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KAM stability estimates in Celestial Mechanics

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Abstract. A review of KAM stability estimates in Celestial Mechanics is presented. Rotational and librational invariant surfaces are constructed to ensure confinement in the phase space of a model obtained in the spin-orbit coupling between the revolutional and rotational motions of a satellite around a primary body. Stability of invariant tori for the restricted, circular, planar three body problem is also presented. Finally, an application of Arnold's theorem to the inclined planetary problem is briefly discussed. © 1998 Elsevier Science Ltd. All rights reserved

1. Introduction

One of the central problems in Celestial Mechanics is certainly the study of periodic or, more in general, quasi-periodic motions. On a technical level, the study of quasi-periodic trajectories leads naturally to the famous "small divisor problem". Even though such problems have been put in a rigorous setting and thoroughly investigated by Henri Poincaré (Poincaré, 1892), the modern mathematical techniques have been developed in the so-called KAM theory (Kolmogorov, 1954; Arnold, 1963a; Moser, 1962). One of the main results of such a theory ensures, under suitable hypotheses, the existence of quasi-periodic motions for conservative dynamical systems which are a small perturbation of an integrable system. The principal motivation for KAM theory, which nowadays has been successfully applied to a huge variety of problems ranging from applied physics to abstract partial differential equations, comes undoubtly from Celestial Mechanics. However, in view of the typical degeneracies of Celestial Mechanical problems and of the relevant parameter ranges, actual applications of KAM theory to Celestial Mechanical models turned out to be particularly difficult.

Here we shall focus on the quantitative aspects of the KAM theory of Celestial Mechanics and review a few results obtained in the last ten years. As mentioned above, the existence of quasi-periodic motions may be established provided certain parameters (measuring the size of the "perturbation") are small enough. KAM techniques do provide (already in the original formulations of the founders) explicit estimates on the allowed range of the perturbing parameters ("KAM stability estimates"); however the "classical" estimates are so stringent to be of no practical interest for any more or less concrete Celestial Mechanical models.

The technique and results reviewed in this paper show, instead, that a systematic reorganizations of KAM tools (which include new developments due to Moser, Salamon and Zehnder, see (Salamon and Zehnder, 1989) and (Celletti and Chierchia, 1995)) joined with (rigorous) computer implementations may lead to stability estimates which are in reasonable agreement with experimental expectations.

The paper is organized as follows. In Section 2 the "spin-orbit coupling" (namely the motion of an ellipsoidal satellite revolving around a central body on a Keplerian orbit and rotating, at the same time, around an internal spin-axis) is considered and *rotational* invariant surfaces are constructed in order to trap the motion of the satellite in a compact region of the phase space; the construction of trapping *librational* surfaces is also presented. In Section 3 KAM estimates for a "planar circular restricted three-body problem", modelled on the Sun–Jupiter–Ceres system, are considered; ("planar circular restricted" means that it is assumed that the ratio of the masses of the secondary bodies is much less than one, that one of the secondary bodies—i.e., the bigger one—moves on a circle (zero eccentricity), that the motion of the three bodies takes place on the same plane). Finally, in Section 4, we briefly discuss a different type of result concerning the "spatial planetary three-body problem", where it is assumed that the masses of the secondary bodies are of the same order of magnitude (so that one cannot neglect their mutual interaction) and that the orbits might have a mutual inclination in space ("different" means that no

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stability bounds are included in such a result). The paper concludes with a brief appendix where the mathematical techniques needed to use computers in order to establish rigorous mathematical estimates are discussed.

2. Spin-orbit coupling

Consider an ellipsoidal satellite S moving around a central body P. Let $T_{\rm rev}$ be the period of revolution of S around P and let $T_{\rm rot}$ be the period of rotation of S around its spin-axis.

Definition: A spin-orbit resonance of type $p:q$ occurs whenever there exist $p, q = \varepsilon\mathbf{Z}\backslash\{0\}$, such that

$$\frac{T_{\rm rev}}{T_{\rm rot}} = \frac{p}{q}.$$

The most familiar example of a 1 : 1 (or synchronous) spin-orbit resonance is provided by our Moon, since the revolutional and rotational periods coincide. Consequently, the Moon always points the same face toward the Earth. As it is well known, all the tidally evolved planets or satellites of the solar system are trapped in a 1 : 1 resonance. The only exception is provided by the Mercury–Sun system, since the ratio between the period of revolution of Mercury around the Sun and the period of rotation about its spin-axis amounts to $\frac{3}{2}(1 \pm 10^{-4})$.

Let us introduce a mathematical model describing an approximation of the spin-orbit problem. Consider a triaxial ellipsoidal satellite S with principal moments of inertia $A < B < C$. We assume that

(i) the centre of mass of the satellite moves on a Keplerian orbit around the primary body P, with semi-major axis a and eccentricity e;
(ii) the spin-axis coincides with the shortest physical axis (i.e., the axis whose moment of inertia is largest);
(iii) the spin-axis is perpendicular to the orbit plane (i.e., we neglect the so-called "obliquity");
(iv) dissipative forces as well as perturbations due to other bodies are neglected.

The equation describing the motion under the above assumptions can be derived from Euler's equations for a rigid body. More precisely, denoting by x the angle between the longest axis of the ellipsoid and the periapsis line, by r the instantaneous orbital radius and f the true anomaly (see Fig. 1), one has

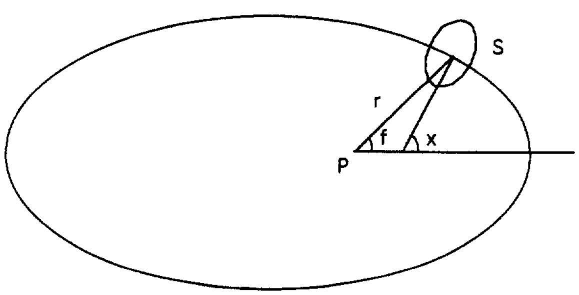

Fig. 1. The spin-orbit geometry

$$\ddot{x} + \frac{3}{2}\frac{B-A}{C}\left(\frac{a}{r}\right)^3 \sin(2x-2f) = 0, \tag{1}$$

where the mean motion has been normalized to one (i.e., $2\pi/T_{\rm rev} = 1$).

We remark that a rotation of the satellite by 180° (i.e., $x \to x+\pi$) gives an equivalent configuration and that the above equation is trivially integrable in the case of equatorial symmetry (i.e., $A = B$) as well as for circular orbits (i.e., $e = 0$).

A spin-orbit resonance of type $p:q$ is therefore a periodic orbit $x \equiv x(t)$ associated to eqn (1), such that

$$x(t+2\pi q) = x(t) + 2\pi p,$$

namely during q orbital revolutions around the primary body P, the satellite S makes p rotations about its spin-axis.

Due to assumption (i), both r and f are known 2π-periodic functions of the time. Introducing the parameter $\varepsilon = \frac{3}{2}[(B-A)/C]$ (proportional to the so-called ellipticity $(B-A)/C$), one can expand (1) as

$$\ddot{x} + \varepsilon \sum_{m\neq 0, m=-\infty}^{\infty} W\left(\frac{m}{2}, e\right) \sin(2x - mt) = 0, \tag{2}$$

where the coefficients $W(m/2, e)$ decay as powers of the eccentricity e, i.e. $W(m/2, e) \propto e^{|m-2|}$. For example, the first few coefficients are given by (see, e.g., Cayley, 1859)

$$W\left(\frac{1}{2}, e\right) = -\frac{e}{2} + \frac{e^3}{16} - \frac{5}{384}e^5 - \frac{143}{18.432}e^7 + O(e^9)$$

$$W(1, e) = 1 - \frac{5}{2}e^2 + \frac{13}{16}e^4 - \frac{35}{288}e^6 + O(e^8)$$

$$W\left(\frac{3}{2}, e\right) = \frac{7}{2} - \frac{123}{16}e^3 + \frac{489}{128}e^5 - \frac{1763}{2048}e^7 + O(e^9)$$

$$W(2, e) = \frac{17}{2}e^2 - \frac{115}{6}e^4 + \frac{601}{48}e^6 + O(e^8).$$

We simplify further the model as follows. According to assumption (iv), we have neglected all the dissipative forces; in particular, the tidal torque, due to the internal non-rigidity of the satellite, provides the strongest contribution. However its magnitude is small compared to the gravitational term and we decide to neglect in the series expansion of (2) those terms whose size is less or equal than the average effect of the tidal torque. Therefore we obtain an equation of the form

$$\ddot{x} + \varepsilon \sum_{m\neq 0, m=N_1}^{N_2} W\left(\frac{m}{2}, e\right) \sin(2x - mt) = 0, \tag{3}$$

where N_1 and N_2 are some integers, which depend on the physical and orbital elements of the satellite.

Let us now investigate the phase-space structure associated to (3): its Poincaré map has a pendulum-like structure, in which the periodic orbit is surrounded by librational curves. A chaotic separatrix divides the librational regime from the region in which rotational invariant curves can be found (Fig. 2).

Due to its conservative character, eqn (3) can be derived

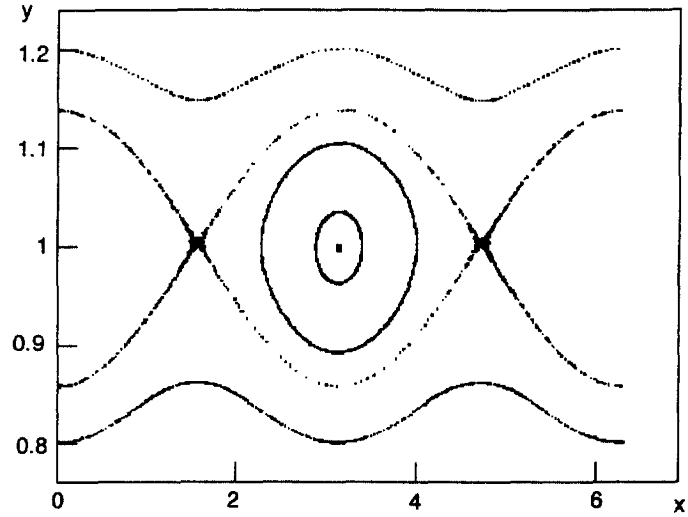

Fig. 2. The Poincaré map around the synchronous periodic orbit associated to (3), with $N_1 = 1$ and $N_2 = 5$, $\varepsilon = 0.01$, $e = 0.0549$. The initial data are: $(x_0, y_0) = (3.14, 1)$, $(3.4, 1)$, $(4, 1)$, $(4.7, 1)$, $(0, 1.2)$, $(0, 0.8)$

from Hamilton's equations associated to the Hamiltonian function

$$H(y,x,t) = \frac{y^2}{2} - \frac{\varepsilon}{2} \sum_{m \neq 0, m = N_1}^{N_2} W\left(\frac{m}{2}, e\right) \cos(2x - mt), \tag{H}$$

where $y \in \mathbf{R}$, $(x,t) \in T^2$. Let us rewrite (H) in the form

$$H(y,x,t) \equiv h(y) + \varepsilon f(x,t) = \frac{y^2}{2} + \varepsilon f(x,t), \tag{$H0$}$$

with the obvious identification of the functions $h(y)$ and $f(x,t)$. In the integrable case Hamilton's equations become

$$\begin{cases} \dot{y} = 0 \\ \dot{x} = \dfrac{dh(y)}{dy} = y \end{cases}$$

and their solution is provided by

$$\begin{cases} y = y_0 \\ x = x_0 + y_0 t, \end{cases}$$

where we refer to the quantity $\omega \equiv [dh(y_0)/dy] = \omega(y_0)$ as the rotation number. The invariant surfaces of the unperturbed system, i.e. $\mathscr{T}_0(\omega) \equiv \{y_0\} \times \mathbf{T}^2$, are described by the planes $y = y_0$ on which periodic or quasi-periodic motions take place, depending on the initial conditions.

When the perturbing parameter ε is not zero, KAM theory provides the existence of an invariant surface for the perturbed system with rotation number ω, say $\mathscr{T}_\varepsilon(\omega)$. This surface will be more deformed and displaced as ε grows, until ε reaches a critical value, say $\varepsilon_c = \varepsilon_c(\omega)$, at which the invariant surface $\mathscr{T}_\varepsilon(\omega)$ breaks-down, bifurcating into a so-called Mather set (a closed invariant Cantor set, lying in a graph of a Lipshitz function, Mather, 1984).

The phase space associated to (H0),

$$\mathscr{S} = \{(y,x,t)/y \in \mathbf{R}, \quad (x,t) \in \mathbf{T}^2\},$$

has dimension 3, while the invariant surfaces $\mathscr{T}_\varepsilon(\omega)$ have dimension 2 and separate $\mathscr{S}$ in two "invariant" regions. At the critical value $\varepsilon = \varepsilon_c(\omega)$, $\mathscr{T}_\varepsilon(\omega)$ breaks down and the orbits can diffuse through the gaps of the Cantor set.

KAM theory provides an explicit constructive algorithm to give an estimate on the perturbing parameter, say $\varepsilon_r = \varepsilon_r(\omega)$ ensuring the existence of $\mathscr{T}_\varepsilon(\omega)$ for any $\varepsilon \leqslant \varepsilon_r(\omega)$. The conditions under which the KAM theorem can be applied are:

(i) the unperturbed Hamiltonian $h(y)$ is not-degenerate, i.e.

$$\frac{d^2 h(y)}{dy^2} \neq 0 \quad \forall y \in \mathbf{R}$$

(notice that, for (H0), $[d^2h(y)/dy^2] \equiv 1$);

(ii) the rotation number ω satisfies the diophantine condition

$$\left|\omega - \frac{p}{q}\right|^{-1} \leqslant Cq^2 \quad \forall p, q \in \mathbf{Z}, \quad q \neq 0, \tag{4}$$

for some positive constant C.

2.1. *Rotational invariant surfaces*

We make use of the confinement property in the 3-dimensional phase space, in order to get the stability of periodic orbits. More precisely, let $\mathscr{P}(p/q)$ be the periodic orbit associated to the $p:q$ resonance. We intend to trap $\mathscr{P}(p/q)$ between invariant surfaces $\mathscr{T}(\omega_1)$ and $\mathscr{T}(\omega_2)$, with $\omega_1 < (p/q) < \omega_2$. To this end, we select the two sequences of irrational rotation numbers

$$\Gamma_k^{(p/q)} \equiv \frac{p}{q} - \frac{1}{k+\alpha}, \quad \Delta_k^{(p/q)} \equiv \frac{p}{q} + \frac{1}{k+\alpha}, \quad k \in \mathbf{Z}, \quad k \geqslant 2$$

($\alpha \equiv [(\sqrt{5}-1)/2]$), which approach p/q from below and above, and satisfy the diophantine condition (4) with the constant $C \equiv C_k = q^2(k+\alpha)$ (see Celletti, 1990).

As a specific example, consider the Moon–Earth system and let us look at the sychronous resonance $\mathscr{P}(1/1)$. According to our simplifications, we are led to study the Hamiltonian

$$\begin{aligned} H(y,x,t) \equiv \frac{y^2}{2} - \varepsilon\Bigg[&\left(-\frac{e}{4}+\frac{e^3}{32}\right)\cos(2x-t) \\ &+\left(\frac{1}{2}-\frac{5}{4}e^2+\frac{13}{32}e^4\right)\cos(2x-2t) \\ &+\left(\frac{7}{4}e-\frac{123}{32}e^3\right)\cos(2x-3t) \\ &+\left(\frac{17}{4}e^2-\frac{115}{12}e^4\right)\cos(2x-4t) \\ &+\left(\frac{845}{96}e^3-\frac{32{,}525}{1536}e^5\right)\cos(2x-5t) \\ &+\frac{533}{32}e^4\cos(2x-6t) \\ &+\frac{228{,}347}{7680}e^5\cos(2x-7t)\Bigg], \end{aligned} \tag{5}$$

where the physical value of the perturbing parameter $\varepsilon \equiv \frac{3}{2}[(B-A)/C]$ is $3.45 \cdot 10^{-4}$, while $e = 0.0549$. The existence of the surface $\mathscr{T}(\Gamma_k^{(1)})$ and $\mathscr{T}(\Delta_k^{(1)})$ for $k = 2, 3, \ldots, 40$ is proven in (Celletti, 1990). Therefore the motion of the Moon, *in the approximation* (5), will be forever trapped in the region enclosed by $\mathscr{T}(\Gamma_{40}^{(1)})$ and $\mathscr{T}(\Delta_{40}^{(1)})$, which is shown to be a subset of $\{(y,x,t):(x,t)\in \mathbf{T}^2,\ 0.97 \leqslant y \leqslant 1.03\}$. The proof is computer-assisted; the rounding-off and truncation errors introduced by the machine are controlled by means of the interval arithmetic technique presented in the Appendix (see also Eckmann and Wittwer, 1985; Lanford, 1984).

Another application is obtained by looking at the Mercury–Sun system. Due to the large eccentricity of Mercury, $e = 0.2056$, we have to retain a larger number of terms in (H):

$$H(y,x,t) \equiv \frac{y^2}{2} - \frac{\varepsilon}{2}\sum_{m\neq 0,m=-11}^{3} W\left(\frac{m}{2},e\right)\cos(2x-mt).$$

By means of KAM theorem, it is proven in (Celletti and Chierchia, 1995) the stability of the 3:2 resonance (in which Mercury is actually observed) for the astronomical value of the perturbing parameter, i.e. $\varepsilon = 1.5 \cdot 10^{-4}$. The closest tori to the periodic orbit $\mathscr{P}(3/2)$ are $\Gamma_{70}^{(3/2)}$ and $\Delta_{70}^{(3/2)}$ and the corresponding trapping region is contained in $\{(y,x,t):(x,t)\in\mathbf{T}^2,\ 1.48 \leqslant y \leqslant 1.52\}$.

2.2. *Librational invariant surfaces*

The confinement of the motion of periodic orbits can also be obtained constructing librational invariant surfaces. Focussing our attention on the 1 : 1 resonance, let us show how one can construct the trapping surfaces (see Celletti, 1994 for further details).

First one centres the Hamiltonian on the 1 : 1 periodic orbit and expand in Taylor series around the new origin. Next one diagonalizes the quadratic terms obtaining a harmonic oscillator plus higher degree (time-dependent) terms. Finally one transforms the Hamiltonian using the action-angle variables (I,ϕ) of the harmonic oscillator. After these symplectic changes of variables one is led to a Hamiltonian of the form

$$H(I,\phi,t) \equiv \omega I + \varepsilon\bar{h}(I) + \varepsilon R(I,\phi,t), \quad I\in\mathbf{R}, \quad (\phi,t)\in T^2,$$

where $\omega \equiv \omega(\varepsilon)$ is the frequency of the harmonic oscillator, while $\bar{h}(I)$ and $R(I,\phi,t)$ are suitable functions, whose expression is given below. As shown in (Celletti and Giorgilli, 1988), in order to improve the results provided by KAM theory, it is convenient to reduce the perturbation to higher orders in ε by a close-to-identity change of variables $(I,\phi,t)\to(I',\phi',t')$. Applying a Birkhoff normalization procedure up to the order k, one gets a Hamiltonian of the type

$$H_k(I',\phi',t) = \omega I' + \varepsilon h_k(I';\varepsilon) + \varepsilon^{k+1}R_k(I',\phi',t). \tag{6}$$

Finally one can apply the KAM estimates provided in (Celletti and Chierchia, 1987) to the Hamiltonian (6).

According to the above strategy, let us start by rewriting the Hamiltonian (*H*) putting into evidence the term related to the synchronous resonance:

$$\begin{aligned} H(y,x,t) = \frac{y^2}{2} &- \varepsilon a\cos(2x-2t) \\ &- \frac{\varepsilon}{2}\sum_{m\neq 0,2,m=N_1}^{N_2} W\left(\frac{m}{2},e\right)\cos(2x-mt), \end{aligned}$$

where $a \equiv \frac{1}{2}W(1,e)$. One can next translate to the origin by means of the coordinate change $x' = 2x-2t$, $y' = \frac{1}{2}(y-1)$. Expanding in Taylor series around the origin and diagonalizing the time independent quadratic terms by means of the symplectic transformation

$$\begin{cases} p = \alpha y' \\ q = \beta x' \end{cases}$$

where $\alpha = [\sqrt{2}/(\varepsilon a)^{1/4}]$, $\beta = [(\varepsilon a)^{1/4}/\sqrt{2}]$, the new Hamiltonian is given by

$$\begin{aligned} H(p,q,t) = \frac{\omega}{2}(p^2+q^2) &- \varepsilon a\left(\frac{q^4}{4!\beta^4}-\frac{q^6}{6!\beta^6}+\cdots\right) \\ &- \frac{\mu}{2}\sum_{m\neq 0,-2}\tilde{W}\left(\frac{m+2}{2},e\right) \\ &\times\Bigg[\cos(mt)\left(1-\frac{q^2}{2\beta^2}+\frac{q^4}{4!\beta^4}+\cdots\right) \\ &+\sin(mt)\left(\frac{q}{\beta}-\frac{q^3}{3!\beta^3}+\frac{q^5}{5!\beta^5}+\cdots\right)\Bigg], \end{aligned}$$

where $\omega = 2\sqrt{\varepsilon a}$ is the frequency of the harmonic oscillation, $\mu \equiv \varepsilon e$, while the coefficients W have been rescaled as $\tilde{W}[(m+2)/2, e] = (1/e)W[(m+2)/2, e]$. Using action-angle variables (I, ϕ), i.e. setting

$$\begin{cases} p = \sqrt{2I}\cos\phi \\ q = \sqrt{2I}\sin\phi \end{cases},$$

the final Hamiltonian is given by

$$\begin{aligned} H(I,\phi,t) = {} & \omega I - \varepsilon a\left(\frac{I^2}{16\beta^4} - \frac{5I^3}{2\cdot 6!\beta^6} + \cdots\right) \\ & -\varepsilon a\left[-\frac{I^2}{12\beta^4}\cos 2\phi + \frac{I^2}{48\beta^4}\cos 4\phi + \frac{I^3}{4\cdot 6!\beta^6}\right. \\ & \left.\times(15\cos 2\phi - 6\cos 4\phi + \cos 6\phi) + \cdots\right] \\ & -\frac{\varepsilon e}{2}\sum_{m\neq 0,-2}\tilde{W}\left(\frac{m+2}{2}, e\right)\Bigg\{\cos(mt) \\ & \times\left[1 - \frac{I}{2!\beta^2}(1-\cos 2\phi) + \frac{I^2}{8\cdot 3!\beta^4}\right. \\ & \times(3 - 4\cos 2\phi + \cos 4\phi) - \frac{I^3}{4\cdot 6!\beta^6} \\ & \left.\times(10 - 15\cos 2\phi + 6\cos 4\phi - \cos 6\phi) + \cdots\right] \\ & +\sin(mt)\left[\frac{\sqrt{2I}}{\beta}\sin\phi - \frac{\sqrt{2}I^{3/2}}{12\beta^3}\right. \\ & \times(3\sin\phi - \sin 3\phi) + \frac{\sqrt{2}I^{5/2}}{4\cdot 5!\beta^5} \\ & \left.\times(10\sin\phi - 5\sin 3\phi + \sin 5\phi) + \cdots\right]\Bigg\}, \end{aligned}$$

which can be rewritten in compact form as

$$H(I,\phi,t) = \omega I + \varepsilon\bar{h}(I) + \varepsilon h(I,\phi) + \varepsilon e f(I,\phi,t).$$

One can next apply a Birkhoff normal form (see, e.g., Gallavotti, 1983) so to reduce the size of the perturbation $R(I,\phi,t) \equiv h(I,\phi) + ef(I,\phi,t)$, obtaining the Hamiltonian

$$H_k(I',\phi',t) = h_k(I';\varepsilon) + \varepsilon^{k+1}R_k(I',\phi',t;\varepsilon),$$

where the functions h_k and R_k can be explicitely determined. The application of the (computer-assisted) KAM estimates developed in (Celletti and Chierchia, 1987) allows to establish the existence of librational invariant surfaces trapping the synchronous resonance.

In particular, considering the Moon–Earth and Rhea–Saturn systems, which are observed in a synchronous spin-orbit resonance as stated in (*The Astronomical Almanac*, 1990), the following results have been obtained in (Celletti, 1994).

Moon–Earth: consider the system described by the Hamiltonian (H) with $N_1 = -1$, $N_2 = 5$ and $e = 0.0549$. Let $\varepsilon_{obs} = 3.45\cdot 10^{-4}$ (i.e., the physical value of the ellipticity of the Moon); then there exists an invariant torus corresponding to a libration of $8^\circ.79$ for any $\varepsilon \leqslant \varepsilon_{obs}/5.26$.

Rhea-Saturn: consider the Hamiltonian function (H) with $N_1 = 1$, $N_2 = 5$ and $e = 0.00098$. Let $\varepsilon_{obs} = 3.45\cdot 10^{-3}$ (i.e., the observed value); then there exists an invariant torus corresponding to a libration of $1^\circ.95$ for any $\varepsilon \leqslant \varepsilon_{obs}$.

Though definite conclusions cannot be drawn about the stability of the Moon for the realistic values of the parameters, we still believe that one can improve the results using a different KAM algorithm. On the contrary, the method applies to the Rhea–Saturn system, providing an insight on the stability of the synchronous resonance.

3. Planar, circular, restricted three-body problem

We consider two bodies P_1 and P_2, with masses m_1 and m_2, respectively, orbiting around a central (primary) body P with mass $M \gg m_1, m_2$. In this section we will focus on KAM stability estimates for the "planar, circular, restricted three-body problem" (hereafter PCRTBP): it is assumed that the ratio of the masses of the secondary bodies is much less than one, i.e. $m_1/m_2 \ll 1$. This hypothesis implies that P_1 does not affect the motion of P_2; therefore we can imagine that the motion of P_2 is ruled by Kepler's law. In particular, we assume that its orbital eccentricity is zero, so that the orbit of P_2 is circular. Moreover we impose that the motion of the three bodies takes place on the same plane.

In order to derive the Hamiltonian function associated to the PCRTBP, let us introduce the classical planar Delaunay variables (Delaunay, 1860). Let λ be the mean anomaly, α the argument of the perihelion and ψ the longitude of the planet P_2. Consider the phase space

$$\mathscr{P} = \{(\lambda,\alpha,\psi)\in\mathbf{T}^3\}\times\{(\Lambda,\Gamma,E)\in\mathbf{R}^3 : \Lambda = 0,\quad |\Gamma| < |\Lambda|\}$$

endowed with the standard symplectic form $d\lambda\wedge d\Lambda + d\alpha\wedge d\Gamma + d\psi\wedge dE$. Choosing the units of measure so that $M + m_1 = 1$ and that the period of P_2 is 2π, the dynamics associated to the PCRTBP is described by the Hamiltonian

$$H_0(\Lambda,\Gamma,E,\lambda,\alpha,\psi) \equiv \frac{1}{2\Lambda^2} + E + \varepsilon R_0(\Lambda,\Gamma,\lambda,\alpha-\psi),$$

where $\varepsilon \equiv m_2/M$ and the perturbation function R_0 is given as follows. Let $v\in\mathbf{T}$ be the eccentric anomaly defined for $|e| < 1$ $(e\in\mathbf{R})$ by the Kepler's equation

$$\lambda = v - e\sin v.$$

Let $\varphi\in\mathbf{T}$ be the true anomaly and r the orbital radius

$$r \equiv \frac{a(1-e^2)}{1+e\cos(\varphi-\alpha)},\quad \text{where } a \equiv L^2.$$

Then the perturbing function R_0 is given by

$$R_0(\Lambda,\Gamma,\lambda,\alpha-\psi) \equiv -\left(r\cos(\varphi-\psi) - \frac{1}{\sqrt{1+r^2-2r\cos(\varphi-\psi)}}\right),$$

where e is defined as

$$e \equiv \sqrt{1-\frac{\Gamma^2}{\Lambda^2}}.$$

Using the Legendre polynomials $P_j \equiv P_j(x)$ one can expand R_0 (for $r < 1$) as

$$R_0 = 1 + \sum_{j=2}^{\infty} r^j P_j(\cos(\varphi - \psi)).$$

Finally the symplectic transformation

$$(l, g, \tau) \equiv (\lambda, \alpha - \psi, \psi), \quad (L, G, T) \equiv (\Lambda, \Gamma, \Gamma + E),$$

reduces H_0 to the form

$$H_1(L, G, l, g) = \frac{1}{2L^2} - G + \varepsilon R_0(L, G, l, g),$$

having omitted the dummy variable T.

As a concrete example we identify P with the Sun, P_1 with the asteroid Ceres and P_2 with Jupiter. We simplify further the problem as follows. We have neglected in our model the gravitational attraction exerted on Ceres by other asteroids or planets as well as indirect perturbations. In particular, one of the strongest contributions among the neglected terms comes from the attraction provided by Saturn. Therefore we decide to neglect in the Fourier expansion of R_0 those terms whose size is comparable to the Ceres–Saturn attraction $\mathscr{G}_{\mathrm{Sa}}$.

Let us evaluate $\mathscr{G}_{\mathrm{Sa}}$ as follows. As found in (*The Astronomical Almanac*, 1990), Ceres moves on a nearly elliptical orbit of average eccentricity $e_0 \equiv 0.0766$ and average semimajor axis $a_0 \equiv 0.532$. In general, for planets whose orbits are external to that of Ceres, the secular term of H_1 is given by ε ($\equiv$ mass of the planet/mass of the Sun) times the term $R_{00} \equiv R_{00}(L; e) \equiv \langle R_0(L, G, l, g)\rangle$ (where $\langle \cdot \rangle$ denotes the average over l, g). Then, for a generic planet $\tilde{P}$

$$\mathscr{G}_{\tilde{P}} \equiv \varepsilon(\tilde{P}) \times R_{00}(L(\tilde{P}); e_0),$$

where $\varepsilon(\tilde{P})$ is the mass ratio between the planet $\tilde{P}$ and the Sun and $L(\tilde{P})$ is the ratio between the semimajor axes of Ceres and the planet $\tilde{P}$. Using the data provided by (*The Astronomical Almanac*, 1990), one finds

$$\mathscr{G}_{Sa} \equiv \mathscr{G}_{\mathrm{Saturn}} = 6.3778 \cdot 10^{-6}.$$

Neglecting in the expansion of εR_0 those terms whose size is smaller than $\mathscr{G}_{\mathrm{Sa}}$, we consider the Hamiltonian

$$H_2(L, G, l, g; \varepsilon) \equiv \frac{1}{2L^2} - G + \varepsilon R(L, G, l, g) \tag{7}$$

with R given by

$$R(L, G, l, g) \equiv \sum_{\substack{n \in \mathbf{Z}^2 \\ 0 \leqslant |n_1| + |n_2| \leqslant 10}} R_n(L, G) \cos(n_1 l + n_2 g) \tag{8}$$

where $R_n \equiv R_{n_1 n_2}$ vanishes unless it belongs to the following list:

$$R_{00} \equiv \frac{L^4}{4}\left(1 + \frac{9}{16}L^4 + \frac{3}{2}e^2\right), \quad R_{10} \equiv -\frac{L^4 e}{2}\left(1 + \frac{9}{8}L^4\right),$$

$$R_{11} \equiv \frac{3}{8}L^6\left(1 + \frac{5}{8}L^4\right), \quad R_{12} \equiv -\frac{L^4 e}{4}(9 + 5L^4),$$

$$R_{22} \equiv \frac{L^4}{4}\left(3 + \frac{5}{4}L^4\right), \quad R_{32} \equiv \frac{3}{4}L^4 e,$$

$$R_{33} \equiv \frac{5}{8}L^6\left(1 + \frac{7}{16}L^4\right), \quad R_{44} \equiv \frac{35}{64}L^8,$$

$$R_{55} \equiv \frac{63}{128}L^{10} \tag{9}$$

(recall that the eccentricity e is related to the action variables by $e = \sqrt{1 - [G^2/L^2]}$).

Our results are based on computer-assisted KAM theory. As mentioned in Section 2, KAM theory requires that the unperturbed Hamiltonian is non-degenerate. In the case of (7), the Hessian matrix associated to be the unperturbed Hamiltonian

$$h_0(L, G) = \frac{1}{2L^2} - G,$$

is not invertible. In order to overcome this problem we adopt Poincaré's trick (Poincaré, 1892), which consists in replacing the Hamiltonian H_2 by its square. We remark that the dynamics associated to a Hamiltonian $h = h(q, p)$ and to h^2 coincide up to a time scale. Finally, we define $H_3 \equiv (H_2)^2$ as

$$H_3(L, G, l, g; \varepsilon) = \left(\frac{1}{2L^2} - G\right)^2 + 2\varepsilon\left(\frac{1}{2L^2} - G\right) R(L, G, l, g) + \varepsilon^2[(R(L, G, l, g)]^2. \tag{10}$$

To be consistent with the criterion adopted to derive (7), we omit the term of order ε^2 in (10) obtaining

$$H(L, G, l, g; \varepsilon) \equiv \left(\frac{1}{2L^2} - G\right)^2 + 2\varepsilon\left(\frac{1}{2L^2} - G\right) R(L, G, l, g) \equiv h(L, G) + \varepsilon f(L, G, l, g). \tag{11}$$

Kepler's third law provides a value of the average frequency of Ceres

$$-\Omega_c \equiv -2.577107.$$

Since $-\Omega_c \simeq (\partial H_1/\partial L)|_{\varepsilon=0} = -L^{-3}$ we take as *reference L-value* the quantity

$$L_0 \equiv 0.729305 \simeq \Omega_c^{1/3}$$

and, since $G_0 = L_0\sqrt{1 - e_0^2}$, we take as *reference G-value* the quantity

$$G_0 \equiv 0.727162.$$

Let B be the analyticity domain associated to (11) defined in

$$B \equiv \{(L, G) \in \mathbf{C}^2 : |L - L_0| \leqslant r_0, \quad |G - G_0| \leqslant r_0\},$$
$$B_0 \equiv B \cap \mathbf{R}^2, \tag{12}$$

with $r_0 = 0.001$. Notice that with our choice of the analyticity radius r_0 one finds that the function $e(L, G)$ satisfies

$$0.019799 < |e(L, G)| < 0.106364, \quad \forall (L, G) \in B.$$

We construct invariant KAM curves trapping the motion of Ceres as follows. Let $a = [(\sqrt{5} - 1)/2]$ and let

$$\Omega_{-} \equiv \frac{5}{2} + \frac{1}{13+a} = 2.573432\ldots,$$

$$\Omega_{+} \equiv \frac{5}{2} + \frac{1}{12+a} = 2.579251\ldots.$$

Since the observed average frequency of Ceres is approximately $-\Omega_c \equiv 2.577107$, one has $\Omega_{-} < \Omega_c < \Omega_{+}$. Let

$$L_{\pm} \equiv \Omega_{\pm}^{-1/3}, \quad G_{\pm} \equiv L_{\pm}\sqrt{1-e_0^2},$$

where $e_0 = 0.0766$. Finally, we let

$$\omega^{(\pm)} \equiv (E_{\pm}\Omega_{\pm}, E_{\pm}), \quad E_{\pm} \equiv -2\left(\frac{1}{2L_{\pm}^2} - G_{\pm}\right), \quad (13)$$

satisfying

$$|\omega^{(\pm)} \cdot n|^{-1} \leqslant C_{\pm}|n|, \quad \forall n \in \mathbf{Z}^2\backslash\{0\},$$

with

$$C_{\pm} = 2|E_{\pm}|(\sqrt{5}+24 \pm 1).$$

Theorem: Let H be as in (11), (8), (9); let B_0 be as in (12); let $\omega^{(\pm)}$ be as in (13). Then, for all $0 \leqslant |\varepsilon| \leqslant 10^{-6}$ there exist (unique) two-dimensional analytic tori $\mathcal{T}_{\varepsilon}(\omega^{(\pm)})[C]B_0 \times \mathbf{T}^2$, depending analytically also on the parameter ε, on which the H-flow is (analytically) conjugated to the linear flow $\theta \in \mathbf{T}^2 \to \theta - \omega^{(\pm)}t$.

We refer to (Celletti and Chierchia, 1997) for the proof of the theorem. We recall that the actual value of the perturbing parameter (i.e. the Jupiter–Sun mass ratio) is $\varepsilon = 10^{-3}$. This result represents a great improvement with respect to previous KAM estimates, which were of the order of 10^{-50} (see Hénon, 1966).

4. Spatial planetary three-body problem

We consider two bodies P_1 and P_2 (with masses m_1 and m_2) orbiting around a central body P (with mass M). We assume that the masses of P_1 and P_2 are of the same order of magnitude, so that it is necessary to consider also the mutual interaction between the secondary bodies, besides their own Keplerian motions around P. In the spatial case, the Hamiltonian function can be written as follows: let $\underline{r}_1$, $\underline{r}_2$ and $\underline{v}_1$, $\underline{v}_2$ be, respectively, the heliocentric positions of the planets and the conjugated momenta with respect to the centre of mass. The Hamiltonian governing the motion of P_1 and P_2 can be decomposed in the form

$$H = H_0 + H_1.$$

The function H_0 is due to the independent Keplerian motions of the planets, i.e.

$$H_0 = \sum_{j=1}^{2}\left(\frac{m_j+M}{m_jM}\|\underline{v}_j\|^2 - G\frac{Mmj}{\|\underline{r}_j\|}\right),$$

where G is the gravitational constant. The perturbing function has the form

$$H_1 = \frac{\underline{v}_1 \cdot \underline{v}_2}{M} - G\frac{m_1m_2}{\|\underline{r}_1 - \underline{r}_2\|},$$

representing the interaction between the two planets P_1 and P_2. Due to the conservation of the angular momentum, the ascending nodes of the planets lie on the invariant plane containing the central body P and perpendicular to the angular momentum.

Stability estimates in the framework of the planetary problem (which presents proper degeneracy) have been investigated by Arnold (Arnold, 1963b), who discussed the existence of invariant tori assuming that the motion of the bodies is planar and that the ratio of the semimajor axes tend to zero. More precisely, Arnold proved that there exists a set of large measure of invariant tori, provided that the planetary masses and the eccentricities are sufficiently small. A rough estimate of the quantities involved in the proof is quoted in (Robutel, 1995) and, precisely, the theorem holds when the perturbing parameter is less than 10^{-300} and the eccentricities are less than 10^{-80}. Later, Robutel improved this result in (Robutel, 1995), showing the applicability of Arnold's theorem in the case of spatial motion and eliminating the restriction to semimajor axes tending to zero. To this end, a normal form defined for any value of the semimajor axes' ratio is explicitely computed in order to overcome the proper degeneracy and the Hamiltonian is expanded using a suitable set of coordinates so to perform the reduction of the angular momentum. The perturbing function is computed according to (Laskar and Robutel, 1995) and the algebraic manipulator TRIP (developed by Laskar in (Laskar, 1990) for Celestial Mechanics purposes) was used to perform the computations. Concrete estimates on the thresholds of the masses or eccentricities are not discussed.

The main results obtained by Robutel (Robutel, 1995) can be summarized as follows:

Assume that the ratio α between the planetary semimajor axes satisfies $10^{-8} \leqslant \alpha \leqslant 0.8$, that $0.01 \leqslant (m_1/m_2) \leqslant 100$ and that the mutual inclination i between the planets satisfies $i < 1°$; then for sufficiently small planetary masses and eccentricities, Arnold's fundamental theorem can be applied.

This statement provides the existence of a large set of invariant tori of maximum dimension. We refer to (Robutel, 1995) for the details of the proof.

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Appendix: Interval arithmetic

The interval arithmetic technique is a rigorous method to control the computer round-off and propagation errors (Eckmann and Wittwer, 1985; Lanford, 1984).

Real numbers are represented by the computer with a sign-exponent-fraction representation, where the number of digits in the fraction and exponent varies with the machine. Since this number is fixed, the result of an "elementary" operation, i.e. sum, subtraction, multiplication and division, usually produces an approximation to the true result. The computer guarantees the result of an elementary operation up to a given precision (i.e., up to a certain decimal digit). The idea of the interval arithmetic is to start with an interval which certainly contains the exact result of an elementary operation and to perform the subsequent computations using subroutines for elementary operations between intervals. For example, let us consider the sum of $[a_1, b_1]$ and $[a_2, b_2]$; the result is the interval $[a_3, b_3] \equiv [a_1+a_2, b_1+b_2]$, such that

$$\text{if } x \in [a_1, b_1] \quad \text{and} \quad y \in [a_2, b_2] \quad \text{then } x+y \in [a_3, b_3].$$

However the end-points a_3, b_3 of the new interval are themselves produced by an elementary operation and therefore we need to consider the round-off error introduced in the computation of a_3 and b_3. The idea is to link the subroutines for operations on intervals to a procedure which provides strict lower and upper bounds on a_3 and b_3, respectively. To this end, we need to know the precision adopted by the computer we use. For example, on a VAX there are F, D, G, H-floating data, which differ in the length of the fractional and exponent representation.

Using G-floating precision, data run in the interval $0.56 \cdot 10^{-308}$ to $0.9 \cdot 10^{308}$ and their precision is guaranteed to about 15 decimal digits (*Vax Architecture Handbook*, p. 35), while the result of an elementary operation is guaranteed up to 1/2 of the least significant bit. In order to obtain strict upper and lower bounds on the result of an elementary operation, we increase or decrease by one unit the least significant bit of the mantissa, taking eventually control of the propagation of the carry.