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On the complex analytic structure of the golden invariant curve for the standard map

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Abstract. We consider the golden mean invariant curve for the standard map and give a strong numerical evidence that its conjugacy to a circle, regarded as a complex analytic function of the nonlinearity parameter, has a natural analyticity boundary found to be a circle of radius equal to the believed breakdown threshold.

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1. Introduction

Since the publication of Greene (1979) [1], which was subsequently supported by a great amount of numerical work (see, e.g., [2,3] and references therein), it is believed that the Chirikov–Greene 'standard map':

$$f_\varepsilon : (x, y) \in \mathbb{T} \times \mathbb{R} \mapsto (x', y') \qquad \mathbb{T} \equiv \mathbb{R}/2\pi\mathbb{Z}$$

$$y' = y + \varepsilon \sin x \qquad x' = x + y' \pmod{2\pi}$$

has a unique smooth (probably analytic [4,5]) golden invariant curve for $0 \leqslant \varepsilon < \varepsilon_c = 0.971\ldots$, while for $\varepsilon > \varepsilon_c$ no invariant curves at all are expected to exist. Here 'invariant curve' will always mean a closed, homotopically non-trivial continuous invariant curve and 'golden' means that the rotation number of such invariant curve is $\omega_g = (\sqrt{5} - 1)\pi$, i.e. 2π times the golden mean (for a general introduction to the dynamics of iterate area-preserving diffeomorphisms, the reader is referred, e.g., to [2,6].

Thus, if one 'follows' the golden invariant curve as ε is increased from zero one would observe a smooth deformation until ε reaches the so-called 'breakdown

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threshold' ε_c. At such a value of the nonlinearity parameter the curve would be non-smooth but probably Hölder continuous [7] and for $\varepsilon > \varepsilon_c$ it would break into Aubry–Mather sets. Such sets are invariant Cantor sets that can be embedded in the graph of a Lipschitz function (Aubry–Mather sets have in fact been proved to exist under very general assumptions [8–11]).

Up to now, not much of this intriguing 'breakdown phenomenon' has been mathematically explained. However, in [12] (see also [13]), it has been proved, by computer-assisted methods, that no invariant curve exists for $\varepsilon > 0.985$ and in [14] it has been proved (also by computer-assisted methods) that the golden invariant curve exists for *complex* values of ε, $|\varepsilon| \leqslant 0.65$ (the meaning of 'invariant curve' when ε is complex will be made clear in the next section).

Here we study numerically the smoothness properties of the function which conjugates the golden mean invariant curve to the rigid rotation for $|\varepsilon| < \varepsilon_c$, with special emphasis on the complex analytic structure *in the nonlinearity parameter* ε.

2. Methods and results

The dynamics of f_ε is easily seen to be described by the following nonlinear finite-difference equation:

$$x_{n+1} - 2x_n + x_{n-1} = \varepsilon \sin x_n \qquad (1)$$

where (x_n, y_n) is the nth iterate of a starting point (x_0, y_0) and $y_n = x_n - x_{n-1}$.

Let Γ be an invariant curve for f_ε; we say that $f_\varepsilon \upharpoonright \Gamma$ is topologically conjugated to a rigid rotation if there exists a continuous embedding of $\mathbb{T}$ into $\mathbb{T} \times \mathbb{R}$:

$$\Phi : \theta \in \mathbb{T} \mapsto \Phi(\theta) = (\Phi_1(\theta), \Phi_2(\theta)) \in \Gamma \subset \mathbb{T} \times \mathbb{R}$$

such that $f_\varepsilon \circ \Phi = \Phi \circ R_\omega$, where $R_\omega(\theta) = \theta + \omega (\operatorname{mod} 2\pi)$; ω is the so-called rotation number of Γ.

Since $y' = x' - x$ we have that $\Phi_2(\theta) = \Phi_1(\theta) - \Phi_1(\theta - \omega)$; therefore, with a slight abuse of language we will call the circle homeomorphism Φ_1 the 'conjugating function'. Notice that if Φ_1 is a conjugating function then $\theta \mapsto \Phi_1(\theta + c)$ is also a conjugating function for any real constant c. We fix this ambiguity by requiring that the continuous periodic function

$$u(\theta) \equiv \Phi_1(\theta) - \theta$$

have vanishing mean value. This function, which is the main object of our investigation, satisfies the following nonlinear equation (cf equation (1)):

$$D^2 u \equiv u(\theta + \omega, \varepsilon) - 2u(\theta, \varepsilon) + u(\theta - \omega, \varepsilon) = \varepsilon \sin(\theta + u(\theta, \varepsilon)). \qquad (2)$$

Conversely, any solution of (2) such that $\theta + u(\theta)$ is strictly monotone yields an invariant curve with rotation number ω, given in parametric representation by:

$$x = \theta + u(\theta, \varepsilon) \qquad y = \omega + u(\theta, \varepsilon) - u(\theta - \omega, \varepsilon) \qquad \theta \in \mathbb{T}.$$

If u is C^k or analytic then so is Γ. In fact, by a theorem of Birkhoff (see e.g. [15]), Γ is the graph of a Lipschitz function $x \mapsto y(x)$, and so one can express derivatives of $y(x)$ in terms of derivatives of the conjugating function $\Phi_1 = \theta + u(\theta)$. For example

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(\mathrm{d}/\mathrm{d}\theta)(u(\theta) - u(\theta - \omega))}{1 + (\mathrm{d}u/\mathrm{d}\theta)}.$$

In [14] it was proved that $u(\theta, \varepsilon) \equiv u_{\omega_g}(\theta, \varepsilon)$ is *jointly analytic* for $|\mathrm{Im}(\theta)| \leq 10^{-4}$ and $|\varepsilon| \leq 0.65$. Thus if one considers equation (1) in $\mathbb{C}^{\mathbb{Z}}$ (i.e. if one allows complex ε and looks for orbits $\{x_n\}_{n\in\mathbb{Z}}$ with $x_n \in \mathbb{C}$) such a result yields complex analytic invariant curves analytically embedded in $\mathbb{C}^2$.

Now, let

$$\sum_{n\geq 1} u_n(\theta)\varepsilon^n \tag{3}$$

be the power series expansion in ε of $u(\theta, \varepsilon)$ and let

$$\rho = \inf_{\theta\in\mathbb{T}} \left[\limsup_{n\to\infty} (|u_n(\theta)|^{1/n})\right]^{-1}$$

(by the above result $\rho > 0.65$), then for any ε with $|\varepsilon| < \rho$, $u(\theta, \varepsilon)$ is also analytic in θ and it is natural to enquire about the relation between ρ and ε_c.

By using the following recursion formulae for the computation of the u_n

$$b_0 = \mathrm{e}^{\mathrm{i}\theta}$$

and, for $n \geq 1$:

$$D^2 u_n = \frac{1}{2\mathrm{i}}(b_{n-1} + b_{n-1}^*) \qquad b_n = \frac{\mathrm{i}}{n}\sum_{k=1}^{n} k u_k b_{n-k}$$

one can compute a few hundreds of the u_n, each as a trigonometric polynomial in θ. These formulae are due to Goroff [16]; for other ways of computing coefficients of the conjugating function see [17, 18].

A plot of $|u_n(\theta)|^{-1/n}$ against n for various real θ suggests that the limiting value is independent of θ and might coincide with ε_c (see figure 1, where $\theta = 1$). However, the slow rate of convergence and the presence of many peaks (quite surely related to small divisors) make a reliable extrapolation practically impossible.

More information can be obtained by means of rational (Padé) approximants. Given a function $f(z)$, analytic near 0, its Padé approximant $[M/N]$ is a rational approximant of f with M and N being respectively the degrees of the numerator and

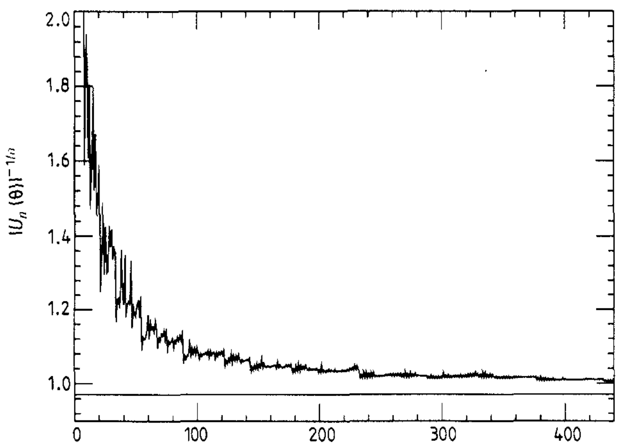
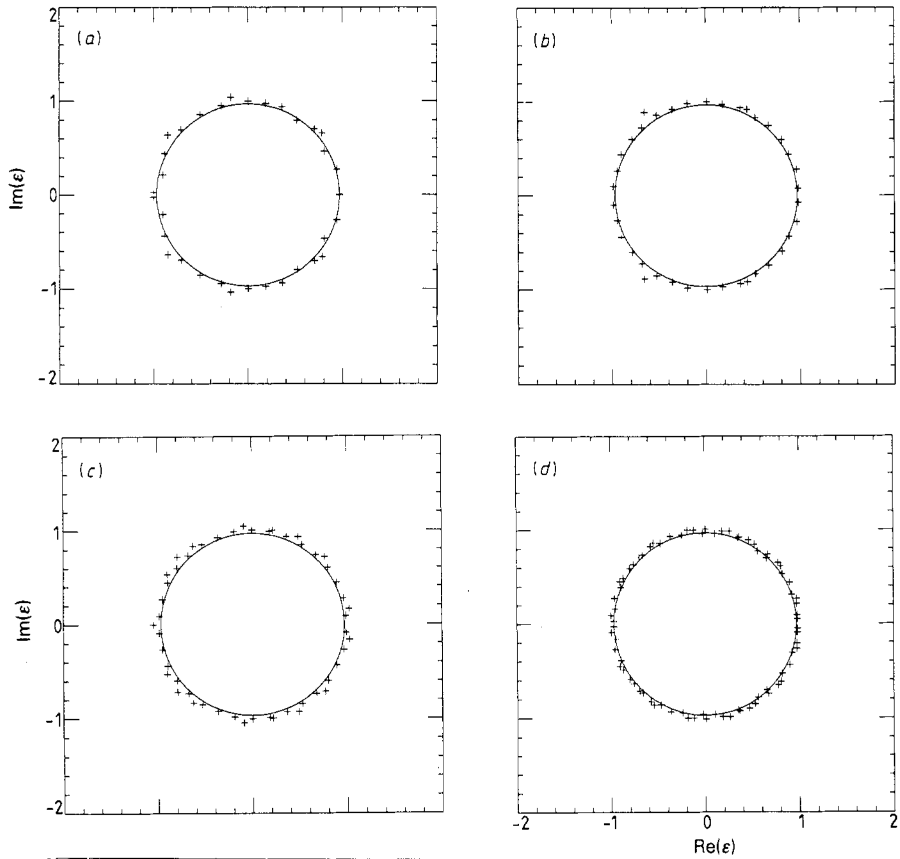


Figure 1. Plot of $|u_n(\theta)|^{-1/n}$ as a function of n, for $\theta = 1$; the horizontal line is at 0.971.

denominator, such that the Taylor expansion of the approximant at order $M+N$ coincides with the Taylor expansion of f at order $M+N$; formal uniqueness is obtained by imposing, for example, that the denominator is 1 at $z=0$. Under suitable non-degeneracy conditions on f such approximants exist, are unique and can be recursively computed from the first $M+N$ coefficients of the Taylor expansion of f (see [19]). In computing the coefficients of the Padé polynomials, we used the formulae given on p77 of [19], which minimise round-off error.

We consider various Padé approximants with $M+N \leqslant 200$ of the series (3), with θ fixed and real, and studied the distribution of the ε-singularities in $\mathbb{C}$. Typically,



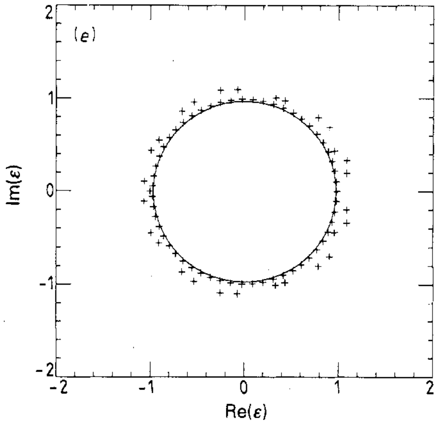


Figure 2. Poles of various Padé approximants in the complex ε plane. (*a*) [38/38] approximant, (*b*) [50/50] approximant, (*c*) [60/60] approximant, (*d*) [85/85] approximant, (*e*) [85/85] approximant. In each of (*a*)–(*d*), $0=1$; in (*e*) $\theta=2$. The circle in each figure has radius 0.971.

we considered the 'diagonal' approximants $[M/M]$ for several choices of θ and $10 \leqslant M \leqslant 90$ and tried to determine the zeros of the denominator; to this end we used the NAG routine CO2AEF, which gives accurate results for polynomials up to degree 100 provided they are not too ill conditioned; the roots found by CO2AEF are tested for really being zero of the polynomial. All calculations have been performed in double precision (64 bits); the calculations of the Padé coefficients have been tested for absence of significant round-off error by computing some of them in quadruple precision and comparing with the same coefficients computed in double precision.

Apart from a small number of unstable poles, we found a phenomenology largely independent of θ (summarised in figure 2) which shows that as the degree M increases, the singularities of the approximants $[M/M]$ tend to fill up densely a circle of radius ε_c. The poles far from the circle of radius ε_c (about 10–20% of the total number M of poles) are very unstable and disappear or move away if any parameter (like θ or M) is changed slightly.

3. Conclusions

The evidence we gather from the above analysis is that the functions $\varepsilon \mapsto u(\theta, \varepsilon)$ have, for typical values of θ, a natural analyticity boundary which is *a circle of radius* ε_c; here typical means that exceptional values of θ might be excluded (e.g. $\theta = 0, \pi$, because $u(\theta)$ vanishes identically there).

The present mathematical technology seems to be rather far from allowing a rigorous explanation of this phenomenon. However, we recall that an important class of analytic functions that shows natural boundaries is the class of 'lacunarity' functions, i.e. functions which admit a series expansion of the type

$$\sum_{i \geqslant 0} a_{n_i} z^{n_i}$$

where the sequence $n_i \uparrow \infty$ fast enough (for a review of the topic see, e.g., [20]). Therefore we tend to believe that a possible explanation of the appearance of circles of singularities could be found by investigating the 'lacunar' structure of the peaks of $\{|u_n|\}$ which, in turn, should be related to the occurrence of particularly small divisors entering the derivation of a coefficient when such a peak is achieved (an analogous mechanism has been considered in [5] in relation to the dependence of the Fourier coefficients $\hat{u}_k = \sum \mathrm{d}\theta \mathrm{e}^{ik\theta} u(\theta, \varepsilon)$ on ε). All this would be very much in line with the 'renormalisation group' picture ([2,3,21]). In this respect, it is suggestive to think of Padé approximants as rational approximations to analytic functions in the same way as truncated continued fraction expansions are rational approximations to irrational numbers.

Finally, it is interesting to ask whether an analogous picture holds for other maps and continuous-time Hamiltonian systems. An investigation of a few other 'typical systems' is currently under way.

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