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Analisi funzionale. — *Nekhoroshev stability for the D'Alembert problem of Celestial Mechanics.* Nota (*) di Luca Biasco e Luigi Chierchia, presentata dal Socio A. Ambrosetti.

Abstract. — The classical D'Alembert Hamiltonian model for a rotational oblate planet revolving near a «day-year» resonance around a fixed star on a Keplerian ellipse is considered. Notwithstanding the strong degeneracies of the model, stability results *a là* Nekhoroshev (*i.e.* for times which are exponentially long in the perturbative parameters) for the angular momentum of the planet hold.

Key words: D'Alembert model; Nekhoroshev estimates; Spin-orbit resonances; Exponential stability.

Riassunto. — *Stabilità di Nekhoroshev per il problema di D'Alembert della meccanica celeste.* Si considera il classico modello hamiltoniano di D'Alembert per un pianeta ruotante e schiacciato ai poli orbitante, vicino ad una risonanza «giorno-anno», attorno a d una stella fissa su un'ellisse kepleriana. Nonostante la forte degenerazione del modello, si provano risultati di stabilità alla Nekhoroshev (cioè per tempi che sono esponenzialmente lunghi nei parametri perturbativi) per il momento angolare del pianeta.

Local (perturbative) methods in the modern theory of Hamiltonian dynamics rely, mainly, on the so-called Kolmogorov-Arnold-Moser (KAM) and Nekhoroshev theory (see [1] for generalities). In order for the techniques beyond such theories to apply, certain non-degeneracy conditions are needed. For example, to establish the existence of a large (in the sense of measure theory) quantity of maximal invariant tori (equivalently, «maximal quasi-periodic solutions») for a nearly-integrable (smooth or analytic) system with hamiltonian $h(I) + \varepsilon f(I, \varphi)$, (I, φ) being standard action-angle symplectic variables and $0 < \varepsilon \ll 1$ a small parameter, one assumes, typically, that h has invertible Hessian on its domain of definition («KAM non-degeneracy»). On the other hand, to establish stability of the perturbed integrals (*i.e.*, of the action-variables) I, on the *whole* phase space for exponentially long times, one assumes that h is a «steep» (or, more restrictively, convex) function (Nekhoroshev).

It has to be noted that the major impulse to the modern theory of conservative Dynamical Systems is certainly due to Poincaré (followed by Birkhoff, Kolmogorov, Siegel, Arnold, Moser, Herman, ...) and his main motivation came from Celestial Mechanics, [6]. Now, typical examples in Celestial Mechanics, such as many-body problems or the D'Alembert planetary model, violate drastically the above mentioned non-degeneracy assumptions. In fact, such models are, typically, *properly-degenerate*, *i.e.*, h does not depend on the whole set of action variables: For this reason the stability conclusions in KAM and Nekhoroshev theory, in such cases, are far from being obvious and a more detailed and delicate analysis is needed.

In this announcement, we consider the D'Alembert planetary model near a day-year (equivalently «spin-orbit») resonance and present Nekhoroshev stability results. Recently,

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KAM type of results (for properly-degenerate Hamiltonian systems with two degrees of freedom) have been established in [2] and [4].

The D'Alembert planetary model is a Hamiltonian model for a rotational planet with polar radius slightly smaller than the equatorial radius, whose center of mass revolves periodically on a given Keplerian ellipse of small eccentricity around a fixed star occupying one of the foci of the ellipse; the planet is subject to the gravitational attraction of the star. This is a Hamiltonian system with two degrees of freedom depending periodically on time (the period being the «year» of the planet). Many planets and satellites of the Solar System are observed in a nearly exact spin-orbit resonance, *i.e.*, the ratio between the period of revolution around the major body and the period of rotation around the spin axis of the planet is (nearly) rational. It is therefore of particular interest to investigate the stability in regions of space surrounding such resonances.

In formulae, the above system is governed by a real-analytic Hamiltonian of the form (see [5, 3])

$$(1)\quad H_{\varepsilon,\mu} \equiv \frac{I_1^2}{2} + \omega(pI_1 - qI_2 + qI_3) + \varepsilon F_0(I_1, I_2, \varphi_1, \varphi_2) + \varepsilon\mu F_1(I_1, I_2, \varphi_1, \varphi_2, \varphi_3; \mu),$$

where: $(I, \varphi) \in A \times \mathbb{T}^3$ are standard symplectic coordinates; the domain $A \subset \mathbb{R}^3$ is given by

$$A \equiv \left\{ |I_1| < \varepsilon^{\ell} \;,\quad |I_2 - \bar{I}_2| < \text{const} \;,\quad I_3 \in \mathbb{R} \right\} \;,$$

with $0 < \ell < 1/2$; $\bar{I}_2$ is a fixed «reference datum» (avoiding certain singularities); ε and μ are two small parameters (measuring, respectively, the oblateness of the planet and the eccentricity of the Keplerian ellipse); p and q are two positive co-prime integers, which identify the spin-orbit resonance (the planet, in the unperturbed regime, revolves q times around the star and p times around its spin axis); ωq is the frequency of the Keplerian motion; the action I_1 measures the displacement from the exact resonance: in these units, $I_1 = 0$ corresponds exactly to a $p:q$ spin-orbit resonance. In fact, $p\omega + I_1$ and I_2 are (in suitable physical units), respectively, the absolute value and the projection onto the polar axis of the planet of the angular momentum of the planet, while I_3 is an artificially introduced variable canonically conjugated to time. The functions F_i are real-analytic funtions in all their arguments, computable via Legendre expansions in the eccentricity μ from the Lagrangian expression of the gravitational (Newtonian) potential (see [5] for explicit computations).

The following stability result holds:

THEOREM 1. *Let* $c > 0$, $0 < \ell < 1/2$ *and* $\kappa < \min\{c, \ell\}$. *Then there exist positive constants* ε_0, κ_i *such that, if* $0 < \varepsilon < \varepsilon_0$ *and* $0 < \mu < \varepsilon^c$, *then*

$$(2)\qquad |I(t) - I(0)| < \varepsilon^{\kappa_1} \;,\qquad \forall\ |t| < \exp(\kappa_2/\varepsilon^{\kappa}) \;,$$

where $(I(t), \varphi(t))$ *denotes the* $H_{\varepsilon,\mu}$*-evolution of an initial datum* $(I(0), \varphi(0)) \in A \times \mathbb{T}^3$.

REMARKS. (*i*) The most important constant in the theorem is κ, the so-called Nekhoroshev exponent. We recall that, for non-degenerate systems with d degrees of freedom, $1/(2d)$ is considered to be the optimal Nekhoroshev exponent. From Theorem 1 it follows that, in the case $\ell \sim 1/2$, one can take $\kappa \sim 1/2$ provided $c \geq \ell$. The reason for which, in such a case, one finds a *better Nekhoroshev exponent* is mainly related to the appearance of *three well separated «times scales»*: a time scale of order 1 (related to the frequency ω), a time scale of order ε^{ℓ} (due to the form of the I_1-action domain), and a time scale of order ε (related to the proper-degeneracy); see, also, next remark.

(*ii*) The relevant stability information concerns the perturbed integral I_2: I_1 is stable by definition of A (as long as trajectories do not leave the domain of definition, a fact that needs to be proved) and I_3 is, in any case, not physically interesting.

We shall now describe briefly the arguments on which is based the proof of the Theorem. For simplicity, we shall discuss the cases for which $(p\,,q) \neq (1\,,1)$, $(2\,,1)$; the (more complicate) proof in the case $q=1$ and $p=1$ or 2 is briefly commented at the end of this *Note*.

Sketch of proof.

STEP 1. Let ϕ_0 be the following linear symplectic map:

$$\phi_0(I'\,,\varphi') \equiv \left(\left(I_1'\,,I_2'\,,-\frac{p}{q}I_1' + I_2' + \frac{1}{q}I_3'\right)\,,(\varphi_1' + p\varphi_3'\,,\varphi_2' - q\varphi_3'\,,q\varphi_3')\right) \quad . \tag{3}$$

Then, ϕ_0 casts the Hamiltonian $H_{\varepsilon\,,\mu}$ into the form

$$\begin{aligned} H^{(0)}(I'\,,\varphi';\varepsilon\,,\mu) &\equiv H_{\varepsilon\,,\mu} \circ \phi_0(I'\,,\varphi') \equiv \\ &\equiv \frac{I_1'^{2}}{2} + \omega I_3' + \varepsilon G_0(I_1'\,,I_2'\,,\varphi_1'\,,\varphi_2'\,,\varphi_3') + \varepsilon\mu G_1(I_1'\,,I_2'\,,\varphi_1'\,,\varphi_2'\,,\varphi_3';\mu)\,, \end{aligned} \tag{4}$$

with G_i real-analytic on $A \times \mathbb{T}^3$, and, exploiting the particular form of F_0 (see, *e.g.*, [5] or [3]) *and using the hypotheses that* $(p\,,q) \neq (1\,,1)\,,(2\,,1)$, one can verify that

$$\int_0^{2\pi} G_0(I_1'\,,I_2'\,,\varphi_1'\,,\varphi_2'\,,\varphi_3')\frac{d\varphi_3'}{2\pi} = f(I_1'\,,I_2') \quad , \tag{5}$$

for a suitable real-analytic function f. Obviously, since ϕ_0 depends upon p and q, also the functions G_i anf f depend upon p and q, but we shall not indicate such dependence in the notation; (in the case $(p\,,q) = (1\,,1)$ or $(2\,,1)$ the function f depends also upon φ_1' making the subsequent analysis more complicate).

We remark that, in general, ϕ_0 is *not* a diffeomorphism of $\mathbb{R}^3 \times \mathbb{T}^3$ (since the induced map on $\mathbb{T}^3$ has determinant equal to q); this fact, however, does not affect the following analysis.

REMARK. Roughly speaking, in the $H^{(0)}$-Hamiltonian flow, the angles φ_i' evolve on different time scales, since $\dot{\varphi}_3' = O(1)$, while $\dot{\varphi}_1' = O(\varepsilon^{\ell})$ and $\dot{\varphi}_2' = O(\varepsilon)$. This intuitive observation can be made rigorous by means of analytic tools borrowed from *normal form theory*, which allow to «average out», up to exponentially small terms, the dependence on the angles. This idea is most conveniently implemented by *subsequent* constructions of normal forms and will be described in the next two steps.

STEP 2. For ε and μ as in Theorem 1, one can find a real-analytic symplectic map, ϕ_1, ε-close to the identity in the action-variables, from a slightly smaller domain $A' \times \mathbb{T}^3$ into $A \times \mathbb{T}^3$, such that

$$
\begin{aligned}
&H^{(1)}(\widehat{I},\widehat{\varphi}\,;\varepsilon,\mu) \equiv H^{(0)} \circ \phi_1(\widehat{I},\widehat{\varphi}) \equiv \\
&\equiv \frac{\widehat{I}_1^2}{2} + \omega \widehat{I}_3 + \varepsilon \overline{f}(\widehat{I}_2) + \varepsilon^a H_1^{(1)}(\widehat{I}_1,\widehat{I}_2,\widehat{\varphi}_1,\widehat{\varphi}_2;\varepsilon,\mu) + O\Big(\exp(-\mathrm{const}/\varepsilon^{\ell})\Big),
\end{aligned}
\tag{6}
$$

where $\overline{f}(\widehat{I}_2) \equiv f(0\,,\widehat{I}_2)$,

$$
a \equiv 1 + \min\{c\,,\ell\} < \frac{3}{2}
$$

and $O(\alpha(\varepsilon))$ denotes, here, a real-analytic function whose (analytic) norm can be bounded by $\alpha(\varepsilon)$, for ε small enough. Thus, *up to an exponentially small term, the flow of the D'Alembert Hamiltonian is equivalent to the two-degrees-of-freedom, properly-degenerate system*

$$
\widehat{H}^{(1)}(\widehat{I}_1\,,\widehat{I}_2\,,\widehat{\varphi}_1\,,\widehat{\varphi}_2;\varepsilon\,,\mu) \equiv \frac{\widehat{I}_1^2}{2} + \varepsilon \overline{f}(\widehat{I}_2) + \varepsilon^a H_1^{(1)}(\widehat{I}_1\,,\widehat{I}_2\,,\widehat{\varphi}_1\,,\widehat{\varphi}_2;\varepsilon\,,\mu)\ ,
\tag{7}
$$

(having dropped the dumb – at this point – parameter $\omega \widehat{I}_3$). This step is described in full details in [3] (see also Appendix B of [2]). From now on, for the purpose of the thesis of Theorem 1, we may consider only the Hamiltonian $\widehat{H}^{(1)}$ in (7).

STEP 3. Also the $\widehat{H}^{(1)}$-flow yields different time scales in the angle evolutions. This allows to exploit normal form theory, in a subregion $\widehat{A} \times \mathbb{T}^2$ of the form

$$
\widehat{A} \equiv \{\varepsilon^{\lambda/2} < |\widehat{I}_1| < \mathrm{const}\ \varepsilon^{\ell}\ \ ,\quad |\widehat{I}_2 - \bar{I}_2| < \mathrm{const}\}\ .
\tag{8}
$$

In fact, let ε, μ ad κ be as in Theorem 1 and let

$$
1 < \lambda < 1 + \Big(\min\{c\,,\ell\} - \kappa\Big)\ .
\tag{9}
$$

Then $\ell < \lambda/2$ (so that the $\widehat{A} \neq \emptyset$) and $\lambda < a$ so that (by normal form theory) one can find a real-analytic symplectic map, ϕ_2, $\varepsilon^{a-\lambda/2}$-close to the identity in the action-variables, from a slightly smaller domain $\widehat{A}' \times \mathbb{T}^2$ into $\widehat{A} \times \mathbb{T}^2$, such that

$$
\begin{aligned}
&H^{(2)}(\widetilde{I}\,,\widetilde{\varphi};\varepsilon\,,\mu) \equiv \widehat{H}^{(1)} \circ \phi_2(\widetilde{I}\,,\widetilde{\varphi}) \equiv \\
&\equiv \frac{\widetilde{I}_1^2}{2} + \varepsilon \overline{f}(\widetilde{I}_2) + \varepsilon^a g(\widetilde{I}\,,\widetilde{\varphi}_2;\varepsilon\,,\mu) + O\Big(\exp(-\mathrm{const}/\varepsilon^{\kappa})\Big)\,,
\end{aligned}
\tag{10}
$$

where $(\widetilde{I}\,,\widetilde{\varphi}) \equiv (\widetilde{I}_1\,,\widetilde{I}_2\,,\widetilde{\varphi}_1\,,\widetilde{\varphi}_2)$ and g is a suitable real-analytic function.

STEP 4. From (10) one sees that, up to exponentially small terms, the Hamiltonian $H^{(2)}$ behaves as a *one dimensional system* (in the variables $\widetilde{I}_2$ and $\widetilde{\varphi}_2$) parameterized by $\widetilde{I}_1$. Therefore one can use (one-dimensional) energy-conservation arguments to check that, with the above choice of the parameters κ and λ, stability holds (up to exponentially long times) also for $\widetilde{I}_2$. From this one concludes exponential stability for the full system.

In the case $q = 1$ and $p = 1$ or 2, in place of (5) one finds

$$(11)\qquad \int_0^{2\pi} G_0(I_1', I_2', \varphi_1', \varphi_2', \varphi_3') \frac{d\varphi_3'}{2\pi} = f(I_1', I_2', \varphi_1') \equiv a_q(I_1', I_2') + b_q(I_1', I_2') \cos j_q \varphi_1' ,$$

where $j_1 \equiv 2$ and $j_2 \equiv 1$. This explicit dependence on φ_1' makes the subsequent analysis more complicate (notice that in a neighborhood of the separatrix of the «pendulum» $I_1'^2/2 + b_q(I_1', I_2') \cos j_q \varphi_1'$ will appear, under the effect of the perturbation, a chaotic zone). The strategy is, then, to construct action-angle variables for the «pendulum» $I_1'^2/2 + b_q(I_1', I_2') \cos j_q \varphi_1'$ in phase space region bounded away from the separatrices and from the stable equilibria by a quantity depending (suitably) on ε. Making use of detailed analytic properties of such action-angle variables (see, for example, Appendix B of [2]) one can show that a scheme analogous to that outlined in Steps $2 \div 4$ above can be carried out so as to obtain the thesis of Theorem 1 also in the present case.

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