \varphi_2(y_2^0, x)\right)\\ &\quad + O_\sigma(\varepsilon^{2a-2} + \varepsilon^{a-1/2} + \varepsilon^{1/2} + \varepsilon^{a_1-1}),\end{aligned} \tag{3.17}$$

$$\Phi_3(x,\varepsilon) = \frac{1}{6}F'''(y_2^0) + \frac{1}{2\lambda^2}F'(y_2^0)\widehat{\Phi} + O_\sigma(\varepsilon^{a-1} + \varepsilon^{1/2}). \tag{3.18}$$

Here $\lambda = \lambda(y_2^0)$ satisfies (3.12), and the functions φ_1 and φ_2 have zero average in x since

$$\begin{aligned}\varphi_1(y_2^0, x) &:= \frac{F'(y_2^0)}{\lambda}\varphi_0(y_2^0, x),\\ \varphi_2(y_2^0, x) &:= \frac{1}{2\lambda}\left(F''(y_2^0) + \frac{3}{\lambda}(F'(y_2^0))^2\right)\varphi_0(y_2^0, x),\\ \varphi_0(y_2^0, x) &:= H_a(0, y_2^0, x) - \overline{F}_a(y_2^0).\end{aligned}$$

Finally,

$$\begin{aligned}\widehat{\Phi}(y_2^0) &:= 2(h - F(y_2^0))F''(y_2^0) + (F'(y_2^0))^2 \\ &= \lambda F''(y_2^0) + (F'(y_2^0))^2 + \varepsilon^{a-1}2H_1(0, y_2^0, x^0)F''(y_2^0) + O_\sigma(\varepsilon^{\frac{1}{2}} + \varepsilon^{a_{1-1}}).\end{aligned} \tag{3.19}$$

Remark 3.1. Obviously, explicit terms in (3.16)–(3.17) of order ε^{a-1} make sense only if $a < 3/2$. Otherwise they are suppressed by the error terms.

The proof of Lemma 3.2 (see Appendix) is based on a direct calculation.

The function

$$Y(\widetilde{y}_2, \tau, x_2, \varepsilon), \quad \tau = x_1,$$

can be regarded as the new Hamiltonian:

$$\frac{dx_2}{d\tau} = \frac{\partial Y}{\partial \widetilde{y}_2}, \qquad \frac{d\widetilde{y}_2}{d\tau} = -\frac{\partial Y}{\partial x_2}.$$

The angular variable τ is fast. Therefore we can perform averaging w.r.t. τ by a $O_\sigma(\sqrt{\varepsilon})$-close to the identity time-dependent canonical transformation $(p, q, \tau) \to (\tilde{y}_2, x_2, \tau)$ with $\widetilde{y}_2 := p + \frac{\sqrt{\varepsilon}}{y_1^*}\widetilde{y}_2^*(p, q, \tau, \varepsilon)$, $x_2 := q + \frac{\sqrt{\varepsilon}}{y_1^*}x_2^*(p, q, \tau, \varepsilon)$. The above-mentioned canonical transformation can be obtained by a composition of four different canonical transformations. Let $p_0 := \widetilde{y}_2$, $q_0 := x_2$. The transformations

$$p_j = p_j(p_{j+1}, q_{j+1}, \tau), \quad q_j = q_j(p_{j+1}, q_{j+1}, \tau) \quad \text{for } j = 0, 1, 2, 3,$$

associated with the time-dependent generating functions

$$p_{j+1}q_j + \frac{\sqrt{\varepsilon}}{y_1^*} S_j(p_{j+1}, q_j, \tau, \varepsilon),$$

with

$$S_0 := S_0^*, \qquad S_1 := p_2 S_1^*, \qquad S_2 := \varepsilon^{a-1} p_3^2 S_2^*, \qquad S_3 := \varepsilon^{2a-2} p_4^3 S_3^*,$$

and

$$S_j^* := S_j^*(q_j, \tau) := \frac{1}{w}\int_0^\tau \left[\frac{1}{2\pi}\int_0^{2\pi} \Phi_j(q, \xi)d\xi - \Phi_j(q, s)\right] ds.$$

The transformations are implicitly given by the equations

$$p_j = p_{j+1} + \frac{\sqrt{\varepsilon}}{y_1^*}\partial_{q_j} S_j(p_{j+1}, q_j, \tau, \varepsilon), \qquad q_{j+1} = q_j + \frac{\sqrt{\varepsilon}}{y_1^*}\partial_{p_{j+1}} S_j(p_{j+1}, q_j, \tau, \varepsilon);$$

notice that the transformed Hamiltonians contains the terms $\frac{\sqrt{\varepsilon}}{y_1^*}\partial_\tau S_j$, which are exactly responsible for the cancellation of the fast oscillating terms. Finally, $p := p_4$ and $q := q_4$.

After this transformation, the new Hamiltonian is

$$\widetilde{Y}(p, q, \tau, \varepsilon) := \frac{\sqrt{\varepsilon}}{w y_1^*}(\phi_0 + \phi_1 p + \varepsilon^{a-1}\phi_2 p^2 + \varepsilon^{2a-2}\phi_3 p^3 + O_\sigma(\varepsilon^{3a-3}p^4)),$$

where

$$\phi_j(q, \varepsilon) := \frac{1}{2\pi}\int_0^{2\pi} \Phi_j(q, \tau)\, d\tau + O_\sigma(\sqrt{\varepsilon}) \qquad \text{for } j = 0, 1, 2, 3. \tag{3.20}$$

We note that, denoting

$$\widetilde{F}_a(y_2^0, q) := \frac{1}{2\pi}\int_0^{2\pi} H_a(0, y_2^0, \tau, q)\, d\tau, \quad \tilde{\varphi}_2(y_2^0, q) := \frac{1}{2\pi}\int_0^{2\pi} \varphi_2(y_2^0, \tau, q)\, d\tau,$$

we have, by (3.19),

$$\phi_1 = F' + O_\sigma(\varepsilon^{a-1} + \varepsilon^{1/2}), \tag{3.21}$$

$$\phi_2 = \frac{\widehat{\Phi}}{2\lambda} + O_\sigma(\varepsilon^{a-1} + \varepsilon^{1/2}) \tag{3.22}$$

$$\begin{aligned} &= \frac{\widehat{\Phi}}{2\lambda} \\ &\quad + \frac{\varepsilon^{a-1}}{2\lambda}\left(-2H_a(0, y_2^0, x^0)F'' + \mu F'' + 3\frac{\mu}{\lambda}(F')^2 + \lambda \widetilde{F}_a'' + 2F'\widetilde{F}_a' + 2\lambda\tilde{\varphi}_2\right) \\ &\quad + O_\sigma(\varepsilon^{2a-2} + \varepsilon^{a-1/2} + \varepsilon^{1/2} + \varepsilon^{a_1-1})), \end{aligned} \tag{3.23}$$

$$\phi_3 = \frac{1}{6}F''' + \frac{1}{2\lambda^2}F'\widehat{\Phi} + O_\sigma(\varepsilon^{a-1} + \varepsilon^{1/2}). \tag{3.24}$$

Suppose now that[7]

$$|F'(y_2^0)| > \varepsilon^\sigma \quad \text{with sufficiently small } \sigma > 0. \tag{3.25}$$

We consider the one-dimensional system with energy

$$E := \phi_0(q) + \phi_1(q)p + \varepsilon^{a-1}\phi_2(q)p^2 + \varepsilon^{2a-2}\phi_3(q)p^3, \tag{3.26}$$

for which we want to introduce a new couple of action-angle variables P, Q such that, after this canonical change of coordinates, the energy E is a function only of the new action P, namely $E = E(P)$. In order to define P, we express p as a function of E and q:

$$\begin{aligned} p &= p(E, q) \\ &= \frac{E - \phi_0}{\phi_1} - \varepsilon^{a-1}\frac{(E-\phi_0)^2\phi_2}{\phi_1^3} \\ &\quad + \varepsilon^{2a-2}\left(\frac{2(E-\phi_0)^3\phi_2^2}{\phi_1^5} - \frac{(E-\phi_0)^3\phi_3}{\phi_1^4}\right) + O_\sigma(\varepsilon^{3a-3}). \end{aligned} \tag{3.27}$$

Let

$$S(E, q) := \int_0^q p(E, q)\, dq.$$

We define

$$P(E) := \frac{1}{2\pi}S(E, 2\pi) = \frac{1}{2\pi}\int_0^{2\pi} p(E, q)\, dq. \tag{3.28}$$

We observe that

$$P(E) = \frac{E - \mu}{F'} + O_\sigma(\varepsilon^{a-1} + \varepsilon^{1/2} + \varepsilon^{a_1 - a}). \tag{3.29}$$

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[7] If $|F'(y_2^0)| \le \varepsilon^\sigma$, slightly changing y_2^0, we can get (3.25) (arguing as in case **1B**).

Moreover we have

$$\dot{P}(E) \;=\; \frac{1}{2\pi}\int_0^{2\pi} \partial_E p(E,q)\,dq \tag{3.30}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\frac{1}{\phi_1} - \varepsilon^{a-1}\frac{2(E-\phi_0)\phi_2}{\phi_1^3} \tag{3.31}$$

$$+\,\varepsilon^{2a-2}\left(\frac{6(E-\phi_0)^2\phi_2^2}{\phi_1^5} - \frac{3(E-\phi_0)^2\phi_3}{\phi_1^4}\right) + O_\sigma(\varepsilon^{3a-3})\Bigg]\,dq, \tag{3.32}$$

and

$$\ddot{P}(E) = \frac{1}{2\pi}\int_0^{2\pi}\left[-\varepsilon^{a-1}\frac{2\phi_2}{\phi_1^3} + 6(E-\phi_0)\varepsilon^{2a-2}\left(\frac{2\phi_2^2}{\phi_1^5} - \frac{\phi_3}{\phi_1^4}\right) + O_\sigma(\varepsilon^{3a-3})\right]dq. \tag{3.33}$$

Thus we have that $P(E)$ is invertible, and the new Hamiltonian is $E(P)$ with $E(P(E)) = E$, $P(E(P)) = P$ with

$$E'(P) \;=\; \frac{1}{\dot{P}(E(p))}, \tag{3.34}$$

$$E''(P) \;=\; -\ddot{P}(E(P))[E'(P)]^3. \tag{3.35}$$

Since, by (3.30)

$$\dot{P}(E) = \frac{1}{F'} + O_\sigma(\varepsilon^{a-1} + \varepsilon^{1/2}),$$

from (3.34) we have

$$E'(P) = F' + O_\sigma(\varepsilon^{a-1}\varepsilon^{1/2}). \tag{3.36}$$

Now we can define the new angle

$$\begin{aligned} Q(P,q) \;&:=\; \partial_P S(E(P),q) = E'(P)\partial_E S(E(P),q) \\ &=\; E'(P)\int_0^q \frac{1}{\phi_1} + O_\sigma(\varepsilon^{a-1}) \\ &=\; (F' + O_\sigma(\varepsilon^{a-1}\varepsilon^{1/2}))\int_0^q \frac{1}{F'} + O_\sigma(\varepsilon^{a-1} + \varepsilon^{1/2}) \\ &=\; q + O_\sigma(\varepsilon^{a-1} + \varepsilon^{1/2}). \end{aligned}$$

We now evaluate

$$-\frac{\ddot{P}(E)}{\varepsilon^{a-1}} = \frac{1}{2\pi}\int_0^{2\pi}\left[-\frac{2\phi_2}{\phi_1^3} + 6(E-\phi_0)\varepsilon^{a-1}\left(\frac{2\phi_2^2}{\phi_1^5} - \frac{\phi_3}{\phi_1^4}\right) + O_\sigma(\varepsilon^{2a-2})\right]dq. \tag{3.37}$$

If $\widehat{\Phi} \neq 0$, we simply have

$$-\frac{\ddot{P}(E)}{\varepsilon^{a-1}} = \frac{1}{\lambda(F')^3}\widehat{\Phi} + O_\sigma(\varepsilon^{a-1} + \varepsilon^{1/2})$$

and

$$E''(P) = \varepsilon^{a-1}\left(\frac{\widehat{\Phi}}{\lambda} + O_\sigma(\varepsilon^{a-1} + \varepsilon^{1/2})\right). \tag{3.38}$$

We remark that the general solution of the equation $\widehat{\Phi} := 2(h-F)F'' + (F')^2 = 0$ is as follows:[8] $F(y) = d(y-r)^2 + h$. Moreover by (3.13) we obtain $dw < 0$, namely, (1.11).

On the other hand, if F has the form $F(y) = d(y-r)^2 + h$, which implies $\widehat{\Phi} \equiv 0$ and $F''' \equiv 0$, we have

$$\begin{aligned} \frac{2\phi_2}{\phi_1^3} &= \frac{1}{\lambda(F')^3}\left(-2H_a(0, y_2^0, x^0)F'' + \mu F'' + 3\mu(F')^2/\lambda + \lambda \widetilde{F}_a{}'' + 2F'\widetilde{F}_a{}'\right) \\ &\quad + O_\sigma(\varepsilon^{a-1} + \varepsilon^{1/2} + \varepsilon^{3/2-a} + \varepsilon^{a_1-a}), \\ \frac{\phi_2^2}{\phi_1^5} &= O_\sigma(\varepsilon^{2a-2} + \varepsilon), \\ \frac{\phi_3}{\phi_1^4} &= O_\sigma(\varepsilon^{a-1} + \varepsilon^{1/2}). \end{aligned}$$

Collecting the previous equalities, we have

$$\begin{aligned} -\frac{\ddot{P}(E)}{\varepsilon^{2a-2}} &= \frac{-2H_a(0, y_2^0, x^0)F'' + \mu F'' + 3\mu(F')^2/\lambda + \lambda\overline{F_a}'' + 2F'\overline{F_a}'}{\lambda(F')^3} \\ &\quad + O_\sigma(\varepsilon^{a-1} + \varepsilon^{1/2} + \varepsilon^{3/2-a} + \varepsilon^{a_1-a}), \end{aligned}$$

from which (and also using (3.35) and (3.36)), we finally obtain

$$\begin{aligned} E''(P) &= \varepsilon^{2a-2}\frac{1}{\lambda}\Big(-2H_a(0, y_2^0, x^0)F'' + \mu F'' + 3\mu(F')^2/\lambda + \lambda\overline{F_a}'' + 2F'\overline{F_a}' \\ &\quad + O_\sigma(\varepsilon^{a-1} + \varepsilon^{1/2} + \varepsilon^{3/2-a} + \varepsilon^{a_1-a})\Big). \end{aligned} \tag{3.39}$$

Finally we note that, after the previous change of variables[9] $p = p(P, Q, \varepsilon)$, $q = q(P, Q, \varepsilon)$, the Hamiltonian $\widetilde{Y}$ is transformed into

$$\widehat{Y}(P, Q, \tau, \varepsilon) := \frac{\sqrt{\varepsilon}}{w y_1^*}(E(P) + O_\sigma(\varepsilon^{3a-3})). \tag{3.40}$$

From Theorem 7, the condition $\widehat{\Phi} \not\equiv 0$, which gives a *twist* term $\partial_{PP}^2 \widehat{Y}$ of order $\varepsilon^{a-1/2}$, is sufficient for the stability of the original system on the energy level (3.10). In fact the evolution of the variable P (and hence of E) remains bounded so that the evolution of p (and, hence of $\tilde{y}_2$) remains bounded by $O(\varepsilon^{-\sigma})$ from (3.27); finally the evolution of y_2 is bounded by $O_\sigma(\varepsilon^{a-1})$ from (3.9). The case in which, for some y_1^{cr},

[8] In fact, if $G := F - h$, the equation becomes $2GG'' = (G')^2 \Leftrightarrow 2G''/G' = G'/G \Leftrightarrow 2\ln G' = \ln G + c_1$ $\Leftrightarrow G'(G)^{-1/2} = c_2 \Leftrightarrow \sqrt{G} = c_2 y + c_3 \Leftrightarrow G = d(y-r)^2$.

[9] This change is $O(1)$ in the sense that $\partial_P p$, $\partial_P q$, $\partial_Q p$, $\partial_Q q$ are quantities of order 0 in ε.

$F(y_2) := H_{01}(y_1^{cr}, y_2)$ is of the type described in (1.11) is more delicate, since the *twist* term is smaller. This case will be discussed in the next section.

2B. If $|y_1^*| < \varepsilon^\sigma$, we change y_2^0 by a quantity of order $\sim \varepsilon^{\sigma_1}$, $\sigma_1 < \sigma/4$. Then $H_{01}(0, y_2^0)$ will change at least by $\sim \varepsilon$. The new y_1^*, satisfying the energy condition (3.10), will differ from the original one at least by $\sim \varepsilon^{4\sigma_1} > 2\varepsilon^\sigma$. Hence, arguing as in case **1B**, we reduced case **2B** to **2A**.

3.3. Proof of Theorem 2

From (1.12), namely $\widehat{\Phi} = 0 \Leftrightarrow (F')^2 = -2(h - F)F''$, we have

$$\mu F'' + \frac{3\mu}{2(h - F)}(F')^2 = -2\mu F'', \tag{3.41}$$

while from (3.12), (3.41), and the definition of μ, we have

$$-2H_a(0, y_2^0, x^0)F'' + \mu F'' + 3\mu(F')^2/\lambda + \lambda\overline{F_a}'' + 2F'\overline{F_a}' = \widehat{\Psi} + O_\sigma(\varepsilon^{a-1} + \varepsilon^{1/2}), \tag{3.42}$$

where

$$\widehat{\Psi} := 2(-F''\overline{F}_a + F'\overline{F}_a' + (h - F)\overline{F}_a'').$$

Finally from (3.39) and (3.42), we obtain

$$E''(P) = \varepsilon^{2a-2}\frac{1}{\lambda}\left(\widehat{\Psi} + O_\sigma(\varepsilon^{a-1} + \varepsilon^{1/2} + \varepsilon^{3/2-a} + \varepsilon^{a_1-a})\right). \tag{3.43}$$

If (1.12) holds, nondegeneracy of the system with Hamiltonian $\widehat{Y}$ appears in the order $\varepsilon^{2a-3/2}$, provided $a < 3/2$ and $\widehat{\Psi} \not\equiv 0$. It remains to note that (1.14) is a general solution of the equation $\widehat{\Psi} = 0$ for F satisfying (1.12). In fact the equation becomes

$$-2d\overline{F}_a(y_2) + 2d(y_2 - r)\overline{F}_a'(y_2) - d(y_2 - r)^2\overline{F}_a''(y_2) = 0,$$

which, after the substitution $\overline{F}_a(y_2) =: \alpha(y_2)(y_2 - r)$, is equivalent to $\alpha''(y_2) = 0$, which has the general solution $\alpha(y_2) = uy_2 + v$.

4. An Application to Celestial Mechanics (After D'Alembert)

In this section we consider the Hamiltonian version of the D'Alembert model for the planetary spin/orbit problem. The model may be described as follows (see [14], [10], [12], [13] for more details).

Let a planet be modelled by a rotational ellipsoid slightly flattened along the symmetry axis (called "north–south" direction); assume that the center of mass of such planet revolves on a slightly eccentric Keplerian ellipse around a *fixed* star occupying one of the foci of the ellipse: The planet is subject to the gravitational attraction of the star and the problem is to study the relative position of the planet and, most notably, the time evolution of its angular momentum.

Such a model may be described using Hamiltonian formalism using action–angle symplectic variables. The Hamiltonian system describing the D'Alembert model results in a two-degrees-of-freedom system depending explicitly and periodically on time (the period being the year of the planet); furthermore such a Hamiltonian system is nearly integrable (with *two* small parameters—the flatness of the planet and the eccentricity of the Keplerian ellipse) and properly degenerate.

As we said in the introduction, we are interested in studying the D'Alembert model in the vicinity of a $p : q$ spin/orbit resonance (p and q positive, coprime integers). For simplicity we will omit in the following formulas the explicit dependence on p and q.

Thanks to a well-known result by Andoyer and Deprit (see, e.g., [1], [16]), the motions of the D'Alembert model are governed, in suitable physical units, by the following Hamiltonian function:[10]

$$\begin{aligned} H_{\varepsilon,\mu}(I,\varphi) \;:=\; & \frac{(\bar{J}_1 + I_1)^2}{2} + \bar{\omega}(I_3 - I_2) \\ & + \varepsilon f_0(I_1, I_2, \varphi_1, \varphi_2) + \varepsilon\mu f_1(I_1, I_2, \varphi_1, \varphi_2, \varphi_3; \mu), \end{aligned} \tag{4.1}$$

where

(a) $\bar{J}_1$ is a constant parameter, which may be interpreted as a "reference datum" in a neighborhood of which the system will be studied;

(b) ε and μ are two *small* nonnegative parameters measuring, respectively, the flatness of the planet and the eccentricity of the Keplerian orbit described by the center of mass of the planet; moreover, we define a constant $c > 0$ such that

$$\mu = \varepsilon^c; \tag{4.2}$$

(c) $(I, \varphi) := (I_1, I_2, I_3, \varphi_1, \varphi_2, \varphi_3) \in A \times \mathbf{T}^3$ are standard symplectic coordinates; the domain $A \subset \mathbf{R}^3$ is given by

$$A := \{|I_1| < d, \quad |I_2 - \bar{J}_2| < d, \quad I_3 \in \mathbf{R}\}, \tag{4.3}$$

where d is a suitable fixed (and small) positive number, while $\bar{J}_2$ is fixed "reference datum" (verifying, together with $\bar{J}_1$, certain assumptions spelled out below);

(d) $2\pi/\bar{\omega}$ is the period of the Keplerian motion ("year of the planet");

(e) the function f_0 is a trigonometric polynomial given by

$$f_0 = \sum_{\substack{j \in \mathbf{Z} \\ |j| \le 2}} c_j \cos(j\varphi_1) + d_j \cos(j\varphi_1 + 2\varphi_2), \tag{4.4}$$

where c_j and d_j are functions of $(\bar{J}_1 + I_1, I_2)$ listed in the following item;

(f) let

$$\begin{aligned} \kappa_1 &:= \kappa_1(I_1) := \frac{L}{\bar{J}_1 + I_1}, \qquad & \kappa_2 &:= \kappa_2(I_1, I_2) := \frac{I_2}{\bar{J}_1 + I_1}, \\ \nu_1 &:= \nu_1(I_1) := \sqrt{1 - \kappa_1^2}, \qquad & \nu_2 &:= \nu_2(I_1, I_2) := \sqrt{1 - \kappa_2^2}, \end{aligned} \tag{4.5}$$

[10] See [14].

where L is a real parameter; the parameters $\bar{J}_i$, L, and the constant d are assumed to satisfy

$$L + d < \bar{J}_1, \qquad |\bar{J}_2| + 2d < \bar{J}_1; \tag{4.6}$$

in this way, $0 < \kappa_i < 1$ (and the ν_i's are well defined on the domain A). Then, the functions c_j and d_j are defined by

$$\begin{aligned}
c_0(I_1, I_2) &:= \frac{1}{4}\left(2\kappa_1^2\nu_2^2 + \nu_1^2(1+\kappa_2^2)\right),\\
d_0(I_1, I_2) &:= -\frac{\nu_2^2}{4}(2\kappa_1^2 - \nu_1^2),\\
c_{\pm 1}(I_1, I_2) &:= \frac{\kappa_1\kappa_2\nu_1\nu_2}{2},\\
d_{\pm 1}(I_1, I_2) &:= \mp\frac{(1\pm\kappa_2)\kappa_1\nu_1\nu_2}{2},\\
c_{\pm 2}(I_1, I_2) &:= -\frac{\nu_1^2\nu_2^2}{8},\\
d_{\pm 2}(I_1, I_2) &:= -\frac{\nu_1^2(1\pm\kappa_2)^2}{8};
\end{aligned} \tag{4.7}$$

(g) the function f_1 is a convergent series in μ of trigonometric polynomials (with increasing degrees); $f_1^0 := f_1|_{\mu=0}$ and $f_1^1 := df_1/d\mu|_{\mu=0}$ are given by

$$\begin{aligned}
f_1^0 &= \sum_{\substack{j\in\mathbf{Z}\\ |j|\le 2}} (-3)c_j\cos(j\varphi_1 + \varphi_3) \\
&\quad + d_j\left(\frac{1}{2}\cos(j\varphi_1 + 2\varphi_2 + \varphi_3) - \frac{7}{2}\cos(j\varphi_1 + 2\varphi_2 - \varphi_3)\right),
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
f_1^1 &= \sum_{\substack{j\in\mathbf{Z}\\ |j|\le 2}} c_j\left(\frac{3}{2}\cos(j\varphi_1) + \frac{9}{2}\cos(j\varphi_1 + 2\varphi_3)\right) \\
&\quad + d_j\left(\frac{17}{2}\cos(j\varphi_1 + 2\varphi_2 - 2\varphi_3) - \frac{5}{2}\cos(j\varphi_1 + 2\varphi_2)\right).
\end{aligned} \tag{4.9}$$

Remark 4.1. (i) Since I_3 appears only linearly with coefficient $\bar{\omega}$, the angle φ_3 corresponds to time t and $H_{\varepsilon,\mu}$ is actually a two-degrees-of-freedom Hamiltonian depending explicitly on time in a periodic way (with period $2\pi/\bar{\omega}$).

(ii) For a physical interpretation of the action variables I_1, I_2, the parameter L, and the angles φ_i, see [14], [12], [13].

We are interested in studying the above system in a neighborhood of a day/year (or "spin/orbit") resonance. Since the daily rotation is measured by the angle φ_1 and since in the unperturbed situation ($\varepsilon = 0$ and $I_1 = 0$) $\varphi_1 = \varphi_1^0 + \bar{J}_1 t$, we see that *an approximate day/year resonance corresponds to taking the "reference datum" $\bar{J}_1$ (which, in our units,* *coincides with the daily frequency) in a rational relation with the year frequency* $\bar{\omega}$, i.e., $\bar{J}_1 = \frac{p}{q}\bar{\omega}$ with p and q coprime positive integers; we shall speak in such a case of a "$p : q$ spin/orbit–resonance."

Setting

$$\bar{J}_1 := \frac{p}{q}\bar{\omega}, \qquad \omega := \frac{\bar{\omega}}{q}, \tag{4.10}$$

we see that *the dynamics near a* $p : q$ *spin/orbit resonance is described by the Hamiltonian*

$$\begin{aligned} H_{\varepsilon,\mu}(I,\varphi) \;:=\; & \frac{I_1^2}{2} + \omega(pI_1 - qI_2 + qI_3) \\ & + \varepsilon f_0(I_1, I_2, \varphi_1, \varphi_2) + \varepsilon\mu f_1(I_1, I_2, \varphi_1, \varphi_2, \varphi_3; \mu), \end{aligned} \tag{4.11}$$

where we have omitted the constant term $\bar{J}_1^2/2$.

Finally, to make the analysis perturbative, we shall take as the action-variable domain an ε–dependent subset of A:

(h) the domain of definition A introduced in item (c) above will, from here on, be replaced by its subset

$$A_\varepsilon := \left\{ |I_1| < r\varepsilon^\ell,\ |I_2 - \bar{J}_2| < r,\ I_3 \in \mathbf{R} \right\}, \tag{4.12}$$

where

$$0 < \ell < \frac{1}{2}, \tag{4.13}$$

and $r > 0$. The parameters $\bar{J}_i$, L and the constant r are assumed to satisfy

$$L + 3r\varepsilon^\ell < \bar{J}_1, \qquad |\bar{J}_2| + 3r(\varepsilon^\ell + 1) < \bar{J}_1, \tag{4.14}$$

so that $0 < \kappa_i < 1$ and the ν_i's are well defined on the domain A.

Let ϕ^* be the following linear symplectic map:

$$\phi^*(I^*, \varphi^*) := \left((I_1^*, I_2^*, -\frac{p}{q}I_1^* + I_2^* + \frac{1}{q}I_3^*), (\varphi_1^* + p\varphi_3^*, \varphi_2^* - q\varphi_3^*, q\varphi_3^*) \right). \tag{4.15}$$

Then, ϕ^* casts the Hamiltonian $H_{\varepsilon,\mu}$ into the form[11]

$$\begin{aligned} H^*(I^*, \varphi^*; \varepsilon, \mu) \;:=\; & H_{\varepsilon,\mu} \circ \phi^*(I^*, \varphi^*) \\ =\; & \frac{(I_1^*)^2}{2} + \omega I_3^* + \varepsilon f_0^* + \varepsilon\mu f_1^* + \varepsilon\mu^2 f_2^* + O(\varepsilon\mu^3) \\ =\; & \frac{(I_1^*)^2}{2} + \omega I_3^* + \varepsilon f_0^* + \varepsilon^{1+c} f_1^* \varepsilon^{1+2c} f_2^* + O(\varepsilon^{1+3c}), \end{aligned} \tag{4.16}$$

[11] In the last equality we have also used the fact that $\mu = \varepsilon^c$ from (4.2).

where c is defined in (4.2) and

$$\begin{aligned} f_0^* = f_0^*(I_1^*, I_2^*, \varphi_1^*, \varphi_2^*, \varphi_3^*) &:= f_0 \circ \phi^*, \\ f_1^* = f_1^*(I_1^*, I_2^*, \varphi_1^*, \varphi_2^*, \varphi_3^*) &:= f_1^0 \circ \phi^*, \\ f_2^* = f_2^*(I_1^*, I_2^*, \varphi_1^*, \varphi_2^*, \varphi_3^*) &:= f_1^1 \circ \phi^*, \end{aligned}$$

namely,

$$f_0^* = \sum_{\substack{j\in\mathbf{Z}\\|j|\le 2}} c_j \cos(j\varphi_1^* + jp\varphi_3^*) + d_j \cos\left(j\varphi_1^* + 2\varphi_2^* + (jp-2q)\varphi_3^*\right), \tag{4.17}$$

$$f_1^* = \sum_{\substack{j\in\mathbf{Z}\\|j|\le 2}} (-3)c_j \cos\left(j\varphi_1^* + (jp+q)\varphi_3^*\right) + \frac{d_j}{2}\left[\cos\left(j\varphi_1^* + 2\varphi_2^* + (jp-q)\varphi_3^*\right) - 7\cos\left(j\varphi_1^* + 2\varphi_2^* + (jp-3q)\varphi_3^*\right)\right], \tag{4.18}$$

$$f_2^* = \sum_{\substack{j\in\mathbf{Z}\\|j|\le 2}} \frac{c_j}{2}\left[3\cos(j\varphi_1^* + jp\varphi_3^*) + 9\cos\left(j\varphi_1^* + (jp+2q)\varphi_3^*\right)\right] + \frac{d_j}{2}\left[17\cos\left(j\varphi_1^* + 2\varphi_2^* + (jp-4q)\varphi_3^*\right) - 5\cos\left(j\varphi_1^* + 2\varphi_2^* + (jp-2q)\varphi_3^*\right)\right]. \tag{4.19}$$

Now, since φ_3^* is a "fast angle," by standard Normal Form Theory (see, for example, [10], [12], [13]) we can find a final close-to-the-identity canonical change of variables $\phi^\sharp(y, x) = (I^*, \varphi^*)$ removing the dependence on the fast angle up to exponentially small terms

$$\begin{aligned} H^\sharp(y, x; \varepsilon) &:= H^* \circ \phi^\sharp(y, x) \\ &= \frac{y_1^2}{2} + \omega y_3 + \varepsilon \bar{f}_0^*(y_1, y_2, x_1, x_2) + \varepsilon^{1+c} \bar{f}_1^*(y_1, y_2, x_1, x_2) \\ &\quad + \varepsilon^{1+2c} \bar{f}_2^*(y_1, y_2, x_1, x_2) + \varepsilon^{a'} g(y_1, y_2, x_1, x_2; \varepsilon) \qquad (4.20) \\ &\quad + O(\varepsilon \exp(-const.\varepsilon^{-\ell})), \qquad (4.21) \end{aligned}$$

where g is a suitable analytic function, the "bar" denotes the average on the third angle, and

$$a' := \min\{2 - \ell, 1 + 3c\}. \tag{4.22}$$

Proposition 4.1. *The motion of the planet in the D'Alembert model is governed, up to an ε-exponentially small term, by the two-degrees-of-freedom properly degenerate* *"effective Hamiltonian"*

$$H_{\rm eff}(y_1, y_2, x_1, x_2; \varepsilon) \tag{4.23}$$
$$:= \frac{y_1^2}{2} + \varepsilon \bar{f}_0^*(y, x) + \varepsilon^{1+c} \bar{f}_1^*(y, x) + \varepsilon^{1+2c} \bar{f}_2^*(y, x) + \varepsilon^{a'} g(y, x; \varepsilon).$$

Hence, up to an ε-exponentially long time, the motion of the planet is well-described by the properly degenerate two-degrees-of-freedom system (4.23). In fact the effective Hamiltonian $H_{\rm eff}$ is of the type (1.1) with

$$H_0 = \frac{y_1^2}{2} \qquad \text{and} \qquad H_1 = \bar{f}_0^* + \varepsilon^c \bar{f}_1^* + \varepsilon^{2c} \bar{f}_2^* + \varepsilon^{a'-1} g. \tag{4.24}$$

If we want that H_1 has the same special structure as in (1.7), we have to exclude some particular spin/orbit resonances. In fact, from (4.17), it is simple to see that, if $(p, q) = (1, 1),$, $(2, 1)$, the function $\bar{f}_0^*$ really depends on the first angle (but not on the second one), so that the perturbation H_1 takes the same form as in (1.5)

$$H_1 = H_{01}(y, x_1) + O(\varepsilon^c), \quad \text{where} \quad H_{01} := \bar{f}_0^*. \tag{4.25}$$

On the other hand,

$$(p, q) \neq (1, 1), (2, 1) \quad \Longrightarrow \quad \bar{f}_0^* = c_0(y_1, y_2), \tag{4.26}$$

and, hence, we have, for $(p, q) \neq (1, 1)$, $(2, 1)$,

$$H_1 = H_{01}(y) + O(\varepsilon^c), \quad \text{with} \quad H_{01} = c_0(y_1, y_2), \tag{4.27}$$

which is of the type described in (1.7). At this point we would apply the stability Theorem 1 to the Hamiltonian $H_{\rm eff}$. Condition (1) is satisfied (recall (4.24)). On the other hand, by the definition of $c_0 = H_{01}$ given in (4.7), it follows that condition (2) means that, for any $|y_1^0| < r\varepsilon^\ell$, the function

$$y_2 \mapsto \frac{(\bar{J}_1 + y_1^0)^2 - 3L^2}{2(\bar{J}_1 + y_1^0)^4}$$

is not identically zero. Such a condition is satisfied if

$$\bar{J}_1 \neq \sqrt{3} L$$

(taking ε small enough). Unfortunately we note that condition (1.11) could be violated. Indeed, for $y_1 = y_1^{cr} = 0$ (which is the only critical point of the unperturbed Hamiltonian $y_1^2/2$), we have

$$H_{01}(y_1^{cr}, y_2) = d(y_2 - r)^2 + h, \tag{4.28}$$

with

$$r := 0, \qquad d := \frac{1}{4\bar{J}_1^2}\left(1 - 3\frac{L^2}{\bar{J}_1^2}\right), \qquad h := \frac{1}{4}\left(1 + \frac{L^2}{\bar{J}_1^2}\right), \tag{4.29}$$

where L and $\bar{J}_1$ are defined in (4.6). Moreover, for[12] $\bar{J}_1 < \sqrt{3}L$, we have $d \le 0$, so that the inequality in (1.11) is also satisfied. Summarizing in order to prove the stability of $H_{\rm eff}$, we can directly apply Theorem 1 only if $\bar{J}_1 > \sqrt{3}L$; otherwise we have to use Theorem 2. Hence we suppose $\bar{J}_1 < \sqrt{3}L$ and try to apply Theorem 2.

In order to do it, we could think of setting $a := 1 + c$, $a_1 := 1 + 2c$, and $H_a := \bar{f}_1^*$ to have in (4.23) the same structure as in (1.10). However if we evaluate $\bar{f}_1^*$, we obtain (remembering that $(p, q) \neq (1, 1), (2, 1)$)

$$\bar{f}_1^* := \begin{cases} -3c_{-2}(y)\cos(2x_1) + \frac{1}{2}d_2(y)\cos(2x_1 + 2x_2) & \text{if } (p, q) = (1, 2), \\ -\frac{7}{2}d_1(y)\cos(x_1 + 2x_2) & \text{if } (p, q) = (3, 1), \\ -\frac{7}{2}d_2(y)\cos(2x_1 + 2x_2) & \text{if } (p, q) = (3, 2), \\ 0 & \text{otherwise.} \end{cases}$$

Hence, when we evaluate $\overline{F}_a(y_2)$ defined in (1.13) for $H_a := \bar{f}_1^*$, we obtain $\overline{F}_a(y_2) = 0$!

So, in order to use Theorem 2, we must require that $\bar{f}_1^* = 0$, and the degeneracy is removed by the term $\bar{f}_2^*$. Hence we take $(p, q) \neq (1, 2), (3, 1), (3, 2)$ so that

$$H_{\rm eff} = \frac{y_1^2}{2} + \varepsilon c_0(y) + \varepsilon^{1+2c}\bar{f}_2^*(y, x) + O(\varepsilon^{a'}), \tag{4.30}$$

which has the form described in (1.10) with

$$H_0 := \frac{y_1^2}{2}, \qquad H_{01} := c_0(y), \qquad H_a := \bar{f}_2^*(y, x), \quad a := 1 + 2c, \quad a_1 := a'. \tag{4.31}$$

Moreover, in order to have $a < 3/2$ (which is a hypothesis of Theorem 2), we assume

$$c < \frac{1}{4}. \tag{4.32}$$

We have

$$\bar{f}_2^* := \begin{cases} \frac{3}{2}c_0(y_1, y_2) + \frac{17}{2}d_{-1}(y_1, y_2)\cos(2x_2 - x_1) & \text{if } (p, q) = (4, 1), \\ \frac{3}{2}c_0(y_1, y_2) & \text{otherwise.} \end{cases}$$

Evaluating $\overline{F}_a(y_2)$ defined in (1.13) for $H_a := \bar{f}_1^*$ defined in (4.31) and for $y_1^{cr} = 0$, we obtain

$$\overline{F}_a(y_2) = \frac{3}{2}c_0(0, y_2) = \frac{3}{2}dy_2^2 + \frac{3}{2}h, \tag{4.33}$$

[12] This is the physically interesting case corresponding to the model in which the equatorial plane of the planet is not too different from its ecliptic plane of revolution around the star.

where d and h are defined in (4.29). Hence, in order to apply Theorem 2, we have to verify condition (1.14), namely that $r = 0$ defined in (4.29) is not a root of $\overline{F}_a(y_2)$. This is true since, from (4.33), we have only to check that $h \neq 0$, which follows from the definition of h given in (4.29). Summarizing, even in the case $\bar{J}_1 < \sqrt{3}L$, we can prove stability by Theorem 2, if we exclude the spin/orbit resonances $(p, q) = (1, 2)$, $(3, 1)$, $(3, 2)$ and we assume (4.32).

Theorem 8. *Suppose that $\bar{J}_1 \neq \sqrt{3}L$, $(p, q) \neq (1, 1), (2, 1)$. Then the effective Hamiltonian of the D'Alembert model defined in (4.23) is stable if the following condition is satisfied:*

$$\bar{J}_1 > \sqrt{3}L \quad \textit{or} \quad (p, q) \neq (1, 2), (3, 1), (3, 2), \quad c < 1/4. \tag{4.34}$$

As a corollary of the previous Theorem and of formula (4.21), we can state the following "Nekhoroshev-type" result:

Theorem 9. *Suppose that $\bar{J}_1 \neq \sqrt{3}L$, $(p, q) \neq (1, 1), (2, 1)$, and that the following condition is satisfied:*

$$\bar{J}_1 > \sqrt{3}L \quad \textit{or} \quad (p, q) \neq (1, 2), (3, 1), (3, 2), \quad c < 1/4. \tag{4.35}$$

Then the action variables of the D'Alembert planetary Hamiltonian $H_{\varepsilon,\mu}$ defined in (4.1) are stable for an exponentially long time; namely, there exist constants c_1, c_2, $c_3 > 0$ such that

$$|I(t) - I(0)| \leq \varepsilon^{c_1}, \qquad \forall\, |t| \leq \exp(c_2 \varepsilon^{-\ell}), \tag{4.36}$$

where $(I(t), \varphi(t))$ denotes the $H_{\varepsilon,\mu}$-evolution of an initial datum $(I(0), \varphi(0)) \in A \times \mathbf{T}^3$.

Remark 4.2. We note that the previous result was already proved in [13] without any assumption[13] on the spin/orbit resonances or on anything else. However, in that generality, the so-called Nekhoroshev exponent in the formula corresponding to (4.36) was lightly worse: a suitable $\gamma_0 < \min\{\ell, c\}$ instead of the present ℓ.

Appendix

Proof of Lemma 3.2. For $\widetilde{y}_1$, satisfying (3.14), equation (3.11) can be rewritten as follows:

$$\begin{aligned} \star &:= H_{00}(y_1^0 + \varepsilon^{a-1}\widetilde{y}_1) - H_{00}(y_1^0) \\ &= -\varepsilon(H_{01}(y_1^0, y_2^0 + \varepsilon^{a-1}\widetilde{y}_2) - H_{01}(y_1^0, y_2^0)) \\ &\quad -\varepsilon(H_{01}(y_1^0 + \varepsilon^{a-1}\widetilde{y}_1, y_2^0 + \varepsilon^{a-1}\widetilde{y}_2) - H_{01}(y_1^0, y_2^0 + \varepsilon^{a-1}\widetilde{y}_2)) \\ &\quad -\varepsilon^a(H_a(0, y_2^0 + \varepsilon^{a-1}\widetilde{y}_2, x) - H_a(0, y_2^0, x^0)) \end{aligned}$$

---

[13] Apart from $\bar{J}_1 \neq \sqrt{3}L$.

$$
\begin{aligned}
& -\varepsilon^a(H_a(y_1^0+\varepsilon^{a-1}\widetilde{y}_1, y_2^0+\varepsilon^{a-1}\widetilde{y}_2, x) - H_a(0, y_2^0+\varepsilon^{a-1}\widetilde{y}_2, x)) \\
& -\varepsilon^a(H_a(0, y_2^0, x^0) - H_a(y_1^0, y_2^0, x^0)) \\
& +\varepsilon^{a_1} H_2(y^0, x^0, \varepsilon^{a-1}\widetilde{y}, x, \varepsilon) \\
= \; & -\varepsilon(H_{01}(y_1^0, y_2^0+\varepsilon^{a-1}\widetilde{y}_2) - H_{01}(y_1^0, y_2^0)) \\
& -\varepsilon^a(H_a(0, y_2^0+\varepsilon^{a-1}\widetilde{y}_2, x) - H_a(0, y_2^0, x^0)) \\
& -\varepsilon(H_{01}(y_1^0+\varepsilon^{a-1}\widetilde{y}_1, y_2^0+\varepsilon^{a-1}\widetilde{y}_2) - H_{01}(y_1^0, y_2^0+\varepsilon^{a-1}\widetilde{y}_2)) \\
& +\varepsilon^b H_3(y^0, x^0, \widetilde{y}_1, \varepsilon^{a-1}\widetilde{y}_2, x, \varepsilon), \qquad (5.1)
\end{aligned}
$$

for some analytic functions H_2, H_3 and

$$b := \min\{2a-1, a+1/2, a_1\}.$$

We have, for

$$|\varepsilon^{a-1}\widetilde{y}_1| \quad \text{small enough}, \qquad (5.2)$$

that

$$
\begin{aligned}
\star &= H_{00}(y_1^0+\varepsilon^{a-1}\widetilde{y}_1) - H_{00}(y_1^0) \\
&= \varepsilon^{a-1/2} w_* y_1^* \widetilde{y}_1 + \frac{w_0}{2}\varepsilon^{2a-2}\widetilde{y}_1^2 + O_\sigma(\varepsilon^{3a-3}\widetilde{y}_1^3), \qquad (5.3) \\
w_* &= \frac{1}{y_1^0}\frac{\partial H_{00}}{\partial y_1}(y_1^0) = w + O_\sigma(\sqrt{\varepsilon}), \\
w_0 &= \frac{\partial^2 H_{00}}{\partial y_1^2}(y_1^0) = w + O_\sigma(\sqrt{\varepsilon}).
\end{aligned}
$$

Now we write Y in the form (3.14), and we want to determine Φ_j for $j = 0, 1, 2, 3$, expanding (5.1) in powers of $\widetilde{y}_2$. First we write

$$\widetilde{y}_1 =: -\frac{\sqrt{\varepsilon}}{w y_1^*}\hat{y}_1,$$

with

$$\hat{y}_1 := \Phi_0 + \Phi_1\widetilde{y}_2 + \varepsilon^{a-1}\Phi_2\widetilde{y}_2^2 + \varepsilon^{2a-2}\Phi_3\widetilde{y}_2^3 + O_\sigma(\varepsilon^{3a-3}\widetilde{y}_2^4).$$

Hence, from (5.3), we have

$$\star = -\varepsilon^a\frac{w_*}{w}\hat{y}_1 + \varepsilon^{2a-1}\frac{w_0}{2w^2(y_1^*)^2}\hat{y}_1^2 + O_\sigma(\varepsilon^{3a-3/2}\hat{y}_1^3). \qquad (5.4)$$

Now we expand the two expressions for $\star$ given in (5.1) and (5.4) in the power of $\widetilde{y}_2$. At zero degree we have

$$-\frac{\varepsilon^a w_*}{w}\Phi_0 + O_\sigma(\varepsilon^{2a-1}) = -\varepsilon^a(H_a(0, y_2^0, x) - H_a(0, y_2^0, x^0)) + O_\sigma(\varepsilon^{a+1/2} + \varepsilon^b).$$

This implies (3.15).

At first degree of $\widetilde{y}_2$ we get

$$-\frac{\varepsilon^a w_*}{w}\Phi_1 + \frac{\varepsilon^{2a-1} w_0}{w^2 (y_1^*)^2}\Phi_0\Phi_1 + O(\varepsilon^{3a-3/2-3\sigma})$$
$$= -\varepsilon^a H'_{01}(y_1^0, y_2^0) - \varepsilon^{2a-1} H'_a(0, y_2^0, x) + O(\varepsilon^{a+1/2-\sigma} + \varepsilon^{b+a-1})$$
$$= -\varepsilon^a F'(y_2^0) - \varepsilon^{2a-1} H'_a(0, y_2^0, x) + O(\varepsilon^{a+1/2-\sigma} + \varepsilon^{b+a-1}),$$

using that $H'_{01}(y_1^0, y_2^0) = F'(y_2^0) + O(\sqrt{\varepsilon})$, from which we have

$$\left(\frac{w_*}{w} - \varepsilon^{a-1}\frac{w_0}{w\lambda}\Phi_0\right)\Phi_1$$
$$= F'(y_2^0) + \varepsilon^{a-1} H'_a(0, y_2^0, x) + O_\sigma(\varepsilon^{2a-3/2} + \varepsilon^{1/2} + \varepsilon^{b-1}).$$

From $|\lambda| \geq \varepsilon^{2\sigma}$ we have

$$\left(\frac{w_*}{w} - \varepsilon^{a-1}\frac{w_0}{w\lambda}\Phi_0\right)^{-1} = \left(1 - \varepsilon^{a-1}\frac{\Phi_0}{\lambda} + O_\sigma(\sqrt{\varepsilon})\right)^{-1}$$
$$= 1 + \varepsilon^{a-1}\frac{\Phi_0}{\lambda} + O_\sigma(\sqrt{\varepsilon} + \varepsilon^{2a-2}),$$

from which we finally obtain

$$\Phi_1 = F'(y_2^0) + \varepsilon^{a-1} H'_a(0, y_2^0, x) + \frac{\varepsilon^{a-1}}{\lambda}\Phi_0 F'(y_2^0) + O_\sigma(\varepsilon^{2a-2} + \varepsilon^{1/2} + \varepsilon^{b-1}).$$

By definition of μ and φ_0, we obtain

$$\Phi_0 = \mu + \varphi_0(y_2^0, x) + O_\sigma(\varepsilon^{1/2} + \varepsilon^{a-1} + \varepsilon^{a_1-1}), \tag{5.5}$$

which implies (3.16).

At second degree of $\widetilde{y}_2$ we get

$$-\frac{\varepsilon^{2a-1} w_*}{w}\Phi_2 + \frac{\varepsilon^{2a-1} w_0}{2w^2 (y_1^*)^2}(\Phi_1^2 + 2\varepsilon^{a-1}\Phi_0\Phi_2) + O_\sigma(\varepsilon^{3a-3/2})$$
$$= -\frac{1}{2}\varepsilon^{2a-1} H''_{01}(y_1^0, y_2^0) - \frac{1}{2}\varepsilon^{3a-2} H''_a(0, y_2^0, x) + O_\sigma(\varepsilon^{2a-1/2} + \varepsilon^{b+2a-2})$$
$$= -\frac{1}{2}\varepsilon^{2a-1} F''(y_2^0) - \frac{1}{2}\varepsilon^{3a-2} H''_a(0, y_2^0, x) + O_\sigma(\varepsilon^{2a-1/2} + \varepsilon^{b+2a-2}),$$

from which we have

$$\frac{w_*}{w}\Phi_2 = \frac{w_0}{2w\lambda}\Phi_1^2 + \varepsilon^{a-1}\frac{w_0}{w\lambda}\Phi_0\Phi_2 + \frac{1}{2}F''(y_2^0) + \varepsilon^{a-1}\frac{1}{2}H''_a(0, y_2^0, x)$$
$$+ O_\sigma(\varepsilon^{a-1/2} + \varepsilon^{1/2} + \varepsilon^{b-1}),$$

and

$$\left(1 - \varepsilon^{a-1}\frac{1}{\lambda}\Phi_0\right)\Phi_2 \;=\; \frac{1}{2\lambda}\Phi_1^2 + \frac{1}{2}F''(y_2^0) + \varepsilon^{a-1}\frac{1}{2}H_a''(0, y_2^0, x) + O_\sigma(\varepsilon^{a-1/2} + \varepsilon^{1/2} + \varepsilon^{b-1}).$$

Since

$$\left(1 - \varepsilon^{a-1}\frac{1}{\lambda}\Phi_0\right)^{-1} = 1 + \varepsilon^{a-1}\frac{1}{\lambda}\Phi_0 + O_\sigma(\varepsilon^{2a-2}),$$

we have

$$\begin{aligned}\Phi_2 \;&=\; \frac{1}{2}F''(y_2^0) + \frac{1}{2}\varepsilon^{a-1}H_a''(0, y_2^0, x) + \frac{1}{2\lambda}\Phi_1^2 + \frac{\varepsilon^{a-1}\Phi_0}{2\lambda}\left(F''(y_2^0) + \frac{1}{\lambda}\Phi_1^2\right)\\ &\quad + O_\sigma(\varepsilon^{2a-2} + \varepsilon^{a-1/2} + \varepsilon^{1/2} + \varepsilon^{b-1})\\ &=\; \frac{1}{2}F''(y_2^0) + \frac{1}{2}\varepsilon^{a-1}H_a''(0, y_2^0, x) + \frac{1}{2\lambda}(F'(y_2^0))^2\\ &\quad + \frac{\varepsilon^{a-1}\Phi_0}{2\lambda}\left(F''(y_2^0) + \frac{3}{\lambda}(F'(y_2^0))^2\right) + \varepsilon^{a-1}\frac{1}{\lambda}F'(y_2^0)H_a'(0, y_2^0, x)\\ &\quad + O_\sigma(\varepsilon^{2a-2} + \varepsilon^{a-1/2} + \varepsilon^{1/2} + \varepsilon^{b-1})\end{aligned}$$

and (3.17) follows from (5.5).

Finally at third degree of $\widetilde{y}_2$ we have that

$$\begin{aligned}&\varepsilon^{3a-2}\left(-\frac{w_*}{w}\Phi_3 + \frac{w_0}{w\lambda}\Phi_1\Phi_2\right) + O_\sigma(\varepsilon^{4a-3} + \varepsilon^{3a-3/2})\\ &\quad = \varepsilon^{3a-2}\left(-\Phi_3 + \frac{1}{\lambda}\Phi_1\Phi_2\right) + O_\sigma(\varepsilon^{4a-3} + \varepsilon^{3a-3/2})\end{aligned}$$

is also equal to

$$-\varepsilon^{3a-2}\frac{1}{6}H_{01}'''(y_1^0, y_2^0) + O_\sigma(\varepsilon^{4a-3}) = -\varepsilon^{3a-2}\frac{1}{6}F'''(y_2^0) + O_\sigma(\varepsilon^{4a-3} + \varepsilon^{3a-3/2}),$$

so that

$$\begin{aligned}\Phi_3 \;&=\; \frac{1}{6}F''' + \frac{1}{\lambda}\Phi_1\Phi_2 + O_\sigma(\varepsilon^{a-1} + \varepsilon^{1/2})\\ &=\; \frac{1}{6}F''' + \frac{1}{\lambda}F'\Phi_2 + O_\sigma(\varepsilon^{a-1} + \varepsilon^{1/2})\\ &=\; \frac{1}{6}F''' + \frac{1}{2\lambda^2}F'\widehat{\Phi} + O_\sigma(\varepsilon^{a-1} + \varepsilon^{1/2}),\end{aligned}$$

where, in the last equation, we have used the fact that $\Phi_2 = \widehat{\Phi}/(2\lambda) + O_\sigma(\varepsilon^{a-1} + \varepsilon^{1/2})$. □

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