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A DIRECT METHOD FOR CONSTRUCTING SOLUTIONS OF THE HAMILTON-JACOBI EQUATION

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SOMMARIO. Si presenta un nuovo metodo che permette di stabilire l'esistenza di soluzioni globali dell'equazione classica di Hamilton-Jacobi per Hamiltoniane $H(y, x)$, periodiche in $(x_1, \ldots, x_d)$.

SUMMARY. A new method that allows to establish the existence of global solutions of the classical Hamilton-Jacobi equation for a Hamiltonian $H(y, x)$, periodic in $(x_1, \ldots, x_d)$, is presented.

1. INTRODUCTION

The purpose of this paper is to describe a direct and constructive procedure that allows to «solve», under suitable assumptions, the classical equation of Hamilton-Jacobi for a Hamiltonian $H(y, x)$ periodic in $x \equiv (x_1, \ldots, x_d)$. The method presented here is particularly suited for computer-assisted implementation (for computer-assisted techniques applied to mechanics see [6]).

Let us consider the phase space $\mathcal{M} \equiv U \times \mathbb{T}^d$, U being an open bounded set of $\mathbb{R}^d$ and $\mathbb{T} \equiv \mathbb{R}/(2\pi\mathbb{Z})$, endowed with the standard symplectic form $\Omega \equiv \Sigma_{i=1}^d \, dy_i \wedge dx_i$, so that the Hamilton equations for H have the usual form

$$\dot{y} = -\partial_x H, \quad \dot{x} = \partial_y H,$$

where ∂ denotes gradient $[\partial_x \equiv (\partial/\partial x_1, \ldots, \partial/\partial x_d)$, etc.].

The Hamilton-Jacobi equation can be written in the form

$$H(\partial_x \phi(\eta, x), x) = h(\eta); \tag{1}$$

the unknowns being the «new» Hamiltonian h, which depends on half of the variables, and the generating function ϕ; (of course, for (1) to make sense one has to require that $\partial_x \phi(\eta, x) \in U$).

Here, «generating function» stands for a $C^\infty(\mathbb{R}^d \times \mathbb{T}^d)$ function satisfying

$$\det \partial^2_{(\eta, x)} \phi \equiv \det \left[\frac{\partial^2 \phi}{\partial \eta_i \partial x_j} \right]_{i,j=1,\ldots,d} \neq 0$$

and such that the map $(y, x) = \mathcal{C}_\phi(\eta, \xi)$ defined via the relations

$$y = \partial_x \phi, \quad \xi = \partial_\eta \phi$$

yields a diffeomorphism of $\mathbb{R}^d \times \mathbb{T}^d$ into itself. Such a diffeomorphism preserves the symplectic 2-form Ω and therefore it preserves also the form of the Hamilton equations (see [1, 13] for general informations). Thus, if there exists a solution satisfying (1) *for all* $x \in \mathbb{T}^d$, the transformed Hamilton equations become simply:

$$\dot{\eta} = -\partial_\xi h \equiv 0, \quad \dot{\xi} = \partial_\eta h,$$

and one sees that the Hamiltonian H admits *global quasi-periodic* solutions

$$(y(t), x(t)) \equiv \mathcal{C}_\phi(\eta, \xi + \omega t), \quad \omega \equiv \partial_\eta h(\eta)$$

running over d-dimensional *invariant tori* $\mathcal{T}_\eta \equiv \{(y, x) = \mathcal{C}_\phi(\eta, \xi), \xi \in \mathbb{T}^d\}$.

However, it is well known ([19] see also [3]) that for general Hamiltonians there *do not exist* any generating function satisfying (1) on some *open set* $U \times \mathbb{T}^d$ (the exceptions being the so-called «integrable systems»). Nevertheless, from KAM theory ([15, 2, 16]; see [4] for a review and [13, 14] for an elementary exposition) it follows that when H has the special form

$$H = H_0(y) + \epsilon H_1(y, x) \tag{2}$$

with

$$\det \partial_y^2 H_0 \equiv \det \left[\frac{\partial^2 H_0}{\partial y_i \partial y_j} \right]_{i,j=1,\ldots,d} \neq 0 \tag{3}$$

then, if ϵ is small enough, one can construct quasi-periodic solutions of the Hamilton equations with *prescribed* frequencies ω satisfying a Diophantine condition of the type

$$|\omega \cdot n| \equiv \left| \sum_{i=1}^d \omega_i n_i \right| \geqslant \frac{1}{\gamma |n|^\tau}$$

for some γ, $\tau > 0$ and any $n \in \mathbb{Z}^d \setminus \{0\}$ (such a vector ω will be called (γ, τ)-Diophantine).

Now, quasi-periodic solutions are closely related to «pointwise» solutions of (1). To be more precise, let us make the following

DEFINITION. *A triple* (Γ, ϕ, h) *will be called a solution (or* Γ*-solution) of (1) if* Γ *is a closed subset of* U, ϕ *a generating function and* h *a* $C^\infty(\mathbb{R}^d)$ *function such that (1) holds identically for* $(\eta, x) \in \Gamma \times \mathbb{T}^d$

In other words, the ξ-derivatives of the new Hamiltonian $H \circ \mathcal{C}_\phi$ vanish on Γ : For any $a \in \mathbb{N}^d \setminus \{0\}$

$$\partial_\xi^a H \circ \mathcal{C}_\phi \equiv \frac{\partial^{|a|}}{\partial \xi_1^{a_1} \ldots \partial \xi_d^{a_d}} H \circ \mathcal{C}_\phi = 0. \qquad (\eta \in \Gamma).$$

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If the Hamiltonian is of the type (2) - (3) and ϵ is small enough it is possible to show that there exist Γ-solutions of (1), Γ being a Cantor set of positive Lebesgue measure ([10, 18]). However the method of [10] and [18] (as well as the original methods of [15, 2, 16]) involves infinitely many changes of variables and the existence of ϕ can be established only in a rather involved way.

Instead, using a suitable *Newton method*, close in spirit to that recently introduced by Moser in [17] (see also [22] and [6]), it is possible (under appropriate assumptions) to construct *directly* solutions of (1). Roughly speaking, by «directly» I mean that, starting from an «approximate solution» of (1), one construct a series (whose first term is the «approximate solution») converging to a true solution in the sense of the above definition.

The smoothness of the solutions constructed here will be proved in a completely elementary way by means of well known tools from real analysis without any use of generalized notions of differentiability (as the notion of «Whitney differentiability», which is used in [10] and [18]).

As of applications, I believe that it is possible to apply the theorem presented below to *numerically constructed* approximate solutions in order to ***rigorously*** establish the existence of actual solutions of (1) close to the numerical approximations. Of course, in such a case, a rigorous control, over the approximations by means of the so-called «interval-arithmetic» would be necessary (see [6] and references therein for more informations on this topic) For example it should be possible to use the Newton scheme, on which the theorem is based, in conjunction with computer assisted estimations (cfr. [6]) to give *effective* stability bounds for the restricted three body problem of celestial mechanics. It would also be interesting to try to apply the theorem below to the work in [7], where the Hamilton-Jacobi equation is numerically solved for several non-integrable systems.

The rest of the paper is organized as follows: In the next section the proper assumptions are made precise and the Theorem is stated. (The formulation of the Theorem is divided in three parts: the first part contains the general statement, the second is a detailed quantitative version of such a statement and in the third part the solution is described via recursive formulae). In Section 3 the method of proof is quickly discussed and in Section 4 details are worked out.

2. ASSUMPTIONS AND THEOREM

Let us now proceed to make the proper assumptions and give a precise formulation of the main result in the *real-analytic* setting.

Let $U_0 \times T_0 \equiv \mathscr{M}_0$ be an open complex neighborhood of $\mathscr{M}$. Assume $H \in C^\infty(\mathbb{R}^d \times \mathbb{T}^d)$ and real-analytic and bounded on $\mathscr{M}_0$ (actually, under suitable growth assumptions on H, non-compact domains can also be treated; see [11]).

Let ϕ be a generating function such that $x_\phi(\eta, \xi)$ [$\equiv$ inverse of $x \to \partial_\eta \phi(\eta, x)$] is real-analytic on

$$Y_\rho(\Gamma) \times \Xi_s \equiv \bigcup_{\eta \in \Gamma} \{\eta' \in \mathbb{C}^d : \|\eta' - \eta\| \leqslant \rho\} \times \{\xi \in \mathbb{C}^d : |\operatorname{Im} \xi_i| \leqslant s\}$$

for some $\rho, s > 0$; furthermore assume that, if

$$X_s \equiv X_s(\Gamma, \phi, \rho) \equiv x_\phi(Y_\rho, \Xi_s), \quad \mathscr{D}_{\Gamma,\rho,s} \equiv Y_\rho(\Gamma) \times X_s,$$

then $\partial_x \phi(\mathscr{D}_{\Gamma,\rho,s}) \subset U_0$ with

$$\delta \equiv \operatorname{dist.}(\partial U_0, \partial_x \phi(\mathscr{D}_{\Gamma,\rho,s})) > 0 \tag{4}$$

«∂U_0» denoting, here, «boundary of U_0».

Let h be a $C^\infty(\mathbb{R}^d)$ function, real-analytic on $Y_\rho(\Gamma)$, such that $\partial_\eta h(\Gamma)$ is a set of uniformly (γ, τ)-Diophantine vectors and let e be the *error function* defined by

$$e(\eta, x) = H(\partial_x \phi, x) - h(\eta). \tag{5}$$

Finally assume that

$$\|(\partial^2_{(\eta,x)}\phi)^{-1}\|_{\Gamma,\rho,s} \equiv \sup_{\mathscr{D}\Gamma,\rho,s} \|(\partial^2_{(\eta,x)}\phi)^{-1}\| < \infty$$

and that

$$\|(\partial^2_\eta h)^{-1}\|_{\Gamma,\rho} \equiv \sup_{Y_\rho(\Gamma)} \|(\partial^2_\eta h)^{-1}\| < \infty.$$

Remark (On the choice of norms): Here we will not compute explicitely the constants appearing in our set of estimates. Therefore we shall not fix explicitely the norms in finite dimensional spaces (of complex or integer vectors, matrices, etc.) since all such norms are, for our purpouses, equivalent. On the other hand, if one is interested in the concrete (and effective) implementation of the method presented here, a careful choice of all norms and parameters is important: see [6].

Remark The quadruple (Γ, ϕ, h, e) should be thought of as an *approximate solution*, the smaller e is, the better is the approximation. For example, in the case (2) - (3) one can take $\phi \equiv \eta \cdot x$, as h any C^∞ extension of H_0 and as Γ the pre-image under the map $\eta \to \omega \equiv \partial_\eta h$ of a set of vectors (γ, τ)-Diophantine (because of (3) $\partial_\eta h$ is invertible on U so that with a suitable choice of the Diophantine constants, $\Gamma \neq \emptyset$). In this case the error function is simply $e(\eta, z) = \epsilon H_1(\eta, x) = \mathscr{O}(\epsilon)$.

THEOREM

Part I: Under the above assumptions, if $\|e\|_{\Gamma,\rho,s}$ *is small enough, then there exist a closed set* $\Gamma_* \subset U$, *a generating function* ϕ_* *and a* C^∞ *function* h_* *such that*

(i) Γ_* *has the same cardinality as* Γ *and*

$$(1 - C)\,\mu(\Gamma) \leqslant \mu(\Gamma_*) \leqslant (1 + C)\,\mu(\Gamma)$$

with a suitable constant $0 < C < 1$;

(ii) $\partial_\eta h_*(\Gamma_*) = \partial_\eta h(\Gamma)$;

(iii) $(\Gamma_*, \phi_*\ h_*)$ *is a* Γ_**-solution of (1).*

Part II: The statements in Part I can be given the following quantitative formulation. Let

$$\epsilon \equiv \gamma\rho^{-1}\|e\|_{\Gamma,\rho,s}, \quad C_1 \equiv \gamma\|\partial_\eta h\|_{\Gamma,\rho},$$

$$C_2 \equiv \|(\partial^2_\eta h)^{-1}\|_{\Gamma,\rho}\|\partial_\eta h\|_{\Gamma,\rho}\,\rho^{-1},$$

$$C_3 \equiv 1 + \gamma\rho \sup_{\mathcal{M}_0} \| \partial_y^2 H \| \| \partial^2_{(\eta,x)}\phi \|_{\Gamma,\rho,s},$$

$$C_4 \equiv \| \partial^2_{(\eta,x)}\phi \|_{\Gamma,\rho,s} \max\{ \| (\partial^2_{(\eta,x)}\phi)^{-1} \|_{\Gamma,\rho,s}, \| (\partial^2_{(\eta,x)}\phi)^{-1} \|_\infty \},$$

[$\| \cdot \|_\infty \equiv \sup_{\mathbb{R}^d \times \mathbb{T}^d} \| \cdot \|$] *and assume (for simplicity) that* $s \leq 1$ *and* $\rho \leq \delta$. *Then there exists a constant* K_*, *depending only on d and* τ, *such that if*

$$2K_* s^{-4(\tau+1)} C_1^2 C_2 C_3 C_4\, \epsilon (\log \epsilon^{-1})^{2(\tau+1)} \leq 1 \tag{6}$$

then (i, ii, iii) above follow with

$$C \equiv K s^{-3(\tau+1)} C_1 C_2 C_3\, \epsilon (\log \epsilon^{-1})^{2(\tau+1)}$$

where $K \equiv K(d, \tau) \leq K_*$. *Furthermore, the following estimates hold*

$$\max\{ \sup_{\eta_* \in \Gamma_*} \inf_{\eta \in \Gamma} \| \eta_* - \eta \|, \sup_{\eta \in \Gamma} \inf_{\eta_* \in \Gamma_*} \| \eta - \eta_* \| \} \leq$$

$$\leq K s^{-2(\tau+1)} C_1 C_2\, \epsilon (\log \epsilon^{-1})^{\tau+1}, \tag{7}$$

$$\sup_{\Gamma_*} \| (\partial_\eta^2 h_*)^{-1} \| \leq 2 \| (\partial_\eta^2 h)^{-1} \|_{\Gamma,\rho}, \tag{8}$$

$$\| (\partial^2_{(\eta,x)} \phi_*)^{-1} \|_\infty \leq 2 \| (\partial^2_{(\eta,x)}\phi)^{-1} \|_\infty$$

and $\forall\, a, b \in \mathbf{N}^d$

$$\| \partial_\eta^a \partial_x^b (\phi_* - \phi) \|_\infty \leq \theta_{a,b}(\epsilon), \sup_{\Gamma_*} \| \partial_\eta^a (h_* - h) \| \leq \theta_a(\epsilon) \tag{9}$$

where the θ*'s are smooth functions (depending on the above constants* C_i*) such that* $\lim_{\epsilon \downarrow 0} \theta(\epsilon)/\epsilon^\alpha = 0$ *for any* $0 \leq \alpha < 1$.

Part III. The Hamilton-Jacobi solution (Γ_*, ϕ_*, h_*) *is given by the following recursive formulae. Let* $\phi_0 \equiv \phi$, $h_0 \equiv h$, $e_0 \equiv e$ *and, for* $j \geq 0$, *let*

$$\phi_{j+1} \equiv \phi_j + \psi_j,\ h_{j+1} \equiv h_j + g_j,\ e_{j+1} \equiv H(\partial_x \phi_{j+1}, x) - h_{j+1}$$

where

$$\psi_j(\eta, x) = - \sum_{\substack{n \in \mathbb{Z}^d \\ 0 < |n| \leq N_j}} \frac{\tilde{e}_{j,n}(\eta)}{i \partial_\eta h_j(\eta) \cdot n} \exp[i\, \partial_\eta \phi_j(\eta, x) \cdot n]$$

with

$$N_j \equiv 4^j \nu, \quad \nu \equiv \frac{2^5}{s} \log \epsilon^{-1},$$

$$\tilde{e}_{j,n}(\eta) \equiv \int_{\mathbb{T}^d} e_j(\eta, x) \exp[-i\, \partial_\eta \phi_j(\eta, x) \cdot n] \det(\partial^2_{(\eta,x)} \phi_j) \frac{dx}{(2\pi)^d}$$

and

$$g_j(\eta) \equiv \tilde{e}_{j,0}(\eta).$$

Then Γ_*, ϕ_* *and* h_* *are given by*

$$\phi_* \equiv \phi_0 + \sum_{j=0}^\infty \psi_j,\ h_* \equiv h_0 + \sum_{j=0}^\infty g_j$$

and

$$\Gamma_* \equiv (\partial_\eta h_*)^{-1} \circ \partial_\eta h(\Gamma).$$

3. IDEA OF THE PROOF

This theorem is based on an iteration of a suitable *Newton algorithm*. Before giving the details of the proof, I will quickly discuss the main points of such an algorithm.

Given an approximate solution (Γ, ϕ, h, e) of (1) [i.e. given e by (5) with η varying in a suitable open neighborhood of Γ], one wants to construct a «better» approximation. As in the classical Newton scheme, the new approximation will be required to reduce quadratically the magnitude of the error. More precisely, we will require that the new approximate solution (Γ', ϕ', h', e') satisfies, for $\eta \in \Gamma'$,

$$\phi' - \phi \equiv \psi = \mathcal{O}(e)$$

$$e' \equiv H(\partial_x \phi', x) - h' = \mathcal{O}(e^2)$$

$$\partial_\eta h'(\Gamma') = \partial_\eta h(\Gamma).$$

To find ϕ' (and h') expand $H(\phi'_x, x)$ near ϕ_x (here and in the following we shall use notations like ϕ_x and $\partial_x \phi$ interchangeably):

$$H(\phi'_x, x) \equiv H(\phi_x + \psi_x, x) = H(\phi_x, x) + H_y(\phi_x, x) \cdot \psi_x + \mathcal{O}(\psi^2)$$

$$= e + h + H_y \cdot \psi_x + \mathcal{O}(\psi^2),$$

the last identity being a consequence of (5). Recalling that we are making the ansatz that $\psi = \mathcal{O}(e)$, we need to solve [up to order $\mathcal{O}(e^2)$] the equation

$$e + H_y(\phi_x, x) \cdot \psi_x = g(\eta) \tag{10}$$

for some function g to be determined.

Taking the η-gradient of (5) one obtains

$$H_y(\phi_x, x) = [(\partial^2_{(\eta,x)} \phi)^T]^{-1} (h_\eta + e_\eta)$$

where the superscript T denotes matrix-transposition.

Therefore, since

$$[(\partial^2_{(\eta,x)}\phi)^T]^{-1} e_\eta \cdot \psi_x = \mathcal{O}(e^2)$$

one sees that, up to order $\mathcal{O}(e^2)$, (10) can be written as

$$e + h_\eta \cdot (\partial^2_{(\eta,x)}\phi)^{-1} \psi_x = g(\eta). \tag{11}$$

Introducing the *change of variables* on $\mathbb{T}^d$ (parametrized by η) $x \to \xi = \phi_\eta(\eta, x)$ and denoting by $\xi \to x_\phi(\eta, \xi)$ its inverse, one checks easily that

$$(\partial^2_{(\eta,x)}\phi)^{-1} \psi_x = \partial_\xi \psi(\eta, x_\phi(\eta, \xi)). \tag{12}$$

Equation (11) can be, now, easily solved for $\eta \in \Gamma$: Set

$$\tilde{e}_n(\eta) = \int_{\mathbb{T}^d} e(\eta, x) \exp[-in \cdot \phi_\eta(\eta, x)] | \partial^2_{(\eta,x)}\phi | \frac{dx}{(2\pi)^d}$$

$$= \int_{\mathbb{T}^d} e(\eta, x_\phi(\eta, \xi)) \exp(-in \cdot \xi) \frac{d\xi}{(2\pi)^d}$$

so that

$$e(\eta, x) = \Sigma\ \tilde{e}_n(\eta) \exp[in \cdot \phi_\eta(\eta, x)].$$

Then one finds immediately

$$\psi(\eta, x) = - \sum_{n \neq 0} \frac{\tilde{e}_n(\eta)}{ih_\eta(\eta) \cdot n} \exp[in \cdot \phi_\eta(\eta, x)] \tag{13}$$

and

$$g(\eta) = \tilde{e}_0(\eta).$$

Notice that ψ is well defined for $\eta \in \Gamma$ since $h_\eta(\Gamma)$ is (γ, τ)-Diophantine, however, in order to have ψ defined on *open sets*, we will follow Arnold's original idea [2] of introducing suitable truncations in Fourier space (cfr. the N_j's in the above Theorem). Once ψ is defined on a neighborhood of Γ, to find a global smooth extension will be elementary.

4. PROOF OF THE THEOREM

(For techniques similar to that used here see, besides the already cited [15, 2, 16, 10, 18, 13, 14] also [12, 20, 21, 23] and, for our purposes, especially [5, 6, 8, 9]).

(i) The iterative step

Let us start by making quantitative the construction of (Γ', ϕ', h', e') out of (Γ, ϕ, h, e). Notice that if $f(\eta, x)$ is real-analytic on $\mathscr{D}_{\Gamma,\rho,s}$ then $\tilde{f}(\eta, \xi) \equiv f(\eta, x_\phi(\eta, \xi))$ is real-analytic on $Y_\rho(\Gamma) \times \Xi_s$ and

$$\| f \|_{\Gamma,\rho,s} \equiv \sup_{(\eta,x) \in \mathscr{D}_{\Gamma,\rho,s}} \| f(\eta, x) \| = \sup_{(\eta,\xi) \in Y_\rho(\Gamma) \times \Xi_s} \| \tilde{f}(\eta, \xi) \| .$$

Fix now an «analyticity-loss» parameter $0 < \sigma < s/2$. As already remarked, to have ψ defined on open sets we need to introduce a truncation of the series expansion of e:

$$e \equiv e^{(T)} + e^{(R)}; \quad e^{(T)} \equiv \sum_{|n| \leqslant N} \tilde{e}_n(\eta) \exp(in \cdot \phi_\eta),$$

the parameter N being chosen so that $e^{(R)} = \mathcal{O}(\| e \|^2_{\Gamma,\rho,s})$. More precisely setting

$$N \equiv \frac{4}{\sigma} \log \epsilon^{-1}, \quad (\epsilon \equiv \gamma\rho^{-1} \| e \|_{\Gamma,\rho,s}),$$

because of the exponentially fast decay of the $\tilde{e}_n$'s [which are nothing else than the Fourier coefficients of $\tilde{e}(\eta, \cdot)$], one has

$$\| e^{(R)} \|_{\Gamma,\rho,s-\sigma} \leqslant \gamma\rho^{-1} \| e \|^2_{\Gamma,\rho,s}$$

provided

$$K_1 \sigma^d \epsilon \leqslant 1 \tag{14}$$

for a suitable constant K_1 depending only on d. Consequently, ψ is defined by replacing the sum over $\mathbb{Z}^d \setminus \{0\}$ in (13) by the sum over $\{n \in \mathbb{Z}^d : 0 < |n| \leqslant N\}$. Now, ψ solves (for $\eta \in \Gamma$)

$$e^{(T)} + h_\eta \cdot (\partial^2_{(\eta,x)}\phi)^{-1} \psi_x = \tilde{e}_0 .$$

Moreover, having a *finite number* of divisors to deal with, one can define ψ (and hence $\phi' = \phi + \psi$) on an open neighborhood of Γ. In fact, setting

$$\tilde{\rho} \equiv \frac{\rho}{4 \| \partial_\eta h \|_{\Gamma,\rho,s} \gamma N^{\tau+1}}$$

one can easily check that $\forall 0 < |n| \leqslant N$ and $\forall \eta \in Y_{\tilde{\rho}}(\Gamma)$

$$| h_\eta(\eta) \cdot n | \geqslant \frac{1}{2\gamma |n|^\tau} \tag{15}$$

whence there exists $K_2 = K_2(d, \tau)$ so that

$$\| \tilde{\psi} \|_{\Gamma,\tilde{\rho},s-\sigma} \equiv \| \psi \|_{\Gamma,\tilde{\rho},s-\sigma} \leqslant K_2 \frac{\gamma}{\sigma^\tau} \| e \|_{\Gamma,\rho,s}.$$

To estimate sums containing small divisors, here and in the following, we will make use of a result of Rüssmann (Theorem 1.1 of [20]).

The new error function e' is given by

$$e' \equiv e^{(R)} + [H(\phi_x + \psi_x, x) - H(\phi_x, x) - H_y(\phi_x, x) \cdot \psi_x)] + + e_\eta \cdot (\partial^2_{(\eta,x)}\phi)^{-1} \psi_x$$

and we want to estimate $\| e' \|$ (for η in a neighborhood of Γ) in terms of $\| e \|_{\Gamma,\rho,s}$. To do this we first show that if e is so small that

$$K_3 \| \partial^2_{(\eta,x)}\phi \|_{\Gamma,\rho,s} \frac{\gamma}{\sigma^{\tau+1}} \| e \|_{\Gamma,\rho,s} \delta^{-1} \leqslant 1 \tag{16}$$

then

$$\phi_x + \psi_x \cdot \mathscr{D}_{\Gamma,\tilde{\rho},s-\sigma} \to Y_\rho(\Gamma). \tag{17}$$

In fact, since

$$\text{dist.}(\partial X_s, \partial X_{s-\sigma}) \geqslant \| \partial^2_{(\eta,x)}\phi \|^{-1}_{\Gamma,\rho,s} \sigma, \tag{18}$$

using Cauchy estimates to bound the supremum of derivatives in terms of the supremum of functions on smaller domains (see, e.g., Lemma 2 of [6]) one has

$$\| \partial_x \psi \|_{\Gamma,\tilde{\rho},s-\sigma} \leqslant K_3 \| \partial^2_{(\eta,x)}\phi \|_{\Gamma,\rho,s} (\gamma/\sigma^{\tau+1}) \| e \|_{\Gamma,\rho,s}.$$

which, together with (16), implies (17). Now, using again Cauchy estimates, $\tilde{\rho} \leqslant \rho/2$ and recalling (12) one finds

$$\| e' \|_{\Gamma,\tilde{\rho},s-\sigma} \leqslant \gamma\rho^{-1} \| e \|^2_{\Gamma,\rho,s} \tag{19}$$

$$+ \frac{1}{2} \sup_{\mathscr{H}_0} \| \partial^2_y H \| \left(\| \partial^2_{(\eta,x)}\phi \|_{\Gamma,\rho,s} K_2 \frac{\gamma}{\sigma^{d+\tau+1}} \| e \|_{\Gamma,\rho,s} \right)^2$$

$$+ 2K_2 \frac{\gamma}{\sigma^{\tau+1}} \rho^{-1} \| e \|^2_{\Gamma,\rho,s}.$$

Let us pass now to define the new set Γ'. Assume that

$$C_0 \equiv K_4 \rho^{-2} \| e \|_{\Gamma,\rho,s} \| (\partial^2_\eta h)^{-1} \|_{\Gamma,\rho} < 1, \tag{20}$$

for a suitable constant $K_4 = K_4(d)$, then by Cauchy estimates,

$$\| (\partial^2_\eta h')^{-1} \|_{\Gamma,\rho/2} = \| [I + (\partial^2_\eta h)^{-1} \partial^2_\eta \tilde{e}_0]^{-1} (\partial^2_\eta h)^{-1} \|_{\Gamma,\rho/2} \leqslant$$

$$\leqslant 2 \| (\partial^2_\eta h)^{-1} \|_{\Gamma,\rho} . \tag{21}$$

Therefore, for any $\eta_0 \in \Gamma$, the map $\eta \to \partial_\eta h'(\eta)$ is invertible on $Y_{\rho/2}(\eta_0)$ [$\equiv Y_{\rho/2}(\{\eta_0\})$]. Moreover, (21) implies that $\partial_\eta h'(Y_{\rho/2}(\eta_0))$ contains a (closed) ball of radius $\rho(4\|(\partial^2_\eta h)^{-1}\|_{\Gamma,\rho})^{-1}$ so that, by (20), $\partial_\eta h(\eta_0) \in \partial_\eta h'(Y_{\rho/2}(\eta_0))$. Also, if $\eta'_0 = (\partial_\eta h')^{-1} \circ \partial_\eta h(\eta_0)$ one check easily

$$\|\eta'_0 - \eta_0\| \leqslant 2\|(\partial^2_\eta h)^{-1}\|_{\Gamma,\rho}\rho^{-1}\|e\|_{\Gamma,\rho,s}. \tag{22}$$

Thus, we can define

$$\Gamma' \equiv (\partial_\eta h')^{-1} \circ \partial_\eta h(\Gamma). \tag{23}$$

As for the measure of Γ' observe that, denoting by J' the Jacobian of $\eta \to (\partial_\eta h')^{-1} \circ \partial_\eta h(\eta)$, $(\eta \in \Gamma)$, one has

$$|\det J' - 1| = |\det[I + (\partial^2_\eta h)^{-1}\partial^2_\eta \tilde{e}_0] - 1| \leqslant \tag{24}$$

$$\leqslant K_4 \rho^{-2}\|e\|_{\Gamma,\rho,s}\|(\partial^2_\eta h)^{-1}\|_{\Gamma,\rho} \equiv C_0, \tag{25}$$

so that

$$(1 - C_0)\mu(\Gamma) \leqslant \mu(\Gamma') \leqslant (1 + C_0)\mu(\Gamma). \tag{26}$$

Now, if e is small enough, we can find a neighborhood of Γ' contained in $Y_\rho(\Gamma)$. More precisely if

$$2\|(\partial^2_\eta h)^{-1}\|_{\Gamma,\rho}\,\rho^{-1}\|e\|_{\Gamma,\rho,s} \leqslant \frac{\tilde{\rho}}{2},$$

i.e., if for a suitable constant K_5

$$K_5 \frac{1}{\sigma^{\tau+1}}\|h_\eta\|_{\Gamma,\rho}\,\rho^{-1}\|(\partial^2_\eta h)^{-1}\|_{\Gamma,\rho}\,\epsilon(\log \epsilon^{-1})^{\tau+1} \leqslant 1 \tag{27}$$

then

$$Y_{\tilde{\rho}/2}(\Gamma') \subset Y_{\tilde{\rho}}(\Gamma).$$

Next, let us discuss the invertibility of $\partial^2_{(\eta,x)}(\phi + \psi)$. Using Cauchy estimates, $\tilde{\rho} \leqslant \rho/2$ and (15), one finds

$$\|\partial^2_{(\eta,x)}\psi\|_{\Gamma,\tilde{\rho},s-\sigma} \leqslant K_6\|\partial^2_{(\eta,x)}\phi\|_{\Gamma,\rho,s}\frac{\gamma^2\|h_\eta\|_{\Gamma,\rho}}{\sigma^{\tau+2}}\rho^{-1}\|e\|_{\Gamma,\rho,s}.$$

Now, an argument similar to that used to derive (21) will show that, if

$$2K_7\|(\partial^2_{(\eta,x)}\phi)^{-1}\|_{\Gamma,\rho,s}\|\partial^2_{(\eta,x)}\phi\|_{\Gamma,\rho,s}\frac{\gamma^2\|h_\eta\|_{\Gamma,\rho}}{\sigma^{\tau+2}}\rho^{-1}\|e\|_{\Gamma,\rho,s} \leqslant 1 \tag{28}$$

then

$$\|(\partial^2_{(\eta,x)}\phi')^{-1}\|_{\Gamma,\tilde{\rho},s-\sigma} \leqslant 2\|(\partial^2_{(\eta,x)}\phi)^{-1}\|_{\Gamma,\rho,s}. \tag{29}$$

Now, observe that (28) implies $\|\psi_\eta\|_{\Gamma,\tilde{\rho},s-\sigma} \leqslant \sigma$; therefore, $\forall \eta \in Y_{\tilde{\rho}/2}(\Gamma') \subset Y_{\tilde{\rho}}(\Gamma)$,

$$\Xi_{s-2\sigma} \subset \phi'_\eta(\eta, x_\phi(\eta, \Xi_{s-\sigma})).$$

This means that

$$x_{\phi'}(Y_{\tilde{\rho}/2}(\Gamma') \times \Xi_{s-2\sigma}) \subset x_\phi(Y_{\tilde{\rho}} \times \Xi_{s-\sigma}).$$

Thus, if one sets

$$s' \equiv s - 2\sigma,$$

one sets that $\partial^2_{(\eta,x)}\phi'$ is invertible on

$$\mathscr{D}'_{\Gamma',\tilde{\rho}/2,s'} \equiv Y_{\tilde{\rho}/2}(\Gamma') \times X'_{s'},\ X'_{s'} \equiv x_{\phi'}(Y_{\tilde{\rho}/2}(\Gamma'), \Xi_{s'})$$

$x_{\phi'}(\eta, \cdot)$ being, of course, the inverse of $\phi'_\eta(\eta, \cdot)$. Moreover

$$\mathscr{D}'_{\Gamma',\tilde{\rho}/2,s'} \subset \mathscr{D}_{\Gamma,\tilde{\rho},s-\sigma}.$$

The last thing we need to discuss, in order to conclude the iterative step, is the smooth extension of ψ.

In the appendix we construct a function $\chi \in C(\mathbb{C}^d) \cap C^\infty(\mathbb{R}^d) : \mathbb{C}^d \to [0, 1]$ with support in $Y_{\tilde{\rho}/4}(\Gamma')$, identically equal to one in $Y_{\tilde{\rho}/8}(\Gamma')$ and such that, $\forall a \in \mathbb{N}^d$,

$$\sup_{\mathbb{R}^d}\|\partial^a_\eta \chi\| \leqslant \alpha\frac{(|a| + 2)!}{\tilde{\rho}^{|a|}}, \tag{30}$$

where α is a suitable universal constant.

We then define $\psi' \in C^\infty(\mathbb{R}^d)$ by setting $\psi' \equiv 0$ if $\eta \in \mathbb{C}^d \setminus Y_{\tilde{\rho}/4}(\Gamma')$ and $\psi' \equiv \chi\psi$ if $\eta \in Y_{\tilde{\rho}/4}(\Gamma')$. Thus, $\psi' \equiv \psi$ in $Y_{\tilde{\rho}/8}(\Gamma')$ and [using once more Cauchy estimates to estimate the derivatives of ψ in $Y_{\tilde{\rho}/4}(\Gamma')$]

$$\|\partial^a_\eta\partial^b_x\psi'\|_\infty \leqslant K_{a,b}\frac{1}{\tilde{\rho}^{|a|}}\left(\frac{\|\partial^2_{(\eta,x)}\phi\|_{\Gamma,\rho,s}}{\sigma}\right)^{|b|}K_2\frac{\gamma}{\sigma^\tau}\|e\|_{\Gamma,\rho,s}. \tag{31}$$

Therefore, $\phi' \equiv \phi + \psi'$, is a (global) canonical transformation if

$$2\|(\partial^2_{(\eta,x)}\phi)^{-1}\partial^2_{(\eta,x)}\psi'\|_\infty \leqslant 1,$$

which, for a suitable constant K_8, is implied by

$$K_8\frac{1}{\sigma^{2\tau+2}}\|\partial^2_{(\eta,x)}\phi\|_{\Gamma,\rho,s}\|(\partial^2_{(\eta,x)}\phi)^{-1}\|_\infty C_1\,\epsilon(\log \epsilon^{-1})^{\tau+1}. \tag{32}$$

To conclude that the iteration step we still need to define the new radius ρ', which in view of the above will be set equal to $\tilde{\rho}/8$.

Finally, one can easily check that all the above smallness conditions on e are fulfilled by requiring

$$K_9\frac{1}{\sigma^{2\tau+2}}C_1C_2C_4\,\epsilon(\log \epsilon^{-1})^{\tau+1} \leqslant 1. \tag{33}$$

(ii) The iteration

The proof goes on by iterating the above step, provided the condition corresponding to (33) is satisfied at each step of the iteration.

Let us label by j the quantities involved at the j^{th} step, i.e., denote by $\{h_j\}_{j\geqslant 0}, \{\phi_j\}_{j\geqslant 0}, \{\Gamma_j\}_{j\geqslant 0}, \ldots (h_0 \equiv h, h_1 \equiv h', \ldots)$, the sequences obtained by iteratively applying the above step. Fix the «analyticity-loss sequence» $\sigma_j \equiv s2^{-(j+3)}$ so that $s_j \equiv s - 2\Sigma_{i=0}^{j-1}\sigma_i \downarrow s/2$ (for a discussion of admissible analyticity-loss sequence, see [9]). Denote by ϵ_j (to be defined in a moment) suitable *upper bounds* on

$$\gamma\rho_j^{-1}\|e\|_j, \quad (\|\cdot\|_j \equiv \|\cdot\|_{\Gamma_j,\rho_j,s_j})$$

and notice that ρ_j will then be defined in terms of ϵ_{j-1}.

Let us proceed by induction. Define

$$N_j \equiv \frac{4}{\sigma_j}2^j\log \epsilon^{-1} = 4^j\frac{2^5}{s}\log \epsilon^{-1},$$

$$\epsilon_0 \equiv \epsilon,\ \epsilon_{j+1} \equiv ab^j\epsilon_j^2(\log \epsilon_j^{-1})^{\tau+1}\ (\forall j \geqslant 0)$$

where $a \equiv K_{10}s^{-2(\tau+1)}C_1C_3$, $b \equiv 2^{\tau+1}$. Assume that, for $0 \leqslant i \leqslant j$,

$(1)_i$ ϵ_i is an upper bound on $\gamma\rho_i^{-1}\|e_i\|_i$;

$(II)_i$ $(33)_i$ holds [$(33)_i$ is (33) with $\sigma, \phi, \Gamma, \ldots$ replaced by σ_i $\phi_i, \Gamma_i \ldots$]:

$(III)_i$ $\| (\partial^2_{(\eta,x)}\phi_i)^{-1} \|_\infty \leq 2 \| (\partial^2_{(\eta,x)}\phi)^{-1} \|_\infty$,

$\| (\partial^2_{(\eta,x)}\phi_i)^{-1} \|_i \leq 2 \| (\partial^2_{(\eta,x)}\phi)^{-1} \|_0$,

$\| \partial^2_{(\eta,x)}\phi_i \|_i \leq 2 \| \partial^2_{(\eta,x)}\phi \|_0$, $\| \partial^2_\eta h_i \|_i \leq 2 \| \partial^2_\eta h \|_0$,

$\| (\partial^2_\eta h_i)^{-1} \|_i \leq 2 \| (\partial^2_\eta h)^{-1} \|_0$.

We are going to see that (6) implies $(I, II, III)_{j+1}$.

Remarks

1) The case $i = 0$ is implied by step (i).

2) Since $(33)_j$ is assumed to hold the $(j+1)^{th}$ quantities are well defined and controlled according to (the analogous of) the estimates in (i).

3) Since for x small enough the functions $x(\log x^{-1})^{\tau+1}$, $x^2(\log x^{-1})^{\tau+1}$ are increasing, (33) implies the corresponding condition with ϵ_i replaced by $\gamma\rho_i \| e_i \|_i$.

4) From the definition of the sequence ϵ_i it follows easily that, for any i,

$$\epsilon^{2^i} \leq \epsilon_i \leq [\beta\epsilon(\log \epsilon^{-1})^{\tau+1}]^{2^i}, \qquad \beta \equiv ab\, 4^{\tau+1}. \tag{34}$$

Thus, assuming that K_* in (6) in such that

$\beta\epsilon(\log \epsilon^{-1})^{\tau+1} \leq 1/2$,

one sees that $\epsilon_i \downarrow 0$ faster than 2^{-2^j}.

From (19) and $(III)_i$, $i = 0, \ldots, j$ it follows easily $(I)_{j+1}$.

Now, I am going to prove in detail the first and the last of $(III)_{j+1}$ (the other inequalities being similar and simpler).

Since $\phi_{j+1} = \phi + \Sigma^j_{i=0} \psi_i$, one has

$$\| (\partial^2_{(\eta,x)}\phi_{j+1})^{-1} \|_\infty \leq$$

$$\leq \| (\partial^2_{(\eta,x)}\phi)^{-1} \|_\infty \left(1 - \| (\partial^2_{(\eta,x)}\phi)^{-1} \|_\infty \sum_{i=0}^{j} \| \partial^2_{(\eta,x)} \psi_i \|_\infty \right)^{-1}.$$

Using the assumptions $(I, III)_i$, (31) and the definition of σ_i, one finds

$$\| \partial^2_{(\eta,x)}\psi_i \|_\infty \leq K_{11} s^{-(\tau+1)} \| \partial^2_{(\eta,x)}\psi \|_0 2^{(\tau+1)i}\epsilon_i.$$

Therefore, by (34) and (6):

$$\| (\partial^2_{(\eta,x)}\phi)^{-1} \|_\infty \sum_{i=0}^{j} \| \partial^2_{(\eta,x)}\psi_i \|_\infty \leq$$

$$\leq K_{11} s^{-(\tau+1)} \| (\partial^2_{(\eta,x)}\phi)^{-1} \|_\infty \| \partial^2_{(\eta,x)}\psi \|_0$$

$$\sum_{i=0}^{\infty} [2^{(\tau+1)i/2^i} \beta\epsilon(\log \epsilon^{-1})^{\tau+1}]^{2^i}$$

$$\leq K_{12} s^{-(\tau+1)} \| (\partial^2_{(\eta,x)}\phi)^{-1} \|_\infty \| \partial^2_{(\eta,x)}\psi \|_0\, \beta\epsilon(\log \epsilon^{-1})^{\tau+1} \leq 1/2,$$

which proves the first of the $(III)_{j+1}$.

As for the last of the $(III)_{j+1}$:

$$\| (\partial^2_\eta h_{j+1})^{-1} \|_i \leq \| (\partial^2_\eta h)^{-1} \|_i \left(1 - \| (\partial^2_\eta h)^{-1} \|_0 \sum_{i=0}^{j} \| \partial^2_\eta \tilde{e}_{i,0} \|_i \right)^{-1}.$$

By Cauchy estimates, by $(III)_i$ and recalling that $\rho_{i+1} \leq \rho_i/2$ one obtains

$$\| (\partial^2_\eta h)^{-1} \|_0 \sum_{i=0}^{j} \| \partial^2_\eta \tilde{e}_{i,0} \|_i \leq \tag{35}$$

$$\leq 4 \| (\partial^2_\eta h)^{-1} \|_0 \sum_{i=0}^{j} \| e_i \|_i \rho_i^{-2}$$

$$= 4 \frac{\| (\partial^2_\eta h)^{-1} \|_0}{\gamma\rho} \sum_{i=0}^{j} \epsilon_i \frac{1}{\rho} \prod_{k=1}^{i} \frac{\rho_{k-1}}{\rho_k}.$$

Now, using $(III)_i$ and the definition of N_i, one obtains

$$\prod_{k=1}^{i} \frac{\rho_{k-1}}{\rho_k} = \prod_{k=1}^{i} 32 \| \partial_\eta h_i \|_i \gamma N_i^{\tau+1} \leq$$

$$\leq (K_{13} s^{-(\tau+1)} (\log \epsilon^{-1})^{\tau+1} \gamma \| \partial_\eta h \|_0)^i\, 2^{(\tau+1)i^2}.$$

Thus, the right-hand-side of (35) is bounded by

$$4 \frac{\| (\partial^2_\eta h)^{-1} \|_0}{\gamma\rho} \sum_{i=0}^{\infty} [(K_{13} s^{-(\tau+1)} (\log \epsilon^{-1})^{\tau+1} \gamma \| \partial_\eta h \|_0)^{i/2^i}$$

$$\times 2^{[(\tau+1)i^2]/2^i} \beta\epsilon (\log \epsilon^{-1})^{\tau+1}]^{2^i} \leq$$

$$\leq K_{14} s^{-5/2(\tau+1)} C_1^{3/2} C_2 C_3\, \epsilon(\log \epsilon^{-1})^{3/2(\tau+1)} \leq \frac{1}{2},$$

where the final bound holds because of (6).

Finally let us prove $(II)_{j+1}$. Using $(I, III)_{j+1}$ and (6): $K_9 \sigma_{j+1}^{-(2\tau+2)} \| \partial^2_{(\eta,x)}\phi_{j+1} \|_{j+1} \max\{\| (\partial^2_{(\eta,x)}\phi_{j+1})^{-1} \|_{j+1},$ $\| (\partial^2_{(\eta,x)}\phi_{j+1})^{-1} \|_\infty\} \times \| (\partial^2_\eta h_{j+1})^{-1} \|_{j+1} \| \partial_\eta h_{j+1} \|_{j+1} \rho_{j+1}^{-1}$ $\gamma \| \partial_\eta h_{j+1} \|_{j+1} \epsilon_{j+1} (\log \epsilon_{j+1}^{-1})^{\tau+1} \leq$

$$\leq K_{15} K_{16}^j C_1 C_2 C_4 (\log \epsilon^{-1})^{\tau+1} \prod_{i=0}^{j} \frac{\rho_i}{\rho_{i+1}} (\beta\epsilon(\log \epsilon^{-1})^{\tau+1})^{2^{j+1}}$$

$$\leq [K_{17} s^{-(2\tau+3)} C_1^2 C_2 C_3 C_4\, \epsilon(\log \epsilon^{-1})^{2(\tau+1)}] \leq \frac{1}{2}.$$

This finishes the proof of the induction. To prove the remaining statements in the Theorem is, at this point, straightforward: (i) of Part I follows from (26) and the inductive estimates, (7) from (22), (8) is included in $(III)_i$ and (9) is a consequence of (31) and the inductive estimates.

APPENDIX

Let $R > 0$ and $\Delta \subset \mathbb{C}^d$ be a compact set. We construct function $\chi \in C(\mathbb{C}^d) \cap C^\infty(\mathbb{R}^d)$: $\mathbb{C}^d \to [0, 1]$ with support in

$$Y_R(\Delta) \equiv \bigcup_{\eta_0 \in \Delta} \{\eta \in \mathbb{C}^d : \| \eta - \eta_0 \| \leq R\},$$

identically equal to one in $Y_{R/2}$ and such that, $\forall a \in \mathbb{N}^d$,

$$\sup_{\mathbb{R}^d} \| \partial^a_\eta \chi \| \leq \alpha \frac{(|a| + 2)!}{R^{|a|}} \tag{36}$$

For $t \in \mathbb{R}$ denote by $\chi_{[0,1/2]}(t)$ the characteristic function of [0, 1/2] and for $\eta \in \mathbb{C}^d$ let

$$g(t) \equiv \chi_{[0,1/2]}(t) \exp\left(-\frac{1}{t^2} - \frac{1}{(t-1/2)^2}\right)$$

$$G(t) \equiv \left(\int_{\mathbb{R}} g(t')\,dt'\right)^{-1} \int_{-\infty}^{t} g(t')\,dt'$$

$$\chi_1(t) \equiv G(1+t)\,G(1-t)$$

$$\chi_d(\eta) \equiv \chi_1\left(\frac{\|\eta\|}{R}\right)$$

It is easy to see that the function χ_d has the desired properties when $\Delta = \{0\}$. In general, if Δ is a compact set, so is $Y_{R/2}(\Delta)$ and since $\cup_{\eta_0 \in \Delta}\{\eta \in \mathbb{C}^d : \|\eta - \eta_0\| < R\}$ is an open cover of $Y_{R/2}(\Delta)$, there exist $\eta_1, \eta_2, \ldots, \eta_m \in \Delta$ such that $Y_{R/2}(\Delta) \subset \cup_{i=1}^{m} \{\eta \in \mathbb{C}^d : \|\eta - \eta_i\| < R\}$. Now, setting $p_1 \equiv \chi_d(\eta - \eta_1)$, $p_2 \equiv (1 - p_1)\,\chi_d(\eta - \eta_2), \ldots,$ $p_m \equiv (1-p_1)(1-p_2)\ldots(1-p_{m-1})\,\chi_d(\eta - \eta_m)$, one realizes that the looked after function χ is given by $\chi \equiv \Sigma_{i=1}^{m} p_i$.

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