Journal of Applied Mathematics and Physics (ZAMP)
Vol. 42, January 1991

0044-2275/91/010080-10 \$ 1.50 + 0.20

On a result by J. N. Mather concerning invariant sets for area-preserving twist maps

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1. Introduction

Let us consider a special class of area-preserving homeomorphisms of the cylinder $\mathscr{C} \equiv S^1 \times \mathbb{R}$ ($S^1 \equiv \mathbb{R}/\mathbb{Z}$), namely,

$$f \equiv f_\lambda : (x, y) \in S^1 \times \mathbb{R} \to (x + y + \lambda v(x), y + \lambda v(x)) \in S^1 \times \mathbb{R} \tag{1.1}$$

where v is a continuous function of period 1 and average 0 and λ is a positive parameter. Such maps are examples of so-called area-preserving, monotone twist mappings which have been extensively studied by, e.g., Poincaré, Birkhoff, Moser, Herman, Aubry and Mather (see [13], [6] and [2] for reviews).

For $\lambda = 0$, f_0 is integrable, i.e., the orbits stay on invariant circles $S^1 \times \{y\}$ and

$$f_0^n(x, y) \equiv \underbrace{f_0 \circ \cdots \circ f_0}_{n\text{-times}}(x, y) = (x + ny, y) \in S^1 \times \{y\}.$$

In general, setting $f^n(x_0, y_0) \equiv (x_n, y_n)$ and observing that $y_n = x_n - x_{n-1}$, one sees that the dynamics of f can be described by the following nonlinear finite-difference equation

$$x_{n+1} - 2x_n + x_{n-1} = \lambda v(x_n).$$

As soon as $\lambda \neq 0$ and v is non trivial such dynamics becomes extremely rich and complicated and one would like to understand the structure of the invariant sets for f_λ. Examples of interesting invariant sets are periodic orbits, invariant circles and Mather (or Aubry-Mather) sets. Here by "circle" we shall always mean "homotopically nontrivial embeddings of S^1 into $\mathscr{C}$" and Mather sets are, roughly speaking (see below for a precise definition), Cantor sets semi-conjugate to rotations.

* This work has been completed while the author was a guest of the ETH, Theoretische Physik, Zürich.

In this note, we extend a result by J. N. Mather [9] and prove a simple and quantitative relation between the location of Mather sets and "relative deviations" of v. As a byproduct we get some new non-existence criteria which involve a condition on the size of v rather than on its derivatives (for non-existence methods based on different ideas see [10], [7] and [4]).

In the next paragraph we recall a few basic facts from Mather's theory. In §3 we formulate our results and in §4 and in the Appendix we prove them.

2. Mather sets

In this section we review some well known aspects of Mather's theory (cf. [8], [9], [11]). Most of the following results were obtained independently and with different, although not unrelated, methods by Aubry and collaborators (see [1] and references therein).

To any area-preserving (right-) twist map it can be associated a *generating function*, namely, a $C^1(\mathbb{R}^2)$ function h such that $h(x+1, x'+1) = h(x, x')$ and

$$(y', x') = f(x, y) \Leftrightarrow y = -h_x(x, x'), y' = h_{x'}(x, x'). \tag{2.1}$$

For f_λ the generating function is given by

$$h(x, x') = \tfrac{1}{2}(x - x')^2 + \lambda V(x), \quad V' = v.$$

For a fixed number ω, Percival [15] introduced the Lagrangian

$$F_\omega(\phi) \equiv \int_0^1 h(\phi, \phi^+)\, dt = \int_0^1 [\tfrac{1}{2}(\phi - \phi^+)^2 + \lambda V(\phi)]\, dt$$

for continuous real functions ϕ such that $\phi(t+1) = \phi(t) + 1$; $\phi^\pm$ denotes the translation of ϕ by $\pm\omega$: $\phi^\pm(t) \equiv \phi(t \pm \omega)$. A formal variation of F_ω yields the following Euler-Lagrange equation

$$E_\phi \equiv h_x(\phi, \phi^+) + h_{x'}(\phi^-, \phi) = -(\phi^+ - 2\phi + \phi^-) + \lambda v(\phi) = 0 \tag{2.2}$$

so that [cf. (2.1)] to any solution of (2.2) corresponds an invariant set

$$\mathcal{M}_\phi \equiv \pi_{\mathscr{C}} \{(\phi, \phi^- - \phi) | t \in \mathbb{R}\}, \tag{2.3}$$

where $\pi_{\mathscr{C}}$ denotes the projection of $\mathbb{R}^2$ onto $\mathscr{C}$. The dynamics on $\mathcal{M}_\phi$ is simply a translation by ω: $f \circ \pi_{\mathscr{C}}(\phi, \phi^- - \phi) = \pi_{\mathscr{C}}(\phi^+, \phi - \phi^+)$. In particular if ϕ is strictly increasing one gets *invariant circles* conjugate to a translation by ω. To actually prove the existence of invariant circles is, in general, possible only under stringent assumptions on ω, λ and on the smoothness of v and proofs are hard (cf. [12], [4, 5], [3]). On the other hand, Mather ([8], [11]) gave a simple proof of the existence of solutions of

(2.2) in the class $Y \equiv \{\phi\colon \mathbb{R} \to \mathbb{R}$ s.t. ϕ is not decreasing, left continuous, i.e., $\phi(t-) = \phi(t)$ and $\phi(t+1) = \phi(t) + 1\}$:

Theorem. (Mather) For any $\omega \in [0, 1]$ there exists a solution $\phi_\omega \in Y$ of (2.2). Moreover, ϕ_ω is strictly increasing whenever ω is irrational.

Remark 1. Originally, Mather considered homeomorphisms of the annulus $S^1 \times [0, 1]$ leaving the boundary circles invariant and later, ([11], §5) adapted his methods to the (actually simpler) case of the cylinder.

Remark 2. In our case f could be also viewed as a homeomorphism of the torus $\mathbb{T}^2 \equiv S^1 \times S^1$. With abuse of language, we shall denote by the same symbol f the lift to $\mathbb{R}^2$ of (1.1) defined by the same formula in (1.1) with $S^1 \times \mathbb{R}$ replaced by $\mathbb{R}^2$.

The above theorem is based on a non-standard variational approach, which we briefly review. One can define a distance on Y by letting $G(\phi)$ denote the "graph" of $\phi \in Y$:

$$G(\phi) \equiv \{(t, x) \in \mathbb{R}^2\colon \phi(t) \le x \le \phi(t+)\}$$

and, for $\phi, \psi \in Y$, letting

$$d(\phi, \psi) \equiv \max\{\sup_{z \in G(\psi)} \inf_{w \in G(\phi)} |z - w|, \sup_{z \in G(\phi)} \inf_{w \in G(\psi)} |z - w|\}.$$

Then one can compactify Y by taking the quotient with respect to translations: $X \equiv Y/\sim$ where $\phi \sim \psi$ if $\exists a \in \mathbb{R}$ s.t. $\phi = \psi T_a$ $[\psi T_a(t) \equiv \psi(t + a)]$. The metric d projects naturally on X by setting for $\phi, \psi \in X$

$$d(\bar{\phi}, \bar{\psi}) \equiv \inf\{d(\phi, \psi)\colon \phi \in \bar{\phi}, \psi \in \bar{\psi}\}$$

and with respect to such a metric X is compact. Then one can easily show that F_ω, which being translation invariant is well defined on X, is continuous on X and therefore has a minimum and a maximum. Finally one can show that the *minimum* verifies the Euler-Lagrange equation (2.2). [Again there is a slight abuse of language here since E_ϕ in (2.2) is defined for functions; however it is clear that $E_\phi(t) \equiv 0$ if and only if $E_\phi T_a(t) \equiv 0$ $\forall a \in \mathbb{R}$ therefore it makes sense to talk about solutions of (2.2) in X.] Such a verification (which does not work for the maximum) is the delicate part of the proof of the above theorem (see also the Appendix below).

Remark 3. Actually in [8] the generating function is taken with opposite sign and therefore F_ω is there maximized rather than minimized. Also the original compactification in [8] is slightly different from that defined above, which is the one used in [9].

Now, it is not difficult to see that if one defines $\mathcal{M} \equiv \mathcal{M}_{\phi_\omega}$ as in (2.3) (with ϕ replaced by ϕ_ω) then $\mathcal{M}$ verifies the following four conditions (cf. §2 of [9]):

(i) $\mathcal{M}$ is minimal for f ("minimal" means that $\mathcal{M}$ is closed, invariant and transitive, i.e., every orbit is dense in $\mathcal{M}$);
(ii) The projection $\pi_{S^1}: \mathscr{C} \to S^1$ is one-to-one on $\mathcal{M}$;
(iii) f (considered on $\mathbb{R}^2$) preserves the $\mathbb{R}$-order on $\pi_{\mathscr{C}}^{-1}(\mathcal{M})$;
(iv) $\lim_{n\to\pm\infty} \pi_{S^1}(f^n(x, y))/n = \omega, \forall (x, y) \in \mathcal{M}$.

A set $\mathcal{M}$ satisfying these conditions will be called a *Mather set* with rotation number ω. In fact Mather proved [9, Proposition at pag. 469] that the *correspondence between Mather sets and solutions of* (2.2) *is one-to-one*.

Remark. As defined in (iv), ω is an element of S^1, however we shall associate to ω the unique real number in (0, 1] in the equivalence class ω and, with the same abuse of language of Remark 2, we shall denote it by the same symbol.

Let us now recall what the situation is in the presence of an invariant circle Γ. By a theorem of Birkhoff (see, e.g., [4]) Γ is the graph of a Lipschitz continuous function $\gamma, \Gamma = \{(x, \gamma(x)) | x \in S^1\}$, therefore f induces a circle homeomorphism defined by

$$\bar{f}: x \in S^1 \to \pi_{S^1} f(x, \gamma(x)),$$

so that the classical theory of Poincaré and Denjoy (see, e.g., [14]) applies. In particular the rotation number

$$\omega = \lim_{n\to\pm\infty} \bar{f}(x)/n \equiv \lim_{n\to\pm\infty} \pi_{S^1}(f^n(x, \gamma(x)))/n$$

exists and is independent of x. Moreover, if ω is irrational and if $\mathcal{K}$ denotes the set of accumulation points of any orbit starting on Γ, *then* $\mathcal{K}$ *is either the whole circle or is the unique minimal set in* Γ *in which case is a Cantor set*. In [11, Proposition 4, pag. 514] it is shown that if ω is irrational, if there exists an invariant circle Γ_ω with rotation number ω and if $\mathcal{M}_\omega$ is a Mather set with the same rotation number then *either* $\mathcal{M}_\omega \equiv \Gamma_\omega$ *or* $\mathcal{M}_\omega$ *is the unique minimal set in* Γ_ω.

3. Results

To state our results we are going to use the following definition of *deviation* for a periodic continuous function with 0 average.

Definition. An interval (ξ_1, ξ_2) will be called a deviation (from 0) if ξ_1 is a relative minimum for v with $v(\xi_1) < 0$ and ξ_2 is a relative maximum with $v(\xi_2) > 0$. The positive number $\sigma \equiv \min\{-v(\xi_1), v(\xi_2)\}$ will be called the (relative) size of the deviation (ξ_1, ξ_2).

Remarks. i) The results presented below *do not* hold for the case of "reverse" deviations, i.e., intervals (x_1, x_2) where $v(x_1) > 0 > v(x_2)$.

ii) By possibly translating v we can always assume that there is no deviation containing integer numbers. In what follows, a deviation will be always taken to be in $(0, 1]$.

iii) There is always a *maximal deviation* (i.e. a deviation with maximal size) given by $\xi_1: v(\xi_1) = \min_{S^1} v$ and $\xi_2: v(\xi_2) = \max_{S^1} v$.

Proposition 1. Let (ξ_1, ξ_2) be a deviation of size σ for v. Then if $\lambda > \sigma^{-1}$, for any ω there exists a Mather set $\mathscr{M}_\omega$ such that $\pi_{S^1}(\mathscr{M}_\omega) \subset (\xi_1, \xi_2)/\mathbb{Z}$.

Corollary 1. Let $M = \max_{S^1} v$, $m = \min_{S^1} v$. Then if $\lambda > \max\{1/M, -1/m\}$ there does not exist any invariant transitive circle for f_λ with irrational rotation number.

Corollary 2. Assume that v has two disjoint deviations of sizes σ_1, σ_2. Then if $\lambda > \max\{1/\sigma_1, 1/\sigma_2\}$ there does not exist any invariant circle for f_λ with irrational rotation number.

Proposition 1 can be extended to the case of many deviations:

Proposition 2. Assume that v has $n \geq 2$ disjoint deviations, $(\xi_1^{(i)}, \xi_2^{(i)})$, with relative sizes $\sigma_1, \ldots, \sigma_n$. Then if $\lambda > \max_k \sigma_k^{-1}$, for any ω there exists a $(n-1)$-parameter family of Mather sets $\mathscr{M}_\omega^a$, $a \equiv (a_1, \ldots, a_{n-1})$, $0 < a_1 < a_2 < \cdots < a_{n-1} < 1$ such that $\pi_{S^1}(\mathscr{M}_\omega^a) \subset \bigcup_{i=1}^n (\xi_1^{(i)}, \xi_2^{(i)})/\mathbb{Z}$ and $\pi_{S^1}(\mathscr{M}_\omega^a) \cap (\xi_1^{(i)}, \xi_2^{(i)})/\mathbb{Z} \neq \emptyset \; \forall i$.

4. Proofs

Corollaries 1 and 2 follow immediately from Propositions 1 and the uniqueness properties of Mather sets in the presence of circles with the same (irrational) rotation numbers (cfr. end of §2). □

Proof of Proposition 1. Following Mather, we let $Y_0 \equiv \{\phi \in Y: \phi((0, 1]) \subset [\xi_1, \xi_2]\}$ and let $X_0 \equiv \pi(Y_0)$, where π denotes the projection onto X. X_0 is a closed subset of X, therefore F_ω has a minimum on it and we have to show that such a minimum satisfies the Euler-Lagrange equation (2.2). To do this we need the following

Lemma. If $\lambda > \sigma^{-1}$ then there exists a $\delta > 0$ and $\phi_\delta \in Y_0$ minimizing F_ω on Y_0 such that $\phi_\delta((0, 1]) \subset [\xi_1 + \delta, \xi_2 - \delta]$.

Remark. Notice that π is a bijection of Y_0 onto X_0, therefore we shall work directly in Y_0 rather than in X_0.

Proof of the Lemma. Let ϕ denote a minimizing element of F_ω in Y_0; we will show that we can find $\phi_\delta \in Y_0$ such that $\phi_\delta((0, 1]) \subset [\xi_1 + \delta, \xi_2 - \delta]$ *and*

$$F_\omega(\phi_\delta) \le F_\omega(\phi) \tag{4.1}$$

so that also ϕ_δ is a minimum for F_0 on Y_0. To show (4.1) we will use the following general inequality: Let $\psi_1, \psi_2 \in Y$ then

$$\left| \int_0^1 [(\psi_1^+ - \psi_1)^2 - (\psi_2^+ - \psi_2)^2]\, dt \right| \le 2 \int_0^1 |\psi_1 - \psi_2|\, dt. \tag{4.2}$$

Remark. This inequality cannot be improved since one gets *equality* by letting $\omega = 1/2$ and $\psi_k \equiv 0$ for $0 < t \le 1/2$ and $\psi_k \equiv k/3$ for $1/2 < t \le 1$. Notice that because of the property $\psi(t+1) = \psi(t) + 1$ it is enough to define elements of Y only in $(0, 1]$.

To check (4.2) let $p \equiv \psi_1^+ - \psi_1 + \psi_2^+ - \psi_2$ and notice that p is periodic and satisfies $0 \le p \le 2$. Then

$$\begin{aligned}
\left| \int_0^1 [(\psi_1^+ - \psi_1)^2 - (\psi_2^+ - \psi_2)^2]\, dt \right| &= \left| \int_0^1 p[(\psi_1^+ - \psi_2^+) - (\psi_1 - \psi_2)]\, dt \right| \\
&= \left| \int_0^1 p(\psi_1^+ - \psi_2^+)\, dt \right. \\
&\qquad \left. - \int_0^1 p(\psi_1 - \psi_2)\, dt \right| \\
&= \left| \int_0^1 (p^- - p)(\psi_1 - \psi_2)\, dt \right| \\
&\le 2 \int_0^1 |\psi_1 - \psi_2|\, dt.
\end{aligned}$$

Now, since $\lambda > \sigma^{-1}$, there exists an $\varepsilon > 0$ such that $\lambda \ge 1/(\sigma - \varepsilon)$; moreover, recalling the definition of σ and the fact that v is the derivative of V, we can pick a δ so that

$$V(x) - V(x') \ge (\sigma - \varepsilon)|x - x'| \tag{4.3}$$

whenever either $\xi_1 \le x \le x' \le \xi_1 + \delta$ or $\xi_2 - \delta \le x' \le x \le \xi_2$. Thus, defining (in (0, 1])

$$\phi_\delta \equiv \begin{cases} \xi_1 + \delta & \text{if } t\colon \xi_1 \le \phi(t) \le \xi_1 + \delta \\ \xi_2 - \delta & \text{if } t\colon \xi_2 - \delta \le \phi(t) \le \xi_2 \\ \phi & \text{otherwise (for } t \text{ in } (0, 1]) \end{cases}$$

(and extending, as above, such definition to all of $\mathbb{R}$) one has that $\phi_\delta((0, 1]) \subset [\xi_1 + \delta, \xi_2 - \delta]$ (hence $\phi_\delta \in Y_0$) and, by (4.2) and (4.3),

$$\begin{aligned} \int_0^1 [(\phi_\delta^+ - \phi_\delta)^2 - (\phi^+ - \phi)^2]\, dt &\le 2 \int_0^1 |\phi_\delta - \phi|\, dt \\ &\le \frac{2}{\sigma - \varepsilon} \int_0^1 [V(\phi) - V(\phi_\delta)]\, dt \\ &\le 2\lambda \int_0^1 [V(\phi) - V(\phi_\delta)]\, dt, \end{aligned}$$

proving (4.1) and finishing the proof of the Lemma. □

Proof of Proposition 1. It remains to prove that the above minimizing element ϕ_δ satisfies the Euler-Lagrange equation (2.2). The argument is given by Mather in [9] (see Proof of Lemma 1 at pag. 472 of [9] and recall Remark 1 and 3 above to adapt Mather's argument to our case). However, for completeness and convenience of the reader we include the detailed argument in the Appendix. □

Proof of Proposition 2. For a vector a as in Proposition 2, introduce the space

$$Y_a \equiv \{\phi \in Y\colon \phi((a_{i-1}, a_i]) \subset [\xi_1^{(i)}, \xi_2^{(i)}],\ i = 1, \ldots, n\}$$

and define $X_a \equiv \pi(Y_a)$. As above, one easily checks that X_a is a closed subset of X, so that F_ω will take a minimum value in X_a and therefore in Y_a for some $\phi_a \in Y_a$. Again as above, one then shows that, since $\lambda > \max_k \sigma_k^{-1}$, there exists a δ and a minimizing element $\phi_\delta \in Y_a$ such that $\phi_\delta((a_{i-1}, a_i]) \subset [\xi_1^{(i)} + \delta, \xi_2^{(i)} - \delta]$. The rest of the argument follows the above pattern and is omitted. □

Appendix

Here, following ideas of Mather, we show that the minimizing element ϕ_δ of Lemma 1 satisfies the Euler-Lagrange equation (2.2).

Since (dropping the subscript δ) ϕ has bounded variation in [0, 1], it has at most a countable number of discontinuities. Therefore, to prove (2.2) it is enough to prove it at points t_0 such that t_0, $t_0 \pm \omega$ are of continuity for ϕ. In order to take variations of F_ω one needs a family of curves in Y_0 rich enough to get the Euler-Lagrange equation out of the variations. Let us introduce the following families of curves. Let $\varrho \in C_0^\infty([0, 1])$ ($\equiv$ the class of C^∞ functions with support in [0, 1]) and extend it periodically to $\mathbb{R}$. Now, let

$$\psi_s \equiv \psi_s^{(\varrho)} \equiv \phi + s\varrho(\phi).$$

Then, if we assume that

$$|s| \leq \min\left\{\frac{\delta}{\|\varrho\|_\infty}, \frac{1}{2\|\varrho'\|_\infty}\right\} \tag{A.1}$$

it follows that $\psi_s \in Y_0$. In fact, clearly ψ_s is left continuous and satisfies $\psi_s(t+1) = \psi_s(t) + 1$; furthermore, $|s| \leq 1/(2\|\varrho'\|_\infty)$ implies that, $\forall t' > t$, one has

$$\psi_s(t') - \psi_s(t) \geq \tfrac{1}{2}[\phi(t') - \phi(t)]$$

hence $\psi_s \in Y_0$. Finally, from $\phi((0, 1]) \subset [\xi_1 + \delta, \xi_2 - \delta]$ and $|s| \leq \delta/\|\varrho\|_\infty$ it follows that $\psi_s \in Y_0$. The curve ψ_s classify for variations and in fact it is easy to see that [recalling the definition of E_ϕ in (2.2)]

$$\left.\frac{d}{ds}\right|_{s=0} F_\omega(\psi_s) = \int_0^1 E_\phi(t)\varrho(\phi)\, dt.$$

Now, let t_0 be as above (i.e. ϕ is continuous at t_0, $t_0 \pm \omega$). There are two cases: either (i) $\phi^{-1}\phi(t_0) = \{t_0\}$, i.e., t_0 is a point of increase for ϕ, or (ii) $\phi^{-1}\phi(t_0)$ is an interval with ends $\alpha < \beta$: such an interval is closed if ϕ is continuous at α otherwise is the interval $(\alpha, \beta]$; in any case, since t_0 is a point of continuity $\alpha < t_0 < \beta$. In the first case, (i), there exist $a < t_0 < b$ such that ϕ is strictly increasing in $[a, b]$. Let $a < a' < t_0 < b' < b$ and pick a ϱ identically equal to 1 on $[\phi(a'), \phi(b')]$ and with support in $[\phi(a), \phi(b)]$. Since ϕ is a minimum for F_ω, $(d/ds)|_{s=0}F_\omega(\psi_s) = 0$ and varying ϱ and a', b' one check easily that $E_\phi(t_0) = 0$.

The case (ii) is the delicate part of the proof [for example, here comes in the fact that Mather's argument works only for minima]. To conclude that $E_\phi(t_0) = 0$ also for points t_0 where ϕ is not increasing one needs to introduce more test curves: Let

$$\psi_s^{(+)} \equiv \begin{cases} \phi & \text{for } 0 < t \leq t_0 \\ \psi_s & \text{for } t_0 < t \leq 1; \end{cases} \quad \psi_s^{(-)} \equiv \begin{cases} \psi_s & \text{for } 0 < t \leq t_0 \\ \phi & \text{for } t_0 < t \leq 1 \end{cases}$$

and extend the definition to $\mathbb{R}$ as usual. Now, from the above discussion one sees that if (A.1) holds *and* $s \geq 0$ for $\psi_s^{(+)}$, $s \leq 0$ for $\psi_s^{(-)}$ *then* $\psi_s^{(\pm)} \in Y_0$ (but $\psi_s^{(\pm)} \notin Y_0$ for s in any neighborhood of 0). The variation of F_ω will now yield

$$\left.\frac{d}{ds}\right|_{s=0} F_\omega(\psi_s^{(+)}) = \int_{t_0}^{1} E_\phi(t)\varrho(\phi)\, dt,$$

$$\left.\frac{d}{ds}\right|_{s=0} F_\omega(\psi_s^{(-)}) = \int_{0}^{t_0} E_\phi(t)\varrho(\phi)\, dt$$

and since ϕ is a minimum one has

$$\left.\frac{d}{ds}\right|_{s=0} F_\omega(\psi_s^{(+)}) \geq 0, \quad \left.\frac{d}{ds}\right|_{s=0} F_\omega(\psi_s^{(-)}) \leq 0.$$

Now, since $h_{xx'} < 0$ (in our case $h_{xx'} \equiv -1$), E_ϕ is *decreasing* ($\equiv$ not increasing) on $(\alpha, \beta]$ and we claim that, by varying ϱ one can show that

$$\begin{aligned} &\left.\frac{d}{ds}\right|_{s=0} F_\omega(\psi_s^{(+)}) \geq 0 \Rightarrow E_\phi(t_0) \geq 0, \\ &\left.\frac{d}{ds}\right|_{s=0} F_\omega(\psi_s^{(-)}) \leq 0 \Rightarrow E_\phi(t_0) \leq 0 \end{aligned} \tag{A.2}$$

so that $E_\phi(t_0) = 0$ also in case (ii).

Let us prove the first of (A.2). If ϕ has a discontinuity at β, i.e. $\phi(\beta+) > \phi(\beta)$, by taking the support of ϱ in $[\phi(0), \phi(\beta+)]$ and $\varrho \equiv 1$ at $\phi(t_0)$, one sees that, because E_ϕ is decreasing in (α, β),

$$0 \leq \int_{t_0}^{1} E_\phi \varrho\, dt = \int_{t_0}^{\beta} E_\phi\, dt \leq E_\phi(t_0)(\beta - t_0),$$

i.e., $E_\phi(t_0) \geq 0$. If $\phi(\beta+) = \phi(\beta)$, one can argue as above by taking $\beta < b' < b$ such that ϕ is strictly increasing in (β, b) and choosing ϱ with support in $[\phi(0), \phi(b)]$ and $\varrho \equiv 1$ on $[\phi(t_0), \phi(b)]$. By taking $(b' - \beta)$ small enough and using the continuity of E_ϕ at t_0 one concludes that $E_\phi(t_0) \geq 0$.

The second implication in (A.2) can be proved analogously. □

Acknowledgements

It is a pleasure to thank J. N. Mather for a useful discussion and J. Fröhlich for his kind hospitality at ETH, Theoretische Physik, Zürich where this work has been completed.

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Abstract

A result by J. N. Mather on the non-uniqueness of solutions of Percival's Euler-Lagrange equations for invariant sets of a certain class of area-preserving twist homeomorphisms of the cylinder is extended and made quantitative. As a by-product a new criterion for the non-existence of invariant circles is found.

(Received: February 28, 1990)