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KAM Stability for a three-body problem of the Solar system[1]

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Abstract. A new (iso-energetic) KAM method is tested on a specific three-body problem "extracted" from the Solar system (Sun-Jupiter + asteroid 12 Victoria). Analytical results in agreement with the observed data are established. This paper is a concise presentation of [2].

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1. Introduction

The stability of the Solar System is among the most important questions in Celestial Mechanics. Its full solution involves understanding the dynamics of a large number of objects such as planets, satellites, comets and asteroids. Due to its intrinsic difficulty, the problem has been usually reduced to the study of models involving a smaller number of objects. In this direction, the simplest (non trivial) model of planetary motion involves the dynamics of three bodies. However, as shown by Poincaré, even the three-body model cannot be solved analytically. In this paper we analyze the stability of the so–called "restricted, circular, planar three-body problem". In particular, we consider the motion of an asteroid under the gravitational attraction of Sun and Jupiter. We assume that the three bodies move on the same plane, that the asteroid does not influence the motion of the primaries, and that the orbit of Jupiter is circular around the Sun–Jupiter barycenter. This problem can be conveniently described in terms of Delaunay action-angle variables. The corresponding Hamiltonian has two degrees of freedom and is nearly–integrable: the integrable part corresponds to the Keplerian motion of the asteroid around the Sun, while the perturbation is due to the gravitational influence of Jupiter. The perturbing parameter ε represents the Jupiter–Sun mass ratio and its astronomical observed value amounts to about 10^{-3}. The perturbing function can be expanded in Fourier–Taylor series and we shall retain only the

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most physically significant terms. The stability of the asteroidal motion can be obtained by fixing an energy level on which we prove the existence of two invariant tori, which trap the motion of the minor body in the phase space. The strategy we follow is based upon a new computer-assisted, iso-energetic KAM theory (which, as well known provides, under suitable general assumptions, the existence of invariant tori).

Before stating the results, we remark that standard applications of KAM theory usually lead to unrealistic estimates of the physical parameters.[2] In our application, we consider the motion of the asteroid 12 Victoria.[3] Exploiting a new iso-energetic KAM theory, we can prove the stability of the motion of the asteroid Victoria by constructing trapping invariant tori on the energy level corresponding to the osculating Keplerian motion. The existence of the trapping invariant surfaces is proved for values of the perturbing parameter less or equal than 10^{-3}. The solutions are proven to be close to the 12^{th} order truncation of (iso-energetic) Lindstedt series.

Due to the length of the computations involved, we make use of a computer, writing a Fortran program consisting of about 12000 lines. The numerical errors are controlled by means of the so–called interval arithmetic (see, e.g., [4], [6]).

2. Iso-energetic KAM theory

We consider a real-analytic Hamiltonian function $H = H(x, y)$, with $x \in \mathbb{T}^d := (\mathbb{R}/2\pi\mathbb{Z})^d$ and $y \in \mathbb{R}^d$ $(d \geq 2)$ being standard symplectic coordinates. Hamilton's equations can be written as

$$\dot{x} = H_y(x, y) \ , \qquad \dot{y} = -H_x(x, y) \ . \tag{2.1}$$

Let $\phi^t_H(x, y)$ be the flow generated by (2.1), representing the solution $x(t)$, $y(t)$ of (2.1) at time t with initial data given by $x(0) = x$ and $y(0) = y$. Given a Diophantine vector,[4] a *KAM torus with frequency* ω is an ϕ^t_H-invariant surface embedded in $\mathbb{T}^d \times \mathbb{R}^d$, which is parametrically described, for $\theta \in \mathbb{T}^d$, by

$$x(\theta) = \theta + \tilde{u}(\theta) \ , \qquad y(\theta) = \tilde{v}(\theta) \ , \qquad \det\Big(I + \tilde{u}_\theta(\theta)\Big) \neq 0 \ , \tag{2.2}$$

[2] In the case of the three-body problem, M. Hénon ([5]) showed that an application of the original version of Arnold's theorem allows to prove the existence of invariant tori for values of the perturbing parameter (corresponding to the planet-Sun mass ratio) less than or equal to 10^{-333}, while an aplication of Moser's theorem yields existence for perturbing parameters up to 10^{-50} (which is approximately the proton/Sun mass ratio). For later improvements, see [1] and references therein.

[3] The number 12 refers to the standard classification of asteroidal bodies as found, e.g., at `http : //ssd.jpl.nasa.gov/sb_elem.html`

[4] I.e. a vector $\omega \in \mathbb{R}^d$ such that $|\omega \cdot n| = |\sum_{j=1}^d \omega_j n_j| \geq \frac{\gamma}{|n|^\tau}$, $\forall\ n \in \mathbb{Z}^d \backslash \{0\}$ for some $\gamma > 0$ and $\tau \geq 1$.

with $\tilde{u}$ and $\tilde{v}$ real-analytic on $\mathbb{T}^d$, and satisfying

$$\phi_H^t(x(\theta), y(\theta)) = (x(\theta+\omega t), y(\theta+\omega t)) \ . \tag{2.3}$$

Inserting $(x(\theta), y(\theta))$ into Hamilton's equations (2.1) (by the rational independence of ω) one obtains a quasi-linear system of PDE's on $\mathbb{T}^d$, which, implemented with the normalization condition[5] $\tilde{u}(0) = 0$ and by the requirement that the torus belongs to the energy surface $H^{-1}(E)$, is given by:

$$\begin{aligned} \omega + D_\omega \tilde{u} - H_y(\theta+\tilde{u}, \tilde{v}) &= 0 \ , \\ D_\omega \tilde{v} + H_x(\theta+\tilde{u}, \tilde{v}) &= 0 \ , \\ \tilde{u}(0) &= 0 \ , \\ H(0, \tilde{v}(0)) - E &= 0 \ , \end{aligned} \tag{2.4}$$

where $D_\omega := \omega \cdot \partial_\theta := \sum_{j=1}^d \omega_j \frac{\partial}{\partial \theta_j}$. Viceversa, given a triple $(\tilde{u}, \tilde{v}, \omega)$ such that (2.4) and the inequality in (2.2) are satisfied, by means of (2.2), one obtains, on the energy level $H^{-1}(E)$, an invariant torus such that (2.3) holds. By a slight abuse of notation, we shall also refer to $(\tilde{u}, \tilde{v}, \omega)$ as a KAM torus with frequency ω.

In order to find solutions of (2.4), we need to define an *approximate KAM torus* which is given by a triple (u, v, ω), where u and v are real-analytic $\mathbb{R}^d$-valued functions on $\mathbb{T}^d$ such that the inequality in (2.2) holds, ω is a Diophantine vector and the normalizing condition $u(0) = 0$ is satisfied. Having fixed an energy level E, an approximate KAM torus satisfies the system of equations

$$\begin{aligned} \omega + D_\omega u - H_y(\theta+u, v) &= f \ , \\ D_\omega v + H_x(\theta+u, v) &= g \ , \\ u(0) &= 0 \ , \\ H(0, v(0)) - E &= h \ , \end{aligned} \tag{2.5}$$

where the "error functions" f, g and the "error number" h are *defined* by these equalities. Obviously, if f and g vanish, and if $h = 0$, the approximate KAM torus, then, corresponds to a KAM torus with frequency ω and energy E.

In the following proposition we show how to construct a new approximate KAM torus given by

$$u' := u + z \ , \qquad v' := v + w \ , \qquad \omega' := (1+a)\omega \ =: \omega_a \ ,$$

starting from an approximate KAM torus (u, v, ω) so that the new "errors" f', g' and h' are "quadratic" in f, g and h. Varying ω by a factor is necessary in order to meet the energy constraint.

Proposition 2.1. *Fix $E \in \mathbb{R}$ and let (u, v, ω) be an approximate KAM torus. Define*[6]

$$\mathcal{T} := \mathcal{M}^{-1} H^0_{yy} \mathcal{M}^{-T} \ , \qquad \mathcal{M} := I + u_\theta \ , \qquad \Big(H^0_{yy} := H_{yy}(\theta + u(\theta), v(\theta))\Big) \tag{2.6}$$

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[5] Such condition corresponds to fix the "origin" of the invariant torus in $\theta = 0$.

[6] In what follows, given an approximate KAM torus (u, v, ω), H^0 (or H^0_x, etc.) will be short for $H(\theta + u(\theta), v(\theta))$ (or for $H_x(\theta + u(\theta), v(\theta))$, etc.)

and, for $a \in \mathbb{R}\backslash\{-1\}$, define

$$\begin{aligned} f_a &:= (1+a)f + a\ H_y^0\ , \qquad g_a := (1+a)g - aH_x^0\ ,\\ b_a &:= v_\theta^T f_a - \mathcal{M}^T g_a\ , \qquad \mathcal{G}_a := f_{a,\theta}^T v_\theta + \mathcal{M}^T g_{a,\theta} - v_\theta^T f_{a,\theta} - g_{a,\theta}^T \mathcal{M}\ . \end{aligned}$$

Then, $\langle b_a \rangle = 0$ and $\langle \mathcal{G}_a \rangle = 0$, where $\langle\cdot\rangle$ denotes average over $\mathbb{T}^d$. Next, assume that there exist $c \in \mathbb{R}^d$ and $a \in \mathbb{R}\backslash\{-1\}$ such that[7]

$$\begin{cases} \langle \mathcal{T}\rangle c + \langle \mathcal{T} D_{\omega_a}^{-1} b_a\rangle - \langle \mathcal{M}^{-1} f_a\rangle\ = 0\ ,\\ H\big(0, v(0) + \mathcal{M}^{-T}(0)\ [c + D_{\omega_a}^{-1} b_a(0)]\big) = E\ , \end{cases} \tag{2.7}$$

where $\omega_a := (1+a)\omega$, $D_{\omega_a} := \omega_a \cdot \partial_\theta$. Define

$$\begin{aligned} \hat z &:= \mathcal{T}c + \mathcal{T}D_{\omega_a}^{-1} b_a - \mathcal{M}^{-1} f_a\ , \qquad \hat z_0 := -\big(D_{\omega_a}^{-1}\hat z\big)(0)\ ,\\ z &:= \mathcal{M} D_{\omega_a}^{-1}\hat z + \mathcal{M}\hat z_0\ , \qquad w := \mathcal{M}^{-T}(v_\theta^T z + c + D_{\omega_a}^{-1} b_a) \end{aligned}$$

and assume that $\det\big(\mathcal{M} + z_\theta\big) \neq 0$. *Then, the triple $\mathcal{K}(u,v,\omega) := (u',v',\omega') = (u+z, v+w, \omega_a)$ is an approximate KAM torus satisfying $H(0,v'(0)) - E = 0$. Furthermore, f' and g', defined by*

$$\begin{aligned} \omega_a + D_{\omega_a} u' - H_y(\theta + u', v') &= f'\ ,\\ D_{\omega_a} v' + H_x(\theta + u', v') &= g'\ , \end{aligned}$$

are quadratic in z, w, $f_{a,\theta}$ and $g_{a,\theta}$, as the following identities show:

$$\begin{aligned} f' &= -[H_y(\theta+u+z, v+w) - H_y^0 - H_{yx}^0 z - H_{yy}^0 w]\\ &\quad + f_{a,\theta}\mathcal{M}^{-1} z + H_{yy}^0 \mathcal{M}^{-T}(D_{\omega_a}^{-1}\mathcal{G}_a)\ \mathcal{M}^{-1} z\ ,\\ g' &= H_x(\theta+u+z, v+w) - H_x^0 - H_{xx}^0 z - H_{xy}^0 w\\ &\quad + \mathcal{M}^{-T} g_{a,\theta}^T z - \mathcal{M}^{-T} f_{a,\theta}^T w\\ &\quad + \mathcal{M}^{-T}\ v_\theta^T\Big(f_{a,\theta}\mathcal{M}^{-1} z + H_{yy}^0\mathcal{M}^{-T}(D_{\omega_a}^{-1}\mathcal{G}_a)\ \mathcal{M}^{-1} z\Big)\ . \end{aligned}$$

It is not difficult to show that a non-degenerate solution (c,a) of the system (2.7) will be proportional to f, g and h so that f' and g' will be quadratic in objects proportional to f, g and h. The map $\mathcal{K} : (u,v,\omega) \to (u',v',\omega')$ in Proposition 2.1 will be called the *KAM map*. The proof of Proposition 2.1 is just a check (see § 2.3 in [2]).

Assume, now, that H is real-analytic on[8] $\mathbb{T}^d_{\bar\xi} \times D^d_r(y_0)$ for some $y_0 \in \mathbb{R}^d$, $\bar\xi > 0$, $r > 0$. Let $E_{p,q}$ be such that $\|\partial_x^p \partial_y^q H\|_{\bar\xi,r} \le E_{p,q}$.

[7] If q is an analytic function on $\mathbb{T}^d$ with $\langle q\rangle = 0$ (and ω is a Diophantine vector), $D_\omega^{-1} q$ denotes the unique analytic function p on $\mathbb{T}^d$ such that $D_\omega p = q$.

[8] **Notations**: $|\cdot|$ =Euclidean norm; $\mathbb{T}^d_\xi := \{y \in \mathbb{C}^d\ :\ |\operatorname{Im} y| \le \xi,\ \ \operatorname{Re} y_i \mod 2\pi\}$; $D^d_r(y_0) := \{y \in \mathbb{C}^d\ :\ |y_i - y_{0i}| \le r,\ \forall i\}$; if f is analytic on $\mathbb{T}^d_\xi$, we set $\|f\|_\xi := \sum_{n\in\mathbb{Z}^d} |f_n| \exp(|n|\xi)$ where f_n are Fourier coefficients; if f is analytic on $\mathbb{T}^d_\xi \times D^d_r(y_0)$, we set $\|f\|_{\xi,r} := \sum_n |f_n|_r \exp(|n|\xi) := \sum_{n,k} |f_{nk}| r^{|k|_1} \exp(|n|\xi)$, where $|k|_1 := \sum_{j=1}^d |k_j|$ and f_{nk} are Taylor–Fourier coefficients (around y_0).

Let (u, v, ω) be an approximate torus as in (2.5), such that

$$\sup_{\mathbb{T}^d_\xi} |\operatorname{Im} u| \le \bar\xi - \xi \ , \qquad \hat r := \sup_{\mathbb{T}^d_\xi} |v(\theta) - y_0|_\infty < r \ , \qquad |\omega| \le \Omega \ , \tag{2.8}$$

for some $0 < \xi < \bar\xi$ and some $\Omega > 0$. Fix $\rho > 0$ and $1 < \kappa \le 2$. Let $\mathcal{M}$ and $\mathcal{T}$ be the matrices defined in (2.6) and assume that the $((d+1)\times(d+1))$-matrix defined as

$$\mathcal{A} := \begin{pmatrix} \langle \mathcal{T} \rangle & -\langle \chi \rangle \\ \chi(0)^T & 0 \end{pmatrix} \ , \quad \chi(\theta) = \chi(\theta;\rho) := \frac{1}{\rho} \mathcal{M}^{-1} H_y(\theta + u(\theta), v(\theta)) \ , \tag{2.9}$$

is invertible[9]. Let F, G, $\overline{h}$, M, $\overline{M}$, U, V, $\tilde V$, $\overline{A}$ be non-negative numbers such that

$$\begin{aligned} &\|f\|_\xi \le F \ , \quad \|g\|_\xi \le G \ , \quad |h| \le \bar h \ , \quad \|\mathcal{M}\|_\xi \le M \ , \quad \|\mathcal{M}^{-1}\|_\xi \le \overline{M} \ , \\ &\sup_{\mathbb{T}^d_\xi} |\operatorname{Im}\ u| \le U \ , \qquad \|v\|_\xi \le V \ , \qquad \|v_\theta\|_\xi \le \tilde V \ , \quad |\mathcal{A}^{-1}| \le \overline{A} \ , \end{aligned} \tag{2.10}$$

and define the following weighted norms:

$$\begin{aligned} &E_1^* := \max\Big\{E_{0,1}, \frac{E_{1,0}}{\rho}\Big\}, \qquad E_2^* := \max\Big\{E_{0,2}, \frac{E_{1,1}}{\rho}\Big\}, \\ &E_3^* := \max\Big\{E_{0,3}, \frac{E_{1,2}}{\rho}, \frac{E_{2,1}}{\rho^2}, \frac{E_{3,0}}{\rho^3}\Big\}, \quad E^* := \max\Big\{E_1^*, E_2^*\rho, E_3^*\rho^2\Big\}, \\ &\mu := \max\left\{\frac{F}{\Omega}, \frac{G}{\Omega\ \rho}, \frac{\overline{h}}{\Omega\ \rho}\right\}, \\ &\Omega^* := \max\Big\{\Omega\ , E^*\Big\} = \max\Big\{\Omega, E_1^*, E_2^*\rho, E_3^*\rho^2\Big\}, \\ &\beta_0 := \max\Big\{1, \frac{\tilde V}{\rho}\Big\}, \qquad \beta_1 := \frac{\Omega^*}{\gamma}, \qquad \alpha := \max\Big\{1, \frac{\overline{A}\Omega^*}{\rho}\Big\}. \end{aligned}$$

Then, iterating the KAM map $\mathcal{K}$, one can prove the following

Theorem 2.1. *Fix $0 < \xi_\infty < \xi$ and let $\xi_* := \min\Big\{1, \xi_\infty, \frac{\xi - \xi_\infty}{4}\Big\}$. Then, there exist constants $\hat c < c_*$ and c_{**} larger than one and depending upon d, τ, κ, such that the following holds. If μ is so small that*

$$c_*\ \overline{M}^{10} M^4\ \xi_*^{-(4\tau+1)}\ \frac{\Omega^*}{\Omega}\alpha^2\beta_0^4\beta_1^4\ \mu \le 1 \ , \quad c_{**}\ \overline{M}^5 M^2\ \xi_*^{-2\tau}\ \alpha\beta_0^2\beta_1^2\ \frac{\rho}{r - \hat r}\ \mu \le 1 \ ,$$

then there exists a (unique) constant $\tilde a \in (-1, 1)$ and (locally unique) real-analytic functions $\tilde u$ and $\tilde v$ satisfying (2.4) with ω replaced by $(1 + \tilde a)\omega$ and

$$\sup_{\mathbb{T}^d_{\xi_\infty}} |\operatorname{Im} \tilde u| < \bar\xi - \xi_\infty \ , \qquad \sup_{\mathbb{T}^d_{\xi_\infty}} |\tilde v(\theta) - y_0|_\infty < r \ .$$

[9] Under such condition, by the IFT, the system (2.7) admits a nondegenerate solution (c, a) proportional to f, g and h.

Furthermore, $|\tilde{a}|$, $\|u-\tilde{u}\|_{\xi_\infty}$ *and* $\|v-\tilde{v}\|_{\xi_\infty}$ *are small with* μ, *i.e.,*

$$\max\Big\{|\tilde{a}|\ ,\ \|\tilde{u}-u\|_{\xi_\infty}\ ,\ \|\tilde{u}_\theta-u_\theta\|_{\xi_\infty}\ ,\ \rho^{-1}\|\tilde{v}-v\|_{\xi_\infty}\ ,\ \rho^{-1}\|\tilde{v}_\theta-v_\theta\|_{\xi_\infty}\Big\} \leq \Big(\hat{c}\ \overline{M}^5 M^2 \xi_*^{-(2\tau+1)}\ \alpha\beta_0^2\beta_1^2\Big)\ \mu\ .$$

The constants $\hat{c}$, c_* and c_{**} can be computed explicitely: for example, in the case $d=2$, $\tau=1$, $\kappa=1.01$, one can take $\hat{c}=111.7$, $c_*=38528.282$ and $c_{**}=49.088$.

3. The restricted, circular, planar three-body problem

Let P_0, P_1, P_2 be three bodies ("point masses") with masses m_0, m_1, m_2, respectively, subject only to the mutual gravitational attraction. Let $u^{(i)}\in\mathbb{R}^3$, $i=0,1,2$, denote the positions of the bodies in an inertial reference frame (without loss of generality, we normalize the gravitational constant to one). Then, Newton equations take the form

$$\frac{d^2u^{(i)}}{dt^2} = -\sum_{\substack{0\leq j\leq 2\\ j\neq i}} m_j \frac{u^{(i)}-u^{(j)}}{|u^{(i)}-u^{(j)}|^3}\ ,\qquad i=0,1,2\ , \tag{3.11}$$

The *restricted three-body problem (with "primary bodies"* P_0 *and* P_1*) is*, by definition, *the problem of studying the bounded motions of the system* (3.11) *with* $m_2=0$. In the restricted three-body problem, the motion of P_0-P_1 becomes an independent (integrable) two-body problem. We assume that such a two-body system revolves on Keplerian circles (*circular* case). Finally, we assume that the position and velocity at a given time (and, hence, for *all* times) of the third body P_2 lie on the P_0-P_1 plane (*planar* case).

If the mass ratio m_1/m_0 is small (as it happens when P_0 is a star and P_1 a planet), then the restricted, circular, planar three-body problem (RCPTBP, for short) may be viewed as a perturbation of the integrable system associated to P_0 and P_1 and, in suitable phase regions, Delaunay action-angle variables may be conveniently used (see, e.g., §3 of [2]).

As a model for a restricted, circular, planar three-body problem, we consider Sun-Jupiter and the asteroid 12 Victoria, a minor body in the asteroidal belt, whose observed osculating data are:[10] $a_{\rm V}\simeq 0.449$, $e_{\rm V}\simeq 0.220$, $\imath_{\rm V}\simeq 1.961\ 10^{-2}$. The size of the (normalized) perturbation parameter ε is given by

$$\varepsilon_{\rm J} := \frac{m_{\rm J}/(m_{\rm S}+m_{\rm J})}{(m_{\rm S}/(m_{\rm S}+m_{\rm J}))^{2/3}} \simeq 0.954\cdot 10^{-3}\ .$$

[10] $a_{\rm V}$ denotes the ratio between the observed semi-major axis of Victoria and that of Jupiter; $e_{\rm V}$ is the observed eccentricity of the osculating ellipse of Victoria and $\imath_{\rm V}$ is the relative inclination of the observed orbital planes of Victoria and Jupiter measured in degrees and normalized to one.

Using Delaunay action-angle variables, one sees that the system $(3.11)|_{m_2=0}$ can be described (in suitable units) by action-angle variables[11] $0 < G < L$, $(g, \ell) \in \mathbb{T}^2$ (with respect to the standard form $d\ell \wedge dL + dg \wedge dG$). Truncating the perturbation in a suitable (physically motivated) way, one is led to the following one-parameter family of Hamiltonians modelling, for $\varepsilon = \varepsilon_{\rm J}$, *the Sun-Jupiter-Victoria system regarded as a RCPTBP*:

$$H_{\rm SJV,\,\varepsilon}(\ell, g, L, G) := -\frac{1}{2L^2} - G - \varepsilon P_{\rm SJV}(\ell, g, L, G) =: H_0(L, G) + \varepsilon H_1(\ell, g, L, G) ,$$

with $(\ell, g) \in \mathbb{T}^2$, $0 < G < L$ and with

$$\begin{aligned}
P_{\rm SJV}(\ell, g, L, G) &:= 1 + \frac{a^2}{4} + \frac{9}{64}\, a^4 + \frac{3}{8}\, a^2 e^2 \\
&-\Big(\frac{1}{2} + \frac{9}{16} a^2\Big)\, a^2 e\, \cos\ell + \Big(\frac{3}{8}\, a^3 + \frac{15}{64} a^5\Big) \cos(\ell + g) \\
&-\Big(\frac{9}{4} + \frac{5}{4} a^2\Big)\, a^2 e\, \cos(\ell + 2g) + \Big(\frac{3}{4}\, a^2 + \frac{5}{16}\, a^4\Big) \cos(2\,\ell + 2\,g) \\
&+\frac{3}{4}\, a^2 e\, \cos(3\,\ell + 2\,g) + \Big(\frac{5}{8}\, a^3 + \frac{35}{128}\, a^5\Big) \cos(3\,\ell + 3\,g) \\
&+\frac{35}{64}\, a^4\, \cos(4\,\ell + 4\,g) + \frac{63}{128}\, a^5\, \cos(5\,\ell + 5\,g) ,
\end{aligned}$$

where $a = L^2$, $e = \sqrt{1 - \frac{G^2}{L^2}}$. For a numerical investigation of the validity of such model, see [3].

We now select the region of the phase space associated to $H_{\rm SJV,\,\varepsilon}$, which can be considered more interesting from an astronomical point of view. From the observed osculating elements of Victoria, we compute the associated "observed" value for the action variable $L_{\rm V} = 0.670$ and by the relation $e = \sqrt{1 - \frac{G^2}{L^2}}$ we compute the corresponding (approximated) value for the action variable $G_{\rm V} = 0.654$. Since the observed astronomical data are provided in terms of osculating Keplerian ellipses, it seems reasonable to define the "osculating energy value" in terms of the Keplerian approximation. However, since the "secular" effects[12] are also important, we take them into account while defining the osculating value of the energy. We therefore let

$$\begin{aligned}
E^{(0)}_{\rm V} &:= H_0(L_{\rm V}, G_{\rm V}) = -\frac{1}{2L^2_{\rm V}} - G_{\rm V} \simeq -1.768 , \\
E^{(1)}_{\rm V} &:= \big\langle H_1(\cdot, L_{\rm V}, G_{\rm V})\big\rangle \\
&\quad = -\Big(1 + \frac{L^4_{\rm V}}{4} + \frac{9}{64}\, L^8_{\rm V} + \frac{3}{8}\, (L^4_{\rm V} - G^2_{\rm V} L^2_{\rm V})\Big) \simeq -1.060 , \\
E_{\rm V}(\varepsilon) &:= E^{(0)}_{\rm V} + \varepsilon E^{(1)}_{\rm V} ,
\end{aligned}$$

[11] Physically, ℓ is the mean-anomaly; $g = \gamma - \psi$, where γ is the argument of the perihelion and ψ is the longitude of Jupiter; $L = \sqrt{a}$ and $G = L\sqrt{1 - e^2}$, where a and e are, respectively, the semi-major axis and the eccentricity of the osculating Sun-Asteroid ellipse.

[12] Roughly speaking, the effects of the averaged perturbing Hamiltonian.

and define the osculating energy level of the Sun-Jupiter-Victoria model as

$$E_{\rm V}(\varepsilon_{\rm J}) = E_{\rm V}^{(0)} + \varepsilon_{\rm J} E_{\rm V}^{(1)} \simeq -1.769 \ .$$

Let $\mathcal{S}_{\varepsilon,{\rm V}} := H^{-1}_{{\rm SJV},\,\varepsilon}\big(E_{\rm V}(\varepsilon)\big)$ and consider the two tori on $\mathcal{S}_{0,{\rm V}}$ having (unperturbed) motion frequencies

$$\tilde\omega_\pm := \frac{\partial H_0}{\partial(L,G)} = \Big(\frac{1}{\tilde L_\pm^3}, -1\Big) =: (\tilde\alpha_\pm, -1) \ ,$$

where $\tilde L_\pm = L_V \pm 0.001$. Such tori trap the osculating torus $(L_{\rm V}, G_{\rm V}) \times \mathbb{T}^2$. In order to apply KAM theory, we need, however, Diophantine frequencies. We, therefore, compute the continued fraction representation of $\tilde\alpha_\pm$ up to order 5 and we modify the frequencies by adding a tail of all one's. This procedure leads to the quadratic "noble" numbers $\alpha_\pm$ given by:

$$\alpha_- := [3; 3, 4, 2, 1^\infty] = 3.30976... \ , \qquad \alpha_+ := [3; 2, 1, 17, 5, 1^\infty] = 3.33955...$$

The frequencies $\omega_\pm := (\alpha_\pm, -1)$ are Diophantine vectors with Diophantine constants (see, e.g., Appendix B in [2]):

$$\tau_\pm := \tau = 1 \ , \qquad \gamma_- := 7.224496 \cdot 10^{-3} \ , \qquad \gamma_+ := 3.324329 \cdot 10^{-2} \ .$$

Let, now, $\mathcal{T}_0^\pm := (L_\pm, G_\pm) \times \mathbb{T}^2 \subset H_0^{-1}(E_V^{(0)})$, where $L_\pm := \frac{1}{\alpha_\pm^{1/3}}$ and $G_\pm := -\frac{1}{2L_\pm^2} - E_V^{(0)}$. Then, we can prove the following stability result.

Theorem 3.1 ([2]). *The tori $\mathcal{T}_0^\pm$ can be analytically continued for $|\varepsilon| \le 10^{-3}$ into invariant tori $\mathcal{T}_\varepsilon^\pm$ on the energy level $\mathcal{S}_{\varepsilon,{\rm V}} = H^{-1}_{{\rm SJV},\,\varepsilon}(E_{\rm V})$ keeping fixed the ratio of the frequencies. Since $H_{{\rm SJV},\,\varepsilon_{\rm J}}$ is a two-degree-of-freedom, iso-energetically non-degenerate Hamiltonian, the tori $\mathcal{T}_{\varepsilon_{\rm J}}^+$ and $\mathcal{T}_{\varepsilon_{\rm J}}^-$ are the boundary of a $\phi^t_{H_{{\rm SJV},\,\varepsilon_{\rm J}}}$-invariant region $\mathcal{J}_{\varepsilon_{\rm J}}$; such region contains the surface $(L_{\rm V}, G_{\rm V}) \times \mathbb{T}^2$, showing, in particular, that the motions*

$$(\ell(t), g(t), L(t), G(t)) := \phi^t_{H_{{\rm SJV},\,\varepsilon_{\rm J}}}(\ell_0, g_0, L_{\rm V}, G_{\rm V}) \tag{3.12}$$

belong for any $t \in \mathbb{R}$ and any $(\ell_0, g_0) \in \mathbb{T}^2$ to the region $\mathcal{J}_{\varepsilon_{\rm J}}$. As a corollary, the values of the perturbed integrals $L(t)$ and $G(t)$ stay close to their initial values $L_{\rm V}$ and $G_{\rm V}$ forever and the actual motion (in the mathematical model) is nearly elliptical with osculating orbital values close to the observed ones.

The proof of this theorem is computer-assisted and is based on the following steps: (1) One finds "starting" approximate tori computing suitable truncations of the Lindstedt iso-energetic series; (2) the parameter values in (2.10) for such approximate tori are evaluated; (3) the KAM map described in Proposition 2.1 is iterated a few times and the norms of the quantities in (2.10) are evaluated; (4) Theorem 2.1 is applied.

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