On the Kolmogorov Set for Many–Body Problems^{*}

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1 Introduction

The many-body problem is the the study of the motion of 1 + N point masses m_0, \dots, m_N interacting through gravity only, hence, whose coordinates $v_0, \dots, v_N \in \mathbb{R}^d$ (where d = 2, 3) obey to the Newton's equations

$$\ddot{v}_i = -\sum_{j \neq i} m_j \frac{v_i - v_j}{|v_i - v_j|^3} \quad \text{for} \quad 0 \le i \le N \;. \tag{1.1}$$

As usual, the "dot" denotes the derivative with respect to time and $|\cdot|$ is the Euclidean norm of \mathbb{R}^d . In the planetary problem, one mass, m_0 (the "Sun"), is much greater than the others (the "planets"). So, it is customary to introduce a small parameter μ and take

$$m_0 = \bar{m}_0$$
 and $m_i = \mu \bar{m}_i$ for $1 \le i \le N$. (1.2)

The Hamiltonian formulation of (1.1) is

$$\left\{ \begin{array}{l} \dot{u}_i = -\partial_{v_i} \hat{\mathcal{H}}_{\text{plt}}(\mu; u, v) \\ \dot{v}_i = \partial_{u_i} \hat{\mathcal{H}}_{\text{plt}}(\mu; u, v) \end{array} \right.$$

with

$$\hat{\mathcal{H}}_{\text{plt}}(\mu; u, v) := \sum_{0 \le i \le N} \frac{|u_i|^2}{2m_i} - \sum_{0 \le i < j \le N} \frac{m_i m_j}{|v_i - v_j|} \ . \tag{1.3}$$

(masses as in (1.2)) which is regular (real-analytic ¹) over the domain ("phase space")

$$\hat{\mathcal{C}}_{\rm cl} = \left\{ (u, v) := \left((u_0, \cdots, u_N), (v_0, \cdots, v_N) \right) \in \mathbb{R}^{\rm dN} \times \mathbb{R}^{\rm dN} : \quad v_i \neq v_j \quad \text{for} \quad 0 \le i < j \le N \right\}$$

Besides the energy, $\hat{\mathcal{H}}_{\text{plt}}$ has, as integrals of the motion (*i.e.*, conserved quantities along its trajectories), the three components of the *linear momentum*

$$\hat{\mathbf{Q}} = \sum_{0 \le i \le N} u_i \tag{1.4}$$

and, for d = 3, also the three components of the angular momentum

$$\hat{\mathcal{C}} = \sum_{0 \le i \le N} v_i \times u_i \tag{1.5}$$

(which are related to the *translation* and *rotation* invariance of $\hat{\mathcal{H}}_{plt}$, respectively), where "×" denotes the standard vector product of \mathbb{R}^3 . Hence, the number of degrees of freedom of the system ² of can be lowered.

The linear momentum reduction is usually performed as follows.

Consider the invariant manifold with dimension $2\mathrm{d}N$

$$\mathcal{M}_{\text{lin}} := \left\{ u, \ v \in \hat{\mathcal{C}}_{\text{cl}} : \quad \sum_{0 \le i \le N} u_i = \sum_{0 \le i \le N} m_i v_i = 0 \right\} \ .$$

¹We recall that a function $f : A \to \mathbb{R}$, where A is an open, connected, bounded subset of \mathbb{R}^n , is realanalytic in $x_0 \in A$ if there exist $\{a_k(x_0)\}_{k \in \mathbb{N}^n}$ and an open neighborhood of x_0 , $U(x_0)$, such that the series $\sum_{k \in \mathbb{N}^n} a_k(x_0)(x-x_0)^k$ converges uniformly to f, for any $x \in U(x_0)$; f is said to be real-analitic in A if it is real-analytic in any point $x_0 \in A$.

²The number of degrees of freedom of an Hamiltonian system is defined as one half of the dimension of the phase space of the Hamiltonian. In the case of (1.3), the number of degrees of freedom is d(1+N)

Let

$$\mathcal{C}_{\rm cl} := \left\{ (y, x) \in \mathbb{R}^{\mathrm{d}N} \times \mathbb{R}^{\mathrm{d}N} : \ 0 \neq x_i \neq x_j \text{ for } 1 \le i < j \le N \right\}$$

be the "collisionless phase space" and define the embedding

$$\phi_{\text{helio}}: (y, x) \in \mathcal{C}_{\text{cl}} \subset \mathbb{R}^{\mathrm{d}N} \times \mathbb{R}^{\mathrm{d}N} \to (u, v) \in \mathcal{M}_{\text{lin}}$$

which acts as

$$\begin{pmatrix} u_0 := -\sum_{1 \le j \le N} y_j \\ v_0 := -\left(\sum_{0 \le j \le N} m_j\right)^{-1} \sum_{1 \le i \le N} m_i x_i \\ u_i := y_i \\ v_i := x_i - \left(\sum_{0 \le j \le N} m_j\right)^{-1} \sum_{1 \le j \le N} m_j x_j$$

Then, it is not difficult to see that the evolution in time for the "relative momenta–coordinates" pairs (y, x) is governed by the Hamilton equations of

$$\hat{\mathcal{H}}_{\mathrm{plt}}(\mu; y, x) := \hat{\mathcal{H}}_{\mathrm{plt}} \circ \phi_{\mathrm{helio}}(\mu; y, x)$$

A suitable rescaling of variables and Hamiltonian (which does not change the motion equations)

$$\mathcal{H}_{\text{plt}}(\mu; y, x) := \mu^{-1} \widetilde{\mathcal{H}}_{\text{plt}}(\mu; \mu y, x)$$

brings finally to

$$\mathcal{H}_{\text{plt}}(\mu; y, x) = \mu^{-1} \widetilde{\mathcal{H}}_{\text{plt}}(\mu; \mu y, x)$$

$$= \sum_{1 \le i \le N} \left(\frac{|y_i|^2}{2\tilde{m}_i} - \frac{\tilde{m}_i \hat{m}_i}{|x_i|} \right) + \mu \sum_{1 \le i < j \le N} \left(\frac{y_i \cdot y_j}{\bar{m}_0} - \frac{\bar{m}_i \bar{m}_j}{|x_i - x_j|} \right) ,$$
(1.6)

where \hat{m}_i , \tilde{m}_i are the "reduced masses"

$$\hat{m}_i := \bar{m}_0 + \mu \, \bar{m}_i \,, \quad \tilde{m}_i := \frac{\bar{m}_0 \bar{m}_i}{\bar{m}_0 + \mu \, \bar{m}_i}$$

and $u \cdot v$ denotes the usual inner product of two vectors u, v of \mathbb{R}^d .

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Notice that the angular momentum (in heliocentric variables)

$$C := \sum_{1 \le i \le N} x_i \times y_i \tag{1.7}$$

is still conserved along the \mathcal{H}_{plt} -trajectories, which is still rotation invariant.

When $\mu = 0$ ("integrable limit"), the \mathcal{H}_{plt} -evolution is the resulting of N independent Keplerian motions for the coordinates x_1, \dots, x_N : each of them, accordingly to the Law of Equal Areas ³ draws

³Law of Equal Areas: the area spanned by $x_i(t)$ on the ellipse \mathcal{E}_i is given by

$$S_i(t) = S_i(0) + \frac{a_i b_i}{2} n_i t$$

where n_i , defined by $n_i^2 a_i^3 = \hat{m}_i$ is the mean motion and a_i , $b_i = a_i \sqrt{1 - e_i^2}$ are the semi-axes of \mathcal{E}_i .

an ellipse in the space, whose position and shape depends only on the initial data (\bar{y}_i, \bar{x}_i) , all the ellipses possessing a common focus (the Sun). The total motion is thus – for $\mu = 0$ – quasi-periodic with N frequencies, provided each two – body energy

$$h_{i} := \frac{|y_{i}|^{2}}{2\tilde{m}_{i}} - \frac{\tilde{m}_{i}\hat{m}_{i}}{|x_{i}|}$$
(1.8)

is negative.

The (analytic, C^{∞}) continuation, for $\mu > 0$, of the quasi-periodic motions (with *N*-frequencies) of (1.8) with quasi-periodic motions with more frequencies has been investigated by several authors. Of great interest is the case of "maximal" continuation, which consists in looking for tori with the maximum number f of frequecies possible, *i.e.*, (analytic, C^{∞}) invariant manifolds for \mathcal{H}_{plt} , diffeomorphic to the standard torus \mathbb{T}^{f} where the angular coordinate evolves with linear low in time.

The pioneering work on this subject is the one by Arnol'd [3], who, in the framework of the KAM theory, stated the existence of a positive measure ("Cantor") set of initial data giving rise to bounded motions. He proved his statement only in the case of the plane three – body problem (d = N = 2) and, for the general case (spatial (1 + N)-body problem), gave only some indications on how to extend the result.

It has been noticed in [17] that such indications contain a flaw. Nonetheless, Arnol'd's proof of existence of quasi-periodic motions in the planar three body problem, is based on a refined KAM theorem – constructed in the framework of real-analytic functions and called by himself *Fundamental Theorem* (quoted below), which could overcome the strong "degeneracy" of the problem. To explain this point, and for future use, we need a bit of preparation.

Let us start by considering the planar case and let us introduce the planar Delaunay–Poincaré variables as follows:

$$\begin{cases} \Lambda_{i} = \tilde{m}_{i}\sqrt{\hat{m}_{i} a_{i}} \\ \lambda_{i} = \ell_{i} + g_{i} \\ \eta_{i} = \sqrt[4]{\hat{m}_{i} a_{i}}\sqrt{2\tilde{m}_{i}(1 - \sqrt{1 - e_{i}^{2}})} \cos g_{i} \\ \xi_{i} = -\sqrt[4]{\hat{m}_{i} a_{i}}\sqrt{2\tilde{m}_{i}(1 - \sqrt{1 - e_{i}^{2}})} \sin g_{i} \end{cases} \qquad (1.9)$$

where, denoting by \mathcal{E}_i the "osculating" ellipse spanned by the solution of the two – body differential problem

$$\begin{cases} \ddot{v} = -\hat{m}_i \frac{v}{|v|^3} , \quad v \in \mathbb{R}^2 \\ \left(\tilde{m}_i \dot{v}(0), v(0) \right) = \left(y_i, x_i \right) \end{cases}$$

 a_i, e_i, g_i, ℓ_i , are the semimajor axis, the eccentricity, the argument of perihelion of \mathcal{E}_i and the mean anomaly of x_i on \mathcal{E}_i (assume that (y, x) varies in a region of \mathcal{C}_{cl} for which each (y_i, x_i) gives rise to an ellipse, *i.e.*, with $0 < e_i < 1$). It is a classical result (see [11], [20], [7]) that the map

$$\phi_{\mathrm{DP}}^{-1}: \quad \left(y, x\right) \to \left(\Lambda, \lambda, \eta, \xi\right) \,,$$

with $\Lambda = (\Lambda_1, \dots, \Lambda_N), \dots$ as in (1.9) is real-analytic 1:1 and symplectic on a suitable open neighborhood of $\{\bar{\Lambda}\} \times \mathbb{T}^N \times \{0\}$.

When expressed in planar Delaunay–Poincaré coordinates, the integrable limit of \mathcal{H}_{plt} becomes

$$h_{\rm plt} := \sum_{1 \leq i \leq N} h_i \circ \phi_{\rm DP} = -\sum_{1 \leq i \leq N} \frac{\tilde{m}_i^3 \hat{m}_i^2}{2\Lambda_i^2}$$

a function of the actions $\Lambda = (\Lambda_1, \dots, \Lambda_N)$, only – a fact usually called "proper degeneracy", which prevents the use of standard KAM (Kolmogorov, Arnol'd, Moser) theory ⁴ in order to construct maximal tori.

It is also well–known, since Laplace, that the "secular perturbation" of the planetary problem, *i.e.* , the mean

$$\bar{f}_{\text{plt}} := \frac{1}{(2\pi)^N} \int_{\mathbb{T}^N} \sum_{1 \le i < j \le N} \left(\frac{y_i \cdot y_j}{\bar{m}_0} - \frac{\bar{m}_i \bar{m}_j}{|x_i - x_j|} \right) \circ \phi_{\text{DP}} \, d\lambda$$

has an *elliptic* equilibrium point at $z := (\eta, \xi) = 0$, for any Λ , *i.e.*, it has an equilibrium point there and can be symplectically put into the form

$$\bar{f}_{\text{plt}} \circ \phi_{\text{diag}} = \overline{f}_{\text{plt}} \circ \phi_{\text{diag}} = \bar{f}_0(\tilde{\Lambda}) + \sum_{1 \le i \le N} \Omega_i(\tilde{\Lambda}) \frac{\tilde{\eta}_i + \xi_i^2}{2} + \mathcal{O}_4 , \qquad (1.10)$$

where Ω_i are usually called *Birkhoff invariants of the first order*. We recall here the Theorem by Arnol'd.

FUNDAMENTAL THEOREM (Arnol'd, 1963, [3]) Assume that

- $(\mathrm{FT}_0) \ \mathcal{H}(I,\varphi,p,q) = h(I) + \varepsilon f(I,\varphi,p,q) \text{ is real-analytic on } U(r_0) := \overline{\mathcal{I}} \times \mathbb{T}^{\overline{n}} \times B^{2\hat{n}}_{r_0}(0), \text{ with } \mathcal{I} \text{ an open, bounded, connected subset of } \mathbb{R}^{\overline{n}};$
- (FT₁) h is a diffeomorphism of an open neighborhood of $\overline{\mathcal{I}}$, with non degenerate Jacobian $\partial \omega = \partial^2 h$ on such neighborhood;
- (FT₂) the mean perturbation $\bar{f}(I, p, q) := \frac{1}{(2\pi)^{\bar{n}}} \int_{\mathbb{T}^{\bar{n}}} f(I, \varphi, p, q) d\varphi$ has the form⁵

$$\bar{f} = f_0(I) + \sum_{1 \le i \le \hat{n}} \Omega_i(I) J_i + \frac{1}{2} \sum_{1 \le i,j \le \hat{n}} A_{ij}(I) J_i J_j + \frac{1}{6} \sum_{1 \le i,j,k \le \hat{n}} B_{ijk}(I) J_i J_j J_k
+ o_6$$
(1.12)
with $J_i := \frac{p_i^2 + q_i^2}{2}$ and $o_6 / |(p,q)|^6 \to 0$;

(FT₃) A is non singular on \mathcal{I} , *i.e.*,

$$\det A(I) \neq 0$$
 for any $I \in \overline{\mathcal{I}}$.

Then, for any $\kappa > 0$, there exists r_* , such that, for any

$$0 < r < r_* \qquad \text{and} \qquad 0 < \varepsilon < r^8 \tag{1.13}$$

an H-invariant ' set $F(r) \subset U(r) = \mathcal{I} \times \mathbb{T}^n \times B_r^{2\hat{n}}$ may be found, with

$$|\Omega(I) \cdot k| \neq 0 \quad \text{for any} \quad I \in \overline{\mathcal{I}}, \quad k = (k_1, \cdots, k_m) \in \mathbb{Z}^{\hat{n}} : \sum_{1 \le i \le \hat{n}} |k_i| \le 6 .$$
(1.11)

⁴The theory, on the persistence, under suitable assumptions, of quasi-periodic motions for nearly– integrable Hamiltonian systems, developed in the late 60's by Moser, (1962, [24]), Arnol'd (1963, [2]) on the basis of an early paper (1954) by Kolmogorov [22]. For a review –and a complete proof–of the original Kolmogorov's Theorem, see [12]. For related references, see also [10], [13], [35].

⁵By Birkhoff theory, a sufficient condition for (1.12) is that \bar{f} has an elliptic equilibrium point at the origin, with non resonant Birkhoff invariants of the first order Ω , that is,

$$\operatorname{meas}\left(U(r) \setminus F(r)\right) < \kappa \operatorname{meas}\left(F(r)\right)$$
(1.14)

consisting of $n(:=\bar{n}+\hat{n})$ -dimensional tori where the \mathcal{H} -flow is $\vartheta \to \vartheta + \nu t$.

Arnol'd's estimate for the tori density in phase space is ⁶

$$\max(F(r)) > (1 - c r^{1/(8(n+4))}) \max(U(r))$$
(1.15)

As told before, Arnol'd applied his theorem to the planar three-body problem, checking, in particular, assumptions (FT_3) , (FT_4) (in fact, (FT_2) is a consequence of (FT_3) and Birkhoff Theory, in view of (1.10)).

The spatial three body problem (d = 3, N = 2) was solved, in 1995, by P. Robutel ([33]; see also [21]). After performing the Jacobi, or nodes (angular momentum) reduction, he checked the assumptions of the FT, proving, so, the existence of (maximal) tori with 4 frequencies.

The first complete proof of the existence of a positive measure set of quasi-periodic motions was given only in 2004, by J. Féjoz ([17],THEORÉME 60), who, completing the investigations of M.Herman [18], in the framework of a refined C^{∞} KAM theory, stated the existence of a positive measure set of initial data giving rise to quasi-periodic motions with 3N - 1 frequencies, with their density going to 1 as $\mu \to 0$.

Another recent proof of Arnol'd's statement, but in the real-analytic framework of 2001 Rüssmann theory [34], may be found in [14] (see also [32]). The real-analytic framework appears more natural for the many-body problem (1.1), which is formulated using real-analytic functions.

Both the proofs presented in [17], [14] are based on the check of "weak" conditions on the first invariants Ω of \bar{f} (suitable non-planarity conditions, sometimes called *Arnol'd- Pyartli*, *Rüssmann* conditions, respectively), which, however, cannot be applied directly to \mathcal{H}_{plt} , due to the presence of two "secular resonances".

Letting, in fact, \mathcal{H}_{plt} in spatial Delaunay–Poincaré variables (definition 4.9), the frequencies Ω correspond to 2N frequencies (related to the motions of perihelia and ascending nodes, respectively) $\sigma = (\sigma_1, \dots, \sigma_N), \zeta = (\zeta_1, \dots, \zeta_N)$ which are found to verify the (unique, [17]) linear relations (up to linear combinations)

$$\zeta_N = 0$$
, $\sum_{1 \le k \le N} \sigma_k + \sum_{1 \le k \le N-1} \zeta_k = 0$. (1.16)

The former relation in (1.16) is known since Laplace; the latter was firstly noticed by M. Herman, so it is usually called *Herman's resonance* (for an interesting investigation on the Herman's resonance, see [1]). Owing to such secular resonances (in particular, the Herman's resonance), both the non–planarity conditions required by the KAM theories used in [17], [14] are violated by \mathcal{H}_{plt} . In order to overcome this problem, in [17] a modified Hamiltonian is introduced, next considered on the symplectic manifold of vertical angular momentum; in [14], the phase space is extended by adding an extra degree of freedom.

Notice that the former relation in (1.16) is actually a resonance of (low) order 1, and also prevents the direct application of FT, making (1.11) false; the Herman's resonance is of higher order, 2N - 1, so, it violates (1.11) only for N = 2, 3.

A direct attack to the problem, in the sense specified by FT, using a good set of coordinates which performs the *angular momentum reduction*, has never been attempted. We outline that such a strategy

⁶This estimate is not explicitly quoted into the statement of FT, but can be deduced as follows. Using the original Arnol'd's notations $(\epsilon, \mu, n_0, n_1) := (r^2, \varepsilon, \bar{n}, \hat{n})$, in the course of the proof, we find the condition $\delta := \epsilon^{1/T} < C \kappa$ with $T = 16(n+4) := 16(n_0 + n_1 + 4)$: see on page 144, eq. (4.2.5) with $\delta^{(3)}$ defined below and p. 145, eq. (4.2.7).

would lead also to a more precise insight into the properties of the quasi-periodic motions (tori measure, frequencies, \cdots).

The problems which this thesis addresses are the following.

- (P₁) (Section 2) Construction of a FT-like KAM Theorem (THEOREM 1 below, for a simplified verson) in the real- analytic class for properly degenerate systems, in order to obtain a fine measure estimate for the "invariant set" (roughly speaking, the set of the KAM quasi-periodic trajectories) of a properly degenerate \mathcal{H} , nearly an equilibrium point so as to
- (P₂) (Section 3) establish the existence of maximal quasi-periodic motions and estimate the measure of Kolmogorov's invariant set for the plane planetary problem;
- (P_3) (Section 4) reduction of the angular momentum in the spatial planetary problem which leads to
- (P_4) (Section 5) a proof of existence of KAM tori with 3N 1 Diophantine frequencies (via a partial reduction of the angular momentum) and measure of the invariant set;
- (P₅) (Section 6) a direct proof of existence of (3N 2)-dimensional KAM tori via analytic theories of [14] (full reduction).

We briefly explain our results.

As for P₁, we prove THEOREM 1 below (for a more general statement, see Theorem 2.1), which may be viewed as a refinement of FT: compare the bound on ε (1.17) and the estimate for the tori measure (1.18) for the invariant set with (1.13), (1.15).

THEOREM 1. Assume (FT_0) , (FT_1) , (FT_3) as in FT and

$$(\operatorname{FT}_{2}') \ \bar{f} = f_{0}(I) + \sum_{1 \leq i \leq m} \Omega_{i}(I)J_{i} + \frac{1}{2} \sum_{1 \leq i,j \leq N} A_{ij}(I)J_{i}J_{j} + o_{4}$$

where $|\Omega(I) \cdot k| \neq 0$ for any $I \in \overline{\mathcal{I}}, \quad k = (k_{1}, \cdots, k_{m}) \in \mathbb{Z}^{m} : \sum_{1 \leq i \leq m} |k_{i}| \leq 4$

where $o_4/|(p,q)|^4 \to 0$. Then, there exist $r_*, 0 < c < 1 < C, b > 0$ such that, for any

$$0 < r < r_*$$
 and $0 < \varepsilon < c(\log r^{-1})^{-2b}$ (1.17)

an invariant set $\mathcal{K}(\varepsilon, r) \subset \mathcal{I} \times \mathbb{T}^{\bar{n}} \times B^{2\hat{n}}_{C(\log r^{-1})^{-1}}(0)$ ("Kolmogorov set") with measure

$$\operatorname{meas}\left(\mathcal{K}(\varepsilon, r)\right) \ge \left(1 - C\varepsilon^{1/2}(\log r^{-1})^b - Cr^{1/2}\right)\operatorname{meas}\left(U(r)\right).$$
(1.18)

consisting of $n = \bar{n} + \hat{n}$ -dimensional invariant tori, with $(\varepsilon r^{5/2}, \tau)$ -Diophantine frequencies ν , where the motion is analytically conjugated to $\vartheta \to \vartheta + \nu t$.

The proof of Theorem 2.1 is made in two steps.

 (s_1) On one side, proof of a quantitative isofrequencial KAM theorem particularly well suited for properly degenerate quasi integrable Hamiltonians in action–angle variables

$$H(J,\psi) = \bar{\mathbf{h}}(\bar{J}) + \bar{\mathbf{h}}(J) + \mathbf{f}(J,\psi)$$
(1.19)

i.e., with integrable part which splits into the sum of two terms: h (thought dominant), which depends only on a part of the action variables \bar{J} and \hat{h} (thought small with respect to \bar{h}) which depends on *all* the actions J. The peculiarity of this theorem is of choosing (the idea goes back to Arnol'd) two different scales for the tori frequencies to be kept fixed.

(S₂) On the other side, we reduce the properly degenerate Hamiltonian $\mathcal{H}(I, \varphi, p, q)$ to the form (1.19), with \bar{h} of order 1, \hat{h} of order εr^2 and the perturbation small ($\varepsilon r^{5/2}$). The reduction is based on a non-standard averaging theory, developed by Biasco et al. [7], and Birkhoff Theory.

As a second step, we apply TH1 to the plane (1 + N)-Body Problem. In order to do that, we compute explicitly the Birkhoff invariants of order 1 and 2, expanding the perturbation of \mathcal{H} in plane Delaunay variables $(\Lambda, \lambda, \eta, \xi)$, up to order 4, after suitable diagonalization and Birkhoff Theory. If \bar{f}_{pl} denotes the mean perturbation of the plane problem in Delaunay variables, the Hessian matrix $\partial^2 \bar{f}_{pl}$ has the form

$$\left(\begin{array}{cc} \mathcal{F}(\Lambda) & 0 \\ 0 & \mathcal{F}(\Lambda) \end{array}\right)$$

with $\mathcal{F}(\Lambda)$ a symmetric $N \times N$ matrix. The first Birkhoff invariants are thus the eigenvalues of $\mathcal{F}(\Lambda)$. We introduce a small parameter, the maximum semimajor axes ratio δ , letting

$$\frac{a_i}{a_{i+1}} = \hat{\alpha}_i \,\delta \;.$$

For small δ , the asymptotics of $\mathcal{F}(\Lambda)$ is

$$\mathcal{F}(\Lambda) \approx \delta^n \begin{pmatrix} f_1 & O(\delta^{n_{12}}) & \cdots & O(\delta^{n_{1k}}) & \cdots \\ & f_2 \delta^{n_{22}} & \cdots & O(\delta^{n_{2k}}) & \cdots \\ & & \ddots & \vdots & \\ & & & f_k \delta^{n_{kk}} & \cdots \\ & & & & \ddots \end{pmatrix}$$

with f_i with order 1 in δ , n_{hk} positive integers verifying

$$n_{kk} < n_{k+1,k+1}$$
 and $n_{h-1,k}, n_{h,k+1} > n_{hk}$.

The eigenvalues of \mathcal{F} are thus $\Omega_k = f_k \delta^{n_k k}$ up to higher orders, and are thus non resonant. The Hamiltonian can be put in Birkhoff normal form

$$\bar{f}_0(\Lambda) + \Omega(\Lambda) \cdot J + \frac{1}{2}J \cdot A(\Lambda)J + \cdots$$
 where $J_i = \frac{\eta_i^2 + \xi_i^2}{2}$

and the Birkhoff invariants of order 2 are the eigenvalues of the symmetric matrix $\mathcal{A}(\Lambda)$. We finally prove that $\mathcal{A}(\Lambda)$ as the asymptotics

$$A(\Lambda) \approx \delta^{p} \begin{pmatrix} \alpha_{11} & \alpha_{12} & O(\delta^{p_{13}}) & \cdots & O(\delta^{p_{1k}}) & \cdots \\ \alpha_{21} & \alpha_{22} & O(\delta^{p_{23}}) & \cdots & O(\delta^{p_{2k}}) & \cdots \\ & & \alpha_{33}\delta^{p_{33}} & \cdots & O(\delta^{p_{3k}}) & \cdots \\ & & & \ddots & \vdots \\ & & & & \alpha_{kk}\delta^{p_{kk}} & \cdots \\ & & & & \ddots & \end{pmatrix}$$

with α_{ij} with order 1 in δ , $p_{k+1,k+1} > p_{kk}$, $\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} \neq 0$, $\alpha_{kk} \neq 0$. This allows us checking that

$$\det A = \delta^{Np} (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}) \prod_{3 \le k \le N} \alpha_{kk} + o(\delta^{Np})$$

is nonvanishing, for small δ , concluding the proof.

The extension of the previous proof to the spatial problem in Delaunay variables is forbidden, by the presence of the above mentioned secular resonances, closely related to the rotation invariance of \mathcal{H}_{plt} . A reduction of the number of degrees of freedom is however possible, with the use of the *Deprit variables* [9], [15].

The remarkable property of this new set of variables is to have, among their conjugated momenta, two coordinates of the angular momentum: the modulus G and the third component C_z . Their conjugated angles will be then cyclic variables. In particular, the conjugated angle ζ of C_z has the meaning of the ascending node longitude of the total angular momentum C, *i.e.*, its third component, so it is an integral of the motion, too.

When expressed in these new variables, only one external parameter will appear (the modulus G) for the reduced problem. This new set of variables can be regularized in a similar way to Poincaré's one for the regularization of the Delaunay variables. The new regularized variables $(\Lambda, \lambda, \eta, \xi, p, q)$ for the reduced problem are of dimension 2N + 2N + 2(N-2) = 2(3N-2). The variables $(\Lambda, \lambda, \eta, \xi)$, of dimension 4N, play the same role as the Poincaré variables in the plane problem. The variables (p, q) are related to the couples (inclinations, nodes): only N - 2 couples may be chosen as independent, having fixed the modulus G of the angular momentum and its verical component C_z . As consequence of the reduction, the D'Alembert symmetries, existing in the plane problem, are broken; the origin of the new secular coordinates $z = (\eta, \xi, p, q)$ is no longer an equilibrium point and an *elliptic singularity* appears, that is a singularity for the perturbation over the manifold

$$G = \sum_{1 \le i \le N} \left(\Lambda_i - \frac{\eta_i^2 + \xi_i^2}{2} \right) - \sum_{1 \le i \le N-2} \frac{p_i^2 + q_i^2}{2}$$

owing to which, the configuration with all zero eccentricities and inclinations (which corresponds to z = 0 and $G = \sum \Lambda_i$) is not allowed. As a consequence, motions arbitrarily close to cocircular and coplanar trajectories cannot be considered, a fact already known in the case of the three body problem.

Nonetheless Deprit's reduction has a partial reduction (partial reductions were also studied in [23]) naturally associated, which consists in using only C_z as generalized momentum, and not G also, making a further symplectic change of variables $(G, g) \leftrightarrow (p_{N-1}, q_{N-1})$, where g is the (cyclic) variable associated to G. In this way, a further inclination is treated as independent, but the number of degrees of freedom is enhanced from 3N - 2 to 3N - 1, having lost the cyclic variable g. Differently from what happens using Delaunay variables, Deprit's partial reduction leaves the mean perturbation regular and even around the secular origin, which is thus an equilibrium point corresponding to zero eccentricities and mutual inclinations. Thus, Deprit's partial reduction allows us to consider a larger region of the phase space than in the case of full reduction, even if at the price of one degree of freedom more.

In Section 5, we show that the set of partially reduced Deprit variables provides a natural proof of the existence of (3N - 1)-dimensional KAM tori via TH1, at least for $N \ge 3$ planets, that is, from the four body problem on. In these variables, the planetary Hamiltonian (1.6)

$$\mathcal{H}_{\rm plt,pr} = h_{\rm plt}(\Lambda) + \mu f_{\rm plt,pr}(\Lambda, \lambda, \eta, \xi, p, q) ,$$

where h_{plt} is the usual Kepler's integrable part, satisfies the following. The "secular perturbation" $\bar{f}_{\text{plt,pr}} := (2\pi)^{-N} \int_{\mathbb{T}^N} f_{\text{plt,pr}} d\lambda$ is even and regular around the secular origin, as said before, and has the form

$$\bar{f}_{\text{plt,pr}} = \bar{f}_{\text{plt,pr}}^{0} + \mathcal{Q}_{h}^{*} \cdot \frac{\eta^{2} + \xi^{2}}{2} + \mathcal{Q}_{v}^{*} \cdot \frac{p^{2} + q^{2}}{2} + \mathfrak{F}(\eta, \xi, p, q) + \cdots$$
(1.20)

with \mathcal{Q}_h^* , \mathcal{Q}_v^* suitable quadratic forms acting only on the "horizontal", "vertical" variables, respectively, ⁷

⁷Following [17]'s notation, in (1.20), the dot "." denotes contraction of indices: $\mathcal{Q} \cdot \eta^2 := \sum_{i,j} \mathcal{Q}_{ij} \eta_i \eta_j$ if $\eta = (\eta_1, \dots,)$.

 \mathfrak{F} a suitable quartic form, all depending parametrically on Λ as well as $\overline{f}_{\text{plt,pr}}^0$ and verifying the following. The respective sets of eigenvalues $s = (s_1, \dots, s_N), z = (z_1, \dots, z_{N-1})$ of $\mathcal{Q}_h^*, \mathcal{Q}_v^*$ (together also with the "mean motions" $n = (n_1, \dots, n_N) := \partial_{\Lambda} h_{\text{plt}}$) are found to satisfy the Herman's resonance

$$\sum_{1 \leq i \leq N} s_i + \sum_{1 \leq i \leq N-1} z_i = 0 \ ,$$

and only that (Proposition 5.1). Since this resonance is of order N + (N-1) = 2N-1, it does not violate the condition FT'_3 of TH1 when $N \geq 3$, and the normal form of FT'_2 can be constructed. This normal form turns out to be non degenerate, *i.e.*, it also satisfies the second order non-degeneracy condition FT_4 (Propositin 5.2). Both the proofs of non degeneracy (of first and second order) are inductive and are developed with similar techniques as in [17]. Then, invoking TH1, we can state the existence of (3N-1)-dimensional KAM tori for the planetary problem and thus estimate the density of the invariant set (Theorem 5.1)

$$1 - \mu^{1/2} (\log \varepsilon^{-1})^b - \varepsilon^{1/2}$$

into a ball with volume $\varepsilon^{2(2N-1)}$, where ε is an upper bound for eccentricities and inclinations. Notice that the partial reduction generates an extra dimension for the KAM tori, relatated to the rotation invariance of $\mathcal{H}_{\text{plt,pr}}$.

Nicely, Deprit's partial reduction, for $N \geq 3$, makes us appear the spatial problem as the natural extension of the plane problem: the set of Birkhoff invariants of order 1 has the planar one as subset, and the same happens at order 2: when comparing the two matrices (planar and spatial) of Birkhoff invariants of order 2, the planar one is a submatrix of the spatial one. This fact cannot be observed in the 3-body problem as treated in [33], because, there, the full reduction is made, and the coinclination of the two planets is expressed as a function of the eccentricities.

In Section 6, we look at the *full* reduction, that is, we use also the modulus of the angular momentum G as generalized momentum. As said before, this makes us gain a cyclic variable, the angle g conjugate to G, lowering the number of degrees of feedom to 3N - 2, but also causes, for $N \ge 3$, an *elliptic singularity* and lack of symmetries (facts already known in Poincaré–Delaunay variables, trying to do a partial reduction, *i.e.*, to eliminate one inclination with the use of the integral C_z). A consequence of the lack of symmetries is that, for $N \ge 3$, the secular origin is no longer an equilibrium point for the mean perturbation. Nonetheless, in the range of small eccentricities and inclinations, it is possible to find a new equilibrium point, which is "small", *i.e.*, consists of almost circular and coplanar orbits, in the region of phase space which is considered, such that after suitable re–centering around it and symplectic diagonalization of the quadratic part, the planetary Hamiltonian is finally put into the form

$$\mathcal{H}_{\rm plt} = h_{\rm plt} + \mu f_{\rm plt}$$

where the mean $\bar{f}_{\text{plt}} := (2\pi)^{-N} \int f_{\text{plt}}$ becomes

$$\bar{f}_{\text{plt}} = \hat{f}_{\text{plt}}^{0}(\Lambda, G) + \sum_{1 \le i \le N} s_{i}(\Lambda, G) \frac{\eta_{i}^{2} + \xi_{i}^{2}}{2} + \sum_{1 \le i \le N-2} z_{i}(\Lambda, G) \frac{p_{i}^{2} + q_{i}^{2}}{2} + O(3)$$

and the first Birkhoff invariants $s = (s_1, \dots, s_N)$, $z = (z_1, \dots, z_{N-2})$, together with the mean motions $n = (n_1, \dots, n_N)$, do not satisfy any linear relation. Then, applying the real-analytic first order theory developed in [14], we can state the existence of (3N-2)-dimensional KAM tori for the planetary problem (Theorem 6.2).

2 Properly Degenerate KAM Theory

We recall some basic notations and definitions.

Let $\bar{n}, \hat{n} \in \mathbb{N}, n := \bar{n} + \hat{n}, \mathcal{I}$ a bounded, connected subset of $\mathbb{R}^{\bar{n}}, \mathbb{T}^{\bar{n}} := \mathbb{R}^{\bar{n}}/2\pi\mathbb{Z}^{\bar{n}}$ the usual real "flat" \bar{n} -torus and $B_r^{\hat{n}}(x)$ the real open \hat{n} -ball in $\mathbb{R}^{\hat{n}}$ with radius r, centered at x.

In order that a compact set $\mathcal{T} \subset V$ is called a $(\gamma, \hat{\gamma}; \tau)$ -Lagrangian torus with frequency ν for a given

$$\mathcal{H}(I,\varphi,p,q) = h(I) + f(I,\varphi,p,q)$$

assumed to be real-analytic on the phase space $V := \mathcal{I} \times \mathbb{T}^{\bar{n}} \times B_r^{2\hat{n}}(0)$, we require that

 (T_1) there exists a real-analytic embedding onto

$$\phi = (\phi_I, \phi_{\varphi}, \phi_p, \phi_q) : \ \vartheta = (\bar{\vartheta}, \hat{\vartheta}) \in \mathbb{T}^{\bar{n}} \times \mathbb{T}^{\hat{n}} \to \phi(\vartheta) \in \mathcal{T}$$

(and, hence, 2π –periodic in each variable) given by ⁸

$$\begin{cases} \phi_I(\vartheta) = \bar{v}(\vartheta) \\ \phi_{\varphi}(\vartheta) = \bar{\vartheta} + \bar{u}(\vartheta) \\ \phi_p(\vartheta) = p_0(\vartheta) + \sqrt{2\hat{v}(\vartheta)} \cos\left(\hat{\vartheta} + \hat{u}(\vartheta)\right) \\ \phi_q(\vartheta) = q_0(\vartheta) + \sqrt{2\hat{v}(\vartheta)} \sin\left(\hat{\vartheta} + \hat{v}(\vartheta)\right) , \end{cases}$$
(2.1)

such that

 (T_2) the map

$$\vartheta \to (\bar{\vartheta} + \bar{u}(\vartheta), \ \hat{\vartheta} + \hat{u}(\vartheta))$$

is a diffeomorphism of $\mathbb{T}^{\bar{n}} \times \mathbb{T}^{\hat{n}}$;

(T₃) \mathcal{T} is invariant under the \mathcal{H} -flow $\phi_t^{\mathcal{H}}$, which, on \mathcal{T} , acts as a translation by ν , i.e.,

$$\phi^{-1} \circ \phi_t^{\mathcal{H}} \circ \phi : \quad \vartheta \to \vartheta + \nu t , \quad \forall \ \vartheta \in \mathbb{T}^{\bar{n}} \times \mathbb{T}^{\hat{n}} ;$$

 (T_4) the torus frequency $\nu \in \mathbb{R}^{\bar{n}+\hat{n}}$ belongs to the generalized Diophantine set

$$\mathcal{D}^{ar{n},\hat{n}} := igcup_{\substack{\gamma,\hat{\gamma}>0\ au>ar{n}+\hat{n}}} \mathcal{D}^{ar{n},\hat{n}}_{\gamma,\hat{\gamma}; au}$$

⁸For shortness, in (2.1), the symbol $\sqrt{2r}\cos\psi$ denotes the \hat{n} -vector

$$(\sqrt{2r_1}\cos\psi_1,\cdots,\sqrt{2r_{\hat{n}}}\cos\psi_{\hat{n}})$$

if $r = (r_1, \dots, r_{\hat{n}}) \in \mathbb{R}^{\hat{n}}, \psi = (\psi_1, \dots, \psi_{\hat{n}}) \in \mathbb{T}^{\hat{n}}$, and similarly for $\sqrt{2r} \sin \psi$.

where

$$\mathcal{D}^{\bar{n},\hat{n}}_{\gamma,\hat{\gamma};\tau} := \bigcap_{0 \neq k \in \mathbb{Z}^{\bar{n}} \times \mathbb{Z}^{\hat{n}}} \mathcal{D}^{\bar{n},\hat{n}}_{\gamma,\hat{\gamma};\tau}(k)$$

with

$$\mathcal{D}^{\bar{n},\hat{n}}_{\gamma,\hat{\gamma};\tau}(k) := \left\{ \nu \in \mathbb{R}^{\bar{n}+\hat{n}} : |\nu \cdot k| \ge \frac{\gamma}{|k|^{\tau}} \quad \text{if} \quad k = (\bar{k},\hat{k}) \quad \text{with} \ \bar{k} \in \mathbb{Z}^{\bar{n}} \setminus \{0\} , \\ |\nu \cdot k| \ge \frac{\hat{\gamma}}{|k|^{\tau}} \quad \text{if} \quad \bar{k} = 0 \in \mathbb{Z}^{\bar{n}} \right\}$$
(2.2)

We shall say that the embedding ϕ as in $(T_1) \div (T_3)$ realizes the Lagrangian torus \mathcal{T} .

We are now ready to quote the following refined version of Arnol'd's Fundamental Theorem [3]. For a simpler formulation, see Remark 2.2.

We assume that

 (D_0)

$$\mathcal{H}(\varepsilon; I, \varphi, p, q) = h(I) + \varepsilon f(I, \varphi, p, q)$$
(2.3)

is real-analytic on $V(\bar{r}) := \overline{\mathcal{I}} \times \mathbb{T}^{\bar{n}} \times B_{\bar{r}}^{2\hat{n}}(0)$, where \mathcal{I} is an open, bounded, connected subset of $\mathbb{R}^{\bar{n}}$;

- (D₁) ∂h is a diffeomorphism of an open neighborhood of $\overline{\mathcal{I}}$, with non singular Jacobian on such neighborhood;
- (D_2) the mean perturbation:

$$\bar{f}(I,p,q) := \frac{1}{(2\pi)^{\bar{n}}} \int_{\mathbb{T}^{\bar{n}}} f(I,\varphi,p,q) d\varphi$$

has the form

$$\bar{f}(I,p,q) = f_0(I) + \sum_{1 \le i \le \hat{n}} \Omega_i(I) \frac{p_i^2 + q_i^2}{2} + \frac{1}{2} \sum_{1 \le i \le j \le \hat{n}} A_{ij}(I) \frac{p_i^2 + q_i^2}{2} \frac{p_j^2 + q_j^2}{2} + o_4 \quad (2.4)$$

where $o_4/|(p,q)|^4 \to 0$ as $(p,q) \to 0$, where

 $(D_3) \ \Omega = (\Omega_1, \cdots, \Omega_{\hat{n}})$ verifies

$$\min_{0 < |k| \le 4} \min_{\overline{\mathcal{I}}} |\Omega(I) \cdot k| > 0 \qquad ("4-non resonance")$$

and

(D₄) $A = (A_{ij})_{1 \le i,j \le N}$ non singular on $\overline{\mathcal{I}}$, *i.e.*,

$$\min_{\overline{\mathcal{I}}} |\det A| > 0 \qquad ("non degeneracy")$$

Theorem 2.1 Let \bar{n} , $\hat{n} \in \mathbb{N}$, $n := \bar{n} + \hat{n}$, $\bar{\tau} > \bar{n}$, $\tau > n$ and assume $D_0 \div D_4$ above. Then, there exists r_* , γ_* , γ^* , $C_* > 0$ such that, for any $0 < r < r_*$ and $\bar{\gamma}$, γ , $\hat{\gamma}$ in the range

$$\begin{array}{l} \gamma_* \max\{\sqrt{\varepsilon}(\log r^{-1})^{\bar{\tau}+1}, \ \sqrt[3]{\varepsilon r}(\log r^{-1})^{\bar{\tau}+1}, \ r^2(\log r^{-1})^{\bar{\tau}+1}\} < \bar{\gamma} < \gamma^* \\ \gamma_* r^{5/2} < \gamma < \gamma^* \\ \gamma_* r^{5/2}(\log_+ (r^5/\gamma^2)^{-1})^{\tau+1} < \hat{\gamma} < \gamma^* r^2 \end{array} \tag{2.5}$$

an invariant set $\mathfrak{K}(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma}) \subset V(\bar{r})$ may be found, with measure

$$\operatorname{meas}\left(\mathfrak{K}(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma})\right) \ge \left[1 - C_*\left(\bar{\gamma} + \gamma + \frac{\hat{\gamma}}{r^2} + r^{\hat{n}/2}\right)\right] \operatorname{meas}\left(V(r)\right)$$
(2.6)

consisting of Lagrangian tori $\{\mathfrak{T}_{\nu}(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma})\}_{\nu}$ with generalized $(\bar{n}, \hat{n}, \gamma, \varepsilon \hat{\gamma}, \tau)$ -Diophantine frequencies ν .

Remark 2.1 From the proof of Theorem 2.1, the following amplifications follow.

(D₁) A Cantor set $\mathfrak{J}_*(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma}) \subseteq \mathcal{I} \times B_{r^2}^{\hat{n}}(0)$ and a bi–Lipschitz homeomorphism (onto)

$$\begin{aligned} \varpi_*(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma}; \cdot) &= (\bar{\varpi}_*(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma}; \cdot), \varepsilon \hat{\varpi}_*(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma}; \cdot) : \\ \mathfrak{J}_*(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma}) &\to \mathfrak{O}_*(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma}) \subset \mathcal{D}^{\bar{n}, \hat{n}}_{(\gamma, \varepsilon \hat{\gamma}, \tau)} \end{aligned}$$

such that

(D₂) for any $\nu \in \mathfrak{O}_*(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma})$, the embedding

$$\begin{aligned} \mathfrak{F}(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma}, \nu; \cdot) &= (\mathfrak{F}_I(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma}, \nu; \cdot), \mathfrak{F}_{\varphi}(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma}, \nu; \cdot), \\ \mathfrak{F}_p(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma}, \nu; \cdot), \mathfrak{F}_q(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma}, \nu; \cdot)) \end{aligned}$$

which realizes $\mathfrak{T}_{\nu}(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma})$ is given by

$$\begin{aligned}
\mathfrak{F}_{I}(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma},\nu;\vartheta) &= \overline{\mathfrak{j}}_{*}(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma},\nu) + \overline{\mathfrak{u}}(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma},\nu;\vartheta) \\
\mathfrak{F}_{\varphi}(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma},\nu;\vartheta) &= \overline{\vartheta} + \overline{\mathfrak{v}}(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma},\nu;\vartheta) \\
\mathfrak{F}_{p}(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma},\nu;\vartheta) &= \mathfrak{p}_{0}(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma},\nu;\vartheta) \\
+ \sqrt{2}\widehat{\mathfrak{j}}_{*}(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma},\nu) + 2\widehat{\mathfrak{u}}(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma},\nu;\vartheta) \\
\times \cos\left[\widehat{\vartheta} + \widehat{\mathfrak{v}}(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma},\nu;\vartheta)\right] \\
\mathfrak{F}_{q}(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma},\nu;\vartheta) &= \mathfrak{q}_{0}(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma},\nu;\vartheta) \\
+ \sqrt{2}\widehat{\mathfrak{j}}_{*}(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma},\nu;\vartheta) \\
+ \sqrt{2}\widehat{\mathfrak{j}}_{*}(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma},\nu;\vartheta)\right]
\end{aligned}$$
(2.7)

where $j_*(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma}; \nu)$ is the $\varpi_*(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma}; \cdot)$ -preimage of ν .

Furtheremore, the unperturbed frequencies ∂h may be chosen $(\bar{\gamma}, \bar{\tau})$ -Diophantine on $\mathfrak{J}_*(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma}, \gamma, \hat{\gamma})$ and the following bounds hold, for $\varpi_*(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma}, \gamma, \hat{\gamma}, \cdot)$, and $\mathfrak{F}(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma}, \cdot, \cdot)$:

$$\begin{split} \sup_{\mathfrak{J}_{*}(\varepsilon,r,\tilde{\gamma},\gamma,\tilde{\gamma})} \left| \bar{\varpi}_{*} - \partial h \right| &\leq C_{*} \min \left\{ \frac{\gamma}{(\log_{+}(r^{5}/\gamma^{2})^{-1})^{\tau+1}}, \frac{\hat{\gamma}}{(\log_{+}(r^{5}/\gamma^{2})^{-1})^{\tau+1}}, \frac{\bar{\gamma}}{(\log_{+}(r^{5}/\gamma^{2})^{-1})^{\tau+1}}, \frac{\bar{\gamma}}{(\log_{\tau}(r^{5}/\gamma^{2})^{-1})^{\tau+1}}, \frac{\bar{\gamma}^{5}(\log_{\tau}(r^{5}/\gamma^{2})^{-1})^{\tau+1}}{\bar{\gamma}}, \frac{\bar{\gamma}^{5}(\log_{\tau}(r^{5}/\gamma^{2})^{-1})^{\tau+1}}{\bar{\gamma}^{3}}, \frac{r^{5}(\log_{\tau}(r^{5}/\gamma^{2})^{-1})^{2(\tau+1)}}{\bar{\gamma}^{2}}, \frac{r^{5}(\log_{\tau}(r^{5}/\gamma^{2})^{-1})^{2(\tau+1)}}{\bar{\gamma}^{3}}, r \right\} \\ \sup_{\mathcal{D}_{*}(\varepsilon,r,\bar{\gamma},\bar{\gamma},\bar{\gamma})\times\mathbb{T}^{n}} |\tilde{v}| \leq C_{*}\max_{\tau} \left\{ r^{5}(\frac{\log_{\tau}(r^{5}/\gamma^{2})^{-1})^{2(\tau+1)}}{\bar{\gamma}^{2}}, \frac{e^{r}(\log r^{-1})^{3\tau+2}}{\bar{\gamma}^{3}}, r \right\} \\ \sup_{\mathcal{D}_{*}(\varepsilon,r,\bar{\gamma},\bar{\gamma},\bar{\gamma},\bar{\gamma})\times\mathbb{T}^{n}} |v_{0}| \leq C_{*}\max_{\tau} \left\{ r^{5}(\log_{\tau}(r^{1})^{\tau+1}}{\bar{\gamma}}, \frac{r^{5}(\log_{\tau}(r^{5}/\gamma^{2})^{-1})^{2(\tau+1)}}{\bar{\gamma}}, \frac{r^{5}(\log_{\tau}(r^{5}/\gamma^{2})^{-1})^{2(\tau+1)}}{\bar{\gamma}^{2}}, 2e^{r}(\log_{\tau}(r^{5}/\gamma^{2})^{-1})^{2(\tau+1)}}, \frac{r^{5}(\log_{\tau}(r^{5}/\gamma^{2})^{-1})^{2(\tau+1)}}{\bar{\gamma}^{2}}, \frac{r^{5}(\log_{\tau}(r^{5}/\gamma^{2})^{-1})^{2(\tau+1)}}{\bar{\gamma}^{2}}, 2e^{r}(\log_{\tau}(r^{5}/\gamma^{2})^{-1})^{2(\tau+1)}}, 2e^{r}(\log_{\tau}(r^{5}/\gamma^{2})^{-1})^{2(\tau+1)}}, r^{5}(\log_{\tau}(r^{5}/\gamma^{2})^{-1})^{2(\tau+1)}}, \frac{r^{5}(\log_{\tau}(r^{5}/\gamma^{2})^{-1})^{2(\tau+1)}}}{\bar{\gamma}^{2}}, 2e^{r}(\log_{\tau}(r^{5}/\gamma^{2})^{-1})^{2(\tau+1)}}, 2e^{r}(\log_{\tau}(r^{5}/\gamma^{2})^{-1})^{2(\tau+1)}}, 2e^{r}(\log_{\tau}(r^{5}/\gamma^{2})^{-1})^{2(\tau+1)}}, 2e^{r}(\log_{\tau}(r^{5}/\gamma^{2})^{-1})^{2(\tau+1)}}, 2e^{r}(\log_{\tau}(r^{5}/\gamma^{2})^{-1})^{2(\tau+1)}}, 2e^{r}(\log_{\tau}(r^{5}/\gamma^{2})^{-1})^{2(\tau+1)}}, 2e^{r}(\log_{\tau}(r^{5}/\gamma^{2})^{-1})^{2(\tau+1)}}, 2e^{r}(\log_{\tau}(r^{5}/\gamma^{2})^{-1})^{2(\tau+1)}}$$

On the proof of Theorem 2.1 and Remark 2.1. The strategy for the proof of Theorem 2.1 is the following. Firstly, we construct an isofrequencial KAM theorem (Theorem 2.2, Section 2.1) which is well suited to properly degenerate quasi integrable Hamiltonians in action–angle variables

$$H(J,\psi) = h(J) + f(J,\psi)$$

i.e., with non degenerate integrable part $(\partial^2 \mathbf{h} \neq 0)$ which splits as

$$\mathbf{h}(J) = \bar{\mathbf{h}}(\bar{J}) + \hat{\mathbf{h}}(J) \qquad J = (\bar{J}, \hat{J})$$

where $\bar{\mathbf{h}}$ (thought dominant) depends only on a part of the action variables \bar{J} and $\bar{\mathbf{h}}$ (thought small) depends on all the actions. The peculiarity of this theorem is the one of choosing (the idea goes back to Arnol'd [3]) generalized $(\gamma, \varepsilon \hat{\gamma}; \tau)$ -Diophantine frequencies (definition) and its smallness condition is

$$c_0 \frac{F \max\{M, \hat{M}^{-1}, N\}}{(\min\{\gamma/M, \varepsilon \hat{\gamma}/\hat{M}, R\})^2} < 1$$
(2.10)

where c_0 is a universal constant, F, M, N, \hat{M} , R, are a measure of ⁹ f, $\partial^2 h$, $(\partial^2 h)^{-1}$, $\partial^2 \hat{h}$ and the strength of \hat{J} , respectively. Next, we reduce the properly degenerate Hamiltonian (2.3) to the form

$$\mathbf{H} = \mathbf{h}_0(\bar{J}) + \varepsilon \mathbf{h}_1(J) + \varepsilon r^5 \mathbf{f}(J, \psi) := \mathbf{h}(J) + \varepsilon r^5 \mathbf{f}(J, \psi) .$$
(2.11)

where h_0 is ε -close to the unperturbed h of \mathcal{H} . This reduction is based on averaging theory, developed by Biasco et al., [7] (which carries itself a smallness condition $\bar{\gamma} > \gamma_* \sqrt{\varepsilon} (\log r^{-1})^{\bar{\tau}+1}$ for the diophantine constant of the unperturbed frequencies ∂h), Birkhoff Theory (Appendix B) and use of symplectic polar coordinates. Taking then $F = C \varepsilon r^5$, M = C, $N = C\varepsilon^{-1}$, $\hat{M} = C\varepsilon$, $R = Cr^2$ (this is due to the use of symplectic polar coordinates), the smallness condition (2.10) essentially becomes

$$c_1 \frac{r^5}{\min\{\gamma^2, \hat{\gamma}^2, r^4\}} < 1 \tag{2.12}$$

and it will be guaranteed as soon as r is small and γ , $\hat{\gamma}$ are chosen not smaller that $\gamma_* r^{5/2}$. The condition $\hat{\gamma} < \gamma^* r^2$ is necessary to find a not empty ∂h -pre image of $\mathcal{D}_{\gamma,\varepsilon\hat{\gamma}}^{\bar{n},\hat{n}}$. Observe the cancellation of ε from (2.10) to (2.12), which makes us take r as perturbative parameter.

Remark 2.2 (Proof of THEOREM 1 and other details) The formulation we have chosen for Theorem 2.1 is very general. Even if the parameters $\bar{\gamma}$, γ , $\hat{\gamma}$, in principle, might assume any value in the ranges (2.5), nonetheless, in order to get the tori density as large as possible, the gamma–constants $\bar{\gamma}$, γ , $\hat{\gamma}$ (which are related to the amount of irrationality of the the unperturbed frequencies ∂h and tori frequencies ν) should be taken as small a possible. The choice

$$\begin{cases} \bar{\gamma} = \gamma_0 \max\left\{\sqrt{\varepsilon} (\log r^{-1})^{\bar{\tau}+1}, \sqrt[3]{\varepsilon r} (\log r^{-1})^{\bar{\tau}+1}, r^2 (\log r^{-1})^{\bar{\tau}+1}\right\} \\ \gamma = \hat{\gamma} = \gamma_0 r^{5/2} \end{cases}$$
(2.13)

⁹As usual, if h is real-analytic on \mathcal{I}_{ρ} , the symbol ∂ h denotes its gradient $(\partial_{I_1} \mathbf{h}, \cdots, \partial_{I_p} \mathbf{h})$; the Hessian $\partial^2 \mathbf{h}$ is the $p \times p$ matrix with entries $\partial^2_{I_i I_j} \mathbf{h}$ (where *i* is the row, *j* the coloumn).

(with a fixed $\gamma_0 > \gamma_*(\log \gamma_*)^{\tau+1}$) leads to an invariant set

$$\mathcal{K}(\varepsilon, r) := \mathfrak{K}|_{(\bar{\gamma}, \gamma, \hat{\gamma}) = (2.13)} \tag{2.14}$$

with density just the one announced in the Introduction (as $\varepsilon r \leq \max{\{\varepsilon^2, r^2\}}$):

$$\begin{aligned} 1 - \gamma_0 \left(\bar{\gamma} + \gamma + \frac{\hat{\gamma}}{r^2} + r^{\hat{n}/2} \right) &\geq & 1 - \hat{C} \Big(\sqrt{\varepsilon} (\log r^{-1})^{\bar{\tau} + 1} + \sqrt[3]{\varepsilon r} (\log r^{-1})^{\bar{\tau} + 1} \\ &+ & r^2 (\log r^{-1})^{\bar{\tau} + 1} + r^{5/2} \\ &+ & \sqrt{r} + r^{\hat{n}/2} \Big) \\ &\geq & 1 - \tilde{C} \Big(\sqrt{\varepsilon} (\log r^{-1})^{\bar{\tau} + 1 + b} + \sqrt{r} \Big) \end{aligned}$$

Furthermore, denoting by

$$\begin{cases} \mathcal{J}_{*}(\varepsilon, r) := \mathfrak{J}_{*}|_{(\bar{\gamma}, \gamma, \hat{\gamma})=(2.13)} \\ \omega_{*}(\varepsilon, r, \cdot) = (\bar{\omega}_{*}(\varepsilon, r, \cdot), \varepsilon \hat{\omega}_{*}(\varepsilon, r, \cdot)) := \varpi = (\bar{\varpi}, \varepsilon \hat{\varpi})|_{(\bar{\gamma}, \gamma, \hat{\gamma})=(2.13)} \\ \mathcal{F}(\varepsilon, r, \cdot, \cdot) = (\mathcal{F}_{I}(\varepsilon, r, \cdot, \cdot), \mathcal{F}_{\varphi}(\varepsilon, r, \cdot, \cdot), \mathcal{F}_{p}(\varepsilon, r, \cdot, \cdot), \mathcal{F}_{q}(\varepsilon, r, \cdot, \cdot)) := \\ \mathfrak{F} = (\mathfrak{F}_{I}, \mathfrak{F}_{\varphi}, \mathfrak{F}_{p}, \mathfrak{F}_{q})|_{(\bar{\gamma}, \gamma, \hat{\gamma})=(2.13)} \end{cases}$$

with

$$\begin{aligned} \mathcal{F}_{I}(\varepsilon,r,\nu;\vartheta) &= \bar{j}_{*}(\varepsilon,r,\nu) + \bar{u}(\varepsilon,r,\nu;\vartheta) \\ \mathcal{F}_{\varphi}(\varepsilon,r,\nu;\vartheta) &= \bar{\vartheta} + \bar{v}(\varepsilon,r,\nu;\vartheta) \\ \mathcal{F}_{p}(\varepsilon,r,\nu;\vartheta) &= p_{0}(\varepsilon,r,\nu;\vartheta) + \sqrt{2\hat{j}_{*}(\varepsilon,r,\nu) + 2\hat{u}(\varepsilon,r,\nu;\vartheta)} \\ &\times \cos\left[\hat{\vartheta} + \hat{v}(\varepsilon,r,\nu;\vartheta)\right] \\ \mathcal{F}_{q}(\varepsilon,r,\nu;\vartheta) &= q_{0}(\varepsilon,r,\nu;\vartheta) + \sqrt{2\hat{j}_{*}(\varepsilon,r,\nu) + 2\hat{u}(\varepsilon,r,\nu;\vartheta)} \\ &\times \sin\left[\hat{\vartheta} + \hat{v}(\varepsilon,r,\nu;\vartheta)\right] \end{aligned}$$

then, the bounds $(2.8) \div (2.9)$ imply

$$\sup_{\mathcal{J}_{*}(\varepsilon,r)} |\bar{\omega}_{*} - \partial h| \leq C_{*} \max\left\{\sqrt{\varepsilon}, \sqrt[3]{\varepsilon r}, r^{5/2}\right\}$$

$$\sup_{\mathcal{J}_{*}(\varepsilon,r)} |\hat{\omega}_{*} - \Omega| \leq \max\left\{\sqrt{\varepsilon}, \sqrt[3]{\varepsilon r}, (\log r^{-1})^{-1}\right\}$$

$$\sup_{\Omega_{*}(\varepsilon,r) \times \mathbb{T}^{n}} |\bar{u}| \leq C_{*} \varepsilon r^{5/2}$$

$$\sup_{\Omega_{*}(\varepsilon,r) \times \mathbb{T}^{n}} |\hat{u}| \leq C_{*} r^{5/2}$$

$$\sup_{\Omega_{*}(\varepsilon,r) \times \mathbb{T}^{n}} |p_{0}|, \sup_{\Omega_{*}(\varepsilon,r) \times \mathbb{T}^{n}} |q_{0}| \leq C_{*} (\log r^{-1})^{-1}$$
(2.15)

Remark 2.3 (A physical comment) The tori in $\mathcal{K}(\varepsilon, r)$ may be thought of the analytic continuation of the tori

$$\begin{cases} I = \bar{I} \\ \varphi \in \mathbb{T}^n \\ (p_i - p_{0i})^2 + (q_i - q_{0i})^2 = \bar{R}_i^2 \qquad Cr^{5/2} \le R_i^2 \le cr^2 \end{cases}$$
(2.16)

being crossed by the system with frequencies $\sqrt{\varepsilon} + (\log r^{-1})^{-1}$ -close to $(\partial h, 0)$. Observe, in particular, that p_0, q_0, \bar{u} go to 0 with ε , for any fixed $0 < r < r_*$.

Notice that if

$$\varepsilon < \operatorname{const} r^2 (\log r^{-1})^{-2(\bar{\tau}+1)},$$
 (2.17)

if $\bar{\gamma}$ is not too small, for instance, $\bar{\gamma} = \text{const} \sqrt{\varepsilon} (\log r^{-1})^{-\bar{\tau}+1}/r$ we than have

$$|(p_0, q_0)| \le \operatorname{const} \sqrt{\varepsilon} (\log r^{-1})^{2(\bar{\tau}+1))} \bar{\gamma}^{-2} = \operatorname{const} r ,$$

i.e., the (p,q) variables into a ball of radius r around the origin (comapare also (2.16)). This agrees with the result obtained in Arnol'd's FUNDAMENTAL THEOREM. When (2.17) is no longer satisfied, we generally have a set of invariant tori for which the (p,q)-variables can stay away from the origin as far as $(\log r^{-1})^{-1}$: compare (2.15) above.

2.1 A Two Times Scale KAM Theorem

In order to state Theorem 2.2 below, we introduce some useful notations and definitions.

(i) The *r*-neighbrhood \mathcal{I}_r of $\mathcal{I} \subset \mathbb{R}^p$ compact and the *s*-neighborhood \mathbb{T}_s^p of \mathbb{T}^p are defined as:

$$\mathcal{I}_r := \bigcup_{I \in \mathcal{I}} D_r^p(I) , \quad \mathbb{T}_s^p := \{ \varphi = (\varphi_1, \cdots, \varphi_p) \in \mathbb{T}_{\mathbb{C}}^p : |\operatorname{Im} \varphi_i| < s) \}$$

where $\mathbb{T}^p_{\mathbb{C}} := \mathbb{C}^p / 2\pi \mathbb{Z}^p$ and

$$D_r^p(I) = \{ I' \in \mathbb{C}^p : |I' - I| < r \}$$

is the usual complex open *p*-ball, where \mathbb{C}^p is equipped with the standard Euclidean norm: $|(I_1, \dots, I_p)| := \sqrt{\sum_{1 \le i \le p} |I_i|^2}$.

- (ii) Real-analytic functions $f: \mathcal{P} \subset \mathbb{R}^{2p} \to \mathbb{R}$ on compact sets $\mathcal{P} = \mathcal{I} \times \mathbb{T}^p \subset \mathbb{R}^{2p}$ $(\mathcal{I} \subset \mathbb{R}^p \text{ compact})$ are identified with their analytic extensions $\bar{f}: \mathcal{P}_{r,s} \subset \mathbb{C}^{2p} \to \mathbb{C}$ over a suitable (r, s)-neighborhood $\mathcal{P}_{r,s} = \mathcal{I}_r \times \mathbb{T}_s^p$ of their real domain.
- (iii) The "sup-Fourier" norm $||f||_{r,s}$ of a real-analytic function f on $\mathcal{P}_{r,s}$ is

$$\|\mathbf{f}\|_{r,s} := \sum_{k \in \mathbb{Z}^p} \sup_{I \in \mathcal{I}_r} |\mathbf{f}_k(I)| e^{|k|s}$$

where |k| is the 1-norm

$$|(k_1,\cdots,k_p)| := \sum_{1 \le i \le p} |k_i|$$

and

$$\mathbf{f}_k(I) := rac{1}{(2\pi)^p} \int_{\mathbb{T}^p} \mathbf{f}(I, \varphi) e^{i k \cdot \varphi} d\varphi$$

is its k^{th} Fourier coefficient.

(iv) If A is a $n \times n$ matrix and $1 \le p, q \le n$ the symbol $A^{[p,q]}$ denotes the $p \times q$ sub–matrix of the first p rows and the last q coloumns of A, *i.e.*, the matrix with elements

$$A_{ij}^{[p,q]} = A_{i,n-q+j} \quad 1 \le i \le p \ , \quad 1 \le j \le q \ .$$

Conversely, $A_*^{[p,q]}$ denotes the $p \times q$ sub–matrix of A of the last p rows and first q coloumns.

(v) We recall that $f: \mathcal{I} \subset \mathbb{R}^p \to \mathbb{R}^p$ is Lipschitz if

$$\mathcal{L}_{\parallel \cdot \parallel}(f) := \sup_{I \neq I' \in \mathcal{I}} \frac{\|f(I) - f(I')\|}{\|I - I'\|} < +\infty$$

(with $\|\cdot\|$ a somewhat norm of \mathbb{R}^p). For a Lipschitz function f, we denote by $\mathcal{L}(f)$ the number

$$\mathcal{L}(f) := \sup_{I \neq I' \in \mathcal{I}} \frac{|f(I) - f(I')|}{|I - I'|}$$

(with respect to the Euclidean norm) and call it Lipschitz constant for f. We define the ρ -Lipschitz norm of f on \mathcal{I}

$$\|f\|_{\rho,\mathcal{I}}^{\operatorname{Lip}} := \rho^{-1} \sup_{\mathcal{I}} |f| + \mathcal{L}(f) .$$

(vi) f is called *bi–Lipschitz* if f is Lipschitz, injective, with Lipschitz inverse, or, equivalently, if there exist $0 < \mathcal{L}_{-}(f) \leq \mathcal{L}_{+}(f)$, called *Lipschitz constants*, such that

$$\mathcal{L}_{-}(f)|I - I'| \le |f(I) - f(I')| \le \mathcal{L}_{+}(f)|I - I'| \quad \text{for all} \quad I, I' \in \mathcal{I} ,$$

where

$$\mathcal{L}_+(f) = \mathcal{L}(f)$$
, $\mathcal{L}_-(f) = \frac{1}{\mathcal{L}(f^{-1})}$

Theorem 2.2 Let $\bar{n}, \hat{n} \in \mathbb{N}, n := \bar{n} + \hat{n}, \tau > n, \gamma \ge \hat{\gamma} > 0, 0 < 2s \le \bar{s} < 1, \bar{\mathcal{I}} \subset \mathbb{R}^{\bar{n}}, \hat{\mathcal{I}} \subset \mathbb{R}^{\hat{n}} \mathcal{I} := \bar{\mathcal{I}} \times \hat{\mathcal{I}}$ such that

$$H(J,\psi) = h(J) + f(J,\psi)$$

real-analytic on $\mathcal{I}_{\rho} \times \mathbb{T}^n_{\overline{s}+s}$. Assume that $\omega := \partial h$ is a diffeomorphism of \mathcal{I}_{ρ} and the Hessian matrix $U := \partial^2 h$ is non singular on \mathcal{I}_{ρ} . Let

$$M \geq \sup_{\mathcal{I}_{\rho}} \|U\|$$
$$\hat{M} \geq \sup_{\mathcal{I}_{\rho}} \|U^{[n,\hat{n}]}\|$$
$$N \geq \sup_{\mathcal{I}_{\rho}} \|T\|$$
$$\bar{N} \geq \sup_{\mathcal{I}_{\rho}} \|T^{[\bar{n},n]}\|$$
$$\hat{N} \geq \sup_{\mathcal{I}_{\rho}} \|T^{[\hat{n},n]}\|$$
$$F \geq \|\mathbf{f}\|_{\rho,s}$$

where $T := U^{-1}$, define

$$\begin{aligned} c &:= \max\left\{2^{11}n, \ \frac{2}{3}(12)^{\tau+1}\right\} \\ K &:= \frac{6}{s} \ \log_+\left(\frac{FM^2 L}{\gamma^2}\right)^{-1} \quad where \quad \log_+(a) := \max\{1, \log a\} \\ \tilde{\rho} &:= \min\left\{\frac{\gamma}{3MK^{\tau+1}} \ , \ \frac{\hat{\gamma}}{3\hat{M}K^{\tau+1}} \ , \ \rho\right\} \\ L &:= \max\{N \ , M^{-1}, \hat{M}^{-1}\} \end{aligned}$$

and assume the "perturbation" f so "small" that the following "KAM condition" holds

$$cE := c\frac{FL}{\tilde{\rho}^2} < 1 .$$

(i) Then, for any frequency $\nu \in \Omega_* := \omega(\mathcal{I}) \cap \mathcal{D}_{\gamma,\hat{\gamma},\tau}^{\bar{n},\hat{n}}$, there exists a unique Lagrangian KAM torus $T_{\nu} \subset \operatorname{Re}(\mathcal{I}_{34\tilde{\rho}E}) \times \mathbb{T}^n$ for H with frequency ν , such that the following holds. There exists a "Cantor" set $\mathcal{I}_* \subset \operatorname{Re}(\mathcal{I}_{32\tilde{\rho}E})$ and a bi-Lipschitz (onto) homeomorphism

$$\omega_* = (\bar{\omega}_*, \hat{\omega}_*): \ \mathcal{I}_* \to \Omega_* \tag{2.19}$$

satisfying

$$\sup_{\mathcal{O}_{*}} |\bar{\omega}_{*}^{-1} - \bar{\omega}^{-1}| \le 2^{5} \frac{\bar{N}}{N} \tilde{\rho} E , \quad \sup_{\mathcal{O}_{*}} |\hat{\omega}_{*}^{-1} - \hat{\omega}^{-1}| \le 2^{5} \frac{\bar{N}}{N} \tilde{\rho} E$$
(2.20)

$$\sup_{\mathcal{I}_*} |\bar{\omega}_* - \bar{\omega}| \le 2^5 M \tilde{\rho} E , \quad \sup_{\mathcal{I}_*} |\hat{\omega}_* - \hat{\omega}| \le 2^5 \hat{M} \tilde{\rho} E$$
(2.21)

$$\|\omega_*^{-1} \circ \omega - \operatorname{id}\|_{\tilde{\rho}, \mathcal{I}_{\gamma, \hat{\gamma}, \tau}}^{\operatorname{Lip}} \leq 2^{11} E , \qquad \mathcal{I}_{\gamma, \hat{\gamma}, \tau} := \omega^{-1}(\mathcal{D}_{\gamma, \hat{\gamma}, \tau}^{n, \hat{n}}) \cap \mathcal{I} .$$
 (2.22)

such that T_{ν} is realized by the real-analytic embedding $\phi_{\nu} = (\phi_{\nu I}, \phi_{\nu \varphi})$ given by

$$\begin{cases} \phi_{\nu I}(\vartheta) = I_*(\nu) + v(\nu, \vartheta) \\ \phi_{\nu \varphi}(\vartheta) = \vartheta + u(\nu, \vartheta) \end{cases} \qquad \qquad \vartheta \in \mathbb{T}^n , \qquad (2.23)$$

where $I_*(\nu) := \omega_*^{-1}(\nu)$ and $v = (\bar{v}, \hat{v}), u = (\bar{u}, \hat{u})$ are bounded as

$$\left|\bar{v}(\nu,\vartheta)\right| \le 2\frac{\gamma}{\gamma}E\,\tilde{\rho}\,\,,\quad \left|\hat{v}(\nu,\vartheta)\right| \le 2\,E\,\tilde{\rho}\,\,,\quad \left|u(\nu,\vartheta)\right| \le 2\,E\,s \tag{2.24}$$

(ii) The measure of the invariant set $\mathbf{K} = \phi_{\Omega_*}(\mathbb{T}^n)$ satisfies

$$\max \left(\mathcal{I} \times \mathbb{T}^n \setminus \mathbf{K} \right) \leq \left(1 + (1 + 2^7 E)^{2n} \right) \left(1 + (1 + 2^{10} E)^n \right) \max \left(\mathcal{I} \setminus \mathcal{I}_{\gamma, \hat{\gamma}, \tau} \times \mathbb{T}^n \right)$$

+ $\left(1 + (1 + 2^7 E)^{2n} \right) (1 + 2^{10} E)^n \max \left(\mathcal{I}_{\rho_1} \setminus \mathcal{I} \times \mathbb{T}^n \right)$
+ $\left(1 + 2^7 E \right)^{2n} \max \left(\mathcal{I}_{\rho_2} \setminus \mathcal{I} \times \mathbb{T}^n \right) .$ (2.25)

where $\rho_1 = 2^6 E \tilde{\rho} / (1 - 2^{10} E), \ \rho_2 = 4 E \tilde{\rho} / (1 - 2^7 E).$

2.1.1 Construction of the Approximating Sequences

The proof of Theorem 2.2 is obtained by infinite iterations of real-analytic symplectomorphisms, converging over a Cantor set. Each iteration is based on Lemma 2.1 below.

Let $\bar{n}, \hat{n}, n = \bar{n} + \hat{n} \in \mathbb{N}, \mathcal{I} \subset \mathbb{R}^{\bar{n}} \times \mathbb{R}^{\hat{n}}$. Following Pöschel [30], we define the \mathcal{P} -norm on $\mathcal{I}_{\rho} \times \mathbb{T}_{s}^{n}$ as

$$|\left((\bar{I},\hat{I}),\varphi\right)|_{\mathcal{P}} := \max\{|\bar{I}|_1, |\hat{I}|_1, |\varphi|_\infty\}$$

where, as usual,

$$|\bar{I}|_1 := \sum_{1 \le i \le \bar{n}} |\bar{I}_i| , \quad |\hat{I}|_1 := \sum_{1 \le i \le \hat{n}} |\hat{I}_i| , \quad |\varphi|_{\infty} := \max_{1 \le i \le n} |\varphi_i|$$

if $\bar{I} = (\bar{I}_1, \dots, \bar{I}_{\bar{n}}), \ \hat{I} = (\hat{I}_1, \dots, \hat{I}_{\hat{n}}), \ \varphi = (\varphi_1, \dots, \varphi_n)$. We also introduce the matrices $W^p_{\alpha,\beta}, W_{\alpha,\beta} \ (1 \le p \le 2n)$ which are defined as the $(2p) \times (2p)$ diagonal matrices

$$W_{\alpha,\beta}^p := (\alpha^{-1} \operatorname{id}_p, \beta^{-1} \operatorname{id}_{2n-p}), \quad W_{\alpha,\beta} := W_{\alpha,\beta}^n = (\alpha^{-1} \operatorname{id}_n, \beta^{-1} \operatorname{id}_n)$$

where id_p is the identity matrix with order p.

Lemma 2.1 (Averaging Theorem) Let $H(I, \varphi) = h(I) + f(I, \varphi)$ real-analytic on $\mathcal{P}_{r,\bar{s}+s} := \mathcal{I}_r \times \mathbb{T}^n_{\bar{s}+s}$. Assume that $\omega := \partial h$ verifies

$$|\omega(I) \cdot k| \ge \begin{cases} \bar{\alpha} & \text{for} \quad k = (\bar{k}, \hat{k}) \in \mathbb{Z}^{\bar{n}} \times \mathbb{Z}^{\hat{n}} \setminus \Lambda \quad \bar{k} \neq 0 , \quad |k| \le K \\ \hat{\alpha} & \text{for} \quad k = (0, \hat{k}) \in \{0\} \times \mathbb{Z}^{\hat{n}} \setminus \Lambda \quad 0 < |\hat{k}| \le K \end{cases}$$
(2.26)

where $\Lambda \subseteq \mathbb{Z}^n$. If

$$\|\mathbf{f}\|_{r,\bar{s}+s} \le \frac{1}{2^7} \,\frac{\alpha \, r}{K} \,\,, \tag{2.27}$$

where $\alpha := \min\{\bar{\alpha}, \hat{\alpha}\}$ and $Ks \ge 6$, then, there exists a real-analytic, symplectic coordinate transformation

$$\Psi: \mathcal{P}_{r/2,\bar{s}+s/6} \to \mathcal{P}_{r,\bar{s}+s}$$

such that

$$\mathrm{H} \circ \Psi(I, \varphi) = \mathrm{h}(I) + g(I, \varphi) + \mathrm{f}_*(I, \varphi)$$

with $g(I,\varphi) = \sum_{k \in \Lambda} g_k(I) e^{ik \cdot \varphi}$ is Λ -completely resonant and

$$\|g - f_0\|_{r/2,\bar{s}+s/6} \le \frac{2^5 K}{\alpha r} \|f\|_{r,\bar{s}+s}^2 , \quad \|f_*\|_{r/2,\bar{s}+s/6} \le e^{-Ks/6} \|f\|_{r,\bar{s}+s}$$

where $f_0 := P_{\Lambda}T_K f$. Moreover, the following bounds hold, uniformly on $\mathcal{P}_{r/2,\bar{s}+s/6}$

$$|W^{\bar{n}}_{\alpha/\bar{\alpha},1}W_{r,s}(\Psi - \mathrm{id})|_{\mathcal{P}} \le \frac{2K \|\mathbf{f}\|_{r,\bar{s}+s}}{\alpha r} , \qquad (2.28)$$

and

$$\|W_{\alpha/\bar{\alpha},1}^{\bar{n}}(W_{r,s}D\Psi W_{r,s}^{-1} - \operatorname{id}_{2n})\|_{\mathcal{P}} \le \frac{2^{6}K}{\alpha r} \|f\|_{r,\bar{s}+s} .$$
(2.29)

where $\|\cdot\|_{\mathcal{P}}$ denotes the operatorial norm induced by $|\cdot|_{\mathcal{P}}$ ¹⁰.

Lemma 2.1 is a useful remake of the Normal Form Lemma of [30]. For sake of completeness, its proof may be found in Appendix A.

Lemma 2.2 (Iterative Lemma) Let $0 < \hat{\gamma} \leq \gamma$, $\overline{\mathcal{I}} \subset \mathbb{R}^{\bar{n}}$, $\hat{\mathcal{I}} \subset \mathbb{R}^{\hat{n}}$ compact sets, put $\mathcal{I} := \overline{\mathcal{I}} \times \hat{\mathcal{I}}$ and let

$$H(J,\psi) = h(J) + f(J,\psi)$$

real-analytic on $\mathcal{I}_{\rho} \times \mathbb{T}^{n}_{\bar{s}+s}$, with $\bar{s} > 0$, 0 < s < 1. Assume that $\omega := \partial h$ is a diffeomorphism of \mathcal{I}_{ρ} with Jacobian matrix $U := \partial^{2}h$ non singular on \mathcal{I}_{ρ} and $\omega(\mathcal{I}) \subseteq \mathcal{D}^{\bar{n},\hat{n}}_{\gamma,\hat{\gamma},\tau}$. Let

$$M \ge \sup_{\mathcal{I}_{\rho}} \|U\|$$
$$\hat{M} \ge \sup_{\mathcal{I}_{\rho}} \|U^{[n,\hat{n}]}\|$$
$$N \ge \sup_{\mathcal{I}_{\rho}} \|T\|$$
$$\bar{N} \ge \sup_{\mathcal{I}_{\rho}} \|T^{[\bar{n},n]}\|$$
$$\hat{N} \ge \sup_{\mathcal{I}_{\rho}} \|T^{\hat{n},n}\|$$
$$F \ge \|f\|_{\rho,\bar{s}+s}$$

¹⁰I.e., $||A||_{\mathcal{P}} := \sup_{z=(I,\varphi), |z|_{\mathcal{P}}=1} |Az|_{\mathcal{P}}$

where $T := U^{-1}$, define

$$\begin{split} K &:= \frac{6}{s} \log_+ \left(\frac{FM^2 L}{\gamma^2}\right)^{-1} \\ \tilde{\rho} &:= \min\left\{\frac{\gamma}{3MK^{\tau+1}} , \frac{\hat{\gamma}}{3\hat{M}K^{\tau+1}} , \rho\right\} \\ L &:= \max\left\{N , M^{-1}, \hat{M}^{-1}\right\} \end{split}$$

and assume

$$2^8 nE := 2^8 n \frac{FL}{\tilde{\rho}^2} \le 1 .$$
 (2.30)

Then, a set $\mathcal{I}_+ \subset \mathcal{I}_{\tilde{\rho}/32}$, two numbers $\rho_+ > 0$, $0 < s_+ < 1$ and a symplectic analytic transformation

$$\Psi : \mathcal{I}_{+\rho_{+}} \times \mathbb{T}^{n}_{\bar{s}+s_{+}} \to \mathcal{I}_{\rho} \times \mathbb{T}^{n}_{\bar{s}+s} ,$$

$$(I_{+}, \varphi_{+}) \to (I, \varphi) = \Psi(I_{+}, \varphi_{+})$$

may be found, putting H into the form

$$H_{+}(I_{+},\varphi_{+}) := H \circ \Psi_{+}(I_{+},\varphi_{+}) = h_{+}(I_{+}) + f_{+}(I_{+},\varphi_{+}) , \qquad (2.31)$$

where $\omega_+ := \partial h_+$ is a diffeomorphism of $\mathcal{I}_{+\rho_+}$ with Jacobian matrix $U_+ := \partial^2 h_+$ non singular on $\mathcal{I}_{+\rho_+}$ such that $\omega_+(\mathcal{I}_+) = \omega(\mathcal{I})$. There also exist suitable constants

$$M_{+} \geq \sup_{\mathcal{I}_{+\rho+}} \|U_{+}\|$$
$$\hat{M}_{+} \geq \sup_{\mathcal{I}_{+\rho+}} \|U_{+}^{[n,\hat{n}]}\|$$
$$N_{+} \geq \sup_{\mathcal{I}_{+\rho+}} \|T_{+}\|$$
$$\bar{N}_{+} \geq \sup_{\mathcal{I}_{+\rho+}} \|T_{+}^{[\bar{n},n]}\|$$
$$\hat{N}_{+} \geq \sup_{\mathcal{I}_{+\rho+}} \|T_{*}^{[\hat{n},n]}\|$$
$$F_{+} \geq \|f_{+}\|_{\rho_{+},\bar{s}+s_{+}}$$

where $T_+ := U_+^{-1}$, such that, defining

$$K_{+} := \frac{6}{s_{+}} \log_{+} \left(\frac{F_{+}M_{+}^{2}L_{+}}{\gamma^{2}} \right)^{-1}$$
$$\tilde{\rho}_{+} := \min \left\{ \frac{\gamma}{3M_{+}K_{+}^{\tau+1}} , \frac{\hat{\gamma}}{3\hat{M}_{+}K_{+}^{\tau+1}} , \rho_{+} \right\}$$
$$L_{+} := \max \left\{ N_{+} , M_{+}^{-1}, \hat{M}_{+}^{-1} \right\}$$

then,

$$E_{+} := \frac{F_{+}L_{+}}{\tilde{\rho}_{+}^{2}} \le E^{2} .$$
(2.32)

More precisely:

(i) The numbers ρ_+ , s_+ and the constants M_+ , \cdots may be taken as

$$\begin{cases} \rho_{+} = \frac{\tilde{\rho}}{8} \\ s_{+} = \frac{\tilde{s}}{6} \\ M_{+} = 2M \\ \hat{M}_{+} = 2\hat{M} \\ N_{+} = 2N \\ \bar{N}_{+} = 2\bar{N} \\ \hat{N}_{+} = 2\hat{N} \\ \hat{N}_{+} = 2\hat{N} \\ F_{+} = \frac{F^{2}LM^{2}}{\gamma^{2}} \end{cases}$$

so as to satisfy

$$\frac{\tilde{\rho}_+}{\tilde{\rho}} \ge \frac{1}{2} \left(\frac{1}{12}\right)^{\tau+1} ;$$

(ii) the set \mathcal{I}_+ may be obtained as $\mathcal{I}_+ = l_+(\mathcal{I}),$ where

(iii) $l_+: \mathcal{I} \to \mathcal{I}_+$ is an injective "isofrequency map", i.e., uniquely defined on \mathcal{I} by

$$\omega_+ \circ l_+ = \omega \quad on \quad \mathcal{I}$$

which satisfies

$$N/\bar{N}\sup_{\mathcal{I}} |\bar{l}_{+} - \mathrm{id}|, \ N/\hat{N}\sup_{\mathcal{I}} |\hat{l}_{+} - \mathrm{id}| \le 2^{4}E\tilde{\rho}, \quad \mathcal{L}_{\mathcal{P}}(l_{+} - \mathrm{id}) \le 2^{8}E; \ (2.33)$$

(iv) the map Ψ satisfies

$$|W^{\bar{n}}_{\hat{\gamma}/\gamma,1}W_{\tilde{\rho},s}(\Psi_{+} - \mathrm{id})|_{\mathcal{P}} \le E$$
(2.34)

and

$$\left\| W^{\bar{n}}_{\hat{\gamma}/\gamma,1}(W_{\bar{\rho},s}D\Psi_{+}W^{-1}_{\bar{\rho},s}-I_{2p}) \right\|_{\mathcal{P}} \le 2^{5} E .$$
(2.35)

Proof. We proceed by steps.

<u>Claim 0.</u>: H is put into the form

 $H_+(I_+,\varphi_+) := H \circ \Psi_{av}(I_+,\varphi_+) = h(I_+) + g(I_+) + f_+(I_+,\varphi_+) := h_+(I_+) + f_+(I_+,\varphi_+)$ where

$$\sup_{\mathcal{I}_{\tilde{\rho}/2}} |\mathbf{g} - \langle \mathbf{f} \rangle_{\varphi}| \le \frac{F}{16}$$
(2.36)

and

$$\|\mathbf{f}_{+}\|_{\tilde{\rho}/2,\bar{s}+s/6} \le \frac{F^2 M^2 L}{\gamma^2} =: F_{+} , \qquad (2.37)$$

by means of a real-analytic symplectomorphism Ψ_{av} defined on $\mathcal{I}_{\tilde{\rho}/2} \times \mathbb{T}^n_{\bar{s}+s/6}$ and verifying

$$|W_{\hat{\gamma}/\gamma,1}^{n}W_{r,s}(\Psi_{\rm av}-I)|_{\mathcal{P}} \leq E , ||W_{\hat{\gamma}/\gamma,1}^{n}(W_{r,s}D\Psi_{\rm av}W_{r,s}^{-1}-I)||_{\mathcal{P}} \leq 2^{5}E .$$
(2.38)

<u>Proof.</u> We are going to construct the transformation Ψ_{av} by means of application of the Averaging Theorem (Lemma 2.1) to H, on the domain $\mathcal{I}_{\tilde{\rho}} \times \mathbb{T}^n_{\bar{s}+s/6}$, with the trivial resonant lattice $\Lambda = \{0\} \in \mathbb{Z}^n$. We first verify the "non resonance" assumption (2.26) out of $\Lambda = \{0\}$. By assumption, for any $I \in \mathcal{I}$, $\omega(I) \in D^{\bar{n},\hat{n}}_{\gamma,\hat{\gamma},\tau}$, which means

$$|\omega(I) \cdot k| \ge \begin{cases} \frac{\gamma}{|k|^{\tau}} & \text{if} \quad k = (\bar{k}, \hat{k}) & \text{with} \quad \bar{k} \neq 0\\ \frac{\hat{\gamma}}{|k|^{\tau}} & \text{if} \quad k = (0, \hat{k}) & \text{with} \quad \hat{k} \neq 0 \end{cases}$$

for any $k \in \mathbb{Z}^{\bar{n}} \times \mathbb{Z}^{\hat{n}} \setminus \{0\}$. In particular, for $0 < |k| \le K$ and $I \in \mathcal{I}$,

$$|\omega(I) \cdot k| \ge \begin{cases} \frac{\gamma}{K^{\tau}} & \text{if} \quad k = (\bar{k}, \hat{k}) & \text{with} \quad \bar{k} \neq 0\\ \frac{\hat{\gamma}}{K^{\tau}} & \text{if} \quad k = (0, \hat{k}) & \text{with} \quad \hat{k} \neq 0 \end{cases}$$

Let, now, $I \in \mathcal{I}_{\tilde{\rho}}$. By definition, there exist $I_0 \in \mathcal{I}$ such that $|I - I_0| < \tilde{\rho}$ and we find (recall that we have prefixed the 1-norm in \mathcal{I})

$$\begin{aligned} |\omega(I) - \omega(I_0)|_{\infty} &:= \max_{1 \le i \le p} |\omega_i(I) - \omega_i(I_0)| \\ &\leq \sup_{\mathcal{I}_{\rho}} ||\partial \omega|| |I - I_0| \\ &\leq M \tilde{\rho} \end{aligned}$$

Hence, if $k = (\bar{k}, \hat{k})$, with $\bar{k} \neq 0$ and $0 < |k| \le K$

$$\begin{aligned} |\omega(I) \cdot k| &\geq |\omega(I_0) \cdot k| - |(\omega(I) - \omega(I_0)) \cdot k| \\ &\geq \frac{\gamma}{K^{\tau}} - |\omega(I) - \omega(I_0)|_{\infty} |k| \\ &\geq \frac{\gamma}{K^{\tau}} - MK\tilde{\rho} \\ &\geq \frac{2}{3}\frac{\gamma}{K^{\tau}} =: \bar{\alpha} \end{aligned}$$

having used $\tilde{\rho} \leq \frac{\gamma}{3MK^{\tau+1}}$. Similarly, taking $k = (0, \hat{k})$, with $0 < |k| = |\hat{k}| \leq K$ and using $\tilde{\rho} \leq \frac{\hat{\gamma}}{3\hat{M}K^{\tau+1}}$, we find that

$$|\omega(I) \cdot k| \ge \frac{2}{3} \frac{\hat{\gamma}}{K^{\tau}} =: \hat{\alpha} \text{ for any } I \in \mathcal{I}_{\tilde{\rho}}, \quad 0 < |k| \le K.$$

But, then, the "smallness condition" (2.27) with $r = \tilde{\rho}$ is also verified in $\mathcal{I}_{\tilde{\rho}}$, because, using

$$\alpha = \min\left\{\frac{2}{3}\frac{\gamma}{K^{\tau}}, \ \frac{2}{3}\frac{\hat{\gamma}}{K^{\tau}}\right\} \ge 2K\min\{M, \hat{M}\}\tilde{\rho} \ge 2K\frac{\tilde{\rho}}{L} ,$$

we find that (2.30) is stronger:

$$2^7 \frac{\|f\|_{\tilde{\rho},\bar{s}+s}K}{\alpha\tilde{\rho}} \le 2^6 \frac{FL}{\tilde{\rho}^2} < 1$$

Therefore, Lemma 2.1 applies (as also, trivially, $Ks = 6 \log_+(FM^2L/\gamma^2) \ge 6$), and H is put into the form

$$H_+:=H\circ\Psi_{av}=h+g+f_+:=h_++f_+$$

by means of a real-analytic symplectomorphism Ψ_{av} defined on $\mathcal{I}_{\tilde{\rho}/2} \times \mathbb{T}^n_{\bar{s}+s/6}$, where g is a $\{0\}$ -completely resonant (which means that g is a function of I only) real-analytic function, suitably close to $f_0 = P_{\{0\}}T_K f = \langle f \rangle_{\varphi}$:

$$\sup_{\mathcal{I}_{\bar{\rho}/2}} |\mathbf{g} - \mathbf{f}_0| = \|\mathbf{g} - \mathbf{f}_0\|_{\bar{\rho}/2, \bar{s}+s/6} \le \frac{2^5 K}{\alpha \, \bar{\rho}} \|\mathbf{f}\|_{\bar{\rho}, \bar{s}+s}^2 \le 2^4 \frac{F^2 L}{\bar{\rho}^2} \le \frac{F}{16}$$

and f_+ is "small":

$$\|\mathbf{f}_{+}\|_{\tilde{\rho}/2,\bar{s}+s/6} \le e^{-Ks/6} \|\mathbf{f}\|_{\tilde{\rho},\bar{s}+s} \le e^{-Ks/6} F = \frac{F^2 M^2 L}{\gamma^2}$$

The bounds (2.38) are an easy consequence of (2.28), (2.29) (recall $\hat{\gamma} < \gamma$):

$$|W^{\bar{n}}_{\hat{\gamma}/\gamma,1}W_{r,s}(\Psi_{\rm av}-I)|_{\mathcal{P}} = |W^{\bar{n}}_{\alpha/\bar{\alpha},1}W_{r,s}(\Psi_{\rm av}-I)|_{\mathcal{P}} \le \frac{2\,K\|\mathbf{f}\|_{\tilde{\rho},\bar{s}+s}}{\alpha\,\tilde{\rho}} \le \frac{FL}{\tilde{\rho}^2} = E \ ,$$

and

$$\begin{split} \|W^{\bar{n}}_{\hat{\gamma}/\gamma,1}(W_{r,s}D\Psi_{\rm av}W^{-1}_{r,s}-I)\|_{\mathcal{P}} &= \|W^{\bar{n}}_{\alpha/\bar{\alpha},1}(W_{r,s}D\Psi_{\rm av}W^{-1}_{r,s}-I)\|_{\mathcal{P}}\\ &\leq \frac{2^{6}K}{\alpha\,\tilde{\rho}}\,\|f\|_{r,\bar{s}+s} \leq 2^{5}E \ , \end{split}$$

Claim 0 is thus proved.

<u>Claim 1</u>: the Jacobian matrix $U_+ := \partial^2 h_+$ is non singular in $\mathcal{I}_{\tilde{\rho}/4}$ and satisfies

$$M_{+} := 2M \ge \sup_{I \in \mathcal{I}_{\rho/4}} \|U_{+}\|$$
(2.39)

$$\hat{M}_{+} := 2\hat{M} \ge \sup_{I \in \mathcal{I}_{q/4}} \|U_{+}^{[n,\hat{n}]}\|$$
(2.40)

$$N_{+} := 2N \ge \sup_{I \in \mathcal{I}_{\rho/4}} \|T_{+}\|$$
(2.41)

$$\bar{N}_{+} := 2\bar{N} \ge \sup_{I \in \mathcal{I}_{\rho/4}} \|T_{+}^{[\bar{n},n]}\|$$
(2.42)

$$\hat{N}_{+} := 2\hat{N} \ge \sup_{I \in \mathcal{I}_{\rho/4}} \|T_{+*}^{[\hat{n},n]}\|$$
(2.43)

where $T_+ := U_+^{-1}$.

<u>*Proof.*</u> : The bound (2.36) gives

$$\sup_{\mathcal{I}_{\tilde{\rho}/2}} |g| \le \sup_{\mathcal{I}_{\tilde{\rho}/2}} |f_0| + \sup_{\mathcal{I}_{\tilde{\rho}/2}} |g - f_0| \le \frac{17}{16} F , \qquad (2.44)$$

whence, applying twice the Pöschel's General Cauchy Inequality (see Appendix D),

$$\sup_{\mathcal{I}_{\tilde{\rho}/4}} \|\partial^2 g\| \le \frac{\sup_{\mathcal{I}_{3\tilde{\rho}/8}} \|\partial g\|}{\tilde{\rho}/8} \le \frac{17/16 F}{(\tilde{\rho}/8)^2} \le 2^7 \frac{F}{\tilde{\rho}^2} < \frac{1}{2L} \le \frac{1}{2} \min\left\{M, \hat{M}, \frac{1}{N}\right\}$$
(2.45)

which suddenly implies

$$\sup_{\mathcal{I}_{\tilde{\rho}/4}} \|U_+\| = \sup_{\mathcal{I}_{\tilde{\rho}/4}} \|\partial^2 \mathbf{h}_+\| = \sup_{\mathcal{I}_{\tilde{\rho}/4}} \|\partial^2 \mathbf{h} + \partial^2 \mathbf{g}\| \le \sup_{\mathcal{I}_{\tilde{\rho}/4}} \|\partial^2 \mathbf{h}\| + \sup_{\mathcal{I}_{\tilde{\rho}/4}} \|\partial^2 \mathbf{g}\| \le M + M = 2M$$

Similarly, one finds

$$\sup_{\mathcal{I}_{\tilde{\rho}/4}} \|U_{+}^{[n,\hat{n}]}\| \le 2\hat{M}$$

But (2.45) also implies

$$\sup_{\mathcal{I}_{\tilde{\rho}/4}} \|\partial^2 g(\partial^2 h)^{-1}\| \leq \sup_{\mathcal{I}_{\tilde{\rho}/4}} \|\partial^2 g\| \sup_{\mathcal{I}_{\tilde{\rho}/4}} \|(\partial^2 h)^{-1}\| \\
\leq \sup_{\mathcal{I}_{\tilde{\rho}/4}} \|\partial^2 g\| N \\
\leq \frac{1}{2}$$
(2.46)

so, the matrix

$$\mathrm{id} \, + \partial^2 \, \mathrm{g} (\partial^2 \, \mathrm{h})^{-1}$$

is non singular on $\mathcal{I}_{\tilde{\rho}/4}$ with

$$\left\| \left(\operatorname{id} + \partial^2 g (\partial^2 h)^{-1} \right)^{-1} \right\| \le 2$$

This implies that

$$\partial^2 h_+ = \partial^2 \mathbf{h} + \partial^2 \mathbf{g} = \left(\operatorname{id} + \partial^2 \, \mathbf{g} (\partial^2 \, \mathbf{h})^{-1} \right) \partial^2 \mathbf{h}$$

is non singular on $\mathcal{I}_{\tilde{\rho}/4}$, with

$$\sup_{\mathcal{I}_{\bar{\rho}/4}} \|T_+\| = \sup_{\mathcal{I}_{\bar{\rho}/4}} \|(\partial^2 h_+)^{-1}\| = \sup_{\mathcal{I}_{\bar{\rho}/4}} \|(\partial^2 h)^{-1} (\operatorname{id} + \partial^2 g(\partial^2 h)^{-1})^{-1}\| \le 2N \ .(2.47)$$

Similarly,

$$\sup_{\mathcal{I}_{\tilde{\rho}/4}} \|T^{[\bar{n},n]}\| \leq 2\,\bar{N} , \qquad \sup_{\mathcal{I}_{\tilde{\rho}/4}} \|T^{[\hat{n},n]}_*\| \leq 2\,\hat{N} ,$$

<u>Claim 2</u>: The new frequency $\omega_+ := \partial \mathbf{h}_+$ is a diffeomorphism of $\mathcal{I}_{3\tilde{\rho}/16}$.

<u>Proof.</u> We want to prove that, if I_+ , $I'_+ \in \mathcal{I}_{3\tilde{\rho}/16}$ verify $\omega_+(I_+) = \omega_+(I'_+)$, then, $I_+ = I'_+$. Let I_+ , $I'_+ \in \mathcal{I}_{\tilde{\rho}/8}$ verify

$$\omega(I'_{+}) + \partial g(I'_{+}) = \omega_{+}(I_{+}) = \omega_{+}(I'_{+}) = \omega(I_{+}) + \partial g(I_{+}) + \partial g(I_{+}) = \omega(I_{+}) + \partial g(I_{+}) = \omega_{+}(I'_{+}) = \omega_{+}($$

Then,

$$|\omega(I'_{+}) - \omega(I_{+})| = |\partial g(I'_{+}) - \partial g(I_{+})| \le 2 \sup_{\mathcal{I}_{\tilde{\rho}/4}} |\partial g| \le 2^4 \frac{F}{\tilde{\rho}}$$

hence,

$$|I_{+} - I'_{+}| = |\omega^{-1}(\omega(I_{+})) - \omega^{-1}(\omega(I'_{+}))| \le \sup \|\partial(\omega^{-1})\| |\omega(I'_{+}) - \omega(I_{+})| \le 2^{4}n \frac{FN}{\tilde{\rho}} \le \frac{\tilde{\rho}}{16}$$

The previous inequality implies that the segment $s(I_+, I'_+)$ from I_+ to \mathcal{I}'_+ lies interely in $\mathcal{I}_{\tilde{\rho}/4}$. So, let τ the curve from $\omega(I_+)$ to $\omega(I'_+)$ defined as $\tau := \omega(s(I_+, I'_+))$; let $F := \partial g \circ \omega^{-1}$ and observe $\sup_{\omega(\mathcal{I}_{\tilde{\rho}/4})} \|\partial F\| = \sup_{\mathcal{I}_{\tilde{\rho}/4}} \|\partial^2 g(\partial h)^{-1}\| \leq 1/2$. Then,

$$0 = |\omega_{+}(I_{+}) - \omega_{+}(I'_{+})| = |\omega(I_{+}) - \omega(I'_{+}) + F(\omega(I_{+})) - F(\omega(I'_{+}))| \geq |\omega(I_{+}) - \omega(I'_{+})| - \left| \int_{\tau} \partial F(\zeta) \cdot d\zeta \right| \geq \frac{1}{2} |\omega(I_{+}) - \omega(I'_{+})|$$

which implies $\omega(I_+) = \omega(I'_+)$, hence, $I_+ = I'_+$.

<u>Claim 3</u>: The new frequency ω_+ maps $\mathcal{I}_{\tilde{\rho}/32}$ over $\omega(\mathcal{I})$, i.e., $\omega_+(\mathcal{I}_{\tilde{\rho}/32}) \supseteq \omega(\mathcal{I})$.

<u>Proof.</u>: we prove that, for any $I_0 \in \mathcal{I}$, $\omega_+(B^n_{\tilde{\rho}/32}(I_0)) \supseteq \omega(I_0)$. If $|I - I_0| = r < \tilde{\rho}/32$, then,

$$\begin{aligned} |\omega_{+}(I) - \omega_{+}(I_{0})| &= |\omega(I) - \omega(I_{0}) + \partial g \circ \omega^{-1}(\omega(I)) - \partial g \circ \omega^{-1}(\omega(I_{0}))| \\ &\geq \left(1 - \|\partial^{2}g(\partial h)^{-1}\|\right) |\omega(I) - \omega(I_{0})| \\ &\geq \frac{1}{2} |\omega(I) - \omega(I_{0})| \\ &\geq \frac{1}{2N} |I - I_{0}| \\ &= \frac{r}{2N} \end{aligned}$$

hence, $\omega_+(B^n_{\tilde{\rho}/32}(I_0)) \supseteq B^n_{\tilde{\rho}/(64N)}(\omega_+(I_0))$. We prove that

$$\omega(I_0) \in B^n_{r/(64N)}(\omega_+(I_0))$$

Using the KAM condition (2.18) and General Cauchy Estimate (for $\sup_{\mathcal{I}} |\partial g|$)

$$|\omega_+(I_0) - \omega(I_0)| = |\partial g(I_0)| \le \sup_{\mathcal{I}} |\partial g| < \frac{17}{16} \frac{F}{\tilde{\rho}/2} < \frac{\tilde{\rho}}{2^6 N}$$

which concludes the proof of the Claim.

<u>Claim 4</u> For any $I \in \mathcal{I}$, equation $\omega_+(I') = \omega(I)$ has a unique solution $I' := l_+(I) = \omega_+^{-1} \circ \omega(I) \in \mathcal{I}_{\tilde{\rho}/32}$ satisfying (2.33).

<u>Proof.</u> Existence and uniqueness of the solution $I' = l_+(I)$ of $\omega_+(I') = \omega(I)$ are consequences of claims 2, 3. We prove (2.33). Let $0 < r < \bar{r} \leq 3\tilde{\rho}/16$, with \bar{r} so small that $\omega(\mathcal{I}_r) \subseteq \omega_+(3\mathcal{I}_{\tilde{\rho}/16})$. For $I \in \mathcal{I}_r$, we find, as ω_+ is a diffeomorphism of \mathcal{I}_r and General Cauchy Inequality,

$$|\overline{\omega_{+}^{-1}}(\omega(I)) - \overline{\omega_{+}^{-1}}(\omega_{+}(I))| \le 2\bar{N}|\omega_{+}(I) - \omega(I)| \le 2\bar{N}\sup_{\mathcal{I}_{r}}|\partial g| \le \frac{2\bar{N}(17F/16)}{3\tilde{\rho}/16 - r}$$

Hence, due to the arbitrariness of r, for $I \in \mathcal{I} = \bigcap_{0 < r < \bar{r}} \mathcal{I}_r$,

$$|\bar{l}_+(I) - \bar{I}| \le \frac{34\bar{N}F}{3\tilde{\rho}} \le 2^4 \frac{\bar{N}F}{\tilde{\rho}}$$

Similarly,

$$|\hat{l}_+(I) - \hat{I}| \le 2^4 \frac{NF}{\tilde{\rho}}$$

Also, using

$$Dl_{+} = D[(\omega + \partial g)^{-1} \circ \omega] = [\mathrm{id}_{n} + (\partial^{2}\mathrm{h})^{-1}\partial^{2}\mathrm{g}]^{-1}$$

we find

$$\begin{aligned} \sup_{\mathcal{I}_r} \|D[(\omega + \partial g)^{-1} \circ \omega] - \mathrm{id}_n\| &= \sup_{\mathcal{I}_r} \|[\mathrm{id}_n + (\partial^2 \mathrm{h})^{-1} \partial^2 \mathrm{g}]^{-1} - \mathrm{id}_n\| \\ &\leq \frac{\sup_{\mathcal{I}_r} \|(\partial^2 \mathrm{h})^{-1} \partial^2 \mathrm{g}\|}{1 - \sup_{\mathcal{I}_r} \|(\partial^2 \mathrm{h})^{-1} \partial^2 \mathrm{g}\|} \\ &\leq 2 \sup_{\mathcal{I}} \|(\partial^2 \mathrm{h})^{-1} \partial^2 \mathrm{g}\| \\ &\leq 2^8 E . \end{aligned}$$

which says

$$\mathcal{L}_{\mathcal{P}}(\omega_{+}^{-1} \circ \omega - \mathrm{id}) \leq 2^{8}E \quad \mathrm{on} \quad \mathcal{I}_{r}$$

and implies, due to the arbitrariness of r,

$$\mathcal{L}_{\mathcal{P}}(l_+ - \operatorname{id}) \le 2^8 E$$
.

<u>Conclusion</u>: Let $\mathcal{I}_+ := l_+(\mathcal{I}), \ \rho_+ := \tilde{\rho}/8, \ s_+ := s/6$. By claim 4, $\mathcal{I}_+ \subset \mathcal{I}_{\tilde{\rho}/32}$, so, the following inclusions hold:

$$\mathcal{I}_{+\rho_+} \subset \mathcal{I}_{5\tilde{
ho}/32} \subset \mathcal{I}_{3\tilde{
ho}/16} \subset \mathcal{I}_{\tilde{
ho}/4}$$

Hence, $\Psi_+ := \Psi_{av}|_{\mathcal{I}_{+\rho_+} \times \mathbb{T}^n_{s+s_+}}$ is well defined and has the desired properties, upon recognizing that

$$E_{+} := \frac{F_{+}M_{+}^{2}L_{+}}{\tilde{\rho}_{+}^{2}} \le E^{2} , \quad \tilde{\rho}_{+} \ge \frac{1}{2} \left(\frac{1}{12}\right)^{\tau+1} \tilde{\rho} .$$

In order to do that, we first prove

$$10 K \le K_+ \le 12 K . \tag{2.48}$$

We find

$$\frac{FLM^2}{\gamma^2} \le \frac{FLM^2}{\gamma^2} \left[\log_+ \left(\frac{FLM^2}{\gamma^2} \right)^{-1} \right]^{2(\tau+1)} \le \frac{1}{9} \left(\frac{s}{6} \right)^{2(\tau+1)} \frac{FL}{\tilde{\rho}^2} = \le \frac{2^8 E}{9 \cdot 6^2 \cdot 2^8} \le \frac{1}{9 \cdot 6^2 \cdot 2^8}$$

which gives

$$2^{16} < x := \left(\frac{FLM^2}{\gamma^2}\right)^{-1} , \qquad (2.49)$$

namely

$$\frac{108}{s}\log 2 < \frac{9}{8}\,K < 2\,K$$

implying immediately (2.48), after using

$$K_{+} = \frac{6}{s_{+}} \log_{+} (x^2/8) = \frac{6}{s_{+}} \log (x^2/8) = 12 K - \frac{108}{s} \log 2$$
.

Now, using

$$\gamma \ge 3 \, M K^{\tau+1} \tilde{\rho} \,\,, \quad \hat{\gamma} \ge 3 \, M K^{\tau+1} \tilde{\rho} \,\,, \quad \frac{K_+}{K} \le 12 \,\,, \quad K \ge \frac{96 \log 2}{s} \,\,, \quad 0 < s \le 1 \,\,,$$

we find

$$E_{+} \leq \max\left\{128\left(\frac{F\,LM}{\gamma\tilde{\rho}}\right)^{2}, \quad 648\left(\frac{FLM}{\gamma^{2}}\right)^{2}K_{+}^{2(\tau+1)}, \quad 648\left(\frac{FLM\hat{M}}{\gamma\hat{\gamma}}\right)^{2}K_{+}^{2(\tau+1)}\right\}$$

$$\leq \max\left\{\frac{128}{9K^{2(\tau+1)}}\left(\frac{F\,L}{\tilde{\rho}^{2}}\right)^{2}, \quad 8\left(\frac{FL}{\tilde{\rho}^{2}}\right)^{2}\frac{K_{+}^{2(\tau+1)}}{K^{4(\tau+1)}}, \quad 8\left(\frac{FL}{\tilde{\rho}^{2}}\right)^{2}\frac{K_{+}^{2(\tau+1)}}{K^{2(\tau+1)}K^{2(\tau+1)}}\right\}$$

$$\leq \max\left\{\frac{128}{9}\left(\frac{s}{96\log 2}\right)^{2(\tau+1)}, \quad 8\left(\frac{s}{8\log 2}\right)^{2(\tau+1)}\right\}E^{2}$$

$$\leq E^{2}.$$

Similarly,

$$\begin{split} \tilde{\rho}_{+} &= \min\left\{\rho_{+}, \frac{\gamma}{3M_{+}K_{+}^{\tau+1}}, \frac{\hat{\gamma}}{3\hat{M}_{+}K_{+}^{\tau+1}}\right\} \\ &= \min\left\{\frac{\rho}{8}, \frac{\gamma}{24MK^{\tau+1}}, \frac{\hat{\gamma}}{24MK^{\tau+1}}, \frac{\gamma}{6MK_{+}^{\tau+1}}, \frac{\hat{\gamma}}{6MK_{+}^{\tau+1}}\right\} \\ &= \min\left\{\frac{\rho}{8}, \frac{\gamma}{6MK_{+}^{\tau+1}}, \frac{\hat{\gamma}}{6MK_{+}^{\tau+1}}\right\} \\ &\geq \frac{1}{2}\left(\frac{1}{12}\right)^{\tau+1}\min\left\{\rho, \frac{\gamma}{3MK^{\tau+1}}, \frac{\hat{\gamma}}{3MK^{\tau+1}}\right\} \\ &= \frac{1}{2}\left(\frac{1}{12}\right)^{\tau+1}\tilde{\rho} \end{split}$$

(use $(M_+, \hat{M}_+) = 2(M, \hat{M}), 12K \ge K_+ \ge 10K$). This concludes the proof.

2.1.2 Lemmas on Measure

We recall the following classical results on Lipschitz functions and measure theory, referring to [16] for their proofs.

Lemma 2.3 (Kirszbraun Theorem) Assume $A \subset \mathbb{R}^n$, and let $f : A \to \mathbb{R}^m$ be Lipschitz. There exists a Lipschitz function $\overline{f} : \mathbb{R}^n \to \mathbb{R}^m$, such that

i) $\bar{f} = f$ on A;

$$ii) \quad \mathcal{L}(\bar{f}) = \mathcal{L}(f) \;.$$

Lemma 2.4 Let $A \subset \mathbb{R}^n$ Lebesgue-measurable, $f : A \to \mathbb{R}^n$ Lipschitz (bi-Lipschitz). Then,

$$\max((f(A)) \leq \mathcal{L}(f)^n \max(A)$$
$$\mathcal{L}_{-}(f)^n \max(A) \leq \max((f(A)) \leq \mathcal{L}_{+}(f)^n \max(A))$$

2.1.3 Proof of Theorem 2.2

Here also, we proceed by steps. <u>Claim 0.</u> ("construction of the sequences"): For each $1 \leq j \in \mathbb{N}$, H is analitically conjugated to

$$\mathbf{H}_j = \mathbf{H} \circ \Phi_j = \mathbf{h}_j + \mathbf{f}_j$$

real-analytic on $\mathcal{P}_j = \mathcal{I}_{j\rho_j} \times \mathbb{T}^n_{\bar{s}+s_j}$, where:

(i) $s_j = s/6^j$ and, letting

$$\begin{cases} M_{j} = 2^{j}M \\ \hat{M}_{j} = 2^{j}\hat{M} \\ N_{j} = 2^{j}N \\ L_{j} = \max\left\{M_{j}^{-1}, \ \hat{M}_{j}^{-1}, \ N_{j}\right\} \\ F_{j} = \frac{F_{j-1}^{2}L_{j-1}M_{j}^{2}}{\gamma^{2}} \\ K_{j} := \frac{6}{s_{j}}\log_{+}\left(\frac{F_{j}L_{j}M_{j}^{2}}{\gamma^{2}}\right) \\ \tilde{\rho}_{j} := \min\left\{\frac{\gamma}{3M_{j}K_{j}^{\tau+1}}, \ \frac{\hat{\gamma}}{3\hat{M}_{j}K_{j}^{\tau+1}}, \ \rho_{j}\right\}\end{cases}$$

then, $\rho_j = \tilde{\rho}_{j-1}/8;$

(ii) $\mathcal{I}_j \subseteq \mathcal{I}_{j-1_{\tilde{\rho}_{j-1}/32}}$ is obtained as $\mathcal{I}_j = l_j(\mathcal{I}_{j-1})$, where l_j is a Lipschitz homeomorphism satisfying

$$\omega_{j} \circ l_{j} = \omega_{j-1} \quad on \quad \mathcal{I}_{j-1}$$
$$\max\left\{\frac{N}{\bar{N}}\sup_{\mathcal{I}_{j}}|\bar{l}_{j+1} - \operatorname{id}_{\bar{n}}|, \quad \frac{N}{\hat{N}}\sup_{\mathcal{I}_{j}}|\hat{l}_{j+1} - \operatorname{id}_{\hat{n}}|\right\} \le 2^{4}\tilde{\rho}_{j}E^{2^{j}} \qquad (2.50)$$

$$\mathcal{L}_{\mathcal{P}}(l_{j+1} - \mathrm{id}) \le 2^8 E^{2^j} := \mu_j$$
 (2.51)

- (iii) $\omega_j := \partial \mathbf{h}_j$ is a diffeomorphism of $\mathcal{I}_{j\rho_j}$ with non singular Jacobian $U_j := \partial^2 \mathbf{h}_j$ such that $\omega_j(\mathcal{I}_j) = \mathcal{D}_{\gamma,\hat{\gamma},\tau}^{\bar{n},\hat{n}} \cap \omega(\mathcal{I});$
- (iv) f_i satisfies

$$\|\mathbf{f}_j\|_{\mathcal{I}_{j\rho_j},\bar{s}+s_j} \le F_j$$

(v) The real-analytic symplectomorphism Φ_j is obtained as $\Phi_j = \Psi_1 \circ \cdots \circ \Psi_j$, where $\Psi_k : \mathcal{P}_k \to \mathcal{P}_{k-1} \ (k \ge 1) \ verifies$

$$\sup_{\mathcal{P}_k} |W^{\bar{n}}_{\hat{\gamma}/\gamma,1} W_{\bar{\rho},s}(\Psi_k - \mathrm{id})|_{\mathcal{P}} \le \left(\frac{1}{6}\right)^{k-1} E^{2^{k-1}}$$

$$(2.52)$$

$$\sup_{\mathcal{P}_{k}} \left\| W_{\tilde{\rho},s} D\Psi_{k} W_{\tilde{\rho},s}^{-1} - \operatorname{id}_{2n} \right\|_{\mathcal{P}} \le 2^{5} \left[\frac{(12)^{\tau+1}}{3} \right]^{k-1} E^{2^{k-1}} =: \zeta_{k} . \quad (2.53)$$

Proof. Starting with

$$\mathbf{H}_0 := \mathbf{H} = \mathbf{h} + \mathbf{f} , \quad \mathcal{P}_0 := \mathcal{I}_{0\rho} \times \mathbb{T}^n_{\bar{s}+s} ,$$

where $\mathcal{I}_0 := \{I \in \mathcal{I} : \omega(I) \in \mathcal{D}_{\gamma,\hat{\gamma},\tau}^{\hat{n},\hat{n}}\}$ and, labeling by "0" the quantities relatively to H_0 , apply (inductively) the Iterative Lemma (Lemma 2.2) to

$$\mathbf{H}_j = \mathbf{h}_j + \mathbf{f}_j , \quad \mathcal{P}_j := \mathcal{I}_{j\rho_j} \times \mathbb{T}^n_{\bar{s}+s_j} , \quad j \ge 0$$

and label by "j + 1" the "+"–quantities of the Iterative Lemma. Next, observe that (2.53) is a consequence of

$$\sup_{\mathcal{P}_{k}} |W_{\hat{\gamma}/\gamma,1}^{\bar{n}} W_{k-1}(\Psi_{k} - \mathrm{id})|_{\mathcal{P}} \leq E^{2^{k-1}}$$
$$\sup_{\mathcal{P}_{k}} \left\| W_{k-1} D\Psi_{k} W_{k-1}^{-1} - \mathrm{id}_{2n} \right\|_{\mathcal{P}} \leq 2^{5} E^{2^{k-1}}$$

where $W_{k-1} := W_{\tilde{\rho}_{k-1}, s_{k-1}}$, using

$$\|W_{\tilde{\rho},s}W_{k-1}^{-1}\|_{\mathcal{P}} = \max\left\{\frac{\tilde{\rho}_{k-1}}{\tilde{\rho}}, \frac{s_{k-1}}{s}\right\} = \left(\frac{1}{6}\right)^{k-1}$$
$$\|W_{k-1}W_{\tilde{\rho},s}^{-1}\|_{\mathcal{P}} = \max\left\{\frac{\tilde{\rho}}{\tilde{\rho}_{k-1}}, \frac{s}{s_{k-1}}\right\} \le [2(12)^{\tau+1}]^{k-1}$$

<u>Claim 1.</u> ("construction of \mathcal{I}_* "): The sequence of Lipschitz homeomorphisms on \mathcal{I}_0

$$\ell_j := l_j \circ l_{j-1} \circ \dots \circ l_1 \tag{2.54}$$

converges uniformly to a bi–Lipschitz homeomorphism $\ell = (\bar{\ell}, \hat{\ell})$ satisfying

$$\frac{N}{\bar{N}}\sup_{\mathcal{I}_0}|\bar{\ell} - \mathrm{id}| , \quad \frac{N}{\hat{N}}\sup_{\mathcal{I}_0}|\hat{\ell} - \mathrm{id}| \le 2^5\tilde{\rho}E$$
(2.55)

$$\mathcal{L}_{\mathcal{P}}(\ell - \mathrm{id}) \le 2^{10}E \tag{2.56}$$

Furthermore, the following holds:

$$\sup_{\mathcal{I}_0} |\ell_j - \ell| \le 2^5 \tilde{\rho} E^{2^j} \tag{2.57}$$

and

$$\mathcal{I}_* := \ell(\mathcal{I}_0) \subseteq \mathcal{I}_{032E\tilde{\rho}} \bigcap \left(\cap_j \mathcal{I}_{j\rho_j} \right)$$

Proof. Using (2.50), the inequality

$$\sup_{\mathcal{I}_0} |\ell_i - \ell_j| \leq \sum_{l=j}^{i-1} \sup_{\mathcal{I}_0} |\ell_{l+1} - \ell_l| \leq 2^4 \tilde{\rho}_j \sum_{l=j}^{i-1} \left(\frac{1}{8}\right)^{l-j} E^{2^l}$$
(2.58)

proves the uniform convergence of ℓ_j . Letting, in (2.58), $i \to +\infty$, we find ¹¹

$$\sup_{\mathcal{I}_0} |\ell_j - \ell| \leq 2^4 \tilde{\rho}_j \sum_{l=j}^{+\infty} \left(\frac{1}{8}\right)^{l-j} E^{2^l}$$

 $^{11}2^l - 2^j \ge l - j$ for any $l \ge j$

$$\leq 2^{4} \tilde{\rho}_{j} E^{2^{j}} \sum_{l=j}^{+\infty} \left(\frac{1}{8}\right)^{l-j} E^{2^{l}-2^{j}}$$

$$\leq 2^{4} \tilde{\rho}_{j} E^{2^{j}} \sum_{0}^{+\infty} \left(\frac{E}{8}\right)^{k}$$

$$\leq 2^{5} \rho_{j} E^{2^{j}}$$

$$< \rho_{j} \qquad (2.59)$$

which also holds for j = 0, with the convention $\ell_0 := \text{ id}$. Eq. (2.59) implies

$$\mathcal{I}_* \subseteq \bigcap_j \mathcal{I}_{j\rho_j}$$
.

In particular, (2.59) with j = 0 gives

$$\sup_{\mathcal{I}} |\ell - \mathrm{id}| \le 2^5 \tilde{\rho} E .$$
(2.60)

which also implies

$$\mathcal{I}_* \subset \mathcal{I}_{032\tilde{\rho}E} . \tag{2.61}$$

With similar techniques, but using (2.60), one proves (2.55). We prove that ℓ is injective on \mathcal{I}_0 . If $I, I' \in \mathcal{I}_0$ are such that $\ell(I) = \ell(I') = I_*$, then, by (2.59)

$$\begin{aligned} |\omega(I) - \omega(I')| &= |\omega_j(\ell_j(I)) - \omega_j(\ell_j(I'))| \\ &\leq M_j |\ell_j(I) - \ell_j(I')| \\ &\leq M_j (|\ell_j(I) - \ell(I)| + |\ell_j(I') - \ell(I')|) \\ &\leq 2M 2^j \rho_j E^j \end{aligned}$$

which gives $\omega(I) = \omega(I')$ (as the r.h.s goes to 0 as $j \to \infty$), hence, I = I'. We prove (2.56). The estimates (2.51) for $\mathcal{L}_{\mathcal{P}}(l_{j+1} - \mathrm{id})$ give ¹²

$$\mathcal{L}_{\mathcal{P}}(\ell_{j+1} - \mathrm{id}) \le \prod_{l=1}^{j} (1 + \mu_l) - 1 \le \prod_{l=1}^{+\infty} (1 + \mu_l) - 1$$
(2.62)

¹²Write

$$i_{j+1} := \ell_{j+1} - \mathrm{id} = (l_{j+1} - \mathrm{id}) \circ (\mathrm{id} + i_j) + i_j$$

to find

$$\mathcal{L}(i_{j+1}) \leq \mathcal{L}(l_{j+1} - \mathrm{id}) \Big(1 + \mathcal{L}(i_j) \Big) + \mathcal{L}(i_j)$$

= $\mathcal{L}(i_j) \Big(\mathcal{L}(l_{j+1} - \mathrm{id}) + 1 \Big) + \mathcal{L}(l_{j+1} - \mathrm{id})$

Iterating the above formula, we find

$$\mathcal{L}(i_{j+1}) \leq \mathcal{L}(l_{j+1} - \mathrm{id}) + (1 + \mathcal{L}(l_{j+1} - \mathrm{id}))\mathcal{L}(l_j - \mathrm{id}) + \cdots + (1 + \mathcal{L}(l_{j+1} - \mathrm{id})) \cdots (1 + \mathcal{L}(l_2 - \mathrm{id}))\mathcal{L}(l_1 - \mathrm{id}) = \prod_{k=1}^{j+1} (1 + \mathcal{L}(l_k - \mathrm{id})) - 1 .$$

where the infinite productory $\prod_{l=1}^{+\infty}(1+\mu_l)$ converges, being bounded by

$$\prod_{l=0}^{+\infty} (1+\mu_l) = \exp\left[\sum_{l} \log(1+\mu_l)\right]$$

$$\leq \exp\left[\sum_{l} \mu_l\right]$$

$$= \exp\left[2^8 E \sum_{l} (E)^l\right]$$

$$\leq \exp\left[2^9 E\right]$$

$$\leq 1+2^{10} E$$
(2.63)

having used the elementary estimate $e^x \leq 1 + 2x$ for $0 \leq x \leq 1$. In follows from (2.62), (2.63)

$$\mathcal{L}_{\mathcal{P}}(\ell - \mathrm{id}) \leq \limsup_{j} \mathcal{L}_{\mathcal{P}}(\ell_{j+1} - \mathrm{id}) \leq 2^{10}E$$
.

<u>Claim 2.</u> ("definition and bounds for ω_* "): The bi-Lipschitz homeomorphism defined on \mathcal{I}_* by $\omega_* = (\bar{\omega}_*, \hat{\omega}_*) := \omega \circ \ell^{-1}$ is onto on $\mathcal{D}_{\gamma,\hat{\gamma},\tau}^{\bar{n},\hat{n}} \cap \omega(\mathcal{I})$ and is subject to the following bounds

$$\begin{split} \sup_{\mathcal{I}_{*}} |\bar{\omega}_{*} - \bar{\omega}| &\leq 2^{5} M \tilde{\rho} E \\ \sup_{\mathcal{I}_{*}} |\hat{\omega}_{*} - \hat{\omega}| &\leq 2^{5} \hat{M} \tilde{\rho} E \\ \sup_{\mathcal{I}_{*}} |\hat{\omega}_{*} - \hat{\omega}| &\leq 2^{5} \hat{M} \tilde{\rho} E \\ \end{split}$$

Proof. Trivially, $\nu \in \mathcal{D}_{\gamma,\hat{\gamma},\tau}^{\bar{n},\hat{n}}$, then, $I_* := \ell \circ \omega^{-1}(\nu)$ is its (unique) preimage, with

$$|\overline{\omega_*^{-1}}(\nu) - \overline{\omega^{-1}}(\nu)| = |\overline{\ell} \circ \omega^{-1}(\nu) - \overline{\omega^{-1}}(\nu)| \le \sup_{\mathcal{I}_0} |\overline{\ell} - \operatorname{id}| \le \frac{N}{N} 2^5 \tilde{\rho} E.$$

Similarly,

$$|\widehat{\omega_*^{-1}}(\nu) - \widehat{\omega^{-1}}(\nu)| \le \frac{\hat{N}}{N} 2^5 \tilde{\rho} E \; .$$

Using (2.55) (recall \bar{N} , $\hat{N} \leq N$), we find

$$\begin{aligned} |\bar{\omega}_*(I_*) - \bar{\omega}(I_*)| &= |\bar{\omega}(I) - \bar{\omega}(\ell(I))| \\ &\leq M |I - \ell(I)| \\ &\leq 2^5 M \tilde{\rho} E , \end{aligned}$$
The proof of

$$\left|\hat{\omega}_*(I_*) - \hat{\omega}(I_*)\right| \le 2^5 \hat{M} \tilde{\rho} E$$

is similar.

<u>Claim 3.</u> ("construction of Φ "): The sequence of real-analytic symplectomorphisms, defined on $\mathcal{P}_j = \mathcal{I}_{j_{\rho_j}} \times \mathbb{T}^n_{\bar{s}+s_j}$,

$$\Phi_j := \Psi_1 \circ \cdots \circ \Psi_j$$

converges uniformly on $\mathcal{P}_* = \mathcal{I}_* \times \mathbb{T}^n_{\bar{s}}$, to an \mathcal{I}_* -family of real-analytic embeddings

$$I_* \in \mathcal{I}_* \to \Phi(I_*, \cdot) : \mathbb{T}^n \to \operatorname{Re}(\mathcal{I}_\rho) \times \mathbb{T}^n$$

where $\Phi(I_*, \vartheta) = (\Phi_I(I_*, \vartheta), \Phi_{\varphi}(I_*, \vartheta))$ is given by

$$\begin{cases} \Phi_I(I_*,\vartheta) = I_* + a(I_*,\vartheta) \\ \Phi_{\varphi}(I_*,\vartheta) = \vartheta + b(I_*,\vartheta) \end{cases}$$

with $a = (\bar{a}, \hat{a})$, b verifying

$$\sup_{\mathcal{I}_* \times \mathbb{T}^n} |\bar{a}| \le 2 \frac{\gamma}{\gamma} E \,\tilde{\rho} \,, \quad \sup_{\mathcal{I}_* \times \mathbb{T}^n} |\hat{a}| \le 2 E \,\tilde{\rho} \,, \quad \sup_{\mathcal{I}_* \times \mathbb{T}^n} |b| \le 2 E \,s \tag{2.64}$$

and $\vartheta \to \vartheta + b(I_*, \vartheta)$ a diffeomorphism of \mathbb{T}^n , for any $I \in \mathcal{I}_*$. Furthermore, the rescaled map

$$\check{\Phi} := W_{\tilde{\rho},s} \Phi \circ W_{\tilde{\rho},s}^{-1} : \quad \tilde{\rho}^{-1} \mathcal{I}_* \times s^{-1} \mathbb{T}_{\bar{s}}^n \to r^{-1} \mathcal{I}_{\rho} \times s^{-1} \mathbb{T}_{\bar{s}+s}^n$$

is bi–Lipschitz, with
$$\mathcal{L}_{\mathcal{P}}(\check{\Phi} - \operatorname{id}) \leq 2^7 E .$$

Proof. The bound (2.53) implies that rescaled maps

$$\check{\Phi}_j := W_{\tilde{\rho},s} \Phi_j \circ W_{\tilde{\rho},s}^{-1} : \quad \tilde{\rho}^{-1} \mathcal{I}_{j\rho_j} \times s^{-1} \mathbb{T}_{\bar{s}+s_j}^n \to r^{-1} \mathcal{I}_{\rho} \times s^{-1} \mathbb{T}_{\bar{s}+s}^n ,$$

are bi–Lipschitz on $\tilde{\rho}^{-1} \mathcal{I}_{j\rho_j} \times s^{-1} \mathbb{T}_{\bar{s}+s_j}^n$ (hence, on $\tilde{\rho}^{-1} \mathcal{I}_* \times s^{-1} \mathbb{T}^n$), with

$$\mathcal{L}_{\mathcal{P}}(\check{\Phi}_j - \mathrm{id}) \le 2^7 \mathrm{E} \le \frac{1}{2} , \qquad (2.65)$$

because

$$\mathcal{L}_{\mathcal{P}}(\check{\Phi}_{j} - \mathrm{id}) \leq \sup_{\mathcal{P}_{j}} \|W_{\tilde{\rho},s} D\Phi_{j} W_{\tilde{\rho},s}^{-1} - \mathrm{id}_{2n}\|_{\mathcal{P}}$$

$$= \|W_{\tilde{\rho},s} D\Psi_{1} \cdots D\Psi_{j} W_{\tilde{\rho},s}^{-1} - \mathrm{id}_{2n}\|_{\mathcal{P}}$$

$$\leq \prod_{0}^{j-1} (1 + \varsigma_{k}) - 1$$

$$\leq \prod_{0}^{+\infty} (1 + \varsigma_{k}) - 1$$

$$\leq 2^{7} E \qquad (2.66)$$

having used

$$\prod_{k=0}^{N} (1+\varsigma_k) = \exp\left[\sum_{k=0}^{N} \log(1+\varsigma_k)\right]$$

$$\leq \exp\left[\sum_{k=0}^{N} \varsigma_k\right]$$

$$= \exp\left[2^5 \sum_{k=0}^{N} \left[\frac{(12)^{\tau+1}}{3}\right]^k E^{2^k}\right]$$

$$\leq \exp\left[2^5 \sum_{k=0}^{+\infty} \left[\frac{(12)^{\tau+1}}{3}\right]^k E^{2^k}\right]$$

$$\leq \exp\left[2^6 E\right]$$

$$\leq 1+2^7 E$$
(2.67)

Hence, the uniform convergence of Φ_j on \mathcal{P}_* easily follows: if $i + 1 \leq j$,

$$\sup_{\mathcal{I}_{*} \times \mathbb{T}_{\tilde{s}}^{n}} |W_{\tilde{\rho},s}(\Phi_{j} - \Phi_{i})|_{\mathcal{P}} = \sup_{\mathcal{I}_{*} \times \mathbb{T}_{\tilde{s}}^{n}} |W_{\tilde{\rho},s}\Phi_{i}(\Psi_{i+1} \circ \cdots \circ \Psi_{j}) - W_{\tilde{\rho},s}\Phi_{i}|_{\mathcal{P}}$$

$$= \sup_{\mathcal{I}_{*} \times \mathbb{T}_{\tilde{s}}^{n}} |\check{\Phi}_{i}(W_{\tilde{\rho},s}\Psi_{i+1} \circ \cdots \circ \Psi_{j}) - \check{\Phi}_{i}(W_{\tilde{\rho},s})|_{\mathcal{P}}$$

$$\leq \mathcal{L}_{\mathcal{P}+}(\check{\Phi}_{i}) \sup_{\mathcal{I}_{*} \times \mathbb{T}_{\tilde{s}}^{n}} |W_{\tilde{\rho},s}(\Psi_{i+1} \circ \cdots \circ \Psi_{j} - \mathrm{id})|_{\mathcal{P}}$$

$$\leq \left(1 + 2^{7}E\right) \sum_{l=i}^{j-1} \left(\frac{1}{6}\right)^{l} E^{2^{l}}$$
(2.68)

having used (2.52), for which

$$\sup_{\mathcal{I}_* \times \mathbb{T}^n_{\bar{s}}} |W^{\bar{n}}_{\hat{\gamma}/\gamma,1} W_{\tilde{\rho},s}(\Psi_{i+1} \circ \dots \circ \Psi_j - \mathrm{id})|_{\mathcal{P}} \le \sum_{l=i}^{j-1} \left(\frac{1}{6}\right)^l E^{2^l} \qquad i+1 \le j$$
(2.69)

Denote then by Φ the uniform limit of Φ_j on \mathcal{P}_* . Taking, in (2.69), i = 0 and letting $j \to \infty$, we find

$$\max\left\{\frac{\gamma}{\hat{\gamma}\tilde{\rho}}\sup_{\mathcal{I}_{*}\times\mathbb{T}^{n}_{\tilde{s}}}|\Phi_{\bar{I}} - \mathrm{id}_{\bar{n}}|, \frac{1}{\tilde{\rho}}\sup_{\mathcal{I}_{*}\times\mathbb{T}^{n}_{\tilde{s}}}|\Phi_{\hat{I}} - \mathrm{id}_{\hat{n}}|, \frac{1}{s}\sup_{\mathcal{I}_{*}\times\mathbb{T}^{n}_{\tilde{s}}}|\Phi_{\varphi} - \mathrm{id}_{n}|\right\}$$

$$= |W^{n}_{\hat{\gamma}/\gamma,1}W_{\tilde{\rho},s}(\Phi - \mathrm{id})|_{\mathcal{P}}$$

$$\leq \sum_{l=0}^{+\infty} \left(\frac{1}{6}\right)^{l} E^{2^{l}}$$

$$\leq \frac{E}{1-\frac{E}{6}} < 2E, \qquad (2.70)$$

which clearly implies (2.64). But (2.70) also implies that, for any fixed I_* , the analytic map

$$\vartheta \to \Phi_{\varphi}(I_*, \vartheta) = \vartheta + b(I_*, \vartheta)$$

is a diffeomorphism of \mathbb{T}^n . In fact, by General Cauchy Inequality,

$$\sup_{\mathcal{I}_* \times \mathbb{T}^n} |\partial_{\vartheta} b| \le \frac{2s}{\bar{s}} E < 1 \quad \text{as} \quad \bar{s} \ge 2s \quad \text{and} \quad E < 1 \; .$$

Finally, by (2.65), the rescaled map

$$\check{\Phi} := W_{\tilde{\rho},s} \Phi \circ W_{\tilde{\rho},s}^{-1} : \quad \tilde{\rho}^{-1} \mathcal{I}_* \times s^{-1} \mathbb{T}_{\bar{s}}^n \to r^{-1} \mathcal{I}_{\rho} \times s^{-1} \mathbb{T}_{\bar{s}+s}^n$$

is bi-Lipschitz, with

$$\mathcal{L}_{\mathcal{P}}(\check{\Phi} - \mathrm{id}) \leq \limsup_{j} \mathcal{L}_{\mathcal{P}}(\check{\Phi}_{j} - \mathrm{id}) \leq 2^{7}E \leq \frac{1}{2}$$

<u>Claim 4.</u> $\Phi(I_*, \mathbb{T}^n)$ is a Lagrangian torus with frequency $\omega_*(I_*)$ for H.:

Proof. We have to prove that the H-flow $\phi_t^{\mathrm{H}}(\Phi(I_*,\vartheta))$ of a point $\Phi(I_*,\vartheta) \in \Phi(I_*,\mathbb{T}^n)$ evolves as

$$\phi_t^{\mathrm{H}}(\Phi(I_*,\vartheta)) = \Phi(I_*,\vartheta+\omega_*(I_*)t) \qquad t \ge 0 \; .$$

We split $|\phi_t^{\mathrm{H}}(\Phi(I_*,\vartheta)) - \Phi(I_*,\vartheta + \omega_*(I_*)t)|$ as

$$\begin{aligned} |\phi_t^{\mathrm{H}}(\Phi(I_*,\vartheta)) - \Phi(I_*,\vartheta + \omega_*(I_*)t)| &\leq |\phi_t^{\mathrm{H}}(\Phi(I_*,\vartheta)) - \phi_t^{\mathrm{H}}(\Phi_j(I_*,\vartheta))| \\ &+ |\phi_t^{\mathrm{H}}(\Phi_j(I_*,\vartheta)) - \Phi_j(I_*,\vartheta + \omega_*(I_*)t)| \\ &+ |\Phi_j(I_*,\vartheta + \omega_*(I_*)t) - \Phi(I_*,\vartheta + \omega_*(I_*)t)| \end{aligned}$$

Due to the uniform convergence of Φ_j to Φ on \mathcal{P}_* and continuous dependence on the initial data, both $|\Phi_j(I_*, \vartheta + \omega_*(I_*)t) - \Phi(I_*, \vartheta + \omega_*(I_*)t)|$ and $|\phi_t^{\mathrm{H}}(\Phi(I_*, \vartheta)) - \phi_t^{\mathrm{H}}(\Phi_j(I_*, \vartheta))|$ go to 0 as $j \to \infty$. We prove that $|\phi_t^{\mathrm{H}}(\Phi_j(I_*, \vartheta)) - \Phi_j(I_*, \vartheta + \omega_*(I_*)t)|$ also goes to 0, which will conclude the proof. As for the canonicity of Φ_j on \mathcal{P}_j ,

$$\phi_t^{\mathrm{H}}(\Phi_j(I_*,\vartheta)) = \Phi_j(\phi_t^{\mathrm{H}_j}(I_*,\vartheta))$$

with

$$\mathbf{H}_j = \mathbf{H} \circ \Phi_j = \mathbf{h}_j + \mathbf{f}_j \ .$$

So, using the Lipschitz property for the rescaled maps $\check{\Phi}_j = W_{\tilde{\rho},\sigma}^{-1} \Phi \circ W_{\tilde{\rho},\sigma}$ on $\tilde{\rho}^{-1} \mathcal{I}_* \times s^{-1} \mathbb{T}^n$, with $\mathcal{L}_{\mathcal{P}}(\check{\Phi}_j) \leq 1 + 2^7 E$, we find

$$\begin{aligned} |\phi_t^{\mathrm{H}}(\Phi_j(I_*,\vartheta)) - \Phi_j(I_*,\vartheta + \omega_*(I_*)t)| &= |\Phi_j(\phi_t^{\mathrm{H}}(I_*,\vartheta)) - \Phi_j(I_*,\vartheta + \omega_*(I_*)t)| \\ &= |W_{\tilde{\rho},\sigma}(\check{\Phi}_j \circ W_{\tilde{\rho},\sigma}^{-1}(\phi_t^{\mathrm{H}_j}(I_*,\vartheta)) \end{aligned}$$

$$- \check{\Phi}_{j} \circ W_{\tilde{\rho},\sigma}^{-1}(I_{*},\vartheta+\omega_{*}(I_{*})t))|$$

$$\leq ||W_{\tilde{\rho},\sigma}||(1+2^{7}E)$$

$$\times |W_{\tilde{\rho},\sigma}^{-1}(\phi_{t}^{\mathrm{H}_{j}}(I_{*},\vartheta)) - W_{\tilde{\rho},\sigma}^{-1}(I_{*},\vartheta+\omega_{*}(I_{*})t))|$$

$$\leq ||W_{\tilde{\rho},\sigma}||(1+2^{7}E)||W_{\tilde{\rho},\sigma}^{-1}||$$

$$\times |\phi_{t}^{\mathrm{H}_{j}}(I_{*},\vartheta) - (I_{*},\vartheta+\omega_{*}(I_{*})t)|$$

$$= ||W_{\tilde{\rho},\sigma}||(1+2^{7}E)||W_{\tilde{\rho},\sigma}^{-1}||$$

$$\times |(I_{j}(t) - I_{*},\varphi_{j}(t) - (\vartheta+\omega_{*}(I_{*})t))|$$

where we have let $(I_j(t), \varphi_j(t)) = \phi_t^{\mathrm{H}_j}(I_*, \vartheta)$ the H_j -evolution of (I_*, ϑ)). Representing $(I_j(t), \varphi_j(t))$ as

$$\begin{cases} I_j(t) = I_* - \int_0^t \partial_{\varphi} \mathbf{H}_j(I_j(\tau), \varphi_j(\tau)) d\tau \\ \varphi_j(t) = \vartheta + \int_0^t \partial_I \mathbf{H}_j(I_j(\tau), \varphi_j(\tau)) d\tau \end{cases}$$

we find, by General Cauchy Inequality,

$$|I_{j}(t) - I_{*}| = \left| \int_{0}^{t} \partial_{\varphi} \mathbf{H}_{j}(I_{j}(\tau), \varphi_{j}(\tau)) d\tau \right|$$

$$= \left| \int_{0}^{t} \partial_{\varphi} f_{j}(I_{j}(\tau), \varphi_{j}(\tau)) d\tau \right|$$

$$\leq \int_{0}^{t} |\partial_{\varphi} f_{j}(I_{j}(\tau), \varphi_{j}(\tau))| d\tau$$

$$\leq \frac{F_{j}}{s_{j}} t \to 0 \quad \text{as} \quad j \to +\infty$$
(2.71)

Now, letting $I_0 := \ell^{-1}(I_*) \in \mathcal{I}_0$ and using $\omega_j(\ell_j(I_0)) = \omega_*(I_*)$, we find

$$\begin{aligned} |\varphi_{j}(t) - \vartheta - \omega_{*}(I_{*})t| &= \left| \int_{0}^{t} \partial_{I} \mathrm{H}_{j}(I_{j}(\tau), \varphi_{j}(\tau)) d\tau - \omega_{*}(I_{*})t \right| \\ &= \left| \int_{0}^{t} \left(\partial_{I} \mathrm{H}_{j}(I_{j}(\tau), \varphi_{j}(\tau)) - \omega_{*}(I_{*}) \right) d\tau \right| \\ &= \left| \int_{0}^{t} \left(\omega_{j}(I_{j}(\tau)) + \partial_{I} \mathrm{f}_{j}(I_{j}(\tau), \varphi_{j}(\tau)) - \omega_{*}(I_{*}) \right) d\tau \right| \\ &\leq \int_{0}^{t} |\omega_{j}(I_{j}(\tau)) + \partial_{I} \mathrm{f}_{j}(I_{j}(\tau), \varphi_{j}(\tau)) - \omega_{*}(I_{*})| d\tau \\ &\leq \int_{0}^{t} |\omega_{j}(I_{j}(\tau)) - \omega_{j}(I_{*})| d\tau + \int_{0}^{t} |\omega_{j}(I_{*}) - \omega_{j}(\ell_{j}(I_{0}))| d\tau \\ &+ \int_{0}^{t} |\partial_{I} \mathrm{f}_{j}(I_{j}(\tau), \varphi_{j}(\tau))| d\tau \end{aligned}$$

$$(2.72)$$

where

$$\int_{0}^{t} |\omega_{j}(I_{j}(\tau)) - \omega_{j}(I_{*})| d\tau \leq M_{j} \sup_{[0,t]} |I_{j}(\tau) - I_{*}| t \leq \frac{M_{j}F_{j}}{s_{j}} t \to 0$$

by (2.71);

$$\int_0^t |\omega_j(I_*) - \omega_j(\ell_j(I_0))| \, d\tau = |\omega_j(\ell(I_0)) - \omega_j(\ell_j(I_0))| \, t \le M_j \sup_{\mathcal{I}_0} |\ell_j - \ell| t \le 2^5 \tilde{\rho} M_j E^{2^j} t \to 0$$

by (2.57), and, finally, by General Cauchy Inequality,

$$\int_0^t \left| \partial_I \mathbf{f}_j(I_j(\tau), \varphi_j(\tau)) \right| d\tau \le \frac{F_j}{\rho_j} t \to 0$$

<u>Claim 5</u>. The "invariant" set $K := \Phi(\mathcal{I}_* \times \mathbb{T}^n)$ satisfies the measure estimate

$$\max \left(\mathcal{I} \times \mathbb{T}^n \setminus \mathbf{K} \right) \leq \left(1 + (1 + 2^7 E)^{2n} \right) \left(1 + (1 + 2^{10} E)^n \right) \max \left(\mathcal{I} \setminus \mathcal{I}_0 \times \mathbb{T}^n \right)$$

+ $\left(1 + (1 + 2^7 E)^{2n} \right) (1 + 2^{10} E)^n \max \left(\mathcal{I}_{\rho_1} \setminus \mathcal{I} \times \mathbb{T}^n \right)$
+ $\left(1 + 2^7 E \right)^{2n} \max \left(\mathcal{I}_{\rho_2} \setminus \mathcal{I} \times \mathbb{T}^n \right) .$

where $\rho_1 = 2^6 E \tilde{\rho} / (1 - 2^{10} E), \ \rho_2 = 4 E \tilde{\rho} / (1 - 2^7 E).$

Proof. Let $\rho_1 = 2^6 E \tilde{\rho}/(1-2^{10}E)$. Extend the Lipschitz function ℓ – id : $\mathcal{I}_0 \to \mathcal{I}_*$ to a Lipschitz function ℓ_e – id on \mathcal{I}_{0r} , with the same Lipschitz constant $\mathcal{L}(\ell_e)(\ell - \mathrm{id}) \leq 2^{10}E$ (this is made possible thanks to Lemma 2.3). Then, ℓ_e is a bi–Lipschitz extension (hence, injective) of ℓ on \mathcal{I}_{0r} , with lower Lipschitz constant $\mathcal{L}_-(\ell_e) \geq 1 - 2^{10}E$. This implies that ℓ_e sends a ball with radius ρ_1 centered at $I_0 \in \mathcal{I}_0$ over a ball with radius $(1 - 2^{10}E)\rho_1 = 2^6E\tilde{\rho} > 2^5E\tilde{\rho} \geq |\ell(I_0) - I_0|$ centered at $\ell(I_0)$, so as to conclude

$$\ell_{\mathrm{e}}(\mathcal{I}_{0\rho_1}) \supset \mathcal{I}_0$$

Then,

$$\begin{split} \operatorname{meas} \begin{pmatrix} \mathcal{I}_0 \setminus \mathcal{I}_* \end{pmatrix} &\leq \operatorname{meas} \begin{pmatrix} \ell_{\mathrm{e}}(\mathcal{I}_{0\rho_1}) \setminus \mathcal{I}_* \end{pmatrix} \\ &= \operatorname{meas} \begin{pmatrix} \ell_{\mathrm{e}}(\mathcal{I}_{0\rho_1}) \setminus \ell_{\mathrm{e}}(\mathcal{I}_0) \end{pmatrix} \\ &\leq \operatorname{meas} \begin{pmatrix} \ell_{\mathrm{e}}(\mathcal{I}_{0\rho_1} \setminus \mathcal{I}_0) \end{pmatrix} \\ &\leq \mathcal{L}(\ell)^n \operatorname{meas} \begin{pmatrix} \mathcal{I}_{0\rho_1} \setminus \mathcal{I}_0 \end{pmatrix} \\ &\leq \mathcal{L}(\ell)^n \operatorname{meas} \begin{pmatrix} \mathcal{I}_{\rho_1} \setminus \mathcal{I}_0 \end{pmatrix} \\ &\leq \mathcal{L}(\ell)^n \left(\operatorname{meas} \begin{pmatrix} \mathcal{I}_{\rho_1} \setminus \mathcal{I} \end{pmatrix} + \operatorname{meas} \begin{pmatrix} \mathcal{I} \setminus \mathcal{I}_0 \end{pmatrix} \right) \end{split}$$

and, finally,

$$\begin{split} \operatorname{meas}(\mathcal{I} \setminus \mathcal{I}_{*}) &\leq \operatorname{meas}(\mathcal{I} \setminus \mathcal{I}_{0}) + \operatorname{meas}(\mathcal{I}_{0} \setminus \mathcal{I}_{*}) \\ &\leq (1 + \mathcal{L}(\ell)^{n}) \operatorname{meas}(\mathcal{I} \setminus \mathcal{I}_{0}) + \mathcal{L}(\ell)^{n} \operatorname{meas}(\mathcal{I}_{\rho_{1}} \setminus \mathcal{I}) \\ &\leq (1 + (1 + 2^{10}E)^{n}) \operatorname{meas}(\mathcal{I} \setminus \mathcal{I}_{0}) + (1 + 2^{10}E)^{n} \operatorname{meas}(\mathcal{I}_{\rho_{1}} \setminus \mathcal{I}) \end{split}$$

$$(2.73)$$

On turn, using

$$\begin{cases} \sup_{\tilde{\rho}^{-1}\mathcal{I}_* \times s^{-1}\mathbb{T}^n} |\check{\Phi} - \operatorname{id}| \le 2E \\ \mathcal{L}(\check{\Phi} - \operatorname{id}) \le 2^7E \end{cases}$$

we can repeat the above argument, with

$$(\check{\Phi}, \check{K} = \check{\Phi}(\tilde{\rho}^{-1}\mathcal{I}_* \times s^{-1}\mathbb{T}^n), \tilde{\rho}^{-1}\mathcal{I}_* \times s^{-1}\mathbb{T}^n, 2n, \mathcal{L}(\check{\Phi}), \check{r} = 2^2 E/(1 - 2^7 E))$$

replacing $(\ell, \mathcal{I}_*, \mathcal{I}_0, n, \mathcal{L}(\ell), r)$ and we find

$$\max\left(\tilde{\rho}^{-1}\mathcal{I} \times s^{-1}\mathbb{T}^n \setminus \check{\mathbf{K}}\right) \leq \left(1 + (1 + 2^7 E)^{2n}\right) \operatorname{meas}\left(\tilde{\rho}^{-1}\mathcal{I} \setminus (\tilde{\rho}^{-1}\mathcal{I}_*) \times s^{-1}\mathbb{T}^n\right) \\ + \left(1 + 2^7 E\right)^{2n} \operatorname{meas}\left((\tilde{\rho}^{-1}\mathcal{I})_{\check{r}} \setminus (\tilde{\rho}^{-1}\mathcal{I}) \times s^{-1}\mathbb{T}^n\right).$$

Hence, rescaling the variables,

$$\max \left(\mathcal{I} \times \mathbb{T}^n \setminus \mathbf{K} \right) \leq \left(1 + (1 + 2^7 E)^{2n} \right) \max \left(\mathcal{I} \setminus \mathcal{I}_* \times \mathbb{T}^n \right)$$
$$+ \left(1 + 2^7 E \right)^{2n} \max \left(\mathcal{I}_{\rho_2} \setminus \mathcal{I} \times \mathbb{T}^n \right) \qquad \rho_2 = \check{r} \tilde{\rho}$$

Finally, taking into accunt (2.73), we find the result.

Conclusion of the proof of Theorem 2.2. Take

$$\phi_{\nu} = \Phi(\omega_*^{-1}(\nu), \cdot)$$

and recognize that

$$\ell = \omega_*^{-1} \circ \omega \quad \text{on} \quad \mathcal{I}_0 = \mathcal{I}_{\gamma,\hat{\gamma},\tau}$$

has Lipschitz norm bounded as in (2.22), by (2.55) and (2.56).

2.1.4 Nondegenerate KAM Theorem via Theorem 2.2

Taking, in Theorem 2.2,

$$\gamma = \hat{\gamma} , \quad \hat{M} = M , \quad \bar{N} = \hat{N} = N$$

gives a standard (nondegenerate, isofrequencial) KAM Theorem:

Theorem 2.3 Let $n \in \mathbb{N}$, $\tau > n$, $\gamma > 0$, $\mathcal{I} \subset \mathbb{R}^n$ compact and let

$$H(J,\psi) = h(J) + f(J,\psi)$$

real-analytic on $\mathcal{I}_{\rho} \times \mathbb{T}_{s}^{n}$, where $\omega := \partial h$ is a diffeomorphism of \mathcal{I}_{ρ} with Jacobian matrix $U := \partial^{2} h$ non singular on \mathcal{I}_{ρ} . Let

$$M \ge \sup_{\mathcal{I}_{\rho}} \|U\|$$
$$N \ge \sup_{\mathcal{I}_{\rho}} \|T\|$$
$$F \ge \|f\|_{\rho,s}$$

where $T := U^{-1}$, define

$$\begin{split} c &:= \max \left\{ 2^{11}n, \ \frac{2}{3}(12)^{\tau+1} \right\} \\ K &:= \frac{6}{s} \ \log_+ \left(\frac{FM^2 L}{\gamma^2} \right)^{-1} \quad where \quad \log_+(a) := \max\{1, \log a\} \\ \tilde{\rho} &:= \min \left\{ \frac{\gamma}{3MK^{\tau+1}} \ , \ \rho \right\} \\ L &:= \max \ \{N \ , M^{-1}\} \end{split}$$

and assume the "perturbation" f so "small" that

$$cE := c\frac{FL}{\tilde{\rho}^2} < 1 .$$

(i) Then, for any frequency $\nu \in \Omega_* := \omega(\mathcal{I}) \cap \mathcal{D}^n_{\gamma,\tau}$, there exists a unique Lagrangian KAM torus $T_{\nu} \subset \operatorname{Re}(\mathcal{I}_{34\tilde{\rho}E}) \times \mathbb{T}^n$ for H with frequency ν , such that the following holds. There exists a "Cantor" set $\mathcal{I}_* \subset \operatorname{Re}(\mathcal{I}_{32\tilde{\rho}E})$ and a bi-Lipschitz (onto) homeomorphism

$$\omega_*: \ \mathcal{I}_* \to \Omega_*$$

satisfying

$$\sup_{\mathcal{O}_*} |\omega_*^{-1} - \omega^{-1}| \le 2^5 \tilde{\rho} E , \quad \sup_{\mathcal{I}_*} |\omega_* - \omega| \le 2^5 M \tilde{\rho} E \|\omega_*^{-1} \circ \omega - \operatorname{id}\|_{\tilde{\rho}, \mathcal{I}_{\gamma, \tau}}^{\operatorname{Lip}} \le 2^{10} E , \qquad \mathcal{I}_{\gamma, \tau} := \omega^{-1}(\mathcal{D}_{\gamma, \tau}^{n, \hat{n}}) \cap \mathcal{I} .$$

such that T_{ν} is realized by the real-analytic embedding $\phi_{\nu} = (\phi_{\nu I}, \phi_{\nu \varphi})$ given by

$$\begin{cases} \phi_{\nu I}(\vartheta) = I_*(\nu) + v(\nu, \vartheta) \\ \phi_{\nu \varphi}(\vartheta) = \vartheta + u(\nu, \vartheta) \end{cases} \qquad \vartheta \in \mathbb{T}^n ,$$

where $I_*(\nu) := \omega_*^{-1}(\nu)$ and v, u are bounded as

$$|v(\nu, \vartheta)| \le 2 E \tilde{\rho}$$
, $|u(\nu, \vartheta)| \le 2 E s$

(ii) The measure of the invariant set $\mathbf{K} = \phi_{\Omega_*}(\mathbb{T}^n)$ satisfies

$$\max \left(\mathcal{I} \times \mathbb{T}^n \setminus \mathbf{K} \right) \leq \left(1 + (1 + 2^7 E)^{2n} \right) \left(1 + (1 + 2^{10} E)^n \right) \operatorname{meas} \left(\mathcal{I} \setminus \mathcal{I}_{\gamma, \tau} \times \mathbb{T}^n \right)$$

+ $\left(1 + (1 + 2^7 E)^{2n} \right) (1 + 2^{10} E)^n \operatorname{meas} \left(\mathcal{I}_{\rho_1} \setminus \mathcal{I} \times \mathbb{T}^n \right)$
+ $\left(1 + 2^7 E \right)^{2n} \operatorname{meas} \left(\mathcal{I}_{\rho_2} \setminus \mathcal{I} \times \mathbb{T}^n \right) .$

with $\rho_1 = 2^6 E \tilde{\rho} / (1 - 2^{10} E), \ \rho_2 = 4 E \tilde{\rho} / (1 - 2^7 E).$

2.2 Properly Degenerate KAM Theory (Proof of Theorem 2.1)

The aim of this section is to prove Theorem 2.1 as an application of Theorem 2.2.

We quote a refined averaging theory by Biasco et al ([7], "full version" on p. 110), in which statement the "sup–Fourier" norm of

$$f(I, \varphi, p, q) = \sum_{k} f_k(I, p, q) e^{ik \cdot \varphi}$$

(with f_k the Fourier coefficient of f), real-analytic on $U_{\bar{r}} \times \mathbb{T}^n_{\bar{s}} \times E_{r_p} \times F_{r_q}$, with $U \subset \mathbb{R}^n$, $E, F \subset \mathbb{R}^m$, is defined as

$$||f||_{\bar{r},\bar{s},r_p,r_q} := \sum_k \sup_{U_{\bar{r}} \times E_{r_p} \times F_{r_q}} |f_k| e^{|k|\bar{s}}$$

In order to avoid confusion with other parameters here introduced, we denote by $a, \bar{\varepsilon}, \bar{r}, \bar{s}, \bar{d}$ the parameters $\alpha, \varepsilon, r, s, d$ of [7], but we do not change the name of the dimensions (n, m) (which correspond to "our" (\bar{n}, \hat{n})) thereby used, letting the reader be aware not to confuse [7]'s n (which corresponds to "our" \bar{n}) with "our" $n = \bar{n} + \hat{n}$.

Proposition 2.1 (Fast Averaging Theorem) Let $H := h(I) + f(I, \varphi, p, q)$ be a realanalytic Hamiltonian on $U_{\bar{r}} \times \mathbb{T}^n_{\bar{s}} \times E_{r_p} \times F_{r_q}$. Denoting $\omega := \partial_I h$ and $c_m := e(1+em)/2$, suppose that

$$|\omega(I) \cdot k| \ge a$$
, for all $I \in U_{\bar{r}}$, $k \in \mathbb{Z}^n$, $k \notin \Lambda$, $|k| \le K$,

where Λ is a \mathbb{Z}^n -module, $K\bar{s} \geq 6$, and

$$\bar{\varepsilon} := \|f\|_{\bar{r},\bar{s},r_p,r_q} < \frac{ad}{2^7 c_m K \bar{s}} , \quad \bar{d} = \min\{\bar{r}\bar{s}, \ r_p r_q\} .$$
(2.75)

Then, there exists a real-analytic, symplectic transformation

$$\Psi: \quad U_{\bar{r}/2} \times \mathbb{T}^n_{s/6} \times E_{r_p/2} \times F_{r_q/2} \quad \to \quad U_{\bar{r}} \times \mathbb{T}^n_{\bar{s}} \times E_{r_p} \times F_{r_q} \\ (I', \varphi', p', q') \quad \to \quad (I, \varphi, p, q) = \Psi(I', \varphi', p', q')$$
(2.76)

such that

$$H_* := H \circ \Psi = h + g + f_* ,$$

with g in normal form:

$$g = \sum_{k \in \Lambda} g_k(I', p', q') e^{ik \cdot \varphi'} .$$

$$(2.77)$$

Moreover, when the projection of $\Psi(I', \varphi', p', q')$ onto the *I*-variables is denoted by $I(I', \varphi', p', q')$, etc,

$$\begin{aligned} \|g - P_{\Lambda} T_{K} f\|_{\bar{r}/2, \bar{s}/6, r_{p}/2, r_{q}/2} &\leq \frac{12}{11} \frac{2^{7} c_{m} \bar{\varepsilon}^{2}}{ad} \leq \frac{\bar{\varepsilon}}{4} , \\ \|f_{*}\|_{\bar{r}/2, \bar{s}/6, r_{p}/2, r_{q}/2} &\leq e^{-K\bar{s}/6} \frac{2^{9} c_{m} \bar{\varepsilon}^{2}}{ad} \leq e^{-K\bar{s}/6} \bar{\varepsilon} , \\ \max \left\{ \bar{s} |I(I', \varphi', p', q') - I'| , \bar{r} |\varphi(I', \varphi', p', q') - \varphi'| , \\ r_{q} |p(I', \varphi', p', q') - p'| , r_{p} |q(I', \varphi', p', q') - q'| \right\} \leq \frac{9\bar{\varepsilon}}{a} . \end{aligned}$$
(2.78)

We are ready to begin our proof.

Let

$$\mathcal{H}(I,\varphi,p,q) = h(I) + \varepsilon f(I,\varphi,p,q)$$

a real-analytic on $\mathcal{I}_{\rho_0} \times \mathbb{T}_{s_0}^{\bar{n}} \times \bar{B}(r_0)_{r_0} \times \bar{B}(r_0)_{r_0}$ Hamiltonian, where $\bar{B}(r) = \overline{B_r^{\hat{n}}(0)}$ and $r_0 \leq \bar{r}/2$. We assume that ρ_0 , r_0 are so small that \bar{f} preserves, on $\mathcal{I}_{\rho_0} \times \mathbb{T}_{s_0}^{\bar{n}} \times \bar{B}(r_0)_{r_0} \times \bar{B}(r_0)_{r_0}$, the form

$$\bar{f} = f_0(I) + \sum_{1 \le i \le \hat{n}} \Omega_i(I) \frac{p_i^2 + q_i^2}{2} + \frac{1}{2} \sum_{1 \le i, j \le \hat{n}} A_{i,j}(I) \frac{p_i^2 + q_i^2}{2} \frac{p_j^2 + q_j^2}{2} + o_4$$

and the non–resonance, non–degeneracy assumptions, i.e.,

$$\min_{0 < |k| \le 4} \inf_{\mathcal{I}_{\rho_0}} |\Omega \cdot k| > 0 ,$$

$$\inf_{\mathcal{I}_{\rho_0}} |\det A| > 0 .$$

$$(2.79)$$

We proceed in 6 steps.

<u>Step 1.</u> ("fast averaging") There exist $0 < c_{\rm av} < 1 < C_{\rm av}$, $r_{\rm av} > 0$ $\gamma_{\rm av}$ such that, for any $0 < r \le r_{\rm av}$, $\bar{\gamma} > \gamma_{\rm av} \sqrt{\varepsilon} (\log r^{-1})^{\bar{\tau}+1}$, \mathcal{H} is put into the form

$$\begin{aligned} \mathcal{H}'(\varepsilon, r, \bar{\gamma}; I', \varphi', p', q') &= \mathcal{H} \circ \Psi_{\mathrm{av}}(\varepsilon, r, \bar{\gamma}; I', \varphi', p', q') \\ &= h(I') + \varepsilon g'(\varepsilon, r, \bar{\gamma}; I', p', q') + \varepsilon r^5 f'(\varepsilon, r, \bar{\gamma}; I', \varphi', p', q') \end{aligned}$$

$$(2.80)$$

by means of a real-analytic symplectomorphism

$$\phi_{\rm av}: \ \bar{\mathcal{I}}_{\bar{\rho}(r)/2} \times \mathbb{T}^{\bar{n}}_{s_0/6} \times \bar{B}(r_0)_{r_0/2} \times \bar{B}(r_0)_{r_{00}/2} \ \to \ \bar{\mathcal{I}}_{\bar{\rho}(r)} \times \mathbb{T}^{\bar{n}}_{s_0} \times \bar{B}(r_0)_{r_0} \times \bar{B}(r_0)_{r_0}$$

where:

(i) $\bar{\mathcal{I}}, \bar{\rho}(r)$ are given by

$$\bar{\mathcal{I}} = \mathcal{I}_{\bar{\gamma},\bar{\tau}} := \left\{ I \in \mathcal{I} : \ \omega_0(I) := \partial h(I) \in \mathcal{D}_{\bar{\gamma},\bar{\tau}}^{\bar{n}} \right\}$$
$$\bar{\rho}(r) := \min \left\{ c_{\mathrm{av}} \frac{\bar{\gamma}}{\left(\log r^{-1}\right)^{\bar{\tau}+1}}, \ \rho_0 \right\}$$

(ii) g', f' satisfy

$$\sup_{\bar{\mathcal{I}}_{\bar{\rho}(r)/2} \times \bar{B}(r_0)_{r_0/2} \times \bar{B}(r_0)_{r_0/2}} |g' - \bar{f}| \le C_{\mathrm{av}} \frac{\varepsilon (\log r^{-1})^{2\bar{\tau}+1}}{\bar{\gamma}^2} ,$$
$$\|f'\|_{\bar{\mathcal{I}}_{\bar{\rho}(r)/2} \times \mathbb{T}^{\bar{n}}_{s_0/6} \times \bar{B}(r_0)_{r_0/2} \times \bar{B}(r_0)_{r_{00}/2}} \le \|f\|_{\rho_0, s_0, r_0} ,$$

 $(\bar{f} = \langle f \rangle_{\varphi})$

(iii) The projections $I(I', \varphi', p', q'), \dots, of \phi_{av}$ over the I, \dots variables satisfy

$$\begin{cases} |I(I', \varphi', p', q') - I'| \leq C_{\mathrm{av}} \frac{\varepsilon(\log r^{-1})^{\bar{\tau}}}{\bar{\gamma}} \\ |\varphi(I', \varphi', p', q') - \varphi'| \leq C_{\mathrm{av}} \frac{\varepsilon(\log r^{-1})^{2\bar{\tau}+1}}{\bar{\gamma}^2} \\ |p(I', \varphi', p', q') - p'| \leq C_{\mathrm{av}} \frac{\varepsilon(\log r^{-1})^{\bar{\tau}}}{\bar{\gamma}} \\ |q(I', \varphi', p', q') - q'| \leq C_{\mathrm{av}} \frac{\varepsilon(\log r^{-1})^{\bar{\tau}}}{\bar{\gamma}} \end{cases}$$
(2.81)

Proof. We apply Proposition 2.1 to our case, taking

$$\Lambda = \{0\}, \quad U = \bar{\mathcal{I}}, \quad E = F = \bar{B}(r_0), \quad \bar{\varepsilon} = \varepsilon ||f||_{\rho_0, s_0, r_0}$$
$$\bar{r} = \bar{\rho}(r) := \min \left\{ \frac{1}{2\bar{M}} \left(\frac{s_0}{30} \right)^{\bar{\tau}+1} \frac{\bar{\gamma}}{\left(\log r^{-1} \right)^{-(\bar{\tau}+1)}}, \rho_0 \right\}, \quad \bar{s} = s_0$$
$$r_p = r_q = r_0, \quad n = \bar{n}, \quad m = \hat{n},$$
$$K = \bar{K}(r) := \frac{30}{s_0} \log r^{-1}, \quad a = \bar{\alpha}(r) := \bar{\gamma}/(2\bar{K}(r)^{\bar{\tau}})$$

Observe, in particular, that K has been chosen such in a way to get a new perturbation f_* of order

$$\|f_*\|_{\rho_0/2, s_0/6, r_0/2} \le \varepsilon e^{-K(r)s_0/6} \|f\|_{\rho_0, s_0, r_0} = \varepsilon r^5 \|f\|_{\rho_0, s_0, r_0} ,$$

so, we will put $f_* := \varepsilon r^5 f'$. We check, then,

(i) $\bar{K}(r)s_0 = 30 \log r^{-1} \ge 6$ (for $0 < r < e^{-1/5}$);

(ii) If $I \in \overline{\mathcal{I}}_{\bar{\rho}(r)}$ and $I_0 \in \overline{\mathcal{I}}$ is such that $|I - I_0| < \bar{\rho}(r)$, then, for $0 < |\bar{k}| \le \bar{K}(r)$,

$$\begin{aligned} \left| \omega_0(I) \cdot \bar{k} \right| &\geq \left| \omega(I_0) \cdot \bar{k} \right| - \left| \left(\omega(I) - \omega(I_0) \right) \cdot \bar{k} \right| \\ &\geq \frac{\bar{\gamma}}{\bar{K}(r)^{\bar{\tau}}} - \bar{M}\bar{K}(r)\bar{\rho}(r) \\ &= \frac{\bar{\gamma}}{2\bar{K}(r)^{\bar{\tau}}} = \bar{\alpha}(r) ; \end{aligned}$$

where

$$\bar{M} = \sup_{\mathcal{I}_{\rho_0}} \|\partial^2 h\| \ .$$

(iii) with

$$\bar{d} = d(r) = \min\left\{\bar{\rho}(r)s_0, \ r_0^2\right\} = \min\left\{\frac{s_0}{2\bar{M}} \left(\frac{s_0}{30}\right)^{\bar{\tau}+1} \frac{\bar{\gamma}}{\left(\log r^{-1}\right)^{\bar{\tau}+1}}, \ s_0\rho_0, \ r_0^2\right\}$$

using $\varepsilon \leq \sqrt{\varepsilon}$ (as $0 \leq \varepsilon \leq 1$), we find a suitable constant $\tilde{\gamma}$ depending on $\bar{\tau}$, s_0 , ρ_0 , \bar{M} , for which

$$\begin{split} \bar{\varepsilon} \frac{2^{7} c_{m} K \bar{s}}{a \bar{d}} &= \varepsilon \| f \|_{\rho_{0}, s_{0}, r_{0}} \frac{2^{7} c_{\hat{n}} \bar{K}(r) s_{0}}{\bar{\alpha}(r) d(r)} \\ &= 2^{8} c_{\hat{n}} s_{0} \| f \|_{\rho_{0}, s_{0}, r_{0}} \left(\frac{30}{s_{0}} \right)^{\bar{\tau}+1} \frac{\varepsilon (\log r^{-1})^{\bar{\tau}+1}}{\bar{\gamma}} \\ &\times \max \left\{ \frac{1}{r_{0}^{2}}, \frac{1}{s_{0} \rho_{0}}, \frac{2 \bar{M}}{s_{0}} \left(\frac{30}{s_{0}} \right)^{\bar{\tau}+1} \frac{\left(\log r^{-1} \right)^{\bar{\tau}+1}}{\bar{\gamma}} \right\} \\ &\leq \tilde{\gamma} \max \left\{ \frac{\sqrt{\varepsilon} \left(\log r^{-1} \right)^{\bar{\tau}+1}}{\bar{\gamma}}, \left(\frac{\sqrt{\varepsilon} \left(\log r^{-1} \right)^{\bar{\tau}+1}}{\bar{\gamma}} \right)^{2} \right\} \\ &< 1 \end{split}$$

provided

$$\bar{\gamma} > \tilde{\gamma} \sqrt{\varepsilon} \left(\log r^{-1} \right)^{\bar{\tau}+1}$$

Hence, Proposition 2.1, applies. Let, then, g, f_*, Ψ as in claimed there, and put $\varepsilon g' := g$, $\varepsilon r^5 f' := f_*, \phi_{av} = := \Psi$. By the definition of $\tilde{\rho}$, we find, then,

$$\begin{aligned} |\varphi(I',\varphi',p',q')-\varphi'| &\leq 18 \|f\|_{\rho_0,s_0,r_0} \left(\frac{30}{s_0}\right)^{\bar{\tau}} \frac{\varepsilon(\log r^{-1})^{\bar{\tau}}}{\tilde{\rho}\bar{\gamma}} \\ &\leq \hat{C} \max\left\{\frac{\varepsilon(\log r^{-1})^{2\bar{\tau}+1}}{\bar{\gamma}^2}, \ \frac{\varepsilon(\log r^{-1})^{\bar{\tau}}}{\bar{\gamma}}\right\} \\ &= \hat{C} \frac{\varepsilon(\log r^{-1})^{2\bar{\tau}+1}}{\bar{\gamma}^2} \end{aligned}$$

as soon as $\sqrt{\varepsilon}(\log r^{-1})^{-1} < 1$ Then, $g', f', \Psi_{\rm av}$ satisfy the claim.

<u>Step 2.</u> ("preparation to Birkhoff Theory") There exist $0 < r_T < 1 < C_T$, $\gamma_T > 0$ such that, for any $0 < r < r_T/(8)$, and

$$\bar{\gamma} > \gamma_{\mathrm{T}} \max\{\sqrt{\varepsilon}(\log r^{-1})^{\bar{\tau}+1}, r^2(\log r^{-1})^{\bar{\tau}+1}, \sqrt[3]{\varepsilon r}(\log r^{-1})^{\bar{\tau}+1}\},\$$

there exists a real-analytic symplectomorphism

$$\phi_{\mathrm{T}} = (I'(\varepsilon, r, \bar{\gamma}; \cdot), \varphi'(\varepsilon, r, \bar{\gamma}; \cdot), p'(\varepsilon, r, \bar{\gamma}; \cdot), p'(\varepsilon, r, \bar{\gamma}; \cdot))$$

on

$$\bar{\mathcal{I}}_{\bar{\rho}(r)/4} \times \mathbb{T}^{\bar{n}}_{s_0/12} \times B(r)_r \times B(r)_r$$

 \mathcal{H}' which puts \mathcal{H}' into the form

$$\mathcal{H}''(I'', \varphi'', p'', q'') = \mathcal{H}' \circ \phi_{\mathrm{T}}(I'', \varphi'', p'', q'')$$

= $h(I'') + \varepsilon g''(I'', p'', q'') + \varepsilon r^5 f''(I'', \varphi'', p'', q'') ,$ (2.82)

where:

(i) g'' has an equilibrium point at (p'',q'') = 0 with Hessian in (p'',q'') = 0 satisfying

$$\sup_{\bar{\mathcal{I}}_{\bar{\rho}(r)/4}} \|\partial^2 g''|_0 - \operatorname{diag}(\Omega_1, \cdots \Omega_{\hat{n}}, \Omega_1, \cdots \Omega_{\hat{n}})\| \le C_{\mathrm{T}} \frac{\varepsilon (\log r^{-1})^{2\bar{\tau}+1}}{\bar{\gamma}^2} ;$$

(iii) f satisfies

$$\|f''\|_{\bar{\mathcal{I}}_{\bar{\rho}(r)/4} \times \mathbb{T}^{\bar{n}}_{s_0/12} \times B(r)_r \times B(r)_r} \le C_{\mathrm{T}}$$

(iv) the following bounds hold for ϕ_{T} , uniformly on $\overline{\mathcal{I}}_{\bar{\rho}(r)/4} \times \mathbb{T}^{\bar{n}}_{s_0/12} \times B(r)_r \times B(r)_r$,

$$\begin{cases}
I'(\varepsilon, r, \bar{\gamma}; \cdot) = I'' \\
|\varphi'(\varepsilon, r, \bar{\gamma}; I'', \varphi'', p'', q'') - \varphi''| \leq C_{\mathrm{T}} \max\left\{\frac{r^{2}(\log r^{-1})^{\bar{\tau}+1}}{\bar{\gamma}}, \frac{\varepsilon r(\log r^{-1})^{3\bar{\tau}+2}}{\bar{\gamma}^{3}}\right\} \\
|p'(\varepsilon, r, \bar{\gamma}; I'', \varphi'', p'', q'') - p''| \leq \frac{C_{\mathrm{T}}}{\bar{\gamma}^{2}} \varepsilon (\log r^{-1})^{2\bar{\tau}+1} \\
|q'(\varepsilon, r, \bar{\gamma}; I'', \varphi'', p'', q'') - q''| \leq \frac{C_{\mathrm{T}}}{\bar{\gamma}^{2}} \varepsilon (\log r^{-1})^{2\bar{\tau}+1}
\end{cases}$$
(2.83)

Proof. Write g' as

$$g'(I', p', q') = \bar{f}(I', p', q') + \tilde{g}(I', p', q')$$

where

$$\bar{f}(I',p',q') := f_0(I') + f_2(I',p',q') + f_4(I',p',q') = f_0(I') + \sum_{1 \le i \le \hat{n}} \Omega_i(I') \frac{{p'_i}^2 + {q'_i}^2}{2} + f_4(I',p',q') = f_0(I') + \sum_{1 \le i \le \hat{n}} \Omega_i(I') \frac{{p'_i}^2 + {q'_i}^2}{2} + f_4(I',p',q') = f_0(I') + \sum_{1 \le i \le \hat{n}} \Omega_i(I') \frac{{p'_i}^2 + {q'_i}^2}{2} + f_4(I',p',q') = f_0(I') + \sum_{1 \le i \le \hat{n}} \Omega_i(I') \frac{{p'_i}^2 + {q'_i}^2}{2} + f_4(I',p',q') = f_0(I') + \sum_{1 \le i \le \hat{n}} \Omega_i(I') \frac{{p'_i}^2 + {q'_i}^2}{2} + f_4(I',p',q') = f_0(I') + \sum_{1 \le i \le \hat{n}} \Omega_i(I') \frac{{p'_i}^2 + {q'_i}^2}{2} + f_4(I',p',q') = f_0(I') + \sum_{1 \le i \le \hat{n}} \Omega_i(I') \frac{{p'_i}^2 + {q'_i}^2}{2} + f_4(I',p',q') = f_0(I') + \sum_{1 \le i \le \hat{n}} \Omega_i(I') \frac{{p'_i}^2 + {q'_i}^2}{2} + f_4(I',p',q') = f_0(I') + \sum_{1 \le i \le \hat{n}} \Omega_i(I') \frac{{p'_i}^2 + {q'_i}^2}{2} + f_4(I',p',q') = f_0(I') + \sum_{1 \le i \le \hat{n}} \Omega_i(I') \frac{{p'_i}^2 + {q'_i}^2}{2} + f_4(I',p',q') = f_0(I') + \sum_{1 \le i \le \hat{n}} \Omega_i(I') \frac{{p'_i}^2 + {q'_i}^2}{2} + f_4(I',p',q') = f_0(I') + \sum_{1 \le i \le \hat{n}} \Omega_i(I') \frac{{p'_i}^2 + {q'_i}^2}{2} + f_4(I',p',q') = f_0(I') + \sum_{1 \le i \le \hat{n}} \Omega_i(I') \frac{{p'_i}^2 + {q'_i}^2}{2} + f_4(I',p',q') = f_0(I') + \sum_{1 \le i \le \hat{n}} \Omega_i(I') \frac{{p'_i}^2 + {q'_i}^2}{2} + f_4(I',p',q') = f_0(I') + \sum_{1 \le i \le \hat{n}} \Omega_i(I') + \sum_{1$$

where, by assumption, f_4 is a power series in (p', q') starting with

$$f_4(I', p', q') = \frac{1}{2} \sum_{1 \le i, j \le \hat{n}} A_{i,j}(I) \frac{{p'_i}^2 + {q'_i}^2}{2} \frac{{p'_j}^2 + {q'_j}^2}{2} + \cdots$$
(2.84)

and, by the previous step,

$$\|\tilde{g}\|_{\bar{\rho}(r)/2, s_0/6, r_0/2} \le C \frac{\varepsilon (\log r^{-1})^{2\bar{\tau}+1}}{\bar{\gamma}^2} \le C (\log r^{-1})^{-1}$$
(2.85)

Then, $F(I',p',q'):=\partial_{(p',q')}g'(I',p',q')$ splits as

$$F(I', p', q') = F_0(I', p', q') + F_1(I', p', q')$$

with

$$F_0 := \begin{pmatrix} \Omega_1(I')p'_1 \\ \vdots \\ \Omega_{\hat{n}}(I')p'_{\hat{n}} \\ \Omega_1(I')q'_1 \\ \vdots \\ \Omega_{\hat{n}}(I')q'_{\hat{n}} \end{pmatrix} \quad \text{and} \quad F_1 := \partial_{(p',q')}(f_4 + \tilde{g})$$

where F_0 is a diffeomorphism of $\mathbb{C}^{\hat{n}}$ sending 0 to 0 and $\det \partial F_0 = \Omega_1(I)^2 \cdots \Omega_{\hat{n}}(I)^2 \neq 0$ on $\overline{\mathcal{I}}_{\bar{\rho}(r)/4}$, thanks to the non resonance condition (first in (2.79)). Furthermore, by (2.84) and Cauchy Estimates on(2.85), the following bound holds for F_1 , uniformly on $\overline{\mathcal{I}}_{\bar{\rho}(r)/4}$:

$$\sup_{B_{r_0/4}^{2\hat{n}}(0)} |F_1| \le C \max\left\{ r_0^3, \ \frac{\varepsilon(\log r^{-1})^{2\bar{\tau}+1}}{\bar{\gamma}^2 r_0/4} \right\}$$

This implies, by Cauchy estimates, that

$$\max\left\{ \sup_{\substack{B_{r_0/4}^{2\hat{n}}(0) \\ r_0/4}(0)} \|\partial F_1\| \sup_{\substack{B_{r_0/4}^{2\hat{n}}(0) \\ \bar{\sigma}_{r_0/4}(0)}} \|(\partial F_0)^{-1}\|, \frac{|\sup_{B_{r_0/4}^{2\hat{n}}(0)} \|(\partial F_0)^{-1}\| \sup_{B_{r_0/4}^{2\hat{n}}(0)} |F_1|}{r_0/8} \right\}$$

$$\leq C \max\left\{ r_0^2, \frac{\varepsilon(\log r^{-1})^{2\bar{\tau}+1}}{\bar{\gamma}^2(r_0/4)^2} \right\}$$

$$\leq \frac{1}{2}$$

as soon as r_0 is small enough and

$$\bar{\gamma} > \tilde{\gamma} \sqrt{\varepsilon} (\log r^{-1})^{\bar{\tau} + 1/2}.$$

Then, by the Quantitative Implicit Function Theorem (Appendix E), for any $I' \in \mathcal{I}_{\bar{\rho}(r)/2}$, we find an equilibrium point $(p_{e}(\varepsilon, r, \bar{\gamma}; I'), q_{e}(\varepsilon, r, \bar{\gamma}; I'))$, with

$$\sup_{\mathcal{I}_{\bar{\rho}(r)/2}} |(p_{\mathrm{e}}(\varepsilon, r, \bar{\gamma}; I'), q_{\mathrm{e}}(\varepsilon, r, \bar{\gamma}; I'))| \le \frac{C}{\bar{\gamma}^{2}} \varepsilon (\log r^{-1})^{2\bar{\tau}+1} \le \frac{r_{0}}{8}$$
(2.86)

for g, *i.e.*, satisfying

$$F(\varepsilon, r, \bar{\gamma}; I', (p_{\mathbf{e}}(\varepsilon, r, \bar{\gamma}; I'), q_{\mathbf{e}}(\varepsilon, r, \bar{\gamma}; I')) = 0 \quad \text{for all} \quad I' \in \mathcal{I}_{\bar{\rho}(r)/2} \ .$$

Define, on

 $\bar{\mathcal{I}}_{\bar{\rho}(r)/4} \times \mathbb{T}^{\bar{n}}_{s_0/12} \times B(r)_r \times B(r)_r \quad \text{where} \quad r < \frac{r_0}{8}$

the transformation $\phi_{\mathrm{T}}(\varepsilon, r, \bar{\gamma}; \cdot) := (I'(\varepsilon, r, \bar{\gamma}, \cdot), \varphi'(\varepsilon, r, \bar{\gamma}, \cdot), p'(\varepsilon, r, \bar{\gamma}, \cdot), q'(\varepsilon, r, \bar{\gamma}, \cdot))$ by means of ¹³

$$\begin{cases}
I' = I'' \\
\varphi' = \varphi'' - \partial_{I''} \left(p'' + p_{e}(\varepsilon, r, \bar{\gamma}; I'') \right) \cdot \left(q' - q_{e}(\varepsilon, r, \bar{\gamma}; I'') \right) \\
p' = p_{e}(\varepsilon, r, \bar{\gamma}; I'') + p'' \\
q' = q_{e}(\varepsilon, r, \bar{\gamma}; I'') + q''
\end{cases}$$
(2.87)

Put

$$D(\varepsilon, r, \bar{\gamma}) := \{ (I'', p'', q') : \{ I'' \in \overline{\mathcal{I}}_{\bar{\rho}(r)/4} , \quad p'' \in B(r)_r , \quad q' - q_e(\varepsilon, r, \bar{\gamma}; I'') \in B(r)_r \}$$

Then, by Cauchy estimate, 14 we find

$$\begin{aligned} |\varphi'(\varepsilon, r, \bar{\gamma}; I'', p'', q'') - \varphi''| &= \left| \partial_{I''} \left(p'' + p_{e}(\varepsilon, r, \bar{\gamma}; I'') \right) \cdot \left(q' - q_{e}(\varepsilon, r, \bar{\gamma}; I'') \right) | \right|_{q'=q_{e}+q''} \\ &\leq \sup_{D(\varepsilon, r, \bar{\gamma})} \left| \partial_{I''} \left(p'' + p_{e}(\varepsilon, r, \bar{\gamma}; I'') \right) \cdot \left(q' - q_{e}(\varepsilon, r, \bar{\gamma}; I'') \right) \right| \\ &\leq \tilde{C} \frac{\sup_{D(\varepsilon, r, \bar{\gamma})} \left| \left(p'' + p_{e}(\varepsilon, r, \bar{\gamma}; I'') \right) \cdot \left(q' - q_{e}(\varepsilon, r, \bar{\gamma}; I'') \right) \right|}{\bar{\rho}(r)/4} \\ &\leq C_{T} \frac{\left(r + \frac{\varepsilon(\log r^{-1})^{2\bar{\tau}+1}}{\bar{\gamma}^{2}} \right) r}{\frac{\bar{\gamma}}{(\log r^{-1})^{\bar{\tau}+1}}} \\ &\leq C_{T} \max \left\{ \frac{r^{2}(\log r^{-1})^{\bar{\tau}+1}}{\bar{\gamma}} , \quad \frac{\varepsilon r(\log r^{-1})^{3\bar{\tau}+2}}{\bar{\gamma}^{3}} \right\} \\ &\leq \frac{s_{0}}{12} \end{aligned}$$

$$(2.88)$$

for a suitably small $\gamma_{\rm T}$. By (2.86) and (2.88), $\phi_{\rm T}$ is well put. By construction, $g'' := g' \circ \phi_{\rm T}$ has an equilibrium point at (p'', q'') = 0. Furthermore, by the splitting

$$g'' = f \circ \phi_{\mathrm{T}} + \tilde{g} \circ \phi_{\mathrm{T}} = f_0 + f_2 \circ \phi_{\mathrm{T}} + f_4 \circ \phi_{\mathrm{T}} + \tilde{g} \circ \phi_{\mathrm{T}}$$

as $\partial^2(f_0) = 0$ and $\partial^2(f_2 \circ \phi_T) = \partial^2 f_2 = \operatorname{diag}(\Omega_1, \cdots \Omega_{\hat{n}}, \Omega_1, \cdots \Omega_{\hat{n}})$, we find

$$\sup_{\bar{\mathcal{I}}_{\bar{\rho}(r)/4}} \|\partial^2 g''|_0 - \operatorname{diag}(\Omega_1, \cdots \Omega_{\hat{n}}, \Omega_1, \cdots \Omega_{\hat{n}})\| = \sup_{\bar{\mathcal{I}}_{\bar{\rho}(r)/4}} \|\partial^2 (f_4 \circ \phi_{\mathrm{T}} + \tilde{g} \circ \phi_{\mathrm{T}})|_0\| \\
\leq C_{\mathrm{T}} \frac{\varepsilon (\log r^{-1})^{2\bar{\tau}+1}}{\bar{\gamma}^2}$$

¹³ $\phi_{\rm T}$ is generated by $S_{\rm T} = I'' \cdot \varphi' + (p'' + p_{\rm e}(\varepsilon, r, \bar{\gamma}; I'') \cdot (q' - q_{\rm e}(\varepsilon, r, \bar{\gamma}; I''))$ ¹⁴Use $a b \leq \max\{a^2, b^2\}$, for any a, b > 0. and the claim is proved.

<u>Step 3.</u> ("Birkhoff Theory") There exists $0 < r_{\rm B} < 1 < C_{\rm B} \ \gamma_{\rm B} > 0$ such that, for any $0 < r < r_{\rm B}$ and

$$\bar{\gamma} > \gamma_{\rm B} \max\{\sqrt{\varepsilon}(\log r^{-1})^{\bar{\tau}+1}, r^2(\log r^{-1})^{\bar{\tau}+1}, \sqrt[3]{\varepsilon r}(\log r^{-1})^{\bar{\tau}+1}\},$$

 $\mathcal{H}^{\prime\prime}$ is put into the form

$$\mathcal{H}'''(I''', \varphi''', p''', q''') = \mathcal{H}'' \circ \phi_{\mathcal{B}}(I''', \varphi''', p''', q''')$$

= $h(I''') + \varepsilon g'''(I''', p''', q''') + \varepsilon r^5 f'''(I''', \varphi''', p''', q''') ,$

where

$$g''' = \tilde{f}_{0}(\varepsilon, r, \bar{\gamma}; I''') + \sum_{1 \le i \le \hat{n}} \tilde{\Omega}_{i}(\varepsilon, r, \bar{\gamma}; I''') \frac{p_{i}^{\prime\prime\prime 2} + q_{i}^{\prime\prime\prime 2}}{2} + \frac{1}{2} \sum_{1 \le i, j \le \hat{n}} \tilde{A}_{i, j}(\varepsilon, r, \bar{\gamma}; I''') \frac{p_{i}^{\prime\prime\prime 2} + q_{i}^{\prime\prime\prime 2}}{2} \frac{p_{j}^{\prime\prime\prime 2} + q_{j}^{\prime\prime\prime 2}}{2}$$
(2.89)

with

$$\begin{split} \tilde{f}_0(\varepsilon,r,\bar{\gamma},\cdot) \ , \quad \tilde{\Omega}(\varepsilon,r,\bar{\gamma},\cdot) &= (\tilde{\Omega}_1(\varepsilon,r,\bar{\gamma},\cdot),\cdots,\tilde{\Omega}_{\hat{n}}(\varepsilon,r,\bar{\gamma},\cdot) \ , \quad \tilde{A}(\varepsilon,r,\bar{\gamma},\cdot) &= (\tilde{A}_{i,j}(\varepsilon,r,\bar{\gamma},\cdot)) \\ \bar{\gamma}^{-2}\varepsilon \ (\log r^{-1})^{2\bar{\tau}+1} - close \ to \ f_0 \ \Omega, \ A, \ respectively \ and \end{split}$$

$$\|f'''\|_{\bar{\mathcal{I}}_{\bar{\rho}(r)/8} \times \mathbb{T}^{\bar{n}}_{s_0/24} \times B(r/2)_{r/2} \times B(r/2)_{r/2}} \le C_{\mathrm{B}}$$

The change of coordinates $\phi_{\rm B} = (I''(\varepsilon, r, \bar{\gamma}; \cdot), \varphi''(\varepsilon, r, \bar{\gamma}; \cdot), p''(\varepsilon, r, \bar{\gamma}; \cdot), p''(\varepsilon, r, \bar{\gamma}; \cdot))$ may be chosen real-analytic on

$$\bar{\mathcal{I}}_{\bar{\rho}(r)/8} \times \mathbb{T}^{\bar{n}}_{s_0/24} \times B(r/2)_{r/2} \times B(r/2)_{r/2}$$

and the following bounds hold, uniformly on $\overline{\mathcal{I}}_{\bar{\rho}(r)/8} \times \mathbb{T}^{\bar{n}}_{s_0/24} \times B(r/2)_{r/2} \times B(r/2)_{r/2}$:

$$\begin{cases} I''(\varepsilon, r, \bar{\gamma}; \cdot) = I''' \\ |\varphi''(\varepsilon, r, \bar{\gamma}; I'', \varphi'', p'', q'') - \varphi'''| \leq C_{\mathrm{B}} \frac{\varepsilon r^{2} (\log r^{-1})^{3\tau+2}}{\bar{\gamma}^{3}} \\ |p''(\varepsilon, r, \bar{\gamma}; I'', \varphi'', p'', q'') - p'''| \leq C_{\mathrm{B}} \frac{\varepsilon r (\log r^{-1})^{2\bar{\tau}+1}}{\bar{\gamma}^{2}} \\ |q''(\varepsilon, r, \bar{\gamma}; I'', \varphi'', p'', q'') - q'''| \leq C_{\mathrm{B}} \frac{\varepsilon r (\log r^{-1})^{2\bar{\tau}+1}}{\bar{\gamma}^{2}} \end{cases}$$

$$(2.90)$$

Proof. For small values of the number $\bar{\gamma}^{-2} \varepsilon (\log r^{-1})^{2\bar{\tau}+1}$, the eigenvalues of $\partial^2 g''|_0$ are purely imaginary, $\bar{\gamma}^{-2} \varepsilon (\log r^{-1})^{2\bar{\tau}+1}$ -close to $(\Omega_1, \dots, \Omega_{\hat{n}}, \Omega_1, \dots, \Omega_{\hat{n}})^{15}$ hence, 4–non resonant on $\bar{\mathcal{I}}_{\bar{\rho}(r)/4}$. Then after a suitable "symplectic diagonalization"

$$\phi_{\mathrm{D}} = (I''(\varepsilon, r, \bar{\gamma}; \cdot), \varphi''(\varepsilon, r, \bar{\gamma}; \cdot), p''(\varepsilon, r, \bar{\gamma}; \cdot), q''(\varepsilon, r, \bar{\gamma}; \cdot))$$

$$I'' = I'' , \quad \varphi'' = \Phi''_j - \frac{\partial_{I_j} t_j}{t_j} P'' \cdot Q'' , \quad p''_j = t_j P''_j , \quad q''_j = \frac{1}{t_j} Q''_j$$

 $^{^{15}\}mathrm{We}$ can always assume that such eigenvalues are pairwise equal, otherwise we perform the change of variables

such that $I''(\varepsilon, r, \bar{\gamma}; \cdot) = I_{\rm D}$, $(p'' - p_{\rm D}, q'' - q_{\rm D})$ is $\varepsilon r \bar{\gamma}^{-2} (\log r^{-1})^{2\bar{\tau}+1}$ -close to the identity, $(\varphi'' - \varphi_{\rm D} \varepsilon r^2 \bar{\gamma}^{-2} (\log r^{-1})^{2\bar{\tau}+1}$ -close to the identity¹⁶ which sends g'' to

$$g_{\rm D} = g_0'' + \sum_{1 \le i \le \hat{n}} \tilde{\Omega}(\varepsilon, r, \bar{\gamma}; I_{\rm D}) \frac{p_{\rm Di}^2 + q_{\rm Di}^2}{2} + o_2 ,$$

we may apply Birkhoff Theory (Appendix B), putting $g_{\rm D}$ into Birkhoff Normal Form

$$g'' \circ \phi_{\mathcal{B}} = g''' + \tilde{o}_4$$

where g''' is as in (2.89) and $|\tilde{o}_4| \leq |(p''', q''')|^5 \leq C_{\rm B} r^5$, by means of a real-analytic symplectomorphism

$$\phi_{\rm B}(\varepsilon, r, \bar{\gamma}; \cdot) = (I_{\rm D}(\varepsilon, r, \bar{\gamma}; \cdot), \varphi_{\rm D}(\varepsilon, r, \bar{\gamma}; \cdot), p_{\rm D}(\varepsilon, r, \bar{\gamma}; \cdot), q_{\rm D}(\varepsilon, r, \bar{\gamma}; \cdot))$$

such that $I_{\rm D} = I'''$, $|p_{\rm D}(\varepsilon, r, \bar{\gamma}, \cdot) - p'''|$, $|q''(\varepsilon, r, \bar{\gamma}, \cdot) - q'''|$ is $\bar{\gamma}^{-2}\varepsilon r^2(\log r^{-1})^{2\bar{\tau}+1}$ -close to the identity and $|\varphi_{\rm D}(\varepsilon, r, \bar{\gamma}, \cdot) - \varphi'''|$ is $\varepsilon r^3(\log r^{-1})^{3\tau+2}/\bar{\gamma}^3$ -close to the identity ¹⁷.

In the following step, we introduce the symplectic polar coordinates. In order to do that, we must stay away from the singularities of these coordinates at $(p_i'', q_i'') = 0$. So, following [29], we introduce a minimum radius r_m for (p''', q'') and later on we will estimate the measure of the descarted zone.

For $0 < r_1 < r_2$, denote

$$A^{p}(r_{1}, r_{2}) := \{ x \in \mathbb{R}^{p} : r_{1} \le |x| \le r_{2} \}$$

the real closed anulus with radii r_1, r_2 .

Step 4. ("the symplectic polar coordinates") There exist C_{pc} , s > 0 such that, for any $\overline{fixed r_m} > 0$, the symplectic ("polar coordinates") transformation ϕ_{pc} defined on the domain

$$\bar{\mathcal{I}}_{\bar{\rho}(r)/8} \times A^{\hat{n}} \left(r_m^2, r^2 \right)_{\hat{\rho}(r)} \times \mathbb{T}_s^{\bar{n}} \times \mathbb{T}_s^{\hat{n}} , \quad \hat{\rho}(r) := \min \left\{ \frac{r_m^2}{2}, r^2 \right\}$$

with

$$t_j = \sqrt[4]{\frac{\tilde{\Omega}_{q_j}}{\tilde{\Omega}_{p_j}}} = 1 + O\left(\frac{\varepsilon(\log r^{-1})^{2\bar{\tau}+1}}{\bar{\gamma}^2}\right)$$

if diag $(\partial^2 g''|_0) = (\tilde{\Omega}_{p_1}, \cdots, \tilde{\Omega}_{p_{\hat{n}}}, \tilde{\Omega}_{q_1}, \cdots, \tilde{\Omega}_{q_{\hat{n}}})$. Such a transformation, generated by $S_{\text{eig}} = \mathbf{I}'' \cdot \varphi'' + \sum_j t_j P_j'' q_j''$ does not change the final estimate (2.90). ¹⁶We may take ϕ_{D} as generated by $S_{\text{D}} = I_{\text{D}} \cdot \varphi'' + s_{\text{D}}(I_{\text{D}}, p_{\text{D}}, q'')$, where $s_{\text{D}}(I_{\text{D}}, p_{\text{D}}, q'')$ is a polynomial

of degree 2 in $(p_{\rm D}, q'')$, the coefficients of which are of order $\varepsilon \bar{\gamma}^{-2} (\log r^{-1})^{2\bar{\tau}+1}$.

¹⁷We may obtain (see Appendix B for details) $\phi_{\rm B}$ in 2 steps (which reduce the diagonalized $g_{\rm D}$ in Birkhoff Normal form of order 3, 4), the first of which generated by $\tilde{I} \cdot \varphi_{\rm D} + \sum_{|\alpha|+|\beta|=3} \tilde{s}_{\alpha,\beta}(\tilde{I})\tilde{p}^{\alpha}q_{\rm D}^{\beta}$, with $\tilde{s}^{i,j} \varepsilon \bar{\gamma}^{-2} (\log r^{-1})^{2\bar{\tau}+1}$ -close to 0; the second one by $I''' \cdot \tilde{\varphi} + \sum_{|\alpha|+|\beta|=4} \check{s}_{\alpha,\beta} (I''') p'''^{\alpha} \tilde{q}^{\beta}$ and $\check{s}^{i,j}$ $\varepsilon \bar{\gamma}^{-2} (\log r^{-1})^{2\bar{\tau}+1}$ -close to 0. Apply then Cauchy estimate.

by

$$\phi_{\rm pc}: \begin{cases} I''' = \bar{J} \\ \varphi''' = \bar{\psi} \\ p''' = \sqrt{2}\,\hat{J}\cos\hat{\psi} \\ q''' = \sqrt{2}\,\hat{J}\sin\hat{\psi} \end{cases}$$
(2.91)

puts $\mathcal{H}'''(\varepsilon, r, \bar{\gamma}, \cdot)$ into the form

$$\begin{aligned} \mathrm{H}\Big(\varepsilon, r, \bar{\gamma}; (\bar{J}, \hat{J}), (\bar{\psi}, \hat{\psi})\Big) &= \mathcal{H}''' \circ \phi_{\mathrm{pc}} \\ &= h(\bar{J}) + \varepsilon \mathrm{h}_1(\varepsilon, r, \bar{\gamma}; (\bar{J}, \hat{J})) + \varepsilon r^5 \mathrm{f}\Big(\varepsilon, r, \bar{\gamma}; (\bar{J}, \hat{J}), (\bar{\psi}, \hat{\psi})\Big) \end{aligned}$$

where

$$\begin{cases} \mathbf{h}_1(\varepsilon, r, \bar{\gamma}; (\bar{J}, \hat{J})) = \tilde{f}_0(\varepsilon, r, \bar{\gamma}; \bar{J}) + \tilde{\Omega}(\varepsilon, r, \bar{\gamma}; \bar{J}) \cdot \hat{J} + \frac{1}{2}\hat{J} \cdot \tilde{A}(\varepsilon, r, \bar{\gamma}; \bar{J})\hat{J}) \\ \|\mathbf{f}\|_{\bar{\mathcal{I}}_{\bar{\rho}(r)/8} \times A(r_m^2, cr^2)_{\hat{\rho}(r)} \times \mathbb{T}_s^{\bar{n}} \times \mathbb{T}_s^{\hat{n}}} \leq C_{\mathrm{pc}} \end{cases}$$

Proof. Obvious.

Remark 2.4 Denote by

$$\phi_{\mathrm{red}}(\varepsilon, r, \bar{\gamma}; \cdot, \cdot) = (\phi_{\mathrm{red},\mathrm{I}}(\varepsilon, r, \bar{\gamma}; \cdot, \cdot), \phi_{\mathrm{red},\varphi}(\varepsilon, r, \bar{\gamma}; \cdot, \cdot), \phi_{\mathrm{red},\mathrm{p}}(\varepsilon, r, \bar{\gamma}; \cdot, \cdot), \phi_{\mathrm{red},\mathrm{q}}(\varepsilon, r, \bar{\gamma}; \cdot, \cdot))$$

the composition of the real-analytic symplectomorphisms described in steps $1 \div 4$. Then, by the estimates (2.81), (2.83), (2.90), we may let

$$\begin{cases} \phi_{\mathrm{red},\mathrm{I}}\left(\varepsilon,r,\bar{\gamma};(\bar{J},\hat{J}),(\bar{\psi},\hat{\psi})\right) = \bar{J} + a\left(\varepsilon,r,\bar{\gamma};(\bar{J},\hat{J}),(\bar{\psi},\hat{\psi})\right) \\ \phi_{\mathrm{red},\varphi}\left(\varepsilon,r,\bar{\gamma};(\bar{J},\hat{J}),(\bar{\psi},\hat{\psi})\right) = \bar{\psi} + b\left(\varepsilon,r,\bar{\gamma};(\bar{J},\hat{J}),(\bar{\psi},\hat{\psi})\right) \\ \phi_{\mathrm{red},\mathrm{p}}\left(\varepsilon,r,\bar{\gamma};(\bar{J},\hat{J}),(\bar{\psi},\hat{\psi})\right) = \sqrt{2\hat{J}}\cos\hat{\psi} + u\left(\varepsilon,r,\bar{\gamma};(\bar{J},\hat{J}),(\bar{\psi},\hat{\psi})\right) \\ \phi_{\mathrm{red},\mathrm{q}}\left(\varepsilon,r,\bar{\gamma};(\bar{J},\hat{J}),(\bar{\psi},\hat{\psi})\right) = \sqrt{2\hat{J}}\sin\hat{\psi} + v\left(\varepsilon,r,\bar{\gamma};(\bar{J},\hat{J}),(\bar{\psi},\hat{\psi})\right) \end{cases}$$

where the functions $a(\varepsilon, r, \bar{\gamma}, \cdot, \cdot)$, $b(\varepsilon, r, \bar{\gamma}, \cdot, \cdot)$, $u(\varepsilon, r, \bar{\gamma}, \cdot, \cdot)$, $v(\varepsilon, r, \bar{\gamma}, \cdot, \cdot)$ satisfy, uniformly on $\bar{\mathcal{I}}_{\bar{\rho}(r)/8} \times A(r_m^2, r^2)_{\hat{\rho}(r)} \times \mathbb{T}_s^{\bar{n}} \times \mathbb{T}_s^{\hat{n}}$,

$$\begin{array}{l} \left| a(\varepsilon, r, \bar{\gamma}, \cdot, \cdot) \right| \leq C \frac{\varepsilon(\log r^{-1})^{\bar{\tau}}}{\bar{\gamma}} \\ \left| b(\varepsilon, r, \bar{\gamma}, \cdot, \cdot) \right| \leq C \max\left\{ \frac{\varepsilon(\log r^{-1})^{2\bar{\tau}+1}}{\bar{\gamma}^2}, \frac{r^2(\log r^{-1})^{2\bar{\tau}+1}}{\bar{\gamma}}, \frac{\varepsilon(\log r^{-1})^{3\bar{\tau}+2}}{\bar{\gamma}^3} \right\} \\ \left| u \right| \leq C \max\left\{ \frac{\varepsilon(\log r^{-1})^{\bar{\tau}}}{\bar{\gamma}}, \frac{\varepsilon(\log r^{-1})^{2\bar{\tau}+1}}{\bar{\gamma}^2} \right\} \\ \left| v \right| \leq C \max\left\{ \frac{\varepsilon(\log r^{-1})^{\bar{\tau}}}{\bar{\gamma}}, \frac{\varepsilon(\log r^{-1})^{2\bar{\tau}+1}}{\bar{\gamma}^2} \right\} \end{array}$$

 $\underline{Step 5.} (``KAM'') Let$

$$\bar{\mathcal{J}} := \bar{\mathcal{I}} \times A^{\hat{n}} \left(r_m^2, r^2 \right)$$

$$\rho := \min\left\{\frac{\bar{\rho}(r)}{16}, \ \hat{\rho}(r)\right\} = \min\left\{r^2, \ \frac{r_m^2}{2}, \ c_* \frac{\bar{\gamma}}{(\log r^{-1})^{\bar{\tau}+1}}, \ \rho_0\right\}$$
(2.92)

There exists r_{KAM} , γ_{KAM} , c_{KAM} such that, for any $0 < r < r_{\text{KAM}}$, $r_m \ge r_{\text{KAM}}^{-1} r^{5/4}$ and

$$\begin{cases} \gamma > \gamma_{\text{KAM}} r^{5/2} \\ \hat{\gamma}_{\varepsilon} > \gamma_{\text{KAM}} \varepsilon r^{5/2} (\log_+ (r^5/\gamma^2)^{-1})^{\tau+1} \\ \bar{\gamma} > \gamma_{\text{KAM}} \max\{\sqrt{\varepsilon} (\log r^{-1})^{\bar{\tau}+1}, \sqrt[3]{\varepsilon r} (\log r^{-1})^{\bar{\tau}+1}, r^2 (\log r^{-1})^{\bar{\tau}+1} \} \end{cases}$$

then the Hamiltonian H satisfies the assumptions of Theorem 2.2 on the domain $\overline{\mathcal{J}}_{\rho} \times \mathbb{T}_{s}^{n}$. <u>Proof.</u>

<u>Claim 1:</u> $\omega = \partial h$ is a diffeomorphism of $\overline{\mathcal{J}}_{\rho}$.

<u>proof of claim 1</u>: Due to the analyticity assumptions, we have only to prove the injectivity for ω . We prove that, for any $\nu = (\bar{\nu}, \varepsilon \hat{\nu}) \in \omega(\varepsilon, r, \bar{\gamma}; \mathcal{I}_{\rho})$, equation $\omega(\varepsilon, r, \bar{\gamma}; (\bar{J}, \hat{J})) = \nu$ has at most one solution on $(\bar{J}, \hat{J}) \in \mathcal{I}_{\rho}$. Let, then,

$$\begin{cases} \bar{\partial} h + \varepsilon (\bar{\partial} \tilde{f}_0 + \bar{\partial} \tilde{\Omega} \cdot \hat{J} + \frac{1}{2} \hat{J} \cdot \bar{\partial} \tilde{A} \hat{J}) = \bar{\nu} \\ \tilde{\Omega} + \tilde{A} \hat{J} = \hat{\nu} \end{cases}$$

where $\bar{\partial}, \hat{\partial}$ denote, respectively, differentiation with respect to \bar{J}, \hat{J} . For any fixed $\bar{J} \in \bar{\mathcal{I}}$, the map

$$\hat{J} \to \tilde{\Omega} + \tilde{A}\hat{J}$$

is injective (as $\tilde{A}(\varepsilon, r, \bar{\gamma}; \bar{J})$ is nonsingular): \bar{J} , we find a unique

$$\hat{J} = \hat{J}_0(\varepsilon, r, \bar{\gamma}, \hat{\nu}, \bar{J}) := \tilde{A}(\varepsilon, r, \bar{\gamma}, \bar{J})^{-1}(\hat{\nu} - \tilde{\Omega}(\varepsilon, r, \bar{\gamma}, \bar{J})$$

solving the second equation. Replacing this value into the equation for the first components, we find an equation of the kind

$$\omega_0(\bar{J}) + \omega_1(\varepsilon, r, \bar{\gamma}, \hat{\nu}; \bar{J})) = \bar{\nu}$$

where $\omega_0 = \bar{\partial}h$ is well defined and analytic up to $\bar{\mathcal{I}}_{\rho_0}$, hence, with $\|(\bar{\partial}\omega_0)^{-1}\|$ uniformly bounded on $\bar{\mathcal{I}}_{\rho}$ by a suitable constant N_0 (which does not depend on $(\varepsilon, r, \bar{\gamma})$) and

$$\omega_1 = \varepsilon \left(\partial \tilde{f}_0 + \bar{\partial} \tilde{\Omega}(\bar{J}) \cdot \hat{J}_0(\varepsilon, r, \bar{\gamma}, \hat{\nu}, \bar{J}) + \frac{1}{2} \hat{J}_0(\varepsilon, r, \bar{\gamma}, \hat{\nu}, \bar{J}) \cdot \bar{\partial} \tilde{A} \hat{J}_0(\varepsilon, r, \bar{\gamma}, \hat{\nu}, \bar{J}) \right)$$

well defined and analytic up to $\overline{\mathcal{I}}_{\overline{\rho}(r)/8}$, hence, by Cauchy estimates,

$$\sup_{\bar{\mathcal{I}}_{\bar{\rho}(r)/16}} \|\bar{\partial}\omega_1\| \le C' \frac{\varepsilon}{\bar{\rho}(r)^2} \le C \frac{\varepsilon(\log r^{-1})^{2(\bar{\tau}+1)}}{\bar{\gamma}^2} \le \frac{1}{2N_0}$$

(recall that \tilde{f}_0 , $\tilde{\Omega}$, \tilde{A} are $\varepsilon (\log r^{-1})^{2\bar{\tau}+1}/\bar{\gamma}^2$ -close to f_0 , Ω , A). This proves the claim. <u>Claim 2</u>: $\omega = \partial h$ has non singular Jacobian on $\bar{\mathcal{J}}_{\rho}$ and $T := (\partial^2 h)^{-1}$ is bounded by

$$\sup_{\bar{\mathcal{J}}_{\rho}} \|T\| \le C\varepsilon^{-1}$$

proof of claim 2: We have

$$\partial \omega = \partial^{2} \mathbf{h}$$

$$= \begin{pmatrix} \partial^{2} h(\bar{J}) & \varepsilon B \\ \varepsilon B^{T} & \varepsilon \tilde{A}(\varepsilon, r, \bar{\gamma}, \bar{J}) \end{pmatrix}$$
(2.93)

where $B = (B_{i,j})$ is the $\bar{n} \times \hat{n}$ matrix with elements

$$B_{i,j} = \partial_{\bar{J}_i} \tilde{\Omega}_j + (\partial_{\bar{J}_i} \tilde{A} \hat{J})_j$$

bounded in norm by (by Cauchy estimates)

$$\sup_{\mathcal{I}_{\rho}} \|B\| \le \sup_{\mathcal{I}_{\bar{\rho}(r)}} \|B\| \le \frac{C'}{\bar{\rho}(r)} \le C \frac{\log r^{-1})^{\bar{\tau}+1}}{\bar{\gamma}}$$
(2.94)

and, for small $\bar{\gamma}^{-2} \varepsilon (\log r^{-1})^{2\bar{\tau}+1}$ and

$$\|\tilde{A}^{-1}\| \le 2\|A^{-1}\|$$

(say). We prove that the matrix

$$\mathcal{M} := \begin{pmatrix} \partial^2 h(\bar{J}) & \varepsilon B \\ B^T & \tilde{A}(\varepsilon, r, \bar{\gamma}, \bar{J}) \end{pmatrix}$$

is non singular which will imply the claim, as, by (2.93),

$$\det(\partial^2 \mathbf{h}) = \varepsilon^{\hat{n}} \det \mathcal{M} \; .$$

We split \mathcal{M} as

$$\mathcal{M} = \mathcal{M}_0 + \mathcal{M}_1 \;,$$

where

$$\mathcal{M}_0 = \begin{pmatrix} \partial^2 h(\bar{J}) & 0 \\ B^T & \tilde{A}(\varepsilon, r, \bar{\gamma}, \bar{J}) \end{pmatrix} , \qquad \mathcal{M}_1 = \begin{pmatrix} 0 & \varepsilon B \\ 0 & 0 \end{pmatrix}$$

The matrix \mathcal{M}_0 is non singular (because $\partial^2 h$, \tilde{A} are so), with inverse matrix

$$\mathcal{M}_0^{-1} = \begin{pmatrix} (\partial^2 h(\bar{J}))^{-1} & 0\\ -(\tilde{A}(\varepsilon, r, \bar{\gamma}, \bar{J}))^{-1} B^T (\partial^2 h(\bar{J}))^{-1} & (\tilde{A}(\varepsilon, r, \bar{\gamma}, \bar{J}))^{-1} \end{pmatrix}$$

Furthermore, the matrix

$$\delta \mathcal{M} := \mathcal{M}_0^{-1} \mathcal{M}_1 = \varepsilon \begin{pmatrix} 0 & (\partial^2 h(\bar{J}))^{-1} B \\ 0 & -(\tilde{A}(\varepsilon, r, \bar{\gamma}, \bar{J}))^{-1} B^T (\partial^2 h(\bar{J}))^{-1} B \end{pmatrix}$$

has norm bounded by

$$\sup_{\bar{\mathcal{J}}_{\rho}} \|\delta \mathcal{M}\| \leq 2\varepsilon \max \left\{ \sup_{\bar{\mathcal{J}}_{\rho}} \|(\partial^{2}h)^{-1}\| \sup_{\bar{\mathcal{J}}_{\rho}} \|B\|, \\ \sup_{\bar{\mathcal{J}}_{\rho}} \|\tilde{A}^{-1}\| \sup_{\bar{\mathcal{J}}_{\rho}} \|\partial^{2}h^{-1}\| (\sup_{\bar{\mathcal{J}}_{\rho}} \|B\|)^{2} \right\} \\ \leq C \frac{\varepsilon (\log r^{-1})^{2(\bar{\tau}+1)}}{\bar{\gamma}^{2}} \\ \leq \frac{1}{2}$$

provided $\bar{\gamma}^{-2} \varepsilon (\log r^{-1})^{2(\bar{\tau}+1)}$ is small. This makes the matrix

$$\mathcal{M} = \mathcal{M}_0(\mathrm{id} + \delta \mathcal{M})$$

invertible. Finally, by (2.93), it is clear that

$$||T|| \leq \varepsilon^{-1} ||\mathcal{M}^{-1}|| \leq C\varepsilon^{-1}$$
.

<u>Claim 3:</u> ("check of the KAM condition") There exist r_{KAM} , γ_{KAM} , $c_{KAM} > 0$ such that, for any $0 < r < r_{KAM}$ and $r_m \ge r_{KAM}^{-1} r^{5/4}$ and

$$\begin{cases} \gamma > \gamma_{\text{KAM}} r^{5/2} \\ \hat{\gamma}_{\varepsilon} > \gamma_{\text{KAM}} \varepsilon r^{5/2} (\log_{+} (r^{5}/\gamma^{2})^{-1})^{\tau+1} \\ \bar{\gamma} > \gamma_{\text{KAM}} \max\{\sqrt{\varepsilon} (\log r^{-1})^{\bar{\tau}+1}, \sqrt[3]{\varepsilon r} (\log r^{-1})^{\bar{\tau}+1}, r^{2} (\log r^{-1})^{\bar{\tau}+1}\} \end{cases}$$
(2.95)

there exist

$$M \ge \sup_{\mathcal{J}_{\rho}} \|U\| , \quad \hat{M} \ge \sup_{\mathcal{J}_{\rho}} \|U^{[n,\hat{n}]}\| , \quad N \ge \sup_{\mathcal{J}_{\rho}} \|T\| , \quad F \ge \|\varepsilon r^{5} f(\varepsilon, r, \bar{\gamma}, \cdot, \cdot)\|_{\rho, s}$$

such that, letting

$$\begin{aligned} c &:= \max\left\{2^{11}n, \ \frac{2}{3}(12)^{\tau+1}\right\} \ , \ \tilde{\rho} &:= \min\left\{\frac{\gamma}{3MK^{\tau+1}} \ , \ \frac{\hat{\gamma}_{\varepsilon}}{3\hat{M}K^{\tau+1}} \ , \ \rho\right\} \ , \\ L &:= \max\left\{N \ , M^{-1}, \hat{M}^{-1}\right\} \ , \quad K &:= \frac{6}{s} \ \log_+\left(\frac{FM^2 L}{\gamma^2}\right)^{-1} \end{aligned}$$

where ρ is as in (2.92), then,

$$E := c \frac{FL}{\tilde{\rho}^2} < 1 .$$

proof of claim 3: To apply Theorem 2.2 to our case, we let, for suitable $c_+ > 1 > c_-$,

$$F = \varepsilon r^5 c_+$$
, $M = c_+$, $\hat{M} = \varepsilon c_+$, $N = c_+ \varepsilon^{-1}$,

so that

$$L = c_{+}\varepsilon^{-1}$$
, $K \le c_{+}(\log (r^{5}/\gamma^{2})^{-1})$.

Taking $\hat{\gamma}_{\varepsilon} = \varepsilon \hat{\gamma}$, we find

$$\tilde{\rho} = \min\left\{\frac{\gamma}{3MK^{\tau+1}}, \frac{\hat{\gamma}_{\varepsilon}}{3\hat{M}K^{\tau+1}}, \rho\right\} \\
\geq c_{-}\min\left\{\frac{\gamma}{(\log_{+}(r^{5}/\gamma^{2})^{-1})^{\tau+1}}, \frac{\hat{\gamma}}{(\log_{+}(r^{5}/\gamma^{2})^{-1})^{\tau+1}}, \frac{\bar{\gamma}_{-}}{(\log r^{-1})^{\bar{\tau}+1}}, \frac{r_{m}^{2}}{2}, r^{2}, \rho_{0}\right\}$$
(2.97)

and hence, the smallness or "KAM" condition

$$cE = c \frac{FL}{\tilde{\rho}^2}$$

$$\leq c_+ \max\left\{ r^5 \frac{(\log_+ (r^5/\gamma^2)^{-1})^{2(\tau+1)}}{\gamma^2} , \frac{r^5 (\log_+ (r^5/\gamma^2)^{-1})^{2(\tau+1)}}{\hat{\gamma}^2} , \frac{r^5 (\log r^{-1})^{2(\bar{\tau}+1)}}{\bar{\gamma}^2} , r , \frac{r^5}{r_m^4} , \frac{r^5}{\rho_0} \right\}$$

$$< 1$$

is fulfilled whenever we choose $r < r_{\text{KAM}}$, then $\bar{\gamma}$, γ , $\hat{\gamma}_{\varepsilon}$ as in (2.95) and finally r_m not less than $r_{\text{KAM}}^{-1} r^{5/4}$, for a suitable small r_{KAM} .

Conclusion of the proof.

Define

$$\begin{cases} \mathfrak{J}_{*}(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma}) \subset \bar{\mathcal{J}} = \bar{\mathcal{I}} \times A^{\hat{n}} \left(r_{m}^{2}, r^{2} \right) \\ \varpi_{*}(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma}, \cdot) : & \mathfrak{J}_{*}(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma}) \to \mathfrak{O}_{*}(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma}) \subset \mathcal{D}_{\gamma, \hat{\gamma}_{\varepsilon}, \tau}^{\bar{n}, \hat{n}} \\ \phi(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma}, \cdot, \cdot) : & \mathfrak{O}_{*} \times \mathbb{T}^{n} \to \bar{\mathcal{J}} \times \mathbb{T}^{n} \\ \mathrm{K}_{*}(\varepsilon, r, \bar{\gamma}, \gamma, \gamma, \hat{\gamma}) \subset \bar{\mathcal{J}} \times \mathbb{T}^{n} \end{cases}$$

as the Cantor set, the Lipschitz homeomorphism onto, the tori embedding and the invariant set which are obtained by Theorem 2.2. Define $\mathfrak{F}(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma}, \cdot, \cdot)$ as the image of $\phi(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma}, \gamma, \hat{\gamma}, \cdot, \cdot)$ under $\phi_{\mathrm{red}}(\varepsilon, r, \bar{\gamma}; \cdot)$; $\mathfrak{U}(\varepsilon, r, \bar{\gamma}) \supset \mathfrak{K}(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma})$, the images of $\bar{\mathcal{J}} \times \mathbb{T}^n \supset \mathrm{K}_*(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma})$, under $\phi_{\mathrm{red}}(\varepsilon, r, \bar{\gamma}; \cdot)$.

Proof of
$$(2.8)$$
. Put

$$\bar{\varpi}_*(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma},\cdot) := \left(\bar{\varpi}_*(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma},\cdot), \ \varepsilon\hat{\varpi}_*(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma},\cdot)\right) \ .$$

By (2.21),

$$\begin{aligned} |\bar{\varpi}_* - \partial h| &\leq |\bar{\varpi}_* - \partial h| + C'\varepsilon \\ &\leq C''\tilde{\rho} + C'\varepsilon \\ &\leq C\min\left\{\frac{\gamma}{(\log_+{(r^5/\gamma^2)^{-1})^{\tau+1}}}, \frac{\hat{\gamma}}{(\log_+{(r^5/\gamma^2)^{-1})^{\tau+1}}}, \frac{\bar{\gamma}}{(\log_+{(r^5/\gamma^2)^{-1})^{\tau+1}}}\right\} \\ &+ C\varepsilon \end{aligned}$$

having used (2.97), for which

$$\tilde{\rho} \leq \tilde{C} \min \left\{ \frac{\gamma}{(\log_+ (r^5/\gamma^2)^{-1})^{\tau+1}}, \frac{\hat{\gamma}}{(\log_+ (r^5/\gamma^2)^{-1})^{\tau+1}}, \frac{\bar{\gamma}}{(\log r^{-1})^{\bar{\tau}+1}}, r^{5/2} \right\}$$

(recall $r_m^2 = \operatorname{const} r^{5/2}$). Similarly,

$$\begin{aligned} |\varepsilon\hat{\varpi}_{*} - \varepsilon\Omega| &\leq |\varepsilon\hat{\varpi}_{*} - \varepsilon\tilde{\Omega} - \varepsilon\tilde{A}\hat{J}| + |\varepsilon\tilde{\Omega} - \varepsilon\Omega| + |\varepsilon(\tilde{A} - A)\hat{J}| + |\varepsilon A\hat{J}| \\ &\leq C\varepsilon\min\left\{\frac{\gamma}{(\log_{+}(r^{5}/\gamma^{2})^{-1})^{\tau+1}}, \frac{\hat{\gamma}}{(\log_{+}(r^{5}/\gamma^{2})^{-1})^{\tau+1}}, \frac{\bar{\gamma}}{(\log r^{-1})^{\bar{\tau}+1}}\right\} \\ &+ C\varepsilon^{2}\frac{(\log r^{-1})^{2\bar{\tau}+1}}{\bar{\gamma}^{2}} + C\varepsilon^{2}r^{2}\frac{(\log r^{-1})^{2\bar{\tau}+1}}{\bar{\gamma}^{2}} + C\varepsilon r^{2} \end{aligned}$$

<u>Proof of (2.9).</u> By Theorem 2.2, we find

$$\begin{split} &\phi(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma},\cdot,\cdot) \\ &= \left(\phi_{\bar{J}}(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma},\cdot,\cdot),\phi_{\hat{J}}(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma},\cdot,\cdot),\phi_{\bar{\psi}}(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma},\cdot,\cdot),\phi_{\hat{\psi}}(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma},\gamma,\hat{\gamma},\cdot,\cdot)\right) \\ &= \left\{ \begin{array}{l} \phi_{\bar{J}}(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma},\cdot,\cdot) = \bar{\mathbf{j}}_{*}(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma};\nu) + \bar{U}(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma};(\bar{\vartheta},\hat{\vartheta})) \\ \phi_{\hat{J}}(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma},\cdot,\cdot) = \bar{\mathbf{j}}_{*}(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma};\nu) + \hat{U}(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma};(\bar{\vartheta},\hat{\vartheta})) \\ \phi_{\bar{\psi}}(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma},\cdot,\cdot) = \bar{\vartheta} + \bar{V}(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma};(\bar{\vartheta},\hat{\vartheta})) \\ \phi_{\hat{\psi}}(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma},\cdot,\cdot) = \hat{\vartheta} + \hat{V}(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma};(\bar{\vartheta},\hat{\vartheta})) \end{split} \end{split}$$

with

$$\begin{cases} |\bar{U}| \leq \check{C}\frac{\hat{\gamma}_{\varepsilon}}{\gamma} \tilde{\rho} \leq C\frac{\hat{\gamma}_{\varepsilon}}{\gamma} \min\left\{\frac{\gamma}{(\log_{+}(r^{5}/\gamma^{2})^{-1})^{\tau+1}}, \frac{\hat{\gamma}}{(\log_{+}(r^{5}/\gamma^{2})^{-1})^{\tau+1}}, \frac{\bar{\gamma}}{(\log_{+}(r^{5}/\gamma^{2})^{-1})^{\tau+1}}, \frac{\bar{\gamma}}{(\log_{+}(r^{5}/\gamma^{2})^{-1})^{\tau+1}}, \frac{\bar{\gamma}}{(\log_{+}(r^{5}/\gamma^{2})^{-1})^{\tau+1}}, r^{5/2}\right\} \\ |\bar{V}|, |\hat{V}| \leq \max\left\{r^{5}\frac{(\log_{+}(r^{5}/\gamma^{2})^{-1})^{2(\tau+1)}}{\gamma^{2}}, \frac{r^{5}(\log_{+}(r^{5}/\gamma^{2})^{-1})^{2(\tau+1)}}{\hat{\gamma}^{2}}, \frac{r^{5}(\log_{r}-1)^{2(\tau+1)}}{\bar{\gamma}^{2}}, r^{5}\right\} \end{cases}$$

Hence, recalling Remark 2.4, the estimates (2.9) follow for $\mathfrak{F}(\varepsilon, \bar{\gamma}, \gamma, \hat{\gamma}; \cdot, \cdot)$ Step 6: proof of (2.6). Let

$$\mathcal{J} := \mathcal{I} \times A^{\hat{n}} \left(r_m^2, \ r^2 \right) \supset \bar{\mathcal{I}} \times A^{\hat{n}} \left(r_m^2, \ r^2 \right) = \bar{\mathcal{J}} \ , \quad r_m \ge r_* r^{5/4}$$

.

We first prove that

$$\operatorname{meas}\left(\mathcal{J}\times\mathbb{T}^n\setminus \mathrm{K}_*(\varepsilon,r,\bar{\gamma},\gamma,\hat{\gamma})\right)\leq C_*\left(\bar{\gamma}+\hat{\gamma}_\varepsilon+\frac{\hat{\gamma}}{r^2}\right)\operatorname{meas}\left(\mathcal{J}\times\mathbb{T}^n\right)$$

The set

$$\mathfrak{K}(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma})$$

is measure–equivalent to

 $\mathrm{K}_*(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma})$

as $\phi_{\text{red}}(\varepsilon, r, \bar{\gamma}; \cdot)$ is, in particular, a real symplectomorphism, hence, area-preserving. The density of $\bar{\mathcal{J}} \times \mathbb{T}^n \setminus K_*(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma})$ in $\bar{\mathcal{J}} \times \mathbb{T}^n$ is estimated by (2.25) of Theorem 2.2:

$$\operatorname{meas}\left(\bar{\mathcal{J}} \times \mathbb{T}^{n} \setminus \mathrm{K}_{*}(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma})\right) \leq \bar{c}\left(\operatorname{meas}(\bar{\mathcal{J}} \setminus \bar{\mathcal{J}}_{\gamma, \hat{\gamma}_{\varepsilon}, \tau} \times \mathbb{T}^{n}) + \operatorname{meas}(\bar{\mathcal{J}}_{\rho_{\max}} \setminus \bar{\mathcal{J}} \times \mathbb{T}^{n})\right)$$

$$(2.98)$$

where (recall (2.97))

$$\rho_{\max} := \max\{\rho_1, \ \rho_2\} \le 2^{-5} \tilde{\rho} \le C \min\{\gamma, \hat{\gamma}\}$$

The second term is easily bounded by $^{\rm 18}$

$$\begin{aligned} \max(\bar{\mathcal{J}}_{\rho_{\max}} \setminus \bar{\mathcal{J}} \times \mathbb{T}^n) &\leq \max(\mathcal{J}_{\rho_{\max}} \setminus \bar{\mathcal{J}} \times \mathbb{T}^n) \\ &\leq \max(\mathcal{J}_{\rho_{\max}} \setminus \mathcal{J} \times \mathbb{T}^n) + \max(\mathcal{J} \setminus \bar{\mathcal{J}} \times \mathbb{T}^n) \\ &\leq \check{c} \max\left\{\frac{\hat{\gamma}}{r^2}, \ \gamma\right\} \operatorname{meas}(\mathcal{J} \times \mathbb{T}^n) + \operatorname{meas}(\mathcal{J} \setminus \bar{\mathcal{J}} \times \mathbb{T}^n) \end{aligned}$$

Inserting this bound into (2.98), we get

$$\operatorname{meas}\left(\bar{\mathcal{J}} \times \mathbb{T}^n \setminus \mathrm{K}_*(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma})\right) \leq \bar{c}\left(\operatorname{meas}(\bar{\mathcal{J}} \setminus \bar{\mathcal{J}}_{\gamma, \hat{\gamma}_{\varepsilon}, \tau} \times \mathbb{T}^n)\right)$$

¹⁸We are using that \mathcal{I} -being an open and bounded set of $\mathbb{R}^{\bar{n}}$ -satisfies the following: there exists $D = D(\mathcal{I}) > 0$, $\bar{\rho} = \bar{\rho}(\mathcal{I})$ such that, for any $0 < \rho < \bar{\rho}$,

$$\operatorname{meas}(\mathcal{I}_{\rho} \setminus \mathcal{I}) \leq \frac{\rho}{D(\mathcal{I})} \operatorname{meas}\mathcal{I} .$$
(2.99)

Then, \mathcal{J} is a product $\mathcal{J} = \mathfrak{A} \times \mathfrak{B}$ where both $\mathfrak{A} = \mathcal{I}$ and $\mathfrak{B} = A^{\hat{n}} \left(r_m^2, r^2 \right)$ have the property (2.99), with $D(\mathfrak{A}) = D_0, D(\mathfrak{B}) = D_0 r^2$. So, using

$$(\mathfrak{A} \times \mathfrak{B})_{\rho} \setminus (\mathfrak{A} \times \mathfrak{B}) = (\mathfrak{A}_{\rho} \setminus \mathfrak{A}) \times \mathfrak{B} \bigcup \mathfrak{A} \times (\mathfrak{B}_{\rho} \setminus \mathfrak{B}) \bigcup (\mathfrak{A}_{\rho} \setminus \mathfrak{A}) \times (\mathfrak{B}_{\rho} \setminus \mathfrak{B})$$

we find

$$\begin{split} \max \! \left(\mathcal{J}_{\rho} \setminus \mathcal{J} \right) &\leq \max (\mathfrak{A}_{\rho} \setminus \mathfrak{A}) \times \mathfrak{B} + \max \mathfrak{A} \times (\mathfrak{B}_{\rho} \setminus \mathfrak{B}) + \max (\mathfrak{A}_{\rho} \setminus \mathfrak{A}) \times (\mathfrak{B}_{\rho} \setminus \mathfrak{B}) \\ &\leq C \left(\rho + \frac{\rho}{r^2} + \frac{\rho^2}{r^2} \right) \max (\mathcal{J}) \end{split}$$

+ meas
$$(\mathcal{J} \setminus \bar{\mathcal{J}} \times \mathbb{T}^n)$$

+ max $\left\{\frac{\hat{\gamma}}{r^2}, \gamma\right\}$ meas $(\mathcal{J} \times \mathbb{T}^n)$)
(2.100)

Now, recalling that

$$\bar{\mathcal{J}} = \bar{\mathcal{I}} \times A^{\hat{n}} \left(r_m^2, \ r^2 \right) \quad \text{with} \quad \bar{\mathcal{I}} = \mathcal{I}_{\bar{\gamma}, \bar{\tau}} = \{ I \in \mathcal{I} : \ \omega(I) \in \mathcal{D}_{\bar{\gamma}, \bar{\tau}}^{\bar{n}} \}$$

we find that first two terms inside the parentheses of (2.100) are similar, and they are simultaneouly estimated by the Lemma 2.5 below.

Lemma 2.5 Let $\hat{n}, \ \bar{n} \in \mathbb{N}, \ \tau > n := \bar{n} + \hat{n}, \ 1 < \alpha < 2, \ 0 < \hat{r} < 1, \ \bar{\mathcal{I}} \ compact, \ ,$ $\hat{\mathcal{I}} := A_{\hat{r}} := A^{\hat{n}} (\hat{r}^{\alpha}, \hat{r})$

$$\omega = (\bar{\omega}, \hat{\omega}) : \quad \mathcal{I} := \bar{\mathcal{I}} \times \hat{\mathcal{I}} \to \mathbb{R}^{\bar{n}} \times \mathbb{R}^{\hat{n}}$$

a diffeomorphism of an open neighborhood of \mathcal{I} , with $\hat{\omega}$ of the form

$$\hat{\omega}(\bar{I},\hat{I}) = \hat{\omega}_0(\bar{I}) + A(\bar{I})\hat{I}$$

where $\bar{I} \to A(\bar{I})$ is non singular on $\bar{\mathcal{I}}$. Let

$$\bar{R} > \max_{\mathcal{I}} |\bar{\omega}| , \quad \mathcal{A} > \max_{\bar{\mathcal{I}}} ||A|| , \quad c(n,\tau) := \sum_{0 \neq k \in \mathbb{Z}^n} \frac{1}{|k|^{\tau}} ,$$

and denote

$$\mathcal{R}_{g,\hat{g},\tau}^{\bar{n},\hat{n}} := \left\{ I = (\bar{I},\hat{I}) \in \mathcal{I} : \quad \omega(I) \notin \mathcal{D}_{g,\hat{g},\tau}^{\bar{n},\hat{n}} \right\} \,.$$

Then, there exists a suitable integer number p such that

$$\operatorname{meas}\left(\mathcal{R}_{g,\hat{g},\tau}^{\bar{n},\hat{n}}\right) \leq \left(\bar{c}g + \hat{c}\,\frac{\hat{g}}{\hat{r}}\right)\operatorname{meas}\left(\mathcal{I}\right)$$

where

$$\begin{cases} \bar{c} := \sup_{\mathcal{I}} \|\omega^{-1}\|^n \frac{\bar{R}^{\bar{n}-1}}{\max(\bar{\mathcal{I}})} \frac{\mathcal{A}^{\hat{n}}}{\max(A_1)} c(n,\tau) p\\ \hat{c} = \frac{\sup_{\bar{\mathcal{I}}} \|A^{-1}\|^{\hat{n}} (\mathcal{A})^{\hat{n}-1}}{\max(A_1)} c(\hat{n},\tau) \end{cases}$$

For continuity reasons, the proof of Lemma 2.5 is postponed at the end of the actual one.

We may then take

$$\begin{cases} \operatorname{meas}(\bar{\mathcal{J}} \setminus \bar{\mathcal{J}}_{\gamma,\hat{\gamma}_{\varepsilon},\tau} \times \mathbb{T}^{n}) \leq c_{*} \left(\gamma + \frac{\hat{\gamma}}{r^{2}}\right) \operatorname{meas}\left(\mathcal{J} \times \mathbb{T}^{n}\right) \\ \operatorname{meas}(\mathcal{J} \setminus \bar{\mathcal{J}} \times \mathbb{T}^{n}) \leq c_{*} \bar{\gamma} \operatorname{meas}\left(\mathcal{J} \times \mathbb{T}^{n}\right) \end{cases}$$
(2.101)

with c_* independent of r. Hence, using (2.101) into (2.100), we finally find

$$\max \left(\mathcal{J} \times \mathbb{T}^n \setminus \mathrm{K}_*(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma}) \right) \leq \max \left(\mathcal{J} \setminus \bar{\mathcal{J}} \times \mathbb{T}^n \right) + \max \left(\bar{\mathcal{J}} \times \mathbb{T}^n \setminus \mathrm{K}_*(\varepsilon, r, \bar{\gamma}, \gamma, \hat{\gamma}) \right) \leq C_* \left(\bar{\gamma} + \gamma + \frac{\hat{\gamma}}{r^2} \right) \max \left(\mathcal{J} \times \mathbb{T}^n \right)$$
(2.102)

which is quite what we meant to prove.

Having now the masure estimate (2.102) and using

$$\operatorname{meas}\left(V(r) \setminus \mathcal{J} \times \mathbb{T}^n\right) \le C_* \left(\frac{r_m}{r}\right)^{2\hat{n}} \operatorname{meas}\left(V(r)\right) \le C_* r^{\hat{n}/2} \operatorname{meas}\left(V(r)\right)$$

(eventually with a different C_*) we easily infer (2.6).

Proof of Lemma 2.5. The first part of the proof uses a compactness argument. Let

$$\bar{R} > \max_{\mathcal{I}} |\bar{\omega}| , \quad \mathcal{A} > \max_{\bar{\mathcal{I}}} ||A|| , \quad \hat{R} := \mathcal{A}\hat{r} > \max_{\mathcal{I}} |\hat{\omega} - \hat{\omega}_0|$$

so that

$$\mathcal{U} := \left\{ B_{\bar{R}}^{\bar{n}}(0) \times B_{\hat{R}}^{\hat{n}}(\omega_0(\bar{I})) , \quad \bar{I} \in \bar{\mathcal{I}} \right\}$$

is an open covering of $\omega(\mathcal{I})$, which is compact, as continuous image of a compact. Then, there exists a finite number of $\bar{I}_1, \dots, \bar{I}_p \in \bar{\mathcal{I}}$ such that

$$\bar{\mathcal{U}} := \bigcup_{1 \le i \le p} U_i , \quad U_i := B^{\bar{n}}_{\bar{R}}(0) \times B^{\hat{n}}_{\hat{R}}(\omega_0(\bar{I}_i))$$

covers $\omega(\mathcal{I})$. Now, the "resonant set" $\mathcal{R}_{g,\hat{g},\tau}^{\bar{n},\hat{n}}$ in \mathcal{I} is

$$\mathcal{R}_{g,\hat{g},\tau}^{\bar{n},\hat{n}} = \bigcup_{k=(\bar{k},\hat{k})\in\mathbb{Z}^{\bar{n}}\times\mathbb{Z}^{\bar{n}},\bar{k}\neq0} \left\{ I: |\omega(I)\cdot k| \le \frac{g}{|k|^{\tau}} \right\} \bigcup_{0\neq\hat{k}\in\mathbb{Z}^{\hat{n}}} \left\{ I: |\hat{\omega}(I)\cdot\hat{k}| \le \frac{\hat{g}}{|\hat{k}|^{\tau}} \right\}$$
(2.103)

The measure of the first set in (2.103) is bounded by

$$\begin{split} & \max\left(\bigcup_{k\in\mathbb{Z}^n,\bar{k}\neq0}\left\{I:\ |\omega(I)\cdot k|\leq\frac{g}{|k|^{\tau}}\right\}\right) \\ & \leq \sup_{\mathcal{I}}\|\omega^{-1}\|^n \max\left(\bigcup_{k\in\mathbb{Z}^n,\bar{k}\neq0}\left\{x\in\omega(\mathcal{I}):\ |x\cdot k|\leq\frac{g}{|k|^{\tau}}\right\}\right) \\ & \leq \sup_{\mathcal{I}}\|\omega^{-1}\|^n \max\left(\bigcup_{k\in\mathbb{Z}^n,\bar{k}\neq0}\bigcup_{i=1}^p\left\{x\in U_i:\ |x\cdot k|\leq\frac{g}{|k|^{\tau}}\right\}\right) \end{split}$$

$$\leq \sup_{\mathcal{I}} \|\omega^{-1}\|^n \sum_{k \in \mathbb{Z}^n, \bar{k} \neq 0} \sum_{1 \leq i \leq p} \max\left(\left\{x \in U_i : |x \cdot k| \leq \frac{g}{|k|^\tau}\right\}\right)$$
$$= \sup_{\mathcal{I}} \|\omega^{-1}\|^n \sum_{k \in \mathbb{Z}^n, \bar{k} \neq 0} \sum_{1 \leq i \leq p} \int_{\mathcal{B}_k^i} d\bar{x} d\hat{x}$$
(2.104)

where

$$\mathcal{B}_{k}^{i} := \left\{ x' = (\bar{x}', \hat{x}') \in U_{i} = B_{\bar{R}}^{\bar{n}}(0) \times B_{\hat{R}}^{\hat{n}}(\omega_{0}(\bar{I}_{i})) : |\bar{x}' \cdot \bar{k} + \hat{x}' \cdot \hat{k}| \le \frac{g}{|k|^{\tau}} \right\}$$

Now, as $\bar{k} \neq 0$, we certainly find $1 \leq j \leq \bar{n}$ with $|\bar{k}_j| \geq 1$. Perform, then, the change of variables

 $z_m = \bar{x}_m$ for $1 \le m \le \bar{n}$, $m \ne j$, $z_j = \bar{x} \cdot \bar{k} + \hat{x} \cdot \hat{k}$, $z_m = \hat{x}_{m-\hat{n}}$ for $\bar{n}+1 \le m \le n$ Then, letting

$$\begin{split} \tilde{\mathcal{B}}_{k}^{i} &:= \{ z' = (z'_{1}, \cdots, z'_{n}) : \\ & \left((z'_{1}, \cdots, z'_{j-1}, \frac{1}{k_{j}} \left(z'_{j} - \sum_{m \neq j} \bar{z}'_{m} \bar{k}_{m} - \sum_{\bar{n}+1 \leq m \leq n} z'_{m} \hat{k}_{m-\bar{n}} \right), z'_{j+1}, \cdots, z'_{\bar{n}} \right) \\ & \in B_{\bar{R}}^{\bar{n}}(0) , \\ & |z'_{j}| \leq \frac{g}{|k|^{\tau}} , \ (z'_{\bar{n}+1}, \cdots, z'_{n}) \in B_{\bar{R}}^{\hat{n}}(\omega_{0}(\bar{I}_{i})) \Big\} \supseteq \\ & \supseteq \ \left\{ z' = (z'_{1}, \cdots, z'_{n}) : \ |z'_{m}| \leq \bar{R} \\ & \text{for } 1 \leq m \neq j \leq \bar{n}, \ |z'_{j}| \leq \frac{g}{|k|^{\tau}}, \\ & |z'_{m} - \omega_{0}(\bar{I}_{i})| \leq \hat{R} \text{ for } m = \bar{n} + 1, \cdots, n \Big\} \\ & =: \ \mathcal{C}_{k}^{i} . \end{split}$$

we find

$$\int_{\mathcal{B}_{k}^{i}} d\bar{x} d\hat{x} = \frac{1}{|k_{j}|} \int_{\tilde{\mathcal{B}}_{k}^{i}} dz \leq \frac{1}{|k_{j}|} \int_{\mathcal{C}_{k}^{i}} dz \leq \int_{\mathcal{C}_{k}^{i}} dz = \bar{R}^{\bar{n}-1} \hat{R}^{\hat{n}} \frac{g}{|k|^{\tau}}$$

Hence, inserting this expression into (2.104), we find

$$\max\left(\bigcup_{k\in\mathbb{Z}^n,\bar{k}\neq0}\left\{I:\ |\omega(I)\cdot k|\leq \frac{g}{|k|^{\tau}}\right\}\right) \leq \|\omega^{-1}\|^n\bar{R}^{\bar{n}-1}\hat{R}^{\hat{n}}pg\sum_{k\in\mathbb{Z}^n,\bar{k}\neq0}\frac{1}{|k|^{\tau}} \\ \leq \sup_{\mathcal{I}}\|\omega^{-1}\|^n\bar{R}^{\bar{n}-1}\hat{R}^{\hat{n}}c(n,\tau)g \\ = \sup_{\mathcal{I}}\|\omega^{-1}\|^n\bar{R}^{\bar{n}-1}\mathcal{A}^{\hat{n}}\hat{r}^{\hat{n}}c(n,\tau)g \\ = \bar{c}g\mathrm{meas}(\mathcal{I})$$

We now estimate the measure of the second set in (2.103). By Fubini's Theorem, we find

$$\operatorname{meas}\left(\bigcup_{0\neq\hat{k}\in\mathbb{Z}^{\hat{n}}}\left\{I: \ |\hat{\omega}(I)\cdot\hat{k}|\leq\frac{\hat{g}}{|\hat{k}|^{\tau}}\right\}\right)=\int_{\bar{\mathcal{I}}}d\bar{I}\int_{\bigcup_{\hat{k}\neq0}\hat{\mathcal{B}}_{k}(\bar{I})}d\hat{I} \qquad (2.105)$$

where

$$\hat{\mathcal{B}}_{k}(\bar{I}) = \bigcup_{0 \neq \hat{k} \in \mathbb{Z}^{\hat{n}}} \left\{ \hat{I} \in B_{\hat{r}}^{\hat{n}}(0) : \ |(\hat{\omega}_{0}(\bar{I}) + A(\bar{I})\hat{I}) \cdot \hat{k}| \le \frac{\hat{g}}{|\hat{k}|^{\tau}} \right\}$$

Perform, in the inner integral, the change of variable

$$\hat{x} = \hat{\omega}_0(\bar{I}) + A(\bar{I})\hat{I}$$

and let

$$\tilde{\mathcal{C}}_{k}(\bar{I}) := \left\{ x \in \mathbb{R}^{\hat{n}} : A(\bar{I})^{-1}(\hat{x} - \omega_{0}(\bar{I})) \in B_{\hat{r}}^{\hat{n}}(0) , \quad |\hat{x} \cdot \hat{k}| \leq \frac{\hat{g}}{|\hat{k}|^{\tau}} \right\} \\
\subseteq \left\{ \hat{x} \in \mathbb{R}^{\hat{n}} : \hat{x} \in B_{\mathcal{A}\hat{r}}^{\hat{n}}(\omega_{0}(\bar{I})) , \quad |\hat{x} \cdot \hat{k}| \leq \frac{\hat{g}}{|\hat{k}|^{\tau}} \right\} \\
=: \hat{\mathcal{C}}_{k}(\bar{I})$$

Then, proceeding as done for the first part of the proof (*i.e.*, with a suitable change of variable, for which $z'_j = \hat{x} \cdot \hat{k}$ if $\hat{k}_j \neq 0$), we find

$$\int_{\bigcup_{\hat{k}\neq 0}\hat{\mathcal{B}}_{k}(\bar{I})} d\hat{I} \leq \sup_{\bar{\mathcal{I}}} \|A^{-1}\|^{\hat{n}} \int_{\bigcup_{\hat{k}\neq 0}\tilde{\mathcal{C}}_{k}(\bar{I})} d\hat{I} \\
\leq \sup_{\bar{\mathcal{I}}} \|A^{-1}\|^{\hat{n}} \int_{\bigcup_{\hat{k}\neq 0}\hat{\mathcal{C}}_{k}(\bar{I})} d\hat{x} \\
\leq \sup_{\bar{\mathcal{I}}} \|A^{-1}\|^{\hat{n}} (\mathcal{A}\hat{r})^{\hat{n}-1} \hat{g} \sum_{\hat{k}\neq 0} \frac{1}{|k|^{\tau}}$$

Hence, inserting this value into (2.105), we find

$$\max\left(\bigcup_{0\neq\hat{k}\in\mathbb{Z}^{\hat{n}}}\left\{I: \ |\hat{\omega}(I)\cdot\hat{k}| \leq \frac{\hat{g}}{|\hat{k}|^{\tau}}\right\}\right) \leq \max(\bar{\mathcal{I}})\sup_{\bar{\mathcal{I}}} ||A^{-1}||^{\hat{n}} (\mathcal{A}\hat{r})^{\hat{n}-1} \hat{g}c(\hat{n},\tau)$$
$$= \frac{\sup_{\bar{\mathcal{I}}} ||A^{-1}||^{\hat{n}} (\mathcal{A})^{\hat{n}-1}}{\max(A_1)} c(\hat{n},\tau) \frac{\hat{g}}{\hat{r}} \operatorname{meas}(\mathcal{I})$$
$$= \hat{c}\frac{\hat{g}}{\hat{r}} \operatorname{meas}(\mathcal{I})$$

because, for small $\hat{r},$

$$\hat{r}^{\hat{n}-1} \le \tilde{c} \, \frac{\operatorname{meas}(\hat{\mathcal{I}})}{\hat{r}}$$

since $\alpha > 1$.

3 Kolmogorov's Set in the Plane Planetary Problem

Let us consider the motion of a system of 1 + N masses m_0, \dots, m_N moving in \mathbb{R}^d (but, soon, we will take d = 2) under the only influence of gravity. As customary, we restrict to the "planetary case": one mass, m_0 (the "Sun") is much greater than m_1, \dots, m_N ("the planets"), namely, we take

$$m_0 = \bar{m}_0$$
, $m_1 = \mu \bar{m}_1$, \cdots , $m_N = \mu \bar{m}_N$ ($\mu \ll 1$). (3.1)

The motion equations are

$$\ddot{v}_i = -\sum_{0 \le j \le N, \ j \ne i} m_j \frac{v_i - v_j}{|v_i - v_j|^3} , \qquad 0 \le i \le N .$$
(3.2)

In the Hamiltonian formalism, equations (3.2) are equivalent to the study of the Hamiltonian

$$\hat{\mathcal{H}}_{\text{plt}}(u,v) = \sum_{0 \le i \le N} \frac{|u_i|^2}{2m_i} - \sum_{0 \le i < j \le N} \frac{m_i m_j}{|v_i - v_j|} \qquad (u_i = m_i \dot{v}_i)$$
(3.3)

on the phase space

,

$$\hat{\mathcal{C}}_{\text{cl,d}} := \left\{ (u, v) = \left((u_0, \cdots, u_N), (v_0, \cdots, v_N) \right) \in (\mathbb{R}^d)^{1+N} \times (\mathbb{R}^d)^{1+N} : v_i \neq v_j \quad \text{for} \quad i \neq j \right\}.$$

The number of degrees of freedom of (3.3) may be reduced (from d(1 + N) to dN) as follows. On the (invariant) symplectic manifold with dimension dN

$$\mathcal{M}_{\text{lin}} = \left\{ u = (u_0, u_1, \cdots, u_N), v = (v_0, v_1, \cdots v_N) \in \hat{\mathcal{C}}_{\text{cl,d}} : \sum_{0 \le i \le N} u_i = \sum_{0 \le i \le N} m_i v_i = 0 \right\}$$

we introduce the relative coordinates

$$\begin{cases} \tilde{x}_i = v_i - v_0 \\ \tilde{y}_i = u_i \end{cases} \quad \text{for} \quad 1 \le i \le N .$$
(3.4)

The motion of m_0 is then recovered by

$$\begin{cases}
 u_0 = -\sum_{1 \le i \le N} \tilde{y}_i \\
 v_0 = -\frac{\sum_{1 \le i \le N} m_i \tilde{x}_i}{\sum_{0 \le i \le N} m_i}
\end{cases}$$
(3.5)

as it results by requiring

$$\begin{cases} 0 = \sum_{0 \le i \le N} u_i = u_0 + \sum_{1 \le i \le N} \tilde{y}_i \\ 0 = \sum_{0 \le i \le N} m_i v_i = m_0 v_0 + \sum_{1 \le i \le N} m_i (\tilde{x}_i + v_0) \end{cases}.$$

The parametrization of \mathcal{M}_{lin} (3.4)÷(3.5), which expresses a point $(u, v) \in \mathcal{M}_{\text{lin}}$ in terms of the coordinates (\tilde{y}, \tilde{x}) , is a homogeneous symplectic embedding, *i.e.*, it preserves the Liouville 1–form:

$$\begin{split} \sum_{0 \le i \le N} u_i \, dv_i &= u_0 \, dv_0 + \sum_{1 \le i \le N} u_i dv_i \\ &= u_0 \, dv_0 + \sum_{1 \le i \le N} \tilde{y}_i d(\tilde{x}_i + v_0) \\ &= \left(u_0 + \sum_{1 \le i \le N} \tilde{y}_i \right) dv_0 + \sum_{1 \le i \le N} \tilde{y}_i d\tilde{x}_i \\ &= \sum_{1 \le i \le N} \tilde{y}_i d\tilde{x}_i \end{split}$$

(because u_0 just coincides with $-\sum_{1 \le i \le N} \tilde{y}_i$).

Then, Hamiltonian (3.3), in terms of the relative coordinates (\tilde{y}, \tilde{x}) , with the masses (3.1), becomes

$$\tilde{\mathcal{H}}_{\text{plt}}(\mu; \tilde{y}, \tilde{x}) := \hat{\mathcal{H}}_{\text{plt}} \circ \phi_{\text{lin}} = \sum_{1 \le i \le N} \left(\frac{|\tilde{y}_i|^2}{2\tilde{m}_i \mu} - \frac{\hat{m}_i \tilde{m}_i \mu}{|\tilde{x}_i|} \right) + \mu \sum_{1 \le i < j \le N} \left(\frac{\tilde{y}_i \cdot \tilde{y}_j}{\bar{m}_0 \mu} - \frac{\bar{m}_i \bar{m}_j \mu}{|\tilde{x}_i - \tilde{x}_j|} \right) ,$$

where

$$\hat{m}_i = \bar{m}_0 + \mu \bar{m}_i , \qquad \tilde{m}_i = \frac{\bar{m}_0 \bar{m}_i}{\bar{m}_0 + \mu \bar{m}_i}$$
(3.6)

are the reduced masses. A rescaling of the variables

$$\begin{cases} \tilde{y} = \mu y \\ \tilde{x} = x \end{cases}$$
(3.7)

joined with the rescaling of the Hamiltonian

$$\mathcal{H}_{\rm plt}(\mu; y, x) := \mu^{-1} \tilde{\mathcal{H}}_{\rm plt}(\mu; \mu y, x)$$

(which does not change Hamiltonian form of the equations of the motion) brings to the Hamiltonian

$$\mathcal{H}_{\text{plt}}(\mu; y, x) = \sum_{1 \le i \le N} \left(\frac{|y_i|^2}{2\tilde{m}_i} - \frac{\hat{m}_i \tilde{m}_i}{|x_i|} \right) + \mu \sum_{1 \le i < j \le N} \left(\frac{y_i \cdot y_j}{\bar{m}_0} - \frac{\bar{m}_i \bar{m}_j}{|x_i - x_j|} \right)$$
(3.8)

with (y, x) varying in the "collisionless" domain

$$\mathcal{C}_{\text{cl,d}} := \left\{ (y, x) = \left((y_1, \cdots, y_N), (x_1, \cdots, x_N) \right) \in \mathbb{R}^{dN} \times \mathbb{R}^{dN} : x_i \neq x_j \neq 0 \quad \forall \ 1 \le i < j \le N \right\}$$
(3.9)

When $\mu = 0$, the Hamiltonian \mathcal{H}_{plt} (3.8) splits into the sum of N Two–Body (integrable) Hamiltonians describing each the interction of a ficticious mass \tilde{m}_i with a fixed star with mass \hat{m}_i .

The aim of this section is the proof of Theorem 3.1 below.

Theorem 3.1 Consider the evolution in time of the coordinates of 1 + N "planetary" masses (3.1) moving on the plane undergoing Newtonian attraction. Let a_i , e_i denote the semimajor axis and the eccentricity of the Keplerian ellipse arising from the two – body interaction of a fictictous mass \tilde{m}_i with a fixed star \hat{m}_i in correspondence of the initial datum (\bar{y}_i, \bar{x}_i) for the coordinates (y_i, x_i) described in $(3.4) \div (3.7)$, where \tilde{m}_i , \hat{m}_i are as in (3.6). Then, there exist b, c, C, $\delta_* > 0$ such that, for any $0 < \delta < \delta_*$, a parameter $\varepsilon_* = \varepsilon_*(\delta)$ may be found such that, for any

$$0 < \varepsilon < \varepsilon_*$$
 and $0 < \mu < (\log \varepsilon^{-1})^{-2i}$

in the set of $(\bar{y}, \bar{x}) = ((\bar{y}_1, \cdots, \bar{y}_N), (\bar{x}_1, \cdots, \bar{x}_N)) \in (\mathbb{R}^2)^N \times (\mathbb{R}^2)^N$ such that

 $a_i = \hat{a}_i \delta^{N-i}$, where $\underline{a} \le \hat{a}_i \le \overline{a}$

there exists a positive Lebesgue measure set \mathcal{K} ("Kolmogorov set"), satisfying

$$c \varepsilon^{2N} > \operatorname{meas} \mathcal{K} > c \left(1 - C(\sqrt{\varepsilon} + \sqrt{\mu}(\log \varepsilon^{-1})^b) \right) \varepsilon^{2N}$$
,

formed by the union of invariant tori of dimension 2N on which the \mathcal{H}_{plt} -flow is linear in time, with Diophantine frequency. Furthermore, the eccentricities on the invariant tori are bounded by $c(\log \varepsilon^{-1})^{-1}$.

3.1 The Plane Delaunay–Poincaré Map

A good set of action-angle variables for the plane problem $(3.8) \div (3.9)$ is the set of Delaunay variables $(L, \mathbf{G}, \ell, \mathbf{g}), L = (L_1, \cdots, L_N), \mathbf{G} = (G_1, \cdots, G_N), \cdots$, with

$$0 < G_i < L_i , \qquad \ell_i, \ g_i \in \mathbb{T}$$

defined as

$$\begin{cases} L_i = \tilde{m}_i \sqrt{\hat{m}_i a_i} \\ G_i = |x_i \times y_i| = \sqrt{1 - e_i^2} L_i \end{cases} \begin{cases} \ell_i = \frac{2\mathcal{A}_i}{a_i^2 \sqrt{1 - e_i^2}} \\ g_i = \text{argument of } P_i \end{cases}$$
(3.10)

where, on the $(\hat{m}_i, \tilde{m}_i; y_i, x_i)$ -"osculating" ellipse ¹⁹ the quantities $a_i, e_i, P_i, \mathcal{A}_i$ denote, respectively, the semimajor axis, eccentricity, perihelion, the area of the elliptic sector from P_i to x_i . In terms of the Delaunay variables, the linear momenta y_i and the positions x_i are recovered by

$$\begin{cases} x_i = x_i^{\mathrm{D}} := a_i \mathcal{R}_{\mathrm{z}}(g_i) \left(\begin{array}{c} \cos u_i - e_i \\ \sqrt{1 - e_i^2} \sin u_i \end{array} \right) & n_i^2 a_i^3 := \hat{m}_i \ , \ (n_i : \text{ "mean motion"}) \ (3.11) \\ y_i = y_i^{\mathrm{D}} := \tilde{m}_i n_i \partial_{\ell_i} x_i \end{cases}$$

¹⁹*I.e.*, the ellipse arising from the initial datum $(\dot{x}(0), x(0)) = (y_i/\tilde{m}_i, x_i)$ for planet in Newtonian interaction with a star with mass \hat{m}_i .

where a_i, e_i are thought as functions of $L_i, G_i, i.e.$,

$$a_i = \frac{1}{\hat{m}_i} \left(\frac{L_i}{\tilde{m}_i}\right)^2 , \quad e_i = \sqrt{1 - \left(\frac{G_i}{L_i}\right)^2}$$
(3.12)

 u_i solves the Kepler's Equation

$$u_i - e_i \sin u_i = \ell_i$$

and $\mathcal{R}_{z}(g)$ denotes a rotation of g in the plane:

$$\mathcal{R}_{\mathbf{z}}(g) = \left(\begin{array}{cc} \cos g & -\sin g\\ \sin g & \cos g \end{array}\right)$$

The plane Delaunay variables (3.10) are well defined whenever

$$e_i \neq 0$$
 for $1 \leq i \leq N$

A "regularization", due to Poincaré, allows the system reaching also zero eccentricities. It is achieved with the symplectic change of variables

$$\phi_{\mathbf{P}}^{-1}: \quad \left\{ \begin{array}{l} \Lambda_i = L_i \\ \lambda_i = l_i + g_i \end{array} \right. \quad \left\{ \begin{array}{l} \eta_i = \sqrt{2(L_i - G_i)} \cos g_i \\ \xi_i = -\sqrt{2(L_i - G_i)} \sin g_i \end{array} \right. \tag{3.13}$$

The regularized variables (3.13) are usually called *Poincaré variables* and, in terms of them, (3.11) become 20

$$\begin{cases} x_i = \hat{x}_i := x_{\rm DP}(\hat{m}_i, \tilde{m}_i; \Lambda_i, \lambda_i, \eta_i, \xi_i) = (x_1^{\rm DP}, x_2^{\rm DP}) \\ y_i = \hat{y}_i := y_{\rm DP}(\hat{m}_i, \tilde{m}_i; \Lambda_i, \lambda_i, \eta_i, \xi_i) = \frac{\hat{m}^2 \tilde{m}^4}{\Lambda^3} \partial_\lambda x_{\rm DP} \end{cases}$$
(3.14)

where

$$\begin{cases} x_1^{\rm DP} = \frac{1}{\hat{m}} \left(\frac{\Lambda}{\tilde{m}}\right)^2 \left[\cos\left(\hat{\zeta} + \lambda\right) - \frac{\xi}{2\Lambda} \left(\eta\sin\left(\hat{\zeta} + \lambda\right) + \xi\cos\left(\hat{\zeta} + \lambda\right)\right) - \frac{\eta}{\sqrt{\Lambda}} \sqrt{1 - \frac{\eta^2 + \xi^2}{4\Lambda}} \right] \\ x_2^{\rm DP} = \frac{1}{\hat{m}} \left(\frac{\Lambda}{\tilde{m}}\right)^2 \left[\sin\left(\hat{\zeta} + \lambda\right) - \frac{\eta}{2\Lambda} \left(\eta\sin\left(\hat{\zeta} + \lambda\right) + \xi\cos\left(\hat{\zeta} + \lambda\right)\right) + \frac{\xi}{\sqrt{\Lambda}} \sqrt{1 - \frac{\eta^2 + \xi^2}{4\Lambda}} \right] \end{cases}$$
(3.15)

and $\hat{\zeta}$ solves the regularized Kepler Equation

$$\zeta = \frac{1}{\sqrt{\Lambda}} \sqrt{1 - \frac{\eta^2 + \xi^2}{4\Lambda}} \left(\eta \sin\left(\zeta + \lambda\right) + \xi \cos\left(\zeta + \lambda\right)\right)$$

²⁰As usual, $\Lambda = (\Lambda_1, \cdots, \Lambda_N), \cdots$.

The semimajor axes a_i and the eccentricities e_i (3.12) become

$$a_i = \frac{1}{\hat{m}_i} \left(\frac{\Lambda_i}{\tilde{m}_i}\right)^2 , \quad e_i = \sqrt{1 - \left(1 - \frac{\eta_i^2 + \xi_i^2}{2\Lambda_i}\right)^2}$$

(zero eccentricities correspond to $(\eta_i, \xi_i) = 0$). In order to avoid collisions we let ²¹

$$\begin{cases} \Lambda \in \mathcal{A}_{\varepsilon} \\ \lambda \in \mathbb{T}^{N} \\ (\hat{\eta}, \hat{\xi}) := \left(\left(\frac{\eta_{1}}{\sqrt{\Lambda_{1}}}, \cdots, \frac{\eta_{N}}{\sqrt{\Lambda_{N}}} \right), \left(\frac{\xi_{1}}{\sqrt{\Lambda_{1}}}, \cdots, \frac{\xi_{N}}{\sqrt{\Lambda_{N}}} \right) \right) \in \mathcal{E}_{\varepsilon} \end{cases}$$
(3.16)

where, for $0 < \varepsilon < 1$,

$$\begin{cases}
\mathcal{A}_{\varepsilon} := \left\{ \Lambda \in \mathbb{R}^{N}_{+} : a_{i}(1+\varepsilon) < a_{i+1}(1-\varepsilon) \quad 1 \le i \le N \right\} \\
\mathcal{E}_{\varepsilon} := \left\{ (\hat{\eta}, \hat{\xi}) \in \mathbb{R}^{N} \times \mathbb{R}^{N} : \frac{\hat{\eta}_{i}^{2} + \hat{\xi}_{i}^{2}}{2} \le 1 - \sqrt{1-\varepsilon^{2}} \quad 1 \le i \le N \right\}
\end{cases}$$
(3.17)

Proposition 3.1 (Delaunay–Poincaré) For any $0 < \varepsilon < 1$, in the domain (3.16) \div (3.17), equations (3.14) \div (3.15) well define a real-analytic symplectomorphism

$$\phi_{\rm DP}: \quad \left(\Lambda, \lambda, \eta, \xi\right) \to \left(\hat{y}, \hat{x}\right) = \left((\hat{y}_1, \cdots \hat{y}_N), (\hat{x}_1, \cdots \hat{x}_N)\right) \tag{3.18}$$

namely, a 1:1 onto, real-analytic and symplectic map, with respect to the standard 2-form

$$\sum_{1 \le i \le N} \left(d\Lambda_i \wedge d\lambda_i + d\eta_i \wedge d\xi_i \right) \,,$$

usually called plane Delaunay–Poincaré map, which carries Hamiltonian of the Plane (1+N)–Body Problem, i.e., the Hamiltonian (3.8) defined on the domain $C_{cl,2}$ (3.9) to

$$\mathcal{H}_{\text{plt}} := -\sum_{1 \le i \le N} \frac{\tilde{m}_i^3 \hat{m}_i^2}{2\Lambda_i^2} + \mu \sum_{1 \le i < j \le N} \left(\frac{\hat{y}_i \cdot \hat{y}_j}{\bar{m}_0} - \frac{\bar{m}_i \bar{m}_j}{|\hat{x}_i - \hat{x}_j|} \right)$$
(3.19)

For a self-contained proof of this Proposition-not easy to be found in literature, see, [11], [20], and also [7].

Sketch of the proof of Theorem 3.1. The proof of Theorem 3.1 consists in applying Theorem 2.1 (in the simplified version of Remark 2.2) to the properly degenerate Hamiltonian \mathcal{H}_{plt} of the plane (1 + N)-Body problem, espressed in Delaunay–Poincaré variables (3.19). We have thus to check all the assumptions thereby involved. We do this in the following steps.

 $\overline{^{21}\mathbb{R}_+} := (0, +\infty).$

(i) Let

$$f := \sum_{1 \le i < j \le N} \left(\frac{\hat{y}_i \cdot \hat{y}_j}{\bar{m}_0} - \frac{\bar{m}_i \bar{m}_j}{|\hat{x}_i - \hat{x}_j|} \right)$$

denote the "perturbation" of \mathcal{H}_{plt} and

$$\bar{f} := \frac{1}{(2\pi)^N} \int_{\mathbb{T}^N} f \, d\lambda$$

its mean (also called "secular" perturbation) with respect to the "fast" angles λ . Due to the *D'Alembert relations* (Lemma 3.2), \bar{f} is even in (η, ξ) , hence, it has an equilibrium point at the origin of the "secular" coordinates, *i.e.*, for $z := (\eta, \xi) = 0$ (Laplace); its quadratic and quartic parts have the form

$$\frac{1}{2}\eta\cdot\mathcal{F}(\Lambda)\eta + \frac{1}{2}\xi\cdot\mathcal{F}(\Lambda)\xi \tag{3.20}$$

and

$$\sum_{1 \le i,j,k,l \le N} q_{i,j,k,l}(\Lambda) \eta_i \eta_j \eta_k \eta_l + r_{i,j,k,l}(\Lambda) \eta_i \eta_j \xi_k \xi_l + q_{i,j,k,l}(\Lambda) \xi_i \xi_j \xi_k \xi_l$$

respectively. In particular, since $\mathcal{F}(\Lambda)$ is a symmetric, in view of (3.20), z = 0 in an *elliptic* equilibrium point, and the eigenvalues $\Omega = (\Omega_1, \dots, \Omega_N)$ of $\mathcal{F}(\Lambda)$ have the meaning of the Birkhoff invariants with order 1. Both the entries of $\mathcal{F}(\Lambda)$ and the tensors $\mathcal{Q} := (q_{ijkl}), \mathcal{R} := (r_{ijkl})$ can be expressed in terms of the Laplace coefficients (Lemmas 3.3, 3.4).

(ii) The diagonalization (by means of a unitary matrix $U(\Lambda)$) of $\mathcal{F}(\Lambda)$ is required, in order to check the 4-non resonance of the Birkhoff invariants with order 1 $\Omega = (\Omega_1, \dots, \Omega_N)$ for \overline{f} . We prove a technical Lemma (Lemma 3.6) which implies that the lowest asymptotics for $\Omega(\Lambda)$ is just the one of the diagonal elements for \mathcal{F} , and so it is given by

$$\begin{split} \Omega_i(\Lambda) &= \delta^{(9-3N)/2} \\ \times & \begin{cases} -\frac{3}{4} \frac{\hat{a}_1^2}{\hat{a}_2^3} \frac{m_1 \bar{m}_2}{\tilde{m}_1 \sqrt{\hat{m}_1 \hat{a}_1}} + O(\delta^2) & \text{for} \quad i = 1 \\ -\frac{3}{4} \frac{\hat{a}_i^2}{\hat{a}_i^3} \frac{\bar{m}_i \bar{m}_{i-1}}{\tilde{m}_i \sqrt{\hat{m}_i \hat{a}_i}} \delta^{(3i-5)/2} + O(\delta^{(3i-3)/2}) & \text{for} \quad 2 \le i \le N-1 \\ -\frac{3}{4} \frac{\hat{a}_i^2}{\hat{a}_N^3} \frac{\bar{m}_N \bar{m}_{N-1}}{\tilde{m}_N \sqrt{\hat{m}_N \hat{a}_N}} \delta^{(3N-5)/2} + O(\delta^{(3N-2)/2}) & \text{for} \quad i = N \end{cases}$$

which immediately implies non resonance up to any finite order, for small δ (Corollary 3.2). We also compute the lowest δ -asymptotics for the entries of $U(\Lambda)$ (Lemma 3.5).

(iii) We diagonalize the quadratic part of \bar{f} with the symplectic transformation

$$\phi_{\text{diag}}: \quad \eta = U(\tilde{\Lambda})\tilde{\eta} \quad \xi = U(\tilde{\Lambda})\tilde{\xi} \quad \Lambda = \tilde{\Lambda} \quad \lambda = \tilde{\lambda} + \varphi(\tilde{\Lambda}, \tilde{\eta}, \tilde{\xi})$$

(where φ a suitable shift of λ which does not change the mean), hence, we put \bar{f} into the form

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$$\begin{split} \tilde{f} &:= \bar{f} \circ \phi_{\text{diag}} = \bar{f}_0(\tilde{\Lambda}) + \sum_{1 \le i \le N} \Omega_i(\tilde{\Lambda}) \frac{\tilde{\eta}_i^2 + \xi_i^2}{2} \\ &+ \sum_{1 \le i, j, k, l \le N} \tilde{q}_{i, j, k, l}(\tilde{\Lambda}) \, \tilde{\eta}_i \tilde{\eta}_j \tilde{\eta}_k \tilde{\eta}_l + \tilde{r}_{i, j, k, l}(\tilde{\Lambda}) \, \tilde{\eta}_i \tilde{\eta}_j \tilde{\xi}_k \tilde{\xi}_l + \tilde{q}_{i, j, k, l}(\tilde{\Lambda}) \, \tilde{\xi}_i \tilde{\xi}_j \tilde{\xi}_k \tilde{\xi}_l \\ &+ o_4 \end{split}$$

(iv) We compute the δ -asymptotics for \tilde{q}_{ijkl} , \tilde{r}_{ijkl} (which involves those of $r(\Lambda)$, $s(\Lambda)$, $U(\Lambda)$) and hence the δ -asymptotics for the entries of the Birkhoff invariants with order 2, which are the entries of the symmetric matrix $A(\bar{\Lambda})$ of the Birkhoff Normal form of \tilde{f}

$$\bar{f}_0(\bar{\Lambda}) + \Omega(\bar{\Lambda}) \cdot J + \frac{1}{2}J \cdot A(\bar{\Lambda})J + o(|J|^3) \quad J_i := \frac{\bar{\eta}_i^2 + \bar{\xi}_i^2}{2}$$

obtained by projections of the entries \tilde{q}_{ijkl} , \tilde{r}_{ijkl} (Lemma 3.10). We check (Lemma 3.9) that $A(\bar{\Lambda})$ has the form

$$A(\Lambda) \approx \delta^{p} \begin{pmatrix} \alpha_{11} & \alpha_{12} & O(\delta^{p_{13}}) & \cdots & O(\delta^{p_{1k}}) & \cdots \\ \alpha_{21} & \alpha_{22} & O(\delta^{p_{23}}) & \cdots & O(\delta^{p_{2k}}) & \cdots \\ & & \alpha_{33}\delta^{p_{33}} & \cdots & O(\delta^{p_{3k}}) & \cdots \\ & & & \ddots & \vdots \\ & & & & \alpha_{kk}\delta^{p_{kk}} & \cdots \\ & & & & \ddots \end{pmatrix}$$

where $\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} \neq 0$ and $p_{k+1,k+1} > p_{kk}$, $\alpha_{kk} \neq 0$ and that this implies (Lemma 3.8)

$$\det A = (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})\delta^q + o(\delta^q) \neq 0 ,$$

concluding the proof.

3.2 Non Resonance and non Degeneracy for the Plane Planetary Problem

3.2.1 Expansion of the Hamiltonian

The perturbation f of the Plane Planetary Problem is composed of two terms

$$f_{\mathbf{p}} := -\sum_{1 \le i < j \le N} \frac{\bar{m}_i \bar{m}_j}{|\hat{x}_i - \hat{x}_j|} , \quad f_{\mathbf{s}} := \frac{1}{\bar{m}_0} \sum_{1 \le i < j \le N} \hat{y}_i \cdot \hat{y}_j$$

usually referred as principal and secondary part, respectively. We are interested to the secular perturbation, *i.e.*, the mean, over $\lambda \in \mathbb{T}$, of f, to which only f_d contributes:

Lemma 3.1 The secondary part of the perturbation has zero mean.

Proof. In fact, as \hat{y}_i is defined as

$$\hat{y}_i := \frac{\hat{m}_i^2 \tilde{m}_i^4}{\Lambda_i^3} \partial_{\lambda_i} \hat{x}_i$$

and does not depend on the variables λ_j with $j \neq i$,

$$\frac{1}{(2\pi)^N} \int_{\mathbb{T}^N} \hat{y}_i \cdot \hat{y}_j \, d\lambda = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \hat{y}_i \cdot \hat{y}_j \, d\lambda_i d\lambda_j$$

$$= \frac{1}{(2\pi)^2} \frac{\hat{m}_i^2 \tilde{m}_i^4}{\Lambda_i^3} \frac{\hat{m}_j^2 \tilde{m}_j^4}{\Lambda_j^3}$$

$$\times \int_{\mathbb{T}^2} \partial_{\lambda_i} \hat{x}_i \cdot \partial_{\lambda_j} \hat{x}_i \, d\lambda_i d\lambda_j$$

$$= \frac{1}{(2\pi)^2} \frac{\hat{m}_i^2 \tilde{m}_i^4}{\Lambda_i^3} \frac{\hat{m}_j^2 \tilde{m}_j^4}{\Lambda_j^3} \int_{\mathbb{T}} \partial_{\lambda_i} \hat{x}_i d\lambda_i$$

$$\cdot \int_{\mathbb{T}} \partial_{\lambda_j} \hat{x}_i \, d\lambda_j$$

$$= 0.$$

Notice that that \hat{x}_i -component of the Plane Delaunay Poincaré depends on Λ_i , η_i , ξ_i as a function of

$$a_i = \frac{1}{\hat{m}_i} \left(\frac{\Lambda_i}{\tilde{m}_i}\right)^2 , \quad (\hat{\eta}_i, \hat{\xi}_i) = \left(\frac{\eta_i}{\sqrt{\Lambda_i}}, \frac{\xi_i}{\sqrt{\Lambda_i}}\right) ,$$

only.

Lemma 3.2 (D'Alembert relations) Let

$$\sum_{(j_1,j_2,i_1,i_2)} a_{j_1,j_2,i_1,i_2}(a_i,a_j) \hat{\eta_i}^{j_1} \, \hat{\eta_j}^{j_2} \, \hat{\xi_i}^{i_1} \, \hat{\xi_j}^{i_2} \tag{3.21}$$

the Taylor expansion of

$$\frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{d\lambda_i d\lambda_j}{|\hat{x}_i - \hat{x}_j|}$$
(3.22)

Then,

Proof. Item (i) is trivial. Items $(ii) \div (iv)$ are related to the following symmetries of the plane Delaunay–Poincaré map

$$\hat{x}_{i}(\hat{m}, \tilde{m}; \Lambda, \pi - \lambda, -\eta, \xi) = \mathcal{R}_{x=0} \hat{x}_{i}(\hat{m}, \tilde{m}; \Lambda, \lambda, \eta, \xi) ,
\hat{x}_{i}(\hat{m}, \tilde{m}; \Lambda, -\lambda, \eta, -\xi) = \mathcal{R}_{y=0} \hat{x}_{i}(\hat{m}, \tilde{m}; \Lambda, \lambda, \eta, \xi) ,
\hat{x}_{i}(\hat{m}, \tilde{m}; \Lambda, \pi/2 - \lambda, \xi, \eta) = \mathcal{R}_{x=y} \hat{x}_{i}(\hat{m}, \tilde{m}; \Lambda, \lambda, \eta, \xi)$$

where $\mathcal{R}_{y=0}$, $\mathcal{R}_{x=0}$, $\mathcal{R}_{x=y}$, denote the reflections in the plane with respect to the axes x, y, x = y, axes.

Remark 3.1 Then, the secular perturbation \overline{f} contains only polynomials $f_{2j}(\Lambda, \eta, \xi)$ with even degree 2j:

$$\bar{f}(\Lambda,\eta,\xi) = f_0(\Lambda) + f_2(\Lambda,\eta,\xi) + f_4(\Lambda,\eta,\xi) + \cdots ; \qquad (3.24)$$

where each f_{2i} is an even function of η , ξ separately. In particular,

Corollary 3.1 (Laplace) The point $(\eta, \xi) = 0$ is an equilibrium point for \overline{f} , for all Λ .

Remark 3.2 The computation of the 0–term $f_0(\Lambda)$ in (3.24) is trivial. When $(\eta, \xi) = 0$, \hat{x}_i reduces to

$$\hat{x}_i|_{(\eta,\xi)=0} = a_i(\cos\lambda_i, \sin\lambda_i)$$
.

Hence,

$$|\hat{x}_i|_{(\eta,\xi)=0} - \hat{x}_j|_{(\eta,\xi)=0}| = \sqrt{a_i^2 + a_j^2 - 2a_i a_j \cos\left(\lambda_i - \lambda_j\right)}$$

and, finally

$$f_{0} = -\sum_{1 \le i < j \le N} \bar{m}_{i} \bar{m}_{j} \frac{1}{(2\pi)^{2}} \int_{\mathbb{T}^{2}} \frac{d\lambda_{i} d\lambda_{j}}{|\hat{x}_{i}|_{(\eta,\xi)=0} - \hat{x}_{j}|_{(\eta,\xi)=0}|}$$

$$= -\sum_{1 \le i < j \le N} \bar{m}_{i} \bar{m}_{j} \frac{1}{2\pi} \int_{\mathbb{T}} \frac{dt}{\sqrt{a_{i}^{2} + a_{j}^{2} - 2a_{i}a_{j}\cos(\lambda_{i} - \lambda_{j})}}$$

$$= -\sum_{1 \le i < j \le N} \frac{\bar{m}_{i} \bar{m}_{j}}{a_{j}} b_{1/2,0}(a_{i}/a_{j})$$

where $b_{s,k}(\alpha)$ is the (s,k)-Laplace coefficient, defined as the k^{th} Fourier coefficient of the function $t \to \left[1 + \alpha - 2\alpha \cos(\lambda_i - \lambda_j)\right]^{-s}$:

$$b_{s,k}(\alpha) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\cos kt}{\left[1 + \alpha - 2\alpha \cos\left(\lambda_i - \lambda_j\right)\right]^s}$$

Regularity properties and expansions (in α) of the Laplace Coefficients are briefly discussed in Appendix F.
Next two lemmas are devoted to the computation of f_2 , f_4 of (3.24). Lemma 3.3 The polynomial with order 2 in the expansion of \bar{f} is

$$f_2(\Lambda,\eta,\xi) = \frac{1}{2}\eta \cdot \mathcal{F}(\Lambda)\eta + \frac{1}{2}\xi \cdot \mathcal{F}(\Lambda)\xi$$

where

$$f_{ij}(\Lambda) = \begin{cases} -2\frac{\bar{m}_i}{\Lambda_i} \left[\sum_{k \neq i} \bar{m}_k a_{2000}(a_i, a_k) \right] & \text{for} \quad i = j \\ -\bar{m}_i \bar{m}_j \frac{a_{1100}(a_i, a_j)}{\sqrt{\Lambda_i \Lambda_j}} & \text{for} \quad i < j \\ -\bar{m}_i \bar{m}_j \frac{a_{1100}(a_j, a_i)}{\sqrt{\Lambda_i \Lambda_j}} & \text{for} \quad i > j \end{cases}$$

with

$$a_{2000}(a,b) = \frac{a}{8b^2} \left[-7a/b \, b_{5/2,0}(a/b) + 4(1+a^2/b^2) \, b_{5/2,1}(a/b) - a/b \, b_{5/2,2}(a/b) \right]$$

$$a_{1100}(a,b) = \frac{a}{8b^2} \left[-17 \, a/b \, b_{5/2,1}(a/b) + 8(1+a^2/b^2) \, b_{5/2,2}(a/b) + a/b \, b_{5/2,3}(a/b) \right]$$

$$(3.25)$$

Proof. Using the symmetries (3.23) outlined in Corollary 3.2, the non vanishing terms with order 2 appearing in the expansion (3.21) of g_{ij} , are only six, and they are individuated by only two independent coefficients, say a_{2000} and a_{1100} :

$$a_{2000}(a_i, a_j) \,\hat{\eta}_i^2 + a_{1100}(a_i, a_j) \,\hat{\eta}_i \hat{\eta}_j + a_{2000}(a_i, a_j) \,\hat{\eta}_j^2 + a_{2000}(a_i, a_j) \,\hat{\xi}_i^2 + a_{1100}(a_i, a_j) \,\hat{\xi}_i \hat{\xi}_j + a_{2000}(a_i, a_j) \,\hat{\xi}_j^2$$

$$(3.26)$$

Thus, multiplying by $-\bar{m}_i\bar{m}_j$ and summing over all $1 \le i < j \le N$ and then symmetrizing the sum, we write f_2 as

$$f_2 = \frac{1}{2}\eta \cdot \mathcal{F}(\Lambda)\eta + \frac{1}{2}\xi \cdot \mathcal{F}(\Lambda)\xi$$

where the matrix $\mathcal{F}(\Lambda) = (f_{i,j}(\Lambda))$ has elements

$$f_{ij}(\Lambda) = \begin{cases} -2 \frac{\bar{m}_i}{\Lambda_i} \left[\sum_{1 \le k < i} \bar{m}_k \, a_{2000}(a_k, a_i) + \sum_{i < k \le N} \bar{m}_k \, a_{2000}(a_i, a_k) \right] & \text{for} \quad i = j \\ -\bar{m}_i \bar{m}_j \frac{a_{1100}(a_i, a_j)}{\sqrt{\Lambda_i \Lambda_j}} & \text{for} \quad i < j \\ -\bar{m}_i \bar{m}_j \frac{a_{1100}(a_j, a_i)}{\sqrt{\Lambda_i \Lambda_j}} & \text{for} \quad i > j \end{cases}$$

The coefficients $a_{2000}(a, b)$, $a_{1100}(a, b)$ coincide with the expressions $(ab/8)\mathcal{I}(a, b)$, $(ab/8)\mathcal{J}(a, b)$ computed in [8], which, written in terms of the Laplace Coefficients are just (3.25). The result then follows taking into account the symmetry of the coefficient $a_{2000}(a_{2000}(a, b) = a_{2000}(b, a))$. **Lemma 3.4** The polynomial with order 4 in the expansion of \overline{f} is

$$f_4 = \sum_{1 \le i, j, k, l \le N} q_{ijkl}(\Lambda) \Big(\eta_i \eta_j \eta_k \eta_l + \xi_i \xi_j \xi_k \xi_l \Big) + \sum_{1 \le i, j, k, l \le N} r_{ijkl}(\Lambda) \eta_i \eta_j \xi_k \xi_l$$

where

$$q_{i,j,k,l}(\Lambda) := \begin{cases} -\frac{\bar{m}_{i}}{\Lambda_{i}^{2}} \sum_{h:h\neq i} \bar{m}_{h} a_{4000}(a_{i}, a_{h}) & \text{for} \quad i = j = k = l \\ -\bar{m}_{i} \bar{m}_{l} \frac{a_{3100}(a_{i}, a_{l})}{\Lambda_{i} \sqrt{\Lambda_{i}} \Lambda_{l}} & \text{for} \quad i = j = k \neq l \\ -\bar{m}_{i} \bar{m}_{l} \frac{a_{2200}(a_{i}, a_{l})}{\Lambda_{i} \Lambda_{l}} & \text{for} \quad i = j < k = l \\ 0 & \text{otherwise}; \\ -\frac{\bar{m}_{i}}{\Lambda_{i}^{2}} \sum_{h:h\neq i} \bar{m}_{h} a_{2020}(a_{i}, a_{h}) & \text{for} \quad i = j = k = l \\ -\bar{m}_{k} \bar{m}_{i} \frac{a_{0220}(a_{k}, a_{i})}{\Lambda_{k} \Lambda_{i}} & \text{for} \quad i = j \neq k = l \\ -\bar{m}_{i} \bar{m}_{j} \frac{a_{1120}(a_{i}, a_{j})}{\Lambda_{i} \sqrt{\Lambda_{i}} \Lambda_{j}} & \text{for} \quad i = j = k \neq l \\ -\bar{m}_{i} \bar{m}_{i} \frac{a_{1120}(a_{i}, a_{l})}{\Lambda_{i} \sqrt{\Lambda_{i}} \Lambda_{j}} & \text{for} \quad i = j = k \neq l \\ -\bar{m}_{i} \bar{m}_{j} \frac{a_{1110}(a_{i}, a_{l})}{\Lambda_{i} \sqrt{\Lambda_{i}} \Lambda_{j}} & \text{for} \quad i = j = k \neq l \\ -\bar{m}_{i} \bar{m}_{j} \frac{a_{1111}(a_{i}, a_{j})}{\Lambda_{i} \Lambda_{j}} & \text{for} \quad i = k < j = l \\ 0 & \text{otherwise}. \end{cases}$$

$$(3.27)$$

where

$$\begin{aligned} a_{4000}(a,b) &= \frac{a}{512b^2} \left[\left(-60(a/b)^5 + 4311(a/b)^3 \right. \\ &- 300(a/b) \right) b_{9/2,0}(a/b) + 8(7(a/b)^6 \\ &- 252(a/b)^4 - 222(a/b)^2 + 7) b_{9/2,1}(a/b) \\ &+ 4(75(a/b)^5 - 503(a/b)^3 + 135(a/b)) b_{9/2,2}(a/b) \\ &+ 24(23(a/b)^4 + 13(a/b)^2) b_{9/2,3}(a/b) \\ &+ 37(a/b)^3 b_{9/2,4}(a/b) \right] \end{aligned}$$

$$a_{3100}(a,b) = -\frac{a}{256b^2} \left[(-744(a/b)^5 + 2014(a/b)^3 - 864(a/b)) b_{9/2,1}(a/b) + 8(28(a/b)^6 - 321(a/b)^4 - 321(a/b)^2 + 28) b_{9/2,2}(a/b) + (552(a/b)^5 + 423(a/b)^3 + 672(a/b)) b_{9/2,3}(a/b) + (1146(a/b)^4 + 1266(a/b)^2) b_{9/2,0}(a/b) \right]$$

$$\begin{array}{rcl} &+& 6(29(a/b)^4+9(a/b)^2)\, b_{9/2,4}(a/b)\\ &-& 5(a/b)^3\, b_{9/2,5}(a/b) \Big]\\ a_{2200}(a,b) &=& \frac{a}{512b^2} \left[(-324(a/b)^5+10584(a/b)^3-324(a/b))\, b_{9/2,0}(a/b)\\ &+& 8(17(a/b)^6-300(a/b)^4-300(a/b)^2+17)\, b_{9/2,1}(a/b)\\ &-& (1272(a/b)^5+6337(a/b)^3+1272(a/b))\, b_{9/2,2}(a/b)\\ &+& (648(a/b)^6+396(a/b)^4+396(a/b)^2\\ &+& 648)\, b_{9/2,3}(a/b)+(348(a/b)^5\\ &+& 800(a/b)^3+348(a/b))\, b_{9/2,4}(a/b)\\ &+& (-60a/b)^4-60(a/b)^2)\, b_{9/2,5}(a/b)\\ &+& 9(a/b)^3\, b_{9/2,6}(a/b) \Big]\\ a_{2020}(a,b) &=& \frac{a}{256b^2} \left[8(7(a/b)^6-252(a/b)^4-222(a/b)^2\\ &+& 7)\, b_{9/2,1}(a/b)+(-60(a/b)^5+4311(a/b)^3\\ &-& 300(a/b))\, b_{9/2,0}(a/b)+4(75(a/b)^5-503(a/b)^3\\ &+& 135(a/b)\, b_{9/2,2}(a/b)+24(23(a/b)^4\\ &+& 13(a/b)^2)\, b_{9/2,3}(a/b)+37(a/b)^4\, b_{9/2,4}(a/b) \Big]\\ a_{1120}(a,b) &=& -\frac{a}{256b^2} \left[(-744(a/b)^5+2014(a/b)^3\\ &-& 864(a/b))\, b_{9/2,1}(a/b)+8(28(a/b)^6-321(a/b)^4\\ &-& 321(a/b)^2+28)\, b_{9/2,2}(a/b)\\ &+& (1552(a/b)^5+423(a/b)^3+672(a/b))\, b_{9/2,3}(a/b)\\ &+& (1146(a/b)^4+1266(a/b)^2)\, b_{9/2,0}(a/b)\\ &+& (629(a/b)^4+9(a/b)^2)\, b_{9/2,4}(a/b)\\ &-& 5(a/b)^3\, b_{9/2,5}(a/b) \right]\\ a_{0220}(a,b) &=& -\frac{3a}{512b^2} \left[(84(a/b)^5-8832(a/b)^3\\ &+& 84(a/b))\, b_{9/2,0}(a/b)-8(5(a/b)^6\\ &-& 652(a/b)^4-652(a/b)^2+5)\, b_{9/2,1}(a/b)\\ &-& 5(328(a/b)^5-561(a/b)^3+328(a/b))\, b_{9/2,2}(a/b)\\ &+& (216(a/b)^6-1020(a/b)^4\\ \end{array}$$

+
$$(216(a/b)^2 + 216) b_{9/2,3}(a/b)$$

+ $(116(a/b)^5 + 200(a/b)^3$

- + $116(a/b)) b_{9/2,4}(a/b)$

$$- (20(a/b)^{4} + 20(a/b)^{2}) b_{9/2,5}(a/b) + 3(a/b)^{3} b_{9/2,6}(a/b)]$$

$$a_{1111}(a,b) = \frac{a}{128b^{2}} \left[(-36(a/b)^{5} - 7956(a/b)^{3} - 36(a/b)) b_{9/2,0}(a/b) + 8((a/b)^{6} + 828(a/b)^{4} + 828(a/b)^{2} + 1) b_{9/2,1}(a/b) + (-3096(a/b)^{5} + 1039(a/b)^{3} - 3096(a/b)) b_{9/2,2}(a/b) + (648(a/b)^{6} - 1332(a/b)^{4} - 1332(a/b)^{2} + 648) b_{9/2,3}(a/b) + (348(a/b)^{5} + 700(a/b)^{3} + 348(a/b)) b_{9/2,4}(a/b) - 60((a/b)^{4} + (a/b)^{2}) b_{9/2,5}(a/b) + 9(a/b)^{3} b_{9/2,6}(a/b) \right]$$
(3.28)

Proof. As in the proof of the previous Lemma, we use the symmetries (3.23) outlined in Corollary 3.2. We find, in the fourth order of the function g_{ij} , only 19 (among the 35 possible ones) non vanishing monomials with degree 4, which are individuated by 7 independent coefficients, say a_{4000} , a_{3100} , a_{2200} , a_{2020} , a_{1120} , a_{0220} and a_{1111} :

$$\begin{aligned} & a_{4000}(a_i,a_j)\,\hat{\eta}_i^4 + a_{3100}(a_i,a_j)\,\hat{\eta}_i^3\hat{\eta}_j + a_{2200}(a_i,a_j)\,\hat{\eta}_i^2\hat{\eta}_j^2 + a_{3100}(a_j,a_i)\,\hat{\eta}_i\hat{\eta}_j^3 \\ &+ a_{4000}(a_j,a_i)\,\hat{\eta}_j^4 + a_{2020}(a_i,a_j)\,\hat{\eta}_i^2\hat{\xi}_i^2 + a_{1120}(a_i,a_j)\,\hat{\eta}_i\hat{\eta}_j\hat{\xi}_i^2 + a_{0220}(a_i,a_j)\,\hat{\eta}_j^2\hat{\xi}_i^2 \\ &+ a_{1120}(a_i,a_j)\,\hat{\eta}_i^2\hat{\xi}_i\hat{\xi}_j + a_{1111}(a_i,a_j)\,\hat{\eta}_i\hat{\eta}_j\hat{\xi}_i\hat{\xi}_j + a_{1120}(a_j,a_i)\,\hat{\eta}_j^2\hat{\xi}_i\hat{\xi}_j + a_{0220}(a_j,a_i)\,\hat{\eta}_i^2\hat{\xi}_j^2 \\ &+ a_{1120}(a_j,a_i)\,\hat{\eta}_i\hat{\eta}_j\hat{\xi}_j^2 + a_{2020}(a_j,a_i)\,\hat{\eta}_j^2\hat{\xi}_j^2 + a_{4000}(a_i,a_j)\,\hat{\xi}_i^4 + a_{3100}(a_i,a_j)\,\hat{\xi}_i^3\hat{\xi}_j \\ &+ a_{2200}(a_i,a_j)\,\hat{\xi}_i^2\hat{\xi}_j^2 + a_{3100}(a_j,a_i)\,\hat{\xi}_i\hat{\xi}_j^3 + a_{4000}(a_j,a_i)\,\hat{\xi}_j^4 \,, \end{aligned}$$

Thus, multiplying by $-\bar{m}_i\bar{m}_j$, and summing over all $1 \le i < j \le N$, we find (3.27).

We perform now the computation of the 7 coefficients (here, a_1, a_2, \cdots are used as "dummy" variables)

$$\begin{aligned} a_{4000}(a_1, a_2) &= \frac{1}{24} \partial_{\hat{\eta}_1^4} \hat{g}(a_1, a_2, \hat{\eta}_1, \hat{\eta}_2, \hat{\xi}_1, \hat{\xi}_2) \Big|_0 \\ a_{3100}(a_1, a_2) &= \frac{1}{6} \partial_{\hat{\eta}_1^3 \hat{\eta}_2} \hat{g}(a_1, a_2, \hat{\eta}_1, \hat{\eta}_2, \hat{\xi}_1, \hat{\xi}_2) \Big|_0 \\ a_{2200}(a_1, a_2) &= \frac{1}{4} \partial_{\hat{\eta}_1^2 \hat{\eta}_2^2} \hat{g}(a_1, a_2, \hat{\eta}_1, \hat{\eta}_2, \hat{\xi}_1, \hat{\xi}_2) \Big|_0 \\ a_{2020}(a_1, a_2) &= \frac{1}{4} \partial_{\hat{\eta}_1^2 \hat{\xi}_1^2} \hat{g}(a_1, a_2, \hat{\eta}_1, \hat{\eta}_2, \hat{\xi}_1, \hat{\xi}_2) \Big|_0 \\ a_{0220}(a_1, a_2) &= \frac{1}{4} \partial_{\hat{\eta}_2^2 \hat{\xi}_1^2} \hat{g}(a_1, a_2, \hat{\eta}_1, \hat{\eta}_2, \hat{\xi}_1, \hat{\xi}_2) \Big|_0 \end{aligned}$$

$$a_{1120}(a_1, a_2) = \frac{1}{2} \partial_{\hat{\eta}_1 \hat{\eta}_2 \hat{\xi}_1^2} \hat{g}(a_1, a_2, \hat{\eta}_1, \hat{\eta}_2, \hat{\xi}_1, \hat{\xi}_2) \Big|_{0}$$

$$a_{1111}(a_1, a_2) = \partial_{\hat{\eta}_1 \hat{\eta}_2 \hat{\xi}_1 \hat{\xi}_2} \hat{g}(a_1, a_2, \hat{\eta}_1, \hat{\eta}_2, \hat{\xi}_1, \hat{\xi}_2) \Big|_{0}$$
(3.29)

where $|_{0}$ stands for $|_{(\hat{\eta}_{1},\hat{\eta}_{2},\hat{\xi}_{1},\hat{\xi}_{2})=0}$ and

$$\hat{g}(a_1, a_2, \hat{\eta}_1, \hat{\eta}_2, \hat{\xi}_1, \hat{\xi}_2) = \frac{1}{(2\pi)^2} \int_{[0, 2\pi]^2} \frac{d\lambda_1 d\lambda_2}{\hat{d}(a_1, a_2, \lambda_1, \lambda_2, \hat{\eta}_1, \hat{\eta}_2, \hat{\xi}_1, \hat{\xi}_2)} .$$

We write

$$\hat{d}(a_1, a_2, \lambda_1, \lambda_2, \hat{\eta}_1, \hat{\eta}_2, \hat{\xi}_1, \hat{\xi}_2)^2 = \sqrt{\hat{d}_2(a_1, a_2, \lambda_1, \lambda_2, \hat{\eta}_1, \hat{\eta}_2, \hat{\xi}_1, \hat{\xi}_2)}$$

where

$$\hat{d}_{2}(a_{1}, a_{2}, \lambda_{1}, \lambda_{2}, \hat{\eta}_{1}, \hat{\eta}_{2}, \hat{\xi}_{1}, \hat{\xi}_{2}) := |\hat{x}(a_{1}, \lambda_{1}, \hat{\eta}_{1}, \hat{\xi}_{1}) - \hat{x}(a_{2}, \lambda_{2}, \hat{\eta}_{2}, \hat{\xi}_{2})|^{2}
= \hat{x}_{1}(a_{1}, \lambda_{1}, \hat{\eta}_{1}, \hat{\xi}_{1})^{2} + \hat{x}_{2}(a_{1}, \lambda_{1}, \hat{\eta}_{1}, \hat{\xi}_{1})^{2}
+ \hat{x}_{1}(a_{2}, \lambda_{2}, \hat{\eta}_{2}, \hat{\xi}_{2})^{2} + \hat{x}_{2}(a_{2}, \lambda_{2}, \hat{\eta}_{2}, \hat{\xi}_{2})^{2}
- 2 \hat{x}_{1}(a_{1}, \lambda_{1}, \hat{\eta}_{1}, \hat{\xi}_{1}) \hat{x}_{1}(a_{2}, \lambda_{2}, \hat{\eta}_{2}, \hat{\xi}_{2})
- 2 \hat{x}_{2}(a_{1}, \lambda_{1}, \hat{\eta}_{1}, \hat{\xi}_{1}) \hat{x}_{2}(a_{2}, \lambda_{2}, \hat{\eta}_{2}, \hat{\xi}_{2})$$
(3.30)

where, for short,

$$(a,\lambda,\hat{\eta},\hat{\xi}) \to \hat{x}(a,\lambda,\hat{\eta},\hat{\xi}) = \left(\hat{x}_1(a,\lambda,\hat{\eta},\hat{\xi}), \hat{x}_2(a,\lambda,\hat{\eta},\hat{\xi})\right)$$

denotes the Delaunay–Poincaré map for N = 1. Then, by usual calculus rules

$$\begin{array}{lcl} \partial_{\zeta_{1}\zeta_{2}\zeta_{3}\zeta_{4}}\frac{1}{\hat{d}} &=& (16\,\hat{d}_{2}^{9/2})^{-1} \\ &\times& \left[105\,\partial_{\zeta_{1}}\hat{d}_{2}\,\partial_{\zeta_{2}}\hat{d}_{2}\,\partial_{\zeta_{3}}\hat{d}_{2}\,\partial_{\zeta_{4}}\hat{d}_{2} - 30\,\hat{d}_{2}\,\partial_{\zeta_{1}\zeta_{2}}\hat{d}_{2}\,\partial_{\zeta_{3}}\hat{d}_{2}\,\partial_{\zeta_{4}}\hat{d}_{2} \\ &-& 30\,\hat{d}_{2}\,\partial_{\zeta_{1}\zeta_{3}}\hat{d}_{2}\,\partial_{\zeta_{2}}\hat{d}_{2}\,\partial_{\zeta_{4}}\hat{d}_{2} - 30\,\hat{d}_{2}\,\partial_{\zeta_{1}\zeta_{4}}\hat{d}_{2}\,\partial_{\zeta_{2}}\hat{d}_{2}\,\partial_{\zeta_{3}}\hat{d}_{2} \\ &-& 30\,\hat{d}_{2}\,\partial_{\zeta_{2}\zeta_{3}}\hat{d}_{2}\,\partial_{\zeta_{1}}\hat{d}_{2}\,\partial_{\zeta_{4}}\hat{d}_{2} - 30\,\hat{d}_{2}\,\partial_{\zeta_{2}\zeta_{4}}\hat{d}_{2}\,\partial_{\zeta_{1}}\hat{d}_{2}\,\partial_{\zeta_{3}} \\ &+& -30\,\hat{d}_{2}\,\partial_{\zeta_{3}\zeta_{4}}\hat{d}_{2}\,\partial_{\zeta_{1}}\hat{d}_{2}\,\partial_{\zeta_{2}}\hat{d}_{2} + 12\,\hat{d}_{2}^{2}\,\partial_{\zeta_{2}\zeta_{3}\zeta_{4}}\hat{d}_{2}\,\partial_{\zeta_{1}}\hat{d}_{2} \\ &+& 12\,\hat{d}_{2}^{2}\,\partial_{\zeta_{2}}\hat{d}_{2}\,\partial_{\zeta_{1}\zeta_{2}\zeta_{3}}\hat{d}_{2} + 12\,\hat{d}_{2}^{2}\,\partial_{\zeta_{1}\zeta_{2}\zeta_{4}}\hat{d}_{2} \\ &+& 12\,\hat{d}_{2}^{2}\,\partial_{\zeta_{1}\zeta_{3}}\hat{d}_{2}\,\partial_{\zeta_{2}\zeta_{4}}\hat{d}_{2} + 12\,\hat{d}_{2}^{2}\,\partial_{\zeta_{1}\zeta_{2}}\hat{d}_{2}\,\partial_{\zeta_{3}\zeta_{4}}\hat{d}_{2} \\ &+& 12\,\hat{d}_{2}^{2}\,\partial_{\zeta_{1}\zeta_{3}}\hat{d}_{2}\partial_{\zeta_{2}\zeta_{4}}\hat{d}_{2} + 12\,\hat{d}_{2}^{2}\,\partial_{\zeta_{1}\zeta_{4}}\hat{d}_{2}\,\partial_{\zeta_{2}\zeta_{3}}\hat{d}_{2} \\ &+& 12\,\hat{d}_{2}^{2}\,\partial_{\zeta_{1}\zeta_{3}}\hat{d}_{2}\partial_{\zeta_{2}\zeta_{4}}\hat{d}_{2} + 12\,\hat{d}_{2}^{2}\,\partial_{\zeta_{1}\zeta_{4}}\hat{d}_{2}\,\partial_{\zeta_{2}\zeta_{3}}\hat{d}_{2} \\ &-& 8\,\hat{d}_{3}^{3}\,\partial_{\zeta_{1}\zeta_{2}\zeta_{3}\zeta_{4}}\hat{d}_{2} \Big] \end{array}$$

We find, then,

$$\begin{aligned} \partial_{\hat{\eta}_{1}^{4}} \frac{1}{\hat{d}} \Big|_{0} &= \left(16 \, \hat{d}_{2}^{9/2} \right)^{-1} \\ &\times \left[105 \, (\partial_{\hat{\eta}_{1}} \hat{d}_{2})^{4} - 180 \, \hat{d}_{2} (\partial_{\hat{\eta}_{1}} \hat{d}_{2})^{2} \, \partial_{\hat{\eta}_{1}^{2}} \hat{d}_{2} \right. \\ &+ \left. 48 \, \hat{d}_{2}^{2} \, \partial_{\hat{\eta}_{1}} \hat{d}_{2} \, \partial_{\hat{\eta}_{1}^{3}} \hat{d}_{2} + 36 \, \hat{d}_{2}^{2} \, (\partial_{\hat{\eta}_{1}^{2}} \hat{d}_{2})^{2} \\ &- \left. 8 \, \hat{d}_{2}^{3} \, \partial_{\hat{\eta}_{1}^{4}} \hat{d}_{2} \, \right]_{0} \end{aligned}$$

$$\begin{aligned} \partial_{\hat{\eta}_{1}^{3}\hat{\eta}_{2}} \frac{1}{\hat{d}} \Big|_{0} &= \left(16 \, \hat{d}_{2}^{9/2} \right)^{-1} \\ &\times \left[105 \, \partial_{\hat{\eta}_{2}} \hat{d}_{2} \, (\partial_{\hat{\eta}_{1}} \hat{d}_{2})^{3} - 90 \, \hat{d}_{2} \, (\partial_{\hat{\eta}_{1}} \hat{d}_{2})^{2} \, \partial_{\hat{\eta}_{1}\hat{\eta}_{2}} \hat{d}_{2} \right. \\ &- \left. 90 \, \hat{d}_{2} \, \partial_{\hat{\eta}_{2}} \hat{d}_{2} \, \partial_{\hat{\eta}_{1}} \hat{d}_{2} \partial_{\hat{\eta}_{1}^{2}} \hat{d}_{2} \right. \\ &+ \left. 36 \, \hat{d}_{2}^{2} \partial_{\hat{\eta}_{1}} \hat{d}_{2} \partial_{\hat{\eta}_{1}^{2}\hat{\eta}_{2}} \hat{d}_{2} + 12 \, \hat{d}_{2}^{2} \partial_{\hat{\eta}_{2}} \hat{d}_{2} \, \partial_{\hat{\eta}_{1}^{3}\hat{\eta}_{2}} \right. \\ &+ \left. 36 \, \hat{d}_{2}^{2} \, \partial_{\hat{\eta}_{1}\hat{\eta}_{2}} \hat{d}_{2} \partial_{\hat{\eta}_{1}^{2}} \hat{d}_{2} - 8 \, \hat{d}_{2}^{3} \, \partial_{\hat{\eta}_{1}^{3}\hat{\eta}_{2}} \hat{d}_{2} \right]_{0} \end{aligned}$$

$$\begin{aligned} \partial_{\hat{\eta}_{1}^{2}\hat{\eta}_{2}^{2}} \frac{1}{\hat{d}} \Big|_{0} &= \left(16 \, \hat{d}_{2}^{9/2} \right)^{-1} \\ \times & \left[105 \, (\partial_{\hat{\eta}_{2}} \hat{d}_{2})^{2} \, (\partial_{\hat{\eta}_{1}} \hat{d}_{2})^{2} - 30 \, \hat{d}_{2} \, \partial_{\hat{\eta}_{2}^{2}} \hat{d}_{2} \, (\partial_{\hat{\eta}_{1}} \hat{d}_{2})^{2} \\ &- 120 \, \hat{d}_{2} \, \partial_{\hat{\eta}_{1}} \hat{d}_{2} \, \partial_{\hat{\eta}_{2}} \hat{d}_{2} \, \partial_{\hat{\eta}_{1}\hat{\eta}_{2}} \hat{d}_{2} - 30 \, \hat{d}_{2} \, (\partial_{\hat{\eta}_{2}} \hat{d}_{2})^{2} \, \partial_{\hat{\eta}_{1}^{2}} \hat{d}_{2} \\ &+ 24 \, \hat{d}_{2}^{2} \, \partial_{\hat{\eta}_{2}} \hat{d}_{2} \, \partial_{\hat{\eta}_{1}^{2}\hat{\eta}_{2}} \hat{d}_{2} + 24 \, \hat{d}_{2}^{2} \, \partial_{\hat{\eta}_{1}\hat{\eta}_{2}} \hat{d}_{2} \\ &+ 12 \, \hat{d}_{2}^{2} \, \partial_{\hat{\eta}_{2}^{2}} \hat{d}_{2} \, \partial_{\hat{\eta}_{1}^{2}} \hat{d}_{2} + 24 \, \hat{d}_{2}^{2} \, (\partial_{\hat{\eta}_{1}\hat{\eta}_{2}} \hat{d}_{2})^{2} \\ &- 8 \, \hat{d}_{2}^{3} \, \partial_{\hat{\eta}_{1}^{2}\hat{\eta}_{2}^{2}} \hat{d}_{2} \, \Big]_{0} \end{aligned}$$

$$\begin{aligned} \partial_{\hat{\eta}_{1}^{2}\hat{\xi}_{1}^{2}}\frac{1}{\hat{d}}\Big|_{0} &= \left(16\,\hat{d}_{2}^{9/2}\right)^{-1} \\ \times & \left[105\,(\partial_{\hat{\xi}_{1}}\hat{d}_{2})^{2}\,(\partial_{\hat{\eta}_{1}}\hat{d}_{2})^{2} - 30\,\hat{d}_{2}\,\partial_{\hat{\xi}_{1}^{2}}\hat{d}_{2}\,(\partial_{\hat{\eta}_{1}}\hat{d}_{2})^{2} \\ & - 120\,\hat{d}_{2}\,\partial_{\hat{\xi}_{1}}\hat{d}_{2}\,\partial_{\hat{\eta}_{1}}\hat{d}_{2}\,\partial_{\hat{\eta}_{1}\hat{\xi}_{1}}\hat{d}_{2} - 30\,\hat{d}_{2}(\partial_{\hat{\xi}_{1}}\hat{d}_{2})^{2}\,\partial_{\hat{\eta}_{1}^{2}}\hat{d}_{2} \\ & + 24\,\hat{d}_{2}^{2}\partial_{\hat{\xi}_{1}}\hat{d}_{2}\,\partial_{\hat{\eta}_{1}^{2}\hat{\xi}_{1}}\hat{d}_{2} + 24\,\hat{d}_{2}^{2}\partial_{\hat{\eta}_{1}}\hat{d}_{2}\,\partial_{\hat{\eta}_{1}\hat{\xi}_{1}^{2}}\hat{d}_{2} \\ & + 12\,\hat{d}_{2}^{2}\,\partial_{\hat{\xi}_{1}^{2}}\hat{d}_{2}\,\partial_{\hat{\eta}_{1}^{2}}\hat{d}_{2} + 24\,\hat{d}_{2}^{2}\,(\partial_{\hat{\eta}_{1}\hat{\xi}_{1}}\hat{d}_{2})^{2} - 8\,\hat{d}_{2}^{3}\,\partial_{\hat{\eta}_{1}^{2}\hat{\xi}_{1}^{2}}\hat{d}_{2}\,\Big]_{0} \end{aligned}$$

$$\begin{aligned} \partial_{\hat{\eta}_{2}^{2}\hat{\xi}_{1}^{2}} \frac{1}{\hat{d}} \Big|_{0} &= \left(16 \, \hat{d}_{2}^{9/2} \right)^{-1} \\ &\times \left[105 \, (\partial_{\hat{\xi}_{1}} \hat{d}_{2})^{2} \, (\partial_{\hat{\eta}_{2}} \hat{d}_{2})^{2} - 30 \, \hat{d}_{2} \, \partial_{\hat{\xi}_{1}^{2}} \hat{d}_{2} \, (\partial_{\hat{\eta}_{2}} \hat{d}_{2})^{2} \\ &- 120 \, \hat{d}_{2} \, \partial_{\hat{\xi}_{1}} \hat{d}_{2} \, \partial_{\hat{\eta}_{2}} \hat{d}_{2} \, \partial_{\hat{\xi}_{1}\hat{\eta}_{2}} \hat{d}_{2} - 30 \, \hat{d}_{2} \, (\partial_{\hat{\xi}_{1}} \hat{d}_{2})^{2} \, \partial_{\hat{\eta}_{2}^{2}} \hat{d}_{2} \\ &+ 24 \, \hat{d}_{2}^{2} \, \partial_{\hat{\xi}_{1}} \hat{d}_{2} \, \partial_{\hat{\eta}_{2}^{2}\hat{\xi}_{1}} \hat{d}_{2} + 24 \, \hat{d}_{2}^{2} \, \partial_{\hat{\eta}_{2}} \hat{d}_{2} \, \partial_{\hat{\eta}_{2}\hat{\xi}_{1}^{2}} \hat{d}_{2} \end{aligned}$$

$$+ 12 \hat{d}_{2}^{2} \partial_{\hat{\xi}_{1}^{2}} \hat{d}_{2} \partial_{\hat{\eta}_{2}^{2}} \hat{d}_{2} + 24 \hat{d}_{2}^{2} (\partial_{\hat{\xi}_{1}\hat{\eta}_{2}} \hat{d}_{2})^{2} - 8 \hat{d}_{2}^{3} \partial_{\hat{\eta}_{2}^{2}\hat{\xi}_{1}^{2}} \hat{d}_{2} \Big]_{0}$$

$$\partial_{\hat{\eta}_{1}\hat{\eta}_{2}\hat{\xi}_{1}^{2}} \frac{1}{\hat{d}}\Big|_{0} = (16 \hat{d}_{2}^{9/2})^{-1}$$

$$\times [105 \partial_{\hat{\eta}_{1}} \hat{d}_{2} \partial_{\hat{\eta}_{2}} \hat{d}_{2} (\partial_{\hat{\xi}_{1}} \hat{d}_{2})^{2} - 30 \hat{d}_{2} \partial_{\hat{\eta}_{1}} \hat{d}_{2} \partial_{\hat{\eta}_{2}} \hat{d}_{2} \partial_{\hat{\xi}_{1}^{2}} \hat{d}_{2}$$

$$- 60 \hat{d}_{2} \partial_{\hat{\xi}_{1}} \hat{d}_{2} \partial_{\hat{\eta}_{2}\hat{\xi}_{1}} \hat{d}_{2} \partial_{\hat{\eta}_{1}} \hat{d}_{2} - 60 \hat{d}_{2} \partial_{\hat{\xi}_{1}} \hat{d}_{2} \partial_{\hat{\eta}_{1}\hat{\xi}_{1}} \hat{d}_{2}$$

$$- 30 \hat{d}_{2} (\partial_{\hat{\xi}_{1}} \hat{d}_{2})^{2} \partial_{\hat{\eta}_{1}\hat{\eta}_{2}} \hat{d}_{2} + 12 \hat{d}_{2}^{2} \partial_{\hat{\eta}_{2}\hat{\xi}_{1}^{2}} \hat{d}_{2} \partial_{\hat{\eta}_{1}\hat{\xi}_{1}} \hat{d}_{2}$$

$$+ 12 \hat{d}_{2}^{2} \partial_{\hat{\eta}_{2}} \hat{d}_{2} \partial_{\hat{\eta}_{1}\hat{\xi}_{1}^{2}} \hat{d}_{2} - 8 \hat{d}_{2}^{3} \partial_{\hat{\eta}_{1}\hat{\eta}_{2}\hat{\xi}_{1}^{2}} \hat{d}_{2} \Big]_{0}$$

$$\begin{aligned} \partial_{\hat{\eta}_{1}\hat{\eta}_{2}\hat{\xi}_{1}\hat{\xi}_{2}}\frac{1}{\hat{d}}\Big|_{0} &= \left(16\,\hat{d}_{2}^{9/2}\right)^{-1} \\ \times \left[105\,\partial_{\hat{\eta}_{1}}\hat{d}_{2}\,\partial_{\hat{\eta}_{2}}\hat{d}_{2}\,\partial_{\hat{\xi}_{1}}\hat{d}_{2}\,\partial_{\hat{\xi}_{2}}\hat{d}_{2} - 30\,\hat{d}_{2}\,\partial_{\hat{\eta}_{1}\hat{\eta}_{2}}\hat{d}_{2}\,\partial_{\hat{\xi}_{1}}\hat{d}_{2}\,\partial_{\hat{\xi}_{2}}\hat{d}_{2} \\ &- 30\,\hat{d}_{2}\,\partial_{\hat{\eta}_{1}\hat{\xi}_{1}}\hat{d}_{2}\,\partial_{\hat{\eta}_{2}}\hat{d}_{2}\,\partial_{\hat{\xi}_{2}}\hat{d}_{2} - 30\,\hat{d}_{2}\,\partial_{\hat{\eta}_{1}\hat{\xi}_{2}}\hat{d}_{2}\,\partial_{\hat{\eta}_{2}}\hat{d}_{2}\,\partial_{\hat{\xi}_{1}}\hat{d}_{2} \\ &- 30\,\hat{d}_{2}\,\partial_{\hat{\eta}_{2}\hat{\xi}_{1}}\hat{d}_{2}\,\partial_{\hat{\eta}_{1}}\hat{d}_{2}\,\partial_{\hat{\xi}_{2}}\hat{d}_{2} - 30\,\hat{d}_{2}\,\partial_{\hat{\eta}_{2}\hat{\xi}_{2}}\hat{d}_{2}\,\partial_{\hat{\eta}_{1}}\hat{d}_{2}\,\partial_{\hat{\xi}_{1}} \\ &- 30\,\hat{d}_{2}\,\partial_{\hat{\eta}_{1}\hat{\xi}_{2}}\hat{d}_{2}\,\partial_{\hat{\eta}_{1}}\hat{d}_{2}\,\partial_{\hat{\eta}_{2}}\hat{d}_{2} + 12\,\hat{d}_{2}^{2}\,\partial_{\hat{\eta}_{2}\hat{\xi}_{1}\hat{\xi}_{2}}\hat{d}_{2}\,\partial_{\hat{\eta}_{1}\hat{\xi}_{1}\hat{\xi}_{2}}\hat{d}_{2} \\ &+ 12\,\hat{d}_{2}^{2}\,\partial_{\hat{\xi}_{1}\hat{\xi}_{2}}\hat{d}_{2}\,\partial_{\hat{\eta}_{1}\hat{\eta}_{2}\hat{\xi}_{2}}\hat{d}_{2} + 12\,\hat{d}_{2}^{2}\,\partial_{\hat{\eta}_{1}\hat{\eta}_{2}\hat{\xi}_{1}}\hat{d}_{2} \\ &+ 12\,\hat{d}_{2}^{2}\,\partial_{\hat{\eta}_{1}\hat{\eta}_{2}}\hat{d}_{2}\,\partial_{\hat{\xi}_{1}\hat{\xi}_{2}}\hat{d}_{2} + 12\,\hat{d}_{2}^{2}\,\partial_{\hat{\eta}_{1}\hat{\eta}_{2}\hat{\xi}_{1}}\hat{d}_{2} \\ &+ 12\,\hat{d}_{2}^{2}\,\partial_{\hat{\eta}_{1}\hat{\eta}_{2}}\hat{d}_{2}\,\partial_{\hat{\eta}_{2}\hat{\xi}_{1}}\hat{d}_{2} - 8\,\hat{d}_{2}^{3}\,\partial_{\hat{\eta}_{1}\hat{\eta}_{2}\hat{\xi}_{1}\hat{\xi}_{2}}\hat{d}_{2}\,\Big|_{0} \end{aligned}$$
(3.31)

where

$$\hat{d}_2\Big|_0 = a_1^2 + a_2^2 - 2a_1a_2\cos(\lambda_1 - \lambda_2)$$

We start by computing the following derivatives in terms of the derivatives of the coordinates

$$\begin{aligned} \partial_{\hat{\eta}_{1}} \hat{d}_{2} \Big|_{0} &= 2 \left[(\hat{x}_{11} - \hat{x}_{12}) \partial_{\hat{\eta}_{1}} \hat{x}_{11} + (\hat{x}_{21} - \hat{x}_{22}) \partial_{\hat{\eta}_{1}} \hat{x}_{21} \right] \Big|_{0} \\ \partial_{\hat{\eta}_{1}^{2}} \hat{d}_{2} \Big|_{0} &= 2 \left[(\partial_{\hat{\eta}_{1}} \hat{x}_{11})^{2} + (\partial_{\hat{\eta}_{1}} \hat{x}_{21})^{2} + (\hat{x}_{11} - \hat{x}_{12}) \partial_{\hat{\eta}_{1}^{2}} \hat{x}_{11} + (\hat{x}_{21} - \hat{x}_{22}) \partial_{\hat{\eta}_{1}^{2}} \hat{x}_{21} \right] \Big|_{0} \\ \partial_{\hat{\eta}_{1}\hat{\eta}_{2}} \hat{d}_{2} \Big|_{0} &= -2 \left[\partial_{\hat{\eta}_{1}} \hat{x}_{11} \partial_{\hat{\eta}_{2}} \hat{x}_{12} + \partial_{\hat{\eta}_{1}} \hat{x}_{21} \partial_{\hat{\eta}_{2}} \hat{x}_{22} \right] \Big|_{0} \\ \partial_{\hat{\eta}_{1}\hat{\xi}_{1}} \hat{d}_{2} \Big|_{0} &= 2 \left[\partial_{\hat{\xi}_{1}} \hat{x}_{11} \partial_{\hat{\eta}_{1}} \hat{x}_{11} + \partial_{\hat{\xi}_{1}} \hat{x}_{21} \partial_{\hat{\eta}_{1}} \hat{x}_{21} + (\hat{x}_{11} - \hat{x}_{12}) \partial_{\hat{\eta}_{1}\hat{\xi}_{1}} \hat{x}_{11} \\ &+ (\hat{x}_{21} - \hat{x}_{22}) \partial_{\hat{\eta}_{1}\hat{\xi}_{1}} \hat{x}_{21} \right] \end{aligned}$$

$$\begin{split} \partial_{\eta_{1}\hat{\xi}_{1}}\hat{d}_{2}\Big|_{0} &= -2\left[\partial_{\xi_{1}}\hat{x}_{11} \partial_{\eta_{2}}\hat{x}_{12} + \partial_{\xi_{1}}\hat{x}_{21} \partial_{\eta_{2}}\hat{x}_{22}\right]\Big|_{0} \\ \partial_{\eta_{1}^{3}}\hat{d}_{2}\Big|_{0} &= 2\left[3\partial_{\eta_{1}}\hat{x}_{11} - \hat{d}_{\eta_{2}^{2}}\hat{x}_{11} + (\hat{x}_{21} - \hat{x}_{22})\partial_{\eta_{1}^{3}}\hat{x}_{21}\right]\Big|_{0} \\ \partial_{\eta_{1}^{3}\hat{\eta}_{2}}\hat{d}_{2}\Big|_{0} &= -2\left[\partial_{\eta_{2}}\hat{x}_{12} \partial_{\eta_{1}^{2}}\hat{x}_{11} + \partial_{\eta_{2}}\hat{x}_{22} \partial_{\eta_{1}^{2}}\hat{x}_{21}\right]\Big|_{0} \\ \partial_{\eta_{1}\hat{\xi}_{1}^{3}}\hat{d}_{2}\Big|_{0} &= 2\left[\partial_{\xi_{1}^{3}}\hat{x}_{11} \partial_{\eta_{1}}\hat{x}_{11} + \partial_{\xi_{1}^{3}}\hat{x}_{21} \partial_{\eta_{1}}\hat{x}_{21} + 2\partial_{\xi_{1}}\hat{x}_{11} \partial_{\xi_{1}}\hat{x}_{11} + 2\partial_{\xi_{1}}\hat{x}_{21} \partial_{\xi_{1}\hat{\eta}_{1}}\hat{x}_{21} \\ &+ (\hat{x}_{11} - \hat{x}_{12})\partial_{\eta_{1}\hat{\xi}_{1}^{2}}\hat{x}_{11} + (\hat{x}_{21} - \hat{x}_{22})\partial_{\eta_{1}\hat{\xi}_{1}^{3}}\hat{x}_{21}\Big|_{0} \\ \partial_{\eta_{2}\hat{\xi}_{1}^{3}}\hat{d}_{2}\Big|_{0} &= -2\left[\partial_{\xi_{1}^{3}}\hat{x}_{11} \partial_{\eta_{2}}\hat{x}_{12} + \partial_{\xi_{1}}\hat{x}_{21} \partial_{\eta_{2}}\hat{x}_{22}\right]\Big|_{0} \\ \partial_{\eta_{2}\hat{\xi}_{1}^{3}}\hat{d}_{2}\Big|_{0} &= -2\left[\partial_{\eta_{2}^{3}}\hat{x}_{11} \partial_{\eta_{2}}\hat{x}_{12} + \partial_{\eta_{1}}\hat{x}_{21} \partial_{\eta_{2}}\hat{x}_{22}\right]\Big|_{0} \\ \partial_{\eta_{1}\hat{\eta}_{2}\hat{\xi}_{1}}\hat{d}_{2}\Big|_{0} &= -2\left[\partial_{\eta_{2}^{3}}\hat{x}_{11}\partial_{\eta_{2}}\hat{x}_{12} + \partial_{\eta_{1}}\hat{x}_{21} \partial_{\eta_{2}}\hat{x}_{22}\right]\Big|_{0} \\ \partial_{\eta_{1}\hat{\eta}_{2}\hat{\xi}_{1}}\hat{d}_{2}\Big|_{0} &= -2\left[\partial_{\eta_{2}}\hat{x}_{11}\partial_{\eta_{2}}\hat{x}_{12} + \partial_{\eta_{1}^{3}}\hat{x}_{21} \partial_{\eta_{2}}\hat{x}_{22}\right]\Big|_{0} \\ \partial_{\eta_{1}^{3}\hat{\eta}_{2}}\hat{d}_{2}\Big|_{0} &= -2\left[\partial_{\eta_{1}^{3}}\hat{x}_{11} + \partial_{\eta_{2}^{3}}\hat{x}_{21} \partial_{\eta_{2}^{3}}\hat{x}_{22}\right]\Big|_{0} \\ \partial_{\eta_{1}^{3}\hat{\eta}_{2}}\hat{d}_{2}\Big|_{0} &= -2\left[\partial_{\eta_{1}^{3}}\hat{x}_{11} \partial_{\eta_{2}}\hat{x}_{12} + \partial_{\eta_{1}^{3}}\hat{x}_{21} \partial_{\eta_{2}^{3}}\hat{x}_{22}\right]\Big|_{0} \\ \partial_{\eta_{1}^{3}\hat{\eta}_{2}^{3}}\hat{d}_{2}\Big|_{0} &= -2\left[\partial_{\eta_{1}^{3}}\hat{x}_{11} \partial_{\eta_{2}^{3}}\hat{x}_{12} + \partial_{\eta_{1}^{3}}\hat{x}_{21} \partial_{\eta_{2}^{3}}\hat{x}_{21} - \partial_{\eta_{1}^{3}\hat{\xi}_{1}^{3}}\hat{x}_{21}\right\Big|_{0} \\ \partial_{\eta_{1}^{3}\hat{\eta}_{2}^{3}}\hat{d}_{2}\Big|_{0} &= -2\left[\partial_{\eta_{1}^{3}}\hat{x}_{11} + \partial_{\eta_{2}^{3}}\hat{x}_{12} + \partial_{\eta_{1}^{3}}\hat{x}_{21} \partial_{\eta_{2}^{3}}\hat{x}_{21}\right\Big|_{0} \\ \partial_{\eta_{1}\hat{\eta}_{2}^{3}\hat{\xi}_{1}}\hat{d}_{2}\Big|_{0} &= -2\left[\partial_{\eta_{1}\hat{\xi}_{1}}\hat{x}_{11} + \partial_{\eta_{2}\hat{\xi}_{1}}\hat{x}_{21} \partial_{\eta_{2}^{3}}\hat{x}_{22}\right\Big|_{0} \\ \partial_{\eta$$

where we have denoted, for shortness,

$$\hat{x}_{ij} := \hat{x}_i(a_j, \lambda_j, \hat{\eta}_j, \hat{\xi}_j) , \qquad i, \ j = 1, 2 .$$
 (3.33)

(3.32)

Recalling the relation

$$\hat{x}_2(a,\lambda,\hat{\eta},\hat{\xi}) = -\hat{x}_1(a,\lambda+\pi/2,\hat{\xi},-\hat{\eta}) ,$$
 (3.34)

the computation is then reduced to the one of the involved derivatives of the coordinates q_1 , which we recall, is defined as

$$\hat{x}_1(a,\lambda,\hat{\eta},\hat{\xi}) = a \left[\cos\left(\hat{\zeta}+\lambda\right) - \frac{\hat{\xi}}{2} \left(\hat{\eta}\sin\left(\hat{\zeta}+\lambda\right) + \hat{\xi}\cos\left(\hat{\zeta}+\lambda\right)\right) - \hat{\eta}\sqrt{1 - \frac{\hat{\eta}^2 + \hat{\xi}^2}{4}} \right]$$
(3.35)

where $\hat{\zeta} = \hat{\zeta}(\lambda, \hat{\eta}, \hat{\xi})$ is implicitely defined as the solution of

$$\hat{\zeta} = \sqrt{1 - \frac{\hat{\eta}^2 + \hat{\xi}^2}{4}} \left[(\hat{\eta} \cos \lambda - \hat{\xi} \sin \lambda) \sin \hat{\zeta} + (\hat{\eta} \sin \lambda + \hat{\xi} \cos \lambda) \cos \hat{\zeta} \right] .$$
(3.36)

To this end, put

$$s(\lambda, \hat{\eta}, \hat{\xi}) := \sqrt{1 - \frac{\hat{\eta}^2 + \hat{\xi}^2}{4}} \left(\hat{\eta} \cos \lambda - \hat{\xi} \sin \lambda\right)$$
$$t(\lambda, \hat{\eta}, \hat{\xi}) := \sqrt{1 - \frac{\hat{\eta}^2 + \hat{\xi}^2}{4}} \left(\hat{\eta} \sin \lambda + \hat{\xi} \cos \lambda\right)$$
(3.37)

Write, from (3.36), $\hat{\zeta}(\lambda, \hat{\eta}, \hat{\xi}) = \tilde{\zeta}(s(\lambda, \hat{\eta}, \hat{\xi}), t(\lambda, \hat{\eta}, \hat{\xi}))$, where $\tilde{\zeta}(s, t)$ is the solution of

$$\zeta = s \, \sin \zeta + t \, \cos \zeta \, . \tag{3.38}$$

The 4–order expansion of $\tilde{\zeta}$ around (s,t) = 0 gives

$$\tilde{\zeta}(s,t) = t + st + s^2t - \frac{1}{2}t^3 + s^3t - \frac{5}{3}st^3 + O_5(s,t)$$
(3.39)

so that, inserting the previous expression into the 4-order developing of

$$\cos(\lambda + \tilde{\zeta}(s, t)) = \cos\lambda - (\sin\lambda)\tilde{\zeta}(s, t) - \frac{1}{2}(\cos\lambda)\tilde{\zeta}(s, t)^2 + \frac{1}{6}(\sin\lambda)\tilde{\zeta}(s, t)^3 + \frac{1}{24}(\cos\lambda)\tilde{\zeta}(s, t)^4 + O_5(\tilde{\zeta}(s, t))$$

we find

$$\cos\left(\lambda + \tilde{\zeta}(s,t)\right) = \cos\lambda - (\sin\lambda)t - (\sin\lambda)st - \frac{1}{2}(\cos\lambda)t^2 - (\sin\lambda)s^2t - (\cos\lambda)st^2 + \frac{2}{3}(\sin\lambda)t^3 - (\sin\lambda)s^3t - \frac{3}{2}(\cos\lambda)s^2t^2 + \frac{13}{6}(\sin\lambda)st^3 + \frac{13}{24}(\cos\lambda)t^4 + O_5(s,t)$$

(3.40)

On the other hand, taking into account the following expansions around $(\eta,\xi)=0$

$$\hat{\eta}\sqrt{1 - \frac{\hat{\eta}^2 + \hat{\xi}^2}{4}} = \hat{\eta} - \frac{\hat{\eta}^3 + \hat{\eta}\hat{\xi}^2}{8} + O_5(\hat{\eta}, \hat{\xi})$$
$$\hat{\xi}\sqrt{1 - \frac{\hat{\eta}^2 + \hat{\xi}^2}{4}} = \hat{\xi} - \frac{\hat{\eta}^2\hat{\xi} + \hat{\xi}^3}{8} + O_5(\hat{\eta}, \hat{\xi})$$
(3.41)

we find, by muliplying the first by $\cos \lambda$ $(\sin \lambda)$ and the second by $-\sin \lambda$ $(\cos \lambda)$, and, then, taking the sum

$$s(\lambda, \hat{\eta}, \hat{\xi}) = \sqrt{1 - \frac{\hat{\eta}^2 + \hat{\xi}^2}{4}} (\hat{\eta} \cos \lambda - \hat{\xi} \sin \lambda)$$

= $(\cos \lambda) \hat{\eta} - (\sin \lambda) \hat{\xi} - \frac{(\cos \lambda) \hat{\eta}^3 - (\sin \lambda) \hat{\eta}^2 \hat{\xi} + (\cos \lambda) \hat{\eta} \hat{\xi}^2 - (\sin \lambda) \hat{\xi}^3}{8}$
+ $O_5(\hat{\eta}, \hat{\xi})$

$$\begin{pmatrix} t(\Lambda, \hat{\eta}, \lambda, \hat{\xi}) &= \sqrt{1 - \frac{\hat{\eta}^2 + \hat{\xi}^2}{4}} \left(\hat{\eta} \sin \lambda + \hat{\xi} \cos \lambda \right) \\ &= (\sin \lambda) \, \hat{\eta} + (\cos \lambda) \, \hat{\xi} - \frac{(\sin \lambda) \, \hat{\eta}^3 + (\cos \lambda) \, \hat{\eta}^2 \, \hat{\xi} + (\sin \lambda) \, \hat{\eta} \, \hat{\xi}^2 + (\cos \lambda) \, \hat{\xi}^3}{8} \\ &+ O_5(\hat{\eta}, \hat{\xi}) \,,$$
 (3.42)

respectively). Consequently, substituting the (3.42), into (3.40), we obtain

$$\begin{aligned} \cos\left(\lambda + \hat{\zeta}(\lambda,\hat{\xi},\hat{\eta})\right) &= \cos\left(\lambda + \tilde{\zeta}(\lambda,s(\lambda,\hat{\xi},\hat{\eta}),t(\lambda,\hat{\xi},\hat{\eta}))\right) \\ &= \cos\lambda + \frac{\cos 2\lambda - 1}{2}\hat{\eta} - \frac{\sin 2\lambda}{2}\hat{\xi} \\ &- 3\frac{\cos\lambda - \cos 3\lambda}{8}\hat{\eta}^2 + \frac{\sin\lambda - 3\sin 3\lambda}{4}\hat{\eta}\hat{\xi} - \frac{\cos\lambda + 3\cos 3\lambda}{8}\hat{\xi}^2 \\ &+ \frac{3 - 19\cos 2\lambda + 16\cos 4\lambda}{48}\hat{\eta}^3 + \frac{9\sin 2\lambda - 16\sin 4\lambda}{16}\hat{\eta}^2\hat{\xi} \\ &+ \frac{1 - \cos 2\lambda - 16\cos 4\lambda}{16}\hat{\eta}\hat{\xi}^2 + \frac{11\sin 2\lambda + 16\sin 4\lambda}{48}\hat{\xi}^3 \\ &+ \frac{46\cos\lambda - 171\cos 3\lambda + 125\cos 5\lambda}{384}\hat{\eta}^4 \\ &- \frac{8\sin\lambda - 99\sin 3\lambda + 125\sin 5\lambda}{96}\hat{\eta}^3\hat{\xi} \\ &+ \frac{10\cos\lambda + 27\cos 3\lambda - 125\cos 5\lambda}{64}\hat{\eta}^2\hat{\xi}^2 \\ &- \frac{8\sin\lambda - 45\sin 3\lambda - 125\sin 5\lambda}{96}\hat{\eta}\hat{\xi}^3 \end{aligned}$$

+
$$\frac{14 \cos \lambda + 117 \cos 3\lambda + 125 \cos 5\lambda}{384} \hat{\xi}^4 + O_5(\hat{\eta}, \hat{\xi})$$

The expansion of $\sin(\lambda + \hat{\zeta}(\lambda, \hat{\eta}, \hat{\xi}))$ is obtained by the previous one, using

$$\sin\left(\lambda + \hat{\zeta}(\lambda, \hat{\eta}, \hat{\xi})\right) = -\cos\left(\lambda + \pi/2 + \hat{\zeta}(\lambda + \pi/2, \hat{\xi}, -\hat{\eta})\right)$$

and gives

$$\begin{split} \sin\left(\lambda + \hat{\zeta}(\lambda, \hat{\eta}, \hat{\xi})\right) &= \sin\lambda + \frac{\cos 2\lambda + 1}{2} \hat{\xi} + \frac{\sin 2\lambda}{2} \hat{\eta} \\ &- 3 \frac{\sin\lambda + \sin 3\lambda}{8} \hat{\xi}^2 + \frac{\cos\lambda + 3\cos 3\lambda}{4} \hat{\eta} \hat{\xi} + \frac{-\sin\lambda + 3\sin 3\lambda}{8} \hat{\eta}^2 \\ &- \frac{3 + 19\cos 2\lambda + 16\cos 4\lambda}{48} \hat{\xi}^3 - \frac{9\sin 2\lambda + 16\sin 4\lambda}{16} \hat{\xi}^2 \hat{\eta} \\ &- \frac{1 + \cos 2\lambda - 16\cos 4\lambda}{16} \hat{\xi} \hat{\eta}^2 + \frac{-11\sin 2\lambda + 16\sin 4\lambda}{48} \hat{\eta}^3 \\ &+ \frac{46\sin\lambda + 171\sin 3\lambda + 125\sin 5\lambda}{384} \hat{\xi}^4 \\ &- \frac{8\cos\lambda + 99\cos 3\lambda + 125\cos 5\lambda}{96} \hat{\xi}^3 \hat{\eta} \\ &- \frac{-10\sin\lambda + 27\sin 3\lambda + 125\sin 5\lambda}{96} \hat{\xi}^2 \hat{\eta}^2 \\ &- \frac{8\cos\lambda + 45\cos 3\lambda - 125\cos 5\lambda}{96} \hat{\xi} \hat{\eta}^3 \\ &- \frac{-14\sin\lambda + 117\sin 3\lambda - 125\sin 5\lambda}{384} \hat{\eta}^4 + O_5(\hat{\eta}, \hat{\xi}) \,. \end{split}$$

Both the previous expansions (of $\cos(\lambda + \hat{\zeta}(\lambda, \hat{\eta}, \hat{\xi}))$) and $\sin(\lambda + \hat{\zeta}(\lambda, \hat{\eta}, \hat{\xi}))$), together with the first line in (3.41), are, next, inserted into the expression of \hat{x}_1 given in (3.35), and one obtains the expansion of \hat{x}_1 to the fourth order:

$$\begin{aligned} \hat{q}_{1}(a,\lambda,\hat{\eta},\hat{\xi}) &= a \left[\cos\lambda + \frac{-3 + \cos 2\lambda}{2} \hat{\eta} - \frac{\sin 2\lambda}{2} \hat{\xi} \right] \\ &- 3 \frac{\cos\lambda - \cos 3\lambda}{8} \hat{\eta}^{2} - \frac{\sin\lambda + 3\sin 3\lambda}{4} \hat{\eta} \hat{\xi} - \frac{5\cos\lambda + 3\cos 3\lambda}{8} \hat{\xi}^{2} \\ &+ \frac{9 - 19\cos 2\lambda + 16\cos 4\lambda}{48} \hat{\eta}^{3} + \frac{5\sin 2\lambda - 16\sin 4\lambda}{16} \hat{\eta}^{2} \hat{\xi} \\ &+ \frac{3 - 9\cos 2\lambda - 16\cos 4\lambda}{16} \hat{\eta} \hat{\xi}^{2} + \frac{23\sin 2\lambda + 16\sin 4\lambda}{48} \hat{\xi}^{3} \\ &+ \frac{46\cos\lambda - 171\cos 3\lambda + 125\cos 5\lambda}{384} \hat{\eta}^{4} \\ &+ \frac{-2\sin\lambda + 81\sin 3\lambda - 125\sin 5\lambda}{96} \hat{\eta}^{3} \hat{\xi} \end{aligned}$$

$$+ \frac{14 \cos \lambda - 9 \cos 3\lambda - 125 \cos 5\lambda}{64} \hat{\eta}^{2} \hat{\xi}^{2} + \frac{-2 \sin \lambda + 99 \sin 3\lambda + 125 \sin 5\lambda}{96} \hat{\eta} \hat{\xi}^{3} + \frac{38 \cos \lambda + 189 \cos 3\lambda + 125 \cos 5\lambda}{384} \hat{\xi}^{4} + O_{5}(\hat{\eta}, \hat{\xi})$$
(3.43)

As outlined before, the expansion for \hat{x}_2 is obtained by

$$\hat{x}_2(a,\lambda,\hat{\eta},\hat{\xi}) = -\hat{x}_1(a,\lambda+\pi/2,\hat{\xi},-\hat{\eta}) ,$$

and the result is

$$\begin{aligned} \hat{x}_{2}(a,\lambda,\hat{\eta},\hat{\xi}) &= a \left[\sin\lambda + \frac{3+\cos 2\lambda}{2} \hat{\xi} + \frac{\sin 2\lambda}{2} \hat{\eta} \right. \\ &- 3 \frac{\sin\lambda + \sin 3\lambda}{8} \hat{\xi}^{2} - \frac{\cos\lambda - 3\cos 3\lambda}{4} \hat{\eta} \hat{\xi} - \frac{5\sin\lambda - 3\sin 3\lambda}{8} \hat{\eta}^{2} \\ &- \frac{9+19\cos 2\lambda + 16\cos 4\lambda}{48} \hat{\xi}^{3} - \frac{5\sin 2\lambda + 16\sin 4\lambda}{16} \hat{\xi}^{2} \hat{\eta} \\ &- \frac{3+9\cos 2\lambda - 16\cos 4\lambda}{16} \hat{\xi} \hat{\eta}^{2} + \frac{-23\sin 2\lambda + 16\sin 4\lambda}{48} \hat{\eta}^{3} \\ &+ \frac{46\sin\lambda + 171\sin 3\lambda + 125\sin 5\lambda}{384} \hat{\xi}^{4} \\ &+ \frac{-2\cos\lambda - 81\cos 3\lambda - 125\cos 5\lambda}{96} \hat{\xi}^{3} \hat{\eta} \\ &+ \frac{14\sin\lambda + 9\sin 3\lambda - 125\sin 5\lambda}{64} \hat{\xi}^{2} \hat{\eta}^{2} \\ &+ \frac{-2\cos\lambda - 99\cos 3\lambda + 125\cos 5\lambda}{96} \hat{\xi} \hat{\eta}^{3} \\ &+ \frac{38\sin\lambda - 189\sin 3\lambda + 125\sin 5\lambda}{384} \hat{\eta}^{4} \right] + O_{5}(\hat{\eta}, \hat{\xi}) \end{aligned}$$

Expansions (3.43) and (3.44) for \hat{x}_1 and \hat{x}_2 give at sight their desired derivatives in $(\eta, \xi) = 0$, which are substituted into (3.32), giving

$$\begin{aligned} \hat{d}_2 \Big|_0 &= a_1^2 + a_2^2 - 2 a_1 a_2 \cos(\lambda_1 - \lambda_2) \\ \partial_{\hat{\eta}_1} \hat{d}_2 \Big|_0 &= -a_1 \left[2 a_1 \cos\lambda_1 + a_2 \left(\cos(2\lambda_1 - \lambda_2) - 3\cos\lambda_2 \right) \right] \\ \partial_{\hat{\eta}_1^2} \hat{d}_2 \Big|_0 &= -\frac{a_1}{2} \left[-6 a_1 + 2 a_1 \cos(2\lambda_1) - 4 a_2 \cos(\lambda_1 - \lambda_2) + 3 a_2 \cos(3\lambda_1 - \lambda_2) \right. \\ &+ a_2 \cos(\lambda_1 + \lambda_2) \right] \\ \partial_{\hat{\eta}_1 \hat{\eta}_2} \hat{d}_2 \Big|_0 &= -\frac{a_1 a_2}{2} \left[9 - 3\cos(2\lambda_1) + \cos(2\lambda_1 - 2\lambda_2) - 3\cos(2\lambda_2) \right] \end{aligned}$$

$$\begin{aligned} \partial_{\hat{\eta}_{1}\hat{\xi}_{1}}\hat{d}_{2}\Big|_{0} &= \frac{a_{1}}{2} \left[2 a_{1} \sin\left(2\lambda_{1}\right) + a_{2} \left(3 \sin\left(3\lambda_{1}-\lambda_{2}\right) + \sin\left(\lambda_{1}+\lambda_{2}\right)\right) \right] \\ \partial_{\hat{\eta}_{2}\hat{\xi}_{1}}\hat{d}_{2}\Big|_{0} &= \frac{a_{1} a_{2}}{2} \left[-3 \sin\left(2\lambda_{1}\right) + \sin\left(2\lambda_{1}-2\lambda_{2}\right) - 3 \sin\left(2\lambda_{2}\right) \right] \\ \partial_{\hat{\eta}_{1}^{3}}\hat{d}_{2}\Big|_{0} &= -\frac{a_{1}}{4} \left[-12 a_{1} \cos\lambda_{1} + 6 a_{1} \cos\left(3\lambda_{1}\right) + a_{2} \left(-21 \cos\left(2\lambda_{1}-\lambda_{2}\right)\right) \right] \\ &+ 16 \cos\left(4\lambda_{1}-\lambda_{2}\right) + 9 \cos\lambda_{2} + 2 \cos\left(2\lambda_{1}+\lambda_{2}\right) \right] \end{aligned}$$

$$\partial_{\hat{\eta}_{1}^{2}\hat{\eta}_{2}}\hat{d}_{2}\Big|_{0} = \frac{a_{1}a_{2}}{4} \left[-9\cos\lambda_{1} + 9\cos(3\lambda_{1}) + 4\cos(\lambda_{1} - 2\lambda_{2}) - 3\cos(3\lambda_{1} - 2\lambda_{2}) - \cos(3\lambda_{1} - 2\lambda_{2})\right]$$

$$\partial_{\hat{\eta}_1 \hat{\xi}_1^2} \hat{d}_2 \Big|_0 = \frac{a_1}{4} \left[4 a_1 \cos \lambda_1 + 6 a_1 \cos (3\lambda_1) + 7 a_2 (\cos (2\lambda_1 - \lambda_2) + 16 \cos (4\lambda_1 - \lambda_2) - 3 \cos \lambda_2 + 2 \cos (2\lambda_1 + \lambda_2)) \right]$$

$$\partial_{\hat{\eta}_{2}\hat{\xi}_{1}^{2}}\hat{d}_{2}\Big|_{0} = \frac{a_{1}a_{2}}{4} \left[-15\cos\lambda_{1} - 9\cos(3\lambda_{1}) + 4\cos(\lambda_{1} - 2\lambda_{2}) + 3\cos(3\lambda_{1} - 2\lambda_{2})\right] + \cos(\lambda_{1} + 2\lambda_{2})\right]$$

$$\partial_{\hat{\eta}_1 \hat{\eta}_2 \hat{\xi}_1} \hat{d}_2 \Big|_{0} = \frac{a_1 a_2}{4} \left[-3 \sin \lambda_1 - 9 \sin (3\lambda_1) + 3 \sin (3\lambda_1 - 2\lambda_2) + \sin (\lambda_1 + 2\lambda_2) \right]$$
$$\partial_{\hat{\eta}_1^4} \hat{d}_2 \Big|_{0} = \frac{a_1}{2} \left[-72 a_1 + 56 a_1 \cos (2\lambda_1) - 32 a_1 \cos (4\lambda_1) - 42 a_2 \cos (\lambda_1 - \lambda_2) \right]$$

$$\begin{aligned} &\hat{\eta}_{1}^{*} d_{2} \Big|_{0} &= \frac{1}{8} \left[-72 \, a_{1} + 50 \, a_{1} \cos\left(2\lambda_{1}\right) - 52 \, a_{1} \cos\left(4\lambda_{1}\right) - 42 \, a_{2} \cos\left(\lambda_{1} - \lambda_{2}\right) \right] \\ &+ 180 \, a_{2} \cos\left(3\lambda_{1} - \lambda_{2}\right) - 125 \, a_{2} \cos\left(5\lambda_{1} - \lambda_{2}\right) - 4 \, a_{2} \cos\left(\lambda_{1} + \lambda_{2}\right) \\ &- 9 \, a_{2} \cos\left(3\lambda_{1} + \lambda_{2}\right) \right] \end{aligned}$$

$$\partial_{\hat{\eta}_1^3 \hat{\eta}_2} \hat{d}_2 \Big|_0 = \frac{a_1 a_2}{8} \left[27 - 57 \cos(2\lambda_1) + 48 \cos(4\lambda_1) + 21 \cos(2\lambda_1 - 2\lambda_2) - 16 \cos(4\lambda_1 - 2\lambda_2) - 9 \cos(2\lambda_2) - 2 \cos(2\lambda_1 + 2\lambda_2) \right]$$

$$\partial_{\hat{\eta}_{1}^{2}\hat{\eta}_{2}^{2}}\hat{d}_{2}\Big|_{0} = \frac{a_{1}a_{2}}{8} \left[12\cos\left(\lambda_{1}-3\lambda_{2}\right)-9\cos\left(3\lambda_{1}-3\lambda_{2}\right)-17\cos\left(\lambda_{1}-\lambda_{2}\right) \right. \\ \left. + 12\cos\left(3\lambda_{1}-\lambda_{2}\right)+8\cos\left(\lambda_{1}+\lambda_{2}\right)-3\cos\left(3\lambda_{1}+\lambda_{2}\right) \right. \\ \left. - 3\cos\left(\lambda_{1}+3\lambda_{2}\right) \right]$$

$$\partial_{\hat{\eta}_{1}^{2}\hat{\xi}_{1}^{2}}\hat{d}_{2}\Big|_{0} = \frac{a_{1}}{8} \left[-24 a_{1} + 32 a_{1} \cos 4\lambda_{1} - 14 a_{2} \cos (\lambda_{1} - \lambda_{2}) + 125 a_{2} \cos (5\lambda_{1} - \lambda_{2}) + 9 a_{2} \cos (3\lambda_{1} + \lambda_{2})\right]$$

$$\partial_{\hat{\eta}_{2}^{2}\hat{\xi}_{1}^{2}}\hat{d}_{2}\Big|_{0} = \frac{3a_{1}a_{2}}{8} \left[4\cos\left(\lambda_{1}-3\lambda_{2}\right)+3\cos\left(3\lambda_{1}-3\lambda_{2}\right)-5\cos\left(\lambda_{1}-\lambda_{2}\right)\right]$$

$$- 4 \cos \left(3\lambda_1 - \lambda_2\right) + \cos \left(3\lambda_1 + \lambda_2\right) + \cos \left(\lambda_1 + 3\lambda_2\right)\right]$$

$$\partial_{\hat{\eta}_1 \hat{\eta}_2 \hat{\xi}_1^2} \hat{d}_2 \Big|_{0} = \frac{a_1 a_2}{8} \left[9 - 27 \cos(2\lambda_1) - 48 \cos(4\lambda_1) \right. \\ \left. + 7 \cos(2\lambda_1 - 2\lambda_2) + 16 \cos(4\lambda_1 - 2\lambda_2) - 3 \cos(2\lambda_2) \right. \\ \left. + 2 \cos(2\lambda_1 + 2\lambda_2) \right]$$

$$\partial_{\hat{\eta}_1 \hat{\eta}_2 \hat{\xi}_1 \hat{\xi}_2} \hat{d}_2 \Big|_{0} = \frac{a_1 a_2}{8} \left[-9 \cos \left(3\lambda_1 - 3\lambda_2 \right) - \cos \left(\lambda_1 - \lambda_2 \right) \right. \\ \left. + 3 \cos \left(3\lambda_1 + \lambda_2 \right) + 3 \cos \left(\lambda_1 + 3\lambda_2 \right) \right]$$

The remaining derivatives involved in (3.32) are found by symmetry (as in the proof of Lemma 3.2):

$$\begin{aligned} \partial_{\hat{\eta}_2} \hat{d}_2 \Big|_0 &= a_2 \left[3 a_1 \cos \lambda_1 - a_1 \cos \left(\lambda_1 - 2\lambda_2\right) - 2 a_2 \cos \lambda_2 \right] \\ \partial_{\hat{\xi}_1} \hat{d}_2 \Big|_0 &= a_1 \left[-3 a_2 \sin \lambda_2 + a_2 \sin \left(2\lambda_1 - \lambda_2\right) + 2 a_1 \sin \lambda_1 \right] \\ \partial_{\hat{\xi}_2} \hat{d}_2 \Big|_0 &= a_2 \left[-3 a_1 \sin \lambda_1 - a_1 \sin \left(\lambda_1 - 2\lambda_2\right) + 2 a_2 \sin \lambda_2 \right] \\ \partial_{\hat{\eta}_2^2} \hat{d}_2 \Big|_0 &= \frac{a_2}{2} \left[6 a_2 - 3 a_1 \cos \left(\lambda_1 - 3\lambda_2\right) + 4 a_1 \cos \left(\lambda_1 - \lambda_2\right) \right. \\ &\quad - 2 a_2 \cos \left(2\lambda_2\right) - a_1 \cos \left(\lambda_1 + \lambda_2\right) \right] \\ \partial_{\hat{\eta}_1 \hat{\xi}_2} \hat{d}_2 \Big|_0 &= \frac{a_1 a_2}{2} \left[-3 \sin \left(2\lambda_2\right) - \sin \left(2\lambda_1 - 2\lambda_2\right) - 3 \sin 2\lambda_1 \right] \\ \partial_{\hat{\eta}_2 \hat{\xi}_2} \hat{d}_2 \Big|_0 &= \frac{a_2}{2} \left[2 a_2 \sin \left(2\lambda_2\right) - 3 a_1 \sin \left(\lambda_1 - 3\lambda_2\right) + a_1 \sin \left(\lambda_2 + \lambda_1\right) \right] \end{aligned}$$

$$\partial_{\xi_1^2} \hat{d}_2 \Big|_0 = \frac{a_1}{2} \left[6 a_1 + 3 a_2 \cos (3\lambda_1 - \lambda_2) + 4 a_2 \cos (\lambda_1 - \lambda_2) + 2 a_1 \cos (2\lambda_1) + a_2 \cos (\lambda_1 + \lambda_2) \right]$$

$$\begin{aligned} \partial_{\hat{\xi}_1\hat{\xi}_2}\hat{d}_2\Big|_0 &= \frac{a_1 a_2}{2} \left[-9 - 3 \cos(2\lambda_1) - \cos(2\lambda_1 - 2\lambda_2) - 3 \cos(2\lambda_2)\right] \\ \partial_{\hat{\eta}_1\hat{\eta}_2^2}\hat{d}_2\Big|_0 &= \frac{a_1 a_2}{4} \left[-9 \cos\lambda_2 + 9 \cos(3\lambda_2) + 4 \cos(2\lambda_1 - \lambda_2) - 3 \cos(2\lambda_1 - 3\lambda_2) \right. \\ &\quad - \cos(2\lambda_1 + \lambda_2)\right] \end{aligned}$$

$$\partial_{\hat{\eta}_{1}^{2}\hat{\xi}_{1}}\hat{d}_{2}\Big|_{0} = \frac{a_{1}}{4} \left[-4 a_{1} \sin \lambda_{1} + 6 a_{1} \sin (3\lambda_{1}) - 7 a_{2} (\sin (2\lambda_{1} - \lambda_{2}) + 16 \sin (4\lambda_{1} - \lambda_{2}) + 3 \sin \lambda_{2} + 2 \sin (2\lambda_{1} + \lambda_{2}))\right]$$

$$\begin{aligned} \partial_{\hat{\eta}_{2}\hat{\xi}_{1}\hat{\xi}_{2}}\hat{d}_{2}\Big|_{0} &= \frac{a_{1}a_{2}}{4} \left[3\cos\lambda_{2} - 9\cos\left(3\lambda_{2}\right) - 3\cos\left(2\lambda_{1} - 3\lambda_{2}\right) + \cos\left(2\lambda_{1} + \lambda_{2}\right) \right] \\ \partial_{\hat{\eta}_{1}\hat{\xi}_{1}\hat{\xi}_{2}}\hat{d}_{2}\Big|_{0} &= \frac{a_{1}a_{2}}{4} \left[3\cos\lambda_{1} - 9\cos\left(3\lambda_{1}\right) - 3\cos\left(3\lambda_{1} - 2\lambda_{2}\right) + \cos\left(\lambda_{1} + 2\lambda_{2}\right) \right] \\ \partial_{\hat{\eta}_{2}^{2}\hat{\xi}_{1}}\hat{d}_{2}\Big|_{0} &= \frac{a_{1}a_{2}}{4} \left[15\sin\lambda_{2} - 9\sin\left(3\lambda_{2}\right) - 4\sin\left(2\lambda_{1} - \lambda_{2}\right) + 3\sin\left(2\lambda_{1} - 3\lambda_{2}\right) \right] \\ &+ \sin\left(2\lambda_{1} + \lambda_{2}\right) \right] \end{aligned}$$

$$\partial_{\hat{\eta}_1 \hat{\eta}_2 \hat{\xi}_1} d_2 \Big|_0 = \frac{1}{4} \left[-3 \sin \lambda_1 - 9 \sin (3\lambda_1) + 3 \sin (3\lambda_1 - 2\lambda_2) + \sin (\lambda_1 + 2\lambda_2) \right]$$

$$\partial_{\hat{\eta}_1 \hat{\eta}_2 \hat{\xi}_2} \hat{d}_2 \Big|_0 = \frac{a_1 a_2}{4} \left[-3 \sin \lambda_2 - 9 \sin (3\lambda_2) - 3 \sin (2\lambda_1 - 3\lambda_2) + \sin (2\lambda_1 + \lambda_2) \right]$$

The last step is to insert the previous list of derivatives into (3.31). The result is

$$\begin{split} \partial_{\tilde{\eta}_{1}^{4}} \frac{1}{\tilde{d}} \bigg|_{0} &= \left[128 \left(a_{1}^{2} + a_{2}^{2} - 2 \, a_{1} a_{2} \cos \left(\lambda_{1} - \lambda_{2} \right) \right)^{9/2} \right]^{-1} \\ &\times \left\{ a_{1} \left[-8 \, a_{1} \left(224 \, a_{1}^{6} - 2964 \, a_{1}^{4} \, a_{2}^{2} + 1305 \, a_{1}^{2} \, a_{2}^{4} - 820 \, a_{2}^{6} \right) \cos 2\lambda_{1} \\ &+ \left(4096 \, a_{1}^{7} - 25080 \, a_{1}^{5} \, a_{2}^{2} - 20178 \, a_{1}^{3} \, a_{2}^{4} - 4952 \, a_{1} \, a_{2}^{6} \right) \cos \left(4\lambda_{1} \right) \\ &+ a_{2} \left(-360 \, a_{1}^{5} \, a_{2} + 25866 \, a_{1}^{3} \, a_{2}^{3} - 1800 \, a_{1} \, a_{2}^{5} \right) - 36 \, a_{1}^{3} \, a_{2}^{4} \cos \left(6\lambda_{1} - 4\lambda_{2} \right) \\ &- 343 \, a_{1}^{3} \, a_{2}^{4} \cos \left(8\lambda_{1} - 4\lambda_{2} \right) + \left(11992 \, a_{1}^{4} \, a_{2}^{3} - 4488 \, a_{1}^{2} \, a_{2}^{5} \right) \cos \left(\lambda_{1} - 3\lambda_{2} \right) \\ &+ \left(232 \, a_{1}^{4} \, a_{2}^{3} + 72 \, a_{1}^{2} \, a_{2}^{5} \right) \cos \left(5\lambda_{1} - 3\lambda_{2} \right) + \left(1676 \, a_{1}^{4} \, a_{2}^{3} \right) \\ &+ \left(232 \, a_{1}^{4} \, a_{2}^{3} + 72 \, a_{1}^{2} \, a_{2}^{5} \right) \cos \left(5\lambda_{1} - 3\lambda_{2} \right) + \left(1676 \, a_{1}^{4} \, a_{2}^{3} \right) \\ &+ \left(232 \, a_{1}^{4} \, a_{2}^{3} + 72 \, a_{1}^{2} \, a_{2}^{5} \right) \cos \left(5\lambda_{1} - 3\lambda_{2} \right) + \left(1676 \, a_{1}^{4} \, a_{2}^{3} \right) \\ &+ \left(232 \, a_{1}^{4} \, a_{2}^{3} + 72 \, a_{1}^{2} \, a_{2}^{5} \right) \cos \left(5\lambda_{1} - 3\lambda_{2} \right) + \left(1676 \, a_{1}^{4} \, a_{2}^{3} \right) \\ &+ \left(232 \, a_{1}^{4} \, a_{2}^{3} + 72 \, a_{1}^{2} \, a_{2}^{5} \right) \cos \left(5\lambda_{1} - 3\lambda_{2} \right) + \left(1676 \, a_{1}^{4} \, a_{2}^{3} \right) \\ &+ \left(-4848 \, a_{1}^{5} \, a_{2}^{2} - 1280 \, a_{1}^{3} \, a_{2}^{4} + 144 \, a_{1} \, a_{2}^{6} \right) \cos \left(2\lambda_{1} - 2\lambda_{2} \right) \\ &+ \left(1800 \, a_{1}^{5} \, a_{2}^{2} - 12072 \, a_{1}^{3} \, a_{2}^{4} + 144 \, a_{1} \, a_{2}^{6} \right) \cos \left(2\lambda_{1} - 2\lambda_{2} \right) \\ &+ \left(1800 \, a_{1}^{5} \, a_{2}^{2} - 12072 \, a_{1}^{3} \, a_{2}^{4} + 3240 \, a_{1} \, a_{2}^{6} \right) \cos \left(2\lambda_{1} - 2\lambda_{2} \right) \\ &+ \left(6568 \, a_{1}^{6} \, a_{2} + 14148 \, a_{1}^{4} \, a_{2}^{3} - 504 \, a_{1}^{2} \, a_{2}^{5} - 1440 \, a_{2}^{7} \right) \cos \left(3\lambda_{1} - \lambda_{2} \right) \\ &+ \left(6568 \, a_{1}^{6} \, a_{2} + 14148 \, a_{1}^{4} \, a_{2}^{3} - 504 \, a_{1}^{2} \, a_{2}^{5} + 1000 \, a_{2}^{7} \right) \cos \left(5\lambda_{1} - \lambda_{2} \right) \\ &+ \left(6568 \, a_{1}^{6} \, a_{2} - 42984 \, a_{1$$

+
$$(-22204 a_1^4 a_2^3 + 2340 a_1^2 a_2^5) \cos(\lambda_1 + 3\lambda_2) \Big| \Big\}$$

$$\begin{split} \partial_{\tilde{\eta}_{1}^{3} \tilde{\eta}_{2}} \frac{1}{\tilde{d}} \bigg|_{0} &= \left[128 \left(a_{1}^{2} + a_{2}^{2} - 2 \, a_{1} a_{2} \cos \left(\lambda_{1} - \lambda_{2} \right) \right)^{9/2} \right]^{-1} \\ \times & \left\{ a_{1} \, a_{2} \left[-3438 \, a_{1}^{4} \, a_{2}^{2} - 3798 \, a_{1}^{2} \, a_{2}^{4} \\ &+ 24 \, a_{1}^{2} \left(76 \, a_{1}^{4} - 197 \, a_{1}^{2} \, a_{2}^{2} + 113 \, a_{2}^{4} \right) \cos \left(2\lambda_{1} \right) \\ &+ \left(-6144 \, a_{1}^{6} + 9462 \, a_{1}^{4} \, a_{2}^{2} + 2904 \, a_{1}^{2} \, a_{2}^{4} \right) \cos \left(4\lambda_{1} \right) \\ &- 343 \, a_{1}^{3} \, a_{2}^{3} \cos \left(\lambda_{1} - 5\lambda_{2} \right) \\ &+ 207 \, a_{1}^{3} \, a_{2}^{3} \cos \left(3\lambda_{1} - 5\lambda_{2} \right) + 25 \, a_{1}^{3} \, a_{2}^{3} \cos \left(7\lambda_{1} - 5\lambda_{2} \right) \\ &- \left(162 \, a_{1}^{4} \, a_{2}^{2} + 18 \, a_{1}^{2} \, a_{2}^{4} \right) \cos \left(6\lambda_{1} - 4\lambda_{2} \right) \\ &+ \left(72 \, a_{1}^{5} \, a_{2} - 18195 \, a_{1}^{3} \, a_{2}^{3} + 2736 \, a_{1} \, a_{2}^{5} \right) \cos \left(\lambda_{1} - 3\lambda_{2} \right) \\ &+ \left(924 \, a_{1}^{5} \, a_{2} + 243 \, a_{1}^{3} \, a_{2}^{3} - 204 \, a_{1} \, a_{2}^{5} \right) \cos \left(5\lambda_{1} - 3\lambda_{2} \right) \\ &+ \left(924 \, a_{1}^{5} \, a_{2} + 243 \, a_{1}^{3} \, a_{2}^{3} - 204 \, a_{1} \, a_{2}^{5} \right) \cos \left(5\lambda_{1} - 3\lambda_{2} \right) \\ &+ \left(924 \, a_{1}^{5} \, a_{2} + 243 \, a_{1}^{3} \, a_{2}^{3} - 204 \, a_{1} \, a_{2}^{5} \right) \cos \left(5\lambda_{1} - 3\lambda_{2} \right) \\ &+ \left(924 \, a_{1}^{5} \, a_{2} + 274 \, a_{1}^{2} \, a_{2}^{4} \right) \cos \left(2\lambda_{1} - 4\lambda_{2} \right) \\ &+ \left(2048 \, a_{1}^{6} - 228 \, a_{1}^{4} \, a_{2}^{2} + 1260 \, a_{1}^{2} \, a_{2}^{4} + 512 \, a_{2}^{6} \right) \cos \left(4\lambda_{1} - 2\lambda_{2} \right) \\ &+ \left(2232 \, a_{1}^{5} \, a_{2} - 6042 \, a_{1}^{3} \, a_{2}^{3} + 2016 \, a_{2} \, a_{2}^{5} \right) \cos \left(4\lambda_{1} - 2\lambda_{2} \right) \\ &+ \left(-672 \, a_{1}^{6} + 7704 \, a_{1}^{4} \, a_{2}^{2} + 7704 \, a_{1}^{2} \, a_{2}^{4} - 672 \, a_{2}^{6} \right) \cos \left(2\lambda_{1} - 2\lambda_{2} \right) \\ &- \left(1656 \, a_{1}^{5} \, a_{2} + 1269 \, a_{1}^{3} \, a_{2}^{3} + 2016 \, a_{2} \, a_{2}^{5} \right) \cos \left(3\lambda_{1} - \lambda_{2} \right) \\ &- \left(1656 \, a_{1}^{5} \, a_{2} + 1269 \, a_{1}^{3} \, a_{2}^{3} + 2016 \, a_{2} \, a_{2}^{5} \right) \cos \left(3\lambda_{1} - \lambda_{2} \right) \\ &- \left(1656 \, a_{1}^{5} \, a_{2} + 1269 \, a_{1}^{3} \, a_{2}^{3} + 300 \, a_{1} \, a_{2}^{5} \right) \cos \left(3\lambda_{1} - \lambda_{2} \right) \\ &- \left(1656 \, a_{1}^{5} \, a_{2} + 1268 \, a_{1}^{3} \, a_{2}^{3} + 300$$

$$\partial_{\hat{\eta}_{1}^{2}\hat{\eta}_{2}^{2}} \frac{1}{\hat{d}} \Big|_{0} = \left[128 \left(a_{1}^{2} + a_{2}^{2} - 2 \, a_{1} a_{2} \cos \left(\lambda_{1} - \lambda_{2} \right) \right)^{9/2} \right]^{-1} \\ \times \left\{ -a_{1} \, a_{2} \left[324 \, a_{1}^{5} \, a_{2} - 10584 \, a_{1}^{3} \, a_{2}^{3} + 324 \, a_{1} \, a_{2}^{5} \right. \\ \left. - 12 \, a_{1} \, a_{2} \left(220 \, a_{1}^{4} - 1067 \, a_{1}^{2} \, a_{2}^{2} + 52 \, a_{2}^{4} \right) \cos \left(2\lambda_{1} \right) \right. \\ \left. - 12 \, a_{1} \, a_{2} \left(599 \, a_{1}^{4} - 277 \, a_{1}^{2} \, \frac{2}{2} - 9 \, a_{2}^{4} \right) \cos \left(4\lambda_{1} \right) - 30 \, a_{1}^{3} \, a_{2}^{3} \cos \left(4\lambda_{1} - 6\lambda_{2} \right) \right]$$

$$\begin{array}{l} + & \left(264\,a_1^4\,a_2^2 - 1176\,a_1^2\,a_2^4\right)\,\cos\left(\lambda_1 - 5\lambda_2\right) \\ + & \left(36\,a_1^4\,a_2^2 + 564\,a_1^2\,a_2^4\right)\,\cos\left(3\lambda_1 - 5\lambda_2\right) \\ - & 30\,a_1^3\,a_2^3\,\cos\left(6\lambda_1 - 4\lambda_2\right) \\ + & \left(96\,a_1^6 - 1920\,a_1^4\,a_2^2 - 14496\,a_1^2\,a_2^4 + 864\,a_2^6\right)\,\cos\left(\lambda_1 - 3\lambda_2\right) \\ + & 75\,a_1^3\,a_2^3\,\cos\left(2\lambda_1 - 6\lambda_2\right) + \left(564\,a_1^4\,a_2^2 + 36\,a_1^2\,a_2^4\right)\,\cos\left(5\lambda_1 - 3\lambda_2\right) \\ + & \left(528\,a_1^5\,a_2 + 26\,a_1^3\,a_2^3 + 4272\,a_1\,a_2^5\right)\,\cos\left(2\lambda_1 - 4\lambda_2\right) \\ + & \left(4272\,a_1^5\,a_2 + 26\,a_1^3\,a_2^3 + 528\,a_1\,a_2^5\right)\,\cos\left(4\lambda_1 - 2\lambda_2\right) \\ + & 75\,a_1^3\,a_2^3\,\cos\left(6\lambda_1 - 2\lambda_2\right) \\ + & \left(-136\,a_1^6 + 2400\,a_1^4\,a_2^2 + 2400\,a_1^2\,a_2^4 - 136\,a_2^6\right)\,\cos\left(\lambda_1 - \lambda_2\right) \\ + & \left(1272\,a_1^5\,a_2 + 6337\,a_1^3\,a_2^3 + 1272\,a_1\,a_2^5\right)\,\cos\left(2\lambda_1 - 2\lambda_2\right) \\ + & \left(-648\,a_1^6 - 396\,a_1^4\,a_2^2 - 396\,a_1^2\,a_2^4 - 648\,a_2^6\right)\,\cos\left(3\lambda_1 - 3\lambda_2\right) \\ + & \left(-348\,a_1^5\,a_2 - 800\,a_1^3\,a_2^3 - 348\,a_1\,a_2^5\right)\,\cos\left(4\lambda_1 - 4\lambda_2\right) \\ + & \left(60\,a_1^4\,a_2^2 + 60\,a_1^2\,a_2^4\right)\,\cos\left(5\lambda_1 - 5\lambda_2\right) - 9\,a_1^3\,a_2^3\,\cos\left(6\lambda_1 - 6\lambda_2\right) \\ + & \left(864\,a_1^6 - 14496\,a_1^4\,a_2^2 - 1920\,a_1^2\,a_2^4 + 96\,a_2^6\right)\,\cos\left(3\lambda_1 - \lambda_2\right) \\ - & \left(1176\,a_1^4\,a_2^2 + 264\,a_1^2\,a_2^4\right)\,\cos\left(5\lambda_1 - \lambda_2\right) \\ + & \left(-624\,a_1^5\,a_2 + 12804\,a_1^3\,a_2^3 - 2640\,a_1\,a_2^5\right)\,\cos\left(2\lambda_2\right) \\ + & \left(108\,a_1^5\,a_2 + 3324\,a_1^3\,a_2^3 - 7188\,a_1\,a_2^5\right)\,\cos\left(\lambda_1 + \lambda_2\right) \\ + & \left(1224\,a_1^5\,a_2 - 45774\,a_1^3\,a_2^3 + 1224\,a_1\,a_2^5\right)\,\cos\left(\lambda_1 + \lambda_2\right) \\ + & \left(-216\,a_1^6 + 29760\,a_1^4\,a_2^2 - 2736\,a_1^2\,a_2^4 - 24\,a_2^6\,\cos\left(\lambda_1 + 3\lambda_2\right)\right] \right\} \\ \end{array}$$

$$\begin{split} \partial_{\hat{\eta}_{1}^{2}\hat{\xi}_{1}^{2}} \frac{1}{\hat{d}} \bigg|_{0} &= \left[128 \left(a_{1}^{2} + a_{2}^{2} - 2 \, a_{1} a_{2} \cos \left(\lambda_{1} - \lambda_{2} \right) \right)^{9/2} \right]^{-1} \\ \times & \left\{ -a_{1} \left[\left(4096 \, a_{1}^{7} - 25080 \, a_{1}^{5} \, a_{2}^{2} - 20178 \, a_{1}^{3} \, a_{2}^{4} - 4952 \, a_{1} \, a_{2}^{6} \right) \cos \left(4\lambda_{1} \right) \right. \\ &+ 120 \, a_{1}^{5} \, a_{2}^{2} - 8622 \, a_{1}^{3} \, a_{2}^{4} + 600 \, a_{1} \, a_{2}^{6} - 343 \, a_{1}^{3} \, a_{2}^{4} \cos \left(8\lambda_{1} - 4\lambda_{2} \right) \\ &+ 4 \, a_{1}^{2} \, a_{2}^{3} \left(419 \, a_{1}^{2} + 339 \, a_{2}^{2} \right) \cos \left(7\lambda_{1} - 3\lambda_{2} \right) \\ &+ \left(-3492 \, a_{1}^{5} \, a_{2}^{2} - 6660 \, a_{1}^{3} \, a_{2}^{4} - 1908 \, a_{1} \, a_{2}^{6} \right) \cos \left(6\lambda_{1} - 2\lambda_{2} \right) \\ &+ \left(-112 \, a_{1}^{6} \, a_{2} + 4032 \, a_{1}^{4} \, a_{2}^{3} + 3552 \, a_{1}^{2} \, a_{2}^{5} - 112 \, a_{2}^{7} \right) \cos \left(\lambda_{1} - \lambda_{2} \right) \\ &+ \left(-600 \, a_{1}^{5} \, a_{2}^{2} + 4024 \, a_{1}^{3} \, a_{2}^{4} - 1080 \, a_{1} \, a_{2}^{6} \right) \cos \left(2\lambda_{1} - 2\lambda_{2} \right) \\ &+ \left(-1104a_{1}^{4} \, a_{2}^{3} - 624 \, a_{1}^{2} \, a_{2}^{5} \right) \cos \left(3\lambda_{1} - 3\lambda_{2} \right) - 74 \, a_{1}^{3} \, a_{2}^{4} \cos \left(4\lambda_{1} - 4\lambda_{2} \right) \\ &+ \left(6568 \, a_{1}^{6} \, a_{2} + 14148 \, a_{1}^{4} \, a_{2}^{3} + 9444 \, a_{1}^{2} \, a_{2}^{5} + 1000 \, a_{2}^{7} \right) \cos \left(5\lambda_{1} - \lambda_{2} \right) \\ &+ \left(29436 \, a_{1}^{5} \, a_{2}^{2} - 18484 \, a_{1}^{3} \, a_{2}^{4} - 468 \, a_{1} \, a_{2}^{6} \right) \cos \left(2\lambda_{1} + 2\lambda_{2} \right) \\ &+ \left(-17784 \, a_{1}^{6} \, a_{2} + 34236 \, a_{1}^{4} \, a_{2}^{3} + 10620 \, a_{1}^{2} \, a_{2}^{5} + 72 \, a_{2}^{7} \right) \cos \left(3\lambda_{1} + \lambda_{2} \right) \end{split}$$

+
$$(-22204 a_1^4 a_2^3 + 2340 a_1^2 a_2^5) \cos(\lambda_1 + 3\lambda_2)]$$

$$\begin{array}{lll} \partial_{\tilde{\eta}_{2}^{2} \tilde{\xi}_{1}^{2}} \left. \frac{1}{\tilde{d}} \right|_{0} &= & \left[128 \left(a_{1}^{2} + a_{2}^{2} - 2 \, a_{1} a_{2} \cos \left(\lambda_{1} - \lambda_{2} \right) \right)^{9/2} \right]^{-1} \\ & \times & \left\{ -3 \, a_{1} \, a_{2} \left[84 \, a_{1}^{5} \, a_{2} - 8832 \, a_{1}^{3} \, a_{2}^{3} + 84 \, a_{1} \, a_{2}^{5} \\ & + \, 4 \, a_{1} \, a_{2} \left(956 \, a_{1}^{4} - 3139 \, a_{1}^{2} \, a_{2}^{2} + 28 \, a_{2}^{4} \right) \cos \left(2\lambda_{1} \right) \\ & + \, 4 \, a_{1} \, a_{2} \left(959 \, a_{1}^{4} - 277 \, a_{1}^{2} \, a_{2}^{2} - 9 \, a_{2}^{4} \right) \cos \left(4\lambda_{1} \right) - 10 \, a_{1}^{3} \, a_{2}^{3} \cos \left(4\lambda_{1} - 6\lambda_{2} \right) \\ & + & \left(88 \, a_{1}^{4} \, a_{2}^{2} + 392 \, a_{1}^{2} \, a_{2}^{4} \right) \cos \left(3\lambda_{1} - 5\lambda_{2} \right) \\ & + & \left(12 \, a_{1}^{4} \, a_{2}^{2} + 188 \, a_{1}^{2} \, a_{2}^{4} \right) \cos \left(3\lambda_{1} - 5\lambda_{2} \right) \\ & + & \left(32 \, a_{1}^{6} - 968 \, a_{1}^{4} \, a_{2}^{2} - 6920 \, a_{1}^{2} \, a_{2}^{4} + 288 \, a_{2}^{6} \right) \cos \left(\lambda_{1} - 3\lambda_{2} \right) \\ & - & 25 \, a_{1}^{3} \, a_{2}^{3} \cos \left(2\lambda_{1} - 6\lambda_{2} \right) - \left(188 \, a_{1}^{4} \, a_{2}^{2} + 12 \, a_{1}^{2} \, a_{2}^{4} \right) \cos \left(5\lambda_{1} - 3\lambda_{2} \right) \\ & + & \left(176 \, a_{1}^{5} \, a_{2} - 158 \, a_{1}^{3} \, a_{2}^{3} - 176 \, a_{1} \, a_{2}^{5} \right) \cos \left(2\lambda_{1} - 2\lambda_{2} \right) \\ & + & \left(-1424 \, a_{1}^{5} \, a_{2} + 158 \, a_{1}^{3} \, a_{2}^{3} - 176 \, a_{1} \, a_{2}^{5} \right) \cos \left(2\lambda_{1} - 2\lambda_{2} \right) \\ & + & \left(-1640 \, a_{1}^{5} \, a_{2} + 2805 \, a_{1}^{3} \, a_{2}^{3} - 1640 \, a_{1} \, a_{2}^{5} \right) \cos \left(2\lambda_{1} - 2\lambda_{2} \right) \\ & + & \left(-1640 \, a_{1}^{5} \, a_{2} + 200 \, a_{1}^{3} \, a_{2}^{3} - 1020 \, a_{1}^{2} \, a_{2}^{4} + 216 \, a_{2}^{6} \right) \cos \left(3\lambda_{1} - 3\lambda_{2} \right) \\ & + & \left(116 \, a_{1}^{5} \, a_{2} + 200 \, a_{1}^{3} \, a_{2}^{3} + 116 \, a_{1} \, a_{2}^{5} \right) \cos \left(2\lambda_{1} - 4\lambda_{2} \right) \\ & - & \left(20 \, a_{1}^{4} \, a_{2}^{2} + 20 \, a_{1}^{2} \, a_{2}^{4} + 268 \, a_{1}^{2} \, a_{2}^{4} - 32 \, a_{2}^{6} \right) \cos \left(3\lambda_{1} - \lambda_{2} \right) \\ & + & \left(-288 \, a_{1}^{6} + 6920 \, a_{1}^{4} \, a_{2}^{2} + 938 \, a_{1}^{2} \, a_{2}^{4} + 32 \, a_{2}^{6} \right) \cos \left(3\lambda_{1} - \lambda_{2} \right) \\ & + & \left(-112 \, a_{1}^{5} \, a_{2} + 12556 \, a_{1}^{3} \, a_{2}^{3} - 2384 \, a_{1} \, a_{2}^{5} \right) \cos \left(2\lambda_{2} \right) \\ & + & \left(-112 \, a_{1}^{5} \, a_$$

$$\begin{aligned} \partial_{\hat{\eta}_{1}\hat{\eta}_{2}\hat{\xi}_{1}^{2}} \frac{1}{\hat{d}} \Big|_{0} &= \left[128 \left(a_{1}^{2} + a_{2}^{2} - 2 \, a_{1} a_{2} \cos \left(\lambda_{1} - \lambda_{2} \right) \right)^{9/2} \right]^{-1} \\ &\times \left\{ a_{1} \, a_{2} \left[-1146 \, a_{1}^{4} \, a_{2}^{2} - 1266 \, a_{1}^{2} \, a_{2}^{4} \right. \\ &+ \left. 144 \, a_{1}^{2} \left(6 \, a_{1}^{4} - 166 \, a_{1}^{2} \, a_{2}^{2} - 31 \, a_{2}^{4} \right) \cos \left(2\lambda_{1} \right) \right. \\ &+ \left. 6 \, a_{1}^{2} \left(1024 \, a_{1}^{4} - 1577 \, a_{1}^{2} \, a - 2^{2} - 349 \, a_{2}^{4} \right) \cos \left(4\lambda_{1} \right) \end{aligned}$$

$$\begin{array}{l} + & 343 \, a_1^3 \, a_2^3 \cos \left(\lambda_1 - 5\lambda_2\right) + 69 \, a_1^3 \, a_2^3 \cos \left(3\lambda_1 - 5\lambda_2\right) \\ - & 25 \, a_1^3 a_2^3 \cos \left(7\lambda_1 - 5\lambda_2\right) \\ + & \left(162 \, a_1^4 \, a_2^2 + 18 \, a_1^2 \, a_2^4\right) \cos \left(6\lambda_1 - 4\lambda_2\right) \\ + & \left(24 \, a_1^5 \, a_2 - 7437 \, a_1^3 \, a_2^3 + 912 \, a_1 \, a_2^5\right) \cos \left(\lambda_1 - 3\lambda_2\right) \\ + & \left(-924 \, a_1^5 \, a_2 - 303 \, a_1^3 \, a_2^3 + 204 \, a_1 \, a_2^5\right) \cos \left(5\lambda_1 - 3\lambda_2\right) \\ + & 75 \, a_1^3 \, a_2^3 \cos \left(7\lambda_1 - 3\lambda_2\right) \\ + & \left(222 \, a_1^4 \, a_2^2 + 1758 \, a_1^2 \, a_2^4\right) \cos \left(2\lambda_1 - 4\lambda_2\right) \\ + & \left(-2048 \, a_1^6 + 2316 \, a_1^4 \, a_2^2 - 900 \, a_1^2 \, a_2^4 - 512 \, a_2^6\right) \cos \left(4\lambda_1 - 2\lambda_2\right) \\ - & \left(486 \, a_1^4 \, a_2^2 + 270 \, a_1^2 \, a_2^4\right) \cos \left(6\lambda_1 - 2\lambda_2\right) \\ + & \left(744 \, a_1^5 \, a_2 - 2014 \, a_1^3 \, a_2^3 + 864 \, a_1 \, a_2^5\right) \cos \left(\lambda_1 - \lambda_2\right) \\ + & \left(-224 \, a_1^6 + 2568 \, a_1^4 \, a_2^2 + 2568 \, a_1^2 \, a_2^4 - 224 \, a_2^6 \cos \left(2\lambda_1 - 2\lambda_2\right) \\ + & \left(-552 \, a_1^5 \, a_2 - 423 \, a_1^3 \, a_2^3 - 672 \, a_1 \, a_2^5\right) \cos \left(3\lambda_1 - 3\lambda_2\right) \\ + & \left(-174 \, a_1^4 \, a_2^2 - 54 \, a_1^2 \, a_2^4\right) \cos \left(4\lambda_1 - 4\lambda_2\right) + 5 \, a_1^3 \, a_2^3 \cos \left(5\lambda_1 - 5\lambda_2\right) \\ + & \left(11280 \, a_1^5 \, a_2 + 294 \, a_2^3 \, a_2^3 + 2616 \, a_1 \, a_2^5\right) \cos \left(3\lambda_1 - \lambda_2\right) \\ + & \left(2772 \, a_1^5 \, a_2 + 1864 \, a_1^3 \, a_2^3 + 300 \, a_1 \, a_2^5\right) \cos \left(5\lambda_1 - \lambda_2\right) \\ + & \left(-450 \, a_1^4 \, a_2^2 - 6492 \, a_1^2 \, a_2^4 + 96 \, a_2^6\right) \cos \left(2\lambda_2\right) \\ + & \left(-4044 \, a_1^5 \, a_2 + 23402 \, a_1^3 \, a_2^3 - 468 \, a_1 \, a_2^5\right) \cos \left(\lambda_1 + \lambda_2\right) \\ + & \left(-64 \, a_1^6 + 41694 \, a_1^4 \, a_2^2 - 4410 \, a_1^2 \, a_2^4 - 64 \, a_2^6\right) \cos \left(2\lambda_1 + 2\lambda_2\right) \\ + & \left(-26088 \, a_1^5 \, a_2 + 10854 \, a_1^3 \, a_2^3 + 360 \, a_1 \, a_2^5\right) \cos \left(\lambda_1 + 3\lambda_2\right) \right] \right\}$$

$$\begin{split} \partial_{\hat{\eta}_{1}\hat{\eta}_{2}\hat{\xi}_{1}\hat{\xi}_{2}} \frac{1}{\hat{d}} \bigg|_{0} &= \left[128 \left(a_{1}^{2} + a_{2}^{2} - 2 \, a_{1}a_{2}\cos\left(\lambda_{1} - \lambda_{2}\right) \right)^{9/2} \right]^{-1} \\ &\times \left\{ a_{1} \, a_{2} \left[-36 \, a_{1}^{5} \, a_{2} - 7956 \, a_{1}^{3} \, a_{2}^{3} - 36 \, a_{1} \, a_{2}^{5} \right. \\ &- 12 \, a_{1} \, a_{2} \left(599 \, a_{1}^{4} - 277 \, a_{1}^{2} \, a_{2}^{2} - 9 \, a_{2}^{4} \right) \cos\left(4\lambda_{1}\right) \\ &- 24 \, a_{1}^{2} \, a_{2}^{2} (11 \, a_{1}^{2} + 49 \, a_{2}^{2}) \cos\left(\lambda_{1} - 5\lambda_{2}\right) + 75 \, a_{1}^{3} \, a_{2}^{3} \cos\left(2\lambda_{1} - 6\lambda_{2}\right) \\ &+ 75 \, a_{1}^{3} \, a_{2}^{3} \cos\left(6\lambda_{1} - 2\lambda_{2}\right) \\ &+ \left(8 \, a_{1}^{6} + 6624 \, a_{1}^{4} \, a_{2}^{2} + 6624 \, a_{1}^{2} \, a_{2}^{4} + 8 \, a_{2}^{6} \right) \cos\left(\lambda_{1} - \lambda_{2}\right) \\ &+ \left(-3096 \, a_{1}^{5} \, a_{2} + 1039 \, a_{1}^{3} \, a_{2}^{3} - 3096 \, a_{1} \, a_{2}^{5} \right) \cos\left(2\lambda_{1} - 2\lambda_{2}\right) \\ &+ \left(648 \, a_{1}^{6} - 1332 \, a_{1}^{4} \, a_{2}^{2} - 1332 \, a_{1}^{2} \, a_{2}^{4} + 648 \, a_{2}^{6} \right) \cos\left(3\lambda_{1} - 3\lambda_{2}\right) \\ &+ \left(348 \, a_{1}^{5} \, a_{2} + 700 \, a_{1}^{3} \, a_{2}^{3} + 348 \, a_{1} \, a_{2}^{5} \right) \cos\left(4\lambda_{1} - 4\lambda_{2}\right) \\ &- \left(60 \, a_{1}^{4} \, a_{2}^{2} + 60 \, a_{1}^{2} \, a_{2}^{4} \right) \cos\left(5\lambda_{1} - 5\lambda_{2}\right) + 9 \, a_{1}^{3} \, a_{2}^{3} \cos\left(6\lambda_{1} - 6\lambda_{2}\right) \\ &- \left(1176 \, a_{1}^{4} \, a_{2}^{2} + 264 \, a_{1}^{2} \, a_{2}^{4} \right) \cos\left(5\lambda_{1} - \lambda_{2}\right) \\ &+ \left(108 \, a_{1}^{5} \, a_{2} + 3324 \, a_{1}^{3} \, a_{2}^{3} - 7188 \, a_{1} \, a_{2}^{5} \right) \cos\left(4\lambda_{2}\right) \end{split}$$

+
$$(1224 a_1^5 a_2 - 45774 a_1^3 a_2^3 + 1224 a_1 a_2^5) \cos(2\lambda_1 + 2\lambda_2)$$

+ $(-216 a_1^6 + 29760 a_1^4 a_2^2 - 2736 a_1^2 a_2^4 - 24 a_2^6) \cos(3\lambda_1 + \lambda_2)$
+ $(-24 a_1^6 - 2736 a_1^4 a_2^2 + 29760 a_1^2 a_2^4 - 216 a_2^6) \cos(\lambda_1 + 3\lambda_2)]$

The previous expressions have all the form of a finite sum

$$\sum_{k_1,k_2} \mathcal{P}_{k_1,k_2}(a_1,a_2) \frac{\cos(k_1\,\lambda_1+k_2\lambda_2)}{[a_1^2+a_2^2-2\,a_1\,a_2\,\cos(\lambda_1-\lambda_2)]^{9/2}}$$

with $\mathcal{P}_{k_1,k_2}(a_1,a_2)$ a homogeneous polynomial in (a_1,a_2) with degree 8. When taking the mean with respect to (λ_1, λ_2) , only the terms with $k_1 + k_2 = 0$ survive. Taking also into account the normalizations factors in front of the r.h.s of (3.29), one finds the expressions (3.28), in terms of the Laplace coefficients $b_{9/2,k}(a_1/a_2)$.

3.2.2 Proof of non Resonance

In this paragraph, we prove

Lemma 3.5 There exists δ_* such that, for any $0 < \delta < \delta_*$, the matrix $\mathcal{F}(\Lambda)$ defining the second order f_2 of the secular perturbation \overline{f} is negative definite in

$$\mathcal{L}_{\underline{\alpha},\overline{\alpha}}(\mu,\delta) = \left\{ \Lambda = (\Lambda_1, \cdots, \Lambda_N) \in \mathbb{R}^N_+ : a_i := \frac{1}{\hat{m}_i} \left(\frac{\Lambda_i}{\tilde{m}_i}\right)^2 \in [\underline{\alpha},\overline{\alpha}] \delta^{N-i} \quad 1 \le i \le N \right\} \,.$$

Denoting, for such values of δ , by $\Omega = (\Omega_1, \dots, \Omega_N)$, $U(\Lambda) = (u_{ij}(\Lambda)$ the eigenvalues of $\mathcal{F}(\Lambda)$ and the unitary matrix through which $\mathcal{F}(\Lambda)$ is put in diagonal form

$$U(\Lambda)^T \mathcal{F}(\Lambda) U(\Lambda) = \operatorname{diag}[\Omega_1, \cdots, \Omega_N], \qquad U(\Lambda)^T U(\Lambda) = \operatorname{id}_N,$$

then, $\Omega(\Lambda)$ and $U(\Lambda)$ satisfy the following asymptotics in δ :

$$\Omega_{i}(\Lambda) = \delta^{(9-3N)/2} \times \begin{cases} -\frac{3}{4} \frac{\hat{a}_{1}^{2}}{\hat{a}_{2}^{3}} \frac{\bar{m}_{1}\bar{m}_{2}}{\tilde{m}_{1}\sqrt{\hat{m}_{1}\hat{a}_{1}}} + O(\delta^{2}) & \text{for } i = 1 \\ \\ -\frac{3}{4} \frac{\hat{a}_{i-1}^{2}}{\hat{a}_{i}^{3}} \frac{\bar{m}_{i}\bar{m}_{i-1}}{\tilde{m}_{i}\sqrt{\hat{m}_{i}\hat{a}_{i}}} \delta^{(3i-5)/2} + O(\delta^{(3i-3)/2}) & \text{for } 2 \le i \le N-1 \\ \\ -\frac{3}{4} \frac{\hat{a}_{N-1}^{2}}{\hat{a}_{N}^{3}} \frac{\bar{m}_{N}\bar{m}_{N-1}}{\tilde{m}_{N}\sqrt{\hat{m}_{N}\hat{a}_{N}}} \delta^{(3N-5)/2} + O(\delta^{(3N-2)/2}) & \text{for } i = N \end{cases}$$

$$u_{ij}(\Lambda) = \begin{cases} 1 - \frac{25}{32} \left[\frac{\hat{a}_{1}\hat{a}_{2}^{3}}{\hat{a}_{2}^{3}} \sqrt{\frac{\hat{m}_{1}}{\hat{m}_{2}}} \sqrt{\frac{\hat{m}_{1}\hat{a}_{1}}{\hat{m}_{2}\hat{a}_{2}}} \right]^{2} \delta^{5/2} + o(\delta^{5/2}) \\ for \quad 1 \le i = j \le 2 \\ 1 - \frac{25}{32} \left[\left(\frac{\bar{m}_{i}}{\bar{m}_{i-2}} \frac{a_{i-1}^{6}}{\hat{a}_{i}^{4}\hat{a}_{i-2}^{2}} \sqrt{\frac{\bar{m}_{i-1}}{\hat{m}_{i}}} \sqrt{\frac{\bar{m}_{i-1}\hat{a}_{i-1}}{\hat{m}_{i}\hat{a}_{i}}} \right)^{2} \\ + \left(\frac{\bar{m}_{i+1}}{\bar{m}_{i-1}} \frac{a_{i}^{6}}{\hat{a}_{i-1}^{4}\hat{a}_{i-1}^{2}} \sqrt{\frac{\bar{m}_{i}}{\hat{m}_{i-1}}} \sqrt{\frac{\bar{m}_{i-1}}{\hat{m}_{i}}} \sqrt{\frac{\bar{m}_{i-1}\hat{a}_{i-1}}{\hat{m}_{i}\hat{a}_{i-1}}} \right)^{2} \right] \delta^{9/2} + o(\delta^{9/2}) \\ for \quad 2 < i = j < N \\ 1 - \frac{25}{32} \left(\frac{\bar{m}_{N}}{\bar{m}_{N-2}} \frac{a_{N}^{6}}{\hat{a}_{N}^{4}\hat{a}_{N-2}^{2}} \sqrt{\frac{\bar{m}_{N-1}}{\hat{m}_{N}}} \sqrt{\frac{\bar{m}_{N-1}\hat{a}_{N-1}}{\hat{m}_{N}\hat{a}_{N}}} \right)^{2} \delta^{9/2} + o(\delta^{9/2}) \\ for \quad i = j = N \\ \frac{5}{for} \quad i = j = N \\ \frac{5}{for} \quad i = j = N \\ \frac{5}{for} \quad 1 = i < j \le N \\ \frac{5}{for} \quad 1 = i < j \le N \\ \frac{5}{for} \quad 1 = i < j \le N \\ \frac{5}{for} \quad 1 = i < j \le N \\ -\frac{5}{fa}\frac{\bar{m}_{i}}{\bar{a}_{i}^{4}} \frac{\hat{a}_{i}^{2}}{\sqrt{\frac{\bar{m}_{1}}{\hat{m}_{i}}}} \sqrt{\frac{\bar{m}_{1}\hat{a}_{i}}{\hat{m}_{j}\hat{a}_{j}}} \delta^{(17j-17i-8)/4} + o(\delta^{(17j-17i-8)/4}) \\ for \quad 1 < i < j \le N \\ -\frac{5}{fa}\frac{\bar{m}_{i}}{\bar{a}_{i}^{2}} \frac{\hat{a}_{i}^{4}\hat{a}}{\sqrt{\frac{\bar{m}_{1}}{\hat{m}_{i}}}} \sqrt{\frac{\bar{m}_{1}\hat{a}_{i}}{\hat{m}_{i}\hat{a}_{i}}} \delta^{(17i-29)/4} + o(\delta^{(17i-29)/4}) \\ for \quad 1 = j < i \le N \\ -\frac{5}{fa}\frac{\bar{m}_{i}}{\bar{m}_{2}} \frac{\hat{a}_{i}^{4}\hat{a}}{\sqrt{\frac{\bar{m}_{1}}{\hat{m}_{i}}}} \sqrt{\frac{\bar{m}_{1}\hat{a}_{i}}}{\hat{m}_{i}\hat{a}_{i}}} \delta^{(17i-29)/4} + o(\delta^{(17i-17j-8)/4}) \\ for \quad 1 < j < i \le N \end{cases}$$

Corollary 3.2 There exists $\delta^* > 0$ such that, for any $0 < \delta \leq \delta_*$, $\Omega(\Lambda)$ is 4-non resonant in $\mathcal{L}_{\underline{\alpha},\overline{\alpha}}(\mu,\delta)$:

$$\min_{1 \le |k| \le 4} \min_{\mathcal{L}_{\underline{\alpha},\overline{\alpha}}(\mu,\delta)} |\Omega(\Lambda) \cdot k| > 0 .$$

We will obtain Lemma 3.5 as a consequence of Lemmas 3.6 and 3.7 below: apply Lemma 3.6 to the matrix $\mathcal{F}(\Lambda)\delta^{-(9-3N)/2}$, hence, with

$$n_{ij} = \begin{cases} 0 \text{ for } i = j = 1\\ \frac{3i-5}{2} \text{ for } 2 \le i = j \le N\\ \frac{17j-11i-18}{4} \text{ for } i < j \end{cases}$$

$$a_{ij}(\delta) = \begin{cases} -\frac{3}{4} \bar{m}_1 \bar{m}_2 \frac{(\hat{a}_1/\hat{a}_2)^2}{\hat{a}_2 \bar{m}_1 \sqrt{\hat{m}_1 \hat{a}_1}} + O(\delta^2) , & \text{for} \quad i = j = 1 ; \\ -\frac{3}{4} \bar{m}_i \bar{m}_{i-1} \frac{(\hat{a}_{i-1}/\hat{a}_i)^2}{\hat{a}_i \bar{m}_i \sqrt{\hat{m}_i \hat{a}_i}} \delta^{(3i-5)/2} + O(\delta^{(3i-3)/2}) \\ \text{for} \quad 2 \le i = j \le N - 1 ; \\ -\frac{3}{4} \bar{m}_N \bar{m}_{N-1} \frac{(\hat{a}_{N-1}/\hat{a}_N)^2}{\hat{a}_N \bar{m}_N \sqrt{\hat{m}_N \hat{a}_N}} \delta^{(3N-5)/2} + O(\delta^{(3N-1)/2}) \\ \text{for} \quad i = j = N , \\ \frac{15}{16} \bar{m}_i \bar{m}_j \frac{(\hat{a}_i/\hat{a}_j)^3}{\hat{a}_j \sqrt{\hat{m}_i \bar{m}_j} \sqrt[4]{\hat{m}_i \hat{m}_j \hat{a}_i \hat{a}_j}} \delta^{(17j-11i-18)/4} + O(\delta^{(25j-19i-18)/4}) \\ \text{for} \quad i < j \end{cases}$$

and next on comparing the remainder terms.

Lemma 3.6 Let $\mathcal{A} = (a_{i,j})_{1 \leq i,j \leq N}$ a real symmetric matrix with order n and elements

$$a_{i,j} = \bar{a}_{i,j}(\delta) \,\delta^{n_{i,j}} , \qquad \text{with} \qquad \bar{a}_{i,j}(0) \neq 0 \tag{3.45}$$

and

$$\begin{cases}
 n_{1,1} = 0; & n_{i,i} < n_{j,j} \quad \text{for } i < j \\
 n_{i,j} < n_{i-1,j}, & n_{i,j} < n_{i,j+1} \quad \text{for } i < j+1.
\end{cases}$$
(3.46)

Then, there exists $\bar{\delta}$ such that, for any $0 < \delta < \bar{\delta}$, the eigenvalues $\lambda_1, \dots, \lambda_n$ of \mathcal{A} satisfy

$$|\lambda_i - a_{i,i}| \le C\delta^{m_i}$$
, where $m_i = 2\min\{n_{i-1,i}, n_{i,i-1}\} - n_{i,i}$

Furthermore, the orthogonal matrix $V = \{v_{i,j}\}_{i,j=1,\dots,N}$ which diagonalizes \mathcal{A}

 $V^T V = \operatorname{id}_n$, $V^T \mathcal{A} V = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$

satisfies

$$v_{i,j} = \delta_{i,j} + \check{v}_{i,j}(\delta) \,\delta^{\nu_{i,j}}$$

where

$$\nu_{i,j} = \begin{cases} n_{i,j} - n_{i,i} & \text{for } i < j \\ 2n_{12} & \text{for } i = j = 1 \\ 2\min\{n_{i-1,i} - n_{i-1,i-1}, n_{i+1,i} - n_{i,i}\} & \text{for } 2 \le i = j \le n - 1 \\ 2(n_{n-1,n} - n_{n-1,n-1}) & \text{for } 2 \le i = j = n \\ n_{i,j} - n_{j,j} & \text{for } i > j \end{cases},$$

$$\check{v}_{i,j}(0) = \begin{cases}
-\frac{\bar{a}_{i,i}(0)}{\bar{a}_{i,i}(0)} \quad for \quad i < j , \\
\frac{\bar{a}_{i,j}(0)}{\bar{a}_{j,j}(0)} \quad for \quad i > j . \\
-\frac{1}{2}\check{v}_{i+1,i}(0)^2 \quad for \quad i = j = 1
\end{cases}$$
(3.47)
$$-\frac{1}{2} \begin{cases}
\check{v}_{i-1,i}(0)^2 \quad if \quad n_{i-1,i} - n_{i-1,i-1} < n_{i+1,i} - n_{i,i} \\
\check{v}_{i+1,i}(0)^2 + \check{v}_{i-1,i}(0)^2 \quad if \quad n_{i-1,i} - n_{i-1,i-1} = n_{i+1,i} - n_{i,i} \\
\check{v}_{i+1,i}(0)^2 \quad if \quad n_{i-1,i} - n_{i-1,i-1} > n_{i+1,i} - n_{i,i} \\
for \quad 2 \le i = j \le n - 1 \\
-\frac{1}{2}\check{v}_{n-1,n}(0)^2 \quad for \quad i = j = n
\end{cases}$$

if $\delta_{i,j}$ is the Kronecker symbol.

This Lemma is purely technical and thus is proved in Appendix C.

Lemma 3.7 The matrix $\mathcal{F}(\Lambda)$ satisfies the following asymptotics:

$$f_{ij} = \begin{cases} -\frac{3}{4} \bar{m}_1 \bar{m}_2 \frac{(\hat{a}_1/\hat{a}_2)^2}{\hat{a}_2 \bar{m}_1 \sqrt{\hat{m}_1} \hat{a}_1} \, \delta^{3(3-N)/2} + O(\delta^{(13-3N)/2}) \ , \ for \ i = j = 1 \ ; \\ -\frac{3}{4} \bar{m}_i \bar{m}_{i-1} \frac{(\hat{a}_{i-1}/\hat{a}_i)^2}{\hat{a}_i \bar{m}_i \sqrt{\hat{m}_i} \hat{a}_i} \, \delta^{(3i+4-3N)/2} + O(\delta^{(3i+6-3N)/2}) \\ for \ 2 \le i = j \le N-1 \ ; \\ -\frac{3}{4} \bar{m}_N \bar{m}_{N-1} \frac{(\hat{a}_{N-1}/\hat{a}_N)^2}{\hat{a}_N \sqrt{\hat{m}_N} \hat{a}_N} \, \delta^2 + O(\delta^4) \ , \ for \ i = j = N \ , \\ \frac{15}{16} \bar{m}_i \bar{m}_j \frac{(\hat{a}_i/\hat{a}_j)^3}{\hat{a}_j \sqrt{\hat{m}_i \bar{m}_j} \sqrt[4]{\hat{m}_i \hat{m}_j} \hat{a}_i \hat{a}_j} }{\delta^{(17j-11i-6N)/4} + O(\delta^{(25j-19i-6N)/4}) \\ for \ i < j \end{cases}$$

Proof. We start with computing the asymptotics for the diagonal elements, which, we write as

$$f_{ii} = -2 \,\frac{\bar{m}_i}{\Lambda_i} \left[\sum_{k < i} \bar{m}_k \, a_{2000}(a_k, a_i) + \sum_{k > i} \bar{m}_k \, a_{2000}(a_i, a_k) \right] \qquad 1 \le i \le N$$

with

$$a_{2000}(a,b) = \frac{a}{8b^2} \left[-7a/b \, b_{5/2,0}(a/b) + 4(1+a^2/b^2) \, b_{5/2,1}(a/b) - a/b \, b_{5/2,2}(a/b) \right]$$

$$= \frac{3 \, a^2}{8 \, b^3} + O\left(\frac{a^4}{b^5}\right)$$
(3.48)

having used the following asymptotics for the involved Laplace coefficients (see Appendix F, Lemma F.1):

$$b_{5/2,0}(\alpha) = 1 + O(\alpha^2)$$

$$b_{5/2,1}(\alpha) = \frac{5}{2}\alpha + O(\alpha^3)$$

$$b_{5/2,2}(\alpha) = O(\alpha^2)$$

$$b_{5/2,3}(\alpha) = O(\alpha^3)$$

Then, letting

$$a_i = \frac{1}{\hat{m}_i} \left(\frac{\Lambda_i}{\tilde{m}_i}\right)^2 = \hat{a}_i \delta^{N-i} , \quad \hat{a}_i \in [\underline{\alpha}, \overline{\alpha}] ,$$

we find

$$f_{ii} = -2 \frac{\bar{m}_i}{\Lambda_i} \left[\sum_{1 \le k < i} \bar{m}_k a_{2000}(a_k, a_i) + \sum_{i < k \le N} \bar{m}_k a_{2000}(a_i, a_k) \right] \\ = -2 \frac{\bar{m}_i}{\tilde{m}_i \sqrt{\hat{m}_i \hat{a}_i \delta^{N-i}}} \left[\sum_{1 \le k < i} \left(\frac{3}{8} \bar{m}_k \frac{\hat{a}_k^2 \delta^{2N-2k}}{\hat{a}_i^3 \delta^{3N-3i}} + O\left(\delta^{5i-4k-N}\right) \right) \right] \\ + \sum_{i < k \le N} \left(\frac{3}{8} \bar{m}_k \frac{\hat{a}_i^2 \delta^{2N-2i}}{\hat{a}_k^3 \delta^{3N-3k}} + O\left(\delta^{5k-4i-N}\right) \right) \right] \\ = -\frac{3}{4} \frac{\bar{m}_i}{\tilde{m}_i \sqrt{\hat{m}_i \hat{a}_i}} \left[\sum_{1 \le k < i} \left(\bar{m}_k \frac{\hat{a}_k^2}{\hat{a}_i^3} \delta^{(7i-4k-3N)/2} + O(\delta^{(11i-8k-3N)/2}) \right) \right] \\ + \sum_{i < k \le N} \left(\bar{m}_k \frac{\hat{a}_i^2}{\hat{a}_k^3} \delta^{(6k-3i-3N)/2} + O(\delta^{(10k-7i-3N)/2}) \right) \right]$$
(3.49)

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So, when i = 1, only the second summation appears and the dominant term is the one with k = 2, namely, the term

$$-\frac{3}{4}\frac{\bar{m}_1\bar{m}_2}{\tilde{m}_1\sqrt{\hat{m}_1\hat{a}_1}}\frac{\hat{a}_1^2}{\hat{a}_2^3}\delta^{3(3-N)/2}$$

and the dominant neglected term is of order $\delta^{(13-3N)/2}$. When i = N, only the first summation appears in (3.49), the dominant term is reached in the sum for k = N - 1:

$$-\frac{3}{4}\frac{\bar{m}_{N-1}\bar{m}_N}{\tilde{m}_N\sqrt{\hat{m}_N\hat{a}_N}}\frac{\hat{a}_{N-1}^2}{\hat{a}_N^3}\delta^2$$

and the dominant neglected term is of order δ^4 .

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When $2 \le i \le N-1$ (for $N \ge 3$ planets), the first summation gives the lowest order term for k = i-1

$$-\frac{3}{4}\frac{\bar{m}_{i-1}\bar{m}_i}{\tilde{m}_i\sqrt{\hat{m}_i\hat{a}_i}}\frac{\hat{a}_{i-1}^2}{\hat{a}_i^3}\delta^{(3i+4-3N)/2}$$

which is dominant with respect to the dominant term of the second summand

$$-\frac{3}{4}\frac{\bar{m}_{i}\bar{m}_{i+1}}{\tilde{m}_{i}\sqrt{\hat{m}_{i}\hat{a}_{i}}}\frac{\hat{a}_{i}^{2}}{\hat{a}_{i+1}^{3}}\delta^{(3i+6-3N)/2}$$

which, on turn, is the dominant neglected term. The evaluation of the off-diagonal elements of $\mathcal{F}(\Lambda)$ is easier. In fact, we find, for i < j,

$$\begin{split} f_{ij} &= -\frac{m_i m_j}{\sqrt{\Lambda_i \Lambda_j}} a_{1100}(a_i, a_j) \\ &= -\frac{\bar{m}_i \bar{m}_j}{\sqrt{\tilde{m}_i \tilde{m}_j \sqrt{\hat{m}_i \hat{m}_j \hat{a}_i \hat{a}_j \delta^{2N-i-j}}}} \left[-\frac{15}{16} \frac{\hat{a}_i^3 \delta^{3N-3i}}{\hat{a}_j^4 \delta^{4N-4j}} + O\left(\frac{\delta^{5N-5i}}{\delta^{6N-6j}}\right) \right] \\ &= \frac{15}{16} \frac{\bar{m}_i \bar{m}_j}{\sqrt{\tilde{m}_i \tilde{m}_j \sqrt{\hat{m}_i \hat{m}_j \hat{a}_i \hat{a}_j}}} \frac{\hat{a}_i^3}{\hat{a}_j^4} \delta^{(17j-11i-6N)/4} + O\left(\delta^{(25j-19i-6N)/4}\right) \end{split}$$

because

$$a_{1100}(a,b) = \frac{a}{8b^2} \left[-17 a/b b_{5/2,1}(a/b) + 8(1+a^2/b^2) b_{5/2,2}(a/b) + a/b b_{5/2,3}(a/b) \right]$$
$$= -\frac{15}{16} \frac{a^3}{b^4} + O\left(\frac{a^5}{b^6}\right)$$

having used

$$b_{5/2,1}(\alpha) = \frac{5}{2}\alpha + O(\alpha^3) b_{5/2,2}(\alpha) = \frac{35}{8}\alpha^2 + O(\alpha^4) b_{5/2,3}(\alpha) = O(\alpha^3)$$

3.2.3 Proof of non Degeneracy

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The aim of this section is to prove the non degeneracy of the Plane Planetary Problem:

Lemma 3.8 There exists $\delta^* > 0$ such that, for any $0 < \delta < \delta^*$ and $0 < \mu < 1$, the matrix of the Birkhoff invariants with order 2 for \overline{f} is non singular on $\mathcal{L}_{\underline{\alpha},\overline{\alpha}}(\mu,\delta)$:

$$\inf_{\mathcal{L}_{\underline{\alpha},\overline{\alpha}}(\mu,\delta)} |\det A| > 0 .$$

We will obtain this result as a consequence of Lemma 3.9 below, which will be obtained by direct check. **Lemma 3.9** The matrix $A_{i,j}(\Lambda)$ of the Birkhoff invariants with order 2 for $\bar{f}(\Lambda, \cdot, \cdot)$ satisfies the following asymptotics

$$A_{i,j} = \delta^{5-2N} \times \begin{cases} +\frac{3}{4} \frac{\bar{m}_{1} \bar{m}_{2}}{\bar{m}_{1}^{2} \hat{m}_{1}} \frac{(\hat{a}_{1}/\hat{a}_{2})^{2}}{\hat{a}_{1} \hat{a}_{2}} (1+\alpha_{11}) & for \quad i=j=1 ; \\ -3 \frac{\bar{m}_{i} \bar{m}_{i-1}}{\bar{m}_{i}^{2} \hat{m}_{i}} \frac{(\hat{a}_{i-1}/\hat{a}_{i})^{2}}{\hat{a}_{i}^{2}} \delta^{2i-3} (1+\alpha_{ii}) & for \quad i=j=2, \cdots, N ; \\ -\frac{9}{4} \frac{\bar{m}_{i} \bar{m}_{j}}{\bar{m}_{i} \bar{m}_{j} \sqrt{\hat{m}_{i}} \hat{m}_{j} \hat{a}_{i} \hat{a}_{j}} \frac{\hat{a}_{i}^{2}}{\hat{a}_{j}^{3}} \delta^{(7j-3i-10)/2} (1+\alpha_{ij}) & for \quad ij . \end{cases}$$
(3.50)

where $\alpha_{ij} = O(\delta)$.

We show here how Lemma 3.8 follows from Lemma 3.9

Proof of Lemma 3.8. Put, for shortness,

$$\tilde{A}_{i,j} := \begin{cases} +\frac{3}{4} \frac{\bar{m}_1 \bar{m}_2}{\tilde{m}_1^2 \hat{m}_1} \frac{(\hat{a}_1/\hat{a}_2)^2}{\hat{a}_1 \hat{a}_2} (1+\alpha_{11}) & \text{for} \quad i=j=1 ; \\ \\ -3 \frac{\bar{m}_i \bar{m}_{i-1}}{\tilde{m}_i^2 \hat{m}_i} \frac{(\hat{a}_{i-1}/\hat{a}_i)^2}{\hat{a}_i^2} (1+\alpha_{ii}) & \text{for} \quad i=j=2,\cdots,N ; \\ \\ -\frac{9}{4} \frac{\bar{m}_i \bar{m}_j}{\sqrt{\hat{m}_i \hat{m}_j \hat{a}_i \hat{a}_j}} \frac{\hat{a}_i^2}{\hat{a}_j^3} (1+\alpha_{ij}) & \text{for} \quad ij , \end{cases}$$

In the case N = 2 (Plane 3–Body Problem), the matrix A is

$$A = \delta \tilde{A}_2 := \delta \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \delta^{1/2} \\ \tilde{A}_{21} \delta^{1/2} & \tilde{A}_{22} \delta \end{pmatrix}$$

 $\mathrm{so},$

$$\det A = \delta^3 (\tilde{A}_{11} \tilde{A}_{22} - \tilde{A}_{12}^2) = -\frac{117}{16} \frac{\bar{m}_1^2 \bar{m}_2^2}{\tilde{m}_1^2 \tilde{m}_2^2 \hat{m}_1 \hat{m}_2} \frac{\hat{a}_1^3}{\hat{a}_2^7} \delta^3 (1+D_2)$$

where $D_2 = O(\delta)$, hence, for a suitable $\delta^* > 0$, det $A_2 \neq 0$ on $\mathcal{L}_{\underline{\alpha},\overline{\alpha}}(\mu,\delta)$ for $0 < \delta < \delta^*$. The following claim concludes the proof.

<u>Claim</u>: For $N \geq 3$, the matrix $\tilde{A} := \delta^{-(5-2N)}A$ has determinant

$$\det \tilde{A} = -\frac{117}{16} \frac{\bar{m}_1^2 \,\bar{m}_2^2}{\tilde{m}_1^2 \,\tilde{m}_2^2 \,\hat{m}_1 \,\hat{m}_2} \frac{\hat{a}_1^3}{\hat{a}_2^7} \prod_{3 \le i \le N} \left(-3 \frac{\bar{m}_i \,\bar{m}_{i-1}}{\tilde{m}_i^2 \,\hat{m}_i} \frac{(\hat{a}_{i-1}/\hat{a}_i)^2}{\hat{a}_i^2} \delta^{2i-3} \right) (1+D)$$

where $D = O(\delta)$.

<u>*Proof.*</u> We prove, by induction, that any matrix with order n the form of \tilde{A}_n has determinant

$$\det \tilde{A}_n = -\frac{117}{16} \frac{\bar{m}_1^2 \,\bar{m}_2^2}{\tilde{m}_1^2 \,\tilde{m}_2^2 \,\hat{m}_1 \,\hat{m}_2} \frac{\hat{a}_1^3}{\hat{a}_2^7} \prod_{3 \le i \le n} \left(-3 \frac{\bar{m}_i \,\bar{m}_{i-1}}{\tilde{m}_i^2 \,\hat{m}_i} \,\frac{(\hat{a}_{i-1}/\hat{a}_i)^2}{\hat{a}_i^2} \right) (1+D_n)$$

where $D_n = O(\delta)$. The claim will be then obtained taking n = N, $D := D_N$. For n = 3, the claim is true by direct computation, since

$$\tilde{A}_{3} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12}\delta^{1/2} & \tilde{A}_{13}\delta^{4} \\ \tilde{A}_{21}\delta^{1/2} & \tilde{A}_{22}\delta & \tilde{A}_{23}\delta^{5/2} \\ \tilde{A}_{31}\delta^{4} & \tilde{A}_{23}\delta^{5/2} & \tilde{A}_{33}\delta^{3} \end{pmatrix}$$

Assume now that the claim is true for n-1 and let us prove it for n. We write

$$\tilde{A}_{n} = \begin{pmatrix} \tilde{A}_{1,n} \delta^{(7n-13)/2} \\ \tilde{A}_{n-1} & \vdots \\ \tilde{A}_{n-1,n} \delta^{(4n-7)/2} \\ \tilde{A}_{n,1} \delta^{(7n-13)/2} & \cdots & \tilde{A}_{n,n-1} \delta^{(4n-7)/2} & \tilde{A}_{nn} \delta^{2n-3} \end{pmatrix}$$

where \tilde{A}_{n-1} is the submatrix of \tilde{A} composed of the first n-1 rows and coloumns, which, by the inductive hypothesis, has determinant

$$\det \tilde{A}_{n-1} = -\frac{117}{16} \frac{\bar{m}_1^2 \,\bar{m}_2^2}{\tilde{m}_1^2 \,\tilde{m}_2^2 \,\hat{m}_1 \,\hat{m}_2} \frac{\hat{a}_1^3}{\hat{a}_2^7} \prod_{3 \le i \le n-1} \left(-3 \frac{\bar{m}_i \,\bar{m}_{i-1}}{\tilde{m}_i^2 \,\hat{m}_i} \frac{(\hat{a}_{i-1}/\hat{a}_i)^2}{\hat{a}_i^2} \delta^{2i-3} \right) (1+D_{n-1})$$

Then, expanding the determinant of \tilde{A}_n along the n^{th} coloumn, we find

$$\det \tilde{A}_{n} = \tilde{A}_{nn} \delta^{2n-3} \det \tilde{A}_{n-1} + R_{n} = -\frac{117}{16} \frac{\bar{m}_{1}^{2} \bar{m}_{2}^{2}}{\tilde{m}_{1}^{2} \tilde{m}_{2}^{2} \hat{m}_{1} \hat{m}_{2}} \frac{\hat{a}_{1}^{3}}{\hat{a}_{2}^{7}} \prod_{3 \le i \le n} \left(-3 \frac{\bar{m}_{i} \bar{m}_{i-1}}{\tilde{m}_{i}^{2} \hat{m}_{i}} \frac{(\hat{a}_{i-1}/\hat{a}_{i})^{2}}{\hat{a}_{i}^{2}} \delta^{2i-3} \right) (1 + D_{n-1}) + R_{n}$$

$$(3.51)$$

where

$$R_n = \sum_{1 \le i \le n-1} (-1)^{n-i} \tilde{A}_{in} \delta^{(7n-3i-10)/2} \det \tilde{\mathcal{A}}_{in}$$
(3.52)

if $\tilde{\mathcal{A}}_{in}$ is the (i, n)-minor of $\tilde{\mathcal{A}}_n$, hence, given by

$$\tilde{\mathcal{A}}_{in} = \begin{pmatrix} \tilde{r}_1 \\ \vdots \\ \tilde{r}_{i-1} \\ \tilde{r}_{i+1} \\ \vdots \\ \tilde{r}_{n-1} \\ \hat{r}_n \end{pmatrix}$$

$$\tilde{r}_i := [\tilde{r}_{i1}, \cdots \tilde{r}_{i,n-1}] \qquad 1 \le i \le n-1$$

is the i^{th} row of \tilde{A}_{n-1} and

$$\hat{r}_n = [\hat{r}_{n1}, \cdots, \hat{r}_{n,n-1}]$$

is the n^{th} row of \tilde{A} , deprivated of its n^{th} component. We prove that the remainder terms appearing in the summation (3.52) are at leas δ times the dominant term

$$-\frac{117}{16}\frac{\bar{m}_1^2 \bar{m}_2^2}{\tilde{m}_1^2 \tilde{m}_2^2 \hat{m}_1 \hat{m}_2}\frac{\hat{a}_1^3}{\hat{a}_2^7}\prod_{3\leq i\leq n} \left(-3\frac{\bar{m}_i \bar{m}_{i-1}}{\tilde{m}_i^2 \hat{m}_i}\frac{(\hat{a}_{i-1}/\hat{a}_i)^2}{\hat{a}_i^2}\delta^{2i-3}\right)$$
(3.53)

.

in (3.51), which will conclude the proof. We distinguish 2 cases.

1. $1 \le i \le N - 2$.

Write $\hat{r}_n = \delta^{(-3n+10+3i)/2}(\tilde{r}_i - \hat{r}_i)$, where the k^{th} component of \hat{r}_i is $\tilde{r}_{ik}(1 - \rho_{ik})$, with

$$\rho_{ik} = \frac{\tilde{A}_{nk}}{\tilde{A}_{ik}} \times \begin{cases} \delta^{10(n-i-1)/2} & \text{for } 1 \le k < i \\ \delta^{(10(n-i)-14)/2} & \text{for } k = i \\ \delta^{(10(n-k-1)/2} & \text{for } i < k \le n-1 \end{cases}$$

Then,

$$\det \tilde{\mathcal{A}}_{in} = \det \begin{pmatrix} \tilde{r}_{1} \\ \vdots \\ \tilde{r}_{i-1} \\ \hat{r}_{i+1} \\ \vdots \\ \tilde{r}_{n-1} \\ \hat{r}_{n} \end{pmatrix}$$

$$= \delta^{(-3n+10+3i)/2} \det \begin{pmatrix} \tilde{r}_{1} \\ \vdots \\ \tilde{r}_{i-1} \\ \tilde{r}_{i} - \hat{r}_{i} \end{pmatrix}$$

$$= \delta^{(-3n+10+3i)/2} \det \begin{pmatrix} \tilde{r}_{1} \\ \vdots \\ \tilde{r}_{n-1} \\ \tilde{r}_{i-1} \\ \tilde{r}_{i+1} \\ \vdots \\ \tilde{r}_{n-1} \\ \tilde{r}_{i} \end{pmatrix} - \delta^{(-3n+10+3i)/2} \det \begin{pmatrix} \tilde{r}_{1} \\ \vdots \\ \tilde{r}_{i-1} \\ \tilde{r}_{i+1} \\ \vdots \\ \tilde{r}_{n-1} \\ \tilde{r}_{i} \end{pmatrix}$$
(3.54)

where both the matrices appearing in (3.54) may be rearranged (changing their determinants at most for a sign) such in a way to take the form of \tilde{A}_{n-1} . So, using the inductive hypothesys we see that each term $\delta^{(7n-3i-10)/2} \det \tilde{A}_{in}$ is of order at least δ^3 times the dominant term (3.53).

2. i = n - 1. In this case, we write

$$\hat{r}_n = (\tilde{r}_{n-1} - \hat{r}_{n-1})\delta^{3/2}$$

where \hat{r}_{n-1} has components $\tilde{r}_{n-1,k}(1-\rho_{n-1,k})$ with

$$\rho_{n-1,k} = \begin{cases} \frac{\tilde{A}_{n,k}}{\tilde{A}_{n-1,k}} \delta^2 & \text{for} \quad 1 \le k \le n-2\\ \frac{\tilde{A}_{n,k}}{\tilde{A}_{n-1,n-1}} & \text{for} \quad k = n-1 \end{cases}$$

So, using, as in the previous case, the inductive hypothesys for

$$\det \tilde{\mathcal{A}}_{n-1,n} = \det \begin{pmatrix} \tilde{r}_{1} \\ \vdots \\ \tilde{r}_{n-2} \\ \hat{r}_{n} \end{pmatrix}$$

$$= \delta^{3/2} \det \begin{pmatrix} \tilde{r}_{1} \\ \vdots \\ \tilde{r}_{i-1} \\ \tilde{r}_{i+1} \\ \vdots \\ \tilde{r}_{n-1} - \hat{r}_{n-1} \end{pmatrix}$$

$$= \delta^{3/2} \det \begin{pmatrix} \tilde{r}_{1} \\ \vdots \\ \tilde{r}_{i-1} \\ \tilde{r}_{i+1} \\ \vdots \\ \tilde{r}_{n-1} \\ \tilde{r}_{n-1} \end{pmatrix} - \delta^{3/2} \det \begin{pmatrix} \tilde{r}_{1} \\ \vdots \\ \tilde{r}_{i-1} \\ \tilde{r}_{i+1} \\ \vdots \\ \tilde{r}_{n-1} \\ \tilde{r}_{n-1} \end{pmatrix}$$

and multiplying by $\delta^{(4n-7)/2}$, we find that the term with i = n - 1 in (3.52) is at least δ times the dominant term (3.53). This completes the proof.

The following Lemmas are devoted to the proof of Lemma 3.9.

Lemma 3.10 Let $U(\Lambda) = (u_{ij}(\Lambda))$ the unitary matrix which diagonalizes $\mathcal{F}(\Lambda)$:

$$U(\Lambda)^T \mathcal{F}(\Lambda) U(\Lambda) = \operatorname{diag}(\Omega_1, \cdots, \Omega_N), \quad U(\Lambda)^T U(\Lambda) = \operatorname{id}_N$$

and let $q_{i,j,k,l}(\Lambda)$, $r_{i,j,k,l}(\Lambda)$ the coefficients of the order 4-expansion of \overline{f} in Delaunay-Poincaré variables:

$$\bar{f}(\Lambda,\eta,\xi) = f_0(\Lambda) + \frac{1}{2} \left(\eta \cdot \mathcal{F}(\Lambda)\eta + \xi \cdot \mathcal{F}(\Lambda)\xi \right) + \sum_{1 \le i,j,k,l \le N} q_{i,j,k,l} \left(\eta_i \eta_j \eta_k \eta_l + \xi_i \xi_j \xi_k \xi_l \right)$$

$$+ \sum_{1 \le i,j,k,l \le N} r_{i,j,k,l}(\Lambda) \eta_i \eta_j \xi_k \xi_l + o_4 .$$

Define

$$\begin{cases}
\tilde{q}_{i,j,k,l}(\Lambda) := \sum_{1 \le i',j',k',l' \le N} q_{i',j',k',l'}(\Lambda) u_{i',i}(\Lambda) u_{j',j}(\Lambda) u_{k',k}(\Lambda) u_{l',l}(\Lambda) \\
\tilde{r}_{i,j,k,l}(\Lambda) := \sum_{1 \le i',j',k',l' \le N} r_{i',j',k',l'}(\Lambda) u_{i',i}(\Lambda) u_{j',j}(\Lambda) u_{k',k} u_{l',l}(\Lambda)
\end{cases}$$
(3.55)

Then, the Birkhoff invariants with order 2 for \overline{f} , namely, the elements of the symmetric matrix $A_{i,j}(\Lambda)$ defining the Birkhoff normal form for \overline{f} of order 2²²

$$f_0(\Lambda) + \sum_{1 \le i \le N} \Omega_i(\Lambda) \frac{p_i^2 + q_i^2}{2} + \frac{1}{2} \sum_{1 \le i,j \le N} A_{i,j}(\Lambda) \frac{p_i^2 + q_i^2}{2} \frac{p_j^2 + q_j^2}{2} , \qquad (3.56)$$

are given by

$$A_{i,j}(\Lambda) := \begin{cases} 6 \,\tilde{q}_{i,i,i,i} + \tilde{r}_{i,i,i,i} & for \quad i = j \\ 2 \tilde{q}_{i,i,j,j} + 2 \tilde{q}_{j,j,i,i} + 2 \tilde{q}_{i,j,i,j} + 2 \tilde{q}_{j,i,j,i} + 2 \tilde{q}_{j,i,j,i} + 2 \tilde{q}_{j,i,i,j} + \tilde{r}_{i,i,j,j} + \tilde{r}_{j,j,i,i} \\ for \quad i \neq j \end{cases}$$
(3.57)

Proof. The transformation

$$\phi_{\mathrm{D}}: \begin{cases} \Lambda = \tilde{\Lambda} \\ \lambda_i = \tilde{\lambda}_i - \tilde{\xi} \cdot U(\tilde{\Lambda})^T \partial_{\Lambda_i} U(\tilde{\Lambda}) \tilde{\eta} \\ \eta = U(\tilde{\Lambda}) \tilde{\eta} \\ \xi = U(\tilde{\Lambda}) \tilde{\xi} \end{cases}$$

diagonalizes the quadratic part of \bar{f} , sending it to

$$\tilde{f}_2 = \sum_{1 \le i \le N} \Omega_i(\tilde{\Lambda}) \frac{\tilde{\eta}_i^2 + \tilde{\xi}_i^2}{2}$$

and puts the order 4 term f_4 to

$$\tilde{f}_4 = \sum_{1 \le i,j,k,l \le N} \tilde{q}_{i,j,k,l} \Big(\tilde{\eta}_i \tilde{\eta}_j \tilde{\eta}_k \tilde{\eta}_l + \tilde{\xi}_i \tilde{\xi}_j \tilde{\xi}_k \tilde{\xi}_l \Big) + \sum_{1 \le i,j,k,l \le N} \tilde{r}_{i,j,k,l} (\tilde{\Lambda}) \tilde{\eta}_i \tilde{\eta}_j \tilde{\xi}_k \tilde{\xi}_l$$
(3.58)

²²The variables Λ , p, q are thought "dummy" in eq (3.56): the existence of the Birkhoff transformaton realizing (3.56) has been proved in the previous section.

with $(\tilde{q}_{ijkl}) := \tilde{\mathcal{Q}}, (\tilde{r}_{ijkl}) := \tilde{\mathcal{R}}$ as in (3.55). Next, as outlined in Appendix B, Remark B.1, the Birkhoff invariants with order 2 may easily be found through the identification

$$\frac{1}{2} \sum_{i,j=1}^{N} A_{i,j}(\tilde{\Lambda}) \frac{p_i^2 + q_i^2}{2} \frac{p_j^2 + q_j^2}{2} := \frac{1}{(2\pi)^N} \int_{[0,2\pi]^N} \tilde{f}_4(\tilde{\Lambda}, \tilde{\eta}, \tilde{\xi}) \big|_{(\tilde{\eta}_h, \tilde{\xi}_h) = \sqrt{p_h^2 + q_h^2} (\cos \varphi_h, \sin \varphi_h)} d\varphi$$
(3.59)

But, replacing \tilde{f}_4 as in (3.58), the r.h.s in (3.59) becomes

$$\frac{1}{(2\pi)^{N}} \int_{[0,2\pi]^{N}} \tilde{f}_{4}(\tilde{\Lambda},\tilde{\eta},\tilde{\xi})|_{(\tilde{\eta}_{h},\tilde{\xi}_{h})=\sqrt{p_{h}^{2}+q_{h}^{2}}(\cos\varphi_{h},\sin\varphi_{h})} d\varphi
= 2 \sum_{1 \leq i,j,k,l \leq N} \tilde{q}_{i,j,k,l} I_{ijkl} \sqrt{(p_{i}^{2}+q_{i}^{2})(p_{j}^{2}+q_{j}^{2})(p_{k}^{2}+q_{k}^{2})(p_{l}^{2}+q_{l}^{2})}
+ \sum_{1 \leq i,j,k,l \leq N} \tilde{r}_{i,j,k,l} J_{ijkl} \sqrt{(p_{i}^{2}+q_{i}^{2})(p_{j}^{2}+q_{j}^{2})(p_{k}^{2}+q_{k}^{2})(p_{l}^{2}+q_{l}^{2})}$$
(3.60)

where we have let

$$I_{ijkl} := \frac{1}{(2\pi)^N} \int_{\mathbb{T}^N} \cos \varphi_i \cos \varphi_j \cos \varphi_k \cos \varphi_l d\varphi$$
$$= \frac{1}{(2\pi)^N} \int_{\mathbb{T}^N} \sin \varphi_i \sin \varphi_j \sin \varphi_k \sin \varphi_l d\varphi$$
$$J_{ijkl} := \frac{1}{(2\pi)^N} \int_{\mathbb{T}^N} \cos \varphi_i \cos \varphi_j \sin \varphi_k \sin \varphi_l d\varphi$$

The elementary integral I_{ijkl} does not vanish only when i = j = k = l, or $i = j \neq k = l$, or $i = k \neq j = l$, or $i = l \neq j = k$. In the first case, it gives

$$I_{iiii} = \frac{1}{2\pi} \int_0^{2\pi} \cos^4 x \, dx = \frac{3}{8} \, .$$

In the three remaining cases, it gives

$$I_{iijj} = I_{ijij} = I_{ijji} = \left(\frac{1}{2\pi} \int_0^{2\pi} \cos^2 x \, dx\right)^2 = \frac{1}{4} \qquad i \neq j$$

So, the first summation in (3.49)

$$\sum_{1 \le i,j,k,l \le N} \tilde{q}_{i,j,k,l} I_{ijkl} \sqrt{(p_i^2 + q_i^2)(p_j^2 + q_j^2)(p_k^2 + q_k^2)(p_l^2 + q_l^2)} = \frac{3}{8} \sum_{1 \le i \le N} \tilde{q}_{i,i,i,i}(\tilde{\Lambda}) (p_i^2 + q_i^2)^2 + \frac{1}{4} \sum_{1 \le i \ne j \le N} (\tilde{q}_{i,i,j,j}(\tilde{\Lambda}) + \tilde{q}_{i,j,i,j}(\tilde{\Lambda}) + \tilde{q}_{i,j,j,i}(\tilde{\Lambda})) (p_i^2 + q_i^2) (p_j^2 + q_j^2)$$
(3.61)

Besides, the integral J_{ijkl} does not vanish only when i = j = k = l or $i = j \neq k = l$. In the first case, it gives

$$J_{iiii} = \frac{1}{2\pi} \int_0^{2\pi} \cos^2 x \sin^2 x \, dx = \frac{1}{8} \; ,$$

in the second case, it gives

$$J_{iijj} = \left(\frac{1}{2\pi} \int_0^{2\pi} \cos^2 x \, dx\right) \left(\frac{1}{2\pi} \int_0^{2\pi} \sin^2 x \, dx\right) = \frac{1}{4} \qquad i \neq j$$

Then, the second summation in (3.60) is

$$\sum_{1 \le i,j,k,l \le N} \tilde{r}_{i,j,k,l} J_{ijkl} \sqrt{(p_i^2 + q_i^2)(p_j^2 + q_j^2)(p_k^2 + q_k^2)(p_l^2 + q_l^2)} \\ \frac{1}{8} \sum_{1 \le i \le N} \tilde{r}_{i,i,i,i}(\tilde{\Lambda}) (p_i^2 + q_i^2)^2 + \frac{1}{4} \sum_{1 \le i \ne j \le N} \tilde{r}_{i,i,j,j}(\tilde{\Lambda}) (p_i^2 + q_i^2)(p_j^2 + q_j^2) .$$
(3.62)

Finally, replacing (3.61) and (3.62) into (3.59) and next simmetrizing the summation, we find the result.

Our next step is the computation of the asymptotics for \mathcal{Q} , \mathcal{R} , which, together with the one for the diagonalization matrix $U(\Lambda)$ (Lemma 3.5), will give the one for the A_{ij} 's.

Lemma 3.11 The δ - asymptotics for the functions $\mathcal{Q} = (q_{ijkl})$, $\mathcal{R} = (r_{ijkl})$ defining the 4-expansion of \overline{f} :

$$\bar{f} = f_0(\Lambda) + \frac{1}{2} \Big(\eta \mathcal{F}(\Lambda) \eta + \xi \cdot \mathcal{F}(\Lambda) \xi \Big) \\ + \sum_{1 \le i, j, k, l \le N} q_{i, j, k, l}(\Lambda) \Big(\eta_i \eta_j \eta_k \eta_l + \xi_i \xi_j \xi_k \xi_l \Big) + \sum_{1 \le i, j, k, l \le N} r_{i, j, k, l}(\Lambda) \eta_i \eta_j \xi_k \xi_l + o_4$$

is

$$q_{i,i,i,i} = \begin{cases} +\frac{3\bar{m}_{1}\bar{m}_{2}}{32\tilde{m}_{1}^{2}\hat{m}_{1}}\frac{(\hat{a}_{1}/\hat{a}_{2})^{2}}{\hat{a}_{1}\hat{a}_{2}}\,\delta^{5-2N} + O(\delta^{7-2N}) & \text{for} \quad i=1 ; \\ -\frac{3\bar{m}_{i}\bar{m}_{i-1}}{8\tilde{m}_{i}^{2}\hat{m}_{i}}\frac{(\hat{a}_{i-1}/\hat{a}_{i})^{2}}{\hat{a}_{i}^{2}}\,\delta^{2i+2-2N} + O\left(\delta^{2i+3-2N}\right) & \text{for} \quad 2 \le i \le N-1 ; \\ -\frac{3\bar{m}_{N}\bar{m}_{N-1}}{8\tilde{m}_{N}^{2}\hat{m}_{N}}\frac{(\hat{a}_{N-1}/\hat{a}_{N})^{2}}{\hat{a}_{N}^{2}}\,\delta^{2} + O\left(\delta^{4}\right) & \text{for} \quad i=N ; \end{cases}$$

$$(3.63)$$

 $q_{i,i,i,j} = r_{i,j,i,i} = r_{i,i,i,j}$

$$= \begin{cases} +\frac{75}{128} \frac{\bar{m}_i \bar{m}_j}{\sqrt{\bar{m}_i^3 \bar{m}_j \sqrt{\bar{m}_i^3 \bar{m}_j \hat{a}_i^3 \hat{a}_j}}} \frac{\hat{a}_i^3}{\hat{a}_j^4} \delta^{(17j-9i-8N)/4} \\ +O\left(\delta^{(25j-17i-8N)/4}\right) \quad for \quad i < j ; \\ +\frac{285}{128} \frac{\bar{m}_i \bar{m}_j}{\sqrt{\bar{m}_i^3 \bar{m}_j \sqrt{\bar{m}_i^3 \bar{m}_j \hat{a}_i^3 \hat{a}_j}}} \frac{\hat{a}_j^3}{\hat{a}_i^4} \delta^{(19i-11j-8N)/4} ; \\ +O\left(\delta^{(27i-19j-8N)/4}\right) \quad for \quad i > j ; \end{cases}$$

$$q_{i,i,j,j} = \begin{cases} -\frac{1}{16} \frac{1}{\tilde{m}_i \tilde{m}_j \sqrt{\tilde{m}_i \tilde{m}_j \hat{a}_i \hat{a}_j}}{\tilde{m}_i \tilde{m}_j \hat{a}_i \hat{a}_j} \frac{1}{\tilde{a}_j^3} \delta^{(i,j)} & d^{(i,j)} + O\left(\delta^{(i,j)} + U(i,j)/2\right), & \text{for } i < j ; \\ \equiv 0 \quad \text{for } i > j ; \end{cases}$$

$$(3.64)$$

$$r_{i,i,i,i} = \begin{cases} +\frac{3\bar{m}_{1}\bar{m}_{2}}{16\bar{m}_{1}^{2}\hat{m}_{1}}\frac{(\hat{a}_{1}/\hat{a}_{2})^{2}}{\hat{a}_{1}\hat{a}_{2}}\delta^{5-2N} + O(\delta^{7-2N}) & \text{for} \quad i=1; \\ -\frac{3\bar{m}_{i}\bar{m}_{i-1}}{4\bar{m}_{i}^{2}\hat{m}_{i}}\frac{(\hat{a}_{i-1}/\hat{a}_{i})^{2}}{\hat{a}_{i}^{2}}\delta^{2i+2-2N} + O\left(\delta^{2i+3-2N}\right) & \text{for} \quad 2 \le i \le N-1; \\ -\frac{3\bar{m}_{N}\bar{m}_{N-1}}{4\bar{m}_{N}^{2}\hat{m}_{N}}\frac{(\hat{a}_{N-1}/\hat{a}_{N})^{2}}{\hat{a}_{N}^{2}}\delta^{2} + O\left(\delta^{4}\right) & \text{for} \quad i=N; \end{cases}$$

$$(3.65)$$

$$r_{i,i,j,j} = \begin{cases} -\frac{9}{16} \frac{\bar{m}_i \bar{m}_j}{\tilde{m}_i \bar{m}_j \sqrt{\hat{m}_i \hat{m}_j \hat{a}_i \hat{a}_j}} \frac{\hat{a}_i^2}{\hat{a}_j} \delta^{(7j-3i-4N)/2} + O\left(\delta^{(11j-7i-4N)/2}\right) & \text{for } i < j ; \\ -\frac{9}{16} \frac{\bar{m}_i \bar{m}_j}{\tilde{m}_i \bar{m}_j \sqrt{\hat{m}_i \hat{m}_j \hat{a}_i \hat{a}_j}} \frac{\hat{a}_j^2}{\hat{a}_i^3} \delta^{(7i-3j-4N)/2} + O\left(\delta^{(11i-7j-4N)/2}\right) & \text{for } i > j . \end{cases}$$

$$(3.66)$$

$$r_{i,j,i,j} = \begin{cases} -\frac{315\,\bar{m}_i\,\bar{m}_j}{64\bar{m}_i\bar{m}_j\sqrt{\hat{m}_i\bar{m}_j\hat{a}_i\hat{a}_j}}\,\frac{\hat{a}_i^4}{\hat{a}_j^5}\,\delta^{(11j-7i-4N)/2} + O(\delta^{(15j-11i-4N)/2}) & for \quad i < j ; \\ \\ \equiv 0 \quad for \quad i > j . \end{cases}$$
(3.67)

Proof.

1. Expansion of
$$q_{i,i,i,i} = -\frac{\bar{m}_i}{\Lambda_i^2} \sum_{h \neq i} \bar{m}_h a_{4000}(a_i, a_h)$$
, $\Lambda_i = \tilde{m}_i \sqrt{\hat{m}_i a_i}$.

The coefficient $a_{4000}(a, b)$ may be written simultaneoulsy as

$$a_{4000}(a,b) = \frac{a}{512b^2} \left[(-60(a/b)^5 + 4311(a/b)^3) \right]$$

$$- 300(a/b)) b_{9/2,0}(a/b) + 8(7(a/b)^{6}
- 252(a/b)^{4} - 222(a/b)^{2} + 7) b_{9/2,1}(a/b)
+ 4(75(a/b)^{5} - 503(a/b)^{3} + 135(a/b)) b_{9/2,2}(a/b)
+ 24(23(a/b)^{4} + 13(a/b)^{2}) b_{9/2,3}(a/b)
+ 37(a/b)^{3} b_{9/2,4}(a/b)]
= \frac{b}{512a^{2}} \left[(-60(b/a) + 4311(b/a)^{3}
- 300(b/a)^{5}) b_{9/2,0}(b/a)
+ 8(7 - 252(b/a)^{2} - 222(b/a)^{4} + 7(b/a)^{6}) b_{9/2,1}(b/a)
+ 4(75(b/a) - 503(b/a)^{3} + 135(b/a)^{5}) b_{9/2,2}(b/a)
+ 24(23(b/a)^{2} + 13(b/a)^{4}) b_{9/2,3}(b/a)
+ 37(b/a)^{3} b_{9/2,4}(b/a) \right]$$

having used

$$b_{9/2,k}(1/\alpha) = \alpha^9 b_{9/2,k}(\alpha)$$
.

The involved Laplace coefficients satisfy the following asymptotics (see Appendix F), for small α

$$\begin{cases}
 b_{9/2,0}(\alpha) = 1 + O(\alpha^2) \\
 b_{9/2,1}(\alpha) = \frac{9}{2} \alpha + O(\alpha^3) \\
 b_{9/2,2}(\alpha) = \frac{99}{8} \alpha^2 + O(\alpha^4) \\
 b_{9/2,k}(\alpha) = O(\alpha^k) , \quad \text{for} \quad k \ge 3 ,
\end{cases}$$
(3.68)

and we find thus the asymptotics for $a_{4000}(a, b)$

$$a_{4000}(a,b) = \begin{cases} -\frac{3}{32} \frac{a^2}{b^3} + O\left(\frac{a^4}{b^5}\right) & \text{for small} & a/b \\ \\ +\frac{3}{8} \frac{b^2}{a^3} + O\left(\frac{b^4}{a^5}\right) & \text{for small} & b/a \end{cases}$$

So, replacing

$$a = a_i = \hat{a}_i \delta^{N-i}$$
, $b = a_h = \hat{a}_h \delta^{N-h}$ (3.69)

we obtain

$$a_{4000}(a_i, a_h) = \begin{cases} -\frac{3}{32} \frac{\hat{a}_i^2}{\hat{a}_h^3} \,\delta^{3h-2i-N} + O\left(\delta^{5h-4i-N}\right) & \text{for} \quad i < h \\ \\ +\frac{3}{8} \frac{\hat{a}_h^2}{\hat{a}_i^3} \,\delta^{3i-2h-N} + O\left(\delta^{5i-4h-N}\right) & \text{for} \quad i > h \end{cases}$$

Finally,

$$\begin{aligned} q_{i,i,i,i} &= -\frac{m_i}{\Lambda_i^2} \sum_{h \neq i} \bar{m}_h a_{4000}(a_i, a_h) \\ &= -\frac{\bar{m}_i}{\tilde{m}_i^2 \hat{m}_i \hat{a}_i} \\ &\times \left\{ \frac{3}{8} \sum_{h < i} \left[\bar{m}_h \frac{\hat{a}_h^2}{\hat{a}_i^3} \delta^{4i - 2h - 2N} + O\left(\delta^{6i - 4h - 2N}\right) \right] \\ &- \frac{3}{32} \sum_{h > i} \left[\bar{m}_h \frac{\hat{a}_i^2}{\hat{a}_h^3} \delta^{3h - i - 2N} + O\left(\delta^{5h - 3i - 2N}\right) \right] \right\} \end{aligned}$$

The lowest order term is reached for h = 2 when i = 1, for h = i - 1 when $i \ge 2$. The first neglected powers of δ are the ones coming from the remainder term with h = 2, for i = 1, from the dominant term with h = i + 1, for $i = 2, \dots, N - 1$, from the remainder term with h = N - 1 when i = N. The final result is then (3.63).

2. Expansion of
$$r_{iiii} = -\frac{m_i}{\Lambda_i^2} \sum_{h \neq i} \bar{m}_h a_{2020}(a_i, a_h).$$

¿From the identity

$$a_{2020}(a,b) = 2 a_{4000}(a,b)$$

one finds $r_{i,i,i,i} = 2 q_{i,i,i,i}$, hence, (3.65).

3. Expansion of
$$q_{i,i,i,j} = r_{i,i,i,j} = r_{i,j,i,i} = -\frac{\bar{m}_i \bar{m}_j a_{3100}(a_i, a_j)}{\Lambda_i \sqrt{\Lambda_i \Lambda_j}} = -\frac{\bar{m}_i \bar{m}_j a_{1120}(a_i, a_j)}{\Lambda_i \sqrt{\Lambda_i \Lambda_j}}$$

for $i \neq j$.

Proceeding similarly to the expansion of q_{iiii} , we write the coefficients a_{3100} , a_{1120} as

$$a_{3100}(a,b) = a_{1120}(a,b) = \begin{cases} -\frac{a}{256b^2} \left[(-744(a/b)^5 + 2014(a/b)^3 \\ -864(a/b) \right) b_{9/2,1}(a/b) + 8(28(a/b)^6 \\ -321(a/b)^4 - 321(a/b)^2 + 28) b_{9/2,2}(a/b) \\ +(552(a/b)^5 + 423(a/b)^3 + 672(a/b)) b_{9/2,3}(a/b) \\ +(1146(a/b)^4 + 1266(a/b)^2) b_{9/2,0}(a/b) \\ +6(29(a/b)^4 + 9(a/b)^2) b_{9/2,4}(a/b) \\ -5(a/b)^3 b_{9/2,5}(a/b) \right] \\ -\frac{b}{256a^2} \left[(-744(b/a) + 2014(b/a)^3 \\ -864(b/a)^5) b_{9/2,1}(b/a) + 8(28 \\ -321(b/a)^2 - 321(b/a)^4 + 28(b/a)^6) b_{9/2,2}(b/a) \\ +(552(b/a) + 423(b/a)^3 + 672(b/a)^5) b_{9/2,3}(b/a) \\ +(1146(b/a)^2 + 1266(b/a)^4) b_{9/2,0}(b/a) \\ +6(29(b/a)^2 + 9(b/a)^4) b_{9/2,4}(b/a) \\ -5(b/a)^3 b_{9/2,5}(b/a) \right] \end{cases}$$

The asymptotics for these coefficients is computed using the asymptotics (3.68) for the involved Laplace cefficients

$$a_{3100}(a,b) = a_{1120}(a,b) = \begin{cases} -\frac{75}{128} \frac{a^3}{b^4} + O\left(\frac{a^5}{b^6}\right) & \text{for small} \quad a/b \\ -\frac{285}{128} \frac{b^3}{a^4} + O\left(\frac{b^5}{a^6}\right) & \text{for small} \quad b/a \end{cases}$$

Hence, replacing $a_i = \hat{a}_i \delta^{N-i}$, $a_j = \hat{a}_i \delta^{N-j}$

$$a_{3100}(a_i, a_j) = a_{1120}(a_i, a_j) = \begin{cases} -\frac{75}{128} \frac{\hat{a}_i^3}{\hat{a}_j^4} \delta^{4j-3i-N} + O\left(\delta^{6j-5i-N}\right) & \text{for } i < j \\ \\ -\frac{285}{128} \frac{\hat{a}_j^3}{\hat{a}_i^4} \delta^{4i-3j-N} + O\left(\delta^{6i-5j-N}\right) & \text{for } i > j \end{cases}$$

and finally, the result, for $q_{iiij},\,r_{ijii},\, {\rm multiplying}$ in front by

$$-\frac{\bar{m}_i\bar{m}_j}{\sqrt{\Lambda_i^3\Lambda_j}} = -\frac{\bar{m}_i\bar{m}_j}{\sqrt{\tilde{m}_i^3\tilde{m}_j\sqrt{\hat{m}_i^3\hat{m}_j\hat{a}_i^3\hat{a}_j}}}\delta^{(3i+j-4N)/4}$$

4. Expansion of

$$q_{iijj} = \begin{cases} -\bar{m}_i \, \bar{m}_j \frac{a_{2200}(a_i, a_j)}{\Lambda_i \, \Lambda_j} & \text{for} \quad i < j \\ 0 & \text{for} \quad i > j \end{cases}$$

In order to compute the coefficient $q_{i,i,j,j}$, we start from the symmetric coefficient

$$\begin{aligned} a_{2200}(a,b) &= a_{2200}(b,a) &= \frac{a}{512b^2} \left[(-324(a/b)^5 + 10584(a/b)^3 - 324(a/b)) \, b_{9/2,0}(a/b) \right. \\ &+ 8(17(a/b)^6 - 300(a/b)^4 - 300(a/b)^2 + 17) \, b_{9/2,1}(a/b) \\ &- (1272(a/b)^5 + 6337(a/b)^3 + 1272(a/b)) \, b_{9/2,2}(a/b) \\ &+ (648(a/b)^6 + 396(a/b)^4 + 396(a/b)^2 \\ &+ 648) \, b_{9/2,3}(a/b) + (348(a/b)^5 \\ &+ 800(a/b)^3 + 348(a/b)) \, b_{9/2,4}(a/b) \\ &+ (-60a/b)^4 - 60(a/b)^2) \, b_{9/2,5}(a/b) \\ &+ 9(a/b)^3 \, b_{9/2,6}(a/b) \right] \end{aligned}$$

and we find

$$a_{2200}(a,b) = \frac{9}{16} \frac{a^2}{b^3} + O\left(\frac{a^4}{b^5}\right)$$
 for small a/b

which gives, for $a = a_i = \hat{a}\delta^{N-i}$, $b = a_j = \hat{a}\delta^{N-j}$ and i < j,

$$a_{2200}(a_i, a_j) = \frac{9}{16} \frac{\hat{a}_i^2}{\hat{a}_j^3} \delta^{3j - 2i - N} + O\left(\delta^{5j - 4i - N}\right) \quad \text{for} \quad i < j$$
hence, (3.64) follows, after multiplying by the factor

$$-\frac{\bar{m}_i \,\bar{m}_j}{\Lambda_i \,\Lambda_j} = -\frac{\bar{m}_i \bar{m}_j}{\tilde{m}_i \tilde{m}_j \sqrt{\hat{m}_i \hat{m}_j \hat{a}_i \hat{a}_j \delta^{2N-i-j}}}$$

5. Expansion of

$$r_{iijj} = -\bar{m}_i \,\bar{m}_j \frac{a_{0220}(a_j, a_i)}{\Lambda_i \,\Lambda_j} \qquad \text{for} \qquad i \neq j \;.$$

We expand the symmetric coefficient

$$\begin{aligned} a_{0220}(a,b) &= -\frac{3a}{512b^2} \left[(84(a/b)^5 - 8832(a/b)^3 \\ &+ 84(a/b)) \, b_{9/2,0}(a/b) - 8(5(a/b)^6 \\ &- 652(a/b)^4 - 652(a/b)^2 + 5) \, b_{9/2,1}(a/b) \\ &- 5(328(a/b)^5 - 561(a/b)^3 + 328(a/b)) \, b_{9/2,2}(a/b) \\ &+ (216(a/b)^6 - 1020(a/b)^4 \\ &- 1020(a/b)^2 + 216) \, b_{9/2,3}(a/b) \\ &+ (116(a/b)^5 + 200(a/b)^3 \\ &+ 116(a/b)) \, b_{9/2,4}(a/b) \\ &- (20(a/b)^4 + 20(a/b)^2) \, b_{9/2,5}(a/b) \\ &+ 3(a/b)^3 \, b_{9/2,6}(a/b) \right] . \end{aligned}$$

and we find

$$a_{0220}(a,b) = \begin{cases} +\frac{9}{16} \frac{a^2}{b^3} + O\left(\frac{a^4}{b^5}\right) & \text{for small} \quad a/b \\ \\ +\frac{9}{16} \frac{b^2}{a^3} + O\left(\frac{b^4}{a^5}\right) & \text{for small} \quad b/a \end{cases}$$

which gives
$$(3.66)$$
 for

$$r_{i,i,j,j} = -\frac{\bar{m}_i \, \bar{m}_j}{\Lambda_i \, \Lambda_j} a_{0220}(a_i, a_j)$$

6. Expansion of

$$r_{ijij} = \begin{cases} -\bar{m}_i \,\bar{m}_j \frac{a_{1111}(a_i, a_j)}{\Lambda_i \,\Lambda_j} & \text{for } i < j \\ 0 & \text{for } i > j \end{cases}$$

We expand the symmetric coefficient

$$a_{1111}(a,b) = \frac{a}{128b^2} \left[(-36(a/b)^5 - 7956(a/b)^3) \right]$$

$$\begin{array}{rcl} &-& 36(a/b)) \, b_{9/2,0}(a/b) + 8((a/b)^6 \\ &+& 828(a/b)^4 + 828(a/b)^2 + 1) \, b_{9/2,1}(a/b) \\ &+& (-3096(a/b)^5 + 1039(a/b)^3 \\ &-& 3096(a/b)) \, b_{9/2,2}(a/b) + (648(a/b)^6 \\ &-& 1332(a/b)^4 - 1332(a/b)^2 + 648) \, b_{9/2,3}(a/b) \\ &+& (348(a/b)^5 + 700(a/b)^3 \\ &+& 348(a/b)) \, b_{9/2,4}(a/b) - 60((a/b)^4 \\ &+& (a/b)^2) \, b_{9/2,5}(a/b) + 9(a/b)^3 \, b_{9/2,6}(a/b) \end{bmatrix}$$

The term of order (a^2/b^3) in the expansion of a_{1111} vanishes, so, we shall go on in the asymptotics for the involved Laplace Coefficients:

$$\begin{cases} b_{9/2,0}(\alpha) = 1 + \frac{81}{4} \alpha^2 + O(\alpha^4) \\ b_{9/2,1}(\alpha) = \frac{9}{2} \alpha + \frac{891}{16} \alpha^3 + O(\alpha^5) \\ b_{9/2,2}(\alpha) = \frac{99}{8} \alpha^2 + O(\alpha^4) \\ b_{9/2,3}(\alpha) = \frac{429}{16} \alpha^3 + O(\alpha^5) \\ b_{9/2,k}(\alpha) = O(\alpha^k) \text{ for } k \ge 4 . \end{cases}$$

We find, for small a/b,

$$\begin{aligned} a_{1111}(a,b) &= \left(-7956 - 36 \cdot \frac{81}{4} + 8 \cdot 828 \cdot \frac{9}{2} + 8 \cdot \frac{891}{16} - 3096 \cdot \frac{99}{8} + 648 \cdot \frac{429}{16}\right) \\ &\times \frac{a^4}{128 \, b^5} + O\left(\frac{a^6}{b^7}\right) \\ &= \frac{315}{64} \frac{a^4}{b^5} + O\left(\frac{a^6}{b^7}\right) , \end{aligned}$$

consequently, taking, for i < j, $a = a_i = \hat{a}_i \delta^{N-i}$, $b = a_j = \hat{a}_j \delta^{N-j}$, we find, for r_{ijij} the expansion (3.67). This completes the proof.

We are ready for the **proof of Lemma 3.9**.

For any i < j, the functions \tilde{q}_{iiii} , \cdots involved into equation (3.57) of Lemma 3.10 may be written as

$$\begin{cases}
\tilde{q}_{i,i,i} = q_{iiii}(1 + \kappa_i) \\
\tilde{r}_{i,i,i} = r_{iiii}(1 + \rho_i) \\
\tilde{q}_{iijj} = q_{iijj}(1 + \kappa_{ij}) \\
\tilde{q}_{jjii} = q_{iijj}\kappa_{ji} \\
\tilde{q}_{i,j,i,j} = q_{iijj}\hat{\kappa}_{ij} \\
\tilde{q}_{j,i,j,i} = q_{iijj}\hat{\kappa}_{ji} \\
\tilde{q}_{jij} = q_{iijj}\tilde{\kappa}_{ji} \\
\tilde{q}_{jij} = q_{iijj}\tilde{\kappa}_{ji} \\
\tilde{q}_{jij} = r_{iijj}(1 + \rho_{ij}) \\
\tilde{r}_{jjii} = r_{jjii}(1 + \rho_{ji})
\end{cases}$$
(3.70)

where

$$\begin{split} \kappa_{i} &:= \left(u_{i,i}^{4} - 1\right) + q_{i,i,i}^{-1} \left[\sum_{k \neq i} q_{k,k,k} \left(u_{k,i}\right)^{4} + \sum_{k \neq l} q_{k,k,k,l} \left(u_{k,i}\right)^{3} u_{l,i} \right. \\ &+ \sum_{k < l} q_{k,k,l,l} \left(u_{k,i}\right)^{2} \left(u_{l,i}\right)^{2} \right] \\ \rho_{i} &:= \left(u_{i,i}^{4} - 1\right) + r_{i,i,i,i}^{-1} \left[\sum_{k \neq i} r_{k,k,k,k} \left(u_{k,i}\right)^{4} + \sum_{k \neq l} r_{k,k,l,l} \left(u_{k,i}\right)^{2} \left(u_{l,i}\right)^{2} \right. \\ &+ \sum_{k \neq l} r_{k,l,k,k} \left(u_{k,i}\right)^{3} u_{l,i} + \sum_{k \neq l} r_{k,k,k,l} \left(u_{k,i}\right)^{3} u_{l,i} + \sum_{k < l} r_{k,l,k,l} \left(u_{k,i}\right)^{2} \left(u_{l,i}\right)^{2} \\ &+ \sum_{k \neq l} r_{k,l,k,k} \left(u_{k,i}\right)^{3} u_{l,i} + \sum_{k \neq l} r_{k,k,k,l} \left(u_{k,i}\right)^{3} u_{l,i} + \sum_{k < l} r_{k,l,k,l} \left(u_{k,i}\right)^{2} \left(u_{l,i}\right)^{2} \\ &+ \sum_{k \neq l} r_{k,l,k,k} \left(u_{k,i}\right)^{2} \left(u_{k,j}\right)^{2} + \sum_{k \neq l} q_{k,k,k,l} \left(u_{k,i}\right)^{2} \left(u_{l,j}\right)^{2} \\ &+ \sum_{k} q_{k,k,k,k} \left(u_{k,i}\right)^{2} \left(u_{k,j}\right)^{2} + \sum_{k \neq l} q_{k,k,k,l} \left(u_{k,i}\right)^{2} \left(u_{k,j}\right)^{2} \\ &+ \sum_{k \neq l} q_{k,k,k,k} \left(u_{k,j}\right)^{2} u_{k,i} u_{l,i} \right] \\ \kappa_{j,i} &:= q_{i,jj}^{-1} \left[\sum_{k = l} q_{k,k,k,l} \left(u_{k,j}\right)^{2} \left(u_{k,j}\right)^{2} + \sum_{k \neq l} q_{k,k,k,l} \left(u_{k,i}\right)^{2} u_{k,j} u_{l,j} \\ &+ \sum_{k < l} q_{k,k,k,l} \left(u_{k,i}\right)^{2} \left(u_{k,j}\right)^{2} + \sum_{k \neq l} q_{k,k,k,l} \left(u_{k,j}\right)^{2} u_{k,i} u_{l,i} \\ &+ \sum_{k < l} q_{k,k,k,l} \left(u_{k,i}\right)^{2} \left(u_{k,j}\right)^{2} + \sum_{k \neq l} q_{k,k,k,l} \left(u_{k,j}\right)^{2} u_{k,i} u_{l,i} \\ &+ \sum_{k < l} q_{k,k,k,l} \left(u_{k,i}\right)^{2} \left(u_{k,j}\right)^{2} + \sum_{k \neq l} q_{k,k,k,l} \left(u_{k,j}\right)^{2} u_{l,i} \\ &+ \sum_{k < l} q_{k,k,k,l} \left(u_{k,i}\right)^{2} \left(u_{k,j}\right)^{2} + \sum_{k \neq l} q_{k,k,k,l} \left(u_{k,j}\right)^{2} u_{l,i} \\ &+ \sum_{k < l} q_{k,k,k,l} \left(u_{k,i}\right)^{2} \left(u_{k,j}\right)^{2} + \sum_{k \neq l} q_{k,k,k,l} \left(u_{k,j}\right)^{2} u_{l,i} \\ &+ \sum_{k < l} q_{k,k,k,l} \left(u_{k,i}\right)^{2} \left(u_{k,j}\right)^{2} + \sum_{k \neq l} q_{k,k,k,l} \left(u_{k,j}\right)^{2} u_{l,i} \\ &+ \sum_{k < l} q_{k,k,k,l} \left(u_{k,i}\right)^{2} \left(u_{k,j}\right)^{2} + \sum_{k \neq l} q_{k,k,k,l} \left(u_{k,j}\right)^{2} u_{l,i} \\ &+ \sum_{k < l} q_{k,k,k,l} \left(u_{k,i}\right)^{2} \left(u_{k,j}\right)^{2} + \sum_{k \neq l} q_{k,k,k,l} \left(u_{k,j}\right)^{2} u_{l,i} \\ &+ \sum_{k < l} q_{k,k,k,l} \left(u_{k,i}\right)^{2} \left(u_{k,j}\right)^{2} \left(u_{k,j}\right)^{2} + \sum_{k \neq$$

$$+ \sum_{k < l} q_{k,k,l,l} u_{k,i} u_{k,j} u_{l,j} u_{l,i} \right]$$

$$\rho_{ij} := (u_{i,i})^{2} (u_{j,j})^{2} - 1$$

$$+ r_{iijj}^{-1} \left[\sum_{k \neq l, \ (k,l) \neq (i,j)} r_{k,k,l,l} (u_{k,i})^{2} (u_{l,j})^{2} + \sum_{k} r_{k,k,k,k} (u_{k,i})^{2} (u_{k,j})^{2} \right]$$

$$+ \sum_{k \neq l} r_{k,l,k,k} u_{k,i} u_{l,i} (u_{k,j})^{2} + \sum_{k \neq l} r_{k,k,k,l} (u_{k,i})^{2} u_{k,j} u_{l,j}$$

$$+ \sum_{k < l} r_{k,l,k,l} u_{k,i} u_{l,i} u_{k,j} u_{l,j} \right]$$

$$\rho_{ji} := (u_{i,i})^{2} (u_{j,j})^{2} - 1$$

$$+ r_{jjii}^{-1} \left[\sum_{k \neq l, \ (k,l) \neq (j,i)} r_{k,k,l,l} (u_{k,j})^{2} (u_{l,i})^{2} + \sum_{k} r_{k,k,k,k} (u_{k,j})^{2} (u_{k,i})^{2} \right]$$

$$+ \sum_{k \neq l} r_{k,l,k,k} u_{k,j} u_{l,j} (u_{k,i})^{2} + \sum_{k \neq l} r_{k,k,k,l} (u_{k,j})^{2} u_{k,i} u_{l,i}$$

$$+ \sum_{k < l} r_{k,l,k,l} u_{k,i} u_{l,i} u_{k,j} u_{l,j} \right]$$

$$(3.71)$$

Let us prove, for istance, the first in (3.70), with κ_i as in (3.71) (since the other ones equalities are similar). Taking into account only the non vanishing components of $Q = q_{i,j,k,l}$, we write $\tilde{q}_{i,i,i,i}$ as

$$\widetilde{q}_{i,i,i,i} = \sum_{i',j',k',l'} q_{i',j',k',l'} u_{i',i} u_{j',i} u_{k',i} u_{l',i}
= \sum_{k} q_{k,k,k,k} (u_{k,i})^4 + \sum_{k \neq l} q_{k,k,k,l} (u_{k,i})^3 u_{l,i} + \sum_{k < l} q_{k,k,l,l} (u_{k,i})^2 (u_{l,i})^2
= q_{i,i,i,i} + \left[q_{i,i,i,i} (u_{i,i}^4 - 1) + \sum_{k \neq i} q_{k,k,k,k} (u_{k,i})^4 + \sum_{k \neq l} q_{k,k,k,l} (u_{k,i})^3 u_{l,i}
+ \sum_{k < l} q_{k,k,l,l} (u_{k,i})^2 (u_{l,i})^2 \right]
= q_{i,i,i,i} (1 + \kappa_i) .$$
(3.72)

<u>Claim</u>: The functions κ_i , ρ_i , κ_{ij} , \cdots defined in (3.71) are $O(\delta)$.

This claim follows by direct computation (throughout the asymptotics for $\mathcal{Q}, \mathcal{R}, U$ given in Lemmas 3.6, 3.11) of the order of the functions appearing in the summations defining the functions κ_i , ρ_i , κ_{ij} , \cdots . The details of this computation, for sake of continuity, are postponed at the end of the present proof: an inspection of such orders, however, shows that they never go under $O(\delta)$. To conclude the proof, we use Lemma 3.10. Using equation (3.70) into equation (3.57) and using the asymptotics (3.63), (3.65) for q_{iiii} , r_{iiii} , we find, for the diagonal elements of A, the asyptotics

$$\begin{split} A_{i,i} &= 6 \, \tilde{q}_{i,i,i,i} + \tilde{r}_{i,i,i,i} \\ &= 6 \, q_{i,i,i,i} (1+\kappa_i) + r_{i,i,i,i} (1+\rho_i) \\ &= (6 \, q_{i,i,i,i} + r_{i,i,i,i}) (1+\alpha_{i,i}) \\ &= (1+\alpha_i) \times \begin{cases} + \frac{3 \, m_1 \, m_2}{4 \, \mu_1^2 \, M_1} \, \frac{(\hat{a}_1/\hat{a}_2)^2}{\hat{a}_1 \hat{a}_2} \, \delta^{5-2N} & \text{for} \quad i=1 \ ; \\ - \frac{3 \, m_i \, m_{i-1}}{\mu_i^2 \, M_i} \, \frac{(\hat{a}_{i-1}/\hat{a}_i)^2}{\hat{a}_i^2} \, \delta^{2i+2-2N} & \text{for} \quad 2 \le i \le N \end{cases} \end{split}$$

where

$$\alpha_{i,i} := \frac{6 \, q_{i,i,i,i} \kappa_i + r_{i,i,i,i} \rho_i}{6 \, q_{i,i,i,i} + r_{i,i,i,i}}$$

is an $O(\delta)$. Similarly, taking into account the asymptotics (3.64), (3.66) for q_{iijj} , r_{iijj} , for for the upper diagonal elements of A, we find

$$\begin{aligned} A_{i,j}(\Lambda) &= 2\tilde{q}_{i,i,j,j} + 2\tilde{q}_{j,j,i,i} + 2\tilde{q}_{i,j,i,j} + 2\tilde{q}_{j,i,j,i} + 2\tilde{q}_{i,j,j,i} + 2\tilde{q}_{j,i,i,j} + \tilde{r}_{i,i,j,j} + \tilde{r}_{j,j,i,i} \\ &= 2q_{iijj}(1+\kappa_{ij}) + r_{i,i,j,j}(1+\rho_{ij}) + r_{j,j,i,i}(1+\rho_{ji}) \\ &+ q_{iijj}(2\kappa_{ji} + 2\hat{k}_{ij} + 2\hat{k}_{ji} + 2\check{k}_{ij} + 2\check{k}_{ji}) \\ &= (2q_{iijj} + 2r_{iijj})(1+\alpha_{ij}) \\ &= -\frac{9}{4}\frac{m_i m_j}{\mu_i \mu_j \sqrt{M_i M_j \hat{a}_i \hat{a}_j}} \frac{\hat{a}_i^2}{\hat{a}_j^3} \delta^{(7j-3i-4N)/2}(1+\alpha_{ij}) \qquad (i < j) \end{aligned}$$

(recall $r_{iijj} = r_{jjii}$) with

$$\alpha_{ij} = \frac{(2\kappa_{ji} + 2\hat{k}_{ij} + 2\hat{k}_{ji} + 2\check{k}_{ij} + 2\check{k}_{ji})q_{iijj} + (\rho_{ij} + \rho_{ji})r_{iijj}}{2\kappa_{ij} + 2q_{iijj} + 2r_{iijj}}$$

an $O(\delta)$ again.

<u>Proof of the Claim</u>: Using the asymptotics for \mathcal{Q} , \mathcal{R} , U given in Lemmas 3.6, 3.11, by direct check, that

1. for $1 \leq i \leq N$,

$$(u_{i,i})^4 - 1$$
, $u_{i,i}^2 u_{j,j}^2 - 1 = O(\delta^{5/2})$;

2. for $k \neq i$

$$\frac{q_{kkkk}}{q_{iiii}}(u_{ki})^4 , \frac{r_{kkkk}}{r_{iiii}}(u_{ki})^4 =$$

$$\begin{array}{ll} O(\delta^{19k-32}) & \quad \mbox{if } 1 = i < k \\ O(\delta^{19(k-i)-8}) & \quad \mbox{if } 1 < i < k \\ O(\delta^{15i-26}) & \quad \mbox{if } 1 = k < i \\ O(\delta^{15(i-k)-8}) & \quad \mbox{if } 1 < k < i \end{array}$$

3. for $k \neq l$, $\frac{q_{kkk}}{q_{iiii}}$

$$\begin{aligned} \frac{kkkl}{iiii} (u_{ki})^3 (u_{li}) , \ \frac{r_{kkkl}}{r_{iiii}} (u_{ki})^3 (u_{li}) , \ \frac{r_{klkk}}{r_{iiii}} (u_{ki})^3 (u_{li}) &= \\ & O(\delta^{7k-8}) & \text{if } 1 = l < i = k \\ & O(\delta^{7(k-l)-4}) & \text{if } 1 < l < i = k \\ & O(\delta^{(17l-29)/2}) & \text{if } i = k = 1 < l \\ & O(\delta^{(17l-29)/2}) & \text{if } i = k = 1 < l \\ & O(\delta^{(17l-k)-8)/2}) & \text{if } 1 < i = k < l = i \\ & O(\delta^{(15l-26)}) & \text{if } 1 = k < l = i \\ & O(\delta^{(15l-15k-8)}) & \text{if } 1 < k < l = i \\ & O(\delta^{(35(k-l)-16)/2}) & \text{if } 1 < i = l < k \\ & O(\delta^{(35(k-l)-16)/2}) & \text{if } 1 = i = l < k \\ & O(\delta^{(35(k+3l-68)/2)}) & \text{if } 1 = i < k < l \\ & O(\delta^{(30(i-k)+17(l-i)-20)/2}) & \text{if } 1 < i < k < l \\ & O(\delta^{(15(i-l)+15(l-k)-10}) & \text{if } 1 < k < l < i \\ & O(\delta^{(35(k-l)+38(l-i)-20)/2}) & \text{if } 1 < l < k < i < l \\ & O(\delta^{(35(k-l)+38(l-i)-20)/2}) & \text{if } 1 < l < k < i \\ & O(\delta^{(35(k-l)+38(l-i)-20)/2}) & \text{if } 1 < l < k < i \\ & O(\delta^{(35(k-l)+38(l-i)-20)/2}) & \text{if } 1 < l < k < i \\ & O(\delta^{(10(k-i)+7(i-l)-10}) & \text{if } 1 < l < k < i \\ & O(\delta^{(10(k-i)+7(i-l)-10}) & \text{if } 1 < l < k < i \\ & O(\delta^{(10(k-i)+7(i-l)-10}) & \text{if } 1 < l < k < i \\ & O(\delta^{(10(k-i)+7(i-l)-10}) & \text{if } 1 < l < k < i \\ & O(\delta^{(10(k-i)+7(i-l)-10}) & \text{if } 1 < l < k < i \\ & O(\delta^{(10(k-i)+7(i-l)-10}) & \text{if } 1 < l < k \\ & O(\delta^{(10(k-i)+7(i-l)-10}) & \text{if } 1 < l < k \\ & O(\delta^{(10(k-i)+7(i-l)-10}) & \text{if } 1 < l < k \\ & O(\delta^{(10(k-i)+7(i-l)-10}) & \text{if } 1 < l < k \\ & O(\delta^{(10(k-i)+7(i-l)-10}) & \text{if } 1 < l < k \\ & O(\delta^{(10(k-i)+7(i-l)-10}) & \text{if } 1 < l < k \\ & O(\delta^{(10(k-i)+7(i-l)-10}) & \text{if } 1 < l < k \\ & O(\delta^{(10(k-i)+7(i-l)-10}) & \text{if } 1 < l < k \\ & O(\delta^{(10(k-i)+7(i-l)-10}) & \text{if } 1 < l < k \\ & O(\delta^{(10(k-i)+7(i-l)-10}) & \text{if } 1 < l < k \\ & O(\delta^{(10(k-i)+7(i-l)-10}) & \text{if } 1 < l < k \\ & O(\delta^{(10(k-i)+7(i-l)-10}) & \text{if } 1 < l < k \\ & O(\delta^{(10(k-i)+7(i-l)-10}) & \text{if } 1 < l < k \\ & O(\delta^{(10(k-i)+7(i-l)-10}) & \text{if } 1 < l < k \\ & O(\delta^{(10(k-i)+7(k-l)-10}) & \text{if } 1 < l < k \\ & O(\delta^{(10(k-i)+7(k-l)-10}) & \text{if } 1 < l < k \\ & O(\delta^{(10(k-i)+7(k-l)-10}) & \text{if } 1 < l < k \\ & O(\delta^{(10$$

4. for k < l,

$$\begin{aligned} \frac{q_{kkll}}{q_{iiii}} (u_{ki})^2 (u_{li})^2 , & \frac{r_{kkll}}{r_{iiii}} (u_{ki})^2 (u_{li})^2 &= \\ & O(\delta^{12l-23}) & \text{if } 1 = i = k < l \\ & O(\delta^{12(l-i)-6}) & \text{if } 1 < i = k < l \\ & O(\delta^{10(i-k)-6}) & \text{if } 1 < k < i = l \\ & O(\delta^{10i-18}) & \text{if } 1 = k < i = l \\ & O(\delta^{12l+7k-34}) & \text{if } 1 = i < k < l \\ & O(\delta^{12(l-k)+19(k-i)-10}) & \text{if } 1 < i < k < l \end{aligned}$$

$$\begin{array}{ll} O(\delta^{12(l-i)+10i-22}) & \text{if } 1=k < i < l \\ O(\delta^{12(l-i)+10(i-k)-10}) & \text{if } 1 < k < i < l \\ O(\delta^{15(i-l)+10l-22}) & \text{if } 1=k < l < i \\ O(\delta^{15(i-l)+10(l-k)-10}) & \text{if } 1 < k < l < i \end{array}$$

5. for k < l

$$\begin{array}{rl} \frac{r_{klkl}}{r_{iiii}}(u_{ki})^2(u_{li})^2 &= \\ & O(\delta^{17l-23}) & \text{if } 1=i=k < l \\ & O(\delta^{14(l-i)-6}) & \text{if } 1 < k=i < l \\ & O(\delta^{14i-22}) & \text{if } 1=k < i=l \\ & O(\delta^{14(i-k)-6}) & \text{if } 1 < k < i=l \\ & O(\delta^{5k+14l-34}) & \text{if } 1=i < k < l \\ & O(\delta^{14(l-k)+17(k-i)-10}) & \text{if } 1 < i < k < l \\ & O(\delta^{14(l-i)+12i-24}) & \text{if } 1=k < i < l \\ & O(\delta^{14(l-i)+12(i-k)-10}) & \text{if } 1 < k < i < l \\ & O(\delta^{15(i-l)+12l-24}) & \text{if } 1=k < l < i \end{array}$$

6. for $i < j, k < l, (i, j) \neq (k, l)$

$$\begin{array}{lll} \frac{q_{kkll}}{q_{iijj}}(u_{k,i})^2(u_{l,j})^2, \ \frac{r_{kkll}}{r_{iijj}}(u_{k,i})^2(u_{l,j})^2 & = \\ & O(\delta^{12(l-k)+19(k-j)+7(j-17)} & \text{if } 1=i < j < k < l \\ O(\delta^{12(l-k)+19(k-j)+7(j-i)-8}) & \text{if } 1 < i < j < k < l \\ O(\delta^{12(l-j)+7(j-i)-8}) & \text{if } 1 < i < k = j < l \\ O(\delta^{12(l-j)+7(j-i)-8}) & \text{if } 1 < i < k = j < l \\ O(\delta^{12(l-j)+7(j-i)-8}) & \text{if } 1 < i < k < j < l \\ O(\delta^{12(l-j)+7(j-k)-8}) & \text{if } 1 < i < k < j < l \\ O(\delta^{12(l-j)+7(j-k)-8}) & \text{if } 1 < i < k < l = j \\ O(\delta^{12(l-j)+7(j-k)-8}) & \text{if } 1 < i < k < l = j \\ O(\delta^{7k-13}) & \text{if } 1 < i < k < l = j \\ O(\delta^{5(j-l)+7k-17}) & \text{if } 1 < i < k < l < j \\ O(\delta^{5(j-l)+7k-17}) & \text{if } 1 < i < k < l < j \\ O(\delta^{5(j-l)+7(k-i)-8}) & \text{if } 1 < i < k < l < j \\ O(\delta^{5(j-l)+7(k-i)-8}) & \text{if } 1 < i < k < l < j \\ O(\delta^{5(j-l)-4}) & \text{if } 1 < k = i < j < l \\ O(\delta^{5(j-l)-4}) & \text{if } 1 < k = i < l < j \\ O(\delta^{5(j-l)-4}) & \text{if } 1 < k = i < l < j \\ O(\delta^{5(j-l)-4}) & \text{if } 1 < k = i < l < j \\ O(\delta^{5(j-l)-4}) & \text{if } 1 < k = i < l < j \\ O(\delta^{5(j-l)-4}) & \text{if } 1 < k = i < l < j \\ O(\delta^{5(j-l)-4}) & \text{if } 1 < k < i < l < j \\ O(\delta^{5(j-l)-4}) & \text{if } 1 < k < i < l < j \\ O(\delta^{5(j-l)-4}) & \text{if } 1 < k < i < l < j \\ O(\delta^{5(j-l)-4}) & \text{if } 1 < k < i < l < j \\ O(\delta^{5(j-l)-4}) & \text{if } 1 < k < i < l < j \\ O(\delta^{5(j-l)-4}) & \text{if } 1 < k < i < l < j \\ O(\delta^{5(j-l)-4}) & \text{if } 1 < k < i < l < j \\ O(\delta^{5(j-l)-4}) & \text{if } 1 < k < i < l < j \\ O(\delta^{5(j-l)-4}) & \text{if } 1 < k < i < l < j \\ O(\delta^{5(j-l)-4}) & \text{if } 1 < k < i < l < j \\ O(\delta^{5(j-l)-4}) & \text{if } 1 < k < i < l < j \\ O(\delta^{5(j-l)-4}) & \text{if } 1 < k < i < l < j \\ O(\delta^{5(j-l)-4}) & \text{if } 1 < k < i < l < j \\ O(\delta^{5(j-l)-4}) & \text{if } 1 < k < i < l < j \\ O(\delta^{5(j-l)-4}) & \text{if } 1 < k < i < l < j \\ O(\delta^{5(j-l)-4}) & \text{if } 1 < k < i < l < j \\ O(\delta^{5(j-l)-4}) & \text{if } 1 < k < i < l < j \\ O(\delta^{5(j-l)-4}) & \text{if } 1 < k < i < l < j \\ O(\delta^{5(j-l)-4}) & \text{if } 1 < k < i < l < j \\ O(\delta^{5(j-l)-4}) & \text{if } 1 < k < i < l < l < j \\ O(\delta^{5(j-l)-4}) & \text{if } 1 < k < i < l < l < l \\ O(\delta^{5(j-l)-4}) & \text{if } 1 < k < i$$

$$\begin{array}{ll} O(\delta^{12(l-j)+10(i-k)-8}) & \text{if } 1 < k < i < j < l \\ O(\delta^{10i-16}) & \text{if } 1 = k < i < l = j \\ O(\delta^{5(j-l)+10(i-k)-8}) & \text{if } 1 < k < i < l = j \\ O(\delta^{5(j-l)+10(i-k)-8}) & \text{if } 1 < k < i < l = j \\ O(\delta^{5(j-l)+10i-20}) & \text{if } 1 < k < i < l = j \\ O(\delta^{5(j-i)+10i-20}) & \text{if } 1 = k < i < l < j \\ O(\delta^{5(j-i)+10(i-k)-8}) & \text{if } 1 < k < l = i < j \\ O(\delta^{5(j-i)+15(i-l)+10(l-k)-8}) & \text{if } 1 = k < l < i < j \\ O(\delta^{5(j-i)+15(j-l)+10(l-k)-8}) & \text{if } 1 < k < l < i < j \\ \end{array}$$

7. for i < j, k < l, $\frac{q_{kkll}}{q_{iijj}}$

$$O(\delta^{5(j-i)+10(i-k)-4}) \quad \text{if } 1 < k < l = i < j$$

$$O(\delta^{5(j-i)+15(i-l)+10l-20}) \quad \text{if } 1 = k < l < i < j$$

$$O(\delta^{5(j-i)+15(j-l)+10(l-k)-8}) \quad \text{if } 1 < k < l < i < j$$

8. for i < j,

$$\begin{aligned} \frac{q_{kkkk}}{q_{iijj}} (u_{ki})^2 (u_{kj})^2, & \frac{r_{kkkk}}{r_{iijj}} (u_{ki})^2 (u_{kj})^2 &= \\ & O(\delta^{5j-8}) & \text{if } 1 = k = i < j \\ & O(\delta^{5(j-i)-2}) & \text{if } 1 < k = i < j \\ & O(\delta^{7(j-i)-2}) & \text{if } k = j > i > 1 \\ & O(\delta^{7j-11}) & \text{if } k = j > i = 1 \\ & O(\delta^{10i+5j-25}) & \text{if } 1 = k \neq i, j \\ & O(\delta^{12(k-j)+7(k-i)-6}) & \text{if } k > j, i > 1 \\ & O(\delta^{12(k-j)+7k-14}) & \text{if } k > j, i = 1 \\ & O(\delta^{7(k-i)+5(j-k)-6}) & \text{if } 1 < i < k < j, k > 1 \\ & O(\delta^{10(i-k)+5(j-k)-6}) & \text{if } 1 < i < k < j, k > 1 \\ & O(\delta^{10(i-k)+5(j-k)-6}) & \text{if } 1 < k < i \end{aligned}$$

9. for $i < j, k \neq l$,

$$\begin{array}{ll} \frac{q_{kkkl}}{q_{iijj}}(u_{ki})^2 u_{kj} u_{lj}, \ \frac{q_{klkl}}{r_{iijj}}(u_{ki})^2 u_{kj} u_{lj} &= \\ & O(\delta^{5j-8}) & \text{if } 1=k=i < j=l \\ & O(\delta^{5(j-k)-2}) & \text{if } 1< k=i < j=l \\ & O(\delta^{10i+5j-24}) & \text{if } 1=k < i < j=l \\ & O(\delta^{15(i-k)+5(j-i)-6} & \text{if } 1=k < i < j=l \\ & O(\delta^{2k+5j-15}) & \text{if } 1=i < k < j=l \\ & O(\delta^{5(j-k)+7(k-i)-6}) & \text{if } 1< i < k < j=l \\ & O(\delta^{(35(k-j)+14j-30)/2}) & \text{if } 1=i < j, \ k>j=l \\ & O(\delta^{(35(k-j)+14(j-i)-12)/2}) & \text{if } 1=i < j, \ k>j=l \\ & O(\delta^{(5j-i)+7(i-l)-4}) & \text{if } 1< i < k < j =l \\ & O(\delta^{5(j-i)+7(i-l)-4}) & \text{if } 1< i < k < j =l \\ & O(\delta^{(5(j-i)+7(i-l)-4}) & \text{if } 1< i < k < j =l \\ & O(\delta^{(5(j-i)+7(i-l)-4}) & \text{if } 1< i < k < j \\ & O(\delta^{(5(j-i)-4)}) & \text{if } 1< i < k < l < j \\ & O(\delta^{(17(l-j)+10(j-2))/2}) & \text{if } 1< i = k < l < j \\ & O(\delta^{(17(l-j)+10(j-i)-8)/2}) & \text{if } 1< i = k < j < l \\ & O(\delta^{(17(l-j)+10(j-i)-8)/2}) & \text{if } 1< i = k < j < l \\ & O(\delta^{(17(l-j)+10(j-i)-8)/2}) & \text{if } 1< i = k < j < l \\ & O(\delta^{(17(l-j)+10(j-i)-8)/2}) & \text{if } 1< i = k < j < l \\ & O(\delta^{(17(l-j)+10(j-i)-8)/2}) & \text{if } 1< i = k < j < l \\ & O(\delta^{(17(l-j)+10(j-i)-8)/2}) & \text{if } 1< i = k < j < l \\ & O(\delta^{(17(l-j)+10(j-i)-8)/2}) & \text{if } 1< i = k < j < l \\ & O(\delta^{(17(l-j)+10(j-i)-8)/2}) & \text{if } 1< i = k < j < l \\ & O(\delta^{(17(l-j)+10(j-i)-8)/2}) & \text{if } 1< i = k < j < l \\ & O(\delta^{(17(l-j)+10(j-i)-8)/2}) & \text{if } 1< i = k < j < l \\ & O(\delta^{(17(l-j)+10(j-i)-8)/2}) & \text{if } 1< i = k < j < l \\ & O(\delta^{(17(l-j)+10(j-i)-8)/2}) & \text{if } 1< i = k < j < l \\ & O(\delta^{(17(l-j)+10(j-i)-8)/2}) & \text{if } 1< i = k < j < l \\ & O(\delta^{(17(l-j)+10(j-i)-8)/2}) & \text{if } 1< i = k < j < l \\ & O(\delta^{(17(l-j)+10(j-i)-8)/2}) & \text{if } 1< i = k < j < l \\ & O(\delta^{(17(l-j)+10(j-i)-8)/2}) & \text{if } 1< i = k < j < l \\ & O(\delta^{(17(l-j)+10(j-i)-8)/2}) & \text{if } 1< i < k < j < l \\ & O(\delta^{(17(l-j)+10(j-i)-8)/2}) & \text{if } 1< i < k < j < l \\ & O(\delta^{(17(l-j)+10(j-i)-8)/2}) & \text{if } 1< i < k < j < l \\ & O(\delta^{(17(l-j)+10(j-i)-8)/2}) & \text{if } 1< i < k < j < l \\ & O(\delta^{(17(l-j)+10(j-i)-8)/2}) & \text{if } 1< i < k < j < l \\ & O($$

 $O(\delta^{14(j-i)+7i-14})$

 $O(\delta^{14j-23}$

 $O(\delta^{(14(j-i)-6)})$

 $O(\delta^{14(j-l)+7l-13})$

 $O(\delta^{14(j-l)+7(l-i)-6}$

 $O(\delta^{(17(l-j)+14j-30)/2})$

 $O(\delta^{(17(l-j)+14(j-i)-12)/2}$

 $O(\delta^{14(j-i)+7(i-l)-6})$

if 1 = l = i < k = j

if 1 < l = i < k = j

if 1 = i < l < k = j

if 1 < i < l < k = j

if 1 = i < k = j < l

if 1 < i < k = j < l

if 1 < k = i < j = l

 $O(\delta^{(1)})$ $O(\delta^{(17(l-j)+27(j-k)+14(k-i)-16)/2}) \\ O(\delta^{(18(k-j)+11j-25)/2})$ if 1 < i < k < j < lif 1 = i = l < j < k

$$\begin{array}{ll} O(\delta^{(27(j-i)+30i-52)/2}) & \text{if } 1=k < i < j=l \\ O(\delta^{(27(j-i)+30(i-k)-12)/2} & \text{if } 1< k < i < j=l \\ O(\delta^{(27(j-k)+14(k-i)-16)/2}) & \text{if } 1=i < k < j=l \\ O(\delta^{(27(j-k)+14(k-i)-16)/2}) & \text{if } 1=i < k < j=l \\ O(\delta^{(27(j-k)+14(k-i)-16)/2}) & \text{if } 1=i < j=l < k \\ O(\delta^{(55(k-j)+14j-34)/2}) & \text{if } 1=i < j=l < k \\ O(\delta^{(55(k-j)+14j-34)/2}) & \text{if } 1=i < k < j \\ O(\delta^{(55(k-j)+14j-34)/2}) & \text{if } 1=i < k < j \\ O(\delta^{(55(j-i)+7(i-l)-6}) & \text{if } 1< l < i = k < j \\ O(\delta^{(55(j-i)+7(i-l)-6}) & \text{if } 1< l < i = k < j \\ O(\delta^{(10(j-1)+27(j-i)-12)/2}) & \text{if } 1=i = k < l < j \\ O(\delta^{(11(j-1)+27(j-i)-12)/2}) & \text{if } 1=i = k < j < l \\ O(\delta^{(11(j-1)+27(j-i)-12)/2}) & \text{if } 1=i < k < j < l \\ O(\delta^{(11(j-1)+44(i-1)-8}) & \text{if } 1< l < i < k = j \\ O(\delta^{(11(j-1)+44(i-1)-8}) & \text{if } 1< l < i < k = j \\ O(\delta^{(11(j-1)+44(i-1)-8}) & \text{if } 1< l < i < k = j \\ O(\delta^{(11(j-1)+44(i-1)-8)/2} & \text{if } 1=i < k < j < l \\ O(\delta^{(11(j-1)+44(j-i)-8)/2} & \text{if } 1=i < k < j < l \\ O(\delta^{(11(j-1)+44(j-i)-8)/2} & \text{if } 1< i < k < j < l \\ O(\delta^{(17(l-j)+14(j-i)-8)/2} & \text{if } 1< i < k < j < l \\ O(\delta^{(17(l-j)+14(j-i)-8)/2} & \text{if } 1< k < i < j < l \\ O(\delta^{(5(j-i)+15(i-k)+7(k-1)-8}) & \text{if } 1< l < k < i < j \\ O(\delta^{(5(j-i)+15(i-k)+7(k-1)-8}) & \text{if } 1< l < k < i < j \\ O(\delta^{(5(j-i)+15(i-k)+7(k-1)-8}) & \text{if } 1< l < k < i < j \\ O(\delta^{(5(j-i)+15(i-k)+7(k-1)-8}) & \text{if } 1< k < i < j, l \neq j, k \\ O(\delta^{(5(j-i)+15(i-k)+7(k-1)-8}) & \text{if } 1< k < i < j, l > k, l \neq j \\ O(\delta^{(10(j-k)+11(k-i)+14(k-2)-2)/2}) & \text{if } 1=i < k < j \\ O(\delta^{(10(j-k)+11(k-i)+14(k-2)-2)/2} & \text{if } 1=i < k < j \\ O(\delta^{(10(j-k)+11(k-i)+14(k-2)-2)/2}) & \text{if } 1=i < k < j \\ O(\delta^{(10(j-k)+11(k-i)+14(k-2)-2)/2}) & \text{if } 1=i < k < j \\ O(\delta^{(10(j-k)+11(k-i)+14(k-2)-16)/2}) & \text{if } 1=i < k < j \\ O(\delta^{(10(j-k)+11(k-i)+14(k-1)-16)/2}) & \text{if } 1=i < k < j \\ O(\delta^{(10(j-k)+11(k-i)+14(k-1)-16)/2}) & \text{if } 1< i < k < i < j \\ O(\delta^{(10(j-k)+11(k-i)+14(k-1)-16)/2}) & \text{if } 1< i < k < i < j \\ O(\delta^{(10(j-k)+11(k-i)+14(k-i)-16)/2}) & \text{if } 1< i < k < i < j \\ O(\delta^{(10(j-k)+11(k-i)+14(k$$

$O(\delta^{(35(k-j)+14j-38)/2})$	if $1 = i < l = j < k$
$O(\delta^{(35(k-j)+11(j-l)+14l-34)/2})$	if $1 = i < l < j < k$
$O(\delta^{(35(k-l)+38(l-j)+14j-34)/2)})$	if $1 = i < j < l < k$
$O(\delta^{(17(l-k)+38(k-j)+14j-34)/2})$	if $1 = i < j < k < l$
$O(\delta^{(35(k-j)+11(j-i)+14i-32)/2})$	if $1 = l < i < j < k$
$O(\delta^{(35(k-j)+11(j-i)-12)/2})$	$if \ 1 < i = l \le j < k$
$O(\delta^{(35(k-j)+28(j-l)+14(l-i)-20)/2})$	if $1 < i < l = j < k$
$O(\delta^{(35(k-j)+11(j-l)+14(l-i)-16)/2})$	if $1 < i \le l \le j < k$
$O(\delta^{(35(k-j)+11(j-i)+14(i-l)-16)/2})$	if $1 < l < i < j < k$
$O(\delta^{(35(k-l)+38(l-j)+14(j-i)-16)/2})$	if $1 < i < j < l < k$
$O(\delta^{(17(l-k)+38(k-j)+14(j-i)-16)/2})$	if $1 < i < j < k < l$

11. for i < j, k < l,

$$\begin{array}{ll} \frac{q_{kkll}}{q_{iijj}} u_{ki} u_{kj} u_{li} u_{lj} \ , \ \delta^{-2(l-k)} \frac{r_{klkl}}{r_{iijj}} u_{ki} u_{kj} u_{li} u_{lj} \ = \\ O(\delta^{12(l-k)+19(k-j)+7j-17}) & \text{if } 1 = i < j < k < l \\ O(\delta^{12(l-k)+19(k-j)+7j-17}) & \text{if } 1 = i < j < k < l \\ O(\delta^{12(l-j)+7j-15}) & \text{if } 1 = i < k = j < l \\ O(\delta^{12(l-j)+7(j-i)-6}) & \text{if } 1 < i < k = j < l \\ O(\delta^{(19(l-j)+12(j-k)+9k-34)/2})) & \text{if } 1 = i < k < j < l \\ O(\delta^{(19(l-j)+12(j-k)+9k-34)/2})) & \text{if } 1 = i < k < j < l \\ O(\delta^{(12(l-j)+17(j-k)+14(k-i)-16)/2}) & \text{if } 1 < i < k < l = j \\ O(\delta^{(10(j-l)+17(j-k)+14(k-i)-12)/2}) & \text{if } 1 < i < k < l = j \\ O(\delta^{(10(j-l)+17(l-k)+14(k-i)-16)/2}) & \text{if } 1 < i < k < l < j \\ O(\delta^{(10(j-l)+17(l-k)+14(k-i)-16)/2}) & \text{if } 1 < i < k < l < j \\ O(\delta^{(10(j-l)+17(l-k)+14(k-i)-16)/2}) & \text{if } 1 < i < k < l < j \\ O(\delta^{(10(j-l)+17(l-k)+14(k-i)-16)/2}) & \text{if } 1 < k < i < j < l \\ O(\delta^{(24(l-j)+17(j-i)-12)/2}) & \text{if } 1 < k = i < j < l \\ O(\delta^{(10(j-l)+17(l-k)+14(k-i)-16)/2}) & \text{if } 1 < k = i < j = l \\ O(\delta^{(10(j-l)+17(l-i)+20(k-k)-12)/2}) & \text{if } 1 < k = i < l < j \\ O(\delta^{(10(j-l)+17(l-i)+20(k-k)-16)/2}) & \text{if } 1 < k < i < j < l \\ O(\delta^{(24(l-j)+17(l-i)+20(k-k)-16)/2}) & \text{if } 1 < k < i < j < l \\ O(\delta^{(10(j-l)+17(l-i)+20(k-k)-16)/2}) & \text{if } 1 < k < i < j < l \\ O(\delta^{(17(j-i)+20(k-k)-12)/2}) & \text{if } 1 < k < i < j < l \\ O(\delta^{(17(j-i)+20(k-k)-12)/2}) & \text{if } 1 < k < i < j < l \\ O(\delta^{(17(j-i)+20(k-k)-12)/2}) & \text{if } 1 < k < i < j < l \\ O(\delta^{(17(j-i)+20(k-k)-12)/2}) & \text{if } 1 < k < i < j < l \\ O(\delta^{(17(j-i)+20(k-k)-12)/2}) & \text{if } 1 < k < i < l < j \\ O(\delta^{(17(j-i)+20(k-k)-12)/2}) & \text{if } 1 < k < i < l < j \\ O(\delta^{(17(j-i)+20(k-k)-12)/2}) & \text{if } 1 < k < i < l < j \\ O(\delta^{(17(j-i)+20(k-k)-12)/2}) & \text{if } 1 < k < i < l < j < l \\ O(\delta^{(10(j-7(k-1)+17(k-k)-12)/2}) & \text{if } 1 < k < i < l < j \\ O(\delta^{(10(j-7(k-1)+17(k-k)-12)/2}) & \text{if } 1 < k < i < l < j \\ O(\delta^{(10(j-7(k-1)+17(k-k)-12)/2}) & \text{if } 1 < k < i < l < j < l \\ O(\delta^{(17(j-k)+17(k-k)-12)/2}) & \text{if } 1 < k < i < l < j \\ O(\delta^{(17(j-k)+17(k-k)-12)/2}) & \text{if } 1 < k$$

if $1 < k < i < l < j$
if $1 = k < l = i < j$
if $1 < k < l = i < j$
if $1 = k < l < i < j$
if $1 < k < l < i < j$

4 Deprit Variables for the Spatial Planetary Problem

Consider the Spatial Planetary Problem

$$\mathcal{H}_{\text{plt}}(\mu; y, x) = \sum_{1 \le i \le N} \left(\frac{|y_i|^2}{2\tilde{m}_i} - \frac{\hat{m}_i \tilde{m}_i}{|x_i|} \right) + \mu \sum_{1 \le i < j \le N} \left(\frac{y_i \cdot y_j}{\bar{m}_0} - \frac{\bar{m}_i \bar{m}_j}{|x_i - x_j|} \right)$$
(4.1)

where $(y, x) = ((y_1, \dots, y_N), (x_1, \dots, x_N))$ varies in the 6N-dimensional collisionless phase space

$$\mathcal{C}_{\mathrm{cl},3} := \left\{ y', \ x' \in \mathbb{R}^{3N} : \ x'_i \neq x'_j \neq 0 \quad \forall \ 1 \le i < j \le N \right\}$$

and, as usual,

$$\hat{m}_i = \bar{m}_0 + \mu \, \bar{m}_i \quad \tilde{m}_i = \frac{\bar{m}_0 \bar{m}_i}{\bar{m}_0 + \mu \, \bar{m}_i}$$

are the reduced masses.

The system (4.1) exhibits three integrals of the motion (besides the energy) related to its rotation invariance: the three components of the total angular momentum

$$C = (C_{\mathbf{x}}, C_{\mathbf{y}}, C_{\mathbf{z}}) = \sum_{1 \le i \le N} x_i \times y_i .$$

$$(4.2)$$

Hence, the number of degrees of freedom of (4.1) can be furtherly reduced. Without performing such a reduction, any attempt of extending to the spatial case the strategy described in the previous section for the plane problem inevitably fails: two well known resonances, called *secular resonances* (one of which with high order 2N - 1 and firstly noticed by M. Herman) appear, preventing the direct application of Theorem 2.1.

This section is devoted to the description of the reduction of the number of degrees of freedom of (4.1), by means of a change of variables essentially discovered, in the case of the Four Body Problem, by Francoise Boigey [9] and then extended to the general case by A. Deprit (1926,2006), [15]. It may be viewed as a natural extension of the *Jacobi* or *nodes reduction*, used in [33], to prove the existence of quasi-periodic motions in the Three–Body Problem.

The three components C_x , C_y , C_z of the total angular momentum do not commute, but they verify the cyclic rules ²³

$$\{C_{\rm x}, C_{\rm y}\} = C_{\rm z}, \{C_{\rm y}, C_{\rm z}\} = C_{\rm x}, \{C_{\rm z}, C_{\rm x}\} = C_{\rm y}.$$

²³As usual, $\{f, g\}$ denotes the usual Poisson brackets of f, g:

$$\left\{f, g\right\} := \sum_{1 \le i \le N} \left(\partial_{x_i} f \partial_{y_i} g - \partial_{y_i} f \partial_{x_i} g\right)$$

However, as well known, starting with C_x , C_y , C_z , it is possible to construct two commuting integrals, for istance

$$C_{\mathrm{z}} \qquad \text{and} \qquad \mathbf{G} := |C| = \sqrt{C_{\mathrm{x}}^2 + C_{\mathrm{y}}^2 + C_{\mathrm{z}}^2} \ .$$

We define then a system of (action-angle) symplectic coordinates, which are adapted to the reduction, since they have C_z and G among their generalized momenta. The angle ζ conjugate to C_z is an integral of the motion too, implying that the Hamiltonian (4.1), when expressed in such variables, does not depend on the couple (C_z, ζ) and the angle g conjugate to G. The constant value $G = G_0$ will appear into the Hamiltonian as an "external parameter", meaning with this that the motion of the remaining 2(3N - 2)variables will take place on a phase space parametrized by G_0 . Owing to the rotation invariance of (4.1), in particular, we find a set of symplectic variables on the manifold of dimension 2(3N - 2)

$$\mathcal{M}_{\text{vert,G}_0} := \left\{ y, \ x \in (\mathbb{R}^3)^N : \ C_{\text{x}} = C_{\text{y}} = C_{N_{\text{x}}} = 0, \ C_{\text{z}} = G_0 \right\} ,$$

where C_{Nx} denotes the first component of the N^{th} angular momentum $C_N = x_N \times y_N$. A further trivial integration will reconstruct the full motion on the full phase space.

Successively, we define a set of regularized variables (analogue to Poincaré's ones) on a larger domain, accordingly to the non–planarity condition.

4.1 Angular Momentum Reduction

Fix an orthonormal 3-ple (k_x, k_y, k_z) in \mathbb{R}^3 . Denote by

$$C_i := x_i \times y_i \qquad 1 \le i \le N$$

the angular momentum of the "body i" and let

$$S_i := \sum_{1 \le j \le i} C_j \qquad 2 \le i \le N$$

the sum of the first *i* angular momenta, so that $S_N \equiv C$ coincides with the total angular momentum (4.2) of the system (S_1 is not defined because it coincides with C_1). Consider also, on the plane orthogonal to C_i , the (\hat{m}_i, \tilde{m}_i) -Keplerian motion evolving from (y_i, x_i) , which is defined as the solution of the differential problem

$$\begin{cases} \ddot{v} = -\hat{m}_i \frac{v}{|v|^3}, \ v \in \mathbb{R}^3\\ (\tilde{m}_i \dot{v}(0), v(0)) = (y_i, x_i) \end{cases}$$
(4.3)

As well known, the curve $t \to v(\hat{m}_i, \tilde{m}_i, y_i, x_i; t)$ solution of (4.3) draws in the space a conic section $\mathcal{E}_i := \mathcal{E}(\hat{m}_i, \tilde{m}_i, y_i, x_i)$ and we denote by $e_i := e(\hat{m}_i, \tilde{m}_i, y_i, x_i)$ its eccentricity.

On the subset \mathcal{C}_* of initial data $(y, x) \in (\mathbb{R}^3)^N \times (\mathbb{R}^3)^N$ for which

$$\begin{cases}
C_1 \times C_2 \neq 0 \\
S_i \times C_{i+1} \neq 0 & 2 \leq i \leq N-1 \\
k_z \times C \neq 0 \\
0 < e_i < 1 & 1 \leq i \leq N
\end{cases}$$
(4.4)

(in particular, each \mathcal{E}_i is an ellipse), we define the set of variables ²⁴

$$\begin{pmatrix} (L, \Gamma, \Psi), (\ell, \gamma, \psi) \end{pmatrix} := \left((L_1, \cdots, L_N, \Gamma_1, \cdots, \Gamma_N, \Psi_1, \cdots, \Psi_N), (\ell_1, \cdots, \ell_N, \gamma_1, \cdots, \gamma_N, \psi_1, \cdots, \psi_N) \right) \\ \in \left(\mathbb{R}^N_+ \times \mathbb{R}^N_+ \times (\mathbb{R}^{N-1}_+ \times \mathbb{R}) \right) \times (\mathbb{T}^N)^3$$

as follows.

(D₁) For $1 \leq i \leq N$, if $a_i := a(\hat{m}_i, \tilde{m}_i, y_i, x_i)$ is the semimajor axis of \mathcal{E}_i , then,

$$L_i := \tilde{m}_i \sqrt{\hat{m}_i a_i} ;$$

(D₂) if $\mathcal{A}_i := \mathcal{A}(\hat{m}_i, \tilde{m}_i, y_i, x_i)$ denotes the area spanned from the perihelion P_i of \mathcal{E}_i to x_i , then, the angle ℓ_i is the mean anomaly

$$\ell_i := 2 \frac{\mathcal{A}_i}{a_i^2 \sqrt{1 - e_i^2}} ;$$

 (D_3) the action Γ_i is

$$\Gamma_i := |C_i| = L_i \sqrt{1 - e_i^2}$$
.

(D₄) For $1 \le i \le N - 1$, the action Ψ_i is

$$\Psi_i = |S_{i+1}| \ .$$

Notice that $\Psi_{N-1} = \mathbf{G} = |C|$ is an integral of the motion.

(D₅) The action Ψ_N is

$$\Psi_N = C_z$$

the third component of C. Also this variable is an integral of the motion.

Now, in order to define the conjugated angles γ , ψ , we introduce the following notations. Given $0 \neq w \in \mathbb{R}^3$, we define the plane π_w orthogonal to w:

$$\pi_w := \{ u \in \mathbb{R}^3 : u \cdot w = 0 \} .$$

 ${}^{24}\mathbb{R}_+ := (0, +\infty) \subset \mathbb{R}$

If u, v are two non vanishing vectors in π_w , we define

$$\mathbf{k}_u := \frac{u}{|u|}$$
, $\mathbf{k}_w := \frac{w}{|w|}$, $\mathbf{k}_{uv} := \mathbf{k}_w \wedge \mathbf{k}_u$

so that the triple $(\mathbf{k}_u, \mathbf{k}_{uw}, \mathbf{k}_w)$ is an orthonormal positively oriented basis ²⁵. We then define the oriented angle seen from w from u to v, and denote it by $\alpha_w(u, v)$, as the angle $t + 2\pi\mathbb{Z}$, where t is the unique number in $[0, 2\pi)$ such that

$$v = \cos t \mathbf{k}_u + \sin t \mathbf{k}_{wu} \; .$$

We can now define the angles γ , ψ . In view of assumptions (4.4), the following "nodes" are non vanishing

$$n_i := \begin{cases} C_2 \times C_1 , & i = 1\\ S_i \times C_i , & 2 \le i \le N \end{cases}$$

$$\bar{n} := \mathbf{k}_{\mathbf{z}} \times C ,$$

hence, the following definitions are well put.

(D₆) For $1 \leq i \leq N$, the angle γ_i locates the perihelion P_i of \mathcal{E}_i :

$$\gamma_i = \alpha_{C_i}(n_i, P_i) \; .$$

(D₇) When $N \geq 3$, the angles $\psi_1, \dots, \psi_{N-2}$ are

$$\psi_i = \alpha_{S_{i+1}}(n_{i+2}, n_{i+1}), \qquad 1 \le i \le N-2.$$

(D₈) The angle ψ_{N-1} is

$$\psi_{N-1} = \mathbf{g} := \alpha_C(\bar{n}, -n_N)$$

(D₉) The angle ψ_N is the longitude of the node ²⁶ of C with respect to (k_x, k_y, k_z) , namely,

$$\psi_N = \zeta := \alpha_{\mathbf{k}_z}(\mathbf{k}_x, \bar{n})$$
.

Notice that this angle, together with the actions G, C_z , is the third component of the total angular momentum C.

The variables $((L, \Gamma, \Psi), (\ell, \gamma, \psi))$ defined via $D_1 \div D_9$ will be referred as action-angle Deprit variables (or, simply, Deprit variables); the map

$$\Phi_*: \ \mathcal{C}_* \to \left(\mathbb{R}^N_+ \times \mathbb{R}^N_+ \times (\mathbb{R}^{N-1}_+ \times \mathbb{R})\right) \times (\mathbb{T}^N)^3$$

 $^{^{25}}$ I. e., the determinant of the matrix with columns the components of the oriented triple (k_u, k_{uw}, k_w) is positive (and in fact 1).

²⁶The longitude of the node of v with respect to the orthonormal 3-ple (e_x, e_y, e_z) is defined as the angle $\alpha_{e_z}(e_x, e_z \times v)$.

which sends a point $(y, x) \in \mathcal{C}_*$ to the Deprit variables Deprit map; their phase space is denoted as \mathcal{D}_* . It corresponds to be the subset of $((L, \Gamma, \Psi), (\ell, \gamma, \psi)) \in (\mathbb{R}^N_+ \times \mathbb{R}^N_+ \times (\mathbb{R}^{N-1}_+ \times \mathbb{R})) \times (\mathbb{T}^N)^3$ defined by the inequalities

$$\begin{cases}
\Gamma_{i} < L_{i} \quad 1 \leq i \leq N \\
|\Gamma_{1} - \Gamma_{2}| < \Psi_{1} < \Gamma_{1} + \Gamma_{2} \\
|\Psi_{i-1} - \Gamma_{i+1}| < \Psi_{i} < \Psi_{i-1} + \Gamma_{i+1} \quad 2 \leq i \leq N-1 \\
|\Psi_{N}| < \Psi_{N-1}
\end{cases}$$
(4.5)

In fact, by definition,

$$\Phi_*(\mathcal{C}_*) \subseteq \mathcal{D}_*$$

and we can prove

Theorem 4.1 The Deprit map Φ_* is a real-analytic symplectomorphism (symplectic diffeomorphism onto) of C_* onto \mathcal{D}_* .

Real-analyticity follows immediately from the definition. To check injectivity and surjectivity, we shall exhibit its inverse transformation. The basis of the inversion formulae is to express the angular momenta C_i $(1 \le i \le N)$, in terms of the Deprit variables : this is done in the following Lemma.

Lemma 4.1 The angular momenta C_1, \dots, C_N can be expressed in terms of the variables (Γ, Ψ, ψ) as follows. First, define N-1 orthonormal triples (e_x^i, e_y^i, e_z^i) , $2 \le i \le N$ by letting

$$\begin{cases} e_{x}^{N} := -\cos\zeta k_{x} - \sin\zeta k_{y} \\ e_{z}^{N} := \sqrt{1 - \left(\frac{C_{z}}{G}\right)^{2}} \sin\zeta k_{x} - \sqrt{1 - \left(\frac{C_{z}}{G}\right)^{2}} \cos\zeta k_{y} + \frac{C_{z}}{G} k_{x} \\ e_{y}^{N} := e_{z}^{N} \times e_{x}^{N} = \frac{C_{z}}{G} \sin\zeta k_{x} - \frac{C_{z}}{G} \cos\zeta k_{y} - \sqrt{1 - \left(\frac{C_{z}}{G}\right)^{2}} k_{z} \end{cases}$$
(4.6)

then (inductively), given $(e_x^{i+1}, e_y^{i+1}, e_z^{i+1})$, for $3 \le i+1 \le N$, let

$$\begin{cases} e_{z}^{i} := -\frac{r_{i+1}}{\Psi_{i-1}} \sin \psi_{i} e_{x}^{i+1} + \frac{r_{i+1}}{\Psi_{i-1}} \cos \psi_{i} e_{y}^{i+1} + \frac{\tilde{h}_{i}}{\Psi_{i-1}} e_{z}^{i+1} \\ e_{x}^{i} := e_{x}^{i+1} \cos \psi_{i} + e_{y}^{i+1} \sin \psi_{i} \\ e_{y}^{i} := e_{z}^{i} \times e_{x}^{i} = -\frac{\tilde{h}_{i}}{\Psi_{i-1}} \sin \psi_{i} e_{x}^{i+1} + \frac{\tilde{h}_{i}}{\Psi_{i-1}} \cos \psi_{i} e_{y}^{i+1} - \frac{r_{i+1}}{\Psi_{i-1}} e_{z}^{i+1} \end{cases}$$

$$(4.7)$$

Then,

$$C_{i} = \begin{cases} r_{i} \sin \psi_{i-1} \mathbf{e}_{\mathbf{x}}^{i} - r_{i} \cos \psi_{i-1} \mathbf{e}_{\mathbf{y}}^{i} + h_{i} \mathbf{e}_{\mathbf{z}}^{i} & 2 \le i \le N \\ -r_{2} \sin \psi_{1} \mathbf{e}_{\mathbf{x}}^{2} + r_{2} \cos \psi_{1} \mathbf{e}_{\mathbf{y}}^{2} + h_{1} \mathbf{e}_{\mathbf{z}}^{2} & i = 1 \end{cases}$$

where, with the convention $\Psi_0 := \Gamma_1$,

$$r_i = \frac{\sqrt{(\Psi_{i-2}^2 - (\Gamma_i - \Psi_{i-1})^2)((\Gamma_i + \Psi_{i-1})^2 - \Psi_{i-2}^2)}}{2\Psi_{i-1}} \quad for \quad 2 \le i \le N$$

$$h_{i} = \begin{cases} \frac{\Psi_{1}^{2} - \Gamma_{2}^{2} + \Gamma_{1}^{2}}{2\Psi_{1}} & for \quad i = 1\\ \\ \frac{\Gamma_{i}^{2} + \Psi_{i-1}^{2} - \Psi_{i-2}^{2}}{2\Psi_{i-1}} & for \quad 2 \leq i \leq N \end{cases}$$
$$\tilde{h}_{i} := \frac{\Psi_{i}^{2} + \Psi_{i-1}^{2} - \Gamma_{i+1}^{2}}{2\Psi_{i}} & for \quad 2 \leq i \leq N-1 \end{cases}$$

Proof. By definition of ζ , G, C_z , the components of C are

$$C = \begin{pmatrix} \sqrt{\mathbf{G}^2 - C_z^2} \sin \zeta \\ -\sqrt{\mathbf{G}^2 - C_z^2} \cos \zeta \\ C_z \end{pmatrix}$$

Consider the orthonormal 3-ple (4.6), which has e_z^N in the direction of C, e_x^N is in the direction of $-\bar{n} = -k_z \times C$. Then, the modulus, the third component and the longitude of C_N with respect to (e_x^N, e_y^N, e_z^N) are given, respectively, by

$$|C_{N}| = \Gamma_{N}$$

$$C_{N} \cdot e_{z} = \frac{C_{N} \cdot S_{N}}{G}$$

$$= \frac{|C_{N}|^{2} + |S_{N}|^{2} - |S_{N} - C_{N}|^{2}}{2\Psi_{N-1}}$$

$$= \frac{\Psi_{N-1}^{2} + \Gamma_{N}^{2} - \Psi_{N-2}^{2}}{2\Psi_{N-1}}$$

$$= h_{N-1}$$

$$\alpha_{e_{z}}(e_{x}, e_{z} \times C_{N}) = \alpha_{S_{N}}(-\bar{n}, S_{N} \times C_{N})$$

$$= \psi_{N-1} \qquad (4.8)$$

which is equivalent to

$$C_N = r_N \sin \psi_{N-1} e_x^N - r_N \cos \psi_{N-1} e_y^N + h_N e_z^N$$
,

with

$$r_N := \sqrt{\Gamma_N^2 - h_N^2} = \frac{\sqrt{(\Psi_{N-2}^2 - (\Gamma_N - \Psi_{N-1})^2)((\Gamma_N + \Psi_{N-1})^2 - \Psi_{N-2}^2)}}{2\Psi_{N-1}}$$

Assume, now, that

$$C_{i+1} = r_{i+1} \sin \psi_i \, \mathbf{e}_{\mathbf{x}}^{i+1} - r_{i+1} \, \cos \psi_i \, \mathbf{e}_{\mathbf{y}}^{i+1} + h_{i+1} \, \mathbf{e}_{\mathbf{z}}^{i+1} \qquad \text{for} \quad 3 \le i+1 \le N$$

Then,

$$S_{i} := C_{1} + \dots + C_{i}$$

= $S_{i+1} - C_{i+1}$
= $\Psi_{i} e_{z}^{i+1} - \left(r_{i+1} \sin \psi_{i} e_{x}^{i+1} - r_{i+1} \cos \psi_{i} e_{y}^{i+1} + h_{i+1} e_{z}^{i+1}\right)$
= $-r_{i+1} \sin \psi_{i} e_{x}^{i+1} + r_{i+1} \cos \psi_{i} e_{y}^{i+1} + \tilde{h}_{i} e_{z}^{i+1}$,

with

$$\tilde{h}_i := \Psi_i - h_{i+1} = \frac{\Psi_i^2 + \Psi_{i-1}^2 - \Gamma_{i+1}^2}{2\Psi_i}$$

Let, now, (e_x^i, e_y^i, e_z^i) the orthonormal 3-ple with e_z^i in the direction of S_i , e_x^i in the direction of $S_i \times C_{i+1}$:

$$\begin{cases} e_{z}^{i} = \frac{S_{i}}{\Psi_{i-1}} = -\frac{r_{i+1}}{\Psi_{i-1}} \sin \psi_{i} e_{x}^{i+1} + \frac{r_{i+1}}{\Psi_{i-1}} \cos \psi_{i} e_{y}^{i+1} + \frac{\tilde{h}_{i}}{\Psi_{i-1}} e_{z}^{i+1} \\ e_{x}^{i} := \frac{S_{i} \times C_{i+1}}{|S_{i} \times C_{i+1}|} = e_{x}^{i+1} \cos \psi_{i} + e_{y}^{i+1} \sin \psi_{i} \\ e_{y}^{i} := e_{z}^{i} \times e_{x}^{i} = -\frac{\tilde{h}_{i}}{\Psi_{i-1}} \sin \psi_{i} e_{x}^{i+1} + \frac{\tilde{h}_{i}}{\Psi_{i-1}} \cos \psi_{i} e_{y}^{i+1} - \frac{r_{i+1}}{\Psi_{i-1}} e_{z}^{i+1} \end{cases}$$

•

Repeating the argument in (4.8), we find that the modulus, the third component and the longitude of the node of C_i with respect to $(\mathbf{e}_{\mathbf{x}}^i, \mathbf{e}_{\mathbf{y}}^i, \mathbf{e}_{\mathbf{z}}^i)$ are given by

$$\begin{aligned} |C_i| &= \Gamma_i \\ C_i \cdot e_z^i &= \frac{C_i \cdot S_i}{\Psi_{i-1}} \\ &= \frac{|C_i|^2 + |S_i|^2 - |S_i - C_i|^2}{2\Psi_{i-1}} \\ &= \frac{\Gamma_i^2 + \Psi_{i-1}^2 - \Psi_{i-2}^2}{2\Psi_{i-1}} \\ &= h_i \\ \alpha_{e_z^i} \left(e_x^i, e_z^i \times C_i\right) &= \alpha_{S_i} (n_{i+1}, n_i) \\ &= \psi_{i-1} \end{aligned}$$

which is equivalent to

$$C_i = r_i \sin \psi_{i-1} \operatorname{e}_{\mathbf{x}}^i - r_i \cos \psi_{i-1} \operatorname{e}_{\mathbf{y}}^i + h_i \operatorname{e}_{\mathbf{z}}^i$$

hence,

$$n_i = S_i \times C_i = \Psi_{i-1} e_z^i \times C_i = \Psi_{i-1} r_i (\cos \psi_{i-1} e_x^i + \sin \psi_{i-1} e_y^i)$$

where

$$r_i = \sqrt{\Gamma_i^2 - h_i^2} = \frac{\sqrt{(\Psi_{i-2}^2 - (\Gamma_i - \Psi_{i-1})^2)((\Gamma_i + \Psi_{i-1})^2 - \Psi_{i-2}^2)}}{2\Psi_{i-1}} \ .$$

with the convention $\Psi_0 := \Gamma_1$. At the N^{th} step, put

$$C_1 = S_2 - C_2 = -r_2 \sin \psi_1 e_x^2 + r_2 \cos \psi_1 e_y^2 + \tilde{h}_1 e_z^2$$

with

$$h_1 := \Psi_1 - h_2 = \Psi_1 - \frac{\Gamma_2^2 + \Psi_1^2 - \Gamma_1^2}{2\Psi_1} = \frac{\Gamma_1^2 + \Psi_1^2 - \Gamma_2^2}{2\Psi_1} .$$

This completes the proof of the Lemma.

Remark 4.1 The Deprit map may be seen as an "unfolding" of the Jacobi reduction of the nodes, available for N = 2. To well understand this point, we write it in *spatial* Delaunay variables $(L, \mathbf{G}, \Theta, \ell, \mathbf{g}, \vartheta)$, with $L = (L_1, \dots, L_N)$, $\mathbf{G} = (G_1, \dots, G_N)$, \dots , which, we recall, are defined as

$$\begin{cases} L_i = \tilde{m}_i \sqrt{\hat{m}_i a_i} \\ G_i = |C_i| = \sqrt{1 - e_i^2} L_i \\ \Theta_i = C_i \cdot \mathbf{k}_z \end{cases} \begin{cases} \ell_i = \frac{2A_i}{a_i \sqrt{1 - e_i^2}} \\ g_i = \alpha_{C_i}(\bar{n}_i, P_i)) \\ \vartheta_i = \alpha_{\mathbf{k}_z}(\mathbf{k}_x, \bar{n}_i) \end{cases}$$
(4.9)

and they are well defined whenever

$$\bar{n}_i := \mathbf{k}_\mathbf{z} \times C_i \neq 0$$
, $e_i \neq 0$ for $1 \le i \le N$.

The variables L_i , ℓ_i , G_i are then left unchanged. To find the expressions of the remaining Delaunay variables in terms of the Deprit variables, we use the expressions of the angular momenta C_i of Lemma 4.1, in the case N = 2:

$$C_{1} = \begin{pmatrix} r\cos\zeta\sin g + r\frac{C_{z}}{G}\sin\zeta\cos g + h_{1}\frac{\sqrt{G^{2}-C_{z}^{2}}}{G}\sin\zeta}{r\sin\zeta\sin g - r\frac{C_{z}}{G}\cos\zeta\cos g - h_{1}\frac{\sqrt{G^{2}-C_{z}^{2}}}{G}\cos\zeta}{\frac{C_{z}}{G}h_{1} - r\frac{\sqrt{G^{2}-C_{z}^{2}}}{G}\cos g} \end{pmatrix}$$

$$C_{2} = \begin{pmatrix} -r\cos\zeta\sin g - r\frac{C_{z}}{G}\sin\zeta\cos g + h_{2}\frac{\sqrt{G^{2}-C_{z}^{2}}}{G}\sin\zeta}{\frac{C_{z}}{G}\cos\zeta} + h_{2}\frac{\sqrt{G^{2}-C_{z}^{2}}}{G}\cos\zeta}{\frac{C_{z}}{G}h_{2} + r\frac{\sqrt{G^{2}-C_{z}^{2}}}{G}\cos\zeta}{G}\cos\zeta} \end{pmatrix}$$

$$(4.10)$$

with

$$r = \frac{\sqrt{(\Gamma_1^2 - (\Gamma_2 - G)^2)((\Gamma_2 + G)^2 - \Gamma_1^2)}}{2G}$$
$$h_1 = \frac{G^2 + \Gamma_1^2 - \Gamma_2^2}{2G}, \quad h_2 = \frac{G^2 + \Gamma_2^2 - \Gamma_1^2}{2G}$$

This allows us to find the nodes \bar{n} , \bar{n}_i , n_i , and hence, the Delaunay perihelia arguments g_i :

$$g_i = \alpha_{C_i}(\bar{n}_i, P_i) = \alpha_{C_i}(n_i, P_i) + \alpha_{C_i}(\bar{n}_i, n_i) = \gamma_i + \alpha_{C_i}(\bar{n}_i, n_i) \qquad (i = 1, 2)$$

and the Delaunay nodes

$$\vartheta_i = \alpha_{\mathbf{k}_{\mathbf{z}}}(\mathbf{k}_{\mathbf{x}}, \bar{n}_i) = \alpha_{\mathbf{k}_{\mathbf{z}}}(\mathbf{k}_{\mathbf{x}}, \bar{n}) + \alpha_{\mathbf{k}_{\mathbf{z}}}(\bar{n}, \bar{n}_i) = \zeta + \alpha_{\mathbf{k}_{\mathbf{z}}}(\bar{n}, \bar{n}_i)$$

Finally, identifying Θ_1 , Θ_2 with the third components of C_1 , C_2 in (4.10), we complete the inversion formulae of the Delaunay variables in therm of the \mathcal{D} -variables with

$$\begin{cases} \Theta_1 = \frac{C_z}{2} + \frac{C_z}{2G^2} (\Gamma_1^2 - \Gamma_2^2) - \frac{\sqrt{(G^2 - C_z^2)(\Gamma_1^1 - (\Gamma_2 - G)^2)((\Gamma_2 + G)^2 - \Gamma_1^1)}}{2G^2} \cos g \\ \Theta_2 = \frac{C_z}{2} - \frac{C_z}{2G^2} (\Gamma_1^2 - \Gamma_2^2) + \frac{\sqrt{(G^2 - C_z^2)(\Gamma_1^1 - (\Gamma_2 - G)^2)((\Gamma_2 + G)^2 - \Gamma_1^1)}}{2G^2} \cos g \end{cases}$$
(4.11)

However, due to the rotation invariance, the expression of the Hamiltonian is independent on the choice of the reference frame (k_x, k_y, k_z) . If we choose k_z parallel to C and k_x parallel to $C \times C_1 = n_1$, we have $\bar{n}_i = n_i$, hence,

$$g_i = \gamma_i \qquad (i = 1, 2) \; .$$

Also, since $n_1 = -n_2$,

$$\begin{cases} \vartheta_1 = \alpha_{\mathbf{k}_z}(\mathbf{k}_x, \bar{n}_1) = \alpha_{\mathbf{k}_z}(n_1, n_1) = 0\\ \vartheta_2 = \alpha_{\mathbf{k}_z}(\mathbf{k}_x, \bar{n}_2) = \alpha_{\mathbf{k}_z}(n_1, n_2) = \pi \end{cases}$$
("opposition of the nodes")

Finally, when the total angular momentum C is seen vertical, $G = C_z$, hence, (4.11) becomes

$$\left\{ \begin{array}{l} \Theta_1 = \frac{\mathrm{G}}{2} + \frac{\Gamma_1^2 - \Gamma_2^2}{2\mathrm{G}} \\ \Theta_2 = \frac{\mathrm{G}}{2} - \frac{\Gamma_1^2 - \Gamma_2^2}{2\mathrm{G}} \end{array} \right.$$

The previous formulae (completed with the identity on L_i , ℓ_i) are recognized as the classical formulae for the Jacobi's reduction of the nodes.

Proposition 4.1 The Deprit map Φ_* is invertible on \mathcal{D}_* and its inverse Φ_*^{-1} is defined as follows. Let \mathcal{R}_x , \mathcal{R}_z denote the elementary rotations

$$\mathcal{R}_{\mathbf{x}}(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix} , \quad \mathcal{R}_{\mathbf{z}}(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

 I_z the reflection

$$I_z = \begin{pmatrix} -1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1 \end{pmatrix} ;$$

let $i, i_j, i_j \in (0, \pi)$ be defined by

$$\cos i = \frac{C_z}{G}$$
, $\cos i_j = \frac{h_j}{\Gamma_j}$, $1 \le j \le N$, $\cos \tilde{i}_j = \frac{h_j}{\Psi_{j-1}}$, $2 \le j \le N - 1$, (4.12)

with h_j , \tilde{h}_j as in Lemma 4.1 put

$$R_{j} := \begin{cases} \mathcal{R}_{z}(\psi_{j})\mathcal{R}_{x}(\tilde{i}_{j}) & 2 \leq j \leq N-1 \\ \mathcal{R}_{z}(\zeta)\mathcal{R}_{x}(i) & j = N \end{cases},$$

$$S_{j} := \begin{cases} \mathcal{R}_{z}(\psi_{1})\mathcal{R}_{x}(i_{1}) & j = 1 \\ \mathcal{R}_{z}(\psi_{j-1})I_{z}\mathcal{R}_{x}(i_{j}) & 2 \leq j \leq N \end{cases}$$

$$(4.13)$$

and define

$$\mathcal{R}_{i} = \begin{cases} \mathbf{R}_{N} \cdots \mathbf{R}_{2} \mathbf{S}_{1} & i = 1\\ \mathbf{R}_{N} \cdots \mathbf{R}_{i} \mathbf{S}_{i} & 2 \leq i \leq N \end{cases}$$
(4.14)

Denote by

$$\mathsf{D}_i: \ (\bar{L}, \bar{\ell}, \bar{\Gamma}, \bar{\gamma}) \to (Y_i(\bar{L}, \bar{\ell}, \bar{\Gamma}, \bar{\gamma}), X_i(\bar{L}, \bar{\ell}, \bar{\Gamma}, \bar{\gamma}))$$

the plane (\hat{m}_i, \tilde{m}_i) -Delaunay map, defined as the four dimensional map

$$\mathsf{D}_i: ((\bar{L},\bar{\Gamma}),(\bar{\ell},\bar{\gamma})) \in \mathbb{R}^2_+ \times \mathbb{T}^2: \quad \bar{\Gamma} < \bar{L} \to (Y_i,X_i) \in \mathbb{R}^3 \times \mathbb{R}^3$$

given by

$$X_{i} = \frac{1}{\hat{m}_{i}} \left(\frac{\bar{L}}{\tilde{m}_{i}}\right)^{2} \mathcal{R}_{z}(\gamma_{i}) \begin{pmatrix} \cos \bar{u} - \sqrt{1 - \left(\frac{\bar{\Gamma}}{\bar{L}}\right)^{2}} \\ \frac{\bar{\Gamma}}{\bar{L}} \sin \bar{u} \\ 0 \end{pmatrix}, \quad Y_{i} = \frac{\tilde{m}_{i}^{4} \hat{m}_{i}^{2}}{\bar{L}^{3}} \partial_{\bar{\ell}} X_{i}$$

with $\bar{u} := u(\bar{L}, \bar{\Gamma}, \bar{\ell})$ the unique solution of the Kepler's Equation

$$u - \sqrt{1 - \left(\frac{\bar{\Gamma}}{\bar{L}}\right)^2} \sin u = \bar{\ell}$$

Then,

$$\begin{cases} y_i = \mathcal{R}_i(\Gamma, \Psi, \psi) Y_i(L_i, \ell_i, \Gamma_i, \gamma_i) \\ x_i = \mathcal{R}_i(\Gamma, \Psi, \psi) X_i(L_i, \ell_i, \Gamma_i, \gamma_i) \end{cases} \quad 1 \le i \le N \end{cases}$$
(4.15)

Proof. By definition, the (\hat{m}_i, \tilde{m}_i) -plane Delaunay map

$$D_i: (L_i, \ell_i, \Gamma_i, \gamma_i) \rightarrow (Y_i(L_i, \ell_i, \Gamma_i, \gamma_i), X_i(L_i, \ell_i, \Gamma_i, \gamma_i))$$
$$:= ((Y_{ix}, Y_{iy}, 0), (X_{ix}, X_{iy}, 0))$$

gives the coordinates of y_i , x_i on the basis of the "orbital triples", *i.e.* the (orthonormal) triples (f_x^i, f_y^i, f_z^i) , where f_x^i is in the direction of n_i , f_z^i in the direction of C_i (and, hence, $f_y^i = f_z^i \times f_x^i$)

$$\begin{cases} y_i = Y_{ix} f_x^i + Y_{iy} f_y^i \\ x_i = X_{ix} f_x^i + X_{iy} f_y^i \end{cases} \quad 1 \le i \le N .$$
(4.16)

Having the expressions of C_1, \dots, C_N in terms of the Deprit variables allows to find the orbital triples (f_x^i, f_y^i, f_z^i) :²⁷

$$\begin{cases} f_{x}^{1} := \frac{n_{1}}{|n_{1}|} = -\cos\psi_{1}e_{x}^{2} - \sin\psi_{1}e_{y}^{2} \\ f_{z}^{1} := \frac{C_{1}}{\Gamma_{1}} = -\frac{r_{2}}{\Gamma_{1}}\sin\psi_{i-1}e_{x}^{2} + \frac{r_{2}}{\Gamma_{1}}\cos\psi_{1}e_{y}^{2} + \frac{h_{1}}{\Gamma_{1}}e_{z}^{2} \\ f_{y}^{1} := f_{z}^{1} \times f_{x}^{1} = \frac{h_{1}}{\Gamma_{1}}\sin\psi_{1}e_{x}^{2} - \frac{h_{1}}{\Gamma_{1}}\cos\psi_{i-1}e_{y}^{2} + \frac{r_{2}}{\Gamma_{1}}e_{z}^{2} \end{cases} \\ \begin{cases} f_{x}^{i} := \frac{n_{i}}{|n_{i}|} = \cos\psi_{i-1}e_{x}^{i} + \sin\psi_{i-1}e_{y}^{i} \\ f_{z}^{i} := \frac{C_{i}}{\Gamma_{i}} = \frac{r_{i}}{\Gamma_{i}}\sin\psi_{i-1}e_{x}^{i} - \frac{r_{i}}{\Gamma_{i}}\cos\psi_{i-1}e_{y}^{i} + \frac{h_{i}}{\Gamma_{i}}e_{z}^{i} \end{cases} \qquad 2 \le i \le N \\ f_{y}^{i} := f_{z}^{i} \times f_{x}^{i} = -\frac{h_{i}}{\Gamma_{i}}\sin\psi_{i-1}e_{x}^{i} + \frac{h_{i}}{\Gamma_{i}}\cos\psi_{i-1}e_{y}^{i} + \frac{r_{i}}{\Gamma_{i}}e_{z}^{i} \end{cases} \qquad (4.17)$$

 $[\]overline{\frac{27 \text{Use } \Psi_{i-1} e_{z}^{i} = \sum_{1 \le j \le i} C_{i} = S_{i}, \text{ hence, for } 2 \le i \le N, n_{i} = S_{i} \times C_{i} = \Psi_{i-1} e_{z}^{i} \times C_{i}} = \Psi_{i-1} r_{i} (\cos \psi_{i-1} e_{z}^{i} + \sin \psi_{i-1} e_{y}^{i}). \text{ For } i = 1, \text{ recall that } n_{1} = -n_{2}.$

Put, for shortness,

$$\mathbf{F}_{i} := \begin{pmatrix} \mathbf{f}_{\mathbf{x}}^{i} \\ \mathbf{f}_{\mathbf{y}}^{i} \\ \mathbf{f}_{\mathbf{z}}^{i} \end{pmatrix} \quad 1 \le i \le N \ , \quad \mathbf{E}_{i} := \begin{pmatrix} \mathbf{e}_{\mathbf{x}}^{i} \\ \mathbf{e}_{\mathbf{y}}^{i} \\ \mathbf{e}_{\mathbf{z}}^{i} \end{pmatrix} \quad 2 \le i \le N \ , \quad \mathbf{K} := \begin{pmatrix} \mathbf{k}_{\mathbf{x}} \\ \mathbf{k}_{\mathbf{y}} \\ \mathbf{k}_{\mathbf{z}} \end{pmatrix}$$

Then, equations (4.6), (4.7), (4.17) may be written as

$$E_N = \hat{\mathbf{R}}_N^T \mathbf{K} , \quad E_i = \hat{\mathbf{R}}_i^T \mathbf{E}_{i+1} \quad 2 \le i \le N-1 , \quad \mathbf{F}_i = \begin{cases} \hat{\mathbf{S}}_1^T \mathbf{E}_2 & i = 1\\ \hat{\mathbf{S}}_i^T \mathbf{E}_i & 2 \le i \le N \end{cases}$$
(4.18)

where:

$$\hat{\mathbf{R}}_{j} := \begin{cases} \mathbf{I}_{z} \mathcal{R}_{z}(\psi_{j}) \mathcal{R}_{x}(\tilde{i}_{j}) \mathbf{I}_{z} & 2 \leq j \leq N-1 \\ \mathcal{R}_{z}(\zeta) \mathcal{R}_{x}(i) \mathbf{I}_{z} & j = N \end{cases}$$
$$\hat{\mathbf{S}}_{j} := \begin{cases} \mathbf{I}_{z} \mathcal{R}_{z}(\psi_{1}) \mathcal{R}_{x}(i_{1}) & j = 1 \\ \mathcal{R}_{z}(\psi_{j-1}) \mathcal{R}_{x}(i_{j}) & 2 \leq j \leq N \end{cases}$$

Then, in view of (4.18), we can write

$$\mathbf{F}_{i} = \mathcal{R}_{i}^{T}\mathbf{K}$$

$$(4.19)$$

,

with

$$\mathcal{R}_{i} := \begin{cases} \hat{\mathbf{R}}_{N} \cdots \hat{\mathbf{R}}_{2} \hat{\mathbf{S}}_{1} & i = 1\\ \hat{\mathbf{R}}_{N} \cdots \hat{\mathbf{R}}_{i} \hat{\mathbf{S}}_{i} & 2 \le i \le N \end{cases}$$
(4.20)

Equations (4.16), (4.19), with \mathcal{R}_i as in (4.20), give the inversion formulae

$$\begin{cases} y_i = \mathcal{R}_i(\Gamma, \Psi, \psi) Y_i(L_i, \ell_i, \Gamma_i, \gamma_i) \\ x_i = \mathcal{R}_i(\Gamma, \Psi, \psi) X_i(L_i, \ell_i, \Gamma_i, \gamma_i) \end{cases} \quad 1 \le i \le N \end{cases}$$

and this concludes the proof (the definitions (4.13), (4.14) of \mathcal{R}_i are a rewrite of (4.20)).

The frames E_i $(2 \le i \le N)$, F_i $(1 \le i \le N)$ introduced in the previous proof correspond to the frames F_{N-i}^* , F_{N-i+1} of the binary tree of kinetic frames arising from F = K of [15].

There remains to prove symplecticity of Φ_* . This is done by induction. We write then explicitly the dependence on N for the Deprit map, *i.e.*, we write the Deprit map as Φ^N_* : $\mathcal{C}^N_* \to \mathcal{D}^N_*$. The basis for the induction is N = 2.

Lemma 4.2 The 2-Deprit map Φ^2_* : $\mathcal{C}^2_* \to \mathcal{D}^2_*$ is symplectic.

The technique for the proof of Lemma 4.2 is similar to the one presented in [15], apart from the introduction of the plane Delaunay map.

Proof. In the case N = 2, we have

$$\mathcal{R}_1 = \mathrm{RS}_1 = \mathcal{R}_z(\zeta)\mathcal{R}_z(g)\mathcal{R}_z(g)\mathcal{R}_z(i_1)$$
, $\mathcal{R}_2 = \mathrm{RS}_2 = \mathcal{R}_z(\zeta)\mathcal{R}_z(g)\mathrm{I}_z\mathcal{R}_z(g)\mathrm{I}_z\mathcal{R}_z(i_2)$

where $i_1, i_2, i \in (0, \pi)$ are defined by

$$\cos i_1 = \frac{h_1}{\Gamma_1}$$
, $\cos i_2 = \frac{h_2}{\Gamma_2}$, $\cos i = \frac{C_z}{G}$

Differentiating x_i in (4.15), we find

$$dx_i = \mathcal{R}_i(\Gamma, \Psi, \psi) dX(L_i, \ell_i, \Gamma_i, \gamma_i) + (d\mathcal{R}_i(\Gamma, \Psi, \psi)) X(L_i, \ell_i, \Gamma_i, \gamma_i) \qquad i = 1, 2$$

So, since \mathcal{R}_i is unitary and the plane Delaunay map D_i is symplectic,

$$y_i \cdot dx_i = Y_i(L_i, \ell_i, \Gamma_i, \gamma_i) \cdot dX_i(L_i, \ell_i, \Gamma_i, \gamma_i) + y_i \cdot (d\mathcal{R}_i(\Gamma, \Psi, \psi))\mathcal{R}_i(\Gamma, \Psi, \psi)^T x_i$$

= $L_i d\ell_i + \Gamma_i d\gamma_i + y_i \cdot (d\mathcal{R}_i(\Gamma, \Psi, \psi))\mathcal{R}_i(\Gamma, \Psi, \psi)^T x_i$

Thus, summing over i = 1, 2,

$$y \cdot dx = L \cdot d\ell + \Gamma \cdot d\gamma + y_1 \cdot (d\mathcal{R}_1(\Gamma, \Psi, \psi))\mathcal{R}_1(\Gamma, \Psi, \psi)^T x_1 + y_2 \cdot (d\mathcal{R}_2(\Gamma, \Psi, \psi))\mathcal{R}_2(\Gamma, \Psi, \psi)^T x_2$$

$$(4.21)$$

Differentiating $\mathcal{R}_1 = \mathcal{R}_z(\zeta)\mathcal{R}_x(i)\mathcal{R}_z(g)\mathcal{R}_x(i_1)$ and using, as well known,

$$(d\mathcal{R}_{\mathbf{x}}(\alpha))\mathcal{R}_{\mathbf{x}}(\alpha)^{T}q = \mathbf{k}_{\mathbf{x}} \times q \, d\alpha , \quad (d\mathcal{R}_{\mathbf{z}}(\alpha))\mathcal{R}_{\mathbf{z}}(\alpha)^{T}q = \mathbf{k}_{\mathbf{z}} \times q \, d\alpha$$

we find

$$(d\mathcal{R}_1(\Gamma, \Psi, \psi))\mathcal{R}_1(\Gamma, \Psi, \psi)^T x_1 = \mathbf{k}_{\mathbf{z}} \times x_1 \, d\zeta + \mathbf{e}_{\mathbf{z}} \times x_1 \, d\mathbf{g} - \mathbf{e}_{\mathbf{x}} \times x_1 \, d\mathbf{i} + \mathbf{f}_{\mathbf{x}}^1 \times x_1 d\mathbf{i}_1$$

which gives, taking the scalar product with y_1 ,

$$y_{1} \cdot (d\mathcal{R}_{1}(\Gamma, \Psi, \psi))\mathcal{R}_{1}(\Gamma, \Psi, \psi)^{T}x_{1} = C_{1} \cdot \mathbf{k}_{z}d\zeta + C_{1} \cdot \mathbf{e}_{z}dg$$

$$- C_{1} \cdot \mathbf{e}_{x}di + C_{1} \cdot \mathbf{f}_{x}^{1}di_{1}$$

$$= C_{1} \cdot \mathbf{k}_{z}d\zeta + C_{1} \cdot \mathbf{e}_{z}dg$$

$$- C_{1} \cdot \mathbf{e}_{x}di \qquad (4.22)$$

since $C_1 \cdot f_x^1 = 0$. Similarly, differentiating $\mathcal{R}_2 = \mathcal{R}_z(\zeta)\mathcal{R}_x(i)\mathcal{R}_z(g)I_z\mathcal{R}_x(i_2)$, we find

$$y_2 \cdot (d\mathcal{R}_2(\Gamma, \Psi, \psi))\mathcal{R}_2(\Gamma, \Psi, \psi)^T x_1 = C_2 \cdot \mathbf{k}_z d\zeta + C_2 \cdot \mathbf{e}_z dg - C_2 \cdot \mathbf{e}_x di$$
(4.23)

The sum of (4.22) and (4.23) gives then

$$y_1 \cdot (d\mathcal{R}_1(\Gamma, \Psi, \psi))\mathcal{R}_1(\Gamma, \Psi, \psi)^T x_1 + y_2 \cdot (d\mathcal{R}_2(\Gamma, \Psi, \psi))\mathcal{R}_2(\Gamma, \Psi, \psi)^T x_2$$

= $C \cdot \mathbf{k}_z d\zeta + C \cdot \mathbf{e}_z dg - C \cdot \mathbf{e}_x di$
= $C_z d\zeta + \mathbf{G} d\mathbf{g}$

since

$$C = C_1 + C_2 = \operatorname{Ge}_{\mathsf{z}}$$

(which also implies $C \cdot e_x = 0$). The proof is complete, in view of (4.21).

It remains to prove the inductive step.

Lemma 4.3 Assume that the N-Deprit map $\Phi_*^N : \mathcal{C}_*^N \to \mathcal{D}_*^N$, is symplectic for a given $N \geq 2$. Then, the (N+1)- Deprit map $\Phi_*^{N+1} : \mathcal{C}_*^{N+1} \to \mathcal{D}_*^{N+1}$ is symplectic.

Proof. Without loss of generality, we shall restrict to the subset $\hat{\mathcal{C}}_*^{N+1}$ of \mathcal{C}_*^{N+1} where also

$$\hat{n}_i := \mathbf{k}_{\mathbf{z}} \times S_i \neq 0 \quad 1 \le i \le N, \quad \bar{n}_i := \mathbf{k}_{\mathbf{z}} \times C_i \neq 0 \quad 1 \le i \le N+1$$

$$(4.24)$$

and then we will recover the result by continuity. Under the assumption (4.24), we can view the Deprit map Φ_*^{N+1} in *Delaunay variables*, namely, we shall write $\Phi_*^{N+1} = \hat{\Phi}_*^{N+1} \circ \Phi_D^{N+1}$, where $(L, \mathbf{G}, \Theta, \ell, \mathbf{g}, \vartheta) = \Phi_D^N(y, x)$ is the map (4.9) which defines the Delaunay variables. Let $\hat{\mathcal{D}}_*^{N+1} := \Phi_D^{N+1}(\hat{\mathcal{C}}_*^{N+1})$. Then, $\hat{\Phi}_*^2$ is symplectic on $\hat{\mathcal{D}}_*^2$, by Lemma 4.2; $\hat{\Phi}_*^N$ is symplectic on $\hat{\mathcal{D}}_*^N$ by assumption. We equivalently prove that $\hat{\Phi}_*^{N+1}$ is symplectic on $\hat{\mathcal{D}}_*^{N+1}$, which will conclude (since Φ_D^n is symplectic on $\hat{\mathcal{C}}_*^n$, for any n). Neglecting the variables (L, ℓ) (on which $\hat{\Phi}_*^{N+1}$ acts as the identity), the map $\hat{\Phi}_*^{N+1}$ is described by equations

$$\begin{cases} \Gamma_{i} = G_{i} & 1 \leq i \leq N+1 \\ \Psi_{i} = |\sum_{1 \leq j \leq i+1} C_{j}| & 1 \leq i \leq N-1 \\ G = |S_{N+1}| \\ C_{z} = \sum_{1 \leq j \leq N+1} \Theta_{j} \\ \gamma_{i} = g_{i} + \alpha_{C_{i}}(n_{i}, \bar{n}_{i}) & 1 \leq i \leq N+1 \\ \psi_{i} = \alpha_{S_{i+1}}(n_{i+2}, n_{i+1}) & 1 \leq i \leq N-1 \\ g = \alpha_{S_{N+1}}(\bar{n}, -n_{N+1}) \\ \zeta = \alpha_{k_{z}}(k_{x}, \bar{n}) \end{cases}$$

where the C_i 's, hence, $S_i = C_1 + \cdots + C_{i+1}$ and the nodes $\bar{n}_i = k_z \times C_i$, $n_i = S_i \times C_i$, $\bar{n} = k_z \times S_{N+1}$, are thought as functions of the Delaunay variables ²⁸.

²⁸The angular momenta C_i 's, in terms of the Delaunay variables are

$$C_i = \begin{pmatrix} \sqrt{G_i^2 - \Theta_i^2} \sin \vartheta_i \\ -\sqrt{G_i^2 - \Theta_i^2} \cos \vartheta_i \\ \Theta_i \end{pmatrix} .$$

This follows from the definitions of $(G_i, \Theta_i, \vartheta_i)$.

Let us introduce the following notations

$$z_{i} := (L_{i}, G_{i}, \Theta_{i}, \ell_{i}, g_{i}, \vartheta_{i}) , \qquad Z_{i} := (L_{i}, \Gamma_{i}, \Psi_{i}, \ell_{i}, \gamma_{i}, \psi_{i}) \\ \begin{cases} z := (z_{1}, \cdots, z_{N+1}) \\ \hat{z}_{i} := (z_{1}, \cdots, z_{N}) \end{cases} \qquad \begin{cases} Z := (Z_{1}, \cdots, Z_{N+1}) \\ \hat{Z}_{i} := (Z_{1}, \cdots, Z_{N}) \end{cases}$$

Now, if $z \in \hat{\mathcal{D}}_*^{N+1}$, then, the point \hat{z} lies in the domain of definition $\hat{\mathcal{D}}_*^N$ of $\hat{\Phi}_*^N$ and we can set ²⁹

$$\tilde{\Phi}_{*}^{N+1}(z) = \tilde{\Phi}_{*}^{N+1}(\hat{z}, z_{N}) = \hat{\Phi}_{*}^{N}(\hat{z}, z_{N+1}) =: Z' = (\hat{Z}', z_{N+1})$$
(4.25)

i.e., $\tilde{\Phi}_*^{N+1}$ acts as $\hat{\Phi}_*^N$ on \hat{z} , while on the last block z_{N+1} of the Delaunay variables acts as the identity. $\tilde{\Phi}_*^{N+1}$ is thus symplectic since $\hat{\Phi}_*^N$ is, as already outlined. Now, leaving the remaining variables unchanged, we apply $\hat{\Phi}_*^2$ to the two blocks consisting, the former, to the block of variables

$$z'_N := (L'_N, \mathbf{G}' = \Psi'_{N-1}, C'_{\mathbf{z}} = \Psi'_N, \ell_{N+1}, \mathbf{g}' = \psi'_{N-1}, \zeta' = \psi'_N)$$

and, the latter, to the block of variables z_{N+1} left unvaried by $\tilde{\Phi}_*^{N+1}$. We define, then,

$$\tilde{\Phi}^2_*: \ Z' \to Z$$

as follows:

$$\begin{cases} Z_i = Z'_i & 1 \le i \le N-2\\ (L_{N-1}, \Gamma_{N-1}, \Gamma_N, \ell_{N-1}, \gamma_{N-1}, \gamma_N) = (L'_{N-1}, \Gamma'_{N-1}, \Gamma'_N, \ell'_{N-1}, \gamma'_{N-1}, \gamma'_N)\\ ((L_N, \Psi_{N-1}, \Psi_N, \ell_{N+1}, \psi_{N-1}, \psi_N), (L_{N+1}, \Gamma_{N+1}, \Psi_{N+1}, \ell_{N+1}, \gamma_{N+1}, \Psi_{N+1})) = \hat{\Psi}_*^2(z'_N, z_{N+1}) \end{cases}$$

Also $\tilde{\Phi}_*^2$ is symplectic, because it is obtained lifting $\hat{\Phi}_*^2$ with the identity map, and, therefore, so is the composition

$$\tilde{\Phi}_*^2 \circ \tilde{\Phi}_*^{N+1} . \tag{4.26}$$

The claim, now, follows upon recognizing that (4.26) reconstructs $\hat{\Phi}_*^{N+1}$:

$$\tilde{\Phi}_*^2 \circ \tilde{\Phi}_*^{N+1} = \hat{\Phi}_*^{N+1} . \tag{4.27}$$

The key point while checking (4.27) is

$$\begin{split} \psi_{N-1}|_{\tilde{\Phi}^2_* \circ \tilde{\Phi}^{N+1}_*} &= g' + \alpha_{S_N}(S_{N+1} \times S_N, \mathbf{k}_z \times S_N) \\ &= \alpha_{S_N}(\mathbf{k}_z \times S_N, -S_N \times C_N) + \alpha_{S_N}(S_{N+1} \times S_N, \mathbf{k}_z \times S_N) \\ &= \alpha_{S_N}(S_{N+1} \times S_N, -S_N \times C_N) \\ &= \alpha_{S_N}(-S_{N+1} \times C_{N+1}, -S_N \times C_N) \\ &= \alpha_{S_N}(S_{N+1} \times C_{N+1}, S_N \times C_N) \\ &= \psi_{N-1}|_{\hat{\Phi}^{N+1}_*}, \end{split}$$

since, by definition, $g' = \alpha_{S_N}(k_z \times S_N, -S_N \times C_N)$ and $S_N = S_{N+1} - C_{N+1}$.

²⁹In (4.25), let $\hat{Z}' = (Z'_1, \dots, Z'_N)$, with $Z'_i = (L'_i, \Gamma'_i, \Psi'_i, \ell'_i, \gamma'_i, \psi'_i)$, for $1 \le i \le N$.

4.2 Regularization

The action-angle Deprit variables discussed in the previous section become singular when some of (4.4) do not hold. In this paragraph, we discuss a Poincaré regularization. For N = 2, we put

$$\begin{cases} G := G \\ g := g + \zeta \\ P := \sqrt{2(G - C_z)} \cos \zeta \\ Q := -\sqrt{2(G - C_z)} \sin \zeta \end{cases} \begin{cases} \Lambda_i = L_i \\ \lambda_i = \ell_i + \gamma_i \end{cases} \\ \zeta = -\sqrt{2(G - C_z)} \sin \zeta \\ \eta_i = \sqrt{2(L_i - \Gamma_i)} \cos \gamma_i \\ \xi_i = -\sqrt{2(L_i - \Gamma_i)} \sin \gamma_i \end{cases}$$

and we recover the "unfolding" of the Jacobi regularized coordinates. So, we discuss in detail only the case $N \geq 3$.

Let then $N \geq 3$ and let $\mathcal{C} \supset \mathcal{C}_*$ the set of $(y, x) \in \mathbb{R}^{3N} \times \mathbb{R}^{3N}$ where

$$\begin{cases}
e_i < 1 \\
\frac{C}{|C|} \cdot \mathbf{k}_z \neq -1 \\
\frac{C_1}{|C_1|} \cdot \frac{C_2}{|C_2|} \neq -1 \\
\frac{C_i}{|C_i|} \cdot \frac{S_i}{|S_i|} \neq -1 \\
\frac{C_i}{|C|} \cdot \frac{C_i}{|S_i|} \neq -1
\end{cases}$$
(4.28)

i.e., the eccentricities are allowed to go to 0; C_1 is allowed to go parallel to C_2 ; C_i are allowed to go parallel to S_i for $2 \le i \le N - 1$; C is allowed to go parallel to k_z .

Other regularizations than (4.28) (relatively to different choices for the signs of the dot products in (4.28)) might be discussed.

In order to regularize zero eccentricities and the first N-1 mutual inclinations, *i.e.*, in order to define a new set of variables in a region of the phase space where

 $e_i = 0$ for some $1 \le i \le N$, or $S_i \parallel C_i$ for some $2 \le i \le N - 1$,

we assume that the variables Λ , G satisfy

$$\left|\sum_{1 \le i \le N-1} \Lambda_i - \Lambda_N\right| < \mathcal{G} < \sum_{1 \le i \le N} \Lambda_i .$$
(4.29)

This guarantees that the configurations of the phase space corresponding to (simultanously) zero eccentricities and first N-1 mutual inclinations might be reached by the system, being inner points of the phase space. **Theorem 4.2** When Λ , G also satisfy (4.29), define the real-analytic symplectomorphism

$$\begin{split} \Phi_{\mathbf{r}} : & \left((L, \Gamma, \Psi), (\ell, \gamma, \psi) \right) \in \mathcal{D}_{*} \quad \rightarrow \quad \left((\Lambda, \lambda), (\eta, \xi), (p, q), (G, g), (P, Q) \right) \\ &= \quad \left((\Lambda_{1}, \cdots, \Lambda_{N}), (\lambda_{1}, \cdots, \lambda_{N}), (\eta_{1}, \cdots, \eta_{N}, \xi_{1}, \cdots, \xi_{N}) \right) \\ & \left(p_{1}, \cdots, p_{N-2}, q_{1}, \cdots, q_{N-2} \right), (G, g), (P, Q) \right) \\ & \in \left(\mathbb{R}^{N}_{+} \times \mathbb{T}^{N} \right) \times \left(\mathbb{R}^{N} \times \mathbb{R}^{N} \right) \times \left(\mathbb{R}^{N-2} \times \mathbb{R}^{N-2} \right) \times \\ & \times \left(\mathbb{R}_{+} \times \mathbb{T} \right) \times \left(\mathbb{R} \times \mathbb{R} \right) \end{split}$$

as follows. Let

$$\begin{split} H_i &:= L_i - \Gamma_i \qquad 1 \le i \le N \quad K_i := \begin{cases} \Gamma_1 - \Psi_1 + \Gamma_2 & i = 1\\ \Psi_{i-1} - \Psi_i + \Gamma_{i+1} & 2 \le i \le N-2 \end{cases} \\ \kappa_i &:= \sum_{i \le j \le N-2} \psi_j \quad 1 \le i \le N-2 \quad \hat{\kappa}_i := \begin{cases} \kappa_1 & i = 1\\ \kappa_{i-1} & 2 \le i \le N-1\\ 0 & i = N \end{cases} \\ h_i &:= \gamma_i + \hat{\kappa}_i \end{split}$$

and put

$$\begin{cases} \Lambda_i := L_i \\ \lambda_i := \ell_i + h_i \\ \eta_i := \sqrt{2H_i} \cos h_i \\ \xi_i := -\sqrt{2H_i} \sin h_i \end{cases} & 1 \le i \le N , \quad \begin{cases} p_i := \sqrt{2K_i} \cos \kappa_i \\ q_i := -\sqrt{2K_i} \sin \kappa_i \end{cases} & 1 \le i \le N - 2 \\ q_i := -\sqrt{2K_i} \sin \kappa_i \end{cases} & 1 \le i \le N - 2 \\ \begin{cases} G := G \\ g := g + \zeta \\ P := \sqrt{2(G - C_z)} \cos \zeta \\ Q := -\sqrt{2(G - C_z)} \sin \zeta \end{cases} \end{cases}$$

Then, the map $\Phi_{BD} := \Phi_r \circ \Phi_*$ extends to a real-analytic symplectomorphism on C. The variables $((\Lambda, \lambda) \ (\eta, \xi), \ (p, q), \ (G, g), \ (P, Q))$ will be referred as regularized Deprit variables. Observe that, now, the role of cyclic variables for (4.1) is played by (P, Q, g). **Remark 4.2** The inverse $\phi_r := \Phi_r^{-1}$ on $\mathcal{D}_{r*} := \Phi_r(\mathcal{D}_*)$ is given by

$$\begin{cases} L_{i} = \Lambda_{i} \\ \ell_{i} = \lambda_{i} - h_{i} \end{cases}, \quad \begin{cases} \Gamma_{i} = \Lambda_{i} - \frac{\eta_{i}^{2} + \xi_{i}^{2}}{\gamma_{i} = h_{i} - \hat{\kappa}_{i}} \\ \gamma_{i} = h_{i} - \hat{\kappa}_{i} \end{cases}$$
$$\Psi_{i} = \begin{cases} \sum_{1 \le j \le i+1} \left(\Lambda_{j} - \frac{\eta_{j}^{2} + \xi_{j}^{2}}{2}\right) - \sum_{1 \le j \le i} \frac{p_{j}^{2} + q_{j}^{2}}{2} & 1 \le i \le N - 2 \\ G & i = N - 1 \\ G - \frac{P^{2} + Q^{2}}{2} & i = N \end{cases}$$
$$\psi_{i} = \begin{cases} \kappa_{i} - \kappa_{i+1} & 1 \le i \le N - 3 \\ \kappa_{N-2} & i = N - 2 \\ g - \zeta(P, Q) & i = N - 1 \\ \zeta(P, Q) & i = N \end{cases}$$
(4.30)

where

$$h_i = \arg(\eta_i, -\xi_i) \ (1 \le i \le N), \quad k_i = \arg(p_i, -q_i) \ (1 \le i \le N-2), \quad \zeta(P, Q) = \arg(P, -Q)$$

(the previous expressions are well put on \mathcal{D}_{r*}).

Remark 4.3 The domain \mathcal{D}_{r*} is the set of $(\Lambda, \lambda, \eta, \xi, p, q, G, g, P, Q)$ where $\Lambda \in \mathbb{R}^N_+$, $(\lambda, g) \in \mathbb{T}^N \times \mathbb{T}$ and the functions Γ , Ψ as in (4.30) verify (4.5).

Proof. Put

$$\mathcal{D}_0 := \left\{ (P, Q) = 0 \right\} \bigcup_{1 \le i \le N} \left\{ (\eta_i, \xi_i) = 0 \right\} \bigcup_{1 \le i \le N-2} \left\{ (p_i, q_i) = 0 \right\}$$

and observe that

$$\mathcal{D}_{r*} \cap \mathcal{D}_0 = \emptyset$$

We prove that the map

$$\begin{pmatrix} (\Lambda,\lambda), (\eta,\xi), (p,q), (G,g), (P,Q) \end{pmatrix} \in \mathcal{D}_{r*} \rightarrow (y,x) = \begin{pmatrix} (y_1,\cdots,y_N), (x_1,\cdots,x_N) \end{pmatrix} \\ = \phi_* \circ \phi_r \Big((\Lambda,\lambda), (\eta,\xi), (p,q), (G,g), (P,Q) \Big) \\ := \phi_{BD} \Big((\Lambda,\lambda), (\eta,\xi), (p,q), (G,g), (P,Q) \Big)$$

where $\phi_* := \Phi_*^{-1}$, can be bijectively and analytically (hence, symplectically) extended to the domain

$$\mathcal{D}_{\mathrm{r}}:=\mathcal{D}_{\mathrm{r}*}igcup\mathcal{D}_{0}$$
 .

By Proposition 4.1, $\phi_{\rm fr}$ is defined by

$$\begin{cases} x_i = (\mathcal{R}_i \circ \phi_{\mathbf{r}}) X_i \circ \phi_{\mathbf{r}} \\ y_i = (\mathcal{R}_i \circ \phi_{\mathbf{r}}) Y_i \circ \phi_{\mathbf{r}} \end{cases} \quad 1 \le i \le N$$

$$(4.31)$$

We explicit ate the matrices \mathcal{R}_i defined in Lemma 4.1 as 30

$$\mathcal{R}_{i} = \mathcal{R}_{z}(\zeta)\mathcal{R}_{x}(i)\mathcal{R}_{z}(g) \\
\times \begin{cases}
I_{z}\mathcal{R}_{x}(i_{N}) & i = N \\
\mathcal{R}_{x}(\tilde{i}_{N-1}) \left(\prod_{j=2}^{N-i} \mathcal{R}_{z}(\psi_{N-j})\mathcal{R}_{x}(\tilde{i}_{N-j})\right) \mathcal{R}_{z}(\psi_{i-1})I_{z}\mathcal{R}_{x}(i_{i}) & 2 \leq i \leq N-1 \\
\mathcal{R}_{x}(\tilde{i}_{N-1}) \left(\prod_{j=2}^{N-2} \mathcal{R}_{z}(\psi_{N-j})\mathcal{R}_{x}(\tilde{i}_{N-j})\right) \mathcal{R}_{z}(\psi_{1})\mathcal{R}_{x}(i_{1}) & i = 1
\end{cases}$$

$$(4.32)$$

and we think (without changing their names) the "inclinations" i, i_j, \tilde{i}_j expressed in regularized Deprit variables (*i* is a function of *P*, *Q*, *G*; i_i, \tilde{i}_i are functions of Λ , (η, ξ) ,

³⁰Clearly, in the second and third line of (4.32), the productories do not appear when N = 3.

(p,q) and G), then, using the expressions for ψ_i , with $1 \leq i \leq N-2$, given (4.30), we can rewrite \mathcal{R}_i in terms of the regularized Deprit variables as

$$\mathcal{R}_{i} \circ \phi_{\mathrm{r}} = \mathfrak{R}_{0} \times \begin{cases} \mathrm{I}_{\mathrm{z}} \mathcal{R}_{\mathrm{x}}(i_{N}) & i = N \\ \mathcal{R}_{\mathrm{x}}(\tilde{i}_{N-1}) \left(\prod_{j=2}^{N-i} \tilde{\mathcal{S}}_{N-j} \right) \mathrm{I}_{\mathrm{z}} \mathcal{S}_{i} \mathcal{R}_{\mathrm{z}}(\kappa_{i-1}) & 2 \leq i \leq N-1 \\ \mathcal{R}_{\mathrm{x}}(\tilde{i}_{N-1}) \left(\prod_{j=2}^{N-2} \tilde{\mathcal{S}}_{N-j} \right) \mathcal{S}_{1} \mathcal{R}_{\mathrm{z}}(\kappa_{1}) & i = 1 \\ =: \mathfrak{R}_{0} \mathfrak{R}_{i} \mathcal{R}_{\mathrm{z}}(\hat{\kappa}_{i}) \qquad (4.33) \end{cases}$$

where

$$\mathfrak{R}_0 := \mathcal{R}_{\mathbf{z}}(\zeta(P,Q))\mathcal{R}_{\mathbf{x}}(i)\mathcal{R}_{\mathbf{z}}(-\zeta(P,Q)+g)$$
(4.34)

and

$$\tilde{\mathcal{S}}_{j} := \mathcal{R}_{z}(\kappa_{j})\mathcal{R}_{x}(\tilde{i}_{j})\mathcal{R}_{z}(-\kappa_{j}) \quad 2 \leq j \leq N-2
\mathcal{S}_{j} := \begin{cases} \mathcal{R}_{z}(\kappa_{1})\mathcal{R}_{x}(i_{1})\mathcal{R}_{z}(-\kappa_{1}) & j=1 \\ \mathcal{R}_{z}(\kappa_{j-1})\mathcal{R}_{x}(i_{j})\mathcal{R}_{z}(-\kappa_{j-1}) & 2 \leq j \leq N-1 \end{cases}$$
(4.35)

Lemma 4.4 With the convention $(p_0, q_0) := (p_1, q_1)$, $\Psi_0 := \Gamma_1$, on \mathcal{D}_{r*} , the matrices \mathcal{S}_j 's, $\tilde{\mathcal{S}}_j$'s have the following expressions:

$$S_{j} = \begin{pmatrix} 1 - q_{j-1}^{2} \mathfrak{c}_{j} & -p_{j-1} q_{j-1} \mathfrak{c}_{j} & -q_{j-1} \mathfrak{s}_{j} \\ -p_{j-1} q_{j-1} \mathfrak{c}_{j} & 1 - p_{j-1}^{2} \mathfrak{c}_{j} & -p_{j-1} \mathfrak{s}_{j} \\ q_{j-1} \mathfrak{s}_{j} & p_{j-1} \mathfrak{s}_{j} & 1 - (p_{j-1}^{2} + q_{j-1}^{2}) \mathfrak{c}_{j} \end{pmatrix} \quad (1 \le j \le N - 1)$$

$$\tilde{S}_{j} = \begin{pmatrix} 1 - q_{j}^{2} \tilde{\mathfrak{c}}_{j} & -p_{j} q_{j} \tilde{\mathfrak{c}}_{j} & -q_{j} \tilde{\mathfrak{s}}_{j} \\ -p_{j} q_{j} \tilde{\mathfrak{c}}_{j} & 1 - p_{j}^{2} \tilde{\mathfrak{c}}_{j} & -p_{j} \tilde{\mathfrak{s}}_{j} \\ q_{j} \tilde{\mathfrak{s}}_{j} & p_{j} \tilde{\mathfrak{s}}_{j} & 1 - (p_{j}^{2} + q_{j}^{2}) \tilde{\mathfrak{c}}_{j} \end{pmatrix} \quad (2 \le j \le N - 2) \quad (4.36)$$

where

where Γ_i , Ψ_j are thought as functions of $(\Lambda, \eta, \xi, p, q)$:

$$\begin{cases} \Gamma_{i} = \Lambda_{i} - \frac{\eta_{i}^{2} + \xi_{i}^{2}}{2} & 1 \leq i \leq N \\ \Psi_{i} = \sum_{1 \leq i \leq i+1} \left(\Lambda_{j} - \frac{\eta_{j}^{2} + \xi_{j}^{2}}{2} \right) - \sum_{1 \leq j \leq i} \frac{p_{j}^{2} + q_{j}^{2}}{2} & 1 \leq i \leq N-2 \end{cases}$$
(4.38)

Proof. Let us prove, for istance, the expression for \mathcal{S}_1 , since the other ones are similar. We have

$$S_{1} = \mathcal{R}_{z}(\kappa_{1})\mathcal{R}_{x}(i_{1})\mathcal{R}_{z}(-k_{1})$$

$$= \begin{pmatrix} 1 - \sin^{2}k_{1}(1 - \cos i_{1}) & \sin k_{1}\cos k_{1}(1 - \cos i_{1}) & \sin k_{1}\sin i_{1} \\ \sin k_{1}\cos k_{1}(1 - \cos i_{1}) & 1 - \cos^{2}k_{1}(1 - \cos i_{1}) & -\cos k_{1}\sin i_{1} \\ -\sin k_{1}\sin i_{1} & \cos k_{1}\sin i_{1} & \cos i_{1} \end{pmatrix}$$

Now, letting $\kappa_1 = \arg(p_1, -q_1)$, hence,

$$\begin{cases} \cos \kappa_1 = \frac{p_1}{\sqrt{p_1^2 + q_1^2}} \\ \sin \kappa_1 = -\frac{q_1}{\sqrt{p_1^2 + q_1^2}} \end{cases}$$

and recognizing that

$$\mathfrak{c}_1 = \frac{\Gamma_2 - \Gamma_1 + \Psi_1}{4\Gamma_1 \Psi_1} = \frac{1 - \cos i_1}{p_1^2 + q_1^2} , \quad \mathfrak{s}_1 = \sqrt{\mathfrak{c}_1 (2 - (p_1^2 + q_1^2)\mathfrak{c}_1)} = \frac{\sin i_1}{\sqrt{p_1^2 + q_1^2}}$$

the claim follows. Recall, in fact, the definition of i_1 :

$$\cos i_1 := \frac{\Psi_1^2 + \Gamma_1^2 - \Gamma_2^2}{2\Psi_1\Gamma_1} \qquad i_1 \in (0, \pi)$$

which gives

$$1 - \cos i_1 = \frac{\Gamma_2^2 - (\Psi_1 - \Gamma_1)^2}{2\Psi_1\Gamma_1}$$

= $\frac{(\Gamma_1 + \Gamma_2 - \Psi_1)(\Gamma_2 - \Gamma_1 + \Psi_1)}{2\Psi_1\Gamma_1}$
= $\frac{\Gamma_2 - \Gamma_1 + \Psi_1}{4\Psi_1\Gamma_1}(p_1^2 + q_1^2)$
= $\mathfrak{c}_1(p_1^2 + q_1^2)$

since

$$\Gamma_1 + \Gamma_2 - \Psi_1 = \frac{p_1^2 + q_1^2}{2} ,$$

hence (as $i_1 \in (0, \pi)$),

$$\sin i_1 = \sqrt{1 - \cos^2 i_1} = \sqrt{1 - \left(1 - \mathfrak{c}_1(p_1^2 + q_1^2)\right)^2} = \sqrt{\mathfrak{c}_1(2 - (p_1^2 + q_1^2)\mathfrak{c}_1)}\sqrt{p_1^2 + q_1^2} = \mathfrak{s}_1\sqrt{p_1^2 + q_1^2}$$

This completes the proof.

By the previous Lemma, the matrices $\tilde{\mathcal{S}}_j$, \mathcal{S}_j can be analytically extended to \mathcal{D}_r . It is not difficult to prove that the same holds for the matrix \mathfrak{R}_0 , hence, for \mathfrak{R}_j . The transformation Φ^{-1} is thus regularized on \mathcal{D}_r , being given by

$$\begin{cases} y_i = \mathfrak{R}_0 \mathfrak{R}_i \hat{y}_i \\ x_i = \mathfrak{R}_0 \mathfrak{R}_i \hat{x}_i \end{cases}$$

where $(\Lambda_i, \lambda_i, \eta_i, \xi_i) \to (\hat{y}_i, \hat{x}_i)$ is the embedding ³¹ in $\mathbb{R}^3 \times \mathbb{R}^3$ of the (\hat{m}_i, \tilde{m}_i) -Plane Delaunay-Poincaré Map. In fact, using the expressions for L, Γ, ℓ, γ in (4.30), we find that

$$(Y_i \circ \phi_{\mathbf{r}}, X_i \circ \phi_{\mathbf{r}}) = \left(\mathcal{R}_{\mathbf{z}}(-\hat{\kappa}_i)\hat{y}_i(\Lambda_i, \lambda_i, \eta_i, \xi_i), \ \mathcal{R}_{\mathbf{z}}(-\hat{\kappa}_i)\hat{x}_i(\Lambda_i, \lambda_i, \eta_i, \xi_i) \right)$$

hence, use (4.31), (4.33). The bijectivity of this extension is trivial.

In particular, we have proven

Proposition 4.2 Let $\mathcal{D}_{\mathbf{r}}$ is the set of $((\Lambda, \lambda), (\eta, \xi), (p, q), (G, g), (P, Q)) \in (\mathbb{R}^{N}_{+} \times \mathbb{T}^{N}) \times (\mathbb{R}^{N} \times \mathbb{R}^{N}) \times (\mathbb{R}^{N-2} \times \mathbb{R}^{N-2}) \times (\mathbb{R}_{+} \times \mathbb{T}) \times (\mathbb{R} \times \mathbb{R})$ where the functions (4.38) verify

$$\begin{cases} \Lambda_i > 0 \\ 0 < \Gamma_i \le \Lambda_i \ (1 \le i \le N) \\ |\Psi_{i-1} - \Gamma_{i+1}| < \Psi_i \le \Psi_{i-1} + \Gamma_{i+1} \\ |\Psi_{N-2} - \Gamma_N| < G < \Psi_{N-2} + \Gamma_N \\ -G < C_z = G - \frac{P^2 + Q^2}{2} \le G \end{cases}$$
(15)

The (real-analytic and symplectic) "full reduction" change of variable

$$\phi_{\mathrm{BD}} := \Phi_{\mathrm{BD}}^{-1} = \phi_* \circ \phi_{\mathrm{r}} :$$

$$\left((\Lambda, \lambda), (\eta, \xi), (p, q), (G, g), (P, Q) \right) \in \mathcal{D}_{\mathrm{r}} \to (y, x) = \left((y_1, \cdots, y_N), (x_1, \cdots, x_N) \in \mathcal{C} \right)$$

expressing the cartesian coordinates (y, x) in terms of the regularized Deprit variables is given by

$$\begin{cases} y_i = \mathfrak{R}_0 \mathfrak{R}_i \hat{y}_i \\ x_i = \mathfrak{R}_0 \mathfrak{R}_i \hat{x}_i \end{cases} \quad 1 \le i \le N \tag{4.39}$$

where $\Re_0 = \Re_0((P,Q), (G,g))$ is (4.34), $\Re_i = \Re_i(\Lambda, (\eta, \xi), (p,q); G)$ is defined via (4.33), *i.e.*,

$$\mathfrak{R}_{i} = \begin{cases} \mathcal{R}_{\mathbf{x}}(\tilde{i}_{N-1}) \left(\prod_{j=2}^{N-2} \tilde{\mathcal{S}}_{N-j} \right) \mathcal{S}_{1} & i = 1 \\ \mathcal{R}_{\mathbf{x}}(\tilde{i}_{N-1}) \left(\prod_{j=2}^{N-i} \tilde{\mathcal{S}}_{N-j} \right) \mathbf{I}_{\mathbf{z}} \mathcal{S}_{i} & 2 \leq i \leq N-2 \\ \mathcal{R}_{\mathbf{x}}(\tilde{i}_{N-1}) \mathbf{I}_{\mathbf{z}} \mathcal{S}_{N-1} & i = N-1 \\ \mathbf{I}_{\mathbf{z}} \mathcal{R}_{\mathbf{x}}(i_{N}) & i = N \end{cases}$$

$$(4.40)$$

³¹This means taking the third coordinate of (\hat{y}_i, \hat{x}_i) equal to zero.

where \tilde{i}_{N-1} , $i_N \in (0,\pi)$ are the inclinations (well defined and regular on \mathcal{D}_r)

$$\tilde{i}_{N-1} = \cos^{-1}\left(\frac{G^2 - \Psi_{N-2}^2 - \Gamma_N^2}{2G\Psi_{N-2}}\right) , \quad i_N = \cos^{-1}\left(\frac{G^2 - \Gamma_N^2 - \Psi_{N-2}^2}{2G\Gamma_N}\right)$$

with \mathcal{S}_i $(1 \leq i \leq N-1)$, $\tilde{\mathcal{S}}_i$ $(2 \leq i \leq N-2)$ as in Lemma 4.4 and, finally, $(\Lambda_i, \lambda_i, \eta_i, \xi_i) \rightarrow (\hat{y}_i, \hat{x}_i)$ is the embedding in $\mathbb{R}^3 \times \mathbb{R}^3$ of the (\hat{m}_i, \tilde{m}_i) -Plane Delaunay-Poincaré Map.

We will refer to the map ϕ_{BD} defined via Proposition 4.2 as Regularized Deprit Map.

4.3 Partial Reduction

The regularized Deprit map discussed in Proposition 4.2 becomes singular when

$$\sum_{1 \le i \le N} \left(\Lambda_i - \frac{\eta_i^2 + \xi_i^2}{2} \right) - \sum_{1 \le i \le N-2} \frac{p_i^2 + q_i^2}{2} = G , \qquad (4.41)$$

which corresponds to the configuration with $S_{N-1} = \sum_{1 \leq i \leq N-1} C_i$ parallel to C_N (the two rotations $\mathcal{R}_{\mathbf{x}}(\tilde{i}_{N-1})$, $\mathcal{R}_{\mathbf{x}}(i_N)$ loss their regularity). Consider then the transformation, denoted as ϕ_{pr} ("pr" stands for "partial reduction") which acts as

$$\begin{cases} G = \sum_{1 \le k \le N} \left(\Lambda_k - \frac{\bar{\eta}_k^2 + \bar{\xi}_k^2}{2} \right) - \sum_{1 \le k \le N-1} \frac{\bar{p}_k^2 + \bar{q}_k^2}{2} \\ g = \arg\left(\bar{p}_{N-1}, -\bar{q}_{N-1}\right) \end{cases}$$
$$\lambda_i = \bar{\lambda}_i - g \quad 1 \le i \le N \\ \left(\begin{array}{c} \eta_i \\ \xi_i \end{array} \right) = \mathcal{R}_z(g) \left(\begin{array}{c} \bar{\eta}_i \\ \bar{\xi}_i \end{array} \right) \qquad 1 \le i \le N \\ \left(\begin{array}{c} p_j \\ q_j \end{array} \right) = \mathcal{R}_z(g) \left(\begin{array}{c} \bar{p}_j \\ \bar{q}_j \end{array} \right) \qquad 1 \le j \le N-2 \end{cases}$$
(4.42)

leaving the remaining variables unvaried. It not difficult to check that ϕ_{pr} is symplectic, since the Liouville 1–form remains unvaried:

$$\sum_{1 \le i \le N} \Lambda_i d\lambda_i + \sum_{1 \le i \le N} I_i d\varphi_i + \sum_{1 \le i \le N-2} J_i d\psi_i + Gdg + PdQ$$
$$= \sum_{1 \le i \le N} \Lambda_i d\bar{\lambda}_i + \sum_{1 \le i \le N} I_i d\bar{\varphi}_i + \sum_{1 \le i \le N-2} J_i d\bar{\psi}_i + J_{N-1} d\bar{\psi}_{N-1} + PdQ$$

where (I_i, φ_i) , (J_j, ψ_j) , $(I_i, \overline{\varphi_i})$, $(J_k, \overline{\psi_k})$ are the polar coordinates associated to (η_i, ξ_i) , (p_j, q_j) , $(\overline{\eta_i}, \overline{\xi_i})$, $(\overline{p_k}, \overline{q_k})$, with the indices i, j, k running on $1 \le i \le N, 1 \le j \le N-2$, $1 \le k \le N-1$.

In terms of the variables $((\Lambda, \bar{\lambda}), (\bar{\eta}, \bar{\xi}), (\bar{p}, \bar{q}), (P, Q))$, the functions $\Gamma = (\Gamma_1, \dots, \Gamma_N), \Psi = (\Psi_1, \dots, \Psi_N)$ are

$$\begin{cases} \Gamma_{i} = \Lambda_{i} - \frac{\bar{\eta}_{i}^{2} + \bar{\xi}_{i}^{2}}{2} & 1 \leq i \leq N \\ \Psi_{i} = \sum_{1 \leq i \leq i+1} \left(\Lambda_{j} - \frac{\bar{\eta}_{j}^{2} + \bar{\xi}_{j}^{2}}{2} \right) - \sum_{1 \leq j \leq i} \frac{\bar{p}_{j}^{2} + \bar{q}_{j}^{2}}{2} & 1 \leq i \leq N-1 \quad (\Psi_{N-1} = G) \quad (4.43) \\ \Psi_{N} = C_{z} = G - \frac{P^{2} + Q^{2}}{2} = \sum_{1 \leq i \leq N} \left(\Lambda_{j} - \frac{\bar{\eta}_{j}^{2} + \bar{\xi}_{j}^{2}}{2} \right) - \sum_{1 \leq j \leq N-1} \frac{\bar{p}_{j}^{2} + \bar{q}_{j}^{2}}{2} - \frac{P^{2} + Q^{2}}{2} \end{cases}$$

Denote now \mathcal{D}_{pr} the subset of

$$\left((\Lambda,\bar{\lambda}),(\bar{\eta},\bar{\xi}),(\bar{p},\bar{q}),(P,Q)\right) \in \mathbb{R}^{N}_{+} \times \mathbb{T}^{N} \times (\mathbb{R}^{N} \times \mathbb{R}^{N}) \times (\mathbb{R}^{N-1} \times \mathbb{R}^{N-1}) \times \mathbb{R} \times \mathbb{R}^{(4.44)}$$

where

$$\begin{cases}
\Lambda_{i} > 0 \\
0 < \Gamma_{i} \le \Lambda_{i} \ (1 \le i \le N) \\
|\Psi_{i-1} - \Gamma_{i+1}| < \Psi_{i} \le \Psi_{i-1} + \Gamma_{i+1} \quad (1 \le i \le N - 1) \\
-G < \Psi_{N} = C_{z} \le G
\end{cases}$$
(4.45)

(*i.e.*, allow also $S_{N-1} \parallel C_N$, which corresponds to $(p_{N-1}, q_{N-1}) = 0$). Then, the transormations $\phi_{\rm pr}$ regularizes, as the following Proposition claims, the proof of which is omitted.

Proposition 4.3 Let \mathcal{D}_r be defined via (4.44)÷(4.45). The real-analytic and symplectic change of variable

$$\phi_{\mathrm{BD,pr}} := \phi_{\mathrm{BD}} \circ \phi_{\mathrm{pr}} :$$

$$\left((\Lambda, \bar{\lambda}), (\bar{\eta}, \bar{\xi}), (\bar{p}, \bar{q}), (P, Q) \right) \in \mathcal{D}_{\mathrm{pr}} \to (y, x) = \left((y_1, \cdots, y_N), (x_1, \cdots, x_N) \in \mathcal{C} \right)$$

expressing the cartesian coordinates (y, x) in terms of the partially reduced, regularized Deprit variables $((\Lambda, \bar{\lambda}), (\bar{\eta}, \bar{\xi}), (\bar{p}, \bar{q}), (P, Q))$ is given by

$$\begin{cases} y_i = \mathfrak{R}_0^{\mathrm{pr}} \mathfrak{R}_i^{\mathrm{pr}} \hat{y}_i \\ x_i = \mathfrak{R}_0^{\mathrm{pr}} \mathfrak{R}_i^{\mathrm{pr}} \hat{x}_i \end{cases} \quad 1 \le i \le N \tag{4.46}$$

where $(\Lambda_i, \lambda_i, \eta_i, \xi_i) \rightarrow (\hat{y}_i, \hat{x}_i)$ is the embedding in $\mathbb{R}^3 \times \mathbb{R}^3$ of the (\hat{m}_i, \tilde{m}_i) -Plane Delaunay-Poincaré Map and

$$\mathfrak{R}_0^{\mathrm{pr}} = \mathcal{R}_{\mathrm{z}}(\zeta(P,Q))\mathcal{R}_{\mathrm{x}}(i)\mathcal{R}_{\mathrm{z}}(-\zeta(P,Q)) ,$$

with

$$i = \cos^{-1}\left(1 - \frac{P^2 + Q^2}{2\Psi_{N-1}}\right)$$
, $\zeta = \arg(P, -Q)$

and

$$\mathfrak{R}_{i}^{\mathrm{pr}} = \begin{cases} \left(\prod_{j=1}^{N-2} \tilde{\mathcal{S}}_{N-j}^{\mathrm{pr}}\right) \mathcal{S}_{1}^{\mathrm{pr}} & i=1\\ \left(\prod_{j=1}^{N-i} \tilde{\mathcal{S}}_{N-j}^{\mathrm{pr}}\right) \mathrm{I}_{\mathrm{z}} \mathcal{S}_{i}^{\mathrm{pr}} & 2 \leq i \leq N-1\\ \mathrm{I}_{\mathrm{z}} \mathcal{S}_{N}^{\mathrm{pr}} & i=N \end{cases}$$

where the matrices $\mathcal{S}_{j}^{\mathrm{pr}}$'s, $\tilde{\mathcal{S}}_{j}^{\mathrm{pr}}$'s have the following expressions:

$$\mathcal{S}_{j}^{\mathrm{pr}} = \begin{pmatrix} 1 - \bar{q}_{j-1}^{2} \mathfrak{c}_{j} & -\bar{p}_{j-1} \bar{q}_{j-1} \mathfrak{c}_{j} & -\bar{q}_{j-1} \mathfrak{s}_{j} \\ -\bar{p}_{j-1} \bar{q}_{j-1} \mathfrak{c}_{j} & 1 - \bar{p}_{j-1}^{2} \mathfrak{c}_{j} & -\bar{p}_{j-1} \mathfrak{s}_{j} \\ \bar{q}_{j-1} \mathfrak{s}_{j} & \bar{p}_{j-1} \mathfrak{s}_{j} & 1 - (\bar{p}_{j-1}^{2} + \bar{q}_{j-1}^{2}) \mathfrak{c}_{j} \end{pmatrix} \qquad (1 \le j \le N) \\
\tilde{\mathcal{S}}_{j}^{\mathrm{pr}} = \begin{pmatrix} 1 - \bar{q}_{j}^{2} \tilde{\mathfrak{c}}_{j} & -\bar{p}_{j} \bar{q}_{j} \tilde{\mathfrak{c}}_{j} & -\bar{q}_{j} \tilde{\mathfrak{s}}_{j} \\ -\bar{p}_{j} \bar{q}_{j} \tilde{\mathfrak{c}}_{j} & 1 - \bar{p}_{j}^{2} \tilde{\mathfrak{c}}_{j} & -\bar{p}_{j} \tilde{\mathfrak{s}}_{j} \\ \bar{q}_{j} \tilde{\mathfrak{s}}_{j} & \bar{p}_{j} \tilde{\mathfrak{s}}_{j} & 1 - (\bar{p}_{j}^{2} + \bar{q}_{j}^{2}) \tilde{\mathfrak{c}}_{j} \end{pmatrix} \qquad (2 \le j \le N - 1) \\$$

$$(4.47)$$

where

$$\begin{cases} \mathfrak{c}_{1} := \frac{1-\cos i_{1}}{\bar{p}_{1}^{2}+\bar{q}_{1}^{2}} = \frac{\Gamma_{2}-\Gamma_{1}+\Psi_{1}}{4\Gamma_{1}\Psi_{1}} \\ \mathfrak{s}_{1} := \frac{\sin i_{1}}{\sqrt{\bar{p}_{1}^{2}+\bar{q}_{1}^{2}}} = \sqrt{\mathfrak{c}_{1}(2-(\bar{p}_{1}^{2}+\bar{q}_{1}^{2})\mathfrak{c}_{1})} \\ \\ \begin{cases} \mathfrak{c}_{j} := \frac{1-\cos i_{j}}{\bar{p}_{j-1}^{2}+\bar{q}_{j-1}^{2}} = \frac{\Psi_{j-2}-\Gamma_{j}+\Psi_{j-1}}{4\Gamma_{j}\Psi_{j-1}} \\ \mathfrak{s}_{j} := \frac{\sin i_{j}}{\sqrt{\bar{p}_{j-1}^{2}+\bar{q}_{j-1}^{2}}} = \sqrt{\mathfrak{c}_{j}(2-(\bar{p}_{j-1}^{2}+\bar{q}_{j-1}^{2})\mathfrak{c}_{j})} \\ \end{cases} \quad (2 \leq j \leq N) \\ \\ \begin{cases} \tilde{\mathfrak{c}}_{j} := \frac{1-\cos \tilde{i}_{j}}{\bar{p}_{j}^{2}+\bar{q}_{j}^{2}} = \frac{\Gamma_{j+1}+\Psi_{j}-\Psi_{j-1}}{4\Psi_{j}\Psi_{j-1}} \\ \tilde{\mathfrak{s}}_{j} := \frac{\sin i_{j}}{\sqrt{\bar{p}_{j}^{2}+\bar{q}_{j}^{2}}} = \sqrt{\tilde{\mathfrak{c}}_{j}(2-(\bar{p}_{j}^{2}+\bar{q}_{j}^{2})\tilde{\mathfrak{c}}_{j})} \\ \end{cases} \quad (2 \leq j \leq N-1) \end{cases} \end{cases}$$

where, Γ_i , Ψ_j are thought as functions of $(\Lambda, \bar{\eta}, \bar{\xi}, \bar{p}, \bar{q})$:

$$\begin{cases} \Gamma_i = \Lambda_i - \frac{\bar{\eta}_i^2 + \bar{\xi}_i^2}{2} & 1 \le i \le N \\ \Psi_i = \sum_{1 \le i \le i+1} \left(\Lambda_j - \frac{\bar{\eta}_j^2 + \bar{\xi}_j^2}{2} \right) - \sum_{1 \le j \le i} \frac{\bar{p}_j^2 + \bar{q}_j^2}{2} & 1 \le i \le N-1 \quad (\Psi_{N-1} = G) \end{cases}$$
(4.49)

For $1 \leq i \leq N-1$ and $1 \leq j \leq N-2$, the functions $\mathfrak{c}_i, \mathfrak{s}_i, \mathfrak{\tilde{c}}_j, \mathfrak{\tilde{s}}_j$ defined in (4.48) coincide with the corresponding functions related to the full reduction (eq. (4.37)).

We will refer to the maps ϕ_{BD} , $\phi_{BD,pr}$ as regularized full reduction (or, simply, reduction), regularized partial reduction (or, simply, partial reduction), respectively.
5 Kolmogorov's Set in the Space Planetary Problem I (Partial Reduction)

5.1 Non–Degeneracy Conditions $(N \ge 3)$

The construction of KAM tori for the spatial planetary problem with $N \ge 3$ planets via Theorem 2.1 becomes quite natural and direct, with the use of the Deprit variables.

In this section, we show that, for $N \geq 3$, the set of Deprit's partially reduced variables discussed in section 4.3 (which, we recall, corresponds to the reduction of C_z) is a good set of coordinates in order to obtain KAM tori with 3N - 1 frequencies. The pregium of Deprit's partial reduction is that, differently from what happens trying a partial reduction in Poincaré–Delaunay's variables, ³² it leaves the secular perturbation regular and symmetric around the secular origin, which, as in the planar case, turns out to be an elliptic equilibrium point, corresponding to the configurations with all zero eccentricities and mutual inclinations.

As said before, the construction of the KAM tori is obtained as an application of Theorem 2.1, so, it is based on the check of the two non degeneracy conditions thereby involved:

- (i) check of 4-non resonance for the Birkhoff invariants of the first order;
- (ii) check of second order non degeneracy, *i.e.*, proof of non singularity of the second order Birkhoff invariants matrix.

³²In Poincaré–Delaunay variables, the third component of the angular momentum is

$$C_{\rm z} = \sum_{1 \le i \le N} \left(\Lambda_i - \frac{\eta_i^2 + \xi_i^2}{2} - \frac{p_i^2 + q_i^2}{2} \right) \;,$$

which is quite the same of the expression of the modulus of the anguar momentum G in partially reduced Deprit variables (first equation in (4.42)), apart for the dimension (N instead than N-1) of the (p,q)variables. Put then, similarly to (4.42),

$$\begin{cases} C_{\mathbf{z}} = \sum_{1 \le k \le N} \left(\Lambda_k - \frac{\eta_k^2 + \xi_k^2}{2} \right) - \sum_{1 \le k \le N} \frac{p_k^2 + q_k^2}{2} &, \quad \hat{\lambda}_i = \lambda_i - \zeta \quad 1 \le i \le N \\ \zeta = \arg\left(p_N, -q_N\right) &\\ \left(\begin{array}{c} \hat{\eta}_i \\ \hat{\xi}_i \end{array} \right) = \mathcal{R}_{\mathbf{z}}(\zeta) \left(\begin{array}{c} \eta_i \\ \xi_i \end{array} \right) & \quad 1 \le i \le N , \quad \left(\begin{array}{c} \hat{p}_j \\ \hat{q}_j \end{array} \right) = \mathcal{R}_{\mathbf{z}}(\zeta) \left(\begin{array}{c} p_j \\ q_j \end{array} \right) & \quad 1 \le j \le N - 1 \end{cases}$$

The variables $((\Lambda, \hat{\lambda}), (\hat{\eta}, \hat{\xi}), (\hat{p}, \hat{q}), (C_z, \zeta))$ realize a (Delaunay) partial reduction, however, singular, relatively to the configurations with vanishing N^{th} inclination, *i.e.*, when

$$\frac{p_N^2 + q_N^2}{2} = \sum_{1 \le i \le N} \left(\Lambda_i - \frac{\hat{\eta}_i^2 + \hat{\xi}_i^2}{2} \right) - \sum_{1 \le i \le N-1} \frac{\hat{p}_i^2 + \hat{q}_i^2}{2} - C_z = 0 \; .$$

This singularity is sometimes called "elliptic singularity".

Both (i) and (ii) are proved by induction, in the range of well separated semimajor axes.

The restriction to $N \geq 3$ is due to the following. When the secular perturbation is put in Deprit partially reduced variables, as in the case of Poincaré–Delaunay variables, its quadratic part splits into the sum of a "horizontal" part \mathcal{Q}_h^* and a "vertical" part \mathcal{Q}_v^* , of order N, N-1 respectively. Then, using partial reduction, a unique secular resonance (compare Proposition 5.1 below) is exhibited by the respective eigenvalues $s = (s_1, \dots, s_N), z = (z_1, \dots, z_{N-1})$ of $\mathcal{Q}_h^*, \mathcal{Q}_v^*$, once again the Herman's resonance:

$$\sum_{1 \le i \le N} s_i + \sum_{1 \le i \le N-1} z_i = 0 .$$
 (5.1)

This resonance is of order 2N - 1, hence, it prevents the construction of the Birkhoff normal form up to order 4 only when N = 2. When $N \ge 3$, the Herman's resonance (5.1) is of high order $(2N - 1 \ge 5)$, allowing us to use partial reduction for the construction (and proof of non-degeneracy) of the normal form.

The first step consists into the expansion of the secular perturbation up to order 2. We denote

$$\mathcal{H}_{\mathrm{plt,pr}} := \mathcal{H}_{\mathrm{plt}} \circ \phi_{\mathrm{BD,pr}} = h_{\mathrm{plt}} + \mu f_{\mathrm{plt,pr}}$$

the planetary Hamiltonian function, put in regularized, partially reduced Deprit variables, where, as usual

$$h_{\rm plt} = -\sum_{1 \le i \le N} \frac{\tilde{m}_i^3 \hat{m}_i^2}{2\Lambda_i^2}$$

is the Kepler's unperturbed integrable part.

Lemma 5.1 For $N \ge 2$, the mean $\bar{f}_{\text{plt,pr}} := (2\pi)^{-N} \int_{\mathbb{T}^N} f_{\text{plt,pr}} d\bar{\lambda}$ is an even function of the "secular fully regularized variables" $\bar{z} := (\bar{\eta}, \bar{\xi}, \bar{p}, \bar{q})$ and its expansion around $\bar{z} = 0$ is the following. Define:

the constants

where $\alpha \to b_{s,k}(\alpha)$ is the (s,k)-Laplace coefficient;

the quadratic forms

$$\begin{cases} \mathcal{Q}_h \cdot \bar{\eta}^2 := \sum_{1 \le j < k \le N} m_j m_k \left(C_1(a_j, a_k) \left(\frac{\bar{\eta}_j^2}{\Lambda_j} + \frac{\bar{\eta}_k^2}{\Lambda_k} \right) + 2C_2(a_j, a_k) \frac{\bar{\eta}_j \bar{\eta}_k}{\sqrt{\Lambda_j \Lambda_k}} \right) \\ \hat{\mathcal{Q}}_v \cdot \bar{p}^2 := -\sum_{1 \le j < k \le N} m_j m_k C_1(a_j, a_k) \left(\bar{p}_j - \bar{p}_k \right)^2 \end{cases}$$
(5.3)

the linear operator

$$\mathfrak{L}: \quad \bar{p} \in \mathbb{R}^{N-1} \to \mathfrak{L}\bar{p} = \left(\cdots, \mathfrak{L}\bar{p}_i, \cdots\right) \in \mathbb{R}^N$$

which acts as

$$\begin{cases} \mathfrak{L}\bar{p}_{1} := \ell_{1} \cdot \bar{p} := c_{1} \, \bar{p}_{1} + \sum_{2 \leq j \leq N-1}^{*} \tilde{c}_{j} \, \bar{p}_{j} \\ \mathfrak{L}\bar{p}_{i} := \ell_{i} \cdot \bar{p} := c_{i} \, \bar{p}_{i-1} + \sum_{i \leq j \leq N-1}^{*} \tilde{c}_{j} \, \bar{p}_{j} \quad 2 \leq i \leq N-1 \quad (if \quad N \geq 3) \quad , \qquad (5.4) \\ \mathfrak{L}\bar{p}_{N} := \ell_{N} \cdot \bar{p} := c_{N} \, \bar{p}_{N-1} \end{cases}$$

where the summands denoted Σ^* does not appear when N = 2, and

$$\begin{cases} c_1 := -\sqrt{\frac{\Lambda_2}{\Lambda_1 L_2}} \\ c_j := \sqrt{\frac{L_{j-1}}{L_j \Lambda_j}} \quad (2 \le j \le N) \\ \tilde{c}_j := -\sqrt{\frac{\Lambda_{j+1}}{L_{j+1} L_j}} \quad (2 \le j \le N-1) \\ L_j := \sum_{1 \le k \le j} \Lambda_k \end{cases}$$
(5.5)

Then,

$$\bar{f}_{\text{plt,pr}} = C_0(m,a) + \mathcal{Q}_h^* \cdot \frac{\bar{\eta}^2 + \bar{\xi}^2}{2} + \mathcal{Q}_v^* \cdot \frac{\bar{p}^2 + \bar{q}^2}{2} + \bar{f}_{\text{plt,pr}}^4 , \qquad (5.6)$$

where

$$\begin{cases}
\mathcal{Q}_h^* \cdot \eta^2 := \mathcal{Q}_h \cdot \eta_*^2, \quad with \quad \eta_* := (-\bar{\eta}_1, \bar{\eta}_2, \cdots, \bar{\eta}_N) \\
Q_v^* \cdot \bar{p}^2 := \hat{\mathcal{Q}}_v \cdot (\mathfrak{L}\bar{p})^2 \\
\bar{f}_{\text{plt,pr}}^4 = O(4).
\end{cases}$$
(5.7)

The details of the computation of the expansion of $f_{\text{plt,pr}}$ (up to order 4, for future use) are in Section 5.1.3.

5.1.1 First Order Conditions

Proposition 5.1 For any $N \geq 2$, there exists an open set with full measure $\mathcal{U} \subset \mathcal{A}$ where the eigenvalues of \mathcal{Q}_h^* and \mathcal{Q}_v^* are paiwise distinct and verify the following. For any open, simply connected set $\mathcal{V} \subset \mathcal{U}$, they define 2N - 1 holomorphic functions $s_1, \dots, s_N, z_1, \dots, z_{N-1}$ which satisfy the only linear relation

$$\sum_{1 \le i \le N} s_i + \sum_{1 \le i \le N-1} z_i = 0 \tag{5.8}$$

(up to an arbitrary multiplicative constant).

Proof. Let us introduce matricial notations. Let \mathcal{F}_h , \mathcal{F}_v denote the matrices (having both order N) associated to the quadratic forms \mathcal{Q}_h , \mathcal{Q}_v (6.3) of the quadratic part of the secular peturbation in Delaunay–Poincaré variables and let \mathcal{F}_h^* , \mathcal{G}_v^* denote the matrices associated to \mathcal{Q}_h^* , \mathcal{Q}_v^* (having order N, N – 1 respectively) of eq. (5.7):

$$\begin{cases} \mathcal{Q}_h \cdot \eta^2 = \eta \cdot \mathcal{F}_h \eta \\ \mathcal{Q}_v \cdot P^2 = P \cdot \mathcal{F}_v P \end{cases} \begin{cases} \mathcal{Q}_h^* \cdot \eta^2 = \eta \cdot \mathcal{F}_h^* \eta \\ \mathcal{Q}_v^* \cdot p^2 = p \cdot \mathcal{G}_v^* p \end{cases}$$

The matrices \mathcal{F}_h^* , \mathcal{F}_h are related by ³³

$$\mathcal{F}_h^* = \mathrm{I}\mathcal{F}_h\mathrm{I}$$

where I changes the sign of the first coordinate, so, they have the same eigenvalues. Then, in order to prove (5.8), in view, of (6.4), we only need to prove that \mathcal{F}_v , \mathcal{F}_v^* have the same trace. We have

$$\mathcal{F}_{v}^{*} = \ell^{\mathrm{T}} \hat{\mathcal{F}}_{v}^{*} \ell , \quad \mathcal{F}_{v} = \ell_{0}^{\mathrm{T}} \hat{\mathcal{F}}_{v} \ell_{0} , \qquad (5.9)$$

where ℓ denotes the $N \times (N-1)$ matrix associated to the linear operator (5.4)÷(5.5), ℓ_0 the diagonal matrix

$$\ell_0 = \operatorname{diag}\left(1/\sqrt{\Lambda_1}, \cdots, 1/\sqrt{\Lambda_N}\right)$$

and $\hat{\mathcal{F}}_v = (\hat{g}_{ij})$ the $N \times N$ matrix of $\hat{\mathcal{Q}}_v$:

$$\hat{\mathcal{Q}}_{v} \cdot \bar{p}^{2} := -\sum_{1 \le j < k \le N} m_{j} m_{k} C_{1}(a_{j}, a_{k}) \left(\bar{p}_{j} - \bar{p}_{k}\right)^{2} := \bar{p} \cdot \hat{\mathcal{F}}_{v} \bar{p} .$$
(5.10)

Equations (5.9) give

$$\begin{cases} \operatorname{tr}(\mathcal{F}_{v}^{*}) = \operatorname{tr}(\ell^{\mathrm{T}}\hat{\mathcal{F}}_{v}\ell) = \operatorname{tr}(\mathcal{L}\hat{\mathcal{F}}_{v}) \\ \operatorname{tr}(\mathcal{F}_{v}) = \operatorname{tr}(\ell_{0}^{\mathrm{T}}\hat{\mathcal{F}}_{v}\ell_{0}) = \operatorname{tr}(\mathcal{L}_{0}\hat{\mathcal{F}}_{v}) \end{cases}$$
(5.11)

where \mathcal{L}_0 is the diagonal matrix

$$\mathcal{L}_0 = \operatorname{diag}(1/\Lambda_1, \cdots, 1/\Lambda_N)$$

and \mathcal{L} is the symmetric matrix with entries

$$\mathcal{L}_{ij} = \ell_i \cdot \ell_j$$

if

$$\ell_1 = (c_1, \tilde{c}_2, \cdots, \tilde{c}_{N-1}), \ \ell_2 = (c_2, \tilde{c}_2, \cdots, \tilde{c}_{N-1}), \cdots, \ell_i = (0, \cdots, 0, c_i, \tilde{c}_i, \cdots, \tilde{c}_{N-1})$$

is the i^{th} row of ℓ , as in (5.4)÷(5.5). We have

$$\begin{cases} \mathcal{L}_{11} = |\ell_1|^2 = c_1^2 + \sum_{\substack{2 \le j \le N-1 \\ i \le j \le N-1}} \tilde{c}_j^2 \\ \mathcal{L}_{ii} = |\ell_i|^2 = c_1^2 + \sum_{\substack{i \le j \le N-1 \\ i \le j \le N-1}} \tilde{c}_j^2 & 1 \le j \le N-1 \\ \mathcal{L}_{NN} = |\ell_N|^2 = c_N^2 \\ \mathcal{L}_{ij} = \ell_i \cdot \ell_j = c_j \tilde{c}_{j-1} + \sum_{\substack{j \le k \le N-1 \\ j \le k \le N-1}} \tilde{c}_k^2 & 1 \le i < j \le N-1 \\ \mathcal{L}_{iN} = \ell_i \cdot \ell_N = c_N \tilde{c}_{N-1} & 1 \le i \le N-1 \end{cases}$$
(5.12)

³³Using the symplectic variables (η_*, ξ_*) (compare (5.7)), rather than (η, ξ) , namely, with $(\eta_1^*, \xi_1^*) = -(\eta_1, \xi_1)$ related to the aphelion position of the first osculating ellipse, rather than the perihelion, would transform \mathcal{F}_h^* to \mathcal{F}_h . We do not use this change of variables here, because unnecessary, but in the next section, for the computation of the Birkhoff invariants of order 2, in order to have simpler expressions.

Using (5.5), we can write

$$\begin{cases} c_1^2 = \frac{1}{\Lambda_1} - \frac{1}{L_2} \\ c_j^2 = \frac{1}{\Lambda_j} - \frac{1}{L_j} & 2 \le j \le N \\ \tilde{c}_j^2 = \frac{1}{L_j} - \frac{1}{L_{j+1}} & 2 \le j \le N - 1 \\ c_j \tilde{c}_{j-1} = -\frac{1}{L_j} & 2 \le j \le N & (\tilde{c}_1 := c_1) \end{cases}$$

hence, inserting the previous expressions into (5.12), by telescopic arguments,

$$\mathcal{L} = \mathcal{L}_0 - \frac{\mathrm{E}}{\mathrm{L}_N} \; ,$$

where E has entries $E_{ij} = 1$ for any *i*, *j*. Hence, in view of (5.11)

$$\operatorname{tr}\left(\mathcal{F}_{v}^{*}\right) = \operatorname{tr}\left(\left(\mathcal{L}_{0} - \frac{\mathrm{E}}{\mathrm{L}_{N}}\right)\hat{\mathcal{F}}_{v}\right) = \operatorname{tr}\left(\mathcal{L}_{0}\hat{\mathcal{F}}_{v}\right) - \frac{1}{\mathrm{L}_{N}}\operatorname{tr}\left(\mathrm{E}\hat{\mathcal{F}}_{v}\right) = \operatorname{tr}\left(\mathcal{L}_{0}\hat{\mathcal{F}}_{v}\right) = \operatorname{tr}\left(\mathcal{F}_{v}\right)$$

since, from (5.10), it is easy to check

$$\operatorname{tr}\left(\mathrm{E}\hat{\mathcal{F}}_{v}\right) = \sum_{j,k} \hat{g}_{jk} = 0$$

We prove now uniqueness of (5.8), proceeding by induction on the number N of planets. For N = 2, it is a consequence of existence and the fact that the planar eigenvalues $s_1 = \sigma_1$, $s_2 = \sigma_2$ do not satisfy any linear condition, as proved in [17]. Assume now that uniqueness of (5.8) holds for N - 1 and let

$$c = (c_1, \cdots, c_N) \in \mathbb{R}^N$$
, $g = (g_1, \cdots, g_{N-1}) \in \mathbb{R}^{N-1}$

such that

$$c \cdot s + g \cdot z = 0$$

where $s = (s_1, \dots, s_N)$, $z = (z_1, \dots, z_{N-1})$ are the eigenvalues of $\mathcal{F}_h := \mathcal{F}_h^N$, $\mathcal{F}_v := \mathcal{F}_v^{*N}$. We write explicitly the entries of the matrices \mathcal{F}_h^N , \mathcal{F}_v^{*N}

$$(\mathcal{F}_{h}^{N})_{ij} = \begin{cases} \frac{\bar{m}_{i}}{\Lambda_{i}} \sum_{i < k \leq N} \bar{m}_{k} C_{1}(a_{i}, a_{k}) + \frac{\bar{m}_{i}}{\Lambda_{i}} \sum_{1 \leq k < i} \bar{m}_{k} C_{1}(a_{k}, a_{i}) & 1 \leq i = j \leq N \\ \frac{\bar{m}_{i} \bar{m}_{j}}{\sqrt{\Lambda_{i} \Lambda_{j}}} C_{2}(a_{i}, a_{j}) & 1 \leq i < j \leq N \end{cases}$$

$$(\mathcal{F}_{v}^{*N})_{ij} = \begin{cases} -\tilde{c}_{i}^{2} \sum_{1 \le k \le i} \sum_{l \ge i+2} \bar{m}_{k} \bar{m}_{l} C_{1}(a_{k}, a_{l})) - c_{i+1}^{2} \sum_{k \ge i+2} \bar{m}_{i+1} \bar{m}_{k} C_{1}(a_{i+1}, a_{k}) \\ -\bar{c}_{i}^{2} \sum_{1 \le k \le i} \bar{m}_{k} \bar{m}_{i+1} C_{1}(a_{k}, a_{i+1}) & 1 \le i = j \le N-2 \\ -\bar{c}_{i}^{2} \sum_{1 \le k \le N-1} \bar{m}_{k} \bar{m}_{N} C_{1}(a_{k}, a_{N}) & i = j = N-1 \\ -\tilde{c}_{i} \tilde{c}_{j} \sum_{1 \le k \le i, \ l \ge j+2} \bar{m}_{k} \bar{m}_{l} C_{1}(a_{k}, a_{l}) - \bar{c}_{j} \tilde{c}_{i} \sum_{1 \le k \le i} \bar{m}_{k} \bar{m}_{j+1} C_{1}(a_{k}, a_{j+1}) \\ -\bar{c}_{j} c_{i+1} \bar{m}_{i+1} \bar{m}_{j+1} C_{1}(a_{i+1}, a_{j+1}) - \tilde{c}_{j} c_{i+1} \sum_{l \ge j+2} \bar{m}_{i+1} \bar{m}_{l} C_{1}(a_{i+1}, a_{l}) \\ 1 \le i < j \le N-2 \\ -\bar{c}_{N-1} \tilde{c}_{i} \sum_{1 \le k \le i} \bar{m}_{k} \bar{m}_{N} C_{1}(a_{k}, a_{N}) - \bar{c}_{N-1} c_{i+1} \bar{m}_{i+1} \bar{m}_{N} C_{1}(a_{i+1}, a_{N}) \\ 1 \le i < j = N-1 \end{cases}$$

$$(5.13)$$

where c_i , \tilde{c}_j are as in (5.5) and

$$\bar{c}_k := c_{k+1} - \tilde{c}_k = \sqrt{\frac{1}{\Lambda_{k+1}} + \frac{1}{L_k}} \quad 1 \le k \le N - 1$$

It is easy to check that, when $a_N \to \infty$,

$$\mathcal{F}_h^N \to \begin{pmatrix} \hat{\mathcal{F}}_h^{N-1} & 0\\ 0 & 0 \end{pmatrix} \qquad \mathcal{F}_v^{*N} \to \begin{pmatrix} \hat{\mathcal{F}}_v^{*(N-1)} & 0\\ 0 & 0 \end{pmatrix}$$

and that when $a_1 \to 0$,

$$\mathcal{F}_{h}^{N} \to \begin{pmatrix} 0 & 0\\ 0 & \check{\mathcal{F}}_{h}^{N-1} \end{pmatrix} \qquad \mathcal{F}_{v}^{N} \to \begin{pmatrix} 0 & 0\\ 0 & \check{\mathcal{F}}_{v}^{*(N-1)} \end{pmatrix}$$

where $\hat{\mathcal{F}}_{h}^{N-1}$, $\hat{\mathcal{F}}_{v}^{*(N-1)}$ denote the horizontal, vertical quadratic forms related to the "first" N-1 bodies and $\check{\mathcal{F}}_{h}^{N-1}$, $\check{\mathcal{F}}_{v}^{*(N-1)}$, the horizontal, vertical quadratic forms related to the "last" N-1 bodies. Then, as s, z are continuous function of the entries of their respective matrices, when $a_N \to \infty$,

$$s \to (\hat{s}, 0)$$
, $z \to (\hat{z}, 0)$

and, when $a_1 \to 0$,

$$s \to (0, \check{s}) , \quad z \to (0, \check{z})$$

where \hat{s} , \hat{z} are the eigenvalues of $\hat{\mathcal{F}}_{h}^{N-1}$, $\hat{\mathcal{F}}_{v}^{*(N-1)}$; \check{s} , \check{z} are the eigenvalues of $\check{\mathcal{F}}_{h}^{N-1}$, $\check{\mathcal{F}}_{v}^{*(N-1)}$. By the inductive hypothesis, the first limit implies $c_{1} = \cdots = c_{N-1} = g_{1} = \cdots = g_{N-2}$; the second limit implies $c_{2} = \cdots = c_{N} = g_{2} = \cdots = g_{N-1}$, hence, the thesis.

5.1.2 Second Order Conditions ("Torsion")

By Proposition 5.1 and Birkhoff theory, when $N \geq 3$, the secular perturbation $\bar{f}_{\text{plt,pr}}$ of the planetary problem can be put in normal form up to order 4. In this section, we prove the non-degeneracy of this normal form. (i) We call **Birkhoff form** of a given polynomial of 2m variables

$$(y,x) \in \mathbb{R}^m \times \mathbb{R}^m \to \mathcal{P}(y,x) \in \mathbb{R}$$

and even degree $p \geq 4$, the polynomial

$$\mathcal{P}_{\mathrm{B}}(y,x) := \frac{1}{(2\pi)^m} \int_{\mathbb{T}^m} \mathcal{P} \circ \phi_{\mathrm{pc}}(J,\varphi) d\varphi \quad \text{with} \quad J = \left(\cdots, \frac{y_i^2 + x_i^2}{2}, \cdots\right)$$

where ϕ_{pc} is the usual symplectic polar coordinates map.

(ii) When \mathcal{P} has degree 4, we call **Birkhoff matrix** associated to \mathcal{P} the symmetric matrix $A = (A_{ij})$ of order m for which

$$\mathcal{P}_{\mathrm{B}}(y,x) = \frac{1}{2} \sum_{1 \le i,j \le m} A_{ij} J_i J_j \quad \text{where} \quad J_i := \frac{y_i^2 + x_i^2}{2}$$

,

Let ρ_h^* , ρ_v^* denote the matrices which diagonalize the quadratic forms \mathcal{Q}_h^* , \mathcal{Q}_v^* , and let ϕ_{diag} the symplectic transformation³⁴

$$\phi_{\text{diag}}: \begin{cases} \bar{\eta} = \rho_h^* \tilde{\eta} \\ \bar{\xi} = \rho_h^* \tilde{\xi} \end{cases} \Leftrightarrow \begin{cases} \eta_* = \rho_h \tilde{\eta} \\ \xi_* = \rho_h \tilde{\xi} \end{cases}$$
$$\begin{cases} \bar{p} = \rho_v^* \tilde{p} \\ \bar{q} = \rho_v^* \tilde{q} \end{cases}, \quad \Lambda = \tilde{\Lambda}$$

where $\rho_h = I \rho_h^*$ is the matrix which diagonalizes the quadratic form $Q_h = I Q_h^* I$ of the plane problem, as in [17]. Then, the secular perturbation $\bar{f}_{\text{plt,pr}}$ is put into the form

$$\begin{aligned} f_{\text{diag}} &:= f_{\text{plt,pr}} \circ \phi_{\text{diag}} \\ &= C_0(m,a) + \sum_{1 \le i \le N} s_i \frac{\tilde{\eta}_i^2 + \tilde{\xi}_i^2}{2} + \sum_{1 \le i \le N-1} z_i \frac{\tilde{p}_i^2 + \tilde{q}_i^2}{2} + \mathfrak{F}\left(\rho_h \tilde{\eta}, \rho_h \tilde{\xi}, \rho_v^* \tilde{p}, \rho_v^* \tilde{q}\right) \\ &+ O(6) \end{aligned}$$

where \mathfrak{F} is the polynomial of degree 4 in $z_* = (\eta_*, \xi_*, \bar{p}, \bar{q})$ for which

$$\tilde{f}^4_{\text{plt,pr}}(\bar{\eta}, \bar{\xi}, \bar{p}, \bar{q}) = \mathfrak{F}(\eta_*, \xi_*, \bar{p}, \bar{q}) + O(6) \ .$$

We then have

Proposition 5.2 For any $N \ge 2$, the Birkhoff matrix A_{plt} , of order 2N - 1, associated to the polynomial $\mathfrak{F}(\rho_h\eta, \rho_h\xi, \rho_v^*p, \rho_v^*q)$ is non singular, provided the semimajor axes $0 < a_1 < a_2 < \cdots < a_N$ are well separated.

³⁴We neglect to write the action on the $\bar{\lambda}$ -variables, which we do non need.

Remark 5.1 When $N \ge 3$, A_{plt} coincides with the matrix of the Birkhoff invariants of order 2 of the planetary problem.

In order to prove Proposition 5.2, we need the exact expression of \mathfrak{F} , which is computed in Section 5.1.3 and summarized in the following Lemma.

Lemma 5.2 In the expansion for the secular perturbation $\bar{f}_{\text{plt,pr}}$ around the secular origin $\bar{z} = (\bar{\eta}, \bar{\xi}, \bar{p}, \bar{q}) = 0$ described in Lemma 5.1, the term $\bar{f}_{\text{plt,pr}}^4$ of eq. (5.6) begins with $\bar{f}_{\text{plt,pr}}^4 = \mathfrak{F} + O(6)$, where

$$\mathfrak{F} = \mathfrak{F}_h + \mathfrak{F}_{hv} + \mathfrak{F}_v \tag{5.14}$$

and \mathfrak{F}_h , \mathfrak{F}_{hv} , \mathfrak{F}_v are three polynomials of degree 4, defined as follows. The "horizontal" part \mathfrak{F}_h is

$$\mathfrak{F}_h = q \cdot (\eta_*^4 + \xi_*^4) + r \cdot \eta_*^2 \xi_*^2 \tag{5.15}$$

where, as in Lemma 5.1, $\eta_* := (-\eta_1, \eta_2, \cdots)$ and

$$q \cdot \eta^{4} := \sum_{1 \le i \le N} q_{iiii} \eta_{i}^{4} + \sum_{1 \le i < j \le N} \left(q_{iiij} \eta_{i}^{3} \eta_{j} + q_{iijj} \eta_{i}^{2} \eta_{j}^{2} + q_{jjji} \eta_{i} \eta_{j}^{3} \right)$$

$$r \cdot \eta^{2} q^{2} := \sum_{1 \le i \le N} r_{iiii} \eta_{i}^{2} \xi_{i}^{2} + \sum_{1 \le i < j \le N} \left(r_{iiij} \eta_{i}^{2} \xi_{i} \xi_{j} + r_{iijj} \eta_{i}^{2} \xi_{j}^{2} + r_{ijii} \eta_{i} \eta_{i} \xi_{i}^{2} + r_{ijij} \eta_{i} \eta_{j} \xi_{i} \xi_{j} \right)$$

$$+ r_{jijj} \eta_{i} \eta_{j} \xi_{j}^{2} + r_{jjii} \eta_{j}^{2} \xi_{i}^{2} + r_{jjji} \eta_{j}^{2} \xi_{i} \xi_{j} \right)$$
(5.16)

if $q = (q_{ijkl})$, $r = (r_{ijkl})$ are the 4-indices tensors defining the quartic form of the secular perturbation of the plane problem, defined in (3.27)÷(3.28). The "vertical" parts \mathfrak{F}_{hv} , \mathfrak{F}_{v} are

$$\begin{aligned} \mathfrak{F}_{hv} &:= \frac{1}{2} \sum_{1 \le i < j \le N} \bar{m}_i \bar{m}_j \Big(\mathcal{Q}_{ij}^{11} (\mathfrak{L}\bar{q}_j - \mathfrak{L}\bar{q}_i)^2 + \mathcal{Q}_{ij}^{22} (\mathfrak{L}\bar{p}_j - \mathfrak{L}\bar{p}_i)^2 \\ &+ (\mathcal{Q}_{ij}^{12} + \mathcal{Q}_{ij}^{21}) \Big(\mathfrak{L}\bar{p}_j - \mathfrak{L}\bar{p}_i \Big) \Big(\mathfrak{L}\bar{q}_j - \mathfrak{L}\bar{q}_i \Big) \\ &+ (\mathcal{Q}_{ij}^{21} - \mathcal{Q}_{ij}^{12}) \sum_{1 \le h < k \le j-1}^* \ell_h^{ij} \ell_k^{ij} (\bar{p}_h \bar{q}_k - \bar{p}_k \bar{q}_h) \Big) \\ \mathfrak{F}_v &:= \sum_{1 \le i < j \le N} \bar{m}_i \bar{m}_j (\mathfrak{Q}_{11}^{i1} + \mathfrak{Q}_{22}^{i2}) C_1(a_i, a_j) \\ &+ \sum_{1 \le i < j \le N} \bar{m}_i \bar{m}_j \left(\Big(\mathfrak{L}\bar{p}_i - \mathfrak{L}\bar{p}_j \Big)^2 + \Big(\mathfrak{L}\bar{q}_i - \mathfrak{L}\bar{q}_j \Big)^2 \Big)^2 C_{13}(a_i, a_j) \\ &+ \sum_{1 \le i < j \le N} \bar{m}_i \bar{m}_j \left(\sum_{1 \le h < k \le j-1}^* \ell_h^{ij} \ell_k^{ij} (\bar{p}_h \bar{q}_k - \bar{p}_k \bar{q}_h) \right)^2 C_{14}(a_i, a_j) \end{aligned}$$

$$(5.17)$$

where the summands denoted \sum^* do not appear when N = 2 or j = 2 and where:

- for h, $k = 1, 2, \mathcal{Q}_{ij}^{hk}$ are the four quadratic forms acting on $(\eta_i^*, \eta_j^*, \xi_i^*, \xi_j^*)$ as in (5.45), with $C_3(a_i, a_j) \div C_{12}(a_i, a_j)$ as in (5.46) below;

- \mathfrak{L} is the linear operator from \mathbb{R}^{N-1} to \mathbb{R}^N defined in eq. (5.4)÷(5.5);
- $-\ell_k^{ij} := \ell_{jk} \ell_{ik}$ with $(\ell_{ik}) := \ell$ the matrix associated to \mathfrak{L} ;
- $C_{13}(a_i, a_j)$, $C_{14}(a_i, a_j)$ are as in (5.65) below and, finally, :
- $-\mathfrak{Q}_{11}^{ij}, \mathfrak{Q}_{22}^{ij}$ are the two first diagonal entries of the matrix of homogeneous polynomials of degree 4 resulting from the productories (5.52) below, when \mathcal{S}_i, \cdots have the expansions defined in eq. (5.61)÷(5.64) below.

We outline that the main difference (and complexity), with respect to the same computation in Delaunay–Poincaré variables, is represented by the productory form of the "verticalizing" matrices \mathfrak{S}_{ij} (compare equation (5.52) below) which describe the mutual orientation between the planes of the osculating orbits of the planets *i* and *j*. We recall that, in turn, this productory structure is a consequence of the "tree" structure Deprit's kinetic frames.

We are now ready for the Proof of Proposition 5.2. For simplicity of notations, during the proof, we write

$$\eta = (\eta_1, \dots, \eta_N)$$
, $\xi = (\xi_1, \dots, \xi_N)$, $p = (p_1, \dots, p_{N-1})$, $q = (q_1, \dots, q_{N-1})$

for

$$\eta_* = (\eta_1^*, \cdots, \eta_N^*) , \quad \xi_* = (\xi_1^*, \cdots, \xi_N^*) , \quad \bar{p} = (\bar{p}_1, \cdots, \bar{p}_{N-1}) , \quad \bar{q} = (\bar{q}_1, \cdots, \bar{q}_{N-1})$$

believing that no confusion arises with the full reduced variables, which are never used in this section.

Proof of Proposition 5.2. We proceed by induction on the number N of planets, starting with N = 2. We esplicitate the dependence on N marking \mathfrak{F}^N , ρ_h^N , \cdots the quantities \mathfrak{F} , ρ_h , \cdots relatively to N planets.

<u>Proof for N = 2</u>. When N = 2, the two matrices \mathcal{F}_h^2 , \mathcal{F}_v^{*2} of the quadratic forms \mathcal{Q}_h^2 , \mathcal{Q}_v^{*2} , have order 2, 1, respectively, so, their diagonalizing matrices ρ_h^2 , ρ_v^{*2} can be exactly computed: $\rho_v^{*2} =$ id is trivial and, diagonalizing

$$\begin{aligned} \mathcal{F}_{h}^{2} &= \bar{m}_{1}\bar{m}_{2} \begin{pmatrix} \frac{C_{1}(a_{1},a_{2})}{\Lambda_{1}} & \frac{C_{2}(a_{1},a_{2})}{\sqrt{\Lambda_{1}\Lambda_{2}}} \\ \frac{C_{2}(a_{1},a_{2})}{\sqrt{\Lambda_{1},\Lambda_{2}}} & \frac{C_{1}(a_{1},a_{2})}{\Lambda_{2}} \end{pmatrix} \\ &= \bar{m}_{1}\bar{m}_{2} \begin{pmatrix} -\frac{3}{4}\frac{a_{1}^{2}}{a_{2}^{3}\Lambda_{1}} + O\left(\frac{a_{1}^{4}}{a_{2}^{5}\Lambda_{1}}\right) & O\left(\frac{a_{1}^{3}}{a_{2}^{4}\sqrt{\Lambda_{1}\Lambda_{2}}}\right) \\ O\left(\frac{a_{1}^{3}}{a_{2}^{4}\sqrt{\Lambda_{1}\Lambda_{2}}}\right) & -\frac{3}{4}\frac{a_{1}^{2}}{a_{2}^{3}\Lambda_{2}} + O\left(\frac{a_{1}^{4}}{a_{2}^{5}\Lambda_{2}}\right) \end{pmatrix} \end{aligned}$$

we find, for

$$a_1 = O(1)$$
, $a_2 \to \infty$,

$$\rho_h^2 = \begin{pmatrix} \frac{1}{\sqrt{1+\varepsilon^2}} & \frac{\varepsilon}{\sqrt{1+\varepsilon^2}} \\ -\frac{\varepsilon}{\sqrt{1+\varepsilon^2}} & \frac{1}{\sqrt{1+\varepsilon^2}} \end{pmatrix} = \operatorname{id} + \begin{pmatrix} O(a_2^{-5/2}) & O(a_2^{-5/4}) \\ O(a_2^{-5/4}) & O(a_2^{-5/2}) \end{pmatrix}$$
(5.18)

where, letting for shortness,

$$a := \frac{C_1(a_1, a_2)}{\Lambda_1}$$
, $b := \frac{C_1(a_1, a_2)}{\Lambda_2}$, $c := \frac{C_2(a_1, a_2)}{\sqrt{\Lambda_1, \Lambda_2}}$,

then ε denotes

$$\varepsilon := \frac{|a-b|}{2c} \left(\sqrt{1 + \frac{4c^2}{(a-b)^2}} - 1 \right) = O(a_2^{-5/4}) \; .$$

Using these expressions, we compute now the asymptitics (in a_2) of the Birkhoff matrix associated to the three polynomials $\mathfrak{F}_h^2(\rho_h\eta,\rho_h\xi,p,q)$, $\mathfrak{F}_{hv}^2(\rho_h\eta,\rho_h\xi,p,q)$, $\mathfrak{F}_v^2(\rho_h\eta,\rho_h\xi,p,q)$. The dominant contribute of $\mathfrak{F}_h^2(\rho_h\eta,\rho_h\xi,p,q)$ to the Birkhoff matrix is (compare with Arnol'd's computation, [3])

$$\begin{pmatrix} \frac{3}{4} \frac{\bar{m}_1 \bar{m}_2}{\Lambda_1^2 a_2} \frac{a_1^2}{a_2^2} & -\frac{9}{4} \frac{\bar{m}_1 \bar{m}_2}{\Lambda_1 \Lambda_2 a_2} \frac{a_1^2}{a_2^2} & * \\ -\frac{9}{4} \frac{\bar{m}_1 \bar{m}_2}{\Lambda_1 \Lambda_2 a_2} \frac{a_1^2}{a_2^2} & -3 \frac{\bar{m}_1 \bar{m}_2}{\Lambda_2^2 a_2} \frac{a_1^2}{a_2^2} & * \\ * & * & * \end{pmatrix}$$
(5.19)

We recall for completeness the computation, essentially done in the study of the plane problem, which leads to this result, since now we want to expand with respect to the semimajor axes, rather that their ratios. It is a consequence of the expression of the horizontal quartic form

$$\begin{aligned} \mathfrak{F}_{h}^{2} &= q_{1111}\eta_{1}^{4} + q_{1112}\eta_{1}^{3}\eta_{2} + q_{1122}\eta_{1}^{2}\eta_{2}^{2} + q_{2221}\eta_{1}\eta_{2}^{3} + q_{2222}\eta_{2}^{4} \\ &+ q_{1111}\xi_{1}^{4} + q_{1112}\xi_{1}^{3}\xi_{2} + q_{1122}\xi_{1}^{2}\xi_{2}^{2} + q_{2221}\xi_{1}\xi_{2}^{3} + q_{2222}\xi_{2}^{4} \\ &+ r_{1111}\eta_{1}^{2}\xi_{1}^{2} + r_{1112}\eta_{1}^{2}\xi_{1}\xi_{2} + r_{1122}\eta_{1}^{2}\xi_{2}^{2} \\ &+ r_{1211}\eta_{1}\eta_{1}\xi_{1}^{2} + r_{1212}\eta_{1}\eta_{2}\xi_{1}\xi_{2} + r_{2122}\eta_{1}\eta_{2}\xi_{2}^{2} \\ &+ r_{2211}\eta_{2}^{2}\xi_{1}^{2} + r_{2221}\eta_{2}^{2}\xi_{1}\xi_{2} + r_{2222}\eta_{2}^{2}\xi_{2}^{2} \end{aligned}$$

$$(5.20)$$

with the following expansions (based on the expansions of the Laplace coefficients) of the involved entries of the tensors q, r

$$q_{1111} = \frac{3}{32} \frac{\bar{m}_1 \bar{m}_2}{\Lambda_1^2} \frac{1}{a_2} \left(\frac{a_1^2}{a_2^2} + O\left(\frac{a_1^4}{a_2^4}\right) \right)$$

$$q_{2222} = -\frac{3}{8} \frac{\bar{m}_1 \bar{m}_2}{\Lambda_2^2} \frac{1}{a_2} \left(\frac{a_1^2}{a_2^2} + O\left(\frac{a_1^4}{a_2^4}\right) \right)$$

$$q_{1122} = -\frac{9}{16} \frac{\bar{m}_1 \bar{m}_2}{\Lambda_1 \Lambda_2} \frac{1}{a_2} \left(\frac{a_1^2}{a_2^2} + O\left(\frac{a_1^4}{a_2^4}\right) \right)$$

$$\begin{aligned} r_{1111} &= \frac{3}{16} \frac{\bar{m}_1 \bar{m}_2}{\Lambda_1^2} \frac{1}{a_2} \left(\frac{a_1^2}{a_2^2} + O\left(\frac{a_1^4}{a_2^4}\right) \right) \\ r_{2222} &= -\frac{3}{4} \frac{\bar{m}_1 \bar{m}_2}{\Lambda_2^2} \frac{1}{a_2} \left(\frac{a_1^2}{a_2^2} + O\left(\frac{a_1^4}{a_2^4}\right) \right) \\ r_{1122} &= -\frac{9}{16} \frac{\bar{m}_1 \bar{m}_2}{\Lambda_1 \Lambda_2} \frac{1}{a_2} \left(\frac{a_1^2}{a_2^2} + O\left(\frac{a_1^4}{a_2^4}\right) \right) \\ r_{2211} &= -\frac{9}{16} \frac{\bar{m}_1 \bar{m}_2}{\Lambda_1 \Lambda_2} \frac{1}{a_2} \left(\frac{a_1^2}{a_2^2} + O\left(\frac{a_1^4}{a_2^4}\right) \right) \\ q_{1112}, r_{1211}, r_{1112} &= O\left(\frac{a_1^3}{a_2^4 \Lambda_1 \sqrt{\Lambda_1 \Lambda_2}}\right) \\ q_{2221}, r_{2122}, r_{2221} &= O\left(\frac{a_1^3}{a_2^4 \Lambda_2 \sqrt{\Lambda_1 \Lambda_2}}\right) \\ r_{1212} &= O\left(\frac{a_1^4}{a_2^5 \Lambda_1 \Lambda_2}\right) \end{aligned}$$
(5.21)

We compute now the Birkhoff matrix associated to $\mathfrak{F}_{hv}^2(\rho_h\eta,\rho_h\xi,p,q)$. Replacing

$$\begin{aligned} \mathfrak{L}p_{2} - \mathfrak{L}p_{1} &= (\ell_{2} - \ell_{1})p = \bar{c}_{1}p = \sqrt{\left(\frac{1}{\Lambda_{1}} + \frac{1}{\Lambda_{2}}\right)}p \\ \mathcal{Q}_{12}^{11} \cdot (\eta_{1}, \eta_{2}, \xi_{1}, \xi_{2}) &= C_{3}(a_{1}, a_{2})\frac{\eta_{1}^{2}}{\Lambda_{1}} + C_{4}(a_{1}, a_{2})\frac{\eta_{1}\eta_{2}}{\sqrt{\Lambda_{1}\Lambda_{2}}} + C_{5}(a_{1}, a_{2})\frac{\eta_{2}^{2}}{\Lambda_{2}} \\ &+ C_{6}(a_{1}, a_{2})\frac{\xi_{1}^{2}}{\Lambda_{1}} + C_{7}(a_{1}, a_{2})\frac{\xi_{1}\xi_{2}}{\sqrt{\Lambda_{1}\Lambda_{2}}} + C_{8}(a_{1}, a_{2})\frac{\xi_{2}^{2}}{\Lambda_{2}} \\ \mathcal{Q}_{12}^{22} \cdot (\eta_{1}, \eta_{2}, \xi_{1}, \xi_{2}) &= C_{6}(a_{1}, a_{2})\frac{\eta_{1}^{2}}{\Lambda_{1}} + C_{7}(a_{1}, a_{2})\frac{\eta_{1}\eta_{2}}{\sqrt{\Lambda_{1}\Lambda_{2}}} + C_{8}(a_{1}, a_{2})\frac{\eta_{2}^{2}}{\Lambda_{2}} \\ &+ C_{3}(a_{1}, a_{2})\frac{\xi_{1}^{2}}{\Lambda_{1}} + C_{4}(a_{1}, a_{2})\frac{\xi_{1}\xi_{2}}{\sqrt{\Lambda_{1}\Lambda_{2}}} + C_{5}(a_{1}, a_{2})\frac{\xi_{2}^{2}}{\Lambda_{2}} \end{aligned}$$
(5.22)

the polynomial $\mathfrak{F}_{hv}^2(\eta,\xi,p,q)$ (compare eq. (5.17)) reduces to

$$\begin{split} \mathfrak{F}_{hv}^{2} &= \frac{1}{2} \bar{m}_{1} \bar{m}_{2} \Big(\mathcal{Q}_{12}^{11} (\mathfrak{L}q_{2} - \mathfrak{L}q_{1})^{2} + \mathcal{Q}_{12}^{22} (\mathfrak{L}p_{2} - \mathfrak{L}p_{1})^{2} \Big) + \hat{\mathfrak{F}}_{hv}^{2} \\ &= \frac{1}{2} \bar{m}_{1} \bar{m}_{2} \left(\frac{1}{\Lambda_{1}} + \frac{1}{\Lambda_{2}} \right) \\ \times & \left[C_{3}(a_{1}, a_{2}) \frac{\eta_{1}^{2} q^{2} + \xi_{1}^{2} p^{2}}{\Lambda_{1}} + C_{4}(a_{1}, a_{2}) \frac{\eta_{1} \eta_{2} q^{2} + \xi_{1} \xi_{2} p^{2}}{\sqrt{\Lambda_{1} \Lambda_{2}}} + C_{5}(a_{1}, a_{2}) \frac{\eta_{2}^{2} q^{2} + \xi_{2}^{2} p^{2}}{\Lambda_{2}} \\ &+ C_{6}(a_{1}, a_{2}) \frac{\xi_{1}^{2} q^{2} + \eta_{1}^{2} p^{2}}{\Lambda_{1}} + C_{7}(a_{1}, a_{2}) \frac{\xi_{1} \xi_{2} q^{2} + \eta_{1} \eta_{2} p^{2}}{\sqrt{\Lambda_{1} \Lambda_{2}}} + C_{8}(a_{1}, a_{2}) \frac{\xi_{2}^{2} q^{2} + \eta_{2}^{2} p^{2}}{\Lambda_{2}} \\ &+ \hat{\mathfrak{F}}_{hv}^{2} \end{split}$$
(5.23)

(referring to (5.5), (5.46), (5.65) for the definition of $C_1, C_3 \div C_8, C_{13}$) where

$$\hat{\mathfrak{F}}_{hv}^2 := \frac{1}{2} \bar{m}_1 \bar{m}_2 \Big(\mathcal{Q}_{12}^{12} + \mathcal{Q}_{12}^{21} \Big) \Big(\mathfrak{L}p_2 - \mathfrak{L}p_2 \Big) \Big(\mathfrak{L}q_j - \mathfrak{L}q_i \Big)$$

This term gives no contribute to the Birkhoff matrix, since, when computed in $(\rho_h^2 \eta, \rho_h^2 \xi, p, q)$, it contains only monomials of the form $\eta_i \xi_j pq$, hence, with vanishing Birkhoff form.

Then, from (5.23), $\mathfrak{F}_{hv}^2(\rho_h\eta,\rho_h\xi,p,q)$ generates on the entries with place (1,3), (2,3), (3,1), (3,2) of the Birkhoff matrix the dominant terms (in a_1)

$$\begin{pmatrix} * & * & \frac{9}{4} \frac{\bar{m}_1 \bar{m}_2}{\Lambda_1^2 a_2} \frac{a_1^2}{a_2^2} \\ * & * & \frac{9}{4} \frac{\bar{m}_1 \bar{m}_2}{\Lambda_1 \Lambda_2 a_2} \frac{a_1^2}{a_2^2} \\ \frac{9}{4} \frac{\bar{m}_1 \bar{m}_2}{\Lambda_1^2 a_2} \frac{a_1^2}{a_2^2} & \frac{9}{4} \frac{\bar{m}_1 \bar{m}_2}{\Lambda_1 \Lambda_2 a_2} \frac{a_1^2}{a_2^2} & * \end{pmatrix}$$
(5.24)

The result is found taking into account the diagonalizing matrices as in (5.18) and using the expansions for the $C_3 \div C_8$

$$C_{3} = \frac{3}{a_{2}} \left(\frac{a_{1}^{2}}{a_{1}^{2}} + O\left(\frac{a_{1}^{4}}{a_{2}^{4}}\right) \right)$$

$$C_{5} = \frac{9}{8a_{2}} \left(\frac{a_{1}^{2}}{a_{1}^{2}} + O\left(\frac{a_{1}^{4}}{a_{2}^{4}}\right) \right)$$

$$C_{6} = -\frac{3}{4a_{2}} \left(\frac{a_{1}^{2}}{a_{1}^{2}} + O\left(\frac{a_{1}^{4}}{a_{2}^{4}}\right) \right)$$

$$C_{8} = \frac{9}{8a_{2}} \left(\frac{a_{1}^{2}}{a_{1}^{2}} + O\left(\frac{a_{1}^{4}}{a_{2}^{4}}\right) \right)$$

$$C_{4}, C_{7} = O\left(\frac{a_{1}^{3}}{a_{2}^{4}}\right)$$

We finally compute now the Birkhoff matrix associated to (compare eq. (5.17))

$$\begin{aligned} \mathfrak{F}_{v}^{2}(\eta,\xi,p,q) &= \bar{m}_{1}\bar{m}_{2}(\mathfrak{Q}_{11}^{12}+\mathfrak{Q}_{22}^{12})C_{1}(a_{1},a_{2}) \\ &+ \bar{m}_{1}\bar{m}_{2}\left(\left(\mathfrak{L}p_{2}-\mathfrak{L}p_{1}\right)^{2}+\left(\mathfrak{L}q_{2}-\mathfrak{L}q_{1}\right)^{2}\right)^{2}C_{13}(a_{1},a_{2})\right) \,. \end{aligned}$$

We recall that in the case N = 2, $\mathfrak{S}_{12} = \overline{\mathfrak{S}}_1$, so, \mathfrak{Q}_{12}^{11} , \mathfrak{Q}_{12}^{22} coincide with the order 4 terms of the antries at places (1, 1), (2, 2) of $\overline{\mathfrak{S}}_1$, which are (compare $(5.61) \div (5.64)$ below)

$$\mathfrak{Q}_{12}^{11} = -q^2 \bar{\mathcal{C}}_1 = -q^2 \frac{2\Lambda_2^2(\frac{\eta_1^2 + \xi_1^2}{2}) + 2\Lambda_1^2 \frac{\eta_2^2 + \xi_2^2}{2} - \Lambda_1 \Lambda_2 \frac{p^2 + q^2}{2}}{4\Lambda_1^2 \Lambda_2^2}$$
$$\mathfrak{Q}_{12}^{22} = -p^2 \bar{\mathcal{C}}_1 = -p^2 \frac{2\Lambda_2^2(\frac{\eta_1^2 + \xi_1^2}{2}) + 2\Lambda_1^2 \frac{\eta_2^2 + \xi_2^2}{2} - \Lambda_1 \Lambda_2 \frac{p^2 + q^2}{2}}{4\Lambda_1^2 \Lambda_2^2}$$

Hence,

$$\begin{aligned} \mathfrak{F}_{v}^{2}(\eta,\xi,p,q) &= \bar{m}_{1}\bar{m}_{2}(\mathfrak{Q}_{11}^{12} + \mathfrak{Q}_{22}^{12})C_{1}(a_{1},a_{2}) \\ &+ \bar{m}_{1}\bar{m}_{2}\left(\left(\mathfrak{L}p_{2} - \mathfrak{L}p_{1}\right)^{2} + \left(\mathfrak{L}q_{2} - \mathfrak{L}q_{1}\right)^{2}\right)^{2}C_{13}(a_{1},a_{2})\right) \\ &= -\frac{\bar{m}_{1}\bar{m}_{2}}{2}C_{1}(a_{1},a_{2}) \\ &\times \left(\frac{2}{\Lambda_{1}^{2}}\frac{\eta_{1}^{2} + \xi_{1}^{2}}{2}\frac{p^{2} + q^{2}}{2} + \frac{2}{\Lambda_{2}^{2}}\frac{\eta_{2}^{2} + \xi_{2}^{2}}{2}\frac{p^{2} + q^{2}}{2} - \frac{1}{\Lambda_{1}\Lambda_{2}}\left(\frac{p^{2} + q^{2}}{2}\right)^{2}\right) \\ &+ \bar{m}_{1}\bar{m}_{2}\left(\frac{1}{\Lambda_{1}} + \frac{1}{\Lambda_{2}}\right)^{2}(p^{2} + q^{2})^{2}C_{13}(a_{1},a_{2}) \end{aligned}$$
(5.25)

Since (5.25), letting $(\eta, \xi, p, q) \to (\rho_h \eta, \rho_h \xi, p, q)$ with ρ_h as in (5.18) and using the expansions

$$C_1 = -\frac{3}{4}\frac{a_1^2}{a_2^3} + O\left(\frac{a_1^4}{a_2^5}\right) , \quad C_{13} = -\frac{3}{32}\frac{a_1^2}{a_2^3} + O\left(\frac{a_1^4}{a_2^5}\right)$$

we find the contribute of the Birkhoff matrix associated to $\mathfrak{F}_v^2(\rho_h\eta,\rho_h\xi,p,q)$

$$\begin{pmatrix} * & * & \frac{3}{4}\frac{\bar{m}_{1}\bar{m}_{2}}{\Lambda_{1}^{2}a_{2}}\frac{a_{1}^{2}}{a_{2}^{2}} \\ * & * & O\left(\frac{1}{\Lambda_{2}^{2}a_{2}}\frac{a_{1}^{2}}{a_{2}^{2}}\right) \\ \frac{3}{4}\frac{\bar{m}_{1}\bar{m}_{2}}{\Lambda_{1}^{2}a_{2}}\frac{a_{1}^{2}}{a_{2}^{2}} & O\left(\frac{1}{\Lambda_{2}^{2}a_{2}}\frac{a_{1}^{2}}{a_{2}^{2}}\right) & -\frac{3}{4}\frac{\bar{m}_{1}\bar{m}_{2}}{\Lambda_{1}^{2}a_{2}}\frac{a_{1}^{2}}{a_{2}^{2}} \end{pmatrix}$$
(5.26)

,

Finally, on count of (5.19), (5.24), (5.26), we find that the Birkhoff matrix associated to $\mathfrak{F}^2(\rho_h\eta,\rho_h\xi,p,q)$ is

$$\frac{\bar{m}_1\bar{m}_2}{a_2} \frac{a_1^2}{a_2^2} \begin{pmatrix} \frac{3}{4\Lambda_1^2} & -\frac{9}{4\Lambda_1\Lambda_2} & \frac{3}{\Lambda_1^2} \\ -\frac{9}{4\Lambda_1\Lambda_2} & -\frac{3}{\Lambda_2^2} & \frac{9}{4\Lambda_1\Lambda_2} \\ \frac{3}{\Lambda_1^2} & \frac{9}{4\Lambda_1\Lambda_2} & -\frac{3}{4\Lambda_1^2} \end{pmatrix} (1+o(1))$$

hence, it is non singular, having determinant

$$-\left(\frac{\bar{m}_1\bar{m}_2}{a_2}\frac{a_1^2}{a_2^2}\right)^3 \frac{27}{16\Lambda_1^4\Lambda_2^2}(1+o(1)) \neq 0 ,$$

and the basis of induction is proved.

For the proof of the inductive step, we need the following result, due to J. Fejóz, to whom we refer for the proof.

Lemma 5.3 (J. Fejóz, [17], COROLLAIRE 72) Let $\delta_1, \dots, \delta_{n-1} \in \mathbb{R}$, $\delta_n = 0$ such that $\sigma := \min_{1 \le j \ne k \le n} |\delta_j - \delta_k| \ne 0$, $\hat{D} \in \operatorname{Matr}_{n-1}(\mathbb{R})$ a symmetric with eigenvalues $\delta_1, \dots, \delta_{n-1}$, D the symmetric matrix

$$D = \left(\begin{array}{cc} \hat{D} & 0\\ 0 & 0 \end{array}\right)$$

and $A_{\varepsilon} \in \operatorname{Matr}_{n-1}(\mathbb{R})$ a symmetric matrix with last coefficient

$$(A_{\varepsilon})_{nn} = c_1 + c_2 \varepsilon^{\beta} , \quad c_1, \ c_2 \in \mathbb{R} , \quad 0 \le \beta < 2 .$$

Then, when $\varepsilon \to 0$, the matrix $D + \varepsilon A$ has an eigenvalue

$$\sigma_n(\varepsilon) = \varepsilon(c_1 + c_2\varepsilon^\beta) + O(\varepsilon^2)$$
.

Furthermore, if \hat{D} is diagonal, $D + \varepsilon A$ is conjugated to a diagonal matrix through a matrix $\rho \in SO_n(\mathbb{R})$ verifying $\rho = I + O(\varepsilon)$.

We can apply the previous lemma taking \mathcal{F}_h^{N-1} , $\mathcal{F}_v^{*(N-1)}$ for \hat{D} and \mathcal{F}_h^N , \mathcal{F}^{*N} for $D + \varepsilon A$, with $\varepsilon = a_N^{-3}$. Observe in fact that both $\mathcal{F}_h^{N-1} \mathcal{F}_v^{*(N-1)}$ verify the assumptions of the Lemma, since their respective eigenvalues do not satisfy any other linear relation than the Herman's resonance (Proposition 5.1 above) and, furthermore,

$$\mathcal{F}_{h}^{N} = \begin{pmatrix} \mathcal{F}_{h}^{N-1} & 0\\ 0 & 0 \end{pmatrix} + O(a_{N}^{-7/2}) , \quad \mathcal{F}_{v}^{*N} = \begin{pmatrix} \mathcal{F}_{v}^{*(N-1)} & 0\\ 0 & 0 \end{pmatrix} + O(a_{N}^{-3}) \quad (a_{N} \to \infty)$$

Then, the diagonalizing matrices ρ_h^N , ρ_v^{*N} at step N are related to the matrices ρ_h^{N-1} , $\rho_h^{*(N-1)}$ of step N-1 by

$$\rho_h^N = \begin{pmatrix} \rho_h^{N-1} & 0\\ 0 & 1 \end{pmatrix} + O(a_N^{-7/2}) , \quad \rho_v^{*N} = \begin{pmatrix} \rho_v^{*(N-1)} & 0\\ 0 & 1 \end{pmatrix} + O(a_N^{-3}) . \tag{5.27}$$

This result will be used in the inductive step, which we are now ready to prove.

Proof of the inductive step $(N-1) \rightarrow N$. Assume that, when

 $a_1 \ll \cdots \ll a_{N-2} \ll a_{N-1} \to \infty$

the Birkhoff matrix $A_{\rm plt}^{N-1}$ associated to

$$\mathfrak{F}^{N-1}(\rho_h^{N-1}\hat{\eta},\rho_h^{N-1}\hat{\xi},\rho_v^{*(N-1)}\hat{p},\rho_v^{*(N-1)}\hat{q})$$

is non degenerate, and let us prove that, when

$$a_1 \ll \cdots \ll a_{N-1} \ll a_N \to \infty$$

then, the matrix A_{plt}^N associated to

$$\mathfrak{F}^N(\rho_h^N\eta,\rho_h^N\xi,\rho_v^{*N}p,\rho_v^{*N}q)$$

is so.

Let $\mathfrak{F} := \mathfrak{F}^N$ as in (5.14) \div (5.17) and let us split

$$\mathfrak{F} := \mathfrak{F}^N = \hat{\mathfrak{F}}^{N-1} + \mathfrak{F}' \tag{5.28}$$

where

$$\hat{\mathfrak{F}}^{N-1} = \hat{\mathfrak{F}}_h^{N-1} + \hat{\mathfrak{F}}_{hv}^{N-1} + \hat{\mathfrak{F}}_v^{N-1}$$

is 4-order polynomial associated to N-1 bodies, in the variables the variables $(\hat{\eta}, \hat{\xi}, \hat{p}, \hat{q})$, when the variables (η, ξ, p, q) related to N bodies are written as

$$\eta := (\hat{\eta}, \eta_N) := \left((\eta_1, \cdots, \eta_{N-1}), \eta_N \right), \quad p := (\hat{p}, p_{N-1}) := \left((p_1, \cdots, p_{N-2}), p_{N-1} \right), \cdots$$

and

$$\mathfrak{F}' := \mathfrak{F}^N - \hat{\mathfrak{F}}^{N-1} = \mathfrak{F}'_h + \mathfrak{F}'_{hv} + \mathfrak{F}'_v$$

with $\mathfrak{F}'_h = \mathfrak{F}_h - \hat{\mathfrak{F}}_h^{N-1}, \cdots$, and similarly for the definitions of $\mathfrak{F}'_{hv}, \mathfrak{F}'_v$. By inspection of its coefficients, \mathfrak{F}'_h is $O(a_N^{-3})$, so, making use of (5.27), it is not difficult to see that (5.28) implies that the first approximation for the Birkhoff matrix A^N_{plt} (suitably rearranged) is

$$A_{\rm plt}^{N} = \begin{pmatrix} A_{\rm plt}^{N-1} + O(a_{N}^{-3}) & O(a_{N}^{-3}) \\ O(a_{N}^{-3}) & \hat{A} + O(a_{N}^{-6}) \end{pmatrix}$$
(5.29)

where $\hat{A} = O(a_N^{-3})$ is the square matrix of order 2 associated to the quadratic form in the couple of variables

$$\left(\frac{\eta_N^2 + \xi_N^2}{2}, \ \frac{p_{N-1}^2 + q_{N-1}^2}{2}\right)$$

appearing the Birkhoff polynomial of

$$\mathfrak{F}'((\rho_h^{N-1}\hat{\eta},\eta_N),(\rho_h^{N-1}\hat{\xi},\xi_N)(\rho_v^{*(N-1)}\hat{p},p_{N-1})(\rho_v^{*(N-1)}\hat{q},q_{N-1}))$$

By (5.29) and the inductive hypothesis, we only need to prove that \hat{A} is non singular. As for the proof of the iductive basis, this is done by direct computation. We start with the horizontal terms

$$\begin{aligned} \mathfrak{F}'_{h} &= \mathfrak{F}^{N}_{h} - \hat{\mathfrak{F}}^{N-1}_{h} \\ &= q_{NNNN} \eta^{4}_{N} + q_{NNNN} \xi^{4}_{N} + r_{NNNN} \eta^{2}_{N} \xi^{2}_{N} \\ &+ \sum_{1 \leq i \leq N-1} \left(q_{iiiN} \eta^{3}_{i} \eta_{N} + q_{iiNN} \eta^{2}_{i} \eta^{2}_{N} + q_{NNNi} \eta_{i} \eta^{3}_{N} \right. \\ &+ q_{iiiN} \xi^{3}_{i} \xi_{N} + q_{iiNN} \xi^{2}_{i} \xi^{2}_{N} + q_{NNNi} \xi^{3}_{i} \xi^{3}_{N} \\ &+ r_{iiiN} \eta^{2}_{i} \xi_{i} \xi_{N} + r_{iiNN} \eta^{2}_{i} \xi^{2}_{N} \\ &+ r_{iNii} \eta_{i} \eta_{i} \xi^{2}_{i} + r_{iNiN} \eta_{i} \eta_{N} \xi_{i} \xi_{N} + r_{NiNN} \eta_{i} \eta_{N} \xi^{2}_{N} \\ &+ r_{NNii} \eta^{2}_{N} \xi^{2}_{i} + r_{NNNi} \eta^{2}_{N} \xi_{i} \xi_{N} \end{aligned}$$

By the previous discussion, we have to pick the monomials in η_N , ξ_N only, namely

$$q_{NNNN}\eta_N^4 + q_{NNNN}\xi_N^4 + r_{NNNN}\eta_N^2\xi_N^2$$

and then we have to compute the related Birkhoff polynomial. Proceeding as done in the previous step, we use the expansions

$$q_{NNNN} = \sum_{1 \le i \le N-1} \left(-\frac{3}{8} \frac{\bar{m}_i \bar{m}_N}{\Lambda_N^2 a_N} \frac{a_i^2}{a_N^2} + O\left(\frac{a_i^2}{a_N^3}\right) \right) = -\frac{3}{8} \frac{\bar{m}_{N-1} \bar{m}_N}{\Lambda_N^2 a_N} \frac{a_{N-1}^2}{a_N^2} + O\left(\frac{a_{N-1}^2}{a_N^3}\right)$$

and, similarly,

$$r_{NNNN} = \sum_{1 \le i \le N-1} \left(-\frac{3}{16} \frac{\bar{m}_i \bar{m}_N}{\Lambda_N^2 a_N} \frac{a_i^2}{a_N^2} + O\left(\frac{a_i^2}{a_N^3}\right) \right) = -\frac{3}{16} \frac{\bar{m}_{N-1} \bar{m}_N}{\Lambda_N^2 a_N} \frac{a_{N-1}^2}{a_N^2} + O\left(\frac{a_{N-1}^2}{a_N^3}\right)$$

At this point the computation is just the same we have seen for N = 2 and, at the place (1, 1) of \hat{A} we will find

$$\hat{A} = \begin{pmatrix} -3\frac{\bar{m}_{N-1}\bar{m}_N}{\Lambda_N^2 a_N} \frac{a_{N-1}^2}{a_N^2} & * \\ * & * \end{pmatrix}$$

Also the analysis of the vertical part \mathfrak{F}'_{hv} will give, as dominant terms on the off diagonal entries, a similar result as in the cae N = 2

$$\hat{A} = \begin{pmatrix} * & \frac{9}{4} \frac{\bar{m}_{N-1} \bar{m}_N}{\Lambda_{N-1} \Lambda_N a_N} \frac{a_{N-1}^2}{a_N^2} \\ \frac{9}{4} \frac{\bar{m}_{N-1} \bar{m}_N}{\Lambda_{N-1} \Lambda_N a_N} \frac{a_{N-1}^2}{a_N^2} & * \end{pmatrix}$$
(5.30)

This result follows computing the Birkhoff matrix relatively to the the monomials with $p_{N-1}, q_{N-1}, \eta_N, \xi_N$ of

$$\begin{split} \mathfrak{F}_{hv}' &:= \mathfrak{F}_{hv}^{N} - \hat{\mathfrak{F}}_{hv}^{N-1} \\ &= \frac{1}{2} \sum_{1 \le i \le N-1} \bar{m}_{i} \bar{m}_{N} \Big(\mathcal{Q}_{iN}^{11} \left(\mathfrak{L}q_{N} - \mathfrak{L}q_{i} \right)^{2} + \mathcal{Q}_{iN}^{22} \left(\mathfrak{L}p_{N} - \mathfrak{L}p_{i} \right)^{2} \Big) \\ &+ \hat{\mathfrak{F}}_{hv}^{N} \end{split}$$

(where $\hat{\mathfrak{F}}^N$ is a suitable polynomial which gives no contribute to the Birkhoff matrix), noticing that $\mathfrak{L}q_N - \mathfrak{L}q_i$ has the form

$$\mathfrak{L}q_N - \mathfrak{L}q_i = \bar{c}_{N-1}p_{N-1} + \hat{c}_i \cdot \hat{p} , \quad \hat{p} = (p_1, \cdots, p_{N-2})$$

with

$$\bar{c}_{N-1} = \sqrt{\frac{1}{\mathcal{L}_{N-1}} + \frac{1}{\Lambda_N}} = \sqrt{\frac{1}{\Lambda_{N-1}}} + O\left(\frac{1}{\Lambda_N}\right) + O\left(\Lambda_{N-2}\right)$$

and \mathcal{Q}_{iN}^{kk} as in (5.22), with *i*, *N* replacing 1, 2 respectively.

The last step consists in evaluating the contribute to \hat{A} of

$$\mathfrak{F}'_v := \mathfrak{F}^N_v - \hat{\mathfrak{F}}^{N-1}_v$$

$$= \sum_{1 \le i \le N-1} \bar{m}_i \bar{m}_N (\mathfrak{Q}_{11}^{iN} + \mathfrak{Q}_{22}^{iN}) C_1(a_i, a_N)$$

+
$$\sum_{1 \le i \le N-1} \bar{m}_i \bar{m}_N \left(\left(\mathfrak{L}p_i - \mathfrak{L}p_N \right)^2 + \left(\mathfrak{L}q_i - \mathfrak{L}q_N \right)^2 \right)^2 C_{13}(a_i, a_N)$$

+
$$\hat{\mathfrak{F}}_v^N$$
(5.31)

where

$$\hat{\mathfrak{F}}_{v}^{N} := \sum_{1 \le i \le N-1} \bar{m}_{i} \bar{m}_{N} \left(\sum_{1 \le h < k \le N-1}^{*} \ell_{h}^{iN} \ell_{k}^{iN} (p_{h}q_{k} - p_{k}q_{h}) \right)^{2} C_{14}(a_{i}, a_{N})$$

gives only a negligible contribute on \hat{A} . Notice that matrices S_i , \tilde{S}_j appearing in the the productories (5.52) do not involve the variables p_{N-1} , q_{N-1} , η_N , ξ_N , so, the monomials of degree 4 in \mathfrak{Q}_{11}^{iN} , \mathfrak{Q}_{22}^{iN} involving only p_{N-1} , q_{N-1} , η_N , ξ_N coincide with the corresponding monomials of degree 4 of \bar{S}_{N-1} , on the entries (1, 1), (2, 2), which are

$$-q_{N-1}^2 \frac{2L_{N-1}^2 \tau_N - L_{N-1} \Lambda_N \rho_{N-1}}{4L_{N-1}^2 \Lambda_N^2} , \quad -p_{N-1}^2 \frac{2L_{N-1}^2 \tau_N - L_{N-1} \Lambda_N \rho_{N-1}}{4L_{N-1}^2 \Lambda_N^2}$$

Thus, the term

$$\sum_{1 \le i \le N-1} \bar{m}_i \bar{m}_N (\mathfrak{Q}_{11}^{iN} + \mathfrak{Q}_{22}^{iN}) C_1(a_i, a_N)$$

gives

$$\begin{pmatrix} * & O\left(\frac{1}{\Lambda_N^2 a_N} \frac{a_{N-1}^2}{a_N^2}\right) \\ O\left(\frac{1}{\Lambda_N^2 a_N} \frac{a_{N-1}^2}{a_N^2}\right) & O\left(\frac{1}{\Lambda_N \Lambda_{N-1} a_N} \frac{a_{N-1}^2}{a_N^2}\right) \end{pmatrix}$$

As in the case N = 2, the off-diagonal terms above can be neglected with respect to the ones appearing in (5.30), while the diagonal term with place (2, 2) can be neglected with respect to the corresponding term

$$\hat{A} = \begin{pmatrix} * & * \\ * & -\frac{3}{4} \frac{\bar{m}_{N-1} \bar{m}_N}{\Lambda_{N-1}^2 a_N} \frac{a_{N-1}^2}{a_N^2} \end{pmatrix}$$

generated by the second line in (5.31). On count of the previous computations, the final result extends the one found for N = 2, giving

$$\hat{A} = \begin{pmatrix} -3\frac{\bar{m}_{N-1}\bar{m}_N}{\Lambda_N^2 a_N} \frac{a_{N-1}^2}{a_N^2} & \frac{9}{4}\frac{\bar{m}_{N-1}\bar{m}_N}{\Lambda_{N-1}\Lambda_N a_N} \frac{a_{N-1}^2}{a_N^2} \\ \frac{9}{4}\frac{\bar{m}_{N-1}\bar{m}_N}{\Lambda_{N-1}\Lambda_N a_N} \frac{a_{N-1}^2}{a_N^2} & -\frac{3}{4}\frac{\bar{m}_{N-1}\bar{m}_N}{\Lambda_{N-1}^2 a_N} \frac{a_{N-1}^2}{a_N^2} \end{pmatrix}$$

so,

$$\det \hat{A} = -\frac{45}{16} \left(\frac{\bar{m}_{N-1} \bar{m}_N}{\Lambda_{N-1} \Lambda_N a_N} \frac{a_{N-1}^2}{a_N^2} \right)^2 \neq 0$$

which finishes the proof.

5.1.3 Expansion of the Hamiltonian

Lemma 5.4 The "secular perturbation", i.e., the mean

$$\bar{f}_{\text{plt,pr}} := \frac{1}{(2\pi)^N} \int_{\mathbb{T}^N} f_{\text{plt,pr}} \qquad f_{\text{plt,pr}} := \sum_{1 \le i < j \le N} \frac{y_i \cdot y_j}{\bar{m}_0} - \frac{\bar{m}_i \bar{m}_j}{|x_i - x_j|}$$

 $(y_i, x_i \text{ as in } (4.46) \div (4.49))$ coincides with the mean of the Newtonian potential:

$$\bar{f}_{\text{plt,pr}} = -\sum_{1 \le i < j \le N} \frac{\bar{m}_i \bar{m}_j}{4\pi^2} \int_{\mathbb{T}^2} \frac{d\lambda_i d\lambda_j}{|x_i - x_j|} .$$
(5.32)

The proof of this lemma is trivial and so it is omitted.

Lemma 5.5 The secular perturbation $f_{\text{plt,pr}}$ is an even function of the "secular variables" $\bar{z} := (\bar{\eta}, \bar{\xi}, \bar{p}, \bar{q}).$

Proof. Actually, it is even in $(\bar{\eta}, \bar{\xi})$ and (\bar{p}, \bar{q}) separately. In fact, letting $(\bar{\eta}, \bar{\xi}) \rightarrow -(\bar{\eta}, \bar{\xi})$ and simultaneously $\bar{\lambda} \rightarrow \bar{\lambda} + \pi$, the Plane Delaunay–Poincaré Map changes for a sign and the matrices $\mathfrak{R}_i^{\mathrm{pr}}$ do not change (they are even in $(\bar{\eta}, \bar{\xi})$), so, taking the mean over $\bar{\lambda}$ of the Newtonian potential, we find that $\bar{f}_{\mathrm{plt,pr}}$ is even in $(\bar{\eta}, \bar{\xi})$. Observing that $\bar{f}_{\mathrm{plt,pr}}$ depends on (\bar{p}, \bar{q}) only through on the entries of $\mathfrak{R}_i^{\mathrm{pr}}$ with place (1, 1), (1, 2), (2, 1), (2, 2),which are even in (\bar{p}, \bar{q}) (since they are products of the matrices $\mathcal{S}_i, \tilde{\mathcal{S}}_j$'s, the entries of which with place (1, 1), (1, 2), (2, 1), (2, 2), (3, 3) are even in (\bar{p}, \bar{q}) , while the ones with place (1, 3), (2, 3), (3, 1), (3, 2) are odd), we also find that $\bar{f}_{\mathrm{plt,pr}}$ is even in (\bar{p}, \bar{q}) .

We proceed with the expansion of the mean of the Newtonian potential (5.32).

Unsing (4.49), we write the Euclidean distance $|x_i - x_j|$ as

$$|x_i - x_j| = |\mathfrak{R}_i^{\mathrm{pr}} \hat{x}_i - \mathfrak{R}_j^{\mathrm{pr}} \hat{x}_j| = |\mathbf{I}_i \hat{x}_i - \mathfrak{S}_{ij} \hat{x}_j| = |\mathbf{I}_i \hat{x}_i - \hat{\mathfrak{S}}_{ij} \hat{x}_j|,$$

where $\hat{\mathfrak{S}}$ denotes the submatrix of order 2 of a give matrix \mathfrak{S} of order 3,

$$\mathbf{I}_i := \begin{cases} \mathbf{I}_{\mathbf{z}} & i = 1\\ \mathrm{id} & i > 1 \end{cases}$$

changes the sign of \hat{x}_1 and $\mathfrak{S}_{ij} := \mathfrak{S}_i^T \mathfrak{S}_j$, with

$$\mathfrak{S}_{i} := \begin{cases} \left(\prod_{j=1}^{N-2} (\tilde{\mathcal{S}}^{\mathrm{pr}})_{N-j}^{\mathrm{T}}\right) \mathcal{S}_{1}^{\mathrm{pr}T} & i = 1\\ \left(\prod_{j=1}^{N-i} (\tilde{\mathcal{S}}_{N-j}^{\mathrm{pr}})^{\mathrm{T}}\right) \mathcal{S}_{i}^{\mathrm{pr}} & 2 \leq i \leq N-1\\ \mathcal{S}_{N}^{\mathrm{pr}} & i = N \end{cases}$$

$$(5.33)$$

Changing, into the integral (5.34), the integration variable $\bar{\lambda}_1$ with $\bar{\lambda}_1 + \pi$ and making use of the relation

$$\hat{x}_1(\Lambda_1, \bar{\lambda}_1 + \pi, \bar{\eta}_1, \bar{\xi}_1) = -\hat{x}_1(\Lambda_1, \bar{\lambda}_1, -\bar{\eta}_1, -\bar{\xi}_1) =: -\hat{x}_1^*(\Lambda_1, \bar{\lambda}_1, \bar{\eta}_1, \bar{\xi}_1) =: -\hat{x}_1(\Lambda_1, \bar{\lambda}_1, \bar{\eta}_1^*, \bar{\xi}_1^*)$$

where $\eta_* := (-\eta_1, \eta_2, \cdots)$, we can write the secular perturbation as

$$\bar{f}_{\text{plt,pr}} = -\sum_{1 \le i < j \le N} \frac{\bar{m}_i \bar{m}_j}{4\pi^2} \int_{\mathbb{T}^2} \frac{d\lambda_i d\lambda_j}{|x_i - x_j|} \\
= -\sum_{1 \le i < j \le N} \frac{\bar{m}_i \bar{m}_j}{4\pi^2} \int_{\mathbb{T}^2} \frac{d\bar{\lambda}_i d\bar{\lambda}_j}{|\mathbf{I}_i \hat{x}_i - \hat{\mathfrak{S}}_{ij} \hat{x}_j|} \\
= -\sum_{1 \le i < j \le N} \frac{\bar{m}_i \bar{m}_j}{4\pi^2} \int_{\mathbb{T}^2} \frac{d\bar{\lambda}_i d\bar{\lambda}_j}{|\hat{x}_i^* - \hat{\mathfrak{S}}_{ij} \hat{x}_j|}$$
(5.34)

where

$$\hat{x}_i^* = \begin{cases} \hat{x}_1^* & i = 1\\ \hat{x}_i & i > 1 \end{cases}.$$

Let us now think that $\hat{\mathfrak{S}}_{ij}$ is expanded in powers of $(\bar{\eta}, \bar{\xi}, \bar{p}, \bar{q})$, up to order four, that is, let us put

$$\hat{\mathfrak{S}}_{ij} = \mathrm{id} + \hat{\mathfrak{S}}_{ij}^2 + \hat{\mathfrak{S}}_{ij}^4 + \mathrm{O}(6)$$

and let us consequently expand the square distance

$$D_{ij} := |\hat{x}_i^* - \hat{\mathfrak{S}}_{ij}\hat{x}_j|^2$$

= $|\hat{x}_i^*|^2 + |\hat{x}_j^*|^2 - 2\hat{x}_i^* \cdot \hat{\mathfrak{S}}_{ij}\hat{x}_j$
= $|\hat{x}_i^* - \hat{x}_j|^2 - 2\hat{x}_i^* \cdot \hat{\mathfrak{S}}_{ij}^2\hat{x}_j - 2\hat{x}_i^* \cdot \hat{\mathfrak{S}}_{ij}^4\hat{x}_j + O(6)$
= $D_0^{ij} + D_2^{ij} + D_4^{ij} + O(6)$

where

$$\begin{cases}
D_0^{ij} := |\hat{x}_i^* - \hat{x}_j|^2 \\
D_2^{ij} := -2\,\hat{x}_i^* \cdot \hat{\mathfrak{S}}_{ij}^2 \hat{x}_j \\
D_4^{ij} := -2\,\hat{x}_i^* \cdot \hat{\mathfrak{S}}_{ij}^4 \hat{x}_j
\end{cases}$$
(5.35)

Using now the elementary expansion

$$\frac{1}{\sqrt{D}} = \frac{1}{D_0^{1/2}} - \frac{D_2}{2D_0^{3/2}} - \frac{D_4}{2D_0^{3/2}} + \frac{3}{8}\frac{D_2^2}{D_0^{5/2}} + O(6)$$

when D has the expansion

$$D = D_0 + D_2 + D_4 + O(6)$$

with $D_k := D_k^{ij}$ as in (5.35), we write

$$\frac{1}{|\hat{x}_i^* - \hat{\mathfrak{S}}_{ij}\hat{x}_j|} = \frac{1}{|\hat{x}_i^* - \hat{x}_j|} + \frac{\hat{x}_i^* \cdot \hat{\mathfrak{S}}_{ij}^2 \hat{x}_j}{|\hat{x}_i^* - \hat{x}_j|^3} + \frac{\hat{x}_i^* \cdot \hat{\mathfrak{S}}_{ij}^4 \hat{x}_j}{|\hat{x}_i^* - \hat{x}_j|^3} + \frac{3}{2} \frac{(\hat{x}_i^* \cdot \hat{\mathfrak{S}}_{ij}^2 \hat{x}_j)^2}{|\hat{x}_i^* - \hat{x}_j|^5} + O(6)$$

When we multiply by $-\bar{m}_i\bar{m}_j$, sum over all $1 \leq i < j \leq N$, take the mean on $(\bar{\lambda}_i, \bar{\lambda}_j)$, we can split $\bar{f}_{\text{plt,pr}}$ as

$$\bar{f}_{\rm plt,pr} = \bar{f}_{\rm pl}^* + \bar{f}_{\rm two} + \bar{f}_{\rm four} \tag{5.36}$$

where

$$\begin{split} \bar{f}_{pl}^{*} &:= -\sum_{1 \le i < j \le N} \frac{\bar{m}_{i} \bar{m}_{j}}{4\pi^{2}} \int_{\mathbb{T}^{2}} \frac{d\bar{\lambda}_{i} d\bar{\lambda}_{j}}{|\hat{x}_{i}^{*} - \hat{x}_{j}|} \\ \bar{f}_{two} &:= -\sum_{1 \le i < j \le N} \frac{\bar{m}_{i} \bar{m}_{j}}{4\pi^{2}} \int_{\mathbb{T}^{2}} \frac{\hat{x}_{i}^{*} \cdot \hat{\mathfrak{S}}_{ij}^{2} \hat{x}_{j}}{|\hat{x}_{i}^{*} - \hat{x}_{j}|^{3}} d\bar{\lambda}_{i} d\bar{\lambda}_{j} \\ \bar{f}_{four} &:= -\sum_{1 \le i < j \le N} \frac{\bar{m}_{i} \bar{m}_{j}}{4\pi^{2}} \int_{\mathbb{T}^{2}} \left(\frac{\hat{x}_{i}^{*} \cdot \hat{\mathfrak{S}}_{ij}^{4} \hat{x}_{j}}{|\hat{x}_{i}^{*} - \hat{x}_{j}|^{3}} + \frac{3}{2} \frac{(\hat{x}_{i}^{*} \cdot \hat{\mathfrak{S}}_{ij}^{2} \hat{x}_{j})^{2}}{|\hat{x}_{i}^{*} - \hat{x}_{j}|^{5}} \right) d\bar{\lambda}_{i} d\bar{\lambda}_{j} \end{split}$$

and we have now to expand the \hat{x} -coordinate of the plane Delaunay–Poincaré map $(\Lambda, \bar{\lambda}, \bar{\eta}, \bar{\xi}) \rightarrow (\hat{y}, \hat{x})$ in powers of (η, ξ) . In the following, we perform this expansion, collecting only the terms of order 2, 4.

(i) **Expansion of** \bar{f}_{pl}^* . The function

$$\bar{f}_{\rm pl}^* = -\sum_{1 \le i < j \le N} \frac{\bar{m}_i \bar{m}_j}{4\pi^2} \int_{\mathbb{T}^2} \frac{d\lambda_i d\lambda_j}{|\hat{x}_i^* - \hat{x}_j|}$$

coincides with the function $\bar{f}_{\rm pl}(\Lambda, \eta_*, \xi_*)$, where $\bar{f}_{\rm pl}$ is the secular perturbation of the plane problem. So, recalling that the expansion of $\bar{f}_{\rm pl}$ is

$$\bar{f}_{pl} = -\sum_{1 \le i < j \le N} \frac{\bar{m}_i \bar{m}_j}{4\pi^2} \int_{\mathbb{T}^2} \frac{d\lambda_i d\lambda_j}{|\hat{x}_i - \hat{x}_j|} = C_0 + \frac{1}{2} \mathcal{Q}_h \cdot (\eta^2 + \xi^2) + \mathfrak{F}_h(\eta, \xi) + \mathcal{O}(6)$$

where

$$C_0 := \bar{f}_0 := \bar{f}_{\text{pl}}|_{(\eta,\xi)=0} = -\sum_{1 \le i < j \le N} \frac{\bar{m}_i \bar{m}_j}{a_j} b_{1/2,0}(a_i/a_j) ,$$

 \mathcal{Q}_h is the quadratic form associated to the matrix \mathcal{F}_h defined in Lemma 3.3 and 35 , \mathfrak{F}_h is the quartic form

$$\mathfrak{F}_h = q \cdot (\eta^4 + \xi^4) + r \cdot \eta^2 \xi^2$$

where q, r are the 4-tensors of $(3.27) \div (3.28)$, then,

$$\bar{f}_{\rm pl}^* = C_0 + \frac{1}{2} \mathcal{Q}_h \cdot (\eta_*^2 + \xi_*^2) + \mathfrak{F}_h(\eta_*, \xi_*) + \mathcal{O}(6) \; .$$

(ii) **Expansion of** \bar{f}_{two} The function

$$\bar{f}_{\text{two}} = -\sum_{1 \le i < j \le N} \frac{\bar{m}_i \bar{m}_j}{4\pi^2} \int_{\mathbb{T}^2} \frac{\hat{x}_i^* \cdot \hat{\mathfrak{S}}_{ij}^2 \hat{x}_j}{|\hat{x}_i^* - \hat{x}_j|^3} d\bar{\lambda}_i d\bar{\lambda}_j$$
(5.37)

³⁵The entries of the matrix \mathcal{F}_h defined in Lemma 3.3 can be written in terms of the only Laplace coefficients $b_{3/2,0}(\alpha)$, $b_{3/2,1}(\alpha)$ as in (5.3).

is of order 2 in \bar{z} , so, it contributes to the 4–expansion of $\bar{f}_{\text{plt,pr}}$ with terms of order 2 and 4, which we denote $\bar{f}_{\text{two}}|_2$, $\bar{f}_{\text{two}}|_4$:

$$f_{\rm two} = \bar{f}_{\rm two}|_2 + \bar{f}_{\rm two}|_4 + O(6) \; .$$

Let us represent $\hat{\mathfrak{S}}_{ij}^2$ through its entries

$$\hat{\mathfrak{S}}_{ij}^2 = \begin{pmatrix} \mathfrak{q}_{11}^{ij} & \mathfrak{q}_{12}^{ij} \\ \mathfrak{q}_{21}^{ij} & \mathfrak{q}_{22}^{ij} \end{pmatrix}$$
(5.38)

so as to write the integrand function of (5.37) as

$$\frac{\hat{x}_{i}^{*} \cdot \hat{\mathfrak{S}}_{ij}^{2} \hat{x}_{j}}{|\hat{x}_{i}^{*} - \hat{x}_{j}|^{3}} = \frac{\hat{x}_{i}^{*1} \hat{x}_{j}^{1} \mathfrak{q}_{11}^{ij} + \hat{x}_{i}^{*2} \hat{x}_{j}^{2} \mathfrak{q}_{22}^{ij} + \hat{x}_{i}^{*1} \hat{x}_{j}^{2} \mathfrak{q}_{12}^{ij} + \hat{x}_{i}^{*2} \hat{x}_{j}^{1} \mathfrak{q}_{21}^{ij}}{|\hat{x}_{i}^{*} - \hat{x}_{j}|^{3}} , \qquad (5.39)$$

where $(\hat{x}_i^1, \hat{x}_i^2)$ are the components of \hat{x}_i .

Then, the term $f_{\text{two}}|_2$ is found replacing \hat{x}_i with its 0–approximation

$$\hat{x}_i^0 = a_i(\cos\bar{\lambda}_i, \sin\bar{\lambda}_i) \tag{5.40}$$

into (5.37) and next into (5.39); this gives

$$f_{\rm two}|_2 = f_{\rm two}|_{\hat{x}=(5.40)} = -\sum_{1 \le i < j \le N} \bar{m}_i \bar{m}_j (\mathfrak{q}_{11}^{ij} + \mathfrak{q}_{22}^{ij}) \frac{a_i}{2a_j^2} b_{3/2,1}(a_i/a_j) \tag{5.41}$$

For the computation of $f_{\text{two}}|_4$, we denote as \mathcal{Q}_{ij}^{hk} the quadratic forms acting on $(\eta_i, \eta_j, \xi_i, \xi_j)$ which realize the \bar{z} -expansion up to order 2, of the four integrals

$$\mathcal{I}_{ij}^{hk} := \frac{1}{4\pi^2} \int_{\mathbb{T}^2} \frac{\hat{x}_i^{*h} \hat{x}_j^k}{|\hat{x}_i^* - \hat{x}_j|^3} d\bar{\lambda}_i d\bar{\lambda}_j \qquad h, \ k = 1, \ 2 \ , \tag{5.42}$$

i.e., we let

$$\mathcal{I}_{ij}^{hk} = \begin{cases} \frac{a_i}{2a_j^2} b_{3/2,1}(a_i/a_j) + \mathcal{Q}_{ij}^{hh} + \cdots & h = k = 1, 2\\ \mathcal{Q}_{ij}^{hk} + \cdots & h \neq k = 1, 2 \end{cases}$$

Then, in view of $(5.37) \div (5.39)$, we have

$$f_{\text{two}}|_4 = -\sum_{1 \le i < j \le N} \bar{m}_i \bar{m}_j \sum_{1 \le h, k \le 2} \mathcal{Q}_{ij}^{hk} \mathfrak{q}_{hk}^{ij}$$
(5.43)

The computation of the polynomials \mathcal{Q}_{ij}^{hk} is quite lenghty. It is performed using, into (5.42), the approximation of \hat{x}_i up to order 2³⁶ and next isolating the quadratic terms in \bar{z} , whose coefficients, as in the expansion of the secular perturbation of the plane problem, have the form of the mean over $(\bar{\lambda}_i, \bar{\lambda}_j) \in \mathbb{T}^2$ of ratios of trigonometric polynomials in $\bar{\lambda}_i, \bar{\lambda}_j$, with "Laplace" denominators $d_{ij}^s := |\hat{x}_i^0 - \hat{x}_j^0|^s$. The result is

$$\begin{aligned} \mathcal{Q}_{ij}^{11}(\eta_{i}^{*},\eta_{j}^{*},\xi_{i}^{*},\xi_{j}^{*}) &= C_{3}(a_{i},a_{j})\frac{\eta_{i}^{*2}}{\Lambda_{i}} + C_{4}(a_{i},a_{j})\frac{\eta_{i}^{*}\eta_{j}^{*}}{\sqrt{\Lambda_{i}\Lambda_{j}}} + C_{5}(a_{i},a_{j})\frac{\eta_{j}^{*2}}{\Lambda_{j}} \\ &+ C_{6}(a_{i},a_{j})\frac{\xi_{i}^{*2}}{\Lambda_{i}} + C_{7}(a_{i},a_{j})\frac{\xi_{i}^{*}\xi_{j}^{*}}{\sqrt{\Lambda_{i}\Lambda_{j}}} + C_{8}(a_{i},a_{j})\frac{\xi_{j}^{*2}}{\Lambda_{j}} \\ \mathcal{Q}_{ij}^{22}(\eta_{i}^{*},\eta_{j}^{*},\xi_{i}^{*},\xi_{j}^{*}) &= C_{6}(a_{i},a_{j})\frac{\eta_{i}^{*2}}{\Lambda_{i}} + C_{7}(a_{i},a_{j})\frac{\eta_{i}^{*}\eta_{j}^{*}}{\sqrt{\Lambda_{i}\Lambda_{j}}} + C_{8}(a_{i},a_{j})\frac{\eta_{j}^{*2}}{\Lambda_{j}} \\ &+ C_{3}(a_{i},a_{j})\frac{\xi_{i}^{*2}}{\Lambda_{i}} + C_{4}(a_{i},a_{j})\frac{\xi_{i}^{*}\xi_{j}^{*}}{\sqrt{\Lambda_{i}\Lambda_{j}}} + C_{5}(a_{i},a_{j})\frac{\xi_{j}^{*2}}{\Lambda_{j}} \\ \mathcal{Q}_{ij}^{12}(\eta_{i}^{*},\eta_{j}^{*},\xi_{i}^{*},\xi_{j}^{*}) &= C_{9}(a_{i},a_{j})\frac{\eta_{i}^{*}\xi_{i}^{*}}{\Lambda_{i}} + C_{10}(a_{i},a_{j})\frac{\eta_{i}^{*}\xi_{j}^{*}}{\sqrt{\Lambda_{i}\Lambda_{j}}} + C_{11}(a_{i},a_{j})\frac{\eta_{j}^{*}\xi_{i}^{*}}{\sqrt{\Lambda_{i}\Lambda_{j}}} \\ &+ C_{12}(a_{i},a_{j})\frac{\eta_{i}^{*}\xi_{i}^{*}}{\Lambda_{i}} + C_{11}(a_{i},a_{j})\frac{\eta_{i}^{*}\xi_{j}^{*}}{\sqrt{\Lambda_{i}\Lambda_{j}}} + C_{10}(a_{i},a_{j})\frac{\eta_{j}^{*}\xi_{i}^{*}}{\sqrt{\Lambda_{i}\Lambda_{j}}} \\ &+ C_{12}(a_{i},a_{j})\frac{\eta_{i}^{*}\xi_{j}^{*}}{\Lambda_{i}} \\ \end{array}$$

 36 which is

$$\hat{x}_i^1 = a_i \left(\cos \bar{\lambda}_i - \frac{3 - \cos 2\bar{\lambda}_i}{2} \hat{\eta}_i - \frac{\sin 2\bar{\lambda}_i}{2} \hat{\xi}_i \right.$$

$$+ \frac{-3\cos \bar{\lambda}_i + 3\cos 3\bar{\lambda}_i}{8} \hat{\eta}_i^2 - \frac{\sin \bar{\lambda}_i + 3\sin 3\bar{\lambda}_i}{4} \hat{\eta}_i \hat{\xi}_i - \frac{5\cos \bar{\lambda}_i + 3\cos 3\bar{\lambda}_i}{8} \hat{\xi}_i^2 \right)$$

$$+ O(3)$$

$$\begin{aligned} \hat{x}_{i}^{2} &= a_{i} \left(\sin \bar{\lambda}_{i} + \frac{\sin 2\bar{\lambda}_{i}}{2} \hat{\eta}_{i} + \frac{3 + \cos 2\bar{\lambda}_{i}}{2} \hat{\xi}_{i} + \right. \\ &+ \left. \frac{-5 \sin \bar{\lambda}_{i} + 3 \sin 3\bar{\lambda}_{i}}{8} \hat{\eta}_{i}^{2} + \frac{-\cos \bar{\lambda}_{i} + 3 \cos 3\bar{\lambda}_{i}}{4} \hat{\eta}_{i} \hat{\xi}_{i} - \frac{3 \sin \bar{\lambda}_{i} + 3 \sin 3\bar{\lambda}_{i}}{8} \hat{\xi}_{i}^{2} \right) \\ &+ O(3) \quad \text{with} \quad (\hat{\eta}_{i}, \hat{\xi}_{i}) := \left(\frac{\bar{\eta}_{i}}{\sqrt{\Lambda_{i}}}, \frac{\bar{\xi}_{i}}{\sqrt{\Lambda_{i}}} \right) \end{aligned}$$

$$(5.44)$$

where $C_3(a_i, a_j) \div C_{12}(a_i, a_j)$ are defined, in terms of the Laplace coefficient, as

(5.45)

$$\begin{split} C_{3}(a_{i},a_{j}) &:= \frac{(a_{i}/a_{j})^{2}}{32a_{j}} (57(a_{i}/a_{j})^{2} + 117)b_{7/2,0}(a_{i}/a_{j}) \\ &+ \frac{(a_{i}/a_{j})}{64a_{j}} (-12(a_{i}/a_{j})^{4} - 291(a_{i}/a_{j})^{2} - 12)b_{7/2,1}(a_{i}/a_{j})) \\ &+ \frac{(a_{i}/a_{j})^{2}}{32a_{j}} (15(a_{i}/a_{j})^{2} - 45)b_{7/2,2}(a_{i}/a_{j})) + 27\frac{(a_{i}/a_{j})^{3}}{64a_{j}}b_{7/2,3}(a_{i}/a_{j})) \\ C_{4}(a_{i},a_{j}) &:= 277\frac{(a_{i}/a_{j})^{3}}{32a_{j}}b_{7/2,0}(a_{i}/a_{j})) \\ &+ \frac{(a_{i}/a_{j})^{2}}{64a_{j}} (-376 - 376(a_{i}/a_{j})^{2})b_{7/2,1}(a_{i}/a_{j})) \\ &+ \frac{(a_{i}/a_{j})^{2}}{32a_{j}} (16(a_{i}/a_{j})^{4} + 10(a_{i}/a_{j})^{2} + 16)b_{7/2,2}(a_{i}/a_{j}) \\ &+ 56\frac{(a_{i}/a_{j})^{2}}{64a_{j}} (a_{i}/a_{j})^{2} + 1)b_{7/2,3}(a_{i}/a_{j}) + \frac{(a_{i}/a_{j})^{3}}{32a_{j}}b_{7/2,4}(a_{i}/a_{j}) \\ C_{5}(a_{i},a_{j}) &:= \frac{(a_{i}/a_{j})^{2}}{32a_{j}} (117(a_{i}/a_{j})^{2} + 57)b_{7/2,0}(a_{i}/a_{j}) \\ &+ \frac{(a_{i}/a_{j})^{2}}{32a_{j}} (-12(a_{i}/a_{j})^{4} - 291(a_{i}/a_{j})^{2} - 12)b_{7/2,1}(a_{i}/a_{j}) \\ &+ \frac{(a_{i}/a_{j})^{2}}{32a_{j}} (-45(a_{i}/a_{j})^{2} + 15)b_{7/2,2}(a_{i}/a_{j}) + 27\frac{(a_{i}/a_{j})^{3}}{64a_{j}}b_{7/2,3}(a_{i}/a_{j}) \\ &+ \frac{(a_{i}/a_{j})^{2}}{32a_{j}} (-20(a_{i}/a_{j})^{2} + 15)b_{7/2,2}(a_{i}/a_{j}) + 27\frac{(a_{i}/a_{j})^{3}}{64a_{j}}b_{7/2,3}(a_{i}/a_{j}) \\ &+ \frac{(a_{i}/a_{j})^{2}}{32a_{j}} (-79(a_{i}/a_{j})^{2} + 15)b_{7/2,2}(a_{i}/a_{j}) + 27\frac{(a_{i}/a_{j})^{3}}{64a_{j}}b_{7/2,3}(a_{i}/a_{j}) \\ &+ \frac{(a_{i}/a_{j})^{2}}{32a_{j}} (-79(a_{i}/a_{j})^{2} - 19)b_{7/2,2}(a_{i}/a_{j}) \\ &+ \frac{(a_{i}/a_{j})^{2}}{64a_{j}} (-79(a_{i}/a_{j})^{2} - 19)b_{7/2,2}(a_{i}/a_{j}) - 47\frac{(a_{i}/a_{j})^{3}}{64a_{j}}b_{7/2,3}(a_{i}/a_{j}) \\ &+ \frac{(a_{i}/a_{j})^{2}}{32a_{j}} (16(a_{i}/a_{j})^{4} + 118(a_{i}/a_{j})^{2} + 16)b_{7/2,2}(a_{i}/a_{j}) \\ &+ \frac{(a_{i}/a_{j})^{2}}{64a_{j}} (16(a_{i}/a_{j})^{4} + 118(a_{i}/a_{j})^{2} + 16)b_{7/2,2}(a_{i}/a_{j}) \\ &+ 56\frac{(a_{i}/a_{j})^{2}}{64a_{j}} (a_{i}/a_{j})^{2} + 10b_{7/2,3}(a_{i}/a_{j}) + \frac{(a_{i}/a_{j})^{3}}{32a_{j}}b_{7/2,4}(a_{i}/a_{j}) \\ &+ 56\frac{(a_{i}/a_{j})^{2}}{64a_{j}} (a_{i}/a_{j})^{2} + 10b_{7/2,3}($$

$$+ \frac{(a_i/a_j)^2}{16a_j} (-47 + 13(a_i/a_j)^2) b_{7/2,2}(a_i/a_j) - \frac{37}{32a_j} (a_i/a_j)^3 b_{7/2,3}(a_i/a_j)$$
(5.46)

(iii) Expansion of \bar{f}_{four} . The function

$$\bar{f}_{\text{four}} = -\sum_{1 \le i < j \le N} \frac{\bar{m}_i \bar{m}_j}{4\pi^2} \int_{\mathbb{T}^2} \left(\frac{\hat{x}_i^* \cdot \hat{\mathfrak{S}}_{ij}^4 \hat{x}_j}{|\hat{x}_i^* - \hat{x}_j|^3} + \frac{3}{2} \frac{(\hat{x}_i^* \cdot \hat{\mathfrak{S}}_{ij}^2 \hat{x}_j)^2}{|\hat{x}_i^* - \hat{x}_j|^5} \right) d\bar{\lambda}_i d\bar{\lambda}_j$$
(5.47)

is of order 4 in $\bar{z} = (\bar{\eta}, \bar{\xi}, \bar{p}, \bar{q})$, so, it sufficies replace \hat{x}_i with its 0-order approximation (5.40) to find

$$\bar{f}_{\text{four}} = \bar{f}_{\text{four}}|_4 + \mathcal{O}(6)$$

As in the previous step, we represent $\hat{\mathfrak{S}}_{ij}^2$, $\hat{\mathfrak{S}}_{ij}^4$ through their entries: $\hat{\mathfrak{S}}_{ij}^2$ as in eq. (5.38) and $\hat{\mathfrak{S}}_{ij}^4$ as

$$\hat{\mathfrak{S}}_{ij}^4 = \left(\begin{array}{cc} \mathfrak{Q}_{11}^{ij} & \mathfrak{Q}_{12}^{ij} \\ \mathfrak{Q}_{21}^{ij} & \mathfrak{Q}_{22}^{ij} \end{array} \right)$$

Using (5.40), into (5.47), we find

$$\begin{aligned} f_{\text{four}}|_{4} &= f_{\text{four}}|_{\hat{x}=(5.40)} \\ &= -\sum_{1 \leq i < j \leq N} \bar{m}_{i} \bar{m}_{j} (\mathfrak{Q}_{11}^{ij} + \mathfrak{Q}_{22}^{ij}) \frac{a_{i}}{2a_{j}^{2}} b_{3/2,1}(a_{i}/a_{j}) \\ &- \frac{3}{8} \sum_{1 \leq i < j \leq N} \bar{m}_{i} \bar{m}_{j} ((\mathfrak{q}_{11}^{ij})^{2} + (\mathfrak{q}_{12}^{ij})^{2} + (\mathfrak{q}_{21}^{ij})^{2} + (\mathfrak{q}_{22}^{ij})^{2}) \frac{a_{i}^{2}}{a_{j}^{3}} b_{5/2,0}(a_{i}/a_{j}) \\ &- \frac{3}{16} \sum_{1 \leq i < j \leq N} \bar{m}_{i} \bar{m}_{j} ((\mathfrak{q}_{11}^{ij} + \mathfrak{q}_{22}^{ij})^{2} + (\mathfrak{q}_{12}^{ij} - \mathfrak{q}_{21}^{ij})^{2}) \frac{a_{i}^{2}}{a_{j}^{3}} b_{5/2,2}(a_{i}/a_{j}) \end{aligned}$$
(5.48)

(iv) **Computation of** $\hat{\mathfrak{S}}_{ij}^2$. The matrices \mathfrak{S}_i are defined through equation (5.33) as suitable products of the matrices $\mathcal{S}_i^{\mathrm{pr}}$'s, $\tilde{\mathcal{S}}_j^{\mathrm{pr}}$'s, which (recall Proposition 4.3) are easily expanded up to order 2 as

$$\begin{cases} \mathcal{S}_{1}^{\text{prT}} = \Sigma_{c_{1}}(\bar{p}_{1}, \bar{q}_{1}) \\ \mathcal{S}_{i}^{\text{pr}} = \Sigma_{c_{i}}(\bar{p}_{i-1}, \bar{q}_{i-1}) & (2 \le i \le N) \\ \mathcal{\tilde{S}_{i}^{\text{prT}}} = \Sigma_{\tilde{c}_{i}}(\bar{p}_{i}, \bar{q}_{i}) & (2 \le i \le N-1) \end{cases}$$
(5.49)

where c_i 's, \tilde{c}_j 's are the constants (5.5) which define the entries the matrix associated to the operator \mathfrak{L} and $\Sigma_{\hat{c}}(\hat{p}, \hat{q})$ denotes

$$\Sigma_{\hat{c}}(\hat{p},\hat{q}) = \begin{pmatrix} 1 - \frac{1}{2}\hat{c}^{2}\hat{q}^{2} & -\frac{1}{2}\hat{c}^{2}\hat{p}\hat{q} & -\hat{c}\hat{q} \\ -\frac{1}{2}\hat{c}^{2}\hat{p}\hat{q} & 1 - \frac{1}{2}\hat{c}^{2}\hat{p}^{2} & -\hat{c}\hat{p} \\ \hat{c}\hat{q} & \hat{c}\hat{p} & 1 - \frac{1}{2}\hat{c}^{2}(\hat{p}^{2} + \hat{q}^{2}) \end{pmatrix}$$
(5.50)

Taking then the products of $\mathcal{S}_i^{\text{pr}}, \cdots$ as prescribed in (5.33), we have

$$\mathfrak{S}_{i}^{\mathrm{pr}} = \Sigma_{\ell_{i,N-1}}(\bar{p}_{N-1}, \bar{q}_{N-1}) \cdots \Sigma_{\ell_{i,1}}(\bar{p}_{1}, \bar{q}_{1}) + \mathcal{O}(3)$$

and hence,

$$\mathfrak{S}_{ij} = \mathfrak{S}_{i}^{T} \mathfrak{S}_{j} = \Sigma_{-\ell_{i1}}(\bar{p}_{1}, \bar{q}_{1}) \cdots \Sigma_{-\ell_{i,N-1}}(\bar{p}_{N-1}, \bar{q}_{N-1}) \\
\times \Sigma_{\ell_{j,N-1}}(\bar{p}_{N-1}, \bar{q}_{N-1}) \cdots \Sigma_{\ell_{j1}}(\bar{p}_{1}, \bar{q}_{1})$$
(5.51)

where

$$\ell_i = (\ell_{i1}, \cdots, \ell_{i,N-1})$$

is the i^{th} row of the matrix ℓ associated to the operator \mathfrak{L} (eq. 5.4)÷(5.5)) Using (5.33), we can write

$$\mathfrak{S}_{ij} = \mathfrak{S}_i^T \mathfrak{S}_j = \begin{cases} \mathcal{S}_1^{\mathrm{pr}} \quad i = 1, \ j = 2\\ \mathcal{S}_1^{\mathrm{pr}} \bar{\mathcal{S}}_2^{\mathrm{pr}} \quad i = 1, \ j = 3\\ \mathcal{S}_1^{\mathrm{pr}} \tilde{\mathcal{S}}_2^{\mathrm{pr}} \cdots \tilde{\mathcal{S}}_{j-2}^{\mathrm{pr}} \bar{\mathcal{S}}_{j-1}^{\mathrm{pr}} \quad i = 1, \ j \ge 4\\ \mathcal{S}_i^{\mathrm{pr}}^T \bar{\mathcal{S}}_i^{\mathrm{pr}} \quad 2 \le i = j - 1\\ \mathcal{S}_i^{\mathrm{pr}}^T \tilde{\mathcal{S}}_i^{\mathrm{pr}} \cdots \tilde{\mathcal{S}}_{j-2}^{\mathrm{pr}} \bar{\mathcal{S}}_{j-1}^{\mathrm{pr}} \quad 2 \le i \le j - 2 \end{cases}$$
(5.52)

having let

$$\bar{\mathcal{S}}_i^{\mathrm{pr}} := \tilde{\mathcal{S}}_i^{\mathrm{pr}} \mathcal{S}_{i+1}^{\mathrm{pr}} = \Sigma_{\bar{c}_i}(p_i, q_i) + \mathcal{O}(3)$$
(5.53)

with

$$\bar{c}_i = c_{i+1} - \tilde{c}_i \qquad 1 \le i \le N - 1 \quad (\tilde{\mathcal{S}}_1 := \mathcal{S}_1 \ , \quad \tilde{c}_1 := c_1)$$

which, together with (5.49), implies (5.54)÷(5.55). Hence, we may also write the matrices $\mathfrak{S}_{ij} = \mathfrak{S}_i^T \mathfrak{S}_j$ as

$$\mathfrak{S}_{ij} = \prod_{1 \le k \le j-1} \Sigma_{\ell_k^{ij}}(\bar{p}_k, \bar{q}_k) + \mathcal{O}(3)$$
(5.54)

where

$$\ell_k^{ij} = \ell_{jk} - \ell_{ik} \tag{5.55}$$

and then we use the following elementary Lemma.

Lemma 5.6 Let

$$c = (c_1, \dots, c_m)$$
, $p = (p_1, \dots, p_m)$, $q = (q_1, \dots, q_m) \in \mathbb{R}^m$,

let $\Sigma_{\hat{c}}(\hat{p},\hat{q})$ as in (5.50) and put

$$\Pi_c(p,q) := \Sigma_{c_1}(p_1,q_1) \cdots \Sigma_{c_m}(p_m,q_m) .$$

Then, the submatrix $\hat{\Pi}_c(p,q)$ of order 2 of $\Pi_c(p,q)$ is

$$\hat{\Pi}_c(p,q) = \operatorname{id} - \frac{1}{2} \begin{pmatrix} (c \cdot q)^2 & (c \cdot p)(c \cdot q) - \Delta_c(p,q) \\ (c \cdot p)(c \cdot q) + \Delta_c(p,q) & (c \cdot p)^2 \end{pmatrix} + O(4)$$

where $\Delta_c(p,q)$ denotes

$$\Delta_c(p,q) := \sum_{1 \le i < j \le m} c_i c_j (p_i q_j - p_j q_i) \quad (when \quad m \ge 2) \; .$$

In view of the previous Lemma and making use of $(5.51) \div (5.54)$, we find the following expressions for the entries $(\mathfrak{q}_{hk}^{ij})_{hk}$ of \mathfrak{S}_{ij}^2

$$\begin{cases}
\mathfrak{q}_{11}^{ij} = -\frac{1}{2} (\mathfrak{L}\bar{q}_j - \mathfrak{L}\bar{q}_i)^2 \\
\mathfrak{q}_{12}^{ij} = -\frac{1}{2} (\mathfrak{L}\bar{q}_j - \mathfrak{L}\bar{q}_i) (\mathfrak{L}\bar{p}_j - \mathfrak{L}\bar{p}_i) + \frac{1}{2} \sum_{1 \le h < k \le j-1} \ell_h^{ij} \ell_k^{ij} (\bar{p}_h \bar{q}_k - \bar{p}_k \bar{q}_h) \\
\mathfrak{q}_{21}^{ij} = -\frac{1}{2} (\mathfrak{L}\bar{q}_j - \mathfrak{L}\bar{q}_i) (\mathfrak{L}\bar{p}_j - \mathfrak{L}\bar{p}_i) - \frac{1}{2} \sum_{1 \le h < k \le j-1} \ell_h^{ij} \ell_k^{ij} (\bar{p}_h \bar{q}_k - \bar{p}_k \bar{q}_h) \\
\mathfrak{q}_{22}^{ij} = -\frac{1}{2} (\mathfrak{L}\bar{p}_j - \mathfrak{L}\bar{p}_i)^2.
\end{cases}$$
(5.56)

(v) **Computation of** $\hat{\mathfrak{S}}_{ij}^4$. Equation (5.52) gives the expression of \mathfrak{S}_{ij} in terms of the matrices $\mathcal{S}_i^{\mathrm{pr}}$, $\tilde{\mathcal{S}}_j^{\mathrm{pr}}$ (eq. (4.48)) and $\bar{\mathcal{S}}_k^{\mathrm{pr}} = \tilde{\mathcal{S}}_k^{\mathrm{pr}} \mathcal{S}_{k+1}^{\mathrm{pr}}$. Hence, the expression of \mathfrak{S}_{ij}^4 is uniquely determined by the expansion of these matrices up to order 4.

Let us first notice that we can write the matrices $\bar{\mathcal{S}}_k^{\text{pr}}$ in the same form as $\mathcal{S}_i^{\text{pr}}$, $\tilde{\mathcal{S}}_j^{\text{pr}}$. In fact, recalling the definitions (eq. (4.35)) of $\tilde{\mathcal{S}}_k^{\text{pr}}$, $\mathcal{S}_{k+1}^{\text{pr}}$, we find, for $1 \leq k \leq N-1$,

$$\bar{\mathcal{S}}_{k}^{\mathrm{pr}} = \tilde{\mathcal{S}}_{k}^{\mathrm{pr}} \mathcal{S}_{k+1}^{\mathrm{pr}} = \mathcal{R}_{\mathrm{z}}(\kappa_{k}) \mathcal{R}_{\mathrm{x}}(\bar{i}_{k}) \mathcal{R}_{\mathrm{z}}(-\kappa_{k}) \qquad \kappa_{k} := \arg\left(p_{k}, -q_{k}\right) \,, \tag{5.57}$$

where $\bar{i}_k := \tilde{i}_k + i_{k+1}$ has the meaning of the outern angle ³⁷ of Ψ_{k-1} , Γ_{k+1} in the triangle with sides with lenghth Ψ_{k-1} , $\Gamma_{k+1} \Psi_k$, hence,

$$\cos \bar{i}_k = \frac{\Psi_k^2 - \Psi_{k-1}^2 - \Gamma_{k+1}^2}{2\Psi_{k-1}\Gamma_{k+1}} \qquad \bar{i}_k \in (0,\pi)$$
(5.58)

Equations $(5.57) \div (5.58)$ easily imply

$$\bar{\mathcal{S}}_{k}^{\mathrm{pr}} = \begin{pmatrix} 1 - \bar{q}_{k}^{2} \bar{\mathfrak{c}}_{k} & -\bar{p}_{k} \bar{q}_{k} \bar{\mathfrak{c}}_{k} & -\bar{q}_{k} \bar{\mathfrak{s}}_{k} \\ -\bar{p}_{k} \bar{q}_{k} \bar{\mathfrak{c}}_{k} & 1 - \bar{p}_{k}^{2} \bar{\mathfrak{c}}_{k} & -\bar{p}_{k} \bar{\mathfrak{s}}_{k} \\ \bar{q}_{k} \bar{\mathfrak{s}}_{k} & \bar{p}_{k} \bar{\mathfrak{s}}_{k} & 1 - (\bar{p}_{k}^{2} + \bar{q}_{k}^{2}) \bar{\mathfrak{c}}_{k} \end{pmatrix} \qquad 1 \le k \le N - 2$$

$$(5.59)$$

³⁷In fact, by the definition 4.12 of \tilde{i}_k , i_{k+1} given in Proposition 4.1 it is clear that \tilde{i}_k , i_{k+1} have the meaning, respectively, of the inner angles corresponding to the couples of sides with lenghth (Ψ_{k-1}, Ψ_k) and $(\Psi_{k-1}, \Gamma_{k+1})$.

where

$$\bar{\mathfrak{c}}_{k} = \frac{1 - \cos\bar{i}_{k}}{\bar{p}_{k}^{2} + \bar{q}_{k}^{2}} = \frac{\Psi_{k-1} + \Gamma_{k+1} + \Psi_{k}}{4\Psi_{k-1}\Gamma_{k+1}} , \quad \bar{\mathfrak{s}}_{k} = \frac{\sin\bar{i}_{k}}{\sqrt{\bar{p}_{k}^{2} + \bar{q}_{k}^{2}}} = \sqrt{\bar{\mathfrak{c}}_{k} \left(2 - (\bar{p}_{k}^{2} + \bar{q}_{k}^{2})\bar{\mathfrak{c}}_{k}\right)}$$
(5.60)

We are now ready for the expansions of $\mathcal{S}_i^{\mathrm{pr}}, \, \tilde{\mathcal{S}}_j^{\mathrm{pr}}, \, \bar{\mathcal{S}}_k^{\mathrm{pr}}$.

Let us observe that the functions $\mathfrak{c}_i, \mathfrak{s}_j, \overline{\mathfrak{s}}_k, \widetilde{\mathfrak{c}}_j, \overline{\mathfrak{s}}_k, \overline{\mathfrak{s}}_k$ (eq. (4.37)÷(5.60)) are even in \overline{z} , so, the entries of \mathcal{S}_i, \dots , with places (1, 1), (1, 2), (2, 1), (2, 2), (3, 3) are even (as functions of \overline{z}) and the ones with places (1, 3), (2, 3), (3, 1), (3, 2) are odd. Then, in order to obtain expansions of $\mathcal{S}_i^{\mathrm{pr}}, \widetilde{\mathcal{S}}_j^{\mathrm{pr}}, \overline{\mathcal{S}}_k^{\mathrm{pr}}$ up to order 4, it sufficies to expand the functions $\mathfrak{c}_i, \mathfrak{s}_i, \dots$ up to order 2. Making this operation leads to the fllowing expansions (which generalizes (5.49)÷(5.50)). If $\Sigma_{\hat{c},\hat{C},\hat{S}}(\hat{p},\hat{q})$ denotes the matrix

$$\Sigma_{\hat{c},\hat{C},\hat{S}}(\hat{p},\hat{q}) = \begin{pmatrix} 1 - \hat{q}^2 \left(\frac{\hat{c}^2}{2} + \hat{C}\right) & -\hat{p}\hat{q} \left(\frac{\hat{c}^2}{2} + \hat{C}\right) & -\hat{q} \left(\hat{c} + \hat{S}\right) \\ -\hat{p}\hat{q} \left(\frac{\hat{c}^2}{2} + \hat{C}\right) & 1 - \hat{p}^2 \left(\frac{\hat{c}^2}{2} + \hat{C}\right) & -\hat{p} \left(\hat{c} + \hat{S}\right) \\ \hat{q} \left(\hat{c} + \hat{S}\right) & \hat{p} \left(\hat{c} + \hat{S}\right) & 1 - (\hat{p}^2 + \hat{q}^2) \left(\frac{\hat{c}^2}{2} + \hat{C}\right) \end{pmatrix} \end{pmatrix} (5.61)$$

then,

$$\begin{cases} \mathcal{S}_{1}^{\text{prT}} = \Sigma_{c_{1}, \mathcal{C}_{1}, S_{1}}(p_{1}, q_{1}) \\ \mathcal{S}_{i}^{\text{pr}} = \Sigma_{c_{i}, \mathcal{C}_{i}, S_{i}}(p_{i-1}, q_{i-1}) & (2 \le i \le N) \\ \tilde{\mathcal{S}}_{i}^{\text{prT}} = \Sigma_{\tilde{c}_{i}, \tilde{\mathcal{C}}_{i}, \tilde{S}_{i}}(p_{i}, q_{i}) & (2 \le i \le N - 1) \\ \bar{\mathcal{S}}_{i}^{\text{pr}} = \Sigma_{\tilde{c}_{i}, \tilde{\mathcal{C}}_{i}, \tilde{S}_{i}}(p_{i}, q_{i}) & (1 \le i \le N - 1) \end{cases}$$

$$(5.62)$$

where, for $2 \le i \le N$, $2 \le j \le N - 1$, $1 \le k \le N - 1$,

$$\begin{cases} c_{1} := -\sqrt{\frac{\Lambda_{2}}{\Lambda_{1}L_{2}}} \\ C_{1} := \frac{2\Lambda_{2}(2\Lambda_{1} + \Lambda_{2})\tau_{1} - 2\Lambda_{1}^{2}\tau_{2} + \Lambda_{1}(\Lambda_{2} - \Lambda_{1})\rho_{1}}{4\Lambda_{1}^{2}L_{2}^{2}} \\ S_{1} := \frac{1}{c_{1}}\left(\mathcal{C}_{1} - \frac{c_{1}^{4}}{4}\rho_{1}\right) \\ \\ c_{i} := \sqrt{\frac{L_{i-1}}{L_{i}\Lambda_{i}}} \\ C_{i} := \frac{2L_{i-1}(L_{i-1} + 2\Lambda_{i})\tau_{i} + \Lambda_{i}(L_{i-1} - \Lambda_{i})\rho_{i-1} - 2\Lambda_{i}^{2}(T_{i-1} + R_{i-2})}{4L_{i}^{2}\Lambda_{i}^{2}} \\ S_{i} := \frac{1}{c_{i}}\left(\mathcal{C}_{i} - \frac{c_{i}^{4}}{4}\rho_{i-1}\right) \end{cases}$$

$$\begin{cases} \tilde{c}_{j} \coloneqq -\sqrt{\frac{\Lambda_{j+1}}{L_{j+1}L_{j}}} \\ \tilde{C}_{j} \coloneqq \frac{-2L_{j}^{2}\tau_{j+1} + L_{j}(\Lambda_{j+1} - L_{j})\rho_{j} + 2\Lambda_{j+1}(2L_{j} + \Lambda_{j+1})(T_{j} + R_{j-1})}{4L_{j}^{2}L_{j+1}^{2}} \\ \tilde{C}_{j} \coloneqq \frac{1}{\tilde{c}_{j}} \left(\tilde{C}_{j} - \frac{c_{j}^{4}}{4}\rho_{j}\right) \\ \\ \bar{c}_{k} \coloneqq c_{k+1} - \tilde{c}_{k} = \sqrt{\frac{L_{k+1}}{L_{k}\Lambda_{k+1}}} \\ \bar{c}_{k} \coloneqq \frac{2\Lambda_{k+1}^{2}(T_{k} + R_{k-1}) + 2L_{k}^{2}\tau_{k+1} - L_{k}\Lambda_{k+1}\rho_{k}}{4L_{k}^{2}\Lambda_{k+1}^{2}} \\ \bar{c}_{k} \coloneqq \frac{1}{\bar{c}_{k}} \left(\bar{C}_{k} - \frac{\bar{c}_{k}^{4}}{4}\rho_{k}\right) \end{cases}$$

$$(5.63)$$

having let, for shortness,

$$\tau_i := \frac{\bar{\eta}_i^2 + \bar{\xi}_i^2}{2} , \quad \rho_i := \frac{\bar{p}_i^2 + \bar{q}_i^2}{2} , \quad \mathbf{T}_i := \sum_{1 \le j \le i} \tau_j , \quad \mathbf{R}_i := \sum_{1 \le j \le i} \rho_j , \quad \mathbf{L}_i := \sum_{1 \le j \le i} \Lambda_j$$
 with

with

$$\mathbf{R}_0 := 0 \tag{5.64}$$

(vi) Conclusion of the computation. In view of the expressions (5.56) for the diagonal terms \mathfrak{q}_{11}^{ij} , \mathfrak{q}_{22}^{ij} of $\hat{\mathfrak{S}}_{ij}^2$, the function $\bar{f}_{\text{two}}|_2$ (eq. (5.41)) becomes

$$f_{\text{two}}|_{2} = \frac{1}{2} \sum_{1 \le i < j \le N} \bar{m}_{i} \bar{m}_{j} \Big((\mathfrak{L}\bar{q}_{j} - \mathfrak{L}\bar{q}_{i})^{2} + (\mathfrak{L}\bar{p}_{j} - \mathfrak{L}\bar{p}_{i})^{2} \Big) \frac{a_{i}}{2a_{j}^{2}} b_{3/2,1}(a_{i}/a_{j})$$

=: $\mathcal{Q}_{v}^{*} \cdot \frac{\bar{p}^{2} + \bar{q}^{2}}{2}$

Similarly, replacing diagonal and off-diagonal entries \mathfrak{q}_{hk}^{ij} as in (5.56), the function $f_{\text{two}}|_4(\text{eq.}(5.43))$ becomes

$$\begin{aligned} f_{\text{two}}|_{4} &= \frac{1}{2} \sum_{1 \leq i < j \leq N} \bar{m}_{i} \bar{m}_{j} (\mathcal{Q}_{ij}^{11} (\mathfrak{L}\bar{q}_{j} - \mathfrak{L}\bar{q}_{i})^{2} + \mathcal{Q}_{ij}^{22} (\mathfrak{L}\bar{p}_{j} - \mathfrak{L}\bar{p}_{i})^{2} \\ &+ (\mathcal{Q}_{ij}^{12} + \mathcal{Q}_{ij}^{21}) \Big(\mathfrak{L}\bar{p}_{j} - \mathfrak{L}\bar{p}_{i} \Big) \Big(\mathfrak{L}\bar{q}_{j} - \mathfrak{L}\bar{q}_{i} \Big) \\ &+ (\mathcal{Q}_{ij}^{21} - \mathcal{Q}_{ij}^{12}) \sum_{1 \leq h < k \leq j-1} \ell_{h}^{ij} \ell_{k}^{ij} (\bar{p}_{h} \bar{q}_{k} - \bar{p}_{k} \bar{q}_{h}) \\ &=: \mathfrak{F}_{hv}(\eta_{*}, \xi_{*}, \bar{p}, \bar{q}) \end{aligned}$$

and the function $\bar{f}_{\text{four}}|_4$ (eq. (5.48) becomes

$$\bar{f}_{\text{four}}|_4 = \sum_{1 \le i < j \le N} \bar{m}_i \bar{m}_j (\mathfrak{Q}_{11}^{ij} + \mathfrak{Q}_{22}^{ij}) C_1(a_i, a_j)$$

$$+ \sum_{1 \le i < j \le N} \bar{m}_i \bar{m}_j \left(\left(\mathfrak{L}\bar{p}_i - \mathfrak{L}\bar{p}_j \right)^2 + \left(\mathfrak{L}\bar{q}_i - \mathfrak{L}\bar{q}_j \right)^2 \right)^2 C_{13}(a_i, a_j)$$

$$+ \sum_{1 \le i < j \le N} \bar{m}_i \bar{m}_j \left(\sum_{1 \le h < k \le j - 1} \ell_h^{ij} \ell_k^{ij} (\bar{p}_h \bar{q}_k - \bar{p}_k \bar{q}_h) \right)^2 C_{14}(a_i, a_j)$$

$$=: \mathfrak{F}_v(\eta_*, \xi_*, \bar{p}, \bar{q})$$

where

$$C_{13}(a_i, a_j) := -\frac{3}{64} \frac{a_i^2}{a_j^3} (2b_{5/2,0}(a_i/a_j) + b_{5/2,2}(a_i/a_j))$$

$$C_{14}(a_i, a_j) := -\frac{3}{16} \frac{a_i^2}{a_j^3} (b_{5/2,0}(a_i/a_j) + b_{5/2,2}(a_i/a_j))$$
(5.65)

Collecting then the expansions

$$\bar{f}_{pl}^{*} = C_{0} + \mathcal{Q}_{h} \cdot \frac{\eta_{*}^{2} + \xi_{*}^{2}}{2} + \mathfrak{F}_{h}(\eta_{*}, \xi_{*}) + O(6) \quad \eta_{*} = (-\bar{\eta}_{1}, \bar{\eta}_{2}, \cdots)$$

$$\bar{f}_{two} = \mathcal{Q}_{v}^{*} \cdot \frac{\bar{p}^{2} + \bar{q}^{2}}{2} + \mathfrak{F}_{hv}(\eta_{*}, \xi_{*}, \bar{p}, \bar{q}) + O(6)$$

$$\bar{f}_{four} = \mathfrak{F}_{v}(\eta_{*}, \xi_{*}, \bar{p}, \bar{q}) + O(6) .$$

we finally find

$$\bar{f}_{\text{plt,pr}} = \bar{f}_{\text{pl}}^* + \bar{f}_{\text{two}} + \bar{f}_{\text{four}} \\
= C_0 + \mathcal{Q}_h \cdot \frac{\eta_*^2 + \xi_*^2}{2} + \mathcal{Q}_v^* \cdot \frac{\bar{p}^2 + \bar{q}^2}{2} \\
+ \mathfrak{F}_h(\eta_*, \xi_*) + \mathfrak{F}_{hv}(\eta_*, \xi_*, \bar{p}, \bar{q}) + \mathfrak{F}_v((\eta_*, \xi_*, \bar{p}, \bar{q}) + O(6))$$

5.2 (3N-1)-Dimensional KAM Tori and Measure of the Kolmogorov's Set

Having checked, for $N \geq 3$, the assumptions of non resonance up to order 4 for the first Birkhoff invariants and non degeneracy for the second Birkhoff invariants, invoking Theorem 2.1, we can state the following result concerning existence of KAM tori of dimension 3N - 1 for the planetary (1 + N) body problem and measure estimates of the invariant set.

Theorem 5.1 Consider, in \mathbb{R}^3 , a star with mass \overline{m}_0 and $N \geq 3$ planets with masses $\mu \overline{m}_1, \dots, \mu \overline{m}_N$, interacting only through gravity. Let a_i denote the instantaneous major semiaxis of the *i*th planet. Then, there exists $\delta_* > 0$, $\varepsilon_* > 0$, $\mu_* > 0$, b > 0, c > 0, C > 0 such that, if $0 < a_i/a_{i+1} < \delta_*$, $0 < \varepsilon < \varepsilon_*$ and $0 < \mu < \mu_*$ and

$$\mu < c(\log \varepsilon^{-1})^{-2b} ,$$

there exists a positive Lebesgue measure set \mathcal{K} such that

- (i) \mathcal{K} ("Kolmogorov set") is formed by the union of invariant tori of dimension 3N-1 on which the \mathcal{H}_{plt} -flow is linear in time, with Diophantine frequency;
- (ii) the measure of \mathcal{K} satisfies

$$c \varepsilon^{2(2N-1)} > \operatorname{meas} \mathcal{K} > c \left(1 - C(\sqrt{\varepsilon} + \sqrt{\mu}(\log \varepsilon^{-1})^b) \right) \varepsilon^{2(2N-1)}$$
.

Furthermore, the eccentricities and the mutual inclinations on the invariant tori are bounded by $c(\log \varepsilon^{-1})^{-1}$.

6 Kolmogorov's Set in the Space Planetary Problem II (Total Reduction)

The proofs of existence of quasi-periodic motions for the planetary problem presented in [17] and [14] are based on the application of a (C^{∞} , analytic, respectively) KAM theory based on "weak" non-degeneracy conditions, for a given properly degenerate system, nearly an elliptic equilibrium point.

For istance, the proof in [14], in the real-analytic framework, is obtained as an application of Theorem 6.1 below, based, on turn, on 2001 Rüssmann Theory [34] (compare [14], **Theorem 4**), where the following weak non-degeneracy condition is required

Definition 6.1 (Rüssmann nondegeneracy condition) A real-analytic function

$$\omega: \quad y \in \mathcal{B} \subset \mathbb{R}^n \to \omega(y) = (\omega_1(y), \cdots, \omega_m(y)) \in \mathbb{R}^m$$

is called **R**-non degenerate if \mathcal{B} is a non-empty open connected set in \mathbb{R}^n and if for any $c \in \mathbb{R}^m \setminus \{0\}$, the map

$$y \to c \cdot \omega(y) \neq 0 \tag{6.1}$$

Theorem 6.1 Consider a Hamiltonian function

$$\mathcal{H}_{\mu} = h(I) + \mu f(I, \varphi, p, q)$$

which assume to be real-analytic for

$$(I, \varphi, p, q) \in \mathcal{I} \times \mathbb{T}^{\bar{n}} \times B_r^{2\hat{n}}(0) := \mathcal{M}$$

with the mean perturbation $\bar{f} := (2\pi)^{-\bar{n}} \int_{\mathbb{T}^{\bar{n}}} f d\varphi$ of the form

$$\bar{f} = \bar{f}_0(I) + \Omega(I) \cdot J + O(|J|^{3/2}) \quad J = \left(\cdots, \frac{p_i^2 + q_i^2}{2}, \cdots\right) \ . \tag{6.2}$$

Assume also that the "frequency map"

$$I \in \mathcal{I} \to (\partial h(I), \Omega(I)) \in \mathbb{R}^{\bar{n}} \times \mathbb{R}^{\hat{n}}$$

is R-non degenerate. Then, if μ is sufficiently small, there exists a positive measure set of phase space points belonging to real-analytic \mathcal{H}_{μ} -invariant tori which are close to $\mathbb{T}^{\bar{n}} \times \{I_0\} \times_j \{p_j^2 + q_j^2 = \rho_j\}$, with $\rho_j = O(\mu)$. Furtherore, the \mathcal{H}_{μ} -flow on such tori is quasi-periodic with Diophantine frequencies.

Following [17] (who deals with Herman's C^{∞} KAM Theory), the strategy of the proof in [14] consists in applying the previous KAM Theory to a suitably modified Hamiltonian

function, which is obtained from the planetary Hamiltonian expressed in Delaunay– Poincaré variables by adding a commuting Hamiltonian. The necessity of modifying the Hamiltonian function is that when the planetary Hamiltonian is put in Poincaré– Delaunay variables, the frequencies Ω of (6.2) correspond the 2N-dimensional vector of the DP-secular frequencies

$$(\sigma,\zeta) = ((\sigma_1,\cdots,\sigma_N),(\zeta_1,\cdots,\zeta_N))$$

defined, respectively, as the eigenvalues of the two "horizontal" and "vertical" quadratic forms

$$\begin{pmatrix}
\mathcal{Q}_{h} \cdot \eta^{2} := \sum_{1 \leq j < k \leq N} m_{j} m_{k} \left(C_{1}(a_{j}, a_{k}) \left(\frac{\eta_{j}^{2}}{\Lambda_{j}} + \frac{\eta_{k}^{2}}{\Lambda_{k}} \right) + 2C_{2}(a_{j}, a_{k}) \frac{\eta_{j} \eta_{k}}{\sqrt{\Lambda_{j} \Lambda_{k}}} \right) \\
\mathcal{Q}_{v} \cdot p^{2} := -\sum_{1 \leq j < k \leq N} m_{j} m_{k} C_{1}(a_{j}, a_{k}) \left(\frac{p_{j}}{\Lambda_{j}} - \frac{p_{k}}{\Lambda_{k}} \right)^{2}$$
(6.3)

which are been proved [17] to satisfy, together with the mean motions $n := \partial h_{plt}$ the only two independent linear combinations, usually called secular resonances

$$\sum_{1 \le i \le N} (\sigma_i + \zeta_i) = 0 , \quad \zeta_N = 0$$
(6.4)

and the Rüssmann condition (6.1) is clearly violated. Adding a commuting Hamiltonian makes the above non degeneracy condition (6.1) verified. The final result is reached with the use of an abstract argument: invariant ergodic tori for the modified Hamiltonian are recognized to be invariant manifolds for the original Hamiltonian.

The use of the regularized (fully) reduced Deprit variables provides a direct application of the KAM machinery of [14] because no secular resonance appears. We recover then a result already found with a different technique in the 2007 revised version of the paper by J. Féjoz [17].

Theorem 6.2 Consider, in \mathbb{R}^3 a star of mass \bar{m}_0 and $N \ge 2$ planets of mass $\mu \bar{m}_1, \dots, \mu \bar{m}_N$, interacting only through gravity. Let a_i denote the instantaneous major semiaxis of the *i*th planet and let ε be an upper bound of the instantaneous eccentricity and inclination of the planets. Then, there exists $\delta_* > 0$, $\varepsilon_* > 0$ and $\mu_* > 0$ such that, if $a_i/a_{i+1} < \delta_*$, $0 < \varepsilon < \varepsilon_*$ and $0 < \mu < \mu_*$, there exists a positive measure set of phase space points whose time evolution lies on real-analytic, 3N - 2 dimensional invariant tori; the time evolution being quasi-periodic with 3N - 2 Diophantine frequencies. Furthermore, during the motion, eccentricities and inclinations are bounded by $C_{\sqrt{\mu}}$.

Proof. It is a corollary of Theorem 6.1 above and Lemma 6.1 of the following section.

6.1 Rüssmann Non–Degeneracy and (3N-2)–Dimensional KAM Tori

Remarks on notations. Referring especially to paragraphs 4.2, 4.3, throughout all this section,

- (i) We disregard the (cyclic, for \mathcal{H}_{plt}) Deprit variables (P, Q), on which we will always think to lift the maps we will discuss, extending them through the identity map. Quite abusively, we do not change the name of the domains \mathcal{D}_{r} , \mathcal{D}_{pr} of the fully, partially reduced regularized Deprit variables.
- (ii) We denote the set of fully reduced regularized Deprit variables as

$$(\Lambda,\lambda)$$
, (η,ξ) , (p,q) , (G,g)

 $(g \text{ cyclic for } \mathcal{H}_{\text{plt}})$, hence, in particular, p, q have dimension N - 2. The planetary Hamiltonian put in fully reduced regularized Deprit variables is denoted

$$\mathcal{H}_{\text{plt,fr}} = h_{\text{plt}} + \mu f_{\text{plt,fr}}$$

where, as usual

$$h_{\rm plt} = -\sum_{1 \le i \le N} \frac{\tilde{m}_i^3 \hat{m}_i^2}{2\Lambda_i^2}$$

(iii) The set of partially reduced regularized Deprit variables with

$$(\Lambda,\lambda)\;,\quad (ar\eta,\xi)\;,\quad (ar p,ar q)$$

hence, with with \bar{p} , \bar{q} of dimension (N-1). The planetary Hamiltonian put in partially reduced regularized Deprit variables is denoted

$$\mathcal{H}_{\rm plt,pr} = h_{\rm plt} + \mu f_{\rm plt,pr}$$

We start with the planetary Hamiltonian written in fully reduced variables

$$\mathcal{H}_{\text{plt,fr}} = h_{\text{plt}} + \mu f_{\text{plt,fr}}$$
.

Lemma 6.1 For a sufficiently small δ_* , in the set $\tilde{\mathcal{D}}_r$ of $((\Lambda, \lambda), (\eta, \xi), (p, q), (G, g))$ of $(\mathbb{R}^N_+ \times \mathbb{T}^N) \times (\mathbb{R}^N \times \mathbb{R}^N) \times (\mathbb{R}^{N-2} \times \mathbb{R}^{N-2}) \times (\mathbb{R}_+ \times \mathbb{T})$ such that

$$a(\Lambda) \in \mathcal{A}$$
, $\delta^2 := \sum_{1 \le i \le N} \Lambda_i - G < \delta_*^2$, $|(\eta, \xi, p, q)|_2 < 2\delta$,

with \mathcal{A} the set of semimajor axes

$$\mathcal{A} := \left\{ a = (a_1, \cdots, a_N) \in \mathbb{R}^N : \quad 0 < a_1 < a_2 < \cdots < a_N \right\} \,,$$

there exists a symplectic real-analytic change of variable

$$\phi: \mathcal{D}_{\mathrm{r}} \to \mathcal{D}_{\mathrm{r}}$$

which leaves G, Λ unvaried and puts $f_{plt,fr}$ into the form

$$\bar{f}_{\text{plt}} := \bar{f}_{\text{plt,fr}} \circ \phi = \overline{f_{\text{plt,fr}} \circ \phi}$$
$$= \bar{f}_{0}(\Lambda, G) + \sum_{1 \le i \le N} \mathbf{s}_{i}(\Lambda, G) \frac{\eta_{i}^{2} + \xi_{i}^{2}}{2} + \sum_{1 \le i \le N-2} \mathbf{z}_{i}(\Lambda, G) \frac{p_{i}^{2} + q_{i}^{2}}{2} + \mathcal{O}(3)$$

where, for any fixed $G \in \mathbb{R}_+$, the "secular frequencies" $s = (s_1, \dots, s_N)$, $z = (z_1, \dots, z_{N-2})$ together with the mean motions $n = (n_1, \dots, n_N) := \partial h_{\text{plt}}$ do not satisfy any linear relation in any simply connected subset \mathcal{V}_G of a suitable subset \mathcal{U}_G with full measure of

$$\mathcal{A}_G := \left\{ a(\Lambda) \in \mathcal{A} , \sum_{1 \le i \le N} \Lambda_i - G < \delta_*^2 \right\}$$

Proof. We discuss only the case $N \ge 3$, since the case N = 2 is well understood.³⁸

Step 1: partial reduction (or full regularization).

Let $\phi_{\rm pr}$ the map "partial reduction map" $\phi_{\rm pr}$ which acts as described in (eq. (4.42)) Section 4.3. This leads $\mathcal{H}_{\rm plt,fr} = h_{\rm plt} + \mu f_{\rm plt,fr}$ to $\mathcal{H}_{\rm plt,pr} = h_{\rm plt} + \mu f_{\rm plt,pr}$, where $\bar{f}_{\rm plt,pr}$ is as in Lemma 5.1:

$$\bar{f}_{\rm plt,pr} = C_0(m,a) + \mathcal{Q}_h^* \cdot \frac{\bar{\eta}^2 + \bar{\xi}^2}{2} + \mathcal{Q}_v^* \cdot \frac{\bar{p}^2 + \bar{q}^2}{2} + \bar{f}_{\rm plt,pr}^4$$

Step 2: diagonalization of \mathcal{Q}_h^* , \mathcal{Q}_v^* . Let ρ_h^* , ρ_v^* the unitary matrices which leave \mathcal{Q}_h^* , \mathcal{Q}_v^* diagonal:

$$\mathcal{Q}_h^* \cdot \tilde{\eta}^2 = \sum_{1 \le i \le N} s_i \tilde{\eta}_i^2 , \quad \mathcal{Q}_v^* \cdot \tilde{p}^2 = \sum_{1 \le i \le N-1} z_i \tilde{p}_i^2$$

where

$$\tilde{\eta} := \rho_h^{*t} \bar{\eta} , \quad \tilde{p} := \rho_v^{*t} \bar{p} .$$

Then, the transformation ϕ_{diag} which leaves Λ unvaried and acts on $\tilde{p}, \tilde{q}, \tilde{\lambda}$ as

$$\phi_{\text{diag}}: \begin{cases} \bar{\eta} := \rho_h^* \tilde{\eta} \\ \bar{\xi} := \rho_h^* \tilde{\xi} \end{cases}, \begin{cases} \bar{p} := \rho_v^* \tilde{p} \\ \bar{q} := \rho_v^* \tilde{q} \end{cases}, \quad \bar{\lambda} = \tilde{\lambda} + \varphi(\Lambda, \tilde{p}, \tilde{q}) \end{cases}$$
(6.5)

(where $(\Lambda, \bar{p}, \bar{q}) \to \varphi(\Lambda, \bar{p}, \bar{q})$ is a suitable shift which makes ϕ_{diag} symplectic) puts $\bar{f}_{\text{plt,pr}}$ into the form

$$\bar{f}_{\text{diag}} := \bar{f}_{\text{plt,pr}} \circ \phi_{\text{diag}} = \overline{f}_{\text{plt,pr}} \circ \phi_{\text{diag}} = C_0(m,a) + \sum_{1 \le i \le N} s_i \frac{\tilde{\eta}_i^2 + \tilde{\xi}_i^2}{2} + \sum_{1 \le i \le N-1} z_i \frac{\tilde{p}_i^2 + \tilde{q}_i^2}{2} + \tilde{f}_{\text{diag}}^4$$

$$s_1 = 2\sigma_1 + \sigma_2 + O(\delta^2)$$
, $s_2 = \sigma_1 + 2\sigma_2 + O(\delta^2)$

Hence, s_1 , s_2 have the desired property since σ_1 , σ_2 have it, as proved in [17].

³⁸As already remarked, for N = 2, the full Deprit reduction corresponds to the Jacobi reduction and the two spatial secular frequencies s_1 , s_2 of the spatial three body problem are manifestly related (see [33]) to the frequencies of the plane problem σ_1 , σ_2 in Delaunay–Poincaré variables by

where $\tilde{f}_{\text{diag}}^4 = O(4)$ and, as proved in Proposition 5.1, $s = (s_1, \dots, s_N)$ and $z = (z_1, \dots, z_N)$ do not satisfy any other linear condition than the Herman's resonance. Notice that, since ρ_h^* , ρ_v^* are unitary, the angular momentum G, in their terms has just the same expression as in full regularized variables:

$$G = \sum_{1 \le i \le N} \left(\Lambda_i - \frac{\tilde{\eta}_i^2 + \tilde{\xi}_i^2}{2} \right) - \sum_{1 \le i \le N-1} \frac{\tilde{p}_i^2 + \tilde{q}_i^2}{2}$$

We are then "justified" if we do not change the name of the variable G we introduce into the following step.

Step 3: full reduction. Apply now $\phi_{\mathrm{fr}} := \phi_{\mathrm{pr}}^{-1}, \ i.e.$, put

$$\begin{cases} \tilde{p}_{N-1} = \sqrt{2\left(\sum_{1 \le i \le N} \Lambda_i - G - \sum_{1 \le i \le N} \frac{\check{\eta}_i^2 + \check{\xi}_i^2}{2} - \sum_{1 \le i \le N-2} \frac{\check{p}_i^2 + \check{q}_i^2}{2}\right)} \cos\check{g} \\ \tilde{q}_{N-1} = -\sqrt{2\left(G - \sum_{1 \le i \le N} \Lambda_i - \sum_{1 \le i \le N} \frac{\check{\eta}_i^2 + \check{\xi}_i^2}{2} - \sum_{1 \le i \le N-2} \frac{\check{p}_i^2 + \check{q}_i^2}{2}\right)} \sin\check{g} \\ \tilde{\lambda}_i = \check{\lambda}_i + \check{g} \\ \left(\frac{\tilde{\eta}_i}{\check{\xi}_i}\right) = \mathcal{R}_z(-\check{g})\left(\frac{\check{\eta}_i}{\check{\xi}_i}\right) \qquad \left(\frac{\tilde{p}_j}{\check{q}_j}\right) = \mathcal{R}_z(-\check{g})\left(\frac{\check{p}_j}{\check{q}_j}\right) \end{cases}$$

This carries f_{diag} to

$$\bar{f}_{\rm fr} := \bar{f}_{\rm diag} \circ \phi_{\rm fr} = \overline{f}_{\rm diag} \circ \phi_{\rm fr} = \check{C}_0(m, a, G) + \sum_{1 \le i \le N} {\rm s}_i^0 \frac{\check{\eta}_i^2 + \check{\xi}_i^2}{2} + \sum_{1 \le i \le N-2} {\rm z}_i^0 \frac{\check{p}_i^2 + \check{q}_i^2}{2} + \check{f} \ ,$$

where

$$\begin{cases} \check{C}_0(m, a, G) := C_0(m, a) + s_{N-1} \left(\sum_{1 \le i \le N} \Lambda_i - G \right) \\ \check{f} := \tilde{f}_{\text{diag}}^4 \circ \phi_{\text{fr}} \\ s_i^0 := s_i - z_{N-1} & 1 \le i \le N \\ z_i^0 := z_i - z_{N-1} & 1 \le i \le N - 2 \end{cases}$$

Lemma 6.2 The functions $s_1^0, \dots, s_N^0, z_1^0, \dots, z_{N-2}^0$ do not satisfy any linear relation in any open, simply connected set $\mathcal{V} \subset \mathcal{U}$ for a suitable open set $\mathcal{U} \subset \mathcal{A}$ with full measure.

Proof. Let $\mathcal{U} \subset \mathcal{A}$ the open set with full measure where Proposition 5.1 holds and assume that, in some point of some simply connected set $\bar{\mathcal{V}} \subset \mathcal{U}$, we had

$$\sum_{1 \le i \le N} c_i \mathbf{s}_i^0 + \sum_{1 \le i \le N-2} g_i \mathbf{z}_i^0 = \sum_{1 \le i \le N} c_i (s_i - z_{N-1}) + \sum_{1 \le i \le N-2} g_i (z_i - z_{N-1}) = 0 ,$$

with $(c_1, \dots, c_N, g_1, \dots, g_{N-2}) \in \mathbb{R}^{2N-2} \setminus \{0\}$. Then, by Proposition 5.1

$$c_1 = \dots = c_N = g_1 = \dots = g_{N-2} = -\sum_{1 \le i \le N} c_i - \sum_{1 \le i \le N-2} g_i$$
which is a contraddiction.

Step 4: recentering $f_{\rm fr}$ at its equilibrium point. For small values of $\delta^2 := \sum_{1 \le i \le N} \Lambda_i - G$ we find a new equilibrium point $\check{z}_{\rm eq} = (\check{\eta}_{\rm eq}, \check{\xi}_{\rm eq}, \check{p}_{\rm eq}, \check{q}_{\rm eq})$ for $\bar{f}_{\rm fr}$, which is O(δ). Rescale, in fact, the variables as

$$\check{\eta} = 2\delta\,\hat{p}\,,\quad\check{\xi} = 2\delta\,\hat{q}\,,\quad\check{p} = 2\delta\,\hat{p}\,,\quad\check{q} = 2\delta\,\hat{q}$$

and then discuss equation

$$\delta^{-2}\partial_{\check{z}}\bar{f}_{\rm fr} = 0 \quad \text{where} \quad \check{z} = (\check{\eta},\check{\xi},\check{p},\check{q}) \;.$$

by an Implicit Function Theorem argument. Perform then the change of variable

$$\phi_{\rm eq}: \quad \check{z} = z_* + \check{z}_{\rm e} \quad \check{g} = g_* + \check{\psi}(\Lambda, G) \ , \quad \check{\lambda}_i = \lambda_i^* + \check{\varphi}_i(\Lambda, G)$$

leaving the remaining variables unvaried, where $\check{\psi}(\Lambda, G)$, $\check{\varphi}_i(\Lambda, G)$ are suitable shifts which make ϕ_e symplectic.

The result then follows after a suitable symplectic diagonalization of the Hessian matrix of $\bar{f}_{eq} := \bar{f}_{fr} \circ \phi_{eq} = \overline{f_{fr}} \circ \phi_{eq}$, which gives linear invariants $s_1, \dots, s_N, z_1, \dots, z_{N-2} \delta$ -close to $s_1^0, \dots, s_N^0, z_1^0, \dots, z_{N-2}^0$, hence, with the desired property.

A Proof of the Averaging Theorem (Lemma 2.1)

Lemma A.1 Let D, A_0, A_1, \dots, A_N square complex matrices of order n, with D non singular, such that

$$||D(A_i - \operatorname{id}_n)|| \le \varepsilon_i , \quad i = 0, \cdots, N .$$

Then, if $\|\cdot\|$ is a norm on $\operatorname{Mat}_{n\times n}^{\mathbb{C}}$ such that $\|AB\| \leq \|A\| \|B\|$,

$$\|D(A_0A_1A_2\cdots A_N - \operatorname{id}_n)\| \leq \varepsilon_0 + (1+\varepsilon_0\|D^{-1}\|)\varepsilon_1 + \cdots + (1+\varepsilon_0\|D^{-1}\|)\cdots(1+\varepsilon_{N-1}\|D^{-1}\|)\varepsilon_N .$$

Proof. Let

$$T := A_1 \cdots A_N - \operatorname{id}_n$$

Then, writing

$$D(A_0A_1\cdots A_N - \operatorname{id}_n) = D(A_0 - \operatorname{id}_n)(\operatorname{id}_n + T) + DT$$

we find

$$\begin{aligned} \|D(A_0A_1\cdots A_N - \operatorname{id}_n)\| &\leq \|D(A_0 - \operatorname{id}_n)\|(1 + \|T\|) + \|DT\| \\ &\leq \varepsilon_0 + (1 + \varepsilon_0 \|D^{-1}\|)\|DT\| \\ &= \varepsilon_0 + (1 + \varepsilon_0 \|D^{-1}\|)\|D(A_1\cdots A_N - \operatorname{id}_n)\| \quad (A.1) \end{aligned}$$

The Lemma is then proved after N iterations of (A.1).

Lemma A.2 (Iterative Lemma) Let $\bar{n} + \hat{n} = n$, $0 < 2\rho < r$ and $0 < 2\sigma < s$. Suppose that the Hamiltonian

$$H = h + g + f$$

is real-analytic on $\mathcal{P}_{r,s} := \mathcal{I}_r \times \mathbb{T}_s^n$ with $\omega := \partial h$ verifying

$$|\omega(I) \cdot k| \ge \begin{cases} \bar{\alpha} & \text{for} \quad k = (\bar{k}, \hat{k}) \in \mathbb{Z}^{\bar{n}} \times \mathbb{Z}^{\hat{n}} \setminus \Lambda \quad \bar{k} \neq 0 , \quad |k|_1 \le K \\ \hat{\alpha} & \text{for} \quad k = (0, \hat{k}) \in \{0\}_{\mathbb{Z}^{\bar{n}}} \times \mathbb{Z}^{\hat{n}} \setminus \Lambda \quad |k|_1 \le K \end{cases}$$
(A.2)

for $I \in \mathcal{I}_r$, and f so small that

$$||f||_{r,s} < \frac{\alpha \rho \sigma}{2}$$
, where $\alpha := \min\{\bar{\alpha}, \hat{\alpha}\}$. (A.3)

Then, there exists a real-analytic, symplectic transformation

 $\Phi: \quad \mathcal{P}_{r-2\rho,s-2\sigma} \to \mathcal{P}_{r,s}$

which carries H into

$$H_+ := H \circ \Phi(I, \varphi) = h + g_+ + f_+$$

with

$$g_+ = g + P_\Lambda T_K f$$

and

$$\|f_{+}\|_{r-2\rho,s-2\sigma} \le \left(1 - \frac{\|f\|_{r,s}}{\alpha\rho\sigma/2}\right)^{-1} \left(\frac{\|g\|_{*} + \|f\|_{r,s}}{\alpha\rho\sigma} + e^{-K\sigma}\right) \|f\|_{r,s}$$
(A.4)

where

$$\|g\|_{*} = \frac{1}{2} \sum_{j} \left(\frac{\rho}{r_{j} - r + \rho} + \frac{\sigma}{s_{j} - s + \sigma} \right) \|g_{j}\|_{r_{j}, s_{j}} ,$$

if $g = \sum_j g_j$, with terms g_j bounded on $\mathcal{P}_{r_j,s_j} \supset \mathcal{P}_{r,s}$. Moreover,

$$\|W^{\bar{n}}_{\bar{\alpha}^{-1},\alpha^{-1}}W_{\rho,\sigma}(\Phi - \mathrm{id})\|_{\mathcal{P}} , \qquad \frac{1}{2} \|W^{\bar{n}}_{\bar{\alpha}^{-1},\alpha^{-1}}(W_{\rho,\sigma} D\Phi W^{-1}_{\rho,\sigma} - \mathrm{id}_{2n})\|_{\mathcal{P}} \le \frac{\|f\|_{r,s}}{\rho\sigma}$$
(A.5)

uniformly on $\mathcal{P}_{r-2\rho,s-2\sigma}$.

Lemma A.2 is a useful remake of the Iterative Lemma of [30]. We outline the differences.

(i) In [30], instead than (A.2), the real nonresonance for ω up to order K and a "smallness" assumption for r:

$$(a): \ |\omega(I) \cdot k| \ge \alpha \text{ for } I \in \mathcal{I} \ , \ k \in \mathbb{Z}^n \setminus \Lambda \ , \ |k|_1 \le K \ , \ (b): \ r \le \frac{\alpha}{pMK}$$
(A.6)

are required, where p > 1 is a prefixed number. But, in the proof, (a) and (b) are used only to prove

$$|\omega(I) \cdot k| \ge \frac{\alpha}{q}$$
 for $I \in \mathcal{I}_r$, $k \in \mathbb{Z}^n \setminus \Lambda$, $|k|_1 \le K$

which is next needed. So, the Lemma remains true when the assumption

$$|\omega(I) \cdot k| \ge \alpha \quad \text{for} \quad I \in \mathcal{I}_r , \quad k \in \mathbb{Z}^n \setminus \Lambda , \quad |k|_1 \le K$$

replaces (A.6) and α replaces α/q in all its occurences. But, (A.2) obviously implies

$$|\omega(I) \cdot k| \ge \alpha := \min\{\bar{\alpha}, \hat{\alpha}\} \text{ for } I \in \mathcal{I}_r, \quad k \in \mathbb{Z}^n \setminus \Lambda, \quad |k|_1 \le K.$$

It now is enough observing that (A.3), (A.4) are just the same of [30], with α replacing α/q .

(ii) For what concernes (A.5), in [30], taking into account (i), we find

$$|W_{\rho,\sigma}(\Phi - \mathrm{id})|_{\mathcal{P}^*} \le \frac{\|f\|_{r,s}}{\alpha \rho \sigma}$$
(A.7)

where

$$|(y,x)|_{\mathcal{P}^*} := \max\{|y|_1, |x|_\infty\}$$
.

In particular, (A.7) holds when ³⁹

$$((0,\Pi_{\hat{I}}\Phi),\Pi_{\varphi}\Phi)$$

replaces

$$\Phi = ((\Pi_{\bar{I}}\Phi, \Pi_{\hat{I}}\Phi), \Pi_{\varphi}\Phi) \; .$$

In order to obtain an estimate for $|\rho^{-1}(\Pi_{\bar{I}}\Phi - \mathrm{id}_{\bar{n}})|_1$, we recall that, in [30], Φ is constructed as the time 1 map of the flow X^t_{ϕ} of a hamiltonian vectorfield $X_{\phi} = (-\partial_{\varphi}\phi, \partial_{I}\phi)$, where ϕ is the trigonometric polynomial

$$\phi(I,\varphi) = \sum_{k \in \mathbb{Z}^{\bar{n}} \times \mathbb{Z}^{\bar{n}} \setminus \Lambda, \ |k| \le K} \frac{f_k(I)}{i\omega(I) \cdot k} e^{ik \cdot \varphi} .$$

But, taking into account the non resonance assumptions (A.2), we may split ϕ as

$$\phi(I,\varphi) = \bar{\phi}(I,\varphi) + \hat{\phi}(I,\hat{\varphi})$$

where

$$\bar{\phi}(I,\varphi) = \sum_{k = (\bar{k},\hat{k}) \in \mathbb{Z}^{\bar{n}} \times \mathbb{Z}^{\hat{n}} \backslash \Lambda, \ |k| \leq K, \ \bar{k} \neq 0} \frac{f_k(I)}{i\omega(I) \cdot k} e^{ik \cdot \varphi}$$

and $\hat{\phi} = \phi - \bar{\phi}$ does not depend from the $\bar{\phi}$ -variables. Hence, the projection $\bar{X}_{\phi} := \partial_{\bar{\phi}}\phi = \partial_{\bar{\phi}}\bar{\phi}$ of the vectorfield X_{ϕ} over $\mathbb{C}^{\bar{n}} \times \{0\}_{\mathbb{C}^{2n-\bar{n}}}$, by the General Cauchy Inequality, is bounded as

$$|\bar{X}_{\phi}|_{1} \le \frac{\|f\|_{r,s}}{\bar{\alpha}\sigma}$$

uniformly on $V_{r-3\rho/2,s-3\sigma/2}$, so, joining this result with (A.7), we find

$$|W_{\bar{\alpha}^{-1},\alpha^{-1}}^{\bar{n}}W_{\rho,\sigma}(\Phi - I_{2n})|_{\mathcal{P}} \le \frac{\|f\|_{r,s}}{\rho\sigma}$$

The second equation in (A.5), then, follows by the General Cauchy Inequality, uniformly on $V_{r-2\rho,s-2\sigma}$.

Proof of Lemma 2.1. We distinguish two cases:

(a)
$$48 \frac{\|f\|_{r,\bar{s}+s}}{\alpha \, r \, s} < e^{-Ks/6}$$
, (b) $48 \frac{\|f\|_{r,\bar{s}+s}}{\alpha \, r} \ge e^{-Ks/6}$.

The case (a) is trivial, because we apply the Iterative Lemma with $g \equiv 0$ and parameters

$$\rho = \rho' := \frac{r}{4}, \qquad \sigma = \sigma' := \frac{s}{3}.$$

 ${}^{39}\Pi_{\hat{I}}, \cdots$ denotes the projection ove the \hat{I}, \cdots variables.

This is made possible, because (using $Ks \ge 6$)

$$\|f\|_{r,\bar{s}+s} < \frac{1}{2^7} \frac{\alpha \, r \, s}{Ks} \le \frac{1}{6 \cdot 2^7} \alpha \, r \, s = \frac{\alpha \, \rho' \, \sigma'}{2^6} < \frac{\alpha \rho' \sigma'}{2} \; .$$

Letting, then,

$$r' = r - 2\rho' = \frac{r}{2}$$
, $s' = s - 2\sigma' = \frac{s}{3}$

we see that the map Φ : $\mathcal{P}_{r',\bar{s}+s'} \to \mathcal{P}_{r,\bar{s}+s}$ verifies

$$\begin{aligned} & \|W^{\bar{n}}_{\bar{\alpha}^{-1},\alpha^{-1}}W_{\rho',\sigma'}(\Phi - \mathrm{id})\|_{\mathcal{P}} ,\\ & \frac{1}{2} \|W^{\bar{n}}_{\bar{\alpha}^{-1},\alpha^{-1}}(W_{\rho',\sigma'} D\Phi W^{-1}_{\rho',\sigma'} - \mathrm{id}_{2n})\|_{\mathcal{P}} \\ & \leq \frac{12}{rs} \|f\|_{r,\bar{s}+s} \\ & \leq \frac{2K}{r} \|f\|_{r,\bar{s}+s} , \end{aligned}$$

and

$$H \circ \Phi = h + f_0 + f_1$$

where $f_0 = P_{\Lambda} T_K f$. Furthermore,

$$\begin{split} \|f_1\|_{r',\bar{s}+s'} &\leq \left(1 - \frac{\|f\|_{r,\bar{s}+s}}{\alpha\rho'\sigma'/2}\right)^{-1} \left(\frac{\|f\|_{r,\bar{s}+s}}{\alpha\rho'\sigma'} + e^{-Ks/3}\right) \|f\|_{r,\bar{s}+s} \\ &\leq (1 - 2^{-5})^{-1} \left(\frac{\|f\|_{r,\bar{s}+s}}{\alpha\rho'\sigma'} + e^{-Ks/3}\right) \|f\|_{r,\bar{s}+s} \\ &= \frac{32}{31} \left(\frac{\|f\|_{r,\bar{s}+s}}{\alpha\,r\,s/(12)} + e^{-Ks/3}\right) \|f\|_{r,\bar{s}+s} \\ &\leq \frac{32}{31} \left(\frac{1}{4} + \frac{1}{e}\right) e^{-Ks/6} \|f\|_{r,\bar{s}+s} \\ &< e^{-Ks/6} \|f\|_{r,\bar{s}+s} \;. \end{split}$$

The lemma is proved with $\Psi = \Phi$.

In case (b), we move in N + 1 steps. First, we apply the Iterative Lemma with $g \equiv 0$ and parameters

$$\rho = \rho_0 := \frac{r}{8} , \qquad \sigma = \sigma_0 := \frac{s}{6} ,$$

thanks to the inequality

$$\|f\|_{r,\bar{s}+s} \le \frac{1}{2^7} \frac{\alpha \, r \, s}{Ks} \le \frac{1}{6 \cdot 2^7} \alpha \, r \, s = \frac{\alpha \, \rho_0 \, \sigma_0}{16}$$

•

Letting,

$$r_1 = r - 2\rho_0 = \frac{3}{4}r$$
, $s_1 = s - 2\sigma_0 = \frac{2}{3}s$

we find

$$\Phi_0: \quad \mathcal{P}_{r_1,\bar{s}+s_1} \to \mathcal{P}_{r,\bar{s}+s}$$

verifying

$$|W_{\bar{\alpha}^{-1},\alpha^{-1}}^{\bar{n}}W_{0}(\Phi_{0} - \mathrm{id})|_{\mathcal{P}} ,$$

$$\frac{1}{2} ||W_{\bar{\alpha}^{-1},\alpha^{-1}}^{\bar{n}}(W_{0} D\Phi_{0} W_{0}^{-1} - \mathrm{id}_{2n})||_{\mathcal{P}}$$

$$\frac{1}{\rho_{0}\sigma_{0}} ||f||_{r,\bar{s}+s} = \frac{48}{rs} ||f||_{r,\bar{s}+s} ,$$

where $W_0 := W_{\rho_0,\sigma_0}$, and

 \leq

$$H_1 := H \circ \Phi_0 = h + g_1 + f_1$$

with

$$\|f_{1}\|_{r_{1},\bar{s}+s_{1}} \leq \left(1 - \frac{\|f\|_{r,\bar{s}+s}}{\alpha\rho_{0}\sigma_{0}/2}\right)^{-1} \left(\frac{\|f\|_{r,\bar{s}+s}}{\alpha\rho_{0}\sigma_{0}} + e^{-K\sigma_{0}}\right) \|f\|_{r,\bar{s}+s}$$

$$\leq \frac{8}{7} \left(\frac{\|f\|_{r,\bar{s}+s}}{\alpha\rho_{0}\sigma_{0}} + e^{-K\sigma_{0}}\right) \|f\|_{r,\bar{s}+s}$$

$$= \frac{8}{7} \left(\frac{\|f\|_{r,\bar{s}+s}}{\alpha r s/48} + e^{-Ks/6}\right) \|f\|_{r,\bar{s}+s}$$

$$\leq \frac{16}{7} \frac{\|f\|_{r,\bar{s}+s}}{\alpha r s/48} \|f\|_{r,\bar{s}+s}$$

$$= \frac{1}{7} \frac{\|f\|_{r,\bar{s}+s}}{\alpha\rho_{0}\sigma_{0}/(16)} \|f\|_{r,\bar{s}+s}$$

$$\leq \frac{\|f\|_{r,\bar{s}+s}}{7} .$$

$$(A.8)$$

Now, let N an integer number. Our aim is to apply the Iterative Lemma N times, each with parameters

$$\rho = \rho_N := \frac{r}{8N} , \quad \sigma = \sigma_N := \frac{s}{4N} , \qquad (A.9)$$

so as to construct, at each step, a symplectic, analytic transformation

$$\Phi_i: \quad \mathcal{P}_{r_{i+1},\bar{s}+s_{i+1}} \to \mathcal{P}_{r_i,\bar{s}+s_i} , \quad i=1,\cdots,N$$

where

$$r_i = r_1 - 2(i-1)\rho_N$$
, $s_i = s_1 - 2(i-1)\sigma_N$, $i = 1, \dots, N+1$,

verifying

$$|W_{\bar{\alpha}^{-1},\alpha^{-1}}^{\bar{n}}W_{N}(\Phi_{i} - \mathrm{id})|_{\mathcal{P}},$$

$$\frac{1}{2} ||W_{\bar{\alpha}^{-1},\alpha^{-1}}^{\bar{n}}(W_{N} D\Phi_{i} W_{N}^{-1} - \mathrm{id}_{2n})||_{\mathcal{P}}$$

$$\leq \frac{1}{\rho_{N}\sigma_{N}} ||f_{i}||_{r_{i},\bar{s}+s_{i}},$$

where $W_N := W_{\rho_N,\sigma_N}$, and

$$H_{i+1} := H \circ \Phi_0 \circ \dots \circ \Phi_i = h + g_{i+1} + f_{i+1} , \quad i = 1, \dots, N$$

with

$$\|f_i\|_{r_i,\bar{s}+s_i} \le \left(\frac{1}{4}\right)^{i-1} \|f_1\|_{r_1,\bar{s}+s_1} \le \left(\frac{1}{4}\right)^i \|f\|_{r,\bar{s}+s_1}$$

Consequentely, after N steps, we will find $H_{N+1} = H \circ \Psi_N = h + g_{N+1} + f_{N+1}$ analytic on $\mathcal{P}_{r/2,\bar{s}+s/6}$

$$f_{N+1} \le \left(\frac{1}{4}\right)^{N+1} \|f\|_{r,\bar{s}+s} \le e^{-Ks/6} \|f\|_{r,\bar{s}+s}$$

provided N is sufficiently large:

$$N+1 \ge \frac{Ks}{12\log 2}$$

Choice, thus, N such in a way that

$$N < \frac{Ks}{12\log 2} \le N+1 \; .$$

If N = 0, the theorem is proved with $\Psi = \Phi_0$. Otherwise, if $N \ge 1$, we verify, by induction, inequalities

$$\|f_i\|_{r_i,\bar{s}+s_i} \leq \frac{\|f_{i-1}\|_{r_{i-1},\bar{s}+s_{i-1}}}{4}, \quad \|f_i\|_{r_i,\bar{s}+s_i} < \frac{\alpha r s}{2^{13}N^2} = \frac{\alpha \rho_N \sigma_N}{2^8}, \\ \|g_i - g_{i-1}\|_{r_i,\bar{s}+s_i} \leq \|f_{i-1}\|_{r_{i-1},\bar{s}+s_{i-1}}, \quad i = 1, \cdots, N$$
 (A.10)

For i = 1, we have

$$\|g_1 - g_0\|_{r_1,\bar{s}+s_1} = \|g_1\|_{r_1,\bar{s}+s_1} = \|P_\Lambda T_K f\|_{r_1,\bar{s}+s_1} \le \|f\|_{r_1,\bar{s}+s_1}$$

and, by (A.8),

$$\|f_1\|_{r_1,\bar{s}+s_1} \le 2^7 \frac{\|f\|_{r,\bar{s}+s}^2}{\alpha \, r \, s} \le \frac{\alpha \, r \, s}{2^7 (Ks)^2} < \frac{\alpha \, r \, s}{2^{13} N^2} = \frac{\alpha \rho_N \sigma_N}{2^8} \le \frac{\|f\|_{r,\bar{s}+s}}{7} \tag{A.11}$$

provided

$$||f||_{r,\bar{s}+s} \le \frac{\alpha r}{2^7 K}$$
.

Assume, now, that (A.10) hold for a given i < N. Then, Lemma A.2 is applicable once again and we find

$$\Phi_i: \ \mathcal{P}_{r_{i+1},\bar{s}+s_{i+1}} \to \mathcal{P}_{r_i,\bar{s}+s_i}$$

such that

$$H_{i+1} := H_i \circ \Phi_i = h + g_{i+1} + f_{i+1}$$

with g_{i+1} and f_{i+1} verifying

$$\|g_{i+1} - g_i\|_{r_{i+1},\bar{s}+s_{i+1}} = \|P_{\Lambda}T_Kf_i\|_{r_{i+1},\bar{s}+s_{i+1}} \le \|f_i\|_{r_{i+1},\bar{s}+s_{i+1}}$$

and

$$\begin{split} \|f_{i+1}\|_{r_{i+1},\bar{s}+s_{i+1}} &\leq \\ \left(1 - \frac{\|f_i\|_{r_i,\bar{s}+s_i}}{\alpha\rho_N\sigma_N/2}\right)^{-1} \left(\frac{\|f_i\|_{r_i,\bar{s}+s_i} + \|g_i\|_*}{\alpha\rho_N\sigma_N} + e^{-Ks/(4N)}\right) \|f_i\|_{r_i,\bar{s}+s_i} \leq \\ \left(1 - \frac{1}{2^7}\right)^{-1} \left(\frac{1}{2^8} + \frac{1}{2^4} + \frac{1}{8}\right) \|f_i\|_{r_i,\bar{s}+s_i} = \\ \frac{49}{254} \|f_i\|_{r_i,\bar{s}+s_i} \leq \\ \frac{\|f_i\|_{r_i,\bar{s}+s_i}}{4} \end{split}$$

where we have used $Ks > 12 \log 2$ which implies

$$e^{-Ks/(4N)} \le \frac{1}{8}$$

and the telescopic expansion

$$g_i = g_1 + \sum_{k=2}^{i} (g_k - g_{k-1})$$

with $g_1 := P_{\Lambda}T_K f$, $g_k - g_{k-1} = P_{\Lambda}T_K f_{k-1}$ bounded on $\mathcal{P}_{r,\bar{s}+s}$, $\mathcal{P}_{r_k,\bar{s}+s_k} \supset \mathcal{P}_{r_i,\bar{s}+s_i}$, for $1 \leq k \leq i$. This gives, according to the definition of $||g_i||_*$ relatively to this expansion,

$$\begin{split} \|g_i\|_* &= \frac{1}{2} \left(\frac{\rho_N}{r - r_i + \rho_N} + \frac{\sigma_N}{\bar{s} + s - (\bar{s} + s_i) + \sigma_N} \right) \|g_1\|_{r,\bar{s} + s} \\ &+ \frac{1}{2} \sum_{k=2}^i \left(\frac{\rho_N}{r_k - r_i + \rho_N} + \frac{\sigma_N}{(\bar{s} + s_k) - (\bar{s} + s_i) + \sigma_N} \right) \|g_k - g_{k-1}\|_{r_k,\bar{s} + s_k} \\ &= \frac{1}{2} \left(\frac{\rho_N}{r - r_i + \rho_N} + \frac{\sigma_N}{s - s_i + \sigma_N} \right) \|g_1\|_{r,\bar{s} + s} \\ &+ \frac{1}{2} \sum_{k=2}^i \left(\frac{\rho_N}{r_k - r_i + \rho_N} + \frac{\sigma_N}{s_k - s_i + \sigma_N} \right) \|g_k - g_{k-1}\|_{r_k,\bar{s} + s_k} \end{split}$$

$$\leq \frac{1}{4} \left(\frac{\rho_N}{\rho_0} + \frac{\sigma_N}{\sigma_0} \right) \|g_1\|_{r,\bar{s}+s} + \frac{1}{2} \sum_{k=2}^{i} \|f_{k-1}\|_{r_k,\bar{s}+s_k}$$

$$\leq \frac{5}{8N} \|g_1\|_{r,\bar{s}+s} + \frac{2}{3} \|f\|_{r,\bar{s}+s}$$

$$\leq \frac{\alpha\rho_N \sigma_N}{2^5} + \frac{\alpha\rho_N \sigma_N}{2^8}$$

$$\leq \frac{\alpha\rho_N \sigma_N}{2^4}$$

(use the following inequalities:

$$\begin{split} \|f_{k-1}\|_{r_k,\bar{s}+s_k} &\leq \|f_{k-1}\|_{r_{k-1},\bar{s}+s_{k-1}} \leq 4^{-(k-1)} \|f\|_{r,\bar{s}+s} ,\\ \|g_1\|_{r,\bar{s}+s} &\leq \|f\|_{r,\bar{s}+s} \leq \alpha \, r/(2^7 K) \leq \alpha \, rs/(2^{10} N) = N \, \alpha \rho_N \sigma_N/(2^5) ,\\ r_k - r_i \geq \rho_N , \quad s_k - s_i \geq \sigma_N ,\\ r - r_i + \rho_N \geq r - r_i \geq r - r_1 = r/4 = 2\rho_0 ,\\ s - s_i + \sigma_N \geq s - s_i \geq s - s_1 = s/3 = 2\sigma_0) . \end{split}$$

and (A.10) are proved for any $1 \le i \le N$. Let

$$H := H_{N+1} = h + g + f_*$$
, $(g := g_{N+1}, f_* := f_{N+1})$.

Then, by construction,

$$||f_*||_{r/2,\bar{s}+s/6} = ||f_{N+1}||_{r/2,\bar{s}+s/6} \le e^{-Ks/6} ||f||_{r,\bar{s}+s}$$

and, using $Ks \ge 8$, $||f_i||_{r_i,\bar{s}+s_i} \le 4^{-i} ||f||_{r,\bar{s}+s}$ and (A.11),

$$\begin{split} \|g - P_{\Lambda}T_{K}f\|_{r/2,\bar{s}+s/6} &= \|g_{N+1} - g_{1}\|_{r/2,\bar{s}+s/6} \\ &\leq \sum_{k=1}^{N} \|g_{k+1} - g_{k}\|_{r/2,\bar{s}+s/6} \\ &\leq \sum_{k=1}^{N} \|g_{k+1} - g_{k}\|_{r_{k},\bar{s}+s_{k}} \\ &\leq \sum_{k=1}^{N} \|f_{k}\|_{r_{k},\bar{s}+s_{k}} \\ &\leq \frac{4}{3} \|f_{1}\|_{r_{1},\bar{s}+s_{1}} \\ &\leq 2^{8} \frac{\|f\|_{r,\bar{s}+s}^{2}}{\alpha \, r \, s} \\ &\leq \frac{2^{5}K}{\alpha \, r} \|f\|_{r,\bar{s}+s}^{2} \, . \end{split}$$

Furthermore, by the usual telescopic arguments,

$$\begin{aligned} |\Pi_{\bar{I}}(\Psi_{N} - \mathrm{id})|_{1} &\leq \sum_{k=0}^{N} |\Pi_{\bar{I}}(\Phi_{k} - \mathrm{id})|_{1} \\ &= |\Pi_{\bar{I}}(\Phi_{0} - \mathrm{id})|_{1} + \sum_{k=1}^{N} |\Pi_{\bar{I}}(\Phi_{k} - \mathrm{id})|_{1} \\ &\leq \frac{\|f\|_{r,\bar{s}+s}}{\bar{\alpha}\sigma_{0}} + \frac{1}{\bar{\alpha}\sigma_{N}} \sum_{k=1}^{N} \|f_{k}\|_{r_{k},\bar{s}+s_{k}} \\ &\leq \frac{\|f\|_{r,\bar{s}+s}}{\bar{\alpha}\sigma_{0}} + \frac{2}{\bar{\alpha}\sigma_{N}} 2^{7} \frac{\|f\|_{r,\bar{s}+s}^{2}}{\bar{\alpha}\,r\,s} \\ &= \frac{\|f\|_{r,\bar{s}+s}}{\bar{\alpha}\sigma_{0}} \left(1 + 2^{8}N \frac{\|f\|_{r,\bar{s}+s}}{\bar{\alpha}\,r\,s}\right) \\ &\leq \frac{\|f\|_{r,\bar{s}+s}}{\bar{\alpha}\sigma_{0}} \left(1 + 2^{5}K \frac{\|f\|_{r,\bar{s}+s}}{\bar{\alpha}\,r\,}\right) \\ &\leq \frac{2\|f\|_{r,\bar{s}+s}}{\bar{\alpha}\sigma_{0}} \\ &\leq \frac{2^{4} \|f\|_{r,\bar{s}+s}}{\bar{\alpha}\,s} \,. \end{aligned}$$
(A.12)

Similarly, one proves

$$|\Pi_{I}(\Psi_{N} - \mathrm{id})|_{1} \leq \frac{2^{4} \|f\|_{r,s+s}}{\alpha \, s} \,, \quad |\Pi_{\varphi}(\Psi_{N} - \mathrm{id})|_{\infty} \leq \frac{2^{4} \|f\|_{r,\bar{s}+s}}{\alpha \, r} \,, \tag{A.13}$$

which gives (2.28). We prove now (2.29). Writing

 $W^{\bar{n}}_{\bar{\alpha}^{-1},\alpha^{-1}}(W_{r,s}D\Psi_{0}W^{-1}_{r,s} - \operatorname{id}_{2n}) = W_{r,s}W^{-1}_{0}[W^{\bar{n}}_{\bar{\alpha}^{-1},\alpha^{-1}}(W_{0}D\Psi_{0}W^{-1}_{0} - \operatorname{id}_{2n})]W_{0}W^{-1}_{r,s}$

and using

$$\|W_{\bar{\alpha}^{-1},\alpha^{-1}}^{\bar{n}}(W_0 D \Psi_0 W_0^{-1} - \operatorname{id}_{2n})\|_{\mathcal{P}} \le \frac{2}{\rho_0 \sigma_0} \|f\|_{r,\bar{s}+s}$$
$$\|W_{r,s} W_0^{-1}\|_{\mathcal{P}} = \frac{1}{6} , \quad \|W_0 W_{r,s}^{-1}\|_{\mathcal{P}} = 8$$

we find

$$\begin{split} \|W_{\bar{\alpha}^{-1},\alpha^{-1}}^{\bar{n}}(W_{r,s}D\Psi_{0}W_{r,s}^{-1} - \operatorname{id}_{2n})\|_{\mathcal{P}} &\leq \|W_{r,s}W_{0}^{-1}\|_{\mathcal{P}}\|W_{0}W_{r,s}^{-1}\|_{\mathcal{P}} \\ &\times \|W_{\bar{\alpha}^{-1},\alpha^{-1}}^{\bar{n}}(W_{0}D\Psi_{0}W_{0}^{-1} - \operatorname{id}_{2n})\|_{\mathcal{P}} \\ &= \frac{4}{3}\|W_{\bar{\alpha}^{-1},\alpha^{-1}}^{\bar{n}}(W_{0}\Psi_{0}W_{0}^{-1} - \operatorname{id}_{2n})\|_{\mathcal{P}} \\ &\leq \frac{8}{3}\frac{1}{\rho_{0}\sigma_{0}}\|f\|_{r,\bar{s}+s} \leq 1 \;. \end{split}$$
(A.14)

Similarly, writing

$$W_{\bar{\alpha}^{-1},\alpha^{-1}}^{\bar{n}}(W_{r,s}D\Psi_{i}W_{r,s}^{-1} - \operatorname{id}_{2n}) = W_{r,s}W_{N}^{-1}[W_{\bar{\alpha}^{-1},\alpha^{-1}}^{\bar{n}}(W_{N}D\Psi_{i}W_{N}^{-1} - \operatorname{id}_{2n})]W_{N}W_{r,s}^{-1}$$

and using

$$\|W_{\bar{\alpha}^{-1},\alpha^{-1}}^{\bar{n}}(W_N D\Phi_i W_N^{-1} - \operatorname{id}_{2n})\|_{\mathcal{P}} \le \frac{2}{\rho_N \sigma_N} \|f_i\|_{r_i,\bar{s}+s_i}$$
$$\|W_{r,s} W_N^{-1}\|_{\mathcal{P}} = \frac{1}{4N} , \quad \|W_N W_{r,s}^{-1}\|_{\mathcal{P}} = 8N$$

we arrive at

$$\|W_{\bar{\alpha}^{-1},\alpha^{-1}}^{\bar{n}}(W_{r,s}D\Phi_{i}W_{r,s}^{-1} - \operatorname{id}_{2n})\|_{\mathcal{P}} \le \frac{4}{\rho_{N}\sigma_{N}}\|f_{i}\|_{r_{i},\bar{s}+s_{i}} \le \alpha , \quad i = 1, \cdots, N .$$
(A.15)

,

Taking into account (A.14), (A.15) and Lemma A.1 (where $W^{\bar{n}}_{\bar{\alpha}^{-1},\alpha^{-1}}$ plays the role of the invertible matrix D, and $4||f_i||/\rho_N\sigma_N$ the one of ε_i , for $i \neq 0$), we find, for

$$D\Psi_N = D\Phi_0 D\Phi_1 \cdots D\Phi_N$$

the bound

$$\|W_{\bar{\alpha}^{-1},\alpha^{-1}}^{\bar{n}}(W_{r,s}\,D\Psi_{N}W_{r,s}^{-1}-\operatorname{id}_{2n})\|_{\mathcal{P}} \leq \frac{8}{3}\frac{\|f\|_{r,\bar{s}+s}}{\rho_{0}\sigma_{0}} + \frac{8}{\rho_{N}\sigma_{N}}\sum_{i=1}^{N}2^{i-1}\|f_{i}\|_{r_{i},\bar{s}+s_{i}}$$

with

$$\sum_{i=1}^{N} 2^{i-1} \|f_i\|_{r_i,\bar{s}+s_i} \le 2 \|f_1\|_{r_1,\bar{s}+s_1} \le 2^8 \frac{\|f\|_{r,\bar{s}+s}^2}{\alpha \, r \, s}$$

so that

$$\begin{split} \|W_{\bar{\alpha}^{-1},\alpha^{-1}}^{\bar{n}}(W_{r,s} D\Psi_N W_{r,s}^{-1} - \operatorname{id}_{2n})\|_{\mathcal{P}} &\leq \frac{2^5 K}{r} \|f\|_{r,\bar{s}+s} + \left(\frac{2^7 K \|f\|}{\alpha r}\right) \left(\frac{K \|f\|}{r}\right) \\ &\leq \frac{2^6 K}{r} \|f\|_{r,\bar{s}+s} \,. \end{split}$$

This completes the proof.

B Birkhoff Normal Form

In this section, we discuss quantitatively the reduction to Birkhoff Normal Form, for Hamiltonians possessing elliptic equilibrium points. For further references, see also [19].

Proposition B.1 (Birkhoff Normal Form) Let $0 < \theta < 1$; let $\mathcal{D} \subseteq \mathbb{C}^n$ such that

$$\Omega = (\Omega_1, \cdots, \Omega_m) : \mathcal{D} \to \mathbb{C}^m$$

is (α, K) non resonant on \mathcal{D} , with $K \geq 2$, and let

$$f(I,\varphi,p,q) = \sum_{i=1}^{m} \Omega_i(I) \frac{p_i^2 + q_i^2}{2} + o_2(p,q;I) ,$$

where o_2 is real-analytic in $\mathcal{D} \times B_r^{2m}(0)$, and verifies

$$\lim_{(p,q)\to 0} \frac{o_2(p,q;I)}{|(p,q)|^2} = 0 \quad for \ all \quad I \in \mathcal{D} \ .$$

Then, there exists $r_K > 0$ and a symplectic, analytic transformation

$$\pi : \mathcal{D} \times \mathbb{C}^n / (2\pi\mathbb{Z}^n) \times B^{2m}_{r_K}(0) \to \mathcal{D} \times \mathbb{C}^n / (2\pi\mathbb{Z}^n) \times B^{2m}_r(0)$$
$$(J, \vartheta, P, Q) \to (I, \varphi, p, q) = \pi(J, \vartheta, P, Q)$$

with I, $\varphi - \vartheta$, p, q independent from ϑ , which puts f into Birkhoff normal form up to order K. Furthermore, the following holds.

i) The transformation π may be obtained as a product

$$\pi = \mathcal{B}_2 \circ \cdots \circ \mathcal{B}_K \quad (\mathcal{B}_2 = \mathrm{id}) ,$$

where

$$\mathcal{B}_k: \quad \mathcal{D} \times \mathbb{C}^n / 2\pi \mathbb{Z}^n \times B^{2m}_{r_k} \quad \to \quad \mathcal{D} \times \mathbb{C}^n / 2\pi \mathbb{Z}^n \times B^{2m}_{r_{k-1}} , \\ (\tilde{J}, \tilde{\vartheta}, \tilde{P}, \tilde{Q}) \quad \to \quad (\tilde{I}, \tilde{\varphi}, \tilde{p}, \tilde{q})$$

verifies $\tilde{I} = \tilde{J}$ and

$$\begin{split} |\tilde{q} - \tilde{Q}| &\leq \frac{M_1^k}{1 - \theta} |(\tilde{P}, \tilde{Q})|^{k-1} , \quad |\tilde{p} - \tilde{P}| \leq \frac{M_2^k}{1 - \theta} |(\tilde{P}, \tilde{Q})|^{k-1} \\ |\tilde{\varphi} - \tilde{\vartheta}| &\leq 2^k M_0^k \max\left\{ 1, \left(\frac{M_1^k}{1 - \theta}\right)^k |(\tilde{P}, \tilde{Q})|^{k(k-2)} \right\} |(\tilde{P}, \tilde{Q})|^k , \end{split}$$
(B.1)

and, for any $k = 2, \dots, K$, the product $\pi_k := \mathcal{B}_2 \circ \dots \circ \mathcal{B}_k$ puts f in Birkhoff normal form up to order k:

$$f \circ \pi_k(\tilde{J}, \tilde{\vartheta}, \tilde{P}, \tilde{Q}) = \sum_{i=1}^m \ \Omega_i(\tilde{J}) \, \frac{\tilde{P}_i^2 + \tilde{Q}_i^2}{2} + \mathcal{P}_k(\tilde{J}, \tilde{P}, \tilde{Q}) + o_k(\tilde{P}, \tilde{Q}; \tilde{J}) \ .$$

ii) The constants M_j^k , M_{jh}^k are inductively defined as follows. Let, for $K \geq 3$ and $k = 3, \dots, K$,

$$o_{k-1}(\tilde{p},\tilde{q};\tilde{I}) = \sum_{|\alpha|_1+|\beta|_1=k} p_{\alpha,\beta}^k(\tilde{I}) \left(\frac{\tilde{p}+i\tilde{q}}{\sqrt{2}}\right)^\alpha \left(\frac{\tilde{p}-i\tilde{q}}{\sqrt{2}}\right)^\beta + \tilde{o}_k(\tilde{p},\tilde{q};\tilde{I}) \qquad (i=\sqrt{-1})$$

with $\tilde{o}_k(\tilde{p}, \tilde{q}; \tilde{I})/|(\tilde{p}, \tilde{q})|^k \to 0$, as $(\tilde{p}, \tilde{q}) \to 0$; let

$$s_k(\tilde{J}, \tilde{P}, \tilde{q}) := \sum_{\alpha \neq \beta} 2 \, i \, \frac{p_{\alpha,\beta}^k(\tilde{J})}{\Omega(\tilde{J}) \cdot (\alpha - \beta)} \left(\frac{\tilde{P} + i\tilde{q}}{\sqrt{2}}\right)^\alpha \left(\frac{\tilde{P} - i\tilde{q}}{\sqrt{2}}\right)^\beta$$

Then,

$$M_{j}^{k} := \sup_{\mathcal{D} \times B_{1}^{2m}(0)} |\partial_{\mathcal{P}(j)} s_{k}| , \qquad M_{jh}^{k} := \sup_{\mathcal{D} \times B_{1}^{2m}(0)} \|\partial_{\mathcal{P}(j)\mathcal{P}(h)} s_{k}\| , \quad j, \ h = 0, \ 1, \ 2 ,$$

where

$$\mathcal{P}(0) := \tilde{J}$$
, $\mathcal{P}(1) := \tilde{P}$, $\mathcal{P}(2) := \tilde{q}$.

iii) The polynomials \mathcal{P}_k are inductively defined as follows. Starting with $\mathcal{P}_2 \equiv 0$, and given $\mathcal{P}_{k-1}(\tilde{I}, \tilde{p}, \tilde{q})$, $o_{k-1}(\tilde{p}, \tilde{q}; \tilde{I})$, then, for $K \geq 3$ and $k = 3, \dots, K$,

$$\mathcal{P}_k(\tilde{J}, \tilde{P}, \tilde{Q}) = \mathcal{P}_{k-1}(\tilde{J}, \tilde{P}, \tilde{Q}) + \mathcal{Q}_k(\tilde{J}, \tilde{P}, \tilde{Q}) ,$$

where

$$\mathcal{Q}_k(\tilde{J}, \tilde{P}, \tilde{Q}) = \begin{cases} 0 \quad \text{for odd} \quad k\\ \sum_{|\alpha|_1 = k/2} p_{\alpha, \alpha}^k(\tilde{J}) \left(\frac{\tilde{P}_1^2 + \tilde{Q}_1^2}{2}\right)^{\alpha_1} \cdots \left(\frac{\tilde{P}_m^2 + \tilde{Q}_m^2}{2}\right)^{\alpha_m} \quad \text{for even} \quad k. \end{cases}$$
(B.2)

iv) The radii r_k are inductively defined as follows. Starting with $r_2 = r$ and given r_{k-1} , then, for $K \ge 3$ and $k = 3, \dots, K$,

$$r_k = \frac{1-\theta}{\sqrt{2}} \rho_k , \quad \rho_k := \min\left\{ \left(\frac{\theta}{M_{12}^k}\right)^{1/(k-2)}, \left(\frac{\theta}{\sqrt{2}M_1^k}\right)^{1/(k-2)}, \left(\frac{\theta}{\sqrt{2}M_2^k}\right)^{1/(k-2)}, r_{k-1} \right\}.$$

Remark B.1 Observe that, rather than projecting the remainders o_{k+1} with order k+1 of $f_k := f \circ \mathcal{B}_2 \circ \cdots \circ \mathcal{B}_k$ over the spaces $(p + iq)/\sqrt{2}$, $(p - iq)/\sqrt{2}$, a simple algorithm for the computation of \mathcal{P}_{k+1} comes from the identity

$$\mathcal{P}_{k+1} = \frac{1}{(2\pi)^m} \int_{\mathbb{T}^n} o_{k+1}(p,q;I) \big|_{p_i = \sqrt{P_i^2 + Q_i^2} \cos \varphi_i, \ q_i = \sqrt{P_i^2 + Q_i^2} \sin \varphi_i} d\varphi_i$$

Our proof of the previous Proposition is based on the following

Lemma B.1 Let $0 < \theta < 1$, r > 0, $\mathcal{D} \subseteq \mathbb{C}^n$ and, for $J \in \mathcal{D}$, $\varphi \in \mathbb{C}^n/2\pi\mathbb{Z}^n$, $(P,q) \in \mathbb{C}^{2m}$,

$$S(J, P, \varphi, q) := J\varphi + Pq + s(J, P, q)$$

where

$$s(J, P, q) := \sum_{|\alpha|_1 + |\beta|_1 = k} \sigma_{\alpha, \beta}(J) P^{\alpha} q^{\beta}$$

is a polynomial with degree $k \geq 3$ in (P,q), with analytic coefficients $J \to \sigma_{\alpha,\beta}(J)$. Then, S is the generating function of a (symplectic,) analytic transformation

$$\mathcal{B}: \quad \mathcal{D} \times \mathbb{C}^n / 2\pi \mathbb{Z}^n \times B^{2m}_{r'}(0) \to \mathcal{D} \times \mathbb{C}^n / 2\pi \mathbb{Z}^n \times B^{2m}_r(0) ,$$
$$(J, \vartheta, P, Q) \to (I, \varphi, p, q)$$

with

$$r' := \frac{1-\theta}{\sqrt{2}} \rho , \quad \rho := \min\left\{ \left(\frac{\theta}{M_{12}}\right)^{1/(k-2)}, \left(\frac{\theta}{\sqrt{2}M_1}\right)^{1/(k-2)}, \left(\frac{\theta}{\sqrt{2}M_2}\right)^{1/(k-2)}, r \right\} .$$

such that I = J and

$$\begin{aligned} |q-Q| &\leq \frac{M_1}{1-\theta} |(P,Q)|^{k-1}, \quad |p-P| \leq \frac{M_2}{1-\theta} |(P,Q)|^{k-1} \\ |\varphi-\vartheta| &\leq 2^k M_0 \max\left\{ 1, \left(\frac{M_1}{1-\theta}\right)^k |(P,Q)|^{k(k-2)} \right\} |(P,Q)|^k, \end{aligned}$$
(B.3)

where

$$M_j := \sup_{\mathcal{D} \times B_1^{2m}(0)} |\partial_{\mathcal{P}(j)} s| , \quad M_{jh} := \sup_{\mathcal{D} \times B_1^{2m}(0)} \|\partial_{\mathcal{P}(j)\mathcal{P}(h)}^2 s\| ,$$

if \mathcal{P} sends the set $\{0, 1, 2\}$ to the set $\{J, P, q\}$ as

$$\mathcal{P}(0) := J , \quad \mathcal{P}(1) := P , \quad \mathcal{P}(2) := q$$

Proof. Observe preliminarly that, as s(J, P, q), is a homogeneous polynomial in (P, q) with degree k, then, $|\partial_{\mathcal{P}(j)}(J, P, q)|$, is a homogeneous function of (P, q) with degree k or k-1 for $j=0, j \neq 0$, respectively, and

$$|\partial_{\mathcal{P}(j)} s(J, P, q)| \le \begin{cases} M_0 |(P, q)|^k & \text{if } j = 0\\ M_j |(P, q)|^{k-1} & \text{if } j \neq 0 \end{cases}$$
(B.4)

by the definitions of M_j . Similarly, $\|\partial^2_{\mathcal{P}(j)\mathcal{P}(h)} s(I, P, Q)\|$ is a homogeneous function with degree k (for j = h = 0), or k - 1 (for $j = 0 \neq h$), or k - 2 (for $j, h \neq 0$), and

$$\|\partial_{\mathcal{P}(j)\mathcal{P}(h)}^{2} s(I, P, Q)\| \leq \begin{cases} M_{00} |(P, q)|^{k} & \text{if } j = h = 0\\ M_{0h} |(P, q)|^{k-1} & \text{if } j = 0, \ h \neq 0\\ M_{jh} |(P, q)|^{k-2} & \text{if } j, \ h \neq 0 \end{cases}$$
(B.5)

We construct \mathcal{B} by its generating equations, which are are

$$\begin{cases}
I = J \\
\varphi = \vartheta - \partial_J s(J, P, q) \\
p = P + \partial_q s(J, P, q)
\end{cases}$$
(B.6)

where q is obtained by solving, with respect to q, the implicit equation

$$q + \partial_P s(J, P, q) = Q . \tag{B.7}$$

Let

$$\rho := \min\left\{ \left(\frac{\theta}{M_{12}}\right)^{1/(k-2)}, \left(\frac{\theta}{\sqrt{2}M_1}\right)^{1/(k-2)}, \left(\frac{\theta}{\sqrt{2}M_2}\right)^{1/(k-2)}, r \right\}$$

By (B.5) it follows, for $J \in \mathcal{D}$, $(P,q) \in B^m_{\rho/\sqrt{2}}(0) \times B^m_{\rho/\sqrt{2}}(0) (\subseteq B^{2m}_{\rho}(0))$,

$$\|\partial_{Pq}^2 s(J, P, q)\| \le M_{12} \rho^{k-2} \le \theta < 1$$
,

which is enough to assert that the function $q \to q + \partial_P s(J, P, q)$ is injective on $B^m_{\rho/\sqrt{2}}(0)$, for any $(J, P) \in \mathcal{D} \times B^m_{\rho/\sqrt{2}}(0)$. Now, using

$$|\partial_P s| \le M_1 \rho^{k-1} \le \frac{\theta \rho}{\sqrt{2}}$$
 for $(J, P, q) \in \mathcal{D} \times B^m_{\rho/\sqrt{2}}(0) \times B^m_{\rho/\sqrt{2}}(0)$

we also find that, for any $(J, P) \in \mathcal{D} \times B^m_{\rho/\sqrt{2}}(0)$, the map $q \to q + \partial_P s(J, P, q)$ is onto on $B^m_{r'}(0)$, with $r' = (1 - \theta)\rho/\sqrt{2}$. Let $\tilde{q} \in B^m_{\rho/\sqrt{2}}(0)$ the unique solution of (B.7), for $J \in \mathcal{D}$, $(P, Q) \in B^{2m}_{r'}(0) (\subseteq B^m_{r'}(0) \times B^m_{r'}(0))$, and let $p = P + \partial_q s(J, P, q)$. Using

$$|\partial_q s| \le M_2 \rho^{k-1} \le \frac{\theta \rho}{\sqrt{2}}$$

we find $p \in B^m_{\rho/\sqrt{2}}(0)$, namely, $(p,q) \in B^{2m}_{\rho}(0) \subseteq B^{2m}_r(0)$. Taking also $\varphi = \vartheta - \partial_J s(J, P, q)$, we have constructed

$$\mathcal{B}: \quad \mathcal{D} \times \mathbb{C}^n / 2\pi \mathbb{Z}^n \times B^{2m}_{r'}(0) \to \mathcal{D} \times \mathbb{C}^n / 2\pi \mathbb{Z}^n \times B^{2m}_r(0)$$

In order to prove (B.3), using (B.4) and the estimate

$$\begin{aligned} |\partial_P s(J, P, q) - \partial_P s(J, P, Q)| &= \left| \int_0^1 \sum_j \partial_{Pq_j}^2 s(J, P, Q + t(q - Q))(q_j - Q_j) dt \right| \\ &\leq \int_0^1 \left| \sum_j \partial_{Pq_j}^2 s(J, P, Q + t(q - Q))(q_j - Q_j) \right| dt \\ &\leq \sup_{\mathcal{D} \times r((P,Q),(P,q))} \|\partial_{Pq}^2 s\| |q - Q| . \end{aligned}$$
(B.8)

where r(P,Q) denotes the straight line from P to Q, we get

$$|q-Q| = |\partial_P s(J, P, q)|$$

$$\leq |\partial_P s(J, P, Q)| + |\partial_P s(J, P, q) - \partial_P s(J, P, Q)|$$

$$\leq M_1 |(P,Q)|^{k-1} + \sup_{\mathcal{D} \times r((P,Q),(P,q))} \|\partial_{Pq}^2 s\| |q-Q|.$$
(B.9)

As (P,Q), $(P,q) \in B^{2m}_{\rho}(0)$, then, $r((P,Q),(P,q)) \subseteq B^{2m}_{\rho}(0)$, hence,

$$\sup_{\mathcal{D} \times r((P,Q),(P,q))} \|\partial_{Pq}^2 s(J,P,q)\| \le M_{12} \rho^{k-2} \le \theta , \qquad (B.10)$$

giving so, by (B.9),

$$|q - Q| \le M_1 |(P, Q)|^{k-1} + \theta |q - Q|$$

namely,

$$|q - Q| \le \frac{M_1}{1 - \theta} |(P, Q)|^{k-1}$$
 (B.11)

The proof of

$$|p - P| \le \frac{M_2}{1 - \theta} |(P, Q)|^{k - 1}$$

is quite similar and is omitted. Using now (B.11), we obtain

$$\begin{aligned} |\varphi - \vartheta| &\leq M_0 |(P,q)|^k \\ &= M_0 |(P,Q) + (0,q-Q)|^k \\ &\leq M_0 (|(P,Q)| + |(0,q-Q)|)^k \\ &\leq 2^k M_0 \max\left\{ |(P,Q)|^k, \ |q-Q|^k \right\} \\ &\leq 2^k M_0 \max\left\{ |(P,Q)|^k, \ \left(\frac{M_1}{1-\theta}\right)^k |(P,Q)|^{k(k-1)} \right\} \\ &= 2^k M_0 |(P,Q)|^k \max\left\{ 1, \ \left(\frac{M_1}{1-\theta}\right)^k |(P,Q)|^{k(k-2)} \right\} \end{aligned}$$
(B.12)

and the proof of (B.3) is complete.

Proof of Proposition B.1. We proceed by induction on K. For K = 2, f is yet in Birkhoff normal form up to order 2, and the Proposition is proved with $r_2 = r$, $\pi_2 = \mathcal{B}_2 = \text{id}, \mathcal{P}_2 \equiv 0$. Assuming, now, that Proposition B.1 holds when K - 1 replaces K, we want to prove it for K. Assume, then, that \mathcal{D} is (α, K) non resonant for Ω . Obviously, \mathcal{D} is $(\alpha, K - 1)$ non resonant. By the inductive hypothesis, we find

$$\pi_{K-1} = \mathcal{B}_2 \circ \cdots \circ \mathcal{B}_{K-1} : \quad \mathcal{D} \times \mathbb{C}^n / 2\pi \mathbb{Z}^n \times B^{2m}_{r_{K-1}} \quad \to \quad \mathcal{D} \times \mathbb{C}^n / 2\pi \mathbb{Z}^n \times B^{2m}_r$$
$$(\tilde{I}, \tilde{\varphi}, \tilde{p}, \tilde{q}) \quad \to \quad (I, \varphi, p, q) = \pi_{K-1} (\tilde{I}, \tilde{\varphi}, \tilde{p}, \tilde{q})$$

with $\mathcal{B}_2 = \mathrm{id}$ and

$$\mathcal{B}_k: \quad \mathcal{D} \times \mathbb{C}^n / 2\pi \mathbb{Z}^n \times B^{2m}_{r_k} \to \mathcal{D} \times \mathbb{C}^n / 2\pi \mathbb{Z}^n \times B^{2m}_{r_{k-1}}, \quad k = 3, \cdots, K-1,$$

which puts f into Birkhoff normal form up to order K - 1:

$$f_{K-1}(\tilde{I}, \tilde{\varphi}, \tilde{p}, \tilde{q}) := f \circ \pi_{K-1}(\tilde{I}, \tilde{\varphi}, \tilde{p}, \tilde{q}) = \sum_{i=1}^{m} \Omega_i(\tilde{I}) \frac{\tilde{p}_i^2 + \tilde{q}_i^2}{2} + \mathcal{P}_{K-1}(\tilde{I}, \tilde{p}, \tilde{q})$$

+ $o_{K-1}(\tilde{p}, \tilde{q}; \tilde{I})$
(B.13)

where items i), \cdots , iv) hold, for $k \leq K - 1$. We prove that, defining

$$\rho_{K} := \min\left\{ \left(\frac{\theta}{M_{12}^{K}}\right)^{1/(K-2)}, \left(\frac{\theta}{\sqrt{2}M_{1}^{K}}\right)^{1/(K-2)}, \left(\frac{\theta}{\sqrt{2}M_{2}^{K}}\right)^{1/(K-2)}, r_{K-1} \right\}, \\
r_{K} := \frac{1-\theta}{\sqrt{2}} \rho_{K},$$
(B.14)

we find a symplectic, analytic transformation

$$\mathcal{B}_{K}: \quad \mathcal{D} \times \mathbb{C}^{n}/2\pi\mathbb{Z}^{n} \times B^{2m}_{r_{K}} \rightarrow \mathcal{D} \times \mathbb{C}^{n}/2\pi\mathbb{Z}^{n} \times B^{2m}_{r_{K-1}}$$
$$(J, \vartheta, P, Q) \rightarrow (\tilde{I}, \tilde{\varphi}, \tilde{p}, \tilde{q}) = \mathcal{B}_{K}(J, \vartheta, P, Q)$$

with $\tilde{\varphi} - \vartheta$, \tilde{p} , \tilde{q} independent from ϑ and verifying $\tilde{I} = J$ and

$$\begin{aligned} |\tilde{q} - Q| &\leq \frac{M_1^K}{1 - \theta} |(P, Q)|^{K - 1}, \quad |\tilde{p} - P| \leq \frac{M_2^K}{1 - \theta} |(P, Q)|^{K - 1} \\ |\tilde{\varphi} - \vartheta| &\leq 2^K M_0^K |(P, Q)|^K \max\left\{ 1, \left(\frac{M_1^K}{1 - \theta}\right)^K |(P, Q)|^{K(K - 2))} \right\}, \end{aligned}$$
(B.15)

which puts f_{K-1} in Birkhoff normal form up to order K. We construct \mathcal{B}_K by means of a generating function $S_K(J, P, \tilde{\varphi}, \tilde{q})$ of the form

$$S_K(J, P, \tilde{\varphi}, \tilde{q}) = J \cdot \tilde{\varphi} + P \cdot \tilde{q} + s_K(J, P, \tilde{q})$$
(B.16)

where $s_K(J, P, \tilde{q})$ is a homogeneous polynomial in (P, \tilde{q}) with degree K, which we write as:

$$s_K(J, P, \tilde{q}) = \sum_{|\alpha|_1 + |\beta|_1 = K} s_{\alpha, \beta}^K(J) \left(\frac{P + i\tilde{q}}{\sqrt{2}}\right)^{\alpha} \left(\frac{P - i\tilde{q}}{\sqrt{2}}\right)^{\beta} .$$
(B.17)

Splitting, in (B.13), o_{K-1} as

$$o_{K-1}(\tilde{p}, \tilde{q}; \tilde{I}) = \sum_{|a|_1 + |\beta|_1 = K} p_{\alpha, \beta}^K(\tilde{I}) \left(\frac{\tilde{p} + i\tilde{q}}{\sqrt{2}}\right)^\alpha \left(\frac{\tilde{p} - i\tilde{q}}{\sqrt{2}}\right)^\beta + \tilde{o}_K(\tilde{p}, \tilde{q}; \tilde{I})$$

where $\tilde{o}_K(\tilde{p}, \tilde{q}; \tilde{I}/|(\tilde{p}, \tilde{q})|^K \to 0$ as $(\tilde{p}, \tilde{q}) \to 0$, and replacing the generating equations of \mathcal{B}_K

$$\begin{cases} \tilde{I} = J \\ \tilde{\varphi} = \vartheta - \partial_J s(J, P, \tilde{q}) \\ Q = \tilde{q} + \partial_P s(J, P, \tilde{q}) \\ \tilde{p} = P + \partial_{\tilde{q}} s(J, P, \tilde{q}) \end{cases}$$

into the definition (B.13) of f_{K-1} , we find that f_{K-1} changes to

$$f_{K}(J,\vartheta,P,Q) = \sum_{i=1}^{m} \frac{P_{i}^{2} + Q_{i}^{2}}{2} \Omega_{i}(J) + \mathcal{P}_{K-1}(J,P,Q) + \sum_{|\alpha|_{1} + |\beta|_{1} = K} \left[\frac{i}{2} \Omega(J) \cdot (\alpha - \beta) s_{\alpha,\beta}^{K}(J) + p_{\alpha,\beta}^{K}(J) \right] \times \left(\frac{P + iQ}{\sqrt{2}} \right)^{\alpha} \left(\frac{P - iQ}{\sqrt{2}} \right)^{\beta} + o_{K}(P,Q;J),$$
(B.18)

and this leads us to choice, in (B.17),

$$s_{\alpha,\beta}^{K}(J) = \begin{cases} 0 & \text{for } \alpha = \beta \\ \\ 2 i p_{\alpha,\beta}^{K}(J) / (\Omega(J) \cdot (\alpha - \beta)) & \text{for } \alpha \neq \beta . \end{cases}$$
(B.19)

The definition (B.19) is well put because $|\alpha - \beta|_1 \leq K$ (observe $|a_l - \beta_l| \leq \max\{\alpha_l, \beta_l\}$) and \mathcal{D} is (α, K) non resonant for Ω . The choice (B.19) allows us to kill, in the summand in (B.18), all the terms with $\alpha \neq \beta$, and f_K is in Birkhoff normal form up to order K. In particular, when K is odd, no term survives, and $\mathcal{P}_K \equiv \mathcal{P}_{K-1}$. For even values of K, by (B.18), we find

$$f_K(J,\vartheta,P,Q) = \sum_{i=1}^m \frac{P_i^2 + Q_i^2}{2} \Omega_i(J) + \mathcal{P}_K(J,P,Q) + o_K(P,Q;J),$$

where

$$\mathcal{P}_K(J, P, Q) = \mathcal{P}_{K-1}(J, P, Q) + \mathcal{Q}_K(J, P, Q)$$

with

$$\mathcal{Q}_{K}(J,P,Q) = \sum_{|\alpha|_{1}=K/2} p_{\alpha,\alpha}^{K}(J) \left(\frac{P_{1}^{2}+Q_{1}^{2}}{2}\right)^{\alpha_{1}} \cdots \left(\frac{P_{m}^{2}+Q_{m}^{2}}{2}\right)^{\alpha_{m}} .$$

On the other hand, by Lemma B.1, the function

$$S_K(J, P, \tilde{\varphi}, \tilde{q}) = J \cdot \tilde{\varphi} + P \cdot \tilde{q} + s_K(J, P, \tilde{q})$$

with

$$s_K(J, P, \tilde{q}) = \sum_{\alpha \neq \beta} 2 i \frac{p_{\alpha, \beta}^K(J)}{\Omega(J) \cdot (\alpha - \beta)} \left(\frac{P + i\tilde{q}}{\sqrt{2}}\right)^{\alpha} \left(\frac{P - i\tilde{q}}{\sqrt{2}}\right)^{\beta}$$

generates an analytic (symplectic) transformation

$$\mathcal{B}_{K}: \quad \mathcal{D} \times \mathbb{C}^{n}/2\pi\mathbb{Z}^{n} \times B^{2m}_{r_{K}}(0) \quad \to \quad \mathcal{D} \times \mathbb{C}^{n}/2\pi\mathbb{Z}^{n} \times B^{2m}_{r_{K-1}}(0)$$
$$(J, \vartheta, P, Q) \quad \to \quad (\tilde{I}, \tilde{\varphi}, \tilde{p}, \tilde{q}) = \mathcal{B}_{K}(J, \vartheta, P, Q) ,$$

with r_K as in (B.14), with $\tilde{I} = J$ and \tilde{q} , \tilde{p} , $\tilde{\varphi} - \vartheta$ independent from ϑ , such that (B.15) holds. This completes the proof.

C Proof of Lemma 3.6

In this appendix, we prove the Lemma 3.6. For shortness, we will refer to the property $(3.45) \div (3.46)$ for a given matrix \mathcal{A} with order n as (*)-property.

Proof. We proceed by induction on n. The assertion is trivially true for j = 2, by direct computation

$$\lambda_{21} = \frac{1}{2} \left(a_{11} + a_{22} + \sqrt{(a_{11} - a_{22})^2 + 4a_{12}^2} \right)$$

= $a_{11} + O\left(\frac{a_{12}^2}{a_{11} - a_{22}}\right)$
= $a_{11} + O\left(\frac{\delta^{2n_{12}}}{1 - \bar{a}_{22}\delta^{n_{22}}}\right)$
= $a_{11} + O\left(\delta^{2n_{12}}\right)$

and, similarly,

$$\lambda_{22} = \frac{1}{2} \left(a_{11} + a_{22} - \sqrt{(a_{11} - a_{22})^2 + 4a_{12}^2} \right) = a_{22} + O\left(\delta^{2n_{12}}\right)$$

Assume, now, that the Lemma holds for n-1. Let \mathcal{A} a matrix with order n with the (*)-property and let $\mathcal{P}(\lambda)$ its characteristic polynomial. We are interested to solve equation

$$\mathcal{P}(\lambda,\delta) = 0$$

closely to any diagonal element a_{jj} of \mathcal{A} . We use an Implicit Function Theorem argument. We expand the determinant of $\mathcal{A} - \lambda \operatorname{id}_n$ along the j^{th} row, so to split $\mathcal{P}(\lambda, \delta)$ as

$$\mathcal{P}(\lambda) = f(\lambda, \delta) + g(\lambda, \delta)$$

with

$$f(\lambda,\delta) := (a_{jj}(\delta) - \lambda) \det[M_{jj}(\lambda,\delta)] , \quad g(\lambda,\delta) := \sum_{k \neq j} (-1)^{k-j} a_{j,k}(\delta) \det[M_{j,k}(\lambda,\delta)] ,$$

where $M_{j,k}$ is the minor with order n-1 of $\mathcal{A} - \lambda \operatorname{id}_n$ with place (j,k). In particular, if $\tilde{\lambda}_1, \dots, \tilde{\lambda}_{n-1}$ are the eigenvalues of $M_{jj}(\lambda, \delta)$, then, $\det[M_{jj}(\lambda, \delta)]$ is given by

$$\det[M_{jj}(\lambda,\delta)] = \prod_{k=1}^{n-1} (\tilde{\lambda}_k(\delta) - \lambda) .$$

But $M_{jj}(\lambda, \delta)$ has the (*)-property, so, by the inductive hypothesis, its eigenvalues $\tilde{\lambda}_1$, \cdots , $\tilde{\lambda}_{n-1}$, verify

$$|\tilde{\lambda}_k - a_{kk}(\delta)| \le C\delta^{\tilde{m}_k}, \quad k = 1, \cdots, j-1, \quad |\tilde{\lambda}_k - a_{k+1,k+1}(\delta)| \le C\delta^{\tilde{m}_k}, \quad k = j, \cdots, n-1,$$

for suitable \tilde{m}_k , $1 \leq k \leq n-1$. Let $c_n > 0$ so small that

$$\frac{c_n(n-2)(1+c_n)^{n-2}}{(1-c_n)^{n-1}} < 1$$

 δ so small that

$$\min_{k} |\tilde{\lambda}_{k}(\delta) - a_{jj}(\delta)| > 0 .$$

The function $\lambda \to f(\lambda, \delta)$ vanishes for $\lambda = a_{jj}(\delta)$ and, for any λ in the complex ball

$$|\lambda - a_{jj}(\delta)| \le R(\delta) := c_n \min_k |\tilde{\lambda}_k(\delta) - a_{jj}(\delta)|$$

it results

$$\begin{aligned} |\partial_{\lambda}f(\lambda,\delta)| &= \left| \left[\prod_{k=1}^{n-1} (\tilde{\lambda}_{k} - \lambda) + (a_{jj} - \lambda) \sum_{1 \le m \le n-1} \prod_{k \ne m} (\tilde{\lambda}_{k} - \lambda) \right] \right| \\ &\ge (1 - c_{n})^{n-1} \prod_{k=1}^{n-1} |\tilde{\lambda}_{k}(\delta) - a_{jj}(\delta)| \\ &- c_{n} \min_{k} |\tilde{\lambda}_{k}(\delta) - a_{jj}(\delta)| \sum_{1 \le m \le n-1} \prod_{k \ne m} |\tilde{\lambda}_{k}(\delta) - a_{jj}(\delta)| (1 + c) \\ &\ge (1 - c_{n})^{n-1} \left[1 - \frac{c(n-2)(1+c)^{n-2}}{(1-c_{n})^{n-1}} \right] \prod_{k=1}^{n-1} |\tilde{\lambda}_{k}(\delta) - a_{jj}(\delta)| \\ &\ge (1 - c_{n})^{n-1} \left[1 - \frac{c(n-2)(1+c)^{n-2}}{(1-c_{n})^{n-1}} \right] \prod_{k=1}^{j-1} ||\tilde{\lambda}_{k}(\delta)| |a_{jj}(\delta)|^{n-j+1} \end{aligned}$$

having used the inequality

$$(1 - c_n)|\tilde{\lambda}_k(\delta) - a_{jj}(\delta)| \le |\tilde{\lambda}_k - \lambda| \le (1 + c_n)|\tilde{\lambda}_k(\delta) - a_{jj}(\delta)|$$

for $1 \leq k \leq n-1$. On the other hand, any minor M_{jk} appearing in the perturbation $g(\lambda, \delta)$ contains the column

$$c_{jk}^{j} \begin{pmatrix} a_{1j} \\ \vdots \\ a_{j-1,j} \\ a_{j+1,j} \\ \vdots \\ a_{n,j} \end{pmatrix}$$

which is of order $\delta^{\min\{n_{j-1,j}, n_{j,j+1}\}}$. The remining n-2 columns of M_{jk} , c_{jk}^l have only at one place place, a coordinate of the kind $a_{mm} - \lambda$, with $m \neq k, j$, and the other coordinates are a_{pm} , with $p \neq k, j |a_{pm}| \leq |a_{mm}| \leq \max\{|\tilde{\lambda}_m|, |a_{jj}|\}$

$$|a_{mm} - \lambda| \le |a_{mm} - a_{jj}| + |a_{jj} - \lambda| \le |a_{mm} - a_{jj}| + |\tilde{\lambda}_m - a_{jj}| \le 2|\tilde{\lambda}_m - a_{jj}| \le 4 \max\{|\tilde{\lambda}_m|, |a_{jj}|\}$$

so that

$$|\det[M_{jk}]| \le 4^n (n-1)! |a_{jj}|^{n-j} \prod_{m=1}^{j-1} |\tilde{\lambda}_m|$$

we find that

$$|g(\lambda,\delta)| = \left|\sum_{k\neq j} (-1)^{k-j} a_{j,k}(\delta) \det[M_{j,k}(\lambda,\delta)]\right| \le 4^n n! \delta^{2\min\{n_{j-1,j}, n_{j,j+1}\}} |a_{jj}|^{n-j} \prod_{m=1}^{j-1} |\tilde{\lambda}_m|^{n-j}$$

By Cauchy estimate we find

$$|\partial_{\lambda}g(\lambda,\delta)| \le 4^{n} n! \delta^{2\min\{n_{j-1,j}, n_{j,j+1}\}} |a_{jj}|^{n-j-1} \prod_{m=1}^{j-1} |\tilde{\lambda}_{m}|^{n-j}$$

so that the Implicit Function Theorem may be applied provided δ is sufficiently small:

$$\sup_{|\lambda - a_{jj}| \le r(\delta)} \frac{|\partial_{\lambda} g(\lambda, \delta)|}{|\partial_{\lambda} f(\lambda, \delta)|} \le \sup_{|\lambda - a_{jj}| \le r(\delta)} \frac{|g(\lambda, \delta)|}{r(\delta)|\partial_{\lambda} f(\lambda, \delta)|} \le \left(\frac{\delta}{\overline{\delta}}\right)^{2\min\{n_{j-1,j}, n_{j,j+1}\} - 2n_{jj}\}} \le \frac{1}{2}$$

with

$$r(\delta) = |a_{jj}|/2 \le R(\delta)$$

We find then a unique solution $\lambda_j(\delta)$ of

$$\mathcal{P}(\lambda) = f(\lambda, \delta) + g(\lambda, \delta) = 0$$

verifying

$$|\lambda_j(\delta) - a_{jj}(\delta)| \le \sup_{|\lambda - a_{jj}| \le r(\delta)} \frac{|g(\lambda, \delta)|}{|\partial_\lambda f(\lambda, \delta)|} \le C\delta^{2\min\{n_{j-1,j}, n_{j,j+1}\} - n_{jj}}$$

Let us now study the 1-dimensional eigenspace \mathcal{V}_j associated to the aigenvalue λ_j . Let

$$v_j = \left(\begin{array}{c} v_{1j} \\ v_{2j} \\ \vdots \\ v_{nj} \end{array}\right)$$

the unitary eigenvector associated to λ_j , so that $V = (v_{ij})$ is the unitary matrix which diagonaluzes \mathcal{A} :

$$V^T \mathcal{A} V = \operatorname{diag} \left(\lambda_1, \cdots, \lambda_n \right), \quad V^T V = \operatorname{id}_n .$$

Then, the vector \hat{v}_j with dimension n-1 which is obtained by v_j dropping its j^{th} component is the unique solution of

$$(M_{jj} - \lambda_j \operatorname{id}_{n-1})\hat{v}_j = -\hat{a}_j v_{jj}$$

where \hat{a}_j is the j^{th} coloumn of \mathcal{A} deprivated of its j^{th} component. But, as noticed before, M_{jj} is almost diagonal, so, we write $M_{jj} - \lambda_j \operatorname{id}_{n-1}$ as

$$M_{jj} - \lambda_j \operatorname{id}_{n-1} = (D_j - \lambda_j \operatorname{id}_{n-1}) \left[\operatorname{id}_{n-1} - B_j \right]$$

where D_j is the principal diagonal of M_{jj} and B_j is the off-diagonal

$$B_j = (\lambda_j \operatorname{id}_{n-1} - D_j)^{-1} (M_{jj} - D_j)$$

The (non zero) elements b_{hk}^{j} of B_{j} go to 0 with δ because they satisfy the following asymptotics

$$b_{h,k}^j = \kappa_{h,k}^j(\delta) \, \delta^{m_{h,k}^j}$$

with

$$0 < m_{h,k}^{j} := \begin{cases} n_{h,k} - n_{h,h} & \text{if} \quad h \neq k = 1, \cdots, j - 1 \\\\ n_{h,k+1} - n_{h,h} & \text{if} \quad h = 1, \cdots, j - 1 , \quad k = j, \cdots, N - 1 \\\\ n_{h+1,k} - n_{j,j} & \text{if} \quad h = j, \cdots, N - 1 , \quad k = 1, \cdots, j - 1 \\\\ n_{h+1,k+1} - n_{j,j} & \text{if} \quad h \neq k = j, \cdots, N - 1 . \end{cases}$$

by assumption. The matrix $\operatorname{id}_{n-1} - B_j$ is thus non singular for small δ and

$$(\mathrm{id}_{n-1} - B_j)^{-1} = \mathrm{id}_{n-1} + \delta^{m_j} \bar{B}_j(\delta)$$

where

$$m_j = \min_{h,k} \left\{ m_{h,k}^j \right\}$$

It follows,

$$\begin{split} \hat{v}_{j} &= -(M_{jj} - \lambda_{j} \operatorname{id}_{n-1})^{-1} \hat{a}_{j} v_{jj} \\ &= -\left[\operatorname{id}_{n-1} - B_{j}\right]^{-1} (D_{j} - \lambda_{j} \operatorname{id}_{n-1})^{-1} \hat{a}_{j} v_{jj} \\ &= -\left(\operatorname{id}_{n-1} + \delta^{m_{j}} \bar{B}_{j}(\delta)\right) \begin{pmatrix} \frac{a_{j1}}{a_{11} - \lambda_{j}} \\ \vdots \\ \frac{a_{j,j+1}}{a_{j-1,j-1} - \lambda_{j}} \\ \vdots \\ \frac{a_{j,n}}{a_{n,n} - \lambda_{j}} \end{pmatrix} v_{jj} \\ &= \begin{pmatrix} \check{v}_{1j} \, \delta^{\nu_{1j}} \\ \vdots \\ \check{v}_{j-1,j} \, \delta^{\nu_{j-1,j}} \\ \check{v}_{j+1,j} \, \delta^{\nu_{j+1,j}} \\ \vdots \\ \check{v}_{n,j} \, \delta^{\nu_{n,j}} \end{pmatrix} v_{jj}$$

where

$$\nu_{k,j} = \begin{cases} n_{jk} - n_{kk} & \text{for } 1 \le k \le j - 1\\ n_{jk} - n_{jj} & \text{for } j + 1 \le k \le n \end{cases}$$
$$\check{v}_{kj}(0) = \begin{cases} -\frac{a_{jk}(0)}{a_{kk}(0)} & \text{for } 1 \le k \le j - 1\\ \frac{a_{jk}(0)}{a_{jj}(0)} & \text{for } j + 1 \le k \le n \end{cases}$$

By normalization,

$$v_{jj} = \frac{1}{\sqrt{1 + \sum_{k \neq j} \check{v}_{k,j}^2 \, \delta^{2\nu_{k,j}}}} = 1 + \check{v}_{jj} \, \delta^{\nu_{j,j}}$$

where $\check{v}_{jj}, \, \nu_{j,j}$ are determined expanding

$$z \to (1+z)^{-1/2} = 1 - \frac{z}{2} + O(z^2)$$

as in (3.47). The proof is complete.

D The General Cauchy Inequality

We state a Cauchy inequality for the operatorial norm

$$|d_v F|_{B,A} = \max_{u \neq 0} \frac{|d_v F(u)|_B}{|u|_A}$$

of the first derivative $d_v F$, as a linear operator from A to B, of a given analytic map $F: A \to B$, where A, B are complex Banach spaces, with norms $|\cdot|_A, |\cdot|_B$. The present form is due to Pöschel [30], to whom we refer for the proof.

Lemma D.1 Let F be an analytic map from the open ball of radius r around v in A into B, such that $|F|_B \leq M$ into this ball. Then, the inequality

$$|d_v F|_{B,A} \le \frac{M}{r}$$

holds.

E Quantitative Implicit Function Theorem

Theorem E.1 Let F = f + g: $C^1(D^n_R(0), \mathbb{C}^n)$, where:

- (i) f is a diffeomorphism of $D_R^n(0)$ such that f(0) = 0 and Jacobian matrix ∂f non degenerate on $D_R^n(0)$;
- (*ii*) $\sup_{D_R(0)} \|\partial g\| \sup_{D_R(0)} \|(\partial f)^{-1}\| \le \frac{1}{2};$
- (*iii*) $\frac{\sup_{D_R(0)} |g| \sup_{D_R(0)} \|(\partial f)^{-1}\|}{r} \le \frac{1}{2}$, where $0 < r \le R$;

Then, there exists a unique $z_0 \in B_r^n(0)$ such that $F(z_0) = 0$.

F The Laplace Coefficients

The Laplace coefficients $b_{s,k}(\alpha)$, is defined as the k^{th} Fourier coefficients of the function $t \to (1 + \alpha^2 - 2\alpha \cos t)^{-s}$

$$b_{s,k}(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos kt}{(1+\alpha^2 - 2\alpha \cos t)^s} dt , \ \alpha \in \mathbb{C} , \ |\alpha| \neq 1 , \ 0 < s \in \mathbb{R} , \ k \in \mathbb{Z}$$
(F.1)

Lemma F.1 The Laplace coefficients are analytic of α , for $|\alpha| < 1$, and verify

(i)
$$b_{s,k}(-\alpha) = (-1)^k b_{s,k}(\alpha)$$
,

(*ii*)
$$b_{s,-k}(\alpha) = b_{s,k}(\alpha);$$

$$\begin{array}{ll} (iii) \ b_{s,k}(1/\alpha) = \alpha^{2s} \ b_{s,k}(\alpha); \\ (iv) \ b_{s,k+2}(\alpha) = \frac{k+1}{k+2-s} \left(\alpha + \frac{1}{\alpha}\right) \ b_{s,k+1}(\alpha) - \frac{k+s}{k+2-s} \ b_{s,k}(\alpha); \\ (v) \ if \ k \ge 0, \ b_{s,k}(\alpha) = \alpha^k \ \beta_{s,k}(\alpha) \ where \ \beta_{s,k}(\alpha) \ is \ an \ even \ function \ of \ \alpha, \ verifying \ density \$$

$$s(s+1)\dots(s+k-1) = s(s+1)\dots(s+k)$$

$$\beta_{s,k}(\alpha) = \frac{s(s+1)\cdots(s+k-1)}{k!} + s \frac{s(s+1)\cdots(s+k)}{(k+1)!} \alpha^2 + O(\alpha^4) ,$$
(F.2)

where

$$\frac{s(s+1)\cdots(s+k-1)}{k!} := 1 \quad if \quad k = 0 \; .$$

Notice that, by (iv), all the $b_{s,k}(\alpha)'s$ with $|k| \ge 2$ may be expressed as linear functions of $b_{s,0}(\alpha)$, $b_{s,1}(\alpha)$.

Proof. Items $(i) \div (iv)$ are imediate consequences of (F.1); in particular, (iv) is found by integrating twice by parts. In order to prove (v), from (F.1), we introduce the hypergeometric series

$$\frac{1}{(1-w)^s} = \sum_{l \ge 0} \, \frac{s(s+1)\cdots(s+l-1)}{l!} \, w^l$$

with

$$\frac{s(s+1)\cdots(s+l-1)}{l!} \equiv 1 \quad \text{for} \quad l=0 \; .$$

The hypergeometric series is uniformly convergent in every closed disk inside the set $\{|w| < 1\}$, therefore, we may expand, for $\{|\alpha| \le r < 1\}$,

$$\frac{1}{(1+\alpha^2 - 2\alpha \cos t)^s} = \frac{1}{(1-\alpha e^{it})^s (1-\alpha e^{-it})^s}$$
$$= \sum_{l,j} \frac{s(s+1)\cdots(s+l-1)}{l!} \frac{s(s+1)\cdots(s+j-1)}{j!} \alpha^{i+j} e^{i(l-j)t}$$

hence, multiplying by $\cos kt$ and then integrating over $[0, 2\pi]$, we find

$$b_{s,k}(\alpha) = \alpha^k \sum_j \frac{s(s+1)\cdots(s+j+k-1)}{(j+k)!} \frac{s(s+1)\cdots(s+j-1)}{j!} \alpha^{2j} .$$
(F.3)

It follows, in particular, that the $b_{s,k}(\alpha)$'s are analytic for $|\alpha| < 1$, and (v) is obtained by truncation of (F.3).

Index of Notations

The references denote the number of the page of the first occurrence in the text.

 $\mathfrak{K}, \mathcal{K}: \mathbf{Kolmogorov sets}, 18, 66.$

KAM

Sets

N, Z, Q, R, C: usual number sets. $B_r^p(I): p$ -dimensional real ball centered at I, with radius r, 19 $D_r^p(I): p$ -dimensional complex disk centered at I, with radius r, 19 $\mathcal{I}_r: \text{complex } r$ -neighborhood of $\mathcal{I} \subset \mathbb{R}^p$, 19 $\mathcal{D}^{\bar{n},\hat{n}}: \text{generalized Diophantine set, 13}$ $\mathcal{D}^{\bar{n},\hat{n}}_{\gamma,\hat{\gamma};\tau}: (\gamma,\hat{\gamma};\tau)$ -generalized Diophantine set, 14 $\mathbb{T}^p: \text{real standard } n$ -dimensional torus, 19 $\mathbb{T}_s^p: \text{complex } s$ -neighborhood of \mathbb{T}^p , 19 $\mathbb{T}_{\mathbb{C}}^p: \text{complex standard } n$ -dimensional torus, 19 If $\mathcal{I} = \bar{\mathcal{I}} \times \hat{\mathcal{I}}, h: \mathcal{I} \to \mathbb{C}$ is analytic, and $\omega := \partial h, \bar{\omega} \text{ means } \partial_{\bar{l}}h, \hat{\omega} \text{ means } \partial_{\hat{l}}h$, when $I = (\bar{I}, \hat{I})$ is the generic element of $\mathcal{I} = \bar{\mathcal{I}} \times \hat{\mathcal{I}}, \text{ with } \bar{I} \in \bar{\mathcal{I}}, \hat{I} \in \hat{\mathcal{I}}, 20$; If \mathcal{I} is as before, $\omega: \quad \mathcal{I} \to \mathcal{O}$ is onto and $\nu \in \mathcal{O}, \bar{\omega}^{-1}(\nu)$ means the projection over the \bar{I} -coordinate of $\omega^{-1}(\nu)$ and $\hat{\omega}^{-1}(\nu)$ means the projection over the \hat{I} -coordinate, 20;

Differential Operators

D differential operator with respect to (I, φ) , 25; ∂, ∂^2 differential operators with respect to I, 20

Matrices

 $A^{[p,q]}, A^{[p,q]}_*$: submatrices of a matrix A, 19

Norms

a) for numbers |k|: 1-norm of $k \in \mathbb{Z}^p$, 19 $|(I, \varphi)|_{\mathcal{P}}$: \mathcal{P} -norm of $\mathcal{I}_{\rho} \times \mathbb{T}^n_s$, 22

b) for analytic functions

 $||f||_{r,s}$: Sup-Fourier norm of a real-analytic function f on $\mathcal{I}_r \times \mathbb{T}_s^p$, $\mathcal{I} \subset \mathbb{R}^p$ compact, 19

c) for Lipschitz functions $\mathcal{L}(f), \mathcal{L}_{+}(f), \mathcal{L}_{-}(f), \mathcal{L}_{\parallel \cdot \parallel}(f), \|f\|_{\rho, \mathcal{I}}^{\text{Lip}}$ Lipschitz norms, 20

d) for vector and matrix functions

If $\omega : \mathcal{I} \to \mathbb{R}^n$, $|\omega|$ means its operator norm when ω is seen as linear operator from $(\mathcal{I}, ||_1)$ to $(\mathbb{C}, ||)$ (corresponds to $|\omega|_{\infty} := \max |\omega_i|$, with $\omega_i i^{th}$ coordinate of ω), 20; If $U : \mathcal{I} \to \operatorname{Matr}(m \times n)$, ||U|| means its operator norm when U is seen as linear operator from $(\mathcal{I}, ||_1)$ to $(\mathbb{C}^m, ||_\infty)$ (corresponds to $|U|_\infty := \max |U_{ij}|$, with U_{ij} the entries of U), 20;

Celestial Mechanics

 $\mathcal{R}_{\mathbf{x}}, \mathcal{R}_{\mathbf{z}}$: elementary rotations, 66, 130; Elliptic elements, , 66, 123; $\pi_w, \alpha_w(u, v), 124, 125;$ $\mathcal{A}, \mathcal{A}_{\mathbf{G}}, 178.$

Domains and maps

 ϕ_{DP} : plane Delaunay–Poincaré map, 68; \mathcal{C}_* : 124; \mathcal{D}_*, Φ_* : Deprit action–angle map, 126; $\mathcal{D}_r, \Phi_r, \phi_r$: regularization of $(\mathcal{D}_*, \Phi_*, \phi_*)$, 137; Φ_{BD} : full reduction map from cartesian variables to \mathcal{D}_r , 137; $\phi_{\text{BD}} = \Phi_{\text{BD}}^{-1}$, 141; \mathcal{D}_{pr} : domain of regularized partially reduced Deprit variables, 143; $\phi_{\text{BD,pr}}$: full reduction map, from \mathcal{D}_{pr} to cartesian variables, 143.

References

- Abdullah, K., Albouy, A.: On a Strange Resonance Noticed by M. Herman. Regul. Chaotic. Dyn. 6, (2001) 421–432.
- [2] Arnol'd, V.I.: Proof of a Theorem by A. N. Kolmogorov on the Invariance of Quasi Periodic Motions under Small Perturbations of the Hamiltonian. Uspehi Mat. Nauk. 18, no 5(113), (1963) 13–40; Engl. transl. Russ. Math. Surv., (1963), 18 no 5, 9–36.
- [3] Arnol'd, V.I.: Small Denominators and Problems of Stability of Motion in Classical and Celestial Mechanics. Russ. Math. Surv. 18, (1963) 85–193.
- [4] Arnol'd, V.I.: Mathematical Methods of Classical Mechanics. Springer-Verlag (1989).
- [5] Arnol'd, V.I., Kozolov V.V., Neishtadt A.I. Mathematical Aspects of Classical and Celestial Mechanics. Encyclopedia of Mathematical Sciences. 3, Springer-Verlag, 3rd edition (2006).
- [6] Biasco, L.: Private communication.
- [7] Biasco, L., Chierchia, L. and Valdinoci, E.: *Elliptic Two-Dimensional Invariant Tori for the Planetary Three-Body Problem*. Arch. Rational Mech. Anal. **170**, (2003) 91–135.
- [8] Biasco, L., Chierchia, L. and Valdinoci, E.: N-Dimensional Elliptic Invariant Tori for the Planar N+1-Body Problem., SIAM Journal on Mathematical Analysis, 37, Number 5, (2006) 1560-1588.
- [9] Boigey, F.: Elimination des noeuds dans le probleme newtonien des quatre corps.
 27, Celestial Mechanics, (1982) 399-414.
- [10] Celletti, A., Chierchia, L.: KAM Stability and Celestial Mechanics. Memoirs of the AMS, 187(878): (2007)134 pp.
- [11] Chenciner, A.,: Intégration du problème de Kepler par la méthode de Hamilton-Jacobi: coordonées "action-angles" de Delaunay and coordonées de Poincaré.. Notes scientifiques et techniques du B.D.L. S026 (1988).
- [12] Chierchia, L.,: Kolmogorov's 1954 Paper on Nearly-Integrable Hamiltonian Systems. Regu. Chaotic Dyn., 13, Number 2, (2008) 130-139.
- [13] Chierchia, L.,: *Kolmogorov-Arnol'd-Moser (KAM) Theory*. Encyclopedia of Complexity and Systems Science. Springer, to appear.
- [14] Chierchia, L., Pusateri, F. Analytic Lagrangian tori for the planetary many body problem, Erg. Th. Dyn Sys., (2008) to appear.

- [15] Deprit, A. Elimination of the Nodes in Problems of n Bodies. Celestial Mechanics and Dynamical Astronomy, 30:(2), (1983) 181–195.
- [16] Evans, L.C., Gariepy, R.F. Measure Theory and Fine Properties of Functions. Studies in Advanced Mathematics. CRC Press. (1992).
- [17] Féjoz, J. Démonstration du 'théoreme d'Arnol'd' sur la stabilité du système planétaire (d'apres M. Herman). Ergodic Theory & Dynamical System, 24(5): (2004) 1521-1582; Version révisée de Janvier 2007: http://people.math.jussieu.fr/ fejoz/arnold.pdf
- [18] Herman M. R.: Démonstration du 'théoreme de V.I. Arnol'd. Séminaire de Systèmes Dynamiques et manuscripts, 1998.
- [19] Hofer, H., Zehnder, E. Symplectic Invariants and Hamiltonian Dynamics. Birkhäuser Verlag, 1994.
- [20] Laskar, J.: Les Variables de Poincaré et le developpement de la fonction perturbatrice. Notes scientifiques et techniques du B.D.L. S028 (1988).
- [21] Laskar, J., Robutel, P.: Stability of the Planetary Three Body Problem I. Expansion of the Planetary Hamiltonian. Celestial Mechanics and Dynamical Astronomy, 62, (1995) 193–217.
- [22] Kolmogorov, A.N.: On the Conservation of Conditionally Periodic Motions under Small Perturbations of the Hamiltonian. Dokl. akad. nauk. SSSR, 98, (1954) 527– 530. Engl. transl.: Stochastic Behavior in Classical and Quantum Hamiltonian Systems, Volta Memorial Conference, Como,1977, Lecture Notes in Physics, vol. 93, Springer, 1979, pp. 51–56.
- [23] Malige, J.,Robutel, P., F.,Laskar Partial Reduction in the n Body Problem Using the Angular Momentum Integral. Celestial Mechanics and Dynamical Astronomy, 84:(3), (2002) 283–316.
- [24] Moser, J. On Invariant Curves of Area-Preserving Mappings of an Annulus. Nachr. Akad. Wiss. Göttingen, Math.-Phys. Kl II (1962) 1–20.
- [25] Nekhoroshev, N.N.: An Exponential Estimate of the Time of Stability of nearlyintegrable Hamiltonian Systems I. Uspehi Mat Nauk, **32** (1977), 5–66; Russian Math. Surveys, **32**, (1977),1–65
- [26] Nekhoroshev, N.N.: An Exponential Estimate of the Time of Stability of nearlyintegrable Hamiltonian Systems II. Trudy Sem. Pertovsk., 5 (1979), 5–50; Topics in Modern Mathematics, Petrowskii Sem. No 5, O.A. Oleinik Ed., Consultant Bureau, New York, 1985.
- [27] Poincaré, H.: Lecons de mécanique céleste. Gauthier-Villars, Paris, 1905–1907.

- [28] Poincaré,H.: Les méthodes nouvelles de la mécanique céleste. Gauthier-Villars, Paris, 1892–1899.
- [29] Pöschel, J.: Integrability of Hamiltonian Systems on Cantor Sets. Communications on Pure and Applied Mathematics, 35, (1982) 653–695.
- [30] Pöschel, J.: Nekhoroshev Estimates for Quasi-convex Hamiltonian Systems. Math Z. 213, (1993) 187–216.
- [31] Pöschel, J.: A KAM Theorem for Some Nonlinear PDEs. Ann Scuola Norm. Sup. Pisa, Cl. Sci. 23, (1996) 119–148.
- [32] Pusateri,F.: Analytic KAM Tori for the Planetary (n+1)-Body Problem. Master's Thesis, Universitá degli Studi Roma Tre, October 2006. www.mat.uniroma3.it/users/chierchia/TESI
- [33] Robutel, P.: Stability of the Planetary Three Body Problem II. KAM Theory and Existence of Quasiperiodic Motions. Celestial Mechanics and Dynamical Astronomy, 62, (1995) 219–261.
- [34] Rüssmann, H.: Invariant Tori in Non-Degenerate Nearly Integrable Hamiltonian Systems. Regul. Chaotic. Dyn., 2, (6) (2001) 119–203.
- [35] Salamon, D.: The Kolmogorov-Arnol'd-Moser theorem Math. Phys. Electron. J. 10 (2004), Paper 3, 37 pp. (electronic).
- [36] Siegel, C.L., Moser J.K.: Lectures on Celestial Mechanics. Reprint of the 1971 edition, Springer (1995).