Small divisors in dynamics

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Abstract

This thesis will deal with small divisors in dynamics: in the introduction we talk briefly about classical examples of dynamical systems where small divisors gives obstructions and the techniques used to avoid these problems. Then, we study in details the following problems: linearization of Gevrey circle diffeomorphisms and the topology of Diophantine sets.

We explain at first the problem of the linearization:

The dynamics on the circle is characterized by an invariant, that is the rotation number. The simplest diffeomorphisms of the circle are the rotations, whose dynamics is clear.

Moreover, when the rotation number of the diffeomorphism is irrational, and the logarithm of the derivative of the diffeomorphism is of bounded variation, by a classical theorem due to Denjoy, the diffeomorphism is conjugate to a rotation by an homeomorphism. So, the problem is to study the regularity of the diffeomorphism that conjugate to a rotation. This problem depends on the arithmetic features of the rotation number, and it is completely studied in the case in which the diffeomorphism is smooth or analytic.

Our results deals with an ultra-differentiable class of functions, the Gevrey diffeomorphisms. They satisfy similar estimates of analytic functions, but they are less rigid with respect to the analytic ones (for example, there exist Gevrey functions with compact support). So, we show that Gevrey diffeomorphisms of the circle with the rotation number that satisfy a Diophantine condition, are conjugate to a rotation by a Gevrey diffeomorphism. Moreover, we show that, under an arithmetic condition that is weaker then Diophantine, if a Gevrey diffeomorphism is C^1 conjugate to a rotation, then it is C^∞ conjugate to a rotation.

Now we explain the second problem:

Diophantine sets[1] arise in a natural way in dynamics. From a topological point of view, they are closed and totally disconnected. So, the only non-trivial topological question is about isolated points, i.e. if there exist isolated points in these sets.

Our aim is to show that, for any Diophantine number, there exists an equivalent number (equivalent in the sense of continued fractions) that is isolated in another Diophantine set. Moreover, we show that, for large parameters, almost all these sets are Cantor sets (almost all these sets have not isolated points).

[1]Diophantine sets are sets of numbers that are "badly" approximable by rationals.

1 Introduction

In the first part of the introduction we give briefly some classical example of dynamical system in which small divisors gives obstruction.

Then, we concetrate on the topics that we study in the thesis, introducing the results that we prove.

1.1 Stability and linearization: an overview

1.1.1 A brief historical survey

The simplest dynamical systems are the integrable ones, where we have a complete description of the qualitative behavior of the motion.

However, in general, arbitrary perturbations of integrable Hamiltonian systems are no more integrable. The first who studied dynamical systems form this viewpoint was Poincaré:

In the last decade of the nineteenth century, Poincaré published his "Les méthodes nouvelles de la mécanique céleste", where the concept of phase space was introduced for the first time, and the interest to find individual solutions of the motion was replaced by the aim to understand the features of all the possible invariants curves (qualitative description of the motion).

The main object that contributed to the foundation of dynamical systems is of course Celestial mechanics and, in particular, the "n-body problem" that started with the work of Newton.

So, in this Hamiltonian setting, Poincaré proved his triviality's Theorems, which show that in general, Hamiltonian systems are not integrable.

The main problem is that, to develop the Hamiltonian in power series with respect to the perturbative parameter and with the coefficients that depends only on the action variables, composing step by step by symplectic change of variables, one has to solve an "homological equation" that gives obstruction: in fact, the convergence of these series is obstructed by the so called "small divisors". The first work to overcome to this problem is due to Siegel in 1942 (see [69]).

1.1.2 Iteration of analytic maps

The problem is the stability around fixed points of holomorphic germs that, up to the trivial case when the fixed point is attracting, is equivalent to conjugate an holomorphic map of the disk which has an elliptic fixed point in the origin to a rotation, that is of course the simplest dynamic. So, to overcome small divisors problem, Siegel used for the first times Diophantine sets in dynamics.

These sets are defined as follows:

Let $\gamma > 0$, $\tau \geq 1$, and define:

$$D_{\gamma,\tau} := \left\{ \alpha \in \mathbb{R} : |q\alpha - p| \geq \frac{\gamma}{q^\tau} \; \forall p \in \mathbb{Z}, q \in \mathbb{N} \right\}.$$

Then, α is Diophantine if $\alpha \in \bigcup_{\gamma>0,\tau>1} D_{\gamma,\tau}$.

In particular, a Diophantine number is a number that is not too close to rationals.

We explain briefly the problem of stability and the result of Siegel:

Let

$$f(z) := \sum_{k\geq 1} a_k z^k$$

be analytic in $B_R(0)$ for some $R > 0$ (with $B_R(0)$ the ball of radius R centered in 0). The fixed point 0 is called stable if, there exist $0 < r_0 < r_1 \leq R$ such that, for all $n \in \mathbb{N}$:

$$f^n(B_{r_0}(0)) \subseteq B_{r_1}(0),$$

with f^n the composition of f n times.

By Schwarz's lemma, it is clear that, for $|a_1| < 1$, the fixed point is stable. Moreover, for $|a_1| > 1$, the fixed point is unstable. So, we can assume that $|a_1| = 1$.

Then, the stability is equivalent to solve the homological equation:

$$\phi(a_1\zeta) = f(\phi(\zeta))$$

for ϕ that is an holomorphic germ at 0.

If a_1 is an n^{th} root of unity, then: the fixed point is stable if and only if $f^{n-1} = id$.

So, we can assume that $a_1 = e^{2\pi i\alpha}$, with α irrational. In this case, a formal solution of the homological equation always exists:

In fact, suppose that ϕ satisfies the equation above and write ϕ as:

$$\phi(\zeta) = \zeta + \sum_{k\geq 2} b_k \zeta^k.$$

So, we have:

$$\sum_{k\geq 2} b_k(a_1^k - a_1)\zeta^k = \sum_{k\geq 2} a_k \left(\zeta + \sum_{i\geq 2} b_i \zeta^i \right)^k.$$

By our assumption ($a_1 = e^{2\pi i\alpha}$ with α irrational), for every $k \in \mathbb{N}$, $a_1^k \neq a_1$. In particular, for all k, the equation on b_k depends only on the coefficients b_h with

$h < k$ (and the coefficients of f). So, we can define the coefficients of ϕ iteratively such that ϕ is a formal solution of the homological equation.

In particular, the question is about the regularity of the solution. Siegel proved the following:

Theorem (Siegel, [69]) *Let $f(z) := e^{2\pi i\alpha}z + O(z^2)$ be an holomorphic germ, and suppose that α is Diophantine. Then, it is linearizable in 0, i.e. there exists an holomorphic germ $h(z) = z + O(z^2)$ such that:*

$$f(h(z)) = h(e^{2\pi i\alpha}z).$$

As noted above, it is always possible to find a formal solution of the equation for the linearization, so the arithmetic condition gives the regularity of the solution (in this case, the convergence of the formal power series).

Note also, that this condition is verified for almost all real numbers. In fact, the following simple Lemma holds:

Lemma *Let μ be the Lebesgue measure on $\mathbb{R}$. Then, for all $N \in \mathbb{N}$, $\mu(D) \cap (-N.N) = 2N$.*

Proof Let $\gamma > 0, \tau > 1$. If $\alpha \notin D_{\gamma,\tau}$, then there exists $p \in \mathbb{Z}, q \in \mathbb{N}$ such that $\alpha \in (\frac{p}{q} - \frac{\gamma}{q^{\tau+1}}, \frac{p}{q} + \frac{\gamma}{q^{\tau+1}})$. In particular, for all $N \in \mathbb{N}$:

$$D^c_{\gamma,\tau} \cap (-N, N) \subseteq \bigcup_{q\in\mathbb{N}} \bigcup_{|p|\leq Nq} \left(\frac{p}{q} - \frac{\gamma}{q^{\tau+1}}, \frac{p}{q} + \frac{\gamma}{q^{\tau+1}} \right).$$

Then:

$$\mu(D^c_{\gamma,\tau} \cap (-N, N)) \leq \sum_{q\in\mathbb{N}} \sum_{|p|\leq Nq} \frac{2\gamma}{q^{\tau+1}} \leq (2N + 1)2\gamma \sum_{q\in\mathbb{N}} \frac{1}{q^\tau} < \infty$$

because of $\tau > 1$. So, if γ is small, also $\mu(D^c_{\gamma,\tau} \cap (-N, N))$ is small. Then, taking the union over all γ, τ we get that the set of non Diophantine points has measure zero. ∎

However, the complement of these sets are G_δ dense, so they are topologically non-trivial.

1.1.3 Diffeomorphisms of the circle

Let f be a diffeomorphism of the circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. The problem is to find a diffeomorphism h that conjugates f to a rotation, i.e.:

$$h \circ f \circ h^{-1} = R_\alpha, \quad \text{with} \quad R_\alpha(x) := x + \alpha, \tag{1}$$

and to study the regularity of h, according to the regularity of f.

Let [2] $\mathrm{Diff}^{\mathrm{r}}_{+}(\mathbb{T})$ be the set of C^r orientation preserving diffeomorphisms of the circle. The dynamics induced by f on the circle is characterized by the rotation number: Let $g \in \mathrm{Diff}^0_+(\mathbb{T})$, $\bar{g}$ a lift of g over $\mathbb{R}$; the rotation number of g is defined as[3]:

$$\rho(g) := \lim \frac{\bar{g}^n}{n} \pmod 1. \tag{2}$$

In the same way, if f is an homeomorphism over $\mathbb{R}$ such that, for $x \in \mathbb{R}$, $f(x+1) = f(x) + 1$, we define the rotation number of f as:

$$\rho(f) := \lim \frac{f^n}{n}. \tag{3}$$

A homeomorphism $g \in \mathrm{Diff}^0_+(\mathbb{T})$ has a periodic orbit if and only if the rotation number is rational and, in this case, it is topologically conjugate to the rotation $R_{\rho(g)}$ if and only if $g^q = R_p$, with $\rho(g) = \frac{p}{q}$.

Moreover, suppose that: $g \in \mathrm{Diff}^{\mathrm{r}}_{+}(\mathbb{T})$ with $r \geq 1$, $\rho(g)$ is rational and g is conjugate to a rotation. Then, the diffeomorphism h that conjugates g to a rotation (unique, up to composition with a rotation) is of class $C^r(\mathbb{T})$ (see [34], p.25). In fact, it is easy to check that, in this case, the conjugating diffeomorphism is given by:

$$h := \frac{1}{q} \sum_{i=0}^{q-1} \left(g^i - i\frac{p}{q} \right). \tag{4}$$

In particular, in such a case, $h \in C^r(\mathbb{T})$ and there is no loss of regularity. However, in general, diffeomorphisms with rational rotation number are not conjugate to a rotation (see [34], p. 31).

If the rotation number is irrational, g is topologically semi-conjugate to the rotation $R_{\rho(g)}$ (i.e. there exists $h \in C(\mathbb{T})$ that is non-decreasing and surjective, such that: $g \circ h = R_{\rho(g)} \circ h$).

The semi-conjugacy is a conjugacy if and only if the support of the unique invariant probability measure with respect to g is $\mathbb{T}$. Moreover, if we assume $g \in \mathrm{Diff}^1_+(\mathbb{T})$ and Dg of bounded variation, by a theorem by Denjoy (for a simple proof of Denjoy's Theorem, see [72]), g is conjugate to a rotation by an homeomorphism h, that is unique up to composition with rotation.

For more regularity on h, we have to assume also some arithmetical condition on $\rho(g)$, to overcome the so-called small divisors problem.

[2] With $r \geq 1$ or $r \in \{0, +\infty, \omega\}$, $\mathrm{Diff}^{\omega}(\mathbb{T})$ is the set of analytic diffeomorphisms of the circle.

[3] Let f be an homeomorphism of $\mathbb{R}$. For $n > 0$, $f^n = f \circ ... \circ f$ denote the composition of f n times. $f^0 := id$. For $n < 0$, $f^n := (f^{-1})^{-n}$.

We recall the definition given above for the Diophantine sets.

Let $\gamma > 0, \beta \geq 0$, and define:

$$\mathtt{D}_{\gamma,\beta} := \left\{ \alpha \in \mathbb{R} : |\alpha q - p| \geq \frac{\gamma}{q^{\beta+1}} \quad \forall q \in \mathbb{N}, p \in \mathbb{Z} \right\}, \tag{5}$$

$$\mathtt{D}_{\beta} := \bigcup_{\gamma>0} \mathtt{D}_{\gamma,\beta}, \quad D := \bigcup_{\beta\geq 0} \mathtt{D}_{\beta}. \tag{6}$$

We say that α is Diophantine if $\alpha \in D$. Moreover, we define $D^k(\mathbb{T})$ with $k \geq 1$, as the set of orientation preserving C^k diffeomorphisms of $\mathbb{R}$ that commutes with $T(x) := x + 1$.

For $a \in \mathbb{R}$, define $T_a(x) := x + a$ $(x \in \mathbb{R})$.

The first result related to local conjugacy of analytic circle diffeomorphism was given by Arnold, who proved, by a KAM scheme, that analytic circle diffeomorphisms close enough to a rotation and with Diophantine rotation number are conjugate to a rotation by an analytic diffeomorphism.

Then, the first global Theorem (i.e. without assuming that the diffeomorphism is close enough to a rotation) was proved by Herman:

Theorem (Herman, [34]) *Let $f \in D^k(\mathbb{T})$, $k \geq 3$, and suppose that $\alpha := \rho(f) \in \bigcap_{\beta>0} \mathtt{D}_{\beta}$. Then, f is conjugate to T_α by a diffeomorphism of class $C^{k-1-\epsilon}$ for every $\epsilon > 0$.*

Yoccoz generalized Herman's Theorem:

Theorem (Yoccoz, [73]) *Let $f \in D^k(\mathbb{T})$, $k \geq 3$, and suppose that $\alpha := \rho(f) \in \mathtt{D}_{\beta}$ with $k > 2\beta + 1$. Then, f is conjugate to T_α by a diffeomorphism of class $C^{k-1-\epsilon-\beta}$ for all $\epsilon > 0$.*

As a corollary of the Herman-Yoccoz Theorem, by reducing to the case in which f is close to a rotation, and then, using the local Theorem of Arnold, one has:[4]

Theorem (Herman, Yoccoz, [34], [73]) *Let $f \in D^\omega(\mathbb{T})$, $\rho(f) \in \mathtt{D}$. Then, the diffeomorphism that conjugates f to a rotation is analytic.*

The assumption $k \geq 3$ is not really necessary, but it is needed to use Schwarzian derivative to avoid technical difficulties. The first results to overcome to this problem were given by Khanin and Sinai who, in particular, proved the following:

Theorem (Khanin, Sinai, [72]) *Let $f(x) \in D^{2+\nu}(\mathbb{T})$, $\nu > 0$, $\alpha := \rho(f) = [a_1, a_2, ...]$ the continued fraction expansion of the rotation number.*

- *Suppose there exists a constant $K > 0$ such that $|a_n| \leq K$. Then, f is conjugate to T_α by a diffeomorphism of class $C^{1+\nu}$.*

[4] The arithmetical condition of this Theorem is not the optimal one (for the optimal arithmetical condition, see (Yoccoz, [74])).

- *Suppose there exists a constant $\delta > 0$ such that $|a_n| \leq n^\delta$. Then, f is conjugate to T_α by a diffeomorphism of class $C^{1+\nu-\epsilon}$ for all $\epsilon > 0$.*

Finally, Katznelson and Ornstein generalized the Theorem of Khanin and Sinai:

Theorem (Katznelson, Ornstein, [40]) *Let $f \in D^k(\mathbb{T})$, $k > \beta + 2$, and suppose that $\alpha := \rho(f) \in \mathtt{D}_\beta$. Then, f is conjugate to T_α by a diffeomorphism $h \in D^{k-1-\epsilon-\beta}(\mathbb{T})$ for all $\epsilon > 0$.*

The arithmetical conditions of this Theorem are the optimal ones (compare the Appendix in [40]).

1.1.4 Vector fields on the Torus

Another example, is that induced by a vector field on $\mathbb{T}^n$. If the vector field is constant constant one:

$$N_\omega(\theta) := \sum_{j=1}^{n} \omega_j \frac{\partial}{\partial \theta_j},$$

with $\omega := (\omega_1, ..., \omega_n) \in \mathbb{R}^n$, the dynamical system is very clear:

it depends only on the arithmetical nature of ω. In fact, the flow is:

$$\phi^t_\omega(\theta) = \theta + t\bar{\omega},$$

with $\bar{\omega}$ the equivalence class of ω in $\mathbb{T}^n$. If ω is non-resonant (i.e. $\omega \cdot q \neq 0$ for all $q \in \mathbb{Z}^n - \{0\}$), then each orbit is minimal on the torus. More in general, if ω is $n - d$ resonant, with $1 \leq d \leq n - 1$ (so, d is the dimension of the smallest rational subspace of $\mathbb{R}^n$ that contains ω), each orbit is minimal on a d-dimensional torus and, in particular, the flow induce a foliation of $\mathbb{T}^n$ by d-dimensional tori.

In our perturbative setting, consider a vector field that is a perturbation of N_ω. So, let:

$$X = N_\omega + P,$$

with ω that is non resonant and P small.

In it natural to try to conjugate X to a rotation, i.e. to search a diffeomorphism ϕ such that:

$$\phi^* X = N_\omega + N_\lambda,$$

with λ small. However, in general, it is not possible: in fact, for invariant measure μ for the flow, we can associate an invariant that is:

$$\int_{\mathbb{T}^n} X d\mu.$$

Let $Rot(X)$ be the set of such invariants, and note that $Rot(X)$ is invariant by conjugacy. In particular, a necessary condition to conjugate X to a rotation is that $Rot(X)$ contains only a constant while, in general, it is not true.

However, Arnold proved the following:

Theorem (Arnold, [1]) *Assume X is real-analytic and ω is diophantine. Then, if P is sufficiently close to zero, there exists a real-analytic diffeomorphism ϕ close to the identity and $\lambda \in \mathbb{R}^n$ close to zero such that:*

$$\phi^*(X + N_\lambda) = N_\omega.$$

The arithmetic condition ω Diophantine is not the optimal one: in fact, Rüssmann proved the same Theorem under the Bruno-Rüssmann condition (that is weaker than Diophantine).

1.1.5 Twist maps

Let $\mathbb{A} := \mathbb{T} \times \mathbb{R}$ be the annulus.

Let $F = F(x, y)$ be a diffeomorphism of the annulus. Write $F(x, y) = (X(x, y), Y(x, y))$. F is an area preserving monotone twist map of the annulus if the following conditions are satisfied:

1. $\lim_{y\to+\infty} Y(x, y) = +\infty$, $\lim_{y\to-\infty} Y(x, y) = -\infty$ for all $x \in \mathbb{T}$ (i.e. F preserves the boundary).
2. $\frac{\partial F}{\partial y} > 0$, that is the positive monotone twist condition.
3. $\det DF = 1$, so F is area preserving.
4. Let f be a lift of F over $\mathbb{R}^2$, then $f(x + 1, y) = f(x, y) + 1$.

In particular, the fact that F is area preserving implies that F preserves the standard symplectic form, so $dY \wedge dX = dy \wedge dx$.

The monotone twist condition simply tells us that F moves points in the upper part faster than on the lower boundary.

Moreover, the twist condition implies that the map: $(x, y) \to (x, X)$ is an embedding of the annulus in $\mathbb{R}^2$.

The fact that F preserves the boundary and the standard symplectic form, implies that the flux of F is zero and, in particular, F has the intersection property, so the intersection of a topologically non trivial closed curve of the annulus with its image is non empty.

Moreover, by these conditions, it is easy to show that F is also exact symplect, so there exists a generating function $S(x, X)$ such that:

$$dS = YdX - ydx.$$

In particular:

$$\begin{cases} Y = \frac{\partial S}{\partial X} \\ y = -\frac{\partial S}{\partial x} \end{cases} \tag{7}$$

Monotone twist maps are not an artificial construction. Let us show a simple example:

Consider a mechanical system:

$$\ddot{x} = -V'(x),$$

where V is a periodic potential. In particular, the Hamiltonian is:

$$H(x, y) = \frac{y^2}{2} + V(x),$$

and the solutions solve the Hamilton's equations:

$$\begin{cases} \dot{x} = H_y(x, y) \\ \dot{y} = -H_x(x, y) \end{cases} \tag{8}$$

Then:

$$\frac{\partial x(t, x_0, y_0)}{\partial y_0} = \int_0^t \frac{\partial \dot{x}(s, x_0, y_0)}{\partial y_0} ds = \int_0^t \frac{\partial y(s, x_0, y_0)}{\partial y_0} ds > 0,$$

if t is small enough. In particular, the time-t map ϕ_H^t is a twist map.

On the other way, as proved by Moser (see [54]), every twist map can be viewed as a time-1 map of the flux of an Hamiltonian system that satisfy the Legendre condition.

So, it is quite natural to think twist maps as Poincaré sections.

The simplest example of twist map of the annulus is the map whose lift is:

$$f(x, y) = (x + y, y), \ x, y \in \mathbb{R}.$$

In fact, in this case, the map is integrable: each circle $\mathbb{T} \times \{y\}$ is invariant, and the dynamics on this circle is simply a rotation.

However, as we said at the beginning, the feature to be integrable in general is not stable under small perturbations.

The simplest and most studied non integrable twist map is the standard family

$$F_\epsilon(x,y) = (x + y - \frac{\epsilon}{2\pi}\sin(2\pi x), y - \frac{\epsilon}{2\pi}\sin(2\pi x)).$$

However, also in this case, many natural questions are open: for example it is known that, there exists $\epsilon_0 > 0$ such that, for all $\epsilon > \epsilon_0$ F_ϵ has not invariant curves (see [45]), but we don't know what is the last invariant curve and, if it is isolated.

For every invariant curve, there is associated the rotation number, that characterize the dynamics on this circle.

By KAM theory, one should expect that invariant curves with rotation number that is more distant to the rationals (in terms of Diophantine condition), persist with bigger perturbations.

In the case of the standard family, numerical evidences suggest that the last invariant curve is the curve with rotation number that is the golden ratio, that is also the Diophantine number most distant to the rationals.

In particular, it is quite natural to study the topology of Diophantine numbers, because it should be related to the topology of the invariant curves: for example, the golden ratio is the only point in $D_{\gamma,1}$ for γ sufficiently large (and smaller than the best γ for the golden ratio), in particular, for these parameters, it is isolated.

The set of topologically nontrivial invariant curves is closed and, it is continuous with respect the rotation number.

As we will prove later, there are many examples of isolated points in Diophantine sets.

On the other hand, it seems that, until today, there are not examples of isolated invariant curves for the standard family, or more in general, for twist maps.

So, the hope is that, a clearer view of the topology of these sets may help also to understand better the dynamical point of view.

The other natural question is the regularity of the invariant curve. A classical Theorem due to Birkhoff tell us that invariant curves are at least Lipschitz:

Theorem (Birkhoff, [46]) *Let γ be a topologically non-trivial invariant curve for a twist map of the annulus. Then, γ is the graph of a Lipschitz function.*

Moreover, the regularity of the invariant curves and its topology are not two separated problems. In fact, sufficiently smooth curves with Diophantine rotation number are never isolated (note that the only non trivial topological question is if the invariant curve is isolated). In fact, the following holds:

Herman's last Geometric Theorem ([26]) *Let F be a smooth diffeomorphism of the annulus having the intersection property. Then given a smooth curve Γ invariant by F on which the rotation number of F is Diophantine, it holds that is accumulated by a positive measure set of smooth invariant curves on which F is smoothly conjugated to rotation maps.*

We recall that a map of the annulus has the intersection property if, every topologically non trivial closed curve has non empty intersection with its image. As we remarked previously, the twist maps have this property.

It is interesting to note that, as proved in [26], Herman's last Geometric Theorem implies Siegel's Theorem. In particular, we have two different proves:

The first one, due to Siegel, is on the regularity of the formal diffeomorphism that conjugate to a rotation, and the second one, due to Herman, from a topological point of view.

In [51], Moser proved the same Theorem in the smooth category, providing that ω is Diophantine. There is also a proof of Bounemoura, based on rational approximations (see [9], [10]).

1.1.6 The KAM theorem

In the perturbative Hamiltonian setting, the first result around the stability of invariant Tori, was given by Kolmogorov in 1954, in the International Congress of Mathematicians in Amsterdam, following a Newton like scheme. We fix the notations:

Let B a ball in $\mathbb{R}^n$, $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$, and consider a real analytic Hamiltonian:

$$(y, x, \epsilon) \in B \times \mathbb{T}^n \times (-\epsilon_0, \epsilon_0) \to H(y, x; \epsilon) \in \mathbb{R},$$

with the phase space endowed of the standard symplectic form:

$$dy \wedge dx = \sum_i dy_i \wedge dx_i.$$

The flow is generated by the Hamiltonian equations:

$$\begin{cases} \dot{x} = H_y(y, x; \epsilon) \\ \dot{y} = -H_x(y, x; \epsilon) \end{cases}, \tag{9}$$

where H_x, H_y are the partial derivatives of H.

If we have a Lagrangian invariant Torus, we can always suppose that, up to symplectic changes of coordinates, the Hamiltonian is of the form:

$$K(y, x) = E + \omega \cdot y + O(|y|^2).$$

In particular, in this form, $\mathbb{T}^n \times \{0\}$ is an invariant Torus. Then, Kolmogorov proved that, if ω is Diophantine, this Torus is stable under small perturbations.

Theorem (Kolmogorov, [41]) *Let $H(y, x; \epsilon)$ be a real analyitic Hamiltonian as above, and suppose that there exists $E \in \mathbb{R}$, $\omega \in \mathbb{R}^n$ such that, for ϵ=0:*

$$H(y, x, 0) = E + \omega \cdot y + Q(y, x),$$

with $Q(y,x) = O(|y|^2)$, *and the non-degeneracy condition*[5]

$$\det(< Q_{yy}(0,.) >) \neq 0.$$

Suppose also that there exist $\gamma, \tau > 0$ *such that* ω *satisfies the Diophantine condition:*

$$|\omega \cdot m| \geq \frac{\gamma}{|m|^\tau} \quad \forall m \in \mathbb{Z}^n \text{ with } m \neq 0.$$

Then, there exists $\epsilon_* \leq \epsilon_0$, *a ball* $B_* \subseteq B$ *with the center in the origin and a real analytic symplectic transformation:*

$$\phi_* : B_* \times \mathbb{T}^n \to B \times \mathbb{T}^n$$

analytic in $\epsilon \in (-\epsilon_*, \epsilon_*)$ *such that* $\phi_* = id.$ *for* $\epsilon = 0$ *and, for* $\epsilon < \epsilon_*$

$$H \circ \phi_*(y,x) = E_*(\epsilon) + \omega \cdot y + O(|y|^2).$$

So, Kolmogorov's Theorem state that, an invariant Lagrangian Torus with Diophantine frequency for an Hamiltonian that satisfy a non degeneracy condition, is persistent under small perturbations.

However, the first complete proof of the Theorem was due to Arnold (see [1]): in fact, in the paper of Kolmogorov missed the last part of the estimates for the convergence. Then, the first Theorem of the persistence of invariant Tori for smooth Hamiltonians was due to Moser, under the same Diophantine hypothesis.

The usuals KAM proofs are based on two steps:

The first one follows a Newton like scheme, in which we reduce the perturbation from order ϵ to order ϵ^2 solving the homological equation and by fixing the frequence by a classical implicit funtion Theorem. In this step, the effect of small divisors in the analytic case, is to reduce the domain of analicity.

In the second step, that is the iterative step, one has to control the growth of the constants.

In the analytic case, the Diophantine condition is not the optimal one:

For the first part of the standard KAM theorem, one has to solve an homological equation of the form:

$$D_\omega f = g - < g >,$$

with g a real analytic function on $\mathbb{T}^n$, $\omega = (\omega_1, ..., \omega_n) \in \mathbb{R}^n$ and D_ω the operator:

$$D_\omega = \omega_1 \frac{\partial}{\partial x_1} + ... + \omega_n \frac{\partial}{\partial x_n}.$$

[5]For a function $f \in C(\mathbb{T}^n, \mathbb{R})$, $< f >$ denotes its average.

To see where small divisors appear, let us develop g in Fourier series:

$$g(x) = \sum_{\nu \in \mathbb{Z}^n} \hat{g}_\nu e^{2\pi i x \cdot \nu}.$$

If f is a solution of the equation above, writing f in Fourier series:

$$f(x) = \sum_{\nu \in \mathbb{Z}^n} \hat{f}_\nu e^{2\pi i x \cdot \nu},$$

for $\nu \neq 0$, the following equation holds:

$$\hat{f}_\nu = \frac{\hat{g}_\nu}{\omega \cdot 2\pi i \nu}.$$

In particular, when ω is non-resonant (i.e. $\omega \cdot \nu \neq 0$ if $\nu \neq 0$), a formal solution always exists.

However, if ω is only non-resonant, then $\omega \cdot \nu$ can be arbitrary small. So, in order to avoid this problem, one has to impose arithmetic conditions on the frequencies.

The optimal arithmetical condition in the analytic setting to solve this equation is the Rüssmann condition, that is weaker than Diophantine.

However, to have the controll for the convergence in the iteration step, one has to assume a more stringent condition, the Bruno-Rüssmann condition, that is anyway weaker than Diophantine.

The Theorem with this arithmetic conditions was proved by Rüssmann in [60].

However, it is not clear what are the best arithmetic conditions, that are known only for the case of circle diffeomorphisms (so, for an Hamiltonian systems of dimension $n = 2$, by Poincare section). For more details about the optimal condition, see [74].

Instead, in the smooth case, the optimal condition to solve the homological equation is the Diophantine condition. So, in the smooth category, it should be the optimal one.

Finally, in [11], Bounemoura and Fejoz proved the persistence of invariant Tori for Hamiltonians in the Gevrey class:

This is a class of ultra-differentiable functions, where we have estimates of the growth of the derivatives similar to the analytic functions, but with much less rigidity with respect to the analytic ones: for example, in the Gevrey class there exists functions with compact support.

The other positive feature in dynamics is that, not only invariant Tori with this regularity persists under small perturbations (with the same regularity), but also that the arithmetic condition is weaker than Diophantine (that is the condition in the smooth class).

In fact, in [11], Bounemoura and Fejoz proved the Theorem under an α-Rüssmann condition (that is weaker than Diophantine).

For other proof of KAM theorem, see for example [12], [16], [17], [57], [58], [62], [66], [65], [23], [29].

1.1.7 Schrödinger operators and cocycles

Finally, we give a classical example about the reducibility of a one dimensional Schrödinger operator with quasiperiodic potential to a system with constant coefficients.

Consider the Schrödinger operator:

$$Ly := -\ddot{y} + q(\omega t)y = Ey,$$

where q is a real-analytic quasi-periodic potential in a neighbourhood of $|Imz| < r$, $\omega \in \mathbb{T}^d$.

The problem is to extend to these nonlinear equations the Floquet Theory, i.e. to search solutions to the form:

$$y(t) = e^{kt}(p_1(t) + tp_2(t)),$$

with k that is a constant, and p_1, p_2 quasiperiodic with frequency ω or $\frac{\omega}{2}$.

Let $X = (y, \dot{y})^t$. Then:

$$\dot{X} = \begin{pmatrix} 0 & 1 \\ q(\omega t) - E & 0 \end{pmatrix} X.$$

Let:

$$\mathcal{M} := \left\{ \frac{\omega \cdot n}{2} : n \in \mathbb{Z}^d \right\},$$

and let $\rho = \rho(E)$ be the rotation number (for more information about the rotation number, see ([38]). The rotation number ρ is said to be Diophantine with respect to $\mathcal{M}$, if there exist $K, \sigma > 0$ such that, $\forall n \in \mathbb{Z}^d$ with $n \neq 0$:

$$\left| \rho - \frac{n \cdot \omega}{2} \right| \geq \frac{1}{K|n|^\sigma}$$

It is well known that, if E is in the resolvent of L, then the system is reducible (see [53]). The first result for E that is not in the resolvent is due to Dinaburg and Sinai, who proved that, there exists $E_0 \in \mathbb{R}$ and a set $\mathtt{R} \subseteq (E_0, +\infty)$ for which the system is reducible to a constant one. The set for which they proved the Theorem is not of full measure and such that $\rho(E)$ is Diophantine for all $E \in \mathtt{R}$ (see [22]).

Then, Moser and Pöschel extended the Theorem to a set such that $\rho(E)$ is rational is E is in this set (see [53]).

Then, Eliasson proved the following:

Theorem (Eliasson, [24]) *There exists a constant $C = C(,r)$ such that if:*

$$E_0(s) = \begin{cases} \left(\frac{s}{C}\right)^2 \text{ if } s \geq C \\ -\infty \text{ if } s < C \end{cases}, \tag{10}$$

then the following hold for $E > E_0(|q|_r)$:

- *If $\rho(E)$ is diophantine or rational, then there exists a matrix $A = \Lambda(E)$ in $sl(2,\mathbb{R})$ and an analytic matrix valued function $Y : \mathbb{T}^d \to GL(2,\mathbb{R})$, also depending on E, such that*

$$X(t) = Y(\frac{\omega}{2}t)e^{At}.$$

- *If p(E) is neither diophantine nor rational, then:*

$$\liminf_{|t|\to+\infty} |X(t) - X(0)| < \frac{1}{2}|X(0)|,$$

$$\lim_{|t|\to+\infty} \frac{|X(t)|}{t} = 0.$$

The idea of the proof is the following: up to a symplectic changes of variable we can suppose that the equation is:

$$\dot{X} = (A_1 + F_1)X,$$

with A_1 constant, F_1 small.

Then, we want to transform $A_1 + F_1$ to $A_2 + F_2$ with F_2 smaller then F_1. So, the idea is to search a transformation that is not close to the identity, but to sum exponential e^{Bt}. Finally. with the help of Diophantine condition, one can prove that, iterating this process, the composition of these transformations converges on compact subsets of $\mathbb{R}$, and so proving the Theorem (with the Diophantine condition, we need a transformation that is not close to the identity only for a finite number of steps).

Now, we reformulate the problem in terms of cocycles in $SL(2,\mathbb{R})$:

Let $\alpha \in \mathbb{R}$, $A \in C^r(\mathbb{T}, SL(2,\mathbb{R}))$. Then, a cocycle (α, A) is the skew-linear product:

$$\begin{aligned}(\alpha, A) :\ & \mathbb{T} \times \mathbb{R}^2 \to \mathbb{T} \times \mathbb{R}^2 \\ & (x, w) \to (x + \alpha, A(x)w)\end{aligned}$$

The cocycle (α, A) is said to be C^r reducible, if there exists $B \in C^r(\mathbb{R}/2\mathbb{Z}, SL(2,\mathbb{R}))$, $C \in SL(2, \mathbb{R}$ such that:

$$B(x+\alpha)A(x)B(x)^{-1} = C$$

A special class of cocycles are the Schrödinger cocycles:

$$S_{v,E}(x) := \begin{pmatrix} v(x) - E & -1 \\ 1 & 0 \end{pmatrix},$$

where $v \in C^r(\mathbb{T}, \mathbb{R})$ is the potential, $E \in \mathbb{R}$ is the energy.

With a similar proof as in the Theorem above, the following holds[6]:

Theorem (Eliasson, [24]) *Let v be a real-analytic potential, α Diophantine. There exists $\lambda_0 = \lambda_0(v, \alpha) > 0$ such that if $0 < \lambda < \lambda_0$, then for almost every $E \in \mathbb{R}$ $S_{v,E}$ is C^ω-reducible.*

The reducibility of cocycles is very close connected to their Lyapunov exponents, that we now recall:

Let (α, A) be a $SL(2, \mathbb{R})$ cocycle. Let:

$$A_n(x) := A(x + (n-1)\alpha)...A(x).$$

Then, the Lyapunov exponent is:

$$L(\alpha, A) := \lim_{n\to+\infty} \int_{\mathbb{T}} \log ||A_n(x)|| dx.$$

Define RDC as follows: α is in RDC if there exists infinitely many $n \in \mathbb{N}$ such that $G^n(\alpha)$ is Diophantine, where $G(\alpha) := \frac{1}{\{\alpha\}}$ is the Gauss's map.

The following dichotomy holds:

Theorem (Avila, Krikorian [5]) *Let $\alpha \in RDC$ and let v be a C^r potential, with $r = +\infty, \omega$. Then, for almost all $E \in \mathbb{R}$, the cocycle $(\alpha, S_{v,E})$ is either reducible or non-uniformly hyperbolic.*

1.2 Main results of the Thesis

The thesis is divided in two parts: in the first part we prove global conjugacy of Gevrey circle diffeomorphisms and, in the second part, we study the topology of Diophantine sets.

The motivation of the thesis to study the diffeomorphisms of the circle and the topology of Diophantine sets is the following:

[6] It is essentially the analogous of the Theorem above reformulated for cocycles

For the diffeomorphisms of the circle, they are the simplest example where the dynamics depends on arithmetic conditions, and at the moment it is the only one where something was done to the hard problem to find the best arithmetic conditions.

For the topology of Diophantine sets, in addition to be a natural problem, it should help to understand better the topology of invariant curves of twist maps of the annulus.

1.2.1 Linearization of Gevrey circle diffeomorphisms

In the first part, our aim is to extend Herman's Theorem on the linearization of circle diffeomorphisms in the Gevrey class. Let us recall the definition of Gevrey functions:

Let $s \geq 1$, we say that a diffeomorphism $f \in D^{\infty}(\mathbb{T})$ is s-Gevrey if there exists $A > 0$ such that, for $k \geq 1$:

$$|D^k f|_0 := \sup_{x \in [0,1]} |D^k f(x)| \leq A^k (k!)^s, \tag{11}$$

where Df is the derivative of f. So, we prove the global linearization of Gevrey diffeomorphisms when the rotation number is Diophantine:

Theorem 1 *Let $s \geq 1$, $f \in D^{\infty}(\mathbb{T})$ be an s-Gevrey diffeomorphism with $\alpha := \rho(f) \in \mathtt{D}$. Then, there exists a diffeomorphism h that is $s + 1 + \epsilon$-Gevrey for all $\epsilon > 0$, such that:*

$$h \circ f \circ h^{-1} = T_{\alpha}. \tag{12}$$

Note that, by [74], for an analytic diffeomorphism[7] with rotation number satisfying the Herman-Yoccoz condition (which is weaker than Diophantine), the diffeomorphism that conjugates to a rotation is also analytic. So, for $s = 1$, the term "$1 + \epsilon$" is not necessary: in particular, it is not clear what is the correct loss of regularity.

In the perturbative Hamiltonian setting, in [11], Bounemoura and Fejoz prove the persistence of KAM tori with Gevrey regularity with frequencies satisfying a Bruno-Rüssmann condition for small perturbations of integrable Gevrey Hamiltonians.

It seems that the problem of global conjugacy of the circle in the Gevrey class is more similar to the smooth case than the analytic case.

In particular, it is not clear for Gevrey diffeomorphisms how to pass from a local to a global Theorem as in the Herman-Yoccoz Theorem.

If we suppose that the Gevrey function is C^1 conjugate to a rotation, then we can prove the following:

[7] Observe that, for $s = 1$, the 1-Gevrey functions are the real analytic funtions that commute with T.

Theorem 2 *Let f be a Gevrey diffeomorphism of the circle, and suppose that $\alpha := \rho(f)$ is such that:*

$$\log q_{n+1} = O((\log q_n)^s),$$

with some $s < 2$. Then, f is C^∞ conjugate to a rotation.

The proof of Theorem 1,2 are based on the same ingredients involved in Yoccoz, [73].

However, there are two main differences with respect to [73]:

The first difference is in the use of Hadamard's inequality:

By the assumption that the diffeomorphism is not only finite differentiable, but also C^∞, we can use Hadamard's inequality for the numbers of times we want, by getting better estimate of the derivatives of the iterates of f in the convergents . In particular, the assumption that f is smooth allow us to proceed by induction to prove a "good" upper-bound for the derivatives of the iterates in the convergents.

The second difference is in the way we get a priori estimates of the derivatives of all iterates of f (Lemma 2), that permit also to control the growth of the constants to prove Gevrey estimates on h (that will be the main difficulty).

The polynomials E_l^k (see our Lemma 1 in §2), that we use to get our estimates, are defined also in (Yoccoz, [73]). However, Yoccoz does not use these polynomials to estimate the derivatives of the iterates of f.

Theorem 2 is an improvement of the arithmetic condition: in fact, the Diophantine condition is equivalent to:

$$\log q_{n+1} = O(\log q_n).$$

1.2.2 Topology of Diophantine sets

In the second part of the Thesis, we study the topology of Diophantine sets. These sets play an important role in dynamical systems, in particular, in small divisors problems with applications to KAM theory, Aubry-Mather theory, conjugation of circle diffeomorphisms, etc. (see, for example, [4], [10], [14], [12], [16], [23], [25]).

The set $D_{\gamma,\tau}$ is compact and totally disconnected (since $D_{\gamma,\tau} \cap \mathbb{Q} = \emptyset$), however, it is not clear whether, for some γ and τ, there exist isolated points in $D_{\gamma,\tau}$.

In §6, we provide explicit examples of $D_{\gamma,\tau}$ with isolated points, giving, in particular, a partial answer to a a question raised by Broer in [73] (see remark (iii) below).

In §7 we show that, for τ large enough and for almost all γ, $D_{\gamma,\tau}$ is a Cantor set.

Our main results are the following.

Proposition *Let $n \in \mathbb{N}$, $n \geq 2$ and define*

$$\bar{\alpha} := \frac{\sqrt{n^2+4}-n}{2}\ , \qquad \gamma := \frac{1}{\bar{\alpha}+n}\ , \qquad \tau := \frac{\log(\bar{\alpha}+n)}{\log n}\ . \tag{13}$$

Then $\bar{\alpha}$ is an isolated point of $D_{\gamma,\tau}$.

Indeed, we can show that, for all Diophantine numbers, there exists an 'equivalent number' that is isolated in some Diophantine set:

Theorem A *Let $\gamma \in (0, \frac{1}{2})$, $\tau \geq 1$. Define the map:*

$$\Phi_{\gamma,\tau}(z) := \frac{\eta z + 1}{(2\eta+1)z + 2} \tag{14}$$

with

$$\eta := \left\lceil \frac{2^\tau 3}{\gamma} \right\rceil. \tag{15}$$

Then $\Phi(D_{\gamma,\tau}) \subseteq D_\tau := \bigcup_{\gamma>0} D_{\gamma,\tau}$. Moreover, for all $\alpha \in \Phi(D_{\gamma,\tau})$ there exists $\tau_\alpha > \tau$ and $\gamma_\alpha > 0$ such that α is isolated in $D_{\gamma_\alpha,\tau_\alpha}$.

The isolated points constructed in Theorem A depend only on the first coefficients of their continued fraction (that we can change up to an equivalent number).

Then, we show that a Diophantine number may be an isolated point "for infinitely many τ":

Theorem B *Fix $\tau \geq 1$ and a strictly decreasing sequence $\{\tau_n\}_{n\in\mathbb{N}}$ with $\tau_n \searrow \tau$. Then, there exist $\gamma > 0$, $\alpha \in D_{\gamma,\tau}$ and sequences $\{\bar{\tau}_n\}_{n\in\mathbb{N}}$, $\{\gamma_n\}_{n\in\mathbb{N}}$ with, $\bar{\tau}_n \in (\tau_n, \tau_{n+1})$, $\gamma_n \searrow \gamma$ such that α is an isolated point of $D_{\gamma_n,\bar{\tau}_n}$ for all n.*

In the second part we provide conditions such that $D_{\gamma,\tau}$ is a Cantor set.

Theorem C Let $\tau > \frac{3+\sqrt{17}}{2}$. Then, for almost all $\gamma > 0$ $D_{\gamma,\tau}$ is a Cantor set.

Remarks (i) The existence of isolated points of Diophantine sets may be related to isolated tori and KAM stability in two degrees of freedom.

(ii) Our analysis is based on continued fractions and relations with dynamics in higher dimensions are, therefore, not clear.

(iii) The paper [13] is entitled: "*Do Diophantine vectors form a Cantor bouquet?*", namely, is the set $\Delta^N_{\gamma,\tau} \cap \mathbb{S}^{N-1}$, where

$$\Delta^N_{\gamma,\tau} := \{\omega \in \mathbb{R}^N : |\omega \cdot n| \geq \frac{\gamma}{|n|^\tau} \quad \forall n \in \mathbb{Z}^N, n \neq 0\},$$

and $\mathbb{S}^{N-1}$ denotes the unit sphere in $\mathbb{R}^N$, a Cantor set?

In dimension $N = 2$ it is clearly equivalent to consider the intersection of $\Delta^2_{\gamma,\tau}$ with the line $\omega_2 = 1$, which, upon restricting to the unit interval, coincides with the set $D_{\gamma,\tau}$.

Our results, therefore, show that, in general, the answer to such a question is negative, at least, in dimension $N = 2$.

(iv) Following the same proof of Theorem C, we can show that for $\tau > \frac{3+\sqrt{17}}{2}$, for almost all $\gamma > 0$ the following property holds: If $\alpha \in D_{\gamma,\tau}$, for all $\epsilon > 0$:

$$\mu(D_{\gamma,\tau} \cap (\alpha - \epsilon, \alpha + \epsilon)) > 0.$$

(v) The constant $\frac{3+\sqrt{17}}{2}$ is not optimal. Probably a better constant should be obtained putting a better inequality in Lemma 5. For $\tau = 1$ and $\frac{1}{3} < \gamma < \frac{1}{2}$, $D_{\gamma,\tau}$ is a finite set. So Theorem C does not hold for $\tau = 1$. It should be reasonable that the optimal lower bound $\bar{\tau}$ such that Theorem C holds satisfies the following property: For every $1 \leq \tau < \bar{\tau}$ there exists $\gamma > 0$ such that $D_{\gamma,\tau}$ is a non empty finite set.

(vi) In all our examples of isolated points the following holds: if α is an isolated point of $D_{\gamma,\tau}$, then γ is the best constant such that the Diophantine conditions with exponent τ holds. By an amazing Theorem of Roth, for any algebraic numbers α, given $\tau > 1$ there exists $\gamma > 0$ such that $\alpha \in D_{\gamma,\tau}$ (see, for example, [14]). We believe that, for algebraic numbers of degree greater then 2, the statement of Theorem B holds. So, information about isolated points may be in connection with continued fraction properties of algebraic numbers.

2 Linearizarion of Gevrey circle diffeomorphisms

In this section we prove global linearization for Gevrey circle diffeomorphisms. We prove at first some techincal lemmas that will be useful for the main Theorem.

2.1 Technical lemmas

From now on, if not specified, we suppose that $f \in D^\infty(\mathbb{T})$.

To prove Therem 1, it is easier to show that $\log Dh$ is $(s+1+\epsilon)$-Gevrey (with h the diffeomorphism that linearize f to a rotation), insteed of proving that h is $(s+1+\epsilon)$-Gevrey. So, the aim of the first two lemmas is to show the equivalence of such two problems.

Lemma 1 (see Yoccoz, [73]) *For $l \geq 0$:*[8]

$$D^{l+1} f = A_l(D \log Df, ..., D^l \log Df) Df, \tag{16}$$

with $A_l(X_1, ..., X_l)$, homogeneous of degree l if the variable X_i has weight i, that are defined as follows:

$$A_0 := 1,\ A_l := X_1 A_{l-1} + \sum_{i=1}^{l-1} \frac{\partial A_{l-1}}{\partial X_i} X_{i+1} \quad \text{for } l \geq 1. \tag{17}$$

For $l \geq 1$:

$$D^l \log Df = B_l \left(\frac{D^2 f}{Df}, ..., \frac{D^{l+1} f}{Df} \right), \tag{18}$$

with $B_l(X_1, ..., X_l)$, homogeneous of degree l if the variable X_i has weight i, that are defined as follows:

$$B_1 := X_1,\ B_{l+1} := \sum_{i=1}^{l} \frac{\partial B_l}{\partial X_i} X_{i+1} - \sum_{i=1}^{l} \frac{\partial B_l}{\partial X_i} X_i X_1 \quad \text{for } l \geq 1. \tag{19}$$

Lemma 2 *Let $f \in D^\infty(\mathbb{T})$, $s \geq 1$. Then, the following are equivalent:*

1. *There exists $A > 1$ such that, for $k \geq 1$:*

$$|D^k f|_0 := \sup_{x \in [0,1]} |D^k f(x)| \leq A^k (k!)^s.$$

2. *There exists $B > 1$ such that, for $k \geq 0$:*

$$|D^k \log Df|_0 \leq B^k (k!)^s.$$

[8] We recall that Df is the derivative of f.

Proof We prove the first implication, the other one can be proved in a similar way. Let:

$$B := A \max_{x\in[0,1]} \frac{1}{Df(x)}.$$

For $h \geq 0$, define $Y_h := B^h (h!)^s$. We want to show by induction that, for $k \geq 1$:

$$A_k(Y_1, ..., Y_k) \leq 2^k Y_{k+1}. \tag{20}$$

The case $k = 1$ is trivial. So, suppose that, for $1 \leq h \leq k$:

$$A_h(Y_1, ..., Y_h) \leq 2^h Y_{h+1}. \tag{21}$$

We want to prove (21) for $k \to k+1$. Write:

$$\begin{aligned} A_k(X_1, ..., X_k) &= \sum_{i_1+....+ki_k=k} a_{i_1,...,i_k} X_1^{i_1} ... X_k^{i_k}, \\ A_{k+1}(X_1, ..., X_{k+1}) &= \sum_{i_1+....+(k+1)i_{k+1}=k+1} b_{i_1,...,i_{k+1}} X_1^{i_1} ... X_{k+1}^{i_{k+1}} + X_1 A_k, \end{aligned}$$

By (17):

$$b_{i_1,...,i_{k+1}} = (i_1+1)a_{i_1+1,i_2-1,...,i_k} + ... + (i_k+1)a_{i_1,...,i_k+1},$$

with the notation that, $a_{i_1,...,i_k} = 0$ if: $i_j < 0$ for some $1 \leq j \leq k$ or $i_1 + ... + ki_k > k$. Let:

$$\begin{aligned} \overline{\imath_j} &= i_j + 1 \quad \text{if}: \quad \mathrm{i_{j+1}} \geq 1, \quad 1 \leq \mathrm{j} \leq \mathrm{k}-1, \\ \overline{\imath_j} &= 0 \quad \text{if}: \quad \mathrm{i_{j+1}} = 0, \quad 1 \leq \mathrm{j} \leq \mathrm{k}-1. \\ \overline{\imath_k} &= 1 \quad \text{if}: \quad \mathrm{i_j} = 0 \quad \text{for} \quad 1 \leq \mathrm{j} \leq \mathrm{k}, \\ \overline{\imath_k} &= 0 \quad \text{otherwise}. \end{aligned}$$

It is easy to check by induction that:

$$2^s \overline{\imath_1} + ... + \overline{\imath_k}(k+1)^s \leq (k+1)^s.$$

Then:

$$\begin{aligned} b_{i_1,...,i_{k+1}} \quad & Y_1^{i_1} \cdots Y_{k+1}^{i_{k+1}} = 2^s(i_1+1)a_{i_1+1,i_2-1,...,i_k} Y_1^{i_1+1} Y_2^{i_2-1} .... Y_k^{i_k} + ... \\ + \quad & (i_k+1)(k+1)^s a_{i_1,i_2-1,...,i_k+1} Y_1^{i_1} .... Y_k^{i_k+1} \\ \leq \quad & (\overline{\imath_1} 2^s + ... + \overline{\imath_k}(k+1)^s))(a_{i_1+1,i_2-1,...,i_k} Y_1^{i_1+1} Y_2^{i_2-1} .... Y_k^{i_k} \\ + \quad & a_{i_1,...,i_k+1} Y_1^{i_1} .... Y_k^{i_k+1}) \\ \leq \quad & (k+1)^s (a_{i_1+1,i_2-1,...,i_k} Y_1^{i_1+1} Y_2^{i_2-1} .... Y_k^{i_k} + a_{i_1,...,i_k+1} Y_1^{i_1} .... Y_k^{i_k+1}). \end{aligned}$$

So, summing over all $i_1, ..., i_k$ and, by the inductive hypothesis:

$A_{k+1}(Y_1, ..., Y_{k+1}) \leq 2(k+1)^s A_k(Y_1, ..., Y_{k-1}) \leq 2^{k+1} B^{k+2}((k+2)!)^s = 2^{k+1} Y_{k+2}$.

In particular, we get (21): for $k \to k+1$. ∎

The basic technical result, which we will use in proving Theorem 1, is the following "Main Lemma":

Main Lemma *Let $C > 1$, and suppose that $f \in D^\infty(\mathbb{T})$ is such that:*[9]

$$|D^p \log Df|_0 \leq C^p (p!)^s \quad \forall p \in \mathbb{N}, \tag{22}$$

$$\sup_{n \in \mathbb{N}} |Df^n|_0 =: D < +\infty. \tag{23}$$

Let $k, t \in \mathbb{N}$ with $k > 2t$, $\alpha \in \mathbb{R} - \mathbb{Q}$, $A \geq CD$, and suppose that, for $1 \leq m \leq k$, $n \in \mathbb{N}$:

$$|D^m \log Df^n|_0 \leq A^m (m!)^{s+1+\epsilon} ||n\alpha||, \quad ||n\alpha|| := \min_{p \in \mathbb{N}} |n\alpha - p|. \tag{24}$$

Then, there exists $B = B(t, \epsilon)$ such that, for $1 \leq h \leq t$, $n \geq 0$:

$$|D^{k+h} \log Df^n|_0 \leq B A^{k+h} (k!)^{s+1+\epsilon} n^h (\log k). \tag{25}$$

Before proving the main Lemma, we give some useful estimates on the polynomials E_l^k, that are introduced in the following Lemma:

Lemma 3 *For $k, n \geq 1$:*

$$D^k \log Df^n = \sum_{l=0}^{k-1} \sum_{i=0}^{n-1} D^{k-l} \log Df \circ f^i (Df^i)^{k-l} E_l^k (D \log Df^i, ... D^l \log Df^i), \tag{26}$$

where the polynomials $E_l^k = E_l^k(X_1, ..., X_l)$ are defined in the following way:

$E_0^1 := 1$. For $k \geq 1$: $E_k^k := 0$, $E_{-1}^k := 0$.

For $k \in \mathbb{N}$, $1 \leq l < k+1$, $E_l^{k+1} = E_l^{k+1}(X_1, ..., X_l)$ are defined iteratively as follows:

$$E_l^{k+1} := E_l^k + (k-l) E_{l-1}^k X_1 + \sum_{h=1}^{l-1} \frac{\partial E_{l-1}^k}{\partial X_h} X_{h+1}. \tag{27}$$

Moreover, giving weight i to X_i, each monomial of E_l^k has degree l (if $1 \leq l < k$).

[9] We recall that $|f|_0$ is the sup-norm of f, Df is the derivative of f.

Proof We prove Lemma 156 by induction. For $k = 1$:

$$D \log Df^n = D\left(\sum_{i=0}^{n-1} \log Df \circ f^i\right) = \sum_{i=0}^{n-1} D \log Df \circ f^i Df^i.$$

So, for $k = 1$, $E_0^1 = 1$.

Now, suppose that the Lemma holds for some fixed $k \geq 1$ and for all $n \in \mathbb{N}$.

We want to prove (26),(27) for $k + 1$.

By inductive hypothesis, for all $n \geq 1$:

$$\begin{aligned} D^{k+1} \log Df^n &= \sum_{l=0}^{k-1} \sum_{i=0}^{n-1} D^{k+1-l} \log Df \circ f^i (Df^i)^{k+1-l} E_l^k(D \log Df^i, ..., D^l \log Df^i) \\ &+ \sum_{l=0}^{k-1} \sum_{i=0}^{n-1} (k-l) D^{k+1-(l+1)} \log Df \circ f^i (Df^i)^{k+1-(l+1)} D \log Df^i E_l^k \\ &+ \sum_{l=0}^{k-1} \sum_{i=0}^{n-1} D^{k+1-(l+1)} \log Df \circ f^i (Df^i)^{k+1-(l+1)} \sum_{h=1}^{l-1} \frac{\partial E_l^k}{\partial X_h}, \end{aligned}$$

with:

$$E_l^k = E_l^k(D \log Df^i, ..., D^l \log Df^i).$$

So, we have:

$$E_l^{k+1} = E_l^k + (k-l) E_{l-1}^k X_1 + \sum_{h=1}^{l-1} \frac{\partial E_{l-1}^k}{\partial X_h} X_{h+1}.$$

In particular, by inductive hypothesis, E_l^{k+1} is homogeneus of degree l if the variable X_i has weight i. ∎

Let $P(X_1, ..., X_l) = \sum a_{i_1,...i_l} X_1^{i_1} ... X_l^{i_l}$ be a polynomial with real coefficients.

We define:

$$||P|| := \max_{i_1,...,i_l} |a_{i_1,...,i_l}|. \tag{28}$$

Lemma 4 *For $k \in \mathbb{N}, 1 \leq l \leq k$:*

$$||E_l^k|| \leq k! \tag{29}$$

Proof We prove Lemma 4 by induction.
For $k = 1$ we have: $E_1^1 = 0$. So, in this case, the Lemma is trivial.
Now assume that: $||E_l^h|| \leq (h!)$ for all h such that $1 \leq h \leq k$, and for $1 \leq l \leq h$.
We want to prove Lemma 4 for $k \to k+1$, $1 \leq l \leq k+1$.
By Lemma 1, for $1 \leq l \leq k+1$:

$$E_l^{k+1} = E_l^k + (k-l)E_{l-1}^k X_1 + \sum_{h=1}^{l-1} \frac{\partial E_{l-1}^k}{\partial X_h} X_{h+1}.$$

Now, using that, for each term $a_{i_1,\ldots,i_{l-1}} X_1^{i_1} \ldots X_{l-1}^{i_{l-1}}$ of E_{l-1}^k holds:

$$i_1 + \ldots + (l-1)i_{l-1} = l-1,$$

it is easy to check that:

$$||\sum_{h=1}^{l-1} \frac{\partial E_{l-1}^k}{\partial X_h} X_{h+1}|| \leq l||E_{l-1}^k||.$$

In particular:

$$\begin{aligned} ||E_l^{k+1}|| &\leq ||E_l^k|| + (k-l)||E_{l-1}^k X_1|| + ||\sum_{h=1}^{l-1} \frac{\partial E_{l-1}^k}{\partial X_h} X_{h+1}|| \\ &\leq k! + (k-l)k! + l(k)! \\ &\leq (k+1)! \quad \blacksquare \end{aligned}$$

Lemma 5 *Let $A > 1, \epsilon > 0$. For $h \in \mathbb{N}$ define:*

$$Y_h := A^h (h!)^{s+1+\epsilon}. \tag{30}$$

Then, for $k \in \mathbb{N}$ and $1 \leq l \leq k$:

$$E_l^k(Y_1, \ldots, Y_l) \leq A^l (l!)^{s+\epsilon} k! \tag{31}$$

Remark 1 If we estimate at first the growth of the coefficients of E_l^k and then we use the Gevrey estimates, we get:

$$E_l^k(Y_1,...,Y_l) \leq A^h (h!)^{s+1+\epsilon}(k!).$$

The improvement that we get in Lemma 5 will be fundamental in the proof of the estimates in the Main Lemma.

Proof We prove (31) by induction. If $k = 1$, the Lemma is trivial because of $E_1^1 = 0, \mathrm{E}_0^1 = 1$.

Now, suppose that the Lemma is true for some $k \geq 1$, so we assume that (31) holds for $1 \leq l \leq k$. We want to prove (31) for $1 \leq l \leq k+1$.

For $k \in \mathbb{N}, 1 \leq l \leq k$ we denote:

$E_l^k = E_l^k(Y_1,...,Y_l)$,

$E_l^{k+1} = E_l^{k+1}(Y_1,...,Y_l)$,

$E_{l-1}^k = E_{l-1}^k(Y_1,...,Y_{l-1})$,

$\dfrac{\partial E_{l-1}^k}{\partial X_h} = \dfrac{\partial E_{l-1}^k}{\partial X_h}(Y_1,...,Y_{l-1}) \quad \text{for } 1 \leq h \leq l-1.$

We claim that:

$$G_l^k := \sum_{h=1}^{l-1} \frac{\partial E_{l-1}^k}{\partial X_h} X_{h+1} \leq A l^{s+1+\epsilon} E_{l-1}^k. \tag{32}$$

Write the polynomials $E_{l-1}^k(X_1,...,X_l), G_l^k(X_1,...,X_l)$ (with the obvious definition of $G_l^k(X_1,...,X_l)$) as:

$$E_{l-1}^k = \sum_{i_1+...+(l-1)i_{l-1}=l-1} a_{i_1,...,i_{l-1}} X_1^{i_1}...X_{l-1}^{i_{l-1}}, \tag{33}$$

with $a_{i_1,...,i_{l-1}} = 0$ if $i_j < 0$ for some $1 \leq j \leq l-1$ or $i_1 + ... + (l-1)i_{l-1} > l-1$.

$$G_l^k = \sum_{i_1+...+li_l=l} b_{i_1,...,i_l} X_1^{i_1}...X_l^{i_l}, \tag{34}$$

with $b_{i_1,...,i_l} = 0$ if $i_j < 0$ for some $1 \leq j \leq l$.

Note that, by the condition:

$$i_1 + ... + (l-1)i_{l-1} = l-1,$$

if $h > \frac{l}{2}$, then: $i_h = 0$ or $i_h = 1$.

We remark also, that for $1 \leq h \leq l-1$:

$$\frac{Y_{h+1}}{Y_h} = A(h+1)^{s+1+\epsilon}. \tag{35}$$

Then, by (33), (34), (61):

$$\begin{aligned} b_{i_1,...,i_l} Y_1^{i_1} ... Y_l^{i_l} &= AY_1^{i_1} ... Y_{l-1}^{i_{l-1}} (2^{s+1+\epsilon}(i_1+1) a_{i_1+1, i_2-1, ..., i_{l-1}} + ... \\ &+ l^{s+1+\epsilon}(i_{l-1}+1) a_{i_1, ..., i_{l-1}+1}) \\ &\leq AY_1^{i_1} ... Y_{l-1}^{i_{l-1}} (2^{s+1+\epsilon} \overline{\imath_1} + ... + l^{s+1+\epsilon} \overline{\imath_{l-1}}) \\ &\times \left(\sum_{h=1}^{l-2} a_{i_1,...,i_h+1, i_{h+1}-1,...,i_{l-1}} + a_{i_1,...i_{l-1}+1} \right), \end{aligned}$$

with:

$$\begin{aligned} \overline{\imath_j} &= i_j + 1 \quad \text{if}: \quad \mathrm{i_{j+1}} \geq 1, \quad 2 \leq \mathrm{j} \leq \mathrm{l}-2, \\ \overline{\imath_j} &= 0 \quad \text{if}: \quad \mathrm{i_{j+1}} \geq 0, \quad 2 \leq \mathrm{j} \leq \mathrm{l}-2, \\ \overline{\imath_{l-1}} &= 1 \quad \text{if}: \quad \mathrm{i_j} = 0 \quad \text{for} \quad 1 \leq \mathrm{j} \leq \mathrm{l}-1, \\ \overline{\imath_{l-1}} &= 0 \quad \text{otherwise}. \end{aligned}$$

An easy proof by induction shows that:

$$2^{s+1+\epsilon} \overline{\imath_1} + ... + l^{s+1+\epsilon} \overline{\imath_{l-1}} \leq l^{s+1+\epsilon}. \tag{36}$$

So, summing over all $i_1, ..., i_l$ and using (36) , the claim follows.

If $l = k+1$, then: $E_l^{k+1} = 0$. In particular, the Lemma is trivial if $l = k+1$. So, we may assume that $l < k+1$.

In this case:

$$\begin{aligned} E_l^{k+1} &= E_l^k + (k-l)E_{l-1}^k X_1 + \sum_{h=1}^{l-1} \frac{\partial E_{l-1}^k}{\partial X_h} X_{h+1} \\ &\leq A^l (l!)^{s+\epsilon} k! + (k-l) A^l k! (l-1)!^{s+\epsilon} + A l^{s+1+\epsilon} A^{l-1} (l-1)!^{s+\epsilon} k! \\ &= A^l k! (l!)^{s+\epsilon} \left(1 + \frac{k-l}{l^{s+\epsilon}} + l \right) \leq A^l l!^{s+\epsilon} (k+1)!. \end{aligned}$$

∎

Lemma 6 *Write E_l^k as:*

$$E_l^k = \sum_{i_1+...+li_l=l} a_{i_1,...,i_l} X_1^{i_1}...X_l^{i_l}. \quad (37)$$

Suppose that, for some $j > \frac{k}{2}$, $i_j \neq 0$ (so, $i_j = 1$). Then:

$$a_{i_1,...,i_l} \leq 3^{k-j}((k-j)!)^{k-j}. \quad (38)$$

Proof We prove (38) by induction over k.
For $k = 1$, (38) is trivial. So, suppose that (38) holds for $1 \leq h \leq k$.
Write E_l^k, E_l^{k+1}, E_l^{k-1} as:

$$\begin{aligned} E_l^{k-1} &= \sum a_{i_1,...,i_{l-1}} X_1^{i_1}...X_{l-1}^{i_{l-1}}, \\ E_l^k &= \sum b_{i_1,...,i_l} X_1^{i_1}...X_l^{i_l}, \\ E_l^{k+1} &= \sum c_{i_1,...,i_l} X_1^{i_1}...X_l^{i_l}. \end{aligned}$$

By (27), if $l \leq k-1$:

$$c_{i_1...,i_l} = b_{i_1,...,i_l} + (k-l)a_{i_1,...,i_{l-1}} + \bar{i_1} a_{i_1+1,i_2-1...,i_{l-1}} + ... + \bar{i_{l-1}} a_{i_1,...i_{l-1}+1}.$$

Let $c_{i_1,...,i_l}$ such that $i_j \neq 0$ with $j > \frac{k+1}{2}$. If $j = l$, then (38) follows directly by (27). So, suppose that $\frac{k+1}{2} < j < l$, then:

$$\begin{aligned} c_{i_1,...,i_l} &= c_{i_1,...,i_j,0,...,0} = b_{i_1,...,i_j,0,...,0} \\ & +(k-l)a_{i_1-1,...,i_j,0,...,0} + \bar{i_1} a_{i_1+1,i_2-1...,i_j,0,...,0} + ... \\ & +a_{i_1,...i_{j-1},0,1,0,...,0} \leq b_{i_1,...,i_j,0,...,0} + (k-l)a_{i_1-1,...,i_j,0,...,0} \\ & +(\bar{i_1} + \bar{i_2} + ... + 1)3^{k-j}((k-j)!)^{k-j} \\ &\leq 2(3)^{k-j}((k-j)!)^{k-j}(k-l) + (k-j+1)((k-j)!)^{k-j} \\ &\leq 3^{k+1-j}((k+1-j)!)^{k+1-j}. \quad \blacksquare \end{aligned}$$

We state also the following simple lemma:

Lemma 7 *Let $\alpha \in \mathbb{R} - \mathbb{Q}$, $l \in \mathbb{N}$. Then:*

$$\frac{1}{n} \sum_{i=0}^{n-1} ||i\alpha||^l \leq \frac{1}{2^l}. \quad (39)$$

Proof It follows by $||\alpha i||^l \leq \frac{1}{2^l}$. ∎

We are now ready to prove the Main Lemmma.

Proof (Main Lemma) We prove at first (25) for $h = 1$:

By Lemma 3 and by (62) it follows that, for $l, m, n \in \mathbb{N}$ such that: $1 \leq l \leq m \leq k+1$, $l < k+1$:

$$|E_l^m(D \log Df^n, ..., D^l \log Df^n)|_0 \leq A^l (l!)^{s+\epsilon} m! ||n\alpha||^l \tag{40}$$

So, for $n \leq k$:

$$\begin{aligned}|D^{k+1} \log Df^n|_0 &\leq \sum_{l=0}^{k} \sum_{i=0}^{n-1} |D^{k+1-l} \log Df|_0 |(Df^i)^{k+1-l}|_0 |E_l^{k+1}|_0 \\ &\leq (k+1)! \sum_{l=0}^{k} \sum_{i=0}^{n-1} (k+1-l)!^s A^{k+1} (l!)^{s+\epsilon} ||i\alpha||^l,\end{aligned}$$

So, if $B_1 = B_1(\epsilon) > 0$ is such that, for all $k \in \mathbb{N}$, $0 \leq l < k$:

$$(k!)((k+1-l)!)^s (l!)^{s+\epsilon} \left(1 + \frac{1}{k}\right) \leq \left(B_1 - 1 - \frac{1}{k}\right) (k!)^{s+1+\epsilon},$$

then:

$$\begin{aligned}((k+1)!) \sum_{l=0}^{k} \sum_{i=0}^{n-1} ((k+1-l)!)^s A^{k+1} (l!)^{s+\epsilon} ||i\alpha||^l &\leq B_1 A^{k+1} (k!)^{s+1+\epsilon} n \\ &< B_1 A^{k+1} (k!)^{s+1+\epsilon} n (\log k),\end{aligned}$$

so, we have proved the Lemma for $h = 1$.

Now, assume that the Lemma holds for some h with $1 \leq h < t$, i.e. we are assuming that there exists $B_h > 0$ such that, for $m, n \in \mathbb{N}$ with $1 \leq m \leq h$:

$$|D^{k+m} \log Df^n|_0 \leq B_h A^{k+m} n^m (k!)^{s+1+\epsilon} (\log k). \tag{41}$$

Then, we prove (25) for $h + 1$.

By Lemma 3:

$$|D^{k+h+1} \log Df^n|_0 \leq \sum_{l=0}^{k+h} \sum_{i=0}^{n-1} |D^{k+h+1-l} \log Df|_0 |(Df^i)^{k+h+1-l}|_0 |E_l^{k+h+1}|_0.$$

Moreover we remark that, for $1 \leq m \leq h$, the term $D^{k+m} \log Df^i$ appears at most once (and with exponent at most 1) in each monomial $X_1^{i_1}...X_l^{i_l}$ of E_l^{k+h+1} (because of $2t < k, i_1 + ... + li_l = l \leq k + h$).

Write E_l^{k+h+1} as:

$$E_l^{k+h+1}(X_1, ..., X_l) = \sum_{i_1+....+(k+h)i_{k+h}=l} a_{i_1,...,i_{k+h}} X_1^{i_1}...X_{k+h}^{i_{k+h}},$$

and define:

$$P_l^{k+h+1}(X_1, ..., X_k) := \sum_{i_1+....+ki_k=l} a_{i_1,...,i_k,0,...,0} X_1^{i_1}...X_k^{i_k},$$

$$Q_l^{k+h+1}(X_1, ..., X_{k+h}) := E_l^{k+h+1} - P_l^{k+h+1}.$$

Observe that, for $l \leq k$, $Q_l^{k+h+1} = 0$. Moreover, for $l > k$, each monomial of Q_l^{k+h+1} has only one variable X_j that satisfy $j > k$ (and with exponent $i_j = 1$).

So, combining the estimates of Lemma 5 and the estimates in (41), for $k < l \leq k+h$ we claim that:

$$|Q_l^{k+h+1}(D \log Df^i, ..., D^l \log Df^i)|_0 \leq hB_h(k!)^{s+1+\epsilon} A^l n^h 3^h (h!)^{h+s+2+\epsilon}(\log k). \tag{42}$$

We prove (42):

Observe that, Q_l^{k+h+1} has at most $h(h!)$ terms, and each term of Q_l^{k+h+1} has the form:

$$a_{i_1,...,i_k,0,...,1,...,0} X_1^{i_1}...X_k^{i_k} X_j,$$

with $k + 1 \leq j < k + h + 1$.

For $1 \leq m \leq k + h$, let $X_m = D^m \log Df^i$.

Then:

$$\begin{aligned}
|a_{i_1,...,i_k,0,...,1,...,0} X_1^{i_1}...X_k^{i_k} X_j|_0 &\leq |a_{i_1,...,i_k,0,...,1,...,0} X_1^{i_1}...X_k^{i_k}|_0 |X_j|_0 \\
&\leq |a_{i_1,...,i_k,0,...,1,...,0} X_1^{i_1}...X_k^{i_k}|_0 B_h A^j n^j (k!)^{s+1+\epsilon}(\log k).
\end{aligned}$$

Moreover:

$$i_1 + ... + ki_k = l - j \leq h,$$

so, by Lemma 6:

$$|a_{i_1,\dots,i_k,0,\dots,1,\dots,0}X_1^{i_1}\dots X_k^{i_k}|_0 \le 3^h (h!)^h |X_1^{i_1}\dots X_k^{i_k}|_0 \le (h!)^h A^{l-j} (h!)^{s+1+\epsilon}.$$

In particular:

$$|Q_l^{k+h+1}(D\log Df^i, \dots, D^l \log Df^i)|_0 \le h3^h(h!)B_h(k)!^{s+1+\epsilon}A^l n^h (h!)^h (h!)^{s+1+\epsilon}(\log k),$$

so, we have proved (42).

Now, we claim that there exists $C = C(h,\epsilon) > 0$ such that:

$$|P_l^{k+h+1}(D\log Df^i, \dots, D^l \log Df^i)|_0 \le CA^l(k!)^{s+1+\epsilon}(\log k). \tag{43}$$

We prove (43):

The proof is by induction over k. So, the base step ($k = 2h+1$) is trivial (it is sufficient to choose a large enough $C = C(h)$). Then, suppose that (43) holds for k. Similarly, as in Lemma 3:

$$\begin{aligned} P_l^{(k+1)+1+h} &= P_l^{k+h+1} + (k+h-l)P_{l-1}^{k+h+1}X_1 + \sum_{j=1}^{k} \frac{\partial P_{l-1}^{k+h+1}}{\partial X_j} X_{j+1} \\ &+ \sum a_{i_1,\dots,i_k,1,0\dots,0}X_1^{i_1}\dots X_k^{i_k}X_{k+1} \\ &+ (k+h-l)X_1 \sum b_{i_1,\dots,i_k,1,0\dots,0}X_1^{i_1}\dots X_k^{i_k}X_{k+1}, \end{aligned}$$

with:

$$\begin{aligned} E_l^{k+h+1} &= \sum a_{i_1,\dots,i_{k+h+1}} X_1^{i_1}\dots\dots X_{k+h+1}^{i_{k+h+1}}, \\ E_{l-1}^{k+h+1} &= \sum b_{i_1,\dots,i_{k+h+1}} X_1^{i_1}\dots\dots X_{k+h+1}^{i_{k+h+1}}. \end{aligned}$$

By Lemma 5, Lemma 6 and the Gevrey estimates:

$$\begin{aligned} \left|\sum a_{i_1,\dots,i_k,1,0\dots,0}X_1^{i_1}\dots X_k^{i_k}X_{k+1}\right|_0 &\le 3^h(h!)^{h+s+2+\epsilon}A^l(k+1)!^{s+1+\epsilon}, \\ \left|\sum b_{i_1,\dots,i_k,1,0\dots,0}X_1^{i_1}\dots X_k^{i_k}X_{k+1}\right|_0 &\le 3^h(h!)^{h+s+2+\epsilon}A^{l-1}(k+1)!^{s+1+\epsilon}. \end{aligned}$$

Moreover, similarly as in Lemma 5:

$$\left|\sum_{j=1}^{k} \frac{\partial P_{l-1}^{k+h+1}}{\partial X_j}\right|_0 \le A((k+1)^{s+1+\epsilon} + h^{s+1+\epsilon})\, |P_{l-1}^{k+h+1}|_0.$$

Then:

$$\begin{array}{rcl}
|P_l^{(k+1)+1+h}|_0 & \leq & |P_l^{k+h+1}|_0 + (k+h-l)|P_{l-1}^{k+h+1}X_1|_0 \\
& & + \left| \sum_{j=1}^{k} \frac{\partial P^{k+h+1}}{\partial X_j} X_{j+1} \right|_0 + \left| \sum a_{i_1,\ldots,i_k,1,0\ldots,0} X_1^{i_1} \ldots X_k^{i_k} X_{k+1} \right|_0 \\
& & + \left| (k+h-l) \sum b_{i_1,\ldots,i_k,1,0\ldots,0} X_1^{i_1} \ldots X_k^{i_k} X_{k+1} \right|_0 \\
& \leq & CA^l (k!)^{s+1+\epsilon} + CA^l (k+h-l)(k!)^{s+1+\epsilon} \\
& & + CA^l ((k+1)!)^{s+1+\epsilon} + CA^l ((k)!)^{s+1+\epsilon} h^{s+1+\epsilon} \\
& & + 3^h (h!)^{h+s+2+\epsilon} A^l ((k+1)!)^{s+1+\epsilon} ((\log k) \\
& & + (k+h+1-l) 3^h (h!)^{h+s+2+\epsilon} A^l ((k+1)!)^{s+1+\epsilon} \\
& \leq & CA^l ((k+1)!)^{s+1+\epsilon} \\
& & \times \left(\frac{k+h-l+1+h^{s+1+\epsilon}}{(k+1)^{s+1+\epsilon}} + (2h3^h (h!)^{h+s+2+\epsilon}) \log(1+\frac{1}{k}) \right) \\
& \leq & C_{k+1} A^l ((k+1)!)^{s+1+\epsilon} \log(k+1)
\end{array}$$

and, because of:

$$\prod_{k \geq 2h+1} \left(\frac{2h+1+h^{s+1+\epsilon}}{(k+1)^{s+1+\epsilon}} + (2h3^h (h!)^{h+s+2+\epsilon}) \log(1+\frac{1}{k}) \right) < +\infty$$

we have proved (43) for all $k \geq 2h+1$, for some $C = C_h$.

If $l \leq k$, by Lemma 5 and (62):

$$|P_l^{k+h+1}|_0 \leq A^l (l!)^{s+\epsilon} (k+h+1)! ||\alpha i||^l \tag{44}$$

Finally:

$$
\begin{aligned}
|D^{k+h+1}\log Df^n|_0 &\leq \sum_{l=0}^{k+h}\sum_{i=0}^{n-1}|D^{k+h+1-l}\log Df|_0|(Df^i)^{k+h+1-l}|_0|E_l^{k+h+1}|_0 \\
&\leq \sum_{l=0}^{k+h}\sum_{i=0}^{n-1}|D^{k+h+1-l}\log Df|_0|(Df^i)^{k+h+1-l}|_0(|P_l^{k+h+1}|_0+|Q_l^{k+h+1}|_0) \\
&= \sum_{l=k+1}^{k+h}\sum_{i=0}^{n-1}|D^{k+h+1-l}\log Df|_0|(Df^i)^{k+h+1-l}|_0|P_l^{k+h+1}|_0 \\
&\quad +\sum_{l=0}^{k}\sum_{i=0}^{n-1}|D^{k+h+1-l}\log Df|_0|(Df^i)^{k+h+1-l}|_0|P_l^{k+h+1}|_0 \\
&\quad +\sum_{l=0}^{k+h}\sum_{i=0}^{n-1}|D^{k+h+1-l}\log Df|_0|(Df^i)^{k+h+1-l}|_0|Q_l^{k+h+1}|_0 \\
&\leq \sum_{l=k+1}^{k+h}\sum_{i=0}^{n-1}A^{k+h+1}(k+h+1-l)!^s(k!)^{s+1+\epsilon}(\log k)C(h) \\
&\quad +\sum_{l=0}^{k}\sum_{i=0}^{n-1}A^{k+h+1}(k+h+1-l)!^s(k+h+1)!(l!)^{s+\epsilon}||\alpha i||^l(\log k) \\
&\quad +\sum_{l=k+1}^{k+h}\sum_{i=0}^{n-1}A^{k+h+1}(k+h+1-l)!^sB_h(k+h+1)!(l!)^{s+\epsilon}n^h(\log k) \\
&\leq B_{h+1}A^{k+h+1}(\log k)n^{h+1}(k!)^{s+1+\epsilon},
\end{aligned}
$$

for some $B_{h+1} > B_h$ that depends only on $h+1, \epsilon$. ∎

Remark 2 In the proof of the main Lemma it is crucial to put the term "$1+\epsilon$" in the estimates. In fact, the term $1+\epsilon$ gives more weight on the terms of the polynomials E_l^k.

Lemma 8 *Let $k, l \in \mathbb{N}$, $0 \leq l \leq k$, and define the polynomials $G_l^k(X_1, ..., X_l)$ as follows:*

$$G_0^k = G_k^k := 0,$$

$$G_l^{k+1} = G_l^k + (k-l)X_1G_{l-1}^k + \sum_{h=1}^{l-1}\frac{\partial G_{l-1}^k}{\partial X_h}X_{h+1}$$

for $1 \leq l \leq k$.

Fix $k, Q \in \mathbb{N}, A > 1, t \geq 1$ with $Q > k$ and define for $h \leq k$:

$$Y_h := \frac{A^h}{Q}(h!)^t.$$

Then, there exists $C > 0$ that does not depend on k such that, for $l \leq m \leq k$:

$$G_l^m(Y_1, ..., Y_h) \leq \frac{C(\log k)}{Q} A^h (h!)^t. \tag{45}$$

Proof The case $m = 1$ is trivial (because of $G_1^1 = Y_1$). So, suppose that, for some $m \in \mathbb{N}$, with $m < k$, $l \leq m$ holds:

$$G_l^m(Y_1, ..., Y_l) \leq C_m \frac{A^l}{Q}(l!)^t. \tag{46}$$

Then, for $m \to m+1$, if $l = m$ the inequality of the Lemma is trivial. So, suppose $l < m$:

$$\begin{aligned} G_l^{m+1}(Y_1, ..., Y_m) &= G_l^m + (m-l)\frac{A}{Q}G_{l-1}^m + \sum_{h=1}^{l-1} \frac{\partial G_{l-1}^m}{\partial X_h} Y_{h+1} \\ &\leq \frac{1}{Q} C_m A^l (l!)^t + (m-l)\frac{AC_m}{Q^2} A^{l-1}((l-1)!)^t \\ &\quad + C_m \frac{1}{Q^2} A^l (l!)^t, \end{aligned}$$

where, the estimate

$$\sum_{h=1}^{l-1} \frac{\partial G_{l-1}^m}{\partial X_h} Y_{h+1} \leq C_m \frac{1}{Q^2} A^l (l!)^t$$

can be proved in the same way of the estimates in Lemma. In particular, it suffices to define:

$$C_{m+1} := C_m \left(1 + \frac{m+1-l}{Q}\right),$$

and $C_1 := 1$. In particular, for $m = k$:

$$G_l^k(Y_1, ..., Y_{k-1}) \leq \frac{1}{Q}\left(\prod_{j=l}^{k-1}\left(1 + \frac{j+1-l}{Q}\right)\right) A^k (l!)^k \leq C(\log k)\frac{1}{Q} A^k (l!)^k$$

for some $C > 0$ that does not depend on k, Q.

Lemma 9 *Let $k,l \in \mathbb{N}$, $0 \le l \le k$, and let $G_l^k(X_1,...,X_l)$ the polynomials defined in Lemma 13. Fix $k,Q,h_0 \in \mathbb{N}, A > 1, t \ge 1$ with $Q > k$, $4h_0 < k$. Define:*

$$\begin{aligned} Y_h &:= \frac{A^h}{Q}(h!)^t \quad \text{if } h \le k, \\ Y_h &:= A^h Q(h!)^t \quad \text{if } h > k, \end{aligned} \tag{48}$$

Then, there exists $C = C(h_0)$ such that, for $h \le h_0$, $1 \le l \le k+h$:

$$G_l^{k+h}(Y_1,...,Y_{k+h}) \le \frac{C(h_0)(\log k)(k)^h}{Q} A^{k+h}(k!)^t. \tag{49}$$

Proof We consider at first the case $h = 1$. For $l = k$, $G_k^{k+1} = X_k$. In particular (49) is trivial for $l = k, h = 1$. For $l < k$:

$$G_l^{k+1} \le (\log k)\frac{1}{Q}CA^l(l!)^t\left(1 + \frac{k+1-l}{Q}\right) \le 2C(\log k)\frac{1}{Q}A^l(l!)^t,$$

because of $Q > k$. In particular, we have proved (49) for $h = 1$. Now, it is easy to prove iteratively (49) for $h \le h_0$, starting by the case $h = 1$ (that we have just proved). In fact, because of $2h_0 < k$, in each monomial of G_l^{k+h}, the term Y_m with $m > k$ appears at most once and the G_l^{k+h} satisfy estimates similar to that of Lemma 6.

Next, we list some results that will be used to prove our Theorem:

Theorem 3 ([34], p. 52, Theorem 6.3.4) *Let $r \ge 1$ and define:*

$$H_r(f) := \sup_{n\in\mathbb{Z}} |Df^n|_{C^{r-1}} \tag{50}$$

with:

$$|g|_{C^{r-1}} := \sum_{l=0}^{r-1} |D^l g|_0 \tag{51}$$

if $r \in \mathbb{N}$, while, if $r \notin \mathbb{N}$:[10]

$$|g|_{C^{r-1}} := |g|_{C^{[r-1]}} + sup_{x\neq y}\frac{|D^{[r-1]}(g(x)-g(y))|}{|x-y|^{\{r\}}}. \tag{52}$$

Then, the following are equivalent:

[10] $[r]$ is the integral part of r, $\{r\}$ is the fractional part of r.

- *f is C^r conjugate to T_α*
- $H_r(f) < +\infty$
- $\sup_{n\in\mathbb{N}} |\log Df^n|_{C^{r-1}} < +\infty$

Theorem 4 ([34], p. 127) *Suppose that $f \in D^\infty(\mathbb{T})$, $\alpha := \rho(f) \in D$. Then, f is C^∞ conjugate to T_α.*

The proof of Theorem 4 is divided in two parts: in the first part the C^1 conjugacy is proved and, in the second one, it is proved that C^1 conjugacy implies C^∞ conjugacy.

In the Appendix we give a simple prove of the second part (C^1 conjugacy implies C^∞ conjugacy). This proof follows the same scheme of the proof in the Gevrey class.
Let α be the rotation number of $f \in D^\infty(\mathbb{T})$. We denote with q the denominator of some convergent of the continued fraction of α and with Q the denominator of the subsequent convergent. Moreover, we denote with $h \in C^\infty(\mathbb{T})$ the diffeomorphism (unique up to composition by a translation) that conjugate f to T_α, i.e.:

$$h \circ f \circ h^{-1} = T_\alpha. \tag{53}$$

Lemma 10 *For all $k \geq 0$, there exists $C = C(k) > 0$ such that, for all $n \in \mathbb{Z}$:*

$$|D^k(f^n - id - n\alpha)|_0 \leq C(k)||n\alpha||, \quad ||n\alpha|| := \min_{h\in\mathbb{Z}} |n\alpha - h|. \tag{54}$$

In particular, for all $k \geq 0$ there exists $C(k) > 0$ such that:

$$|D^k \log Df^q|_0 \leq \frac{C(k)}{Q}. \tag{55}$$

Proof Let $k \geq 0$,

$$|D^k(h \circ T_{n\alpha} - h - n\alpha)|_0 \leq |D^{k+1}h|_0||n\alpha||. \tag{56}$$

By the identity:

$$f^n - id - n\alpha = (h \circ T_{n\alpha} - h - n\alpha) \circ h^{-1} \tag{57}$$

and, using Faa di Bruno formula, we get (28).

To prove (15), it suffices to note that there exist polynomials $A_k(X_1, ..., X_k)$ homogeneous of degree k if the variable X_i has weight i such that:

$$D^k \log Dg = A_k\left(\frac{D^2g}{Dg}, ..., \frac{D^{k+1}g}{Dg}\right). \tag{58}$$

Moreover, by Theorem 3, $|Df^n|_0$ are bounded uniformly in n. ∎

In particular, as a corollary of (15) we have:

Corollary 1 *There exists $C > 0$ such that, for q a convergent to $\alpha = \rho(f)$, Q the subsequent convergent, $k \in \mathbb{N}$ with $k \leq Q$:*

$$(Df^q)^k \leq \left(1 + \frac{Ck}{Q}\right) \tag{59}$$

Finally, we state Hadamard's inequality, that will be crucial in the subsequent section.

Theorem 5 (Hadamard's inequality) *([36], Appendix A) Let $g \in C^k([0,1])$. For $h, l, s, k \in \mathbb{N}$ with: $0 \leq h \leq l \leq s \leq k, s \neq h$, there exists $C = C(k) > 0$ such that:*

$$|D^l g|_0 \leq C |D^h g|_0^{\frac{s-l}{s-h}} |D^s g|_0^{\frac{l-h}{s-h}}. \tag{60}$$

2.2 Proof of Theorem 1

For $n \in \mathbb{N}$, define:

$$f_n := \frac{1}{n} \sum_{i=0}^{n-1} (f^i - i\alpha).$$

By Theorem 2 and Ascoli-Arzela Theorem, for all $k \in \mathbb{N}$:

$$|D^k f_n - D^k h|_0 \to 0, \tag{61}$$

with h the diffeomorphism that conjugates f to a rotation (we recall that h is unique up to a rotation). In particular, if we prove that for $\epsilon > 0$ there exists $C = C_\epsilon > 0$ such that, for $k \in \mathbb{N}, i \in \mathbb{N}_0$:

$$|D^k f^i|_0 \leq C^k (k!)^{s+1+\epsilon}, \tag{62}$$

then, for $k, n \in \mathbb{N}$:

$$|D^k f_n|_0 \leq \frac{1}{n} \sum_{i=0}^{n-1} |D^k f^i|_0 \leq C^k (k!)^{s+1+\epsilon}.$$

In particular, by (61), the diffeomorphism h is $s + 1 + \epsilon$ Gevrey with constant C_ϵ. So, it suffice to prove (62).

By Lemma 2, it is also equivalent to prove that, there exists $B = B_\epsilon > 0$ such that, for $k, i \in \mathbb{N}$:

$$|D^k \log Df^i|_0 \leq C^k (k!)^{s+1+\epsilon}. \tag{63}$$

So, we will prove (63).

The proof of (63) is based on the following six steps:

1. Fix $\epsilon > 0$. By Lemma 7, for every $k \in \mathbb{N}$, there exists $A = A(k) > 0$ such that, for $l \leq k$, $n \geq 0$:

$$|D^l \log Df^n|_0 \leq A^l (l!)^{s+1+\epsilon} ||n\alpha||, \tag{64}$$

   This follows from (15), providing that $A(k)$ is large enough (at this point the factor $(l!)^{s+1+\epsilon}$ is irrelevant, in fact it suffices to take $A(k) = C(k)$, with $C(k)$ as in (28)).

2. We fix $k_0 \in \mathbb{N}$ with $k_0 > 8(\beta + 2)$ (with $\alpha := \rho(f) \in \mathtt{D}_\beta$) and take $A = A(k_0)$ big enough, such that equation (64) holds for $0 < l \leq k_0$.

   By the Main Lemma , if $h_0 \in \mathbb{N}$ with $2h_0 < k_0$, there exists $B = B(h_0, \epsilon) > 0$ such that, for $h \leq h_0$, $n \in \mathbb{N}$:

$$|D^{k_0+h} \log Df^n|_0 \leq BA^{k_0+h}(k_0!)^{s+1+\epsilon}(\log k_0)(n)^h. \tag{65}$$

3. In the third step, using (65), we prove the following estimates:

$$|D^{k_0+h} \log Df^{aq}|_0 \leq C_1 BA^{k_0+h}(k_0!)^{s+1+\epsilon}(k_0)^{h-1} aq(\log k_0), \tag{66}$$

   with[11] $h < h_0 := [\frac{k_0}{4}]$, $C = C(h_0, \alpha)$, $a \leq \frac{Q}{q}$.

4. Using Hadamard's inequality, we show that, for $h = 1, 2$:

$$|D^{k_0+h} \log Df^{aq}|_0 \leq C_2 B \frac{A^{k_0+h} a}{Q^{1-\delta}} (k_0!)^{s+1+\epsilon}(k_0)^h (\log k_0), \tag{67}$$

   with $C_2 = C_2(k_0, \epsilon, \alpha) > C_1$ that depend also on the constant of Hadamard's inequality, $a < \frac{Q}{q}$, $\frac{1}{\delta} > 2\tau + 1$.

5. By good estimates on the convergents we obtain estimates for all the iterates of f. This is the only step where we use that the rotation number is Diophantine.

   In particular, we prove the following:

$$|D^{k_0+h} \log Df^n|_0 \leq C_3 BA^{k_0+h}(k_0!)^{s+1+\epsilon}(k_0)^h (\log k_0), \tag{68}$$

   for $h = 1, 2$.

6. With the help of equations (57), (68) and, up to chose $A \geq C$ we get estimates of step 1 for $h = k_0 + 1$ (using that $k^2 \log k \leq C(\epsilon)(k+1)^{s+1+\epsilon}$, because of $s \geq 1$). Then we proceed by induction.

[11] For $t \in \mathbb{R}$, $[t]$ is the integral part of t.

In fact, we have proved that, if (64) holds for $l = k \geq k_0$, $A > C = C(\epsilon, \alpha)$, then:

$$|D^{k+1} \log Df^n|_0 \leq CA^k((k+1)!)^{s+1+\epsilon} \leq A^{k+1}((k+1)!)^{s+1+\epsilon}||n\alpha||. \quad (69)$$

So, by induction, we prove (64), for all $l \in \mathbb{N}$ (for $l < k_0$ the estimate are trivial for a sufficently large A). In particular, Theorem 1 follows.

Remark 3 The loss of regularity of $1+\epsilon$ appear in the last step, in which we have to use also estimates on the $k+1$ derivative, but, actually the loss of regularity is hidden in the Main Lemma.

Remark 4 The choice of h_0 depends only on ϵ.

Step 1: It is a consequence of Theorem 3, Theorem 4, Lemma 7. ∎

Step 2: Step 2 follows directly by the Main Lemma. ∎

Step 3: We will use the following identity:

For $a, b, k \in \mathbb{N}$:

$$\begin{aligned} D^k \log Df^{(a+b)} &= D^k \log Df^a \circ f^b + D^k \log Df^b \\ &+ \sum_{l=1}^{k-1} D^{k-l} \log Df^a \circ f^b (Df^b)^{k-l} G_l^k(D \log Df^b, ..., D^l \log Df^b), \end{aligned}$$

with $G_l^k(X_1, ..., X_l)$ homogeneous of degree l if the variable X_i has weight i. Moreover, we recall that the polynomials G_l^k satisfy the following relation:

$$G_l^{k+1} = G_l^k + (k-l)X_1 G_{l-1}^k + \sum_{h=1}^{l-1} \frac{\partial G_{l-1}^k}{\partial X_h} X_{h+1},$$

with:

$$G_0^k = G_k^k := 0 \; \forall k \in \mathbb{N}.$$

The proof of this identity is the same of the proof of Lemma 3, so it is omitted.

By (65), for $h \leq h_0$, $n \leq k_0$:

$$|D^{k_0+h} \log Df^n|_0 \leq BA^{k_0+h}(k_0!)^{s+1+\epsilon}(\log k_0)(k_0)^{h-1}n.$$

We show at first that, there exists $C > 0$ such that, if $q \in \mathbb{N}$ is a convergent of α, then:

$$|D^{k_0+1} \log Df^q|_0 \leq CA^{k_0+1}(k_0!)^{s+1+\epsilon}(\log k_0)q. \quad (70)$$

Let $\{q_n\}_{n\in\mathbb{N}_0}$ be the convergents to $\alpha = [a_0, a_1, ...] := a_0 + \frac{1}{a_1+\frac{1}{a_2+\dots}}$, so that $q_{n+1} = a_{n+1}q_n + q_{n-1}$. If $q = q_n \le k_0$, then (70) follows by (65).

Now, suppose that for $i < n$:

$$|D^{k_0+1}\log Df^{q_i}|_0 \le BA^{k_0+h}(k_0!)^{s+1+\epsilon}(\log k_0)q_i,$$

and $q_n > k$.

Then, for $i < n$, $q_{i+1} > k$, $1 < a \le \frac{q_{i+1}}{q_i}$:

$$\begin{aligned}
|D^{k_0+1}\log Df^{aq_i}|_0 &= |D^{k_0+1}\log Df^{(a-1)q_i}\circ f^{q_i} + D^{k_0+1}\log Df^{q_i} \\
&\quad + \sum_{l=1}^{k_0} D^{k_0+1-l}\log Df^{(a-1)q_i}\circ f^{q_i}(Df^{q_i})^{k+1-l}G_l^{k_0+1}|_0 \\
&\le |D^{k_0+1}\log Df^{(a-1)q_i}|_0 + |D^{k_0+1}\log Df^{q_i}|_0 \\
&\quad + \sum_{l=1}^{k_0}\frac{a-1}{q_{i+1}}A^{k_0+1-l}((k_0+1-l)!)^{s+1+\epsilon}\left(1+\frac{C(k+1-l)}{q_{i+1}}\right)|G_l^{k_0+1}|_0
\end{aligned}$$

where, in the last sums, we have used that for $1 \le l \le k_0$:

$$\begin{aligned}
|D^{k_0+1-l}\log Df^{(a-1)q_i}|_0 &\le \frac{a-1}{q_{i+1}}A^{k_0+1-l}((k_0+1-l)!)^{s+1+\epsilon}, \\
(Df^{q_i})^{k+1-l} &\le 1+\frac{C(k+1-l)}{q_{i+1}}.
\end{aligned}$$

The first inequality follows by (64), the second by Corollary 1 and by $q_i > k$. Then, by Lemma 9:

$$|G_l^{k_0+1}|_0 \le (\log k)\frac{C}{q_{i+1}}A^l((l)!)^{s+1+\epsilon}$$

So:

$$\begin{aligned}
&\sum_{l=1}^{k_0}\frac{a-1}{q_{i+1}}A^{k_0+1-l}((k_0+1-l)!)^{s+1+\epsilon}\left(1+\frac{C(k+1-l)}{q_{i+1}}\right)|G_l^{k_0+1}|_0 \\
&\qquad \le C(\log k_0)\sum_{l=1}^{k_0}\frac{a-1}{q_{i+1}^2}A^{k_0+1}((k_0+1-l)!)^{s+1+\epsilon} \\
&\qquad\qquad \times\left(1+\frac{C(k+1-l)}{q_{i+1}}\right)(l!)^{s+1+\epsilon} \\
&\qquad \le C\frac{(a-1)k_0}{q_{i+1}^2}A^{k_0+1}((k_0)!)^{s+1+\epsilon}(\log k)
\end{aligned}$$

So, iterating, we have:

$$\begin{aligned}
|D^{k_0+1}\log Df^{aq_i}|_0 &\leq a|D^{k_0+1}\log Df^{q_i}|_0\left(1+\frac{Ck}{q_{i+1}}\right)^a \\
&\quad +C\frac{a(a-1)k_0}{q_{i+1}^2}A^{k_0+1}((k_0)!)^{s+1+\epsilon}(\log k_0) \\
&\leq Caq_iA^{k_0+1}((k_0)!)^{s+1+\epsilon}(\log k_0)\left(1+\frac{Cak}{q_{i+1}}\right)
\end{aligned}$$

Moreover:

$$\prod_{i\geq n}\left(1+\frac{Ca_ik}{q_{i+1}}\right)\leq\prod_{i\geq n}\left(1+\frac{Ck}{q_i}\right)\leq\prod_{i\geq 0}\left(1+\frac{C}{2^i}\right),$$

that is a constant that does not depend on k.

Then, write q_n as: $q=q_n=a_nq_{n-1}+q_{n-2}$:

$$\begin{aligned}
D^{k_0+1}\log Df^{q_n} &= D^{k_0+1}\log Df^{(a_nq_{n-1}+q_{n-2})}=D^{k_0+1}\log Df^{a_nq_{n-1}}\circ f^{q_{n-2}}(Df^{q_{n-2}})^{k_0+1} \\
&\quad +D^{k_0+1}\log Df^{q_{n-2}} \\
&\quad +\sum_{l=1}^{k_0}D^{k_0+1-l}\log Df^{a_nq_{n-1}}\circ f^{q_{n-2}}(Df^{q_{n-2}})^{k+1-l}G_l^{k_0+1},
\end{aligned}$$

with

$$G_l^{k_0+1}=G_l^{k_0+1}(D\log Df^{q_{n-2}},...,D^l\log Df^{q_{n-2}}).$$

By Lemma 8:

$$\begin{aligned}
|D^{k_0+1}\log Df^{q_n}|_0 &\leq |D^{k_0+1}\log Df^{a_nq_{n-1}}\circ f^{q_{n-2}}|_0(|Df^{q_{n-2}}|_0)^{k_0+1} \\
&\quad +|D^{k_0+1}\log Df^{q_{n-2}}|_0 \\
&\quad +|\sum_{l=1}^{k_0}D^{k_0+1-l}\log Df^{a_nq_{n-1}}\circ f^{q_{n-2}}(Df^{q_{n-2}})^{k+1-l}G_l^{k_0+1}|_0 \\
&\leq C(a_nq_{n-1})A^{k_0+1}((k_0)!)^{s+1+\epsilon}(\log k_0)k_0\left(1+\frac{Ck}{q_{n-1}}\right) \\
&\quad +C(q_{n-2})A^{k_0+1}((k_0)!)^{s+1+\epsilon}(\log k_0)k_0 \\
&\quad +\left(\frac{Ck}{q_{n-1}}\right)C(a_nq_{n-1})A^{k_0+1}((k_0)!)^{s+1+\epsilon}(\log k_0)k_0 \\
&\leq C(q_n)A^{k_0+1}((k_0)!)^{s+1+\epsilon}(\log k_0)\left(1+\frac{Ck}{q_{n-1}}\right).
\end{aligned}$$

In particular, we have proved (70). Finally, we want to show that, for $h\leq h_0$, $q>k$:

$$|D^{k_0+h}\log Df^q|_0\leq BA^{k_0+h}(k_0!)^{s+1+\epsilon}(\log k_0)(k_0)^{h-1}q.$$

So, we suppose that for $1 \leq h < h_0$, $q > k$, $a < \frac{Q}{q}$ holds:

$$|D^{k_0+h} \log Df^{aq}|_0 \leq BA^{k_0+h}(k_0!)^{s+1+\epsilon}(\log k_0)(k_0)^{h-1}aq. \tag{71}$$

Note that we have just proved the case $h = 1$. We prove (71) for $h+1$. So, suppose that (71) is true for all $q_i < q_n$ with $q_{n+1} > k_0$. Then:

$$\begin{aligned}
|D^{k_0+h+1} \log Df^{q_n}|_0 &= |D^{k_0+h+1} \log Df^{(a_n q_{n-1}+q_{n-2})}|_0 \\
&\leq |D^{k_0+h+1} \log Df^{q_{n-2}} \circ f^{a_n q_{n-1}}(Df^{a_n q_{n-1}})^{k_0+h+1}|_0 \\
&\quad +|D^{k_0+h+1} \log Df^{a_n q_{n-1}}|_0 \\
&\quad +|\sum_{l=1}^{k_0+h} D^{k_0+h+1-l} \log Df^{q_{n-2}} \circ f^{a_n q_{n-1}}(Df^{a_n q_{n-1}})^{k_0+h+1-l} G_l^{k_0+h+1}|_0 \\
&\leq \left(1 + \frac{C(k_0+h)}{q_n}\right) BA^{k_0+h+1}(k_0!)^{s+1+\epsilon}(\log k_0)(k_0)^h q_{n-2} \\
&\quad +BA^{k_0+h+1}(k_0!)^{s+1+\epsilon}(\log k_0)(k_0)^h a_{n-1} q_{n-1} \\
&\quad +|\sum_{l=1}^{k_0+h} D^{k_0+h+1-l} \log Df^{q_{n-2}} \circ f^{a_n q_{n-1}}(Df^{a_n q_{n-1}})^{k_0+h+1-l} G_l^{k_0+h+1}|_0.
\end{aligned}$$

We estimate the last term: note that, by Lemma 8, for $k_0 < l \leq h$:

$$|G_l^{k_0+h+1}(D \log Df a_n q_{n-1}, ..., D^l \log Df^{a_n q_{n-1}})|_0 (\log(k)_0)(k_0)^l a_n q_{n-1} A^{k+l}(k!)^{s+1+\epsilon},$$

while, for $l \leq k_0$:

$$|G_l^{k_0+h+1}(D \log Df a_n q_{n-1}, ..., D^l \log Df^{a_n q_{n-1}})|_0 \leq C(k_0)^l \frac{1}{a_n q_{n-1}} A^{k+l}(l!)^{s+1+\epsilon}.$$

Moreover, for $l \geq h+1$:

$$|D^{k_0+h+1-l} \log Df^{q_{n-2}}|_0 \leq \frac{1}{q_{n-1}} A^{k_0+h+1-l}((k_0+h+1-l)!)^{s+1+\epsilon},$$

and, for $l < h+1$:

$$|D^{k_0+h+1-l} \log Df^{q_{n-2}}|_0 \leq Cq_{n-2} A^{k_0+h+1-l} k_0^{h+1-l}((k_0+h+1-l)!)^{s+1+\epsilon}(\log k_0).$$

In paricular:

$$|\sum_{l=1}^{k_0+h} D^{k_0+h+1-l} \log Df^{q_{n-2}} \circ f^{a_n q_{n-1}}(Df^{a_n q_{n-1}})^{k_0+h+1-l} G_l^{k_0+h+1}|_0 \leq$$

$$2ha_n CA^{k_0+h}(k_0!)^{s+1+\epsilon}(\log k_0)(k_0)^h + \frac{k}{a_n q_{n-1}^2} CA^{k_0+h}(k_0!)^{s+1+\epsilon}(\log k_0)(k_0)^h.$$

In particular:

$$|D^{k_0+h+1}\log Df^{q_n}|_0 \leq C(\log(k)_0)(k_0)^l a_n q_{n-1} A^{k+l}(k!)^{s+1+\epsilon}.$$

Step 4: Let $n \in \mathbb{N}$. Following Herman, we can write:

$$n = \sum_{i=0}^{s} b_s q_s, \tag{72}$$

with q_i the denominators of the convergents of α, $n \leq q_{s+1}$, $b_s \leq \frac{q_{s+1}}{q_s}$.
Using (72) we get, for $i = 1, 2$:

$$|D^{k+i}\log Df^n|_0 \leq CA^{k+i}(k!)^{s+1+\epsilon}k^i \sum_{s\geq 0}\frac{b_s}{q_{s+1}^{1-\epsilon}} \leq CA^{k+i}(k!)^{s+1+\epsilon}k^i \sum_{s\geq 0}\frac{q_{s+1}^{\epsilon}}{q_s}.$$

Because α is Diophantine, we have:

$$q_{s+1} \leq \frac{q_s^{\tau}}{\gamma}$$

for some $\gamma > 0, \tau \geq 1$. In particular, for ϵ small enough we get the convergence of the series (in fact it converges for ϵ small enough if and only if α is Diophantine). So, there exists $C > 0$ such that, for $i = 1, 2$:

$$|D^{k+i}\log Df^n|_0 \quad \leq \quad CA^{k+i}(k!)^{s+1+\epsilon}k^i. \quad \blacksquare$$

Step 5: We use the following identity (the proof follows easily by induction):

$$\begin{aligned} D^{k+1}\log Df^q \quad &= \quad D^{k+1}\log Dh^{-1} - \left(D^{k+1}\log Dh^{-1}\circ f^q\right)(Df^q)^{k+1} \\ &\quad -\sum_{l=1}^{k} D^{k+1-l}\log Dh^{-1}\circ f^q (Df^q)^{k+1-l}E_l^{k+1}(D\log f^q, ..., D^l \log Df^q), \end{aligned}$$

with $G_l^{k+1}(X_1, ..., X_l)$ polynomials homogeneous of degree l if X_i has weight i. By this identity we get:

$$\begin{aligned} |D^{k+1}\log Df^q|_0 \quad \leq \quad & |D^{k+1}\log Dh^{-1} - D^{k+1}\log Dh^{-1}\circ f^q|_0 \\ & +|D^{k+1}\log Dh^{-1}\circ f^q \left(1-(Df^q)^{k+1}\right)|_0 \\ & +\sum_{l=1}^{k} |D^{k+1-l}\log Dh^{-1}\circ f^q (Df^q)^{k+1-l}E_l^{k+1}(D\log f^q, ..., D^l \log Df^q)|_0. \end{aligned}$$

We recall that:

$$f_n := \frac{1}{n}\sum_{i=0}^{n-1} f^i. \tag{73}$$

Observe that:

$$f_n \circ f \circ f_n^{-1} = id + \frac{f^n - id}{n} \circ f_n^{-1}. \tag{74}$$

So, by our assumption on f we know that f_n converges to h in norm C^r for all $r \in \mathbb{N}$ and f_n^{-1} converges to h^{-1} in norm C^r for all $r \in \mathbb{N}$.

In particular, for some $C > 0$ we get the estimates:

$$\begin{aligned} |D^{k+1}\log Dh^{-1}|_0 &\leq CA^{k+1}(k!)^{s+1+\epsilon}k\log k, \\ |D^{k+2}\log Dh^{-1}|_0 &\leq CA^{k+2}(k!)^{s+1+\epsilon}k^2\log k. \end{aligned}$$

In particular:

$$\begin{aligned} |D^{k+1}\log Dh^{-1} - D^{k+1}\log Dh^{-1}\circ f^q|_0 &\leq |D^{k+2}\log Dh^{-1}|_0|Df^q - 1|_0 \\ \leq C|D^{k+2}\log Dh^{-1}|_0|\log Df^q|_0 &\leq \frac{C}{Q}A^{k+2}(k!)^{s+1+\epsilon}k^2\log k \\ &\leq \frac{C}{Q}A^{k+1}((k+1)!)^{s+1+\epsilon}. \\ |D^{k+1}\log Dh^{-1}\circ f^q\left(1-(Df^q)^{k+1}\right)|_0 &\leq C(k+1)A^{k+1}(k!)^{s+1+\epsilon}k\log k|\log Df^q|_0 \\ \leq \frac{C}{Q}A^{k+1}(k!)^{s+1+\epsilon}k^2\log k &\leq \frac{C}{Q}A^{k}((k+1)!)^{s+1+\epsilon}. \end{aligned}$$

It remains to estimate the third term. Observe that:

$$|(Df^q)^{k+1-l}|_0 \leq \frac{C}{Q}(k+1).$$

Moreover, similarly as in Lemma 5 we have:

$$|G_l^{k+1}(D\log f^q, ..., D^l\log Df^q)|_0 \leq Ck!A^l(l!)^{s+\epsilon}.$$

Putting together all these estimates:

$$\sum_{l=1}^{k}|D^{k+1-l}\log Dh^{-1}\circ f^q(Df^q)^{k+1-l}G_l^{k+1}|_0 \leq \frac{C}{Q}A^k((k+1)!)^{s+1+\epsilon},$$

so, also:

$$|D^{k+1}\log Df^q|_0 \leq \frac{C}{Q}A^k(k+1)!^{s+1+\epsilon}.$$

Choosing $A > C$ we have proved the estimate for $k+1$. Then, the proof of Theorem 1 follows by induction. ∎

2.3 Sketch of the proof of Theorem 2

Here we give a sketch of the proof of Theorem 2. We proceed by induction as in Theorem 1. So, suppose that we have that f is a s-Gevrey function that is C^k conjugate to a rotation: we want to prove that f is C^{k+1} conjugate to a rotation. In particular, we want to prove by induction that for $k \geq 1$:

$$\sup_{n\in\mathbb{N}} |D^{k+1}\log Df^n|_0 \leq C_{k+1}||\alpha n||, \tag{75}$$

for some $C > 0$ and with $\alpha = \rho(f)$. So, the proof can be divided in four steps:

Step 1 Suppose that (75) holds for some $k \in \mathbb{N}$. Then, in a similar way of the Main Lemma, there exist $A > 0$, $t > 0$ such that, for $h \in \mathbb{N}$:

$$|D^{k+h}\log Df^n|_0 \leq A^{k+h}((k+h)!)^{t+h}n^h.$$

Step 2 In Step 2 we improve in the estimate of Step 1, the growth of the part that depends on the iterates of f. In particular, if q is a convergent of α, then there exist $C > 0, \nu > t$ such that, for $h \geq 1$:

$$|D^{k+h}\log Df^q|_0 \leq A^{k+h}((k+h)!)^{\nu+h}q^{\frac{h}{4}}.$$

To prove this, note that for h small this is true: in fact, we have proved above that, for $h < \frac{k}{2}$, the term $q^{\frac{h}{4}}$ can be replaced by the term q in the inequality. The inequality is true also for $q < k^2$ (it suffices to choose ν big enough).

Then, we prove that, if the inequality is true for $l < h$, and if it is true for $l = h$, $q_i < q_n$ for some $n \in \mathbb{N}$, then it is true also for q_n.

It follows by:

$$\begin{aligned} D^{k+h}\log Df^{q_n} &= D^{k+h}\log Df^{a_nq_{n-1}+q_{n-2}} \\ &= D^{k+h}\log Df^{a_nq_{n-1}} \circ f^{q_{n-2}}(Df^{q_{n-2}})^{k+h} + D^{k+h}\log Df^{q_{n-2}} \\ &\quad + \sum_{l=1}^{k+h-1} D^{k+h-l}\log Df^{a_nq_{n-1}} \circ f^{q_{n-2}}(Df^{q_{n-2}})^{k+h-l}G_l^{k+h}, \end{aligned}$$

and by the usual estimates for the polynomials G_l^{k+h} and for $D^{k+h}\log Df^{aq_{n-1}}$.

Step 3 Let $s < 1$, and define $\delta_n := s\frac{\log q_{n+1}}{\log q_n}$. Using Hadamard's inequality, we prove that, for $h = 1, 2$:

$$|D^{k+h}\log Df^{q_n}|_0 \leq A^{k+h}((k+h)!)^{\nu+h}\frac{1}{q_{n+1}^{1-\delta_n}}2^{(\log\frac{1}{\delta_n})^2}. \tag{76}$$

In fact, let $t_n \in \mathbb{N}$ such that $2^{t_n} = [\frac{1}{\delta_n}] + 1$. Then, by Hadamard's inequality and Step 2:

$$|D^{k+2^{t_n}}\log Df^{q_n}|_0 \leq 4^tA^{k+2^{t_n}}((k+2^{t_n})!)^{\nu+2^{t_n}}\frac{q_n^{\frac{2^t_n}{4}}}{q_{n+1}^{\frac{1}{2}}},$$

$$|D^{k+2^{t_n-1}} \log Df^{q_n}|_0 \leq 4^{t_n+(t_n-1)} A^{k+2^{t_n}} ((k+2^{t_n})!)^{\nu+2^{t_n}} \frac{q_n^{\frac{2^{t_n-1}}{4}}}{q_{n+1}^{\frac{1}{2}+\frac{1}{4}}},$$

and so, iterating we prove (76).

Step 4 By the assumption

$$\log q_{n+1} = O((\log q_n)^s)$$

with $s < 2$, we have:

$$\sup_{n\in\mathbb{N}} \frac{1}{q_{n+1}^{1-\delta_n}} 2^{(\log \frac{1}{\delta_n})^2} < +\infty, \tag{77}$$

$$\sum_{n\geq 0} \frac{q_{n+1}^{\delta}}{q_n} < +\infty. \tag{78}$$

Let $n \in \mathbb{N}$, and write n as:

$$n = \sum_{0\leq i\leq j} b_i q_i,$$

with $b_i < \frac{q_{i+1}}{q_i}$, $n < q_{j+1}$. Then, similarly as in Theorem 1, for $h = 1, 2$, by (77) and (78):

$$|D^{k+h} \log Df^n|_0 \leq C \sum_{i\geq 0} q_i |D^{k+h} \log Df^{q_i}|_0 < \infty$$

In particular, f is C^{k+2} conjugate to a rotation.

Step 5 As in Theorem 1, the fact that f is C^{k+2} conjugate to a rotation, implies that:

$$\sup_{n\in\mathbb{N}} |D^{k+1} \log Df^n|_0 \leq C_{k+1} ||\alpha n||.$$

Then, we proceed by induction. ∎

2.4 Questions

We prove the theorem using a Diophantine arithmetical condition. Moreover, using the existence of Gevrey functions of compact support, proceeding as in [39], it is easy to find arithmetical conditions such that in general the C^1 conjugacy does not hold (for example imposing $\limsup \frac{\log q_{n+1}}{q_n} = \infty$). However, a natural question regard the best arithmetical condition.

In the proof we have a loss of regularity of type $1+\epsilon$. However, we don't know if it is the optimal one. For example, in the analytic case the term $1+\epsilon$ is not necessary (the diffeomorphism h is also analytic).

3 Topology of Diophantine sets

In this second part, we study the topolgy of Diophantine sets, constructing many examples of isolated points in these sets, and showing that, for large parameters, Diophantine sets are Cantor sets.

3.1 Basic definitions and remarks

3.1.1 Definitions

- $\mathbb{N} := \{1, 2, 3, ...\}$, $\mathbb{N}_0 := \{0, 1, 2, 3, ...\}$
- Given $a, b \in \mathbb{Z} - \{0\}$, we indicate with (a, b) the maximum common divisor of a and b.
- Let α be a real number. We indicate with $[\alpha]$ the integral part of α, with $\{\alpha\}$ the fractional part of α .
- Given E$\subseteq \mathbb{R}$, we indicate with $\mathcal{I}$(E) the set of isolated points of E.
- Given E$\subseteq \mathbb{R}$, we indicate with $\mathcal{A}$(E) the set of accumulation points of E.
- We say that E$\subseteq \mathbb{R}$ is perfect if $\mathcal{A}$(E)=E.
- Given a Borel set E$\subseteq \mathbb{R}$ we denote with μ(E) the Lebesgue measure of E.
- A topological space X is a totally disconnected space if the points are the only connected subsets of X.
- $X \subseteq \mathbb{R}$ is a Cantor set if it is closed, totally disconnected and perfect.
- For $E \subseteq \mathbb{R}^n$, $\dim_H E$ is the Hausdorff dimension of E.

- Given $\alpha \in \mathbb{R}$ we define:
$$||\alpha|| := \min_{p\in\mathbb{Z}} |\alpha - p|$$

- Given $\gamma > 0, \tau \geq 1$, we define the (γ, τ) Diophantine points in $(0;1)$ as the numbers in the set:
$$D_{\gamma,\tau} := \{\alpha \in (0;1) : ||q\alpha|| \geq \frac{\gamma}{q^\tau} \quad \forall q \in \mathbb{N}\}$$

-
$$D^{\mathbb{R}}_{\gamma,\tau} := \{\alpha \in \mathbb{R} : ||q\alpha|| \geq \frac{\gamma}{q^\tau} \quad \forall q \in \mathbb{N}\},$$
$$D_\tau := \bigcup_{\gamma>0} D_{\gamma,\tau}, \quad D := \bigcup_{\tau\geq 1} D_\tau.$$
We call D the set of Diophantine numbers.

- Given $\tau \geq 1, \alpha \in \mathbb{R}$, we define:
$$\gamma(\alpha, \tau) := \inf_{q\in\mathbb{N}} q^\tau ||q\alpha||$$

- Given $\alpha \in \mathbb{R}$ we define:
$$\tau(\alpha) := \inf\{\tau \geq 1 : \gamma(\alpha, \tau) > 0\}$$

- Given an irrational number $\alpha = [a_0; a_1, ...] := a_0 + \frac{1}{a_1 + \frac{1}{a_2 + ...}}$, we denote with $\{\frac{p_n}{q_n}\}_{n\in\mathbb{N}_0}$ the convergents of α, $\alpha_n := [a_n; a_{n+1}, ...]$[12].

- We indicate with $[a_1, a_2, a_3, ...] := \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + ...}}}$.

- Let α be an irrational number. We define:
$$\gamma_n(\alpha, \tau) := q_n^\tau ||q_n\alpha|| = q_n^\tau |q_n\alpha - p_n|$$

- Let $\tau \geq 1$,
$$\gamma_-(\alpha, \tau) := \inf_{n\in 2\mathbb{N}_0} \gamma_n(\alpha, \tau),$$
$$\gamma_+(\alpha, \tau) := \inf_{n\in 2\mathbb{N}_0+1} \gamma_n(\alpha, \tau),$$
$$\mathcal{D}_\tau := \{\alpha \in D_\tau : \tau(\alpha) = \tau\},$$
$$\mathcal{I}_\tau := \{\alpha \in D_\tau : \exists n \not\equiv m \pmod 2, \gamma_n(\alpha, \tau) = \gamma_m(\alpha, \tau) = \gamma(\alpha, \tau)\}.$$
$$\mathcal{I} := \cup_{\tau\geq 1} \mathcal{I}_\tau$$

[12] for information about continued fractions see [4],[8],[15]

- Let $p \in \mathbb{Z}, q \in \mathbb{N}$, $\gamma > 0, \tau \geq 1$. We define: $I_{\gamma,\tau}(p,q) := \left(\frac{p}{q} - \frac{\gamma}{q^{\tau+1}}; \frac{p}{q} + \frac{\gamma}{q^{\tau+1}}\right)$.
- Let $\tau \geq 1$,

$$\mathcal{D}_\tau := \{\alpha \in D_\tau : \tau(\alpha) = \tau\},$$

$$\mathcal{I}^1_{\gamma,\tau} := \{\alpha \in D_{\gamma,\tau} : \exists n \not\equiv m \pmod 2, \gamma_n(\alpha,\tau) = \gamma_m(\alpha,\tau) = \gamma(\alpha,\tau)\},$$

$$\mathcal{I}^2_{\gamma,\tau} := \{\alpha \in D_{\gamma,\tau} : \exists n \in \mathbb{N}_0, \gamma_n(\alpha,\tau) = \gamma(\alpha,\tau)\} \cap (\mathcal{I}^1_{\gamma,\tau})^c,$$

$$\mathcal{I}^3_{\gamma,\tau} := \mathcal{I}(D_{\gamma,\tau}) \cap (\mathcal{I}^1_{\gamma,\tau} \cup \mathcal{I}^2_{\gamma,\tau})^c,$$

$$\mathcal{I}^1_\tau := \bigcup_{\gamma>0} \mathcal{I}^1_{\gamma,\tau},$$

$$\mathcal{I}^2_\tau := \bigcup_{\gamma>0} \mathcal{I}^2_{\gamma,\tau},$$

$$\mathcal{I}^3_\tau := \bigcup_{\gamma>0} \mathcal{I}^3_{\gamma,\tau}.$$

3.1.2 Remarks

(a) $\alpha \in D_{\gamma,\tau} \iff 1-\alpha \in D_{\gamma,\tau}$.

(b) $\gamma(\alpha,\tau) \leq \min\{\alpha, 1-\alpha\}$.

(c) Fixed $\tau \geq 1$, $\gamma(.,\tau) : D_\tau \to (0, \frac{1}{2})$.

(d) $D^{\mathbb{R}}_{\gamma,\tau} = \bigcup_{n\in\mathbb{Z}}(D_{\gamma,\tau} + n)$, thus we can restrict to study the Diophantine points in $(0,1)$.

(e)

$$\begin{cases} \gamma_n(\alpha,\tau) = \frac{q_n^\tau}{\alpha_{n+1}q_n + q_{n-1}}, \\ \frac{1}{\gamma_n(\alpha,\tau)} = \frac{q_{n+1}}{q_n^\tau} + \frac{1}{\alpha_{n+2}q_n^{\tau-1}} \end{cases} \tag{79}$$

(f) $\gamma(\alpha,\tau) = \inf_{n\in\mathbb{N}_0} \gamma_n(\alpha,\tau)$.

(g) If $\tau < \tau(\alpha)$, then $\gamma(\alpha,\tau) = 0$; if $\tau > \tau(\alpha)$ then $\gamma(\alpha,\tau) > 0$. Moreover, for $\tau > \tau(\alpha)$ the inf is a minimum.

(h) $\alpha \in \mathcal{D}_\tau \iff \tau(\alpha) = \tau$ and $\gamma(\alpha,\tau) > 0$.

(i) If $\alpha \in \mathcal{I}_\tau$, then α is an isolated point of $D_{\gamma,\tau}$.

(j) The cardinality of $\mathcal{I}_\tau$ is at most countable.

(k) $\mu(\mathcal{D}_\tau) = 0$ for all $\tau \geq 1$.

(l) $\gamma_0(\alpha,\tau) = \{\alpha\}$, in particular $\gamma_0(\alpha,\tau)$ does not depend on τ.

(m) Let $\frac{p}{q}$ a rational number.

$$\alpha \in D_\tau \iff \left\{\alpha + \frac{p}{q}\right\} \in D_\tau, \tag{80}$$

$$\alpha \in \mathcal{D}_\tau \iff \left\{\alpha + \frac{p}{q}\right\} \in \mathcal{D}_\tau. \tag{81}$$

(n) If $\tau > \tau(\alpha)$, $\gamma_-(\alpha,\tau) = \gamma_+(\alpha,\tau)$, then $\alpha \in \mathcal{I}_\tau$.

(o) $\alpha \in D_\tau \iff q_{n+1} = O(q_n^\tau)$.

(p) Let α be an irrational number. We define:

$$\gamma_n(\alpha,\tau) := q_n^\tau ||q_n\alpha|| = q_n^\tau |q_n\alpha - p_n|$$

Proof (a), (d) are clear, (b) follows by definition of $\gamma(\alpha,\tau)$ and by remark (a). (c) follows by (b) (α is in $(0;1)$).

(e): the first formula follows by properties of continued fractions, moreover:

$$\frac{1}{\gamma_n(\alpha,\tau)} = \frac{\alpha_{n+1}q_n + q_{n-1}}{q_n^\tau} = \frac{(a_{n+1}q_n + q_{n-1}) + \frac{q_n}{\alpha_{n+2}}}{q_n^\tau} = \frac{q_{n+1}}{q_n^\tau} + \frac{1}{\alpha_{n+2}q_n^{\tau-1}}. \tag{82}$$

(f): follows by:

$$||q_n\alpha|| = \min_{1\le q\le q_n} ||q\alpha|| \tag{83}$$

and by definition of $\gamma(\alpha,\tau)$.

(g): The first part is clear. To prove that for $\tau > \tau(\alpha)$ the inf is a minimum, take $\tau(\alpha) < \tau' < \tau$, then $\gamma(\alpha,\tau') > 0$ and

$$\lim_{n\to+\infty} q_n^\tau ||q_n\alpha|| = \lim_{n\to+\infty} q_n^{\tau-\tau'} q_n^{\tau'} ||q_n\alpha|| \ge \lim_{n\to+\infty} \gamma(\alpha,\tau') q_n^{\tau-\tau'} = +\infty. \tag{84}$$

By (84) there exists $N \in \mathbb{N}$ such that for all $n \ge N$,

$$\gamma_n(\alpha,\tau) > \gamma(\alpha,\tau). \tag{85}$$

Therefore the inf is reached and it is a minimum.

(h): It is obvious.

(i): If α is in $\mathcal{I}_\tau$, there exist n even and m odd such that:

$$\gamma(\alpha,\tau) = \gamma_n(\alpha,\tau) = \gamma_m(\alpha,\tau). \tag{86}$$

So α is separated by the two intervals $I_{\gamma,\tau}(p_n,q_n)$ and $I_{\gamma,\tau}(p_m,q_m)$. Then, noting that $I_{\gamma,\tau}(p,q) \subseteq D^c_{\gamma,\tau}$ for all $p \in \mathbb{Z}, q \in \mathbb{N}$, we get (i).

(j): If $\gamma_n(\alpha,\tau) = \gamma_m(\alpha,\tau) = \gamma(\alpha,\tau)$, with n even, m odd, then:

$$\alpha = \frac{p_n}{q_n} + \frac{\frac{p_m}{q_m} - \frac{p_n}{q_n}}{1 + (\frac{q_n}{q_m})^{\tau+1}}, \tag{87}$$

$$\gamma = \frac{\frac{p_m}{q_m} - \frac{p_n}{q_n}}{\frac{1}{q_n^{\tau+1}} + \frac{1}{q_m^{\tau+1}}}, \tag{88}$$

so $\mathcal{I}_\tau$ is at most countable.

(k): $\mu(D_1) = 0$ (D_1 is the set of numbers with bounded coefficients of the continued fraction). Moreover $\mu(D_\tau) = 1$ for all $\tau > 1$ (because of $\mu(D^c_{\gamma,\tau}) = O(\gamma)$). For $1 < \tau' < \tau$ we have $\mathcal{D}_\tau \subseteq D^c_{\tau'}$. So, for $\tau > 1$: $\mu\left(\mathcal{D}_\tau\right) = 0$.

(l), (m): They are obvious.

(n): Because of $\tau > \tau(\alpha)$, as in the proof of (g) we get that $\gamma_-(\alpha,\tau)$ and $\gamma_+(\alpha,\tau)$ are reached, so there exist n even and m odd with $\gamma_-(\alpha,\tau) = \gamma_n(\alpha,\tau)$, $\gamma_+(\alpha,\tau) = \gamma_m(\alpha,\tau)$. Now (n) follows by definition of $\mathcal{I}_\tau$ and by $\gamma_-(\alpha,\tau) = \gamma_+(\alpha,\tau) = \gamma(\alpha,\tau)$.

(o): It follows by (e) and (f).

3.2 Basic properties of Diophantine sets

Let us recall some simple facts about Diophantine sets. The case $\tau = 1$ is quite different to the others.

Remark 5 If $0 < \gamma' \leq \gamma$, $\tau' \geq \tau \geq 1$, then $D_{\gamma,\tau} \subseteq D_{\gamma',\tau'}$. Moreover, $D_{\gamma,\tau}$ is compact and totally disconnected (because of $D_{\gamma,\tau} \cap \mathbb{Q} = \emptyset$).

Remark 6 D_1 is the set of irrational numbers with bounded coefficients of their continued fractions.

Proof It follows by (79). ∎

Theorem 6 (Hurwitz) *(see [39]) Let α be an irrational number. There exist infitely many $q \in \mathbb{N}$ such that*

$$q||q\alpha|| < \frac{1}{\sqrt{5}q}. \tag{89}$$

Theorem 7 (Borel) *(see [34]) Given a function $\psi : \mathbb{N} \to \mathbb{N}$, define*

$$A(\psi) := \{[a_0; a_1, ..., a_n, ...] : 0 < a_n < \psi(n)\}.$$

Then:

$$\sum_{n\in\mathbb{N}} \frac{1}{\psi(n)} < \infty \Rightarrow \quad \mu(A) > 0, \tag{90}$$

$$\sum_{n\in\mathbb{N}} \frac{1}{\psi(n)} = \infty \Rightarrow \quad \mu(A) = 0. \tag{91}$$

Remark 7 By Hurwitz's theorem, if $\gamma > \frac{1}{\sqrt{5}}$, then $D_{\gamma,1} = \emptyset$.

Remark 8 For all $\gamma \in (0, \frac{1}{2})$ we have $\mu(D_{\gamma,1}) = 0$. In particular $\mu(D_1) = 0$.

Proof It follows by (79) and Borel's theorem. ∎

Unless D_1 has zero measure, it has positive Hasdorff dimension. In fact, the following holds:

Theorem 8 (Jarnik) *(see [71])* $\dim_H(D_1) = 1$.

Theorem 9 *(see [36]) Let $\gamma > \frac{1}{3}$. Then the set:*

$$\{\alpha \in (0,1) : \liminf q||q\alpha|| \geq \gamma\} \tag{92}$$

is at most countable. In particular, for $\gamma > \frac{1}{3}$ $D_{\gamma,1}$ is at most countable.

The case $\tau > 1$ is quite different.

Remark 9 Let $\tau > 1$. Then, for $\gamma > 0$ we have

$$\mu(D^c_{\gamma,\tau}) = O(\gamma). \tag{93}$$

In particular, $\mu(D_\tau) = 1$ for all $\tau > 1$.

Proof For $\tau > 1$:

$$\mu(D^c_{\gamma,\tau}) \leq \sum_{q\in\mathbb{N}} \sum_{0\leq p\leq q-1} \frac{2\gamma}{q^{\tau+1}} = 2\gamma \sum_{q\in\mathbb{N}} \frac{1}{q^\tau} = O(\gamma). \tag{94}$$

∎

Corollary 2

$$\mu\left(\bigcap_{\tau>1} D_\tau\right) = 1. \tag{95}$$

3.3 Isolated points of Diophantine sets

In this section we give the proof of the results. We start by proving the Proposition we state in the introduction.

Proof Fix $\alpha := \bar{\alpha} + n$. It is easy to verify that α is such that:

$$\begin{cases} \alpha = \dfrac{1}{\alpha} + n\ , \qquad n^\tau = \alpha\ , \\[2ex] \alpha = [n; n, n, n, ...] := n + \dfrac{1}{n + \frac{1}{n + \dots}}\ , \\[2ex] p_0 = n,\ q_0 = 1,\ p_1 = n^2 + 1,\ q_1 = n,\ \alpha_k = \alpha \quad \forall k \geq 1,\ q_{k+1} = p_k\ (\forall k \geq 0)\ . \end{cases} \tag{96}$$

For $k = 0$:

$$\left|\alpha - \frac{p_0}{q_0}\right| \overset{(96)}{=} \alpha - n \overset{(96)}{=} \frac{1}{\alpha} = \gamma. \tag{97}$$

For $k \geq 1$, from (96) and the fact that $p_k/q_k \leq p_1/q_1$ and $q_k \geq q_1$, we obtain:

$$\begin{aligned} \frac{q_{k+1}}{q_k^\tau} + \frac{1}{a_{k+2} q_k^{\tau-1}} &= \frac{p_k}{q_k} \frac{1}{q_k^{\tau-1}} + \frac{1}{\alpha q_k^{\tau-1}} \\ &\leq \frac{p_1}{q_1} \frac{1}{q_1^{\tau-1}} + \frac{1}{\alpha q_1^{\tau-1}} = \frac{n^2+1}{n^\tau} + \frac{1}{n^{\tau-1}\alpha} \\ &= \frac{n^2+1}{\alpha} + \frac{n}{\alpha^2} = \frac{1}{\alpha}\left(n^2 + 1 + \frac{n}{\alpha}\right) \\ &= \frac{1}{\alpha}\,(\alpha n + 1) = n + \frac{1}{\alpha} = \alpha \\ &= \frac{1}{\gamma}\ , \end{aligned}$$

that, togheter with (97), it shows that $\alpha \in D_{\gamma,\tau} + n$.
From (96),

$$\begin{aligned} \left|\alpha - \frac{p_1}{q_1}\right| &= \frac{p_1}{q_1} - \alpha = \frac{n^2+1}{n} - \alpha = \frac{1}{n} + n - \alpha \\ &= \frac{1}{n} - \frac{1}{\alpha} = \frac{1}{n\alpha^2} = \frac{1}{\alpha}\ \frac{1}{q_1 n^\tau} = \frac{1}{\alpha q_1^{\tau+1}} \\ &= \frac{\gamma}{q_1^{\tau+1}}\ , \end{aligned}$$

that shows, togheter with (97), that α divides the two intervals $I_{\gamma,\tau}(p_0, q_0)$ and $I_{\gamma,\tau}(p_1, q_1)$, with $I_{\gamma,\tau}(p,q) := \left(\frac{p}{q} - \frac{\gamma}{q^{\tau+1}}; \frac{p}{q} + \frac{\gamma}{q^{\tau+1}}\right)$. So $\alpha \in D_{\gamma,\tau} + n$ implies that α is an isolated point of $D_{\gamma,\tau} + n$, i.e. $\bar{\alpha}$ is an isolated point of $D_{\gamma,\tau}$. ▮

Before proving Theorem A we need some simple lemma. So we prove at first the continuity of the functions $\gamma(\alpha,\tau)$, $\gamma_-(\alpha,\tau), \gamma_+(\alpha,\tau)$ as functions of τ.

Lemma 11 *Let $a \in \mathbb{R}$, $f_n \geq 0$ be continuous and increasing functions in $[a, +\infty)$ such that:*

$$\forall x > a, \quad \lim_{n\to+\infty} f_n(x) = +\infty. \tag{98}$$

Define

$$f(x) := \inf_{n\in\mathbb{N}} f_n(x). \tag{99}$$

If f is bounded, then $f \in C([a, +\infty))$.

Proof Observe that f is increasing because f_n are increasing. Let $C > 0$ be such that $f(x) \leq C$ for all $x \in [a, +\infty)$. Take $x \in \mathbb{R}$ such that $a < x$. By (29) there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $f_n(x) > C > 0$. For $y \geq x$, $f(y) = \min_{0\leq n<N} f_n(y)$, so f is continuous and increasing in $(x, +\infty)$ and $f \in C((a, +\infty))$. It remains to show that f is continuous in a, i.e. $f(a) = \lim_{x\to a} f(x)$. In fact, for all $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that

$$0 < f_n(a) - f(a) < \epsilon \tag{100}$$

and by continuity of f_n there exists $\delta > 0$ such that for $0 < x - a < \delta$ we have:

$$0 < f_n(x) - f_n(a) < \epsilon. \tag{101}$$

So, for $0 < x - a < \delta$:

$$0 \leq f(x) - f(a) \leq f_n(x) - f_n(a) + f_n(a) - f(a) < 2\epsilon, \tag{102}$$

that proves the continuity in a.

Corollary 3 *Fixed $\alpha \in D$, the functions $\gamma(\alpha,\tau), \gamma_-(\alpha,\tau), \gamma_+(\alpha,\tau)$ are continuous and increasing for $\tau \geq \tau(\alpha)$.*

Proof We prove the corollary for $\gamma(\alpha,\tau)$ (the proof for $\gamma_-(\alpha,\tau), \gamma_+(\alpha,\tau)$ are similar). Observe that $\gamma_n(\alpha,\tau) \leq \frac{1}{2}$. Consider the $\gamma_n(\alpha,\tau)$ as functions of τ. For $\tau > \tau(\alpha)$ we have

$$\lim_{n\to+\infty} \gamma_n(\alpha,\tau) = +\infty \tag{103}$$

Moreover the $\gamma_n(\alpha,\tau)$ are increasing with respect to τ, so the hypothesis of Lemma 12 are satisfied. ▮

Now we give a simple sufficient condition such that a Diophantine number belongs to $\mathcal{I}_\tau$ for some $\tau \geq \tau(\alpha)$.

Lemma 12 *Let $\alpha \in D \cap (0; \frac{1}{2})$ be such that there exists $\tau' > \tau(\alpha)$ with:*

$$\gamma_-(\alpha, \tau') \geq \gamma_+(\alpha, \tau') \tag{104}$$

Then there exists $\tau \geq \tau'$ such that $\alpha \in \mathcal{I}_\tau$

Proof If:

$$\gamma_-(\alpha, \tau') = \gamma_+(\alpha, \tau') \tag{105}$$

then $\alpha \in \mathcal{I}_{\tau'}$ by remark (g) and because of $\tau' > \tau(\alpha)$. Now consider the case:

$$\gamma_-(\alpha, \tau') > \gamma_+(\alpha, \tau') \tag{106}$$

Observe that:

$$\gamma_-(\alpha, \tau) \leq \gamma_0(\alpha, \tau) \leq \max\{\alpha, 1-\alpha\}. \tag{107}$$

Moreover

$$\lim_{\tau \to +\infty} \gamma_+(\alpha, \tau) = +\infty \tag{108}$$

because it is an increasing function and because of $\alpha \in (0, \frac{1}{2})$. So, by continuity of $\gamma_-(\alpha, \tau), \gamma_+(\alpha, \tau)$ and by (107), (108) there exists $\tau > \tau'$ such that $\gamma_-(\alpha, \tau) = \gamma_+(\alpha, \tau)$, so $\alpha \in \mathcal{I}_\tau$ by remark (g). ∎

Remark 10 Note that the condition (104) is satisfied for $\bar{\alpha}$ defined in the Proposition. Moreover, for this $\bar{\alpha}$ there exists a unique τ such that $\gamma_-(\bar{\alpha}, \tau) = \gamma_+(\bar{\alpha}, \tau)$.

Proof (Theorem A) Fixed $\tau \geq 1, \gamma \in (0; \frac{1}{2})$, consider the map $\Phi_{\gamma,\tau}$ defined in the statement of Teorem A. Let $\alpha \in D_{\gamma,\tau}$. Observe that, if $\alpha = [a_1, a_2, ...]$ then:

$$\Phi(\alpha) = [2, \left[2^\tau \frac{3}{\gamma}\right], a_1, a_2, ...] =: [b_1, b_2, b_3, ...]. \tag{109}$$

We denote with q_n the denominator of the n-th convergent to $\Phi(\alpha)$, with β_n the n-th residue of $\Phi(\alpha)$ and with q'_n the denominator of the n-th convergent to α. We recall that:

$$\frac{1}{\gamma_n(\Phi(\alpha), \tau)} = \frac{q_{n+1}}{q_n^\tau} + \frac{1}{\beta_{n+2} q_n^{\tau+1}}, \tag{110}$$

and

$$\frac{q_{n+1}}{q_n^\tau} + \frac{1}{\beta_{n+2} q_n^{\tau+1}} = \frac{q_{n-1}}{q_n^\tau} + \frac{b_{n+1}}{q_n^{\tau-1}} + \frac{1}{\beta_{n+2} q_n^{\tau+1}}. \tag{111}$$

So, by (111):

$$\begin{cases} \dfrac{1}{\gamma_0(\Phi(\alpha), \tau)} < \left[\dfrac{3}{\gamma}\right] \\[2ex] \dfrac{1}{\gamma_1(\Phi(\alpha), \tau)} > \left[\dfrac{3}{\gamma}\right] \\[2ex] \dfrac{1}{\gamma_n(\Phi(\alpha), \tau)} < \dfrac{2}{\gamma} \quad for \quad n \geq 2. \end{cases} \tag{112}$$

In fact:

$$\frac{1}{\gamma_0(\Phi(\alpha),\tau)} = q_1 + \frac{1}{\beta_2} = 2 + \frac{1}{\beta_2} < 3 < \left[\frac{3}{\gamma}\right], \tag{113}$$

$$\frac{1}{\gamma_1(\Phi(\alpha),\tau)} > \frac{q_2}{q_1^\tau} = \frac{2\left[2^\tau \frac{3}{\gamma}\right]+1}{2^\tau} \geq \left[\frac{3}{\gamma}\right], \tag{114}$$

while, for $n \geq 2$:

$$\frac{1}{\gamma_n(\Phi(\alpha),\tau)} = \frac{q_{n-1}}{q_n^\tau} + \frac{a_{n-1}}{q_n^{\tau-1}} + \frac{1}{\alpha_{n-2}q_n^{\tau+1}} < \tag{115}$$

$$< 1 + \frac{a_{n-1}}{q_{n-2}^{\prime(\tau-1)}} < 1 + \frac{1}{\gamma} < \left[\frac{3}{\gamma}\right], \tag{116}$$

by $q_n > q'_{n-2}$. By (112), for all $\alpha \in D_{\gamma,\tau}$, $\Phi(\alpha)$ satisfies the hypothesis of Lemma 15. In fact the first coefficient of $\Phi(\alpha)$ is greater then 1, moreover:

$$\gamma_-(\Phi(\alpha),\tau) > \left[\frac{\gamma}{3}\right] > \gamma_+(\Phi(\alpha),\tau). \tag{117}$$

So, given $\alpha \in D_{\gamma,\tau}$, $\Phi(\alpha)$ is a Diophantine number equivalent to α that is in $\mathcal{I}_{\tau'}$ for some $\tau' > \tau$. From the arbitrariness of γ, τ, Theorem A follows. ∎

Corollary 4 *For all $\tau \geq 1$ we have:*

$$\mu\Big(\bigcup_{\tau' \geq \tau} \mathcal{I}_{\tau'}\Big) > 0. \tag{118}$$

Proof It suffices to note that for all $\gamma \in (0, \frac{1}{2}), \tau \geq 1$, the map: $\Phi_{\gamma,\tau} : D_{\gamma,\tau} \to D$ is Lipschitz and that $\mu(D_{\gamma,\tau}) > 0$ for small γ. ∎

Remark 11 Suppose that $\alpha \in D$ such that $\gamma_-(\alpha,\tau) = \gamma_+(\alpha,\tau)$ for some $\tau > \tau(\alpha)$. Then α is an isolated point of $D_{\gamma(\alpha,\tau),\tau}$.

Proof In fact, for $\tau > \tau(\alpha)$ $\gamma_-(\alpha,\tau)$ and $\gamma_+(\alpha,\tau)$ are achieved for some n even and m odd. ∎

Remark 12 If $\gamma_-(\alpha,\tau) = \gamma_+(\alpha,\tau)$ with $\alpha \in D$ and $\tau = \tau(\alpha)$, in general α is not an isolated point of $D_{\gamma(\alpha,\tau),\tau}$.

Proof For example, take $\tau = 2, \gamma = \frac{1}{4}$. We define $\alpha = [a_1, a_2, ...]$ iteratively. $a_1 := 2$, and for $n \geq 1$:

$$a_{n+1} := \frac{q_n^{\tau-1}}{\gamma} - 3 \tag{119}$$

with $q_{-1} = 0, q_0 = 1$, $q_n = a_{n-1}q_{n-1} + q_{n-2}$ for $n \geq 1$. Then it is easy to check that the a_n are strictly increasing, moreover $\tau(\alpha) = \tau = 2$, $\gamma(\alpha, \tau(\alpha)) = \gamma = \frac{1}{4}$. For $n \geq 2$ define:

$$\delta_n := [a_1, a_2, ..., a_{n-1}, a_n + 1, 1, 1, 1, ...]. \tag{120}$$

We show that $\delta_k \in D_{\gamma,\tau}$ and $\delta_k \to \alpha$. For $n < k-1$ we have:

$$\frac{1}{\gamma_n(\delta_k, \tau)} < \frac{a_{n+1}}{q_n^{\tau-1}} + \frac{q_{n-1}}{q_n^{\tau}} + \frac{1}{q_n^{\tau-1}} = \frac{1}{\gamma} + \frac{q_{n-1}}{q_n^{\tau}} - \frac{2}{q_n^{\tau-1}} < \frac{1}{\gamma} \tag{121}$$

For $n > k-1$ it is clear that

$$\frac{1}{\gamma_n(\delta_k, \tau)} < 2. \tag{122}$$

For $n = k-1$:

$$\frac{1}{\gamma_n(\delta_k, \tau)} < \frac{a_{n+1}+1}{q_n^{\tau-1}} + \frac{q_{n-1}}{q_n^{\tau}} + \frac{1}{q_n^{\tau-1}} = \frac{1}{\gamma} + \frac{q_{n-1}}{q_n^{\tau}} - \frac{1}{q_n^{\tau-1}} < \frac{1}{\gamma} \tag{123}$$

So we have proved that $\delta_k \in D_{\gamma,\tau}$ for all $k \geq 2$. Moreover $\delta_k \to \alpha$, so α is not an isolated point of $D_{\gamma,\tau}$. ∎

The number constructed in the proof of Remark (12) is not an isolated point because the sequence $\frac{1}{\gamma_n(\alpha,\tau)}$ converges too slowly to $\frac{1}{\gamma}$. Moreover, observe that $\gamma(\alpha, \tau)$ is not achieved ($\gamma_n(\alpha, \tau) < \gamma$ for all n).

Proof (Theorem B) We construct $\alpha = [a_1, a_2, ...]$ with a_n defined iteratively. We fix:

$$\begin{cases} a_1 = 3, \quad a_2 = \left[3^{\tau_1+1}\right], \\ \\ q_0 = 1, \quad q_1 = a_1, \quad q_2 = a_1a_2 + 1 \end{cases} \tag{124}$$

Define:

$$C_1 := \max_{k=0,1} \frac{q_{k+1}}{q_k^{\tau_2}} = \frac{q_2}{q_1^{\tau_2}} > 3. \tag{125}$$

For $n \geq 3$ let:

$$b_n^{(1)} := \left[({C_1}^2 q_{n-1})^{\tau_2 - 1}\right]. \tag{126}$$

As long as n is even or

$$\frac{b_n^{(1)}}{q_{n-1}^{\tau_1-1}} \geq C_1 - 1, \tag{127}$$

define

$$a_n = 1. \tag{128}$$

Because of $q_{n-1} > 2^{n-1}$ and $\tau_1 > \tau_2$, there exists n_1 such that:

$$\frac{b_{n_1}^{(1)}}{q_{n_1-1}^{\tau_1-1}} < C_1 - 1. \tag{129}$$

For such n_1, define

$$a_{n_1} = b_{n_1}. \tag{130}$$

Define:

$$C_2 := \max_{k \le n_1} \frac{a_k}{q_{k-1}^{\tau_3-1}} = \frac{a_{n_1}}{q_{n_1-1}^{\tau_3-1}} > {C_1}^2 - 1 \tag{131}$$

For $n > n_1$, define:

$$b_n^{(2)} := \Big[({C_2}^2 q_{n-1})^{\tau_3-1}\Big]. \tag{132}$$

As long as n is odd or

$$\frac{b_n^{(2)}}{q_{n-1}^{\tau_2-1}} \ge C_2 - 1 \tag{133}$$

or

$$\frac{b_n^{(2)}}{q_{n-1}^{\tau_1-1}} \ge C_1 - 1, \tag{134}$$

define $a_n := 1$. define $a_n = 1$. Because of $q_n > 2^n$ and $\tau_3 < \tau_2 < \tau_1$, there exists $n_2 > n_1$ such that all these condition are not satisfied For this n_2 define

$$a_{n_2} = b_{n_2}. \tag{135}$$

So, iterating this costruction, we define $\alpha := [a_1, a_2, ...]$. By definition of a_n we get that, for n even:

$$\gamma_-(\alpha, \tau_n) < \gamma_+(\alpha, \tau_n), \tag{136}$$

and for n odd:

$$\gamma_-(\alpha, \tau_n) > \gamma_+(\alpha, \tau_n). \tag{137}$$

In fact, for n even we have:

$$\gamma(\alpha, \tau_n) = \gamma_-(\alpha, \tau_n) \ge C_{n-1} > \gamma_+(\alpha, \tau_n) \tag{138}$$

and, for n odd:

$$\gamma(\alpha, \tau_n) = \gamma_+(\alpha, \tau_n) \ge C_{n-1} > \gamma_-(\alpha, \tau_n) \tag{139}$$

Moreover, it is easy to verify that $\tau(\alpha) = \tau$ (using remark (o)), so $\alpha \in D_{\bar{\tau}}$ for all $\bar{\tau} > \tau$. By Lemma 15, there is a sequence $\{\bar{\tau}_n\}_{n\in\mathbb{N}}$ with $\tau_{n+1} < \bar{\tau}_n < \tau_n$ with $\alpha \in \mathcal{I}_{\bar{\tau}_n}$. ∎

As an immediate consequence of Theorem B we have the following:

Corollary 5 *The set*

$$\mathcal{T} := \left\{ \tau \geq 1 : \mathcal{I}_\tau \neq \emptyset \right\} \tag{140}$$

is dense in $[1, +\infty)$.

Remark 13 $\mathcal{I}_\tau = \emptyset$ for all $\tau \in \mathbb{Q}$.

Proof It follows by (87). ∎

Remark 14 $\mathcal{I}$ is strictly contained in D.

Proof Define $\alpha := [3, 1, 1, 1, ...]$, so $\alpha \in D_1$. For $\tau \geq 1, n \geq 1$:

$$\frac{1}{\gamma_0(\alpha, \tau)} = \frac{1}{\gamma_0(\alpha, 1)} > 3, \tag{141}$$

$$\frac{1}{\gamma_n(\alpha, \tau)} = \frac{1}{q_n^{\tau-1}} + \frac{q_{n-1}}{q_n^\tau} + \frac{1}{\alpha_{n+2} q_n^{\tau-1}} < \frac{3}{q_n^{\tau-1}} \tag{142}$$

because of $q_n < q_{n-1}$. So, for $\tau \geq 1$ we have:

$$\gamma_-(\alpha, \tau) < \frac{1}{3} \leq \gamma_+(\alpha, \tau) \tag{143}$$

Then, for all $\tau \geq 1$ we have $\alpha \notin \mathcal{I}_\tau$. ∎

Remark 15 Given $\alpha \in D$, the set:

$$\mathcal{E}(\alpha) := \{\tau \geq 1 : \alpha \in \mathcal{I}_\tau\} \tag{144}$$

is discrete.

Proof Suppose $\tau \in \mathcal{E}(\alpha)$. Let $n := \min\{h \in \mathbb{N}_0 : \gamma_h(\alpha, \tau) = \gamma(\alpha, \tau)\}$ Because of $\gamma_+(\alpha, -), \gamma_-(\alpha, -) \in C([\tau(\alpha), +\infty))$, it is easy to verify that there exists $\delta > 0$ such that

$$\gamma(\alpha, \tau') = \gamma_n(\alpha, \tau') < \gamma_k(\alpha, \tau') \tag{145}$$

for all $\tau' \in (\tau, \tau + \delta)$, $k \neq n$. If $\tau = \tau(\alpha)$, then it is clear that $\alpha \notin \mathcal{I}_{\tau'}$ for all $\tau' < \tau$. If $\tau > \tau(\alpha)$, it is well defined also:

$$m := \max\left\{h \in \mathbb{N}_0 : \gamma_h(\alpha, \tau) = \gamma(\alpha, \tau)\right\}. \tag{146}$$

Then, it is easy to check that there exists $\delta' > 0$ such that:

$$\gamma(\alpha, \tau') = \gamma_m(\alpha, \tau') < \gamma_k(\alpha, \tau') \tag{147}$$

for all $\tau' \in (\tau - \delta', \tau)$, $k \neq m$. So, by definition of $\mathcal{I}_\tau$ we have $\alpha \notin \mathcal{I}_{\tau'}$ for all $\tau' \in (\tau - \delta', \tau) \cup (\tau, \tau + \delta)$. ∎

Remark 16 If $\alpha \in D$, $\tau = \tau(\alpha)$ and there exists a strictly decreasing sequence $\{\tau_n\}_{n \in \mathbb{N}}$ with $\tau_n \searrow \tau$ and with $\alpha \in \mathcal{I}_{\tau_n}$ for all $n \in \mathbb{N}$, then $\alpha \notin \mathcal{I}_\tau$.

Proof It follows directly by Remark (15). ∎

3.4 Diophantine sets in general are Cantor sets

In the first part of this section, we suppose without loss of generality that n is always even. In fact, for n odd it suffices to consider $1-\alpha$ We want to prove Theorem C, i.e. for $\tau > \frac{3+\sqrt{17}}{2}$:

$$\mu\left(\left\{0<\gamma<\frac{1}{2} : \mathcal{I}(D_{\gamma,\tau}) \neq \emptyset\right\}\right) = 0.$$

By Remark (j) it is enough to prove it for $\mathcal{I}^2_{\gamma,\tau}$ and $\mathcal{I}^3_{\gamma,\tau}$. Observe that the isolated points of type 2,3 are obtained by infinitely many intersections of intervals centered in rational numbers $\frac{p}{q}$ with length $\frac{2\gamma}{q^{\tau+1}}$. Thus, the first step is to show that, given $\alpha \in D_{\gamma,\tau}$, it is enough (up to a set of measure zero and for τ big enough) to control the intersection of intervals centred in the convergents. The second step will be to show that, if intervals centred in the convergents intersects, then the coefficients of the continued fractions cannot grow too. In the final step we prove that, when intervals centred in the convergents do not intersect and for big convergents, the interval between two subsequent convergentes (with the same parity) contains a diophantine subset with positive mesure.

Lemma 13 *Let $\gamma > 0, \tau > 1, \alpha \in D_{\gamma,\tau}$, $\frac{p_n}{q_n}$ the convergents to α,*

$$I_n := \left(\frac{p_n}{q_n}, \frac{p_{n+2}}{q_{n+2}}\right).$$

Suppose that $\exists N \in \mathbb{N}$ such that, for all $n > N$ even:

$$\frac{p_n}{q_n} + \frac{\gamma}{q_n^{\tau+1}} < \frac{p_{n+2}}{q_{n+2}} - \frac{\gamma}{q_{n+2}^{\tau+1}}. \tag{148}$$

For $n > N$ define

$$A_n := \left(\frac{p_n}{q_n} + \frac{\gamma}{q_n^{\tau+1}}, \frac{p_{n+2}}{q_{n+2}} - \frac{\gamma}{q_{n+2}^{\tau+1}}\right).$$

Moreover, suppose that for every n (even):

$$\alpha - \frac{p_n}{q_n} > \frac{\gamma}{q_n^{\tau+1}} \tag{149}$$

Then, there exists $N_1 \in \mathbb{N}$ such that, for all $n > N_1$:

$$\frac{p}{q} \notin I_n \implies \frac{p}{q} + \frac{\gamma}{q^{\tau+1}}, \frac{p}{q} - \frac{\gamma}{q^{\tau+1}} \notin A_n.$$

Proof Note that it is enough to verify the inequality when $\frac{p}{q} < \alpha$. In fact the inequality is trivial if $\frac{p}{q} > \alpha$ (because of $\alpha \in D_{\gamma,\tau}$ implies $\frac{p}{q} - \frac{\gamma}{q^{\tau+1}} \geq \alpha > \frac{p_{n+2}}{q_{n+2}} + \frac{\gamma}{q_{n+2}^{\tau+1}}$ by (12)). By (148) it follows that $A_n \cap A_m = \emptyset$ for $n \neq m$, with $n, m > N$ even. By

$$\alpha - \frac{p_n}{q_n} > \frac{\gamma}{q_n^{\tau+1}}$$

for n even, we get
$$\max_{2n\le N}\frac{p_{2n}}{q_{2n}}+\frac{\gamma}{q_{2n}^{\tau+1}}=:C<\alpha,$$
from which it follows that there exists $N_1\in\mathbb{N}$ such that for n even, $n>N_1$:
$$\frac{p_n}{q_n}-\frac{\gamma}{q_n^{\tau+1}}>C.$$
If $\frac{p}{q}=\frac{p_m}{q_m}\notin I_n$ is an even convergent to α with $n>N_2:=\max\{N,N_1\}$ then, for $m\le N$ even:
$$\frac{p_m}{q_m}<\frac{p_n}{q_n}.$$
Moreover, by definition of N_1 it follows that:
$$\frac{p_m}{q_m}+\frac{\gamma}{q_m^{\tau+1}}\le C<\frac{p_n}{q_n}-\frac{\gamma}{q_n^{\tau+1}},$$
from which it follows that the Lemma holds if $\frac{p}{q}=\frac{p_m}{q_m}$ is an even convergent to α with $m\le N$. If $m>N$ and $n>m$ is even:
$$\frac{p_m}{q_m}+\frac{\gamma}{q_m^{\tau+1}}<\frac{p_{m+2}}{q_{m+2}}-\frac{\gamma}{q_{m+2}^{\tau+1}}\le\frac{p_n}{q_n}+\frac{\gamma}{q_n^{\tau+1}}$$
while, for $n<m$ even:
$$\frac{p_m}{q_m}-\frac{\gamma}{q_m^{\tau+1}}>\frac{p_{m-2}}{q_{m-2}}+\frac{\gamma}{q_{m-2}^{\tau+1}}\ge\frac{p_{n+2}}{q_{n+2}}+\frac{\gamma}{q_{n+2}^{\tau+1}}.$$
So Lemma 13 is true if $\frac{p}{q}$ is an even convergent to α. Thus, Lemma 13 remains to be verified when $\frac{p}{q}$ is not a convergent to α. It is no restrictive to suppose that there exists $m\ne n$ even for which $\frac{p}{q}\in I_m$, otherwise Lemma 13 is trivial. Now we show that, for m big enough:
$$\frac{p}{q}+\frac{\gamma}{q^{\tau+1}},\frac{p}{q}-\frac{\gamma}{q^{\tau+1}}\in\left(\frac{p_m}{q_m}-\frac{\gamma}{q_m^{\tau+1}},\frac{p_{m+2}}{q_{m+2}}+\frac{\gamma}{q_{m+2}^{\tau+1}}\right)$$
from which Lemma 13 follows immediately by (12). By the properties of Farey sequence, for the rationals $\frac{p}{q}\in I_m$ we have $q>q_m$, so the inequality:
$$\frac{p}{q}-\frac{\gamma}{q^{\tau+1}}>\frac{p_m}{q_m}-\frac{\gamma}{q_m^{\tau+1}}$$
holds. It remains to show that:
$$\frac{p}{q}+\frac{\gamma}{q^{\tau+1}}<\frac{p_{m+2}}{q_{m+2}}+\frac{\gamma}{q_{m+2}^{\tau+1}}.$$
This inequality holds for $q\ge\frac{q_{m+2}}{2}$ and m big enough. In fact, in that case:
$$\frac{p_{m+2}}{q_{m+2}}-\frac{p}{q}\ge\frac{1}{qq_{m+2}}>\frac{\gamma}{q^{\tau+1}}-\frac{\gamma}{q_{m+2}^{\tau+1}},$$

that is true for m big enough (because of $\tau > 1$). So, we can assume that $q_m < q < \frac{q_{m+2}}{2}$. Because we have assumed that $\frac{p}{q}$ is not a convergent, by Legendre's Theorem (see [39]), we have:

$$\alpha - \frac{p}{q} > \frac{1}{2q^2},$$

while, because $\frac{p_m}{q_m}$ is a convergent, we have:

$$\alpha - \frac{p_{m+2}}{q_{m+2}} < \frac{1}{q_{m+2}^2}.$$

So, putting together the two inequalities, if $q < \frac{q_{m+2}}{2}$:

$$\frac{p_{m+2}}{q_{m+2}} - \frac{p}{q} = \frac{p_{m+2}}{q_{m+2}} - \alpha + \alpha - \frac{p}{q} > \frac{1}{2q^2} - \frac{1}{q_{m+2}^2} > -\frac{\gamma}{q_{m+2}^{\tau+1}} + \frac{\gamma}{q^{\tau+1}} \iff$$

$$\frac{1}{2q^2} - \frac{\gamma}{q^{\tau+1}} > \frac{1}{q_{m+2}^2} - \frac{\gamma}{q_{m+2}^{\tau+1}},$$

that is true for m big enough (it follows by $q_m < q < \frac{q_{m+2}}{2}$). So Lemma 13 is proved. ∎

We know by Farey's sequence that for $\frac{p}{q} \in I_n$, $q > q_{n+1}$. So, there are a finite numbers of $\frac{p}{q} \in I_n$ with $q < q_{n+2}$. In the next Lemma we want to control the distance between these numbers and $\frac{p_{n+2}}{q_{n+2}} - \frac{\gamma}{q_{n+2}^{\tau+1}}$.

Lemma 14 *Let $\gamma > 0$, $\tau > 3, \alpha \in D_{\gamma,\tau}$, $\frac{p_n}{q_n}$ the convergents to α. There exists $N_1 \in \mathbb{N}$ such that, for $n > N_1$:*

$$\frac{p}{q} \in I_n, q < q_{n+2} \implies \frac{p}{q} + \frac{\gamma}{q^{\tau+1}} < \frac{p_{n+2}}{q_{n+2}} - \frac{\gamma}{q_{n+2}^{\tau+1}} - \frac{2\gamma}{q_{n+2}^{\tau-1}}.$$

Proof Let $n > N$, $\frac{p}{q} \in I_n$, so by definition of convergents and the fact that $\frac{p_n}{q_n} < \frac{p}{q} < \frac{p_{n+2}}{q_{n+2}}$ we get that $\frac{p}{q}$ is not a convergent. If $q \geq \frac{q_{n+2}}{2}$ we get:

$$\frac{p_{n+2}}{q_{n+2}} - \frac{p}{q} \geq \frac{1}{qq_{n+2}} \geq \frac{1}{q_{n+2}^2} > \frac{\gamma 2^{\tau+1}}{q_{n+2}^{\tau+1}} + \frac{\gamma}{q_{n+2}^{\tau+1}} + \frac{2\gamma}{q_{n+2}^{\tau-1}} \geq \frac{\gamma}{q^{\tau+1}} + \frac{\gamma}{q_{n+2}^{\tau+1}} + \frac{2\gamma}{q_{n+2}^{\tau-1}}$$

for n big enough (because of $\tau > 3$). So, for n big enough, the inequality remain to be proved for $q < \frac{q_{n+2}}{2}$. In that case:

$$\frac{p_{n+2}}{q_{n+2}} - \frac{p}{q} = \frac{p_{n+2}}{q_{n+2}} - \alpha + \alpha - \frac{p}{q} > \frac{1}{2q^2} - \frac{1}{q_{n+2}^2} > \frac{\gamma}{q^{\tau+1}} + \frac{\gamma}{q_{n+2}^{\tau+1}} + \frac{2\gamma}{q_{n+2}^{\tau-1}} \iff$$

$$\frac{1}{2q^2} - \frac{\gamma}{q^{\tau+1}} > \frac{1}{q_{n+2}^2} + \frac{\gamma}{q_{n+2}^{\tau+1}} + \frac{2\gamma}{q_{n+2}^{\tau-1}}.$$

From the fact that

$$G(x) := \frac{1}{2x^2} - \frac{\gamma}{x^{\tau+1}}$$

is a decreasing function for x big enough, it is enough to show the inequality for $q = [\frac{q_{n+2}}{2}]$. In this case we get:

$$\frac{1}{2q^2} - \frac{1}{q_{n+2}^2} \geq \frac{2}{q_{n+2}^2} - \frac{1}{q_{n+2}^2} = \frac{1}{q_{n+2}^2} > \frac{\gamma}{q^{\tau+1}} + \frac{\gamma}{q_{n+2}^{\tau+1}} + \frac{2\gamma}{q_{n+2}^{\tau-1}}$$

for n big enough (for $\tau > 3$), so $\exists N_1 \in \mathbb{N}$ such that, when $n > N_1$ is even the inequality is verified. ∎

Lemma 15 *Let* $\tau > 3$ *,* $\alpha = [a_1, a_2, ...] \in D_{\gamma,\tau}, \frac{p_n}{q_n}$ *the convergents to* α*, then* $\exists N \in \mathbb{N}$ *such that for all* $n > N$ *even:*

$$\mu\left(\bigcup_{\frac{p}{q}\in I_n, q\geq q_{n+2}} \left(\frac{p}{q} - \frac{\gamma}{q^{\tau+1}}, \frac{p}{q} + \frac{\gamma}{q^{\tau+1}}\right)\right) < \frac{2\gamma}{q_{n+2}^{\tau-1}}$$

Proof

$$\mu\left(\bigcup_{\frac{p}{q}\in I_n, q\geq q_{n+2}} \left(\frac{p}{q} - \frac{\gamma}{q^{\tau+1}}, \frac{p}{q} + \frac{\gamma}{q^{\tau+1}}\right)\right)$$

$$< \sum_{q\geq q_{n+2}} \sum_{q\frac{p_n}{q_n} < p < q\frac{p_{n+2}}{q_{n+2}}} \frac{2\gamma}{q^{\tau+1}} < 2\gamma \left(\frac{p_{n+2}}{q_{n+2}} - \frac{p_n}{q_n}\right) \sum_{q\geq q_{n+2}} \frac{1}{q^\tau}$$

$$< 2\gamma C \left(\frac{p_{n+2}}{q_{n+2}} - \frac{p_n}{q_n}\right) \frac{1}{q_{n+2}^{\tau-1}} = o\left(\frac{2\gamma}{q_{n+2}^{\tau-1}}\right)$$

for some constant $C > 0$. ∎

Lemma 16 *Let* $\tau > 1, \gamma > 0$, $\alpha = [a_1, a_2, ...] \in D_{\gamma,\tau}, \frac{p_n}{q_n}$ *be the convergents to* α*. Then:*

$$\frac{p_n}{q_n} + \frac{\gamma}{q_n^{\tau+1}} < \frac{p_{n+2}}{q_{n+2}} - \frac{\gamma}{q_{n+2}^{\tau+1}} \iff \tag{150}$$

$$a_{n+2} > \frac{q_n}{\gamma q_{n+1}} \frac{1}{(\frac{1}{\gamma} - \frac{q_{n+1}}{q_n^\tau}) - \frac{q_n q_{n+1}}{q_{n+2}^{\tau+1}}} - \frac{q_n}{q_{n+1}} \tag{151}$$

Proof (150) is true if and only if:

$$\frac{p_{n+2}}{q_{n+2}} - \frac{p_n}{q_n} = \frac{p_{n+2}}{q_{n+2}} - \frac{p_{n+1}}{q_{n+1}} + \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} =$$

$$\frac{1}{q_n q_{n+1}} - \frac{1}{q_{n+1}q_{n+2}} > \frac{\gamma}{q_{n+2}^{\tau+1}} + \frac{\gamma}{q_n^{\tau+1}} \iff$$

$$\frac{1}{q_{n+2}q_{n+1}} < \frac{1}{q_n q_{n+1}} - \frac{\gamma}{q_n^{\tau+1}} - \frac{\gamma}{q_{n+2}^{\tau+1}} \iff$$

$$\frac{1}{q_{n+2}q_{n+1}} < \frac{\gamma}{q_n q_{n+1}}\left(\frac{1}{\gamma} - \frac{q_{n+1}}{q_n^\tau}\right) - \frac{\gamma}{q_{n+2}^{\tau+1}} \iff$$

$$\frac{1}{q_{n+2}} < \frac{\gamma}{q_n}\left(\frac{1}{\gamma} - \frac{q_{n+1}}{q_n^\tau}\right) - q_{n+1}\frac{\gamma}{q_{n+2}^{\tau+1}} \iff$$

$$\begin{cases} \dfrac{1}{\gamma} - \dfrac{q_{n+1}}{q_n^\tau} > \dfrac{q_n q_{n+1}}{q_{n+2}^{\tau+1}}, \\ \\ q_{n+2} > \dfrac{q_n}{\gamma} \dfrac{1}{\left(\frac{1}{\gamma} - \frac{q_{n+1}}{q_n^\tau}\right) - \frac{q_n q_{n+1}}{q_{n+2}^{\tau+1}}} \end{cases} \tag{152}$$

The first inequality is always true because of:

$$\frac{1}{\gamma} - \frac{q_{n+1}}{q_n^\tau} > \frac{1}{\alpha_{n+2} q_n^{\tau-1}} > \frac{q_n q_{n+1}}{q_{n+2}^{\tau+1}}.$$

So Lemma 16 follows from the fact that $q_{n+2} = a_{n+2}q_{n+1} + q_n$. ∎

Lemma 17 *Let $\tau > 1$, for almost all $\gamma \in (0, \frac{1}{2})$ (for $\gamma \geq \frac{1}{2}$ $D_{\gamma,\tau} = \emptyset$), given $\epsilon > 0$ there exists $C = C(\epsilon, \gamma) > 0$ such that:*

$$\left|\frac{1}{\gamma} - \frac{p}{q^\tau}\right| \geq \frac{C}{q^{\tau+1+\epsilon}}$$

for all $\frac{p}{q} \in \mathbb{Q}$.

Proof Define $B_{C,k} := \left\{\alpha : |\alpha - \frac{p}{q^\tau}| \geq \frac{C}{q^k} \quad \forall \frac{p}{q} \in \mathbb{Q}\right\}$, so $\alpha \in B_{C,k}^c \iff$ there exists $\frac{p}{q}$ such that $\alpha \in \left(\frac{p}{q} - \frac{C}{q^k}, \frac{p}{q} + \frac{C}{q^k}\right)$. So, given $N \in \mathbb{N}$ we get:

$$\mu\left(B_{C,k}^c \cap (-N, N)\right) < \sum_{q>0} \sum_{-Nq^\tau < p < Nq^\tau} \frac{2C}{q^k} < \sum_{q>0} \frac{4NC}{q^{k-\tau}}$$

and for $k > \tau + 1$, C that tends to zero, also

$$\mu\left(B^c_{C,k} \cap (-N, N)\right)$$

goes to zero. From the arbitrariness of N we obtain:

$$\mu\left(\bigcap_{C>0} B^c_{C,k}\right) = 0$$

for $k > \tau + 1$, from which follows Lemma 17. ∎

Lemma 18 *Let $\tau > 1$, $\alpha = [a_1, a_2, ...] \in D_{\gamma,\tau}$, $\frac{p_n}{q_n}$ the convergents to α. The inequality:*

$$\frac{p_n}{q_n} + \frac{\gamma}{q_n^{\tau+1}} < \frac{p_{n+2}}{q_{n+2}} - \frac{\gamma}{q_{n+2}^{\tau+1}} - \frac{2\gamma}{q_{n+2}^{\tau-1}} \tag{153}$$

is definitively verified if and only if definitively:

$$a_{n+2} > \frac{q_n}{\gamma q_{n+1}} \frac{1}{\left(\frac{1}{\gamma} - \frac{q_{n+1}}{q_n^\tau}\right) - \frac{q_n q_{n+1}}{q_{n+2}^{\tau+1}} - \frac{2q_n q_{n+1}}{q_{n+2}^{\tau-1}}} - \frac{q_n}{q_{n+1}} \tag{154}$$

Remark 17 Observe that (154) is definitively true if:

$$\limsup \frac{q_{n+1}}{q_n^\tau} < \frac{1}{\gamma},$$

because in that case:

$$\limsup \frac{q_n}{\gamma q_{n+1}} \frac{1}{\left(\frac{1}{\gamma} - \frac{q_{n+1}}{q_n^\tau}\right) - \frac{q_n q_{n+1}}{q_{n+2}^{\tau+1}} - \frac{2q_n q_{n+1}}{q_{n+2}^{\tau-1}}} - \frac{q_n}{q_{n+1}} < 1.$$

Thus, if for infinitely many n even (154) is not verified, for this n, with n big enough:

$$\frac{q_{n+1}}{q_n^\tau} \sim \frac{1}{\gamma},$$

so $q_{n+1} \sim \frac{q_n^\tau}{\gamma}$.

Proof

In a similar way of Lemma 16, (153) is verified if and only if:

$$\begin{cases} \dfrac{1}{\gamma} - \dfrac{q_{n+1}}{q_n^\tau} > \dfrac{q_n q_{n+1}}{q_{n+2}^{\tau+1}} + \dfrac{2q_n q_{n+1}}{q_{n+2}^{\tau-1}}, \\ \\ q_{n+2} > \dfrac{q_n}{\gamma} \dfrac{1}{\left(\frac{1}{\gamma} - \frac{q_{n+1}}{q_n^\tau}\right) - \frac{q_n q_{n+1}}{q_{n+2}^{\tau+1}} - \frac{2q_n q_{n+1}}{q_{n+2}^{\tau-1}}} \end{cases} \tag{155}$$

Because of $\alpha \in D_{\gamma,\tau}$, the first of the two conditions is definitively verified, in fact, for n big enough:

$$\frac{q_nq_{n+1}}{q_{n+2}^{\tau+1}}+\frac{2q_nq_{n+1}}{q_{n+2}^{\tau-1}}<\frac{1}{\alpha_{n+2}q_n^{\tau-1}}<\frac{1}{\gamma}-\frac{q_{n+1}}{q_n^{\tau}}$$

So, from the fact that $q_{n+2}=a_{n+2}q_{n+1}+q_n$ we are done. ∎

Lemma 19 *Let $\tau>\frac{3+\sqrt{17}}{2}$. For almost all $\gamma\in(0,\frac{1}{2})$, if $\alpha=[a_0,a_1,...]\in D_{\gamma,\tau}$,for n even big enough: (150) is true if and only if (153) is true.*

Proof If (153) is true, then trivially (150) is true. So we have to show that for almost all $\gamma\in(0,\frac{1}{2})$ and for all $\alpha\in D_{\gamma,\tau}$ (with $\tau>\frac{3+\sqrt{17}}{2}$) holds the converse. So, suppose by contradiction that exists $A\subseteq(C_1,C_2)$, with $0<C_1<C_2<\frac{1}{2}$, $\mu(A)>0$ such that, for all $\gamma\in A$ there exists $\alpha\in D_{\gamma,\tau}$ that satisfies (150) but not (153) for infinitely many n even. By Lemma 16 and Lemma 18 it follows that for all γ in A there exists $\alpha\in D_{\gamma,\tau}$ such that for infinitely many n even:

$$\frac{q_n}{\gamma q_{n+1}}\frac{1}{(\frac{1}{\gamma}-\frac{q_{n+1}}{q_n^{\tau}})-\frac{q_nq_{n+1}}{q_{n+2}^{\tau+1}}-\frac{2q_nq_{n+1}}{q_{n+2}^{\tau-1}}}-\frac{q_n}{q_{n+1}}\geq a_{n+2}>\frac{q_n}{\gamma q_{n+1}}\frac{1}{(\frac{1}{\gamma}-\frac{q_{n+1}}{q_n^{\tau}})-\frac{q_nq_{n+1}}{q_{n+2}^{\tau+1}}}-\frac{q_n}{q_{n+1}},$$

and by Remark 17 it follows that, for this n:

$$q_{n+1}\sim\frac{q_n^{\tau}}{\gamma}.$$

So, for n big enough such that (150) holds but (153) doesn't hold we get:

$$\frac{q_n^{\tau}}{C_2}<q_{n+1}<\frac{q_n^{\tau}}{C_1}.$$

Moreover:

$$a_{n+2}>\frac{q_n}{\gamma q_{n+1}}\frac{1}{(\frac{1}{\gamma}-\frac{q_{n+1}}{q_n^{\tau}})-\frac{q_nq_{n+1}}{q_{n+2}^{\tau+1}}}-\frac{q_n}{q_{n+1}}\iff$$

$$\frac{a_{n+2}q_{n+1}}{q_n}+1=\frac{q_{n+2}}{q_n}>\frac{1}{1-\frac{\gamma q_{n+1}}{q_n^{\tau}}-\frac{\gamma q_nq_{n+1}}{q_{n+2}^{\tau+1}}}\iff$$

$$1-\frac{\gamma q_{n+1}}{q_n^{\tau}}-\frac{\gamma q_nq_{n+1}}{q_{n+2}^{\tau+1}}>\frac{q_n}{q_{n+2}}\iff$$

$$\gamma<\frac{1-\frac{q_n}{q_{n+2}}}{\frac{q_{n+1}}{q_n^{\tau}}+\frac{q_nq_{n+1}}{q_{n+2}^{\tau+1}}}$$

In a similar way:

$$\frac{q_n}{\gamma q_{n+1}}\frac{1}{(\frac{1}{\gamma}-\frac{q_{n+1}}{q_n^{\tau}})-\frac{q_nq_{n+1}}{q_{n+2}^{\tau+1}}-\frac{2q_nq_{n+1}}{q_{n+2}^{\tau-1}}}-\frac{q_n}{q_{n+1}}\geq a_{n+2}\iff$$

$$\gamma \geq \frac{1-\frac{q_n}{q_{n+2}}}{\frac{q_{n+1}}{q_n^\tau}+\frac{q_nq_{n+1}}{q_{n+2}^{\tau+1}}+\frac{2q_nq_{n+1}}{q_{n+2}^{\tau-1}}}.$$

Thus:

$$\frac{1-\frac{q_n}{q_{n+2}}}{\frac{q_{n+1}}{q_n^\tau}+\frac{q_nq_{n+1}}{q_{n+2}^{\tau+1}}+\frac{2q_nq_{n+1}}{q_{n+2}^{\tau-1}}} \leq \gamma < \frac{1-\frac{q_n}{q_{n+2}}}{\frac{q_{n+1}}{q_n^\tau}+\frac{q_nq_{n+1}}{q_{n+2}^{\tau+1}}}$$

for infinitely many n even, so for all $\gamma \in A$ there exist infinitely many $q \in \mathbb{N}$ such that:

$$\frac{1-\frac{q}{Np+q}}{\frac{p}{q^\tau}+\frac{qp}{(Np+q)^{\tau+1}}+\frac{2qp}{(Np+q)^{\tau-1}}} \leq \gamma < \frac{1-\frac{q}{Np+q}}{\frac{p}{q^\tau}+\frac{qp}{(Np+q)^{\tau+1}}}$$

for some $N \in \mathbb{N}$ and some $\frac{q^\tau}{C_2} < p < \frac{q^\tau}{C_1}$. So for all $M \in \mathbb{N}$:

$$A \subseteq \bigcup_{q>M} \bigcup_{\frac{q^\tau}{C_2}<p<\frac{q^\tau}{C_1}} \bigcup_{N>0} \left(\frac{1-\frac{q}{Np+q}}{\frac{p}{q^\tau}+\frac{qp}{(Np+q)^{\tau+1}}+\frac{2qp}{(Np+q)^{\tau-1}}}, \frac{1-\frac{q}{Np+q}}{\frac{p}{q^\tau}+\frac{qp}{(Np+q)^{\tau+1}}} \right),$$

moreover:

$$\frac{1-\frac{q}{Np+q}}{\frac{p}{q^\tau}+\frac{qp}{(Np+q)^{\tau+1}}} - \frac{1-\frac{q}{Np+q}}{\frac{p}{q^\tau}+\frac{qp}{(Np+q)^{\tau+1}}+\frac{2qp}{(Np+q)^{\tau-1}}} <$$

$$\frac{2qp}{(Np+q)^{\tau-1}} \left(\frac{1}{\frac{p}{q^\tau}+\frac{qp}{(Np+q)^{\tau+1}}} \right)^2 < \frac{2qC_2^2}{N^{\tau-1}p^{\tau-2}}$$

so we obtain:

$$m(A) \leq \sum_{q>M} \sum_{\frac{q^\tau}{C_2}<p<\frac{q^\tau}{C_1}} \sum_{N>0} \frac{2qC_2^2}{N^{\tau-1}p^{\tau-2}} <$$

$$\beta \sum_{q>M} \frac{q^{\tau+1}}{q^{\tau^2-2\tau}} = \beta \sum_{q>M} \frac{1}{q^{\tau^2-3\tau-1}}$$

for some constant $\beta > 0$. From the hypothesis ($\tau > \frac{3+\sqrt{17}}{2}$) we have that the series converge, so for M that goes to infinity we get that $\mu(A) = 0$, that contradicts the hypothesis $\mu(A) > 0$. Thus, for almost all $\gamma \in (C_1, C_2)$ we have that: if (150) holds, then (153) holds, and from the arbitrariness of C_1, C_2 Lemma 19 follows. ∎

Proposition 1 *Let $\tau > \frac{3+\sqrt{17}}{2}$. For almost every $0 < \gamma < \frac{1}{2}$: if $\alpha \in D_{\gamma,\tau}$, $\frac{p_n}{q_n}$ are the convergents to α, $\alpha - \frac{p_n}{q_n} > \frac{\gamma}{q_n^{\tau+1}}$, and definitively:*

$$\frac{p_n}{q_n} + \frac{\gamma}{q_n^{\tau+1}} < \frac{p_{n+2}}{q_{n+2}} - \frac{\gamma}{q_{n+2}^{\tau+1}},$$

then α is an accumulation point of $D_{\gamma,\tau}$ and in particular, for n even big enough:

$$\mu\left(D_{\gamma,\tau}\cap\left(\frac{p_n}{q_n},\frac{p_{n+2}}{q_{n+2}}\right)\right)>0$$

Proof By Lemma 13 it follows that $\exists N_1\in\mathbb{N}$ such that for $n>N_1$ even:

$$\frac{p}{q}\notin I_n \implies \frac{p}{q}+\frac{\gamma}{q^{\tau+1}},\frac{p}{q}-\frac{\gamma}{q^{\tau+1}}\notin A_n,$$

and by Lemma 19 for almost all $\gamma\in(0,\frac{1}{2})$:

$$\frac{p_n}{q_n}+\frac{\gamma}{q_n^{\tau+1}}<\frac{p_{n+2}}{q_{n+2}}-\frac{\gamma}{q_{n+2}^{\tau+1}} \implies \frac{p_n}{q_n}+\frac{\gamma}{q_n^{\tau+1}}<\frac{p_{n+2}}{q_{n+2}}-\frac{\gamma}{q_{n+2}^{\tau+1}}-\frac{2\gamma}{q_{n+2}^{\tau-1}},$$

therefore, up to a set of measure zero we can suppose that γ satisfies this property. Moreover, by Lemma 14, for n even big enough, if $\frac{p}{q}\in I_n$, $q<q_{n+2}$ then:

$$\frac{p}{q}+\frac{\gamma}{q^{\tau+1}}<\frac{p_{n+2}}{q_{n+2}}-\frac{\gamma}{q_{n+2}^{\tau+1}}-\frac{2\gamma}{q_{n+2}^{\tau-1}}.$$

So, if we define:

$$c_n:=\max_{\frac{p}{q}\in[\frac{p_n}{q_n},\frac{p_{n+2}}{q_{n+2}}),q<q_{n+2}}\frac{p}{q}+\frac{\gamma}{q^{\tau+1}},$$

we obtain:

$$c_n<\frac{p_{n+2}}{q_{n+2}}-\frac{2\gamma}{q_{n+2}^{\tau-1}}-\frac{\gamma}{q_{n+2}^{\tau+1}}.$$

By Lemma 13, if $n>N_1$ is even and $\frac{p}{q}\notin I_n$, then

$$\frac{p}{q}+\frac{\gamma}{q^{\tau+1}},\frac{p}{q}-\frac{\gamma}{q^{\tau+1}}\notin A_n,$$

so, if

$$\frac{p}{q}<\frac{p_n}{q_n} \implies \frac{p}{q}+\frac{\gamma}{q^{\tau+1}}<\frac{p_n}{q_n}+\frac{\gamma}{q_n^{\tau+1}}\le c_n,$$

while for $\frac{p}{q}>\frac{p_{n+2}}{q_{n+2}}$ we get $q>q_{n+2}$, so:

$$\frac{p}{q}-\frac{\gamma}{q^{\tau+1}}>\frac{p_{n+2}}{q_{n+2}}-\frac{\gamma}{q_{n+2}^{\tau+1}},$$

but from:

$$\beta\in D_{\gamma,\tau}^c \iff \exists\frac{p}{q}\in(0,1):\beta\in\left(\frac{p}{q}-\frac{\gamma}{q^{\tau+1}},\frac{p}{q}+\frac{\gamma}{q^{\tau+1}}\right)$$

we get that for $n > N_1$ even, holds:

$$\mu\left(D^c_{\gamma,\tau} \cap I_n\right) \le \mu\left(\bigcup_{\frac{p}{q} \in [\frac{p_n}{q_n}, \frac{p_{n+2}}{q_{n+2}}), q < q_{n+2}} \left(\frac{p}{q} - \frac{\gamma}{q^{\tau+1}}, \frac{p}{q} + \frac{\gamma}{q^{\tau+1}}\right) \cap I_n \right)$$

$$+\mu\left(\bigcup_{\frac{p}{q} \in I_n, q \ge q_{n+2}} \left(\frac{p}{q} - \frac{\gamma}{q^{\tau+1}}, \frac{p}{q} + \frac{\gamma}{q^{\tau+1}}\right)\right) + \mu\left(\frac{p_{n+2}}{q_{n+2}} - \frac{\gamma}{q_{n+2}^{\tau+1}}, \frac{p_{n+2}}{q_{n+2}}\right).$$

So by Lemma 15:

$$\mu(D^c_{\gamma,\tau} \cap I_n) \le c_n - \frac{p_n}{q_n} + \frac{2\gamma}{q_{n+2}^{\tau-1}} + \frac{\gamma}{q_{n+2}^{\tau+1}} < \mu(I_n) = \frac{p_{n+2}}{q_{n+2}} - \frac{p_n}{q_n} \iff$$

$$c_n < \frac{p_{n+2}}{q_{n+2}} - \frac{\gamma}{q_{n+2}^{\tau+1}} - \frac{2\gamma}{q_{n+2}^{\tau-1}},$$

that follows from the definition of c_n. ∎

So, given $\tau > 3$, for almost all $\gamma > 0$: if $\alpha \in D_{\gamma,\tau}$ is not an isolated point of the first type and definitively the intervals centered in the convergents have an empty intersection, then α is an accumulation point in $D_{\gamma,\tau}$. The second step is to show that: if $\tau > 3$, $\gamma > 0$, $\alpha \in D_{\gamma,\tau}$ but α is not an isolated point of the first type and $\tau > \tau(\alpha)$, then α is an accumulation point in $D_{\gamma,\tau}$.

Lemma 20 *Let $\tau > 3$. For almost all $\gamma \in (0, \frac{1}{2})$: given $\alpha \in D_{\gamma,\tau}$, if for infinitely many n even:*

$$\frac{p_n}{q_n} + \frac{\gamma}{q_n^{\tau+1}} > \frac{p_{n+2}}{q_{n+2}} - \frac{\gamma}{q_{n+2}^{\tau+1}},$$

then there exists $C > 0$ such that for this n:

$$a_{n+2} \le C q_n^{2+\epsilon},$$

with $\epsilon > 0$ arbitrarily small.

Proof By Lemma 16 it follows that, given $\alpha \in D_{\gamma,\tau}$ that satisfies the hypothesis of Lemma 20, for n even big enough:

$$a_{n+2} \le \frac{q_n}{\gamma q_{n+1}} \frac{1}{(\frac{1}{\gamma} - \frac{q_{n+1}}{q_n^{\tau}}) - \frac{q_n q_{n+1}}{q_{n+2}^{\tau+1}}} - \frac{q_n}{q_{n+1}},$$

so, up to a set of measure zero, by Lemma 17 we can suppose that there exist $\epsilon > 0, C > 0$ such that $\frac{1}{\gamma} \in B_{C,\tau+1+\epsilon}$ with $\tau + 1 + \epsilon < \tau^2 - 1$, from which it follows that:

$$\frac{q_n}{\gamma q_{n+1}} \frac{1}{(\frac{1}{\gamma} - \frac{q_{n+1}}{q_n^{\tau}}) - \frac{q_n q_{n+1}}{q_{n+2}^{\tau+1}}} - \frac{q_n}{q_{n+1}} \le \frac{q_n}{\gamma q_{n+1}} \frac{1}{\frac{C}{q_n^{\tau+1+\epsilon}} - \frac{q_n q_{n+1}}{q_{n+2}^{\tau+1}}} - \frac{q_n}{q_{n+1}},$$

moreover, by Remark 17 it follows that $q_{n+1} \sim \frac{q_n^\tau}{\gamma}$, from which we obtain:

$$\frac{q_n q_{n+1}}{q_{n+2}^{\tau+1}} < \frac{q_n}{q_{n+1}^\tau} \sim \frac{\gamma^\tau}{q_n^{\tau^2-1}},$$

so, if n is big enough, by $\tau + 1 + \epsilon < \tau^2 - 1$ we have:

$$\frac{C}{q_n^{\tau+1+\epsilon}} - \frac{q_n q_{n+1}}{q_{n+2}^{\tau+1}} > \frac{C}{2q_n^{\tau+1+\epsilon}}.$$

So we obtain:

$$a_{n+2} < \frac{q_n}{q_{n+1}} \frac{2q_n^{\tau+1+\epsilon}}{C} \sim \frac{2\gamma}{C} q_n^{2+\epsilon} < \frac{4\gamma}{C} q_n^{2+\epsilon} = C' q_n^{2+\epsilon}$$

definitively, from which we get Lemma 20. ∎

Lemma 21 *Let $\tau > \frac{3+\sqrt{17}}{2}, \gamma > 0$, $\alpha \in D_{\gamma,\tau}$. If for infinitely many m even, for $n < m$ even holds:*

$$\frac{p_n}{q_n} + \frac{\gamma}{q_n^{\tau+1}} < \frac{p_m}{q_m} - \frac{\gamma}{q_m^{\tau+1}} - \frac{2\gamma}{q_m^{\tau-1}}, \tag{156}$$

and $\alpha - \frac{p_n}{q_n} > \frac{\gamma}{q_n^{\tau+1}}$ for all n even, then α is in $\mathcal{A}(D_{\gamma,\tau})$.

Proof Let $\frac{p_n}{q_n} < \frac{p}{q} < \frac{p_{n+2}}{q_{n+2}}$ with n even and $n < m - 2$, for $\frac{q_{n+2}}{2} \leq q$:

$$\frac{p}{q} + \frac{\gamma}{q^{\tau+1}} < \frac{p_{n+2}}{q_{n+2}} + \frac{\gamma}{q_{n+2}^{\tau+1}}$$

is definitively true, while for $q < \frac{q_{n+2}}{2}$:

$$\frac{p_{n+2}}{q_{n+2}} - \frac{p}{q} = \frac{p_{n+2}}{q_{n+2}} - \alpha + \alpha - \frac{p}{q} > \frac{1}{2q^2} - \frac{1}{q_{n+2}^2} > \frac{\gamma}{q^{\tau+1}} - \frac{\gamma}{q_{n+2}^{\tau+1}} \iff$$

$$\frac{1}{2q^2} - \frac{\gamma}{q^{\tau+1}} > \frac{1}{q_{n+2}^2} - \frac{\gamma}{q_{n+2}^{\tau+1}},$$

that is true for q big enough, so $\exists T \in \mathbb{N}$ such that the inequality is verified for $q \geq T$ (from the fact that $G(x) := \frac{1}{2x^2} - \frac{\gamma}{x^{\tau+1}}$ is definitively decreasing and $\tau > 3 > 1$). From the hypothesis that $\alpha - \frac{p_n}{q_n} > \frac{\gamma}{q_n^{\tau+1}}$ for all n even:

$$v := \max_{\frac{p}{q} < \alpha, q \leq T} \frac{p}{q} + \frac{\gamma}{q^{\tau+1}} < \alpha,$$

so there exists $T_1 \in \mathbb{N}$ such that for $n > T_1$:

$$\frac{p_n}{q_n} + \frac{\gamma}{q_n^{\tau+1}} > v.$$

By Lemma 14, for m big enough, $\frac{p}{q} \in I_n$, with $n < m-2$ even:

$$\frac{p}{q} + \frac{\gamma}{q^{\tau+1}} \leq \max\left\{\frac{p_{n+2}}{q_{n+2}} + \frac{\gamma}{q_{n+2}^{\tau+1}}, v\right\} \leq \frac{p_{m-2}}{q_{m-2}} + \frac{\gamma}{q_{m-2}^{\tau+1}} < \frac{p_m}{q_m} - \frac{\gamma}{q_m^{\tau+1}} - \frac{2\gamma}{q_m^{\tau+1}},$$

while by Lemma 14, for m big enough:

$$\frac{p}{q} \in I_{m-2}, q < q_{m-2} \implies \frac{p}{q} + \frac{\gamma}{q^{\tau+1}} < \frac{p_m}{q_m} - \frac{\gamma}{q_m^{\tau+1}} - \frac{2\gamma}{q_m^{\tau-1}},$$

so if we define:

$$c_m := \max\left\{\max_{\frac{p}{q}\in I_{m-2}, q<q_m}\left(\frac{p}{q} + \frac{\gamma}{q^{\tau+1}}\right), \max_{\frac{p}{q}\leq\frac{p_{m-2}}{q_{m-2}}}\left(\frac{p}{q} + \frac{\gamma}{q^{\tau+1}}\right)\right\},$$

for m even big enough:

$$c_m < \frac{p_m}{q_m} - \frac{\gamma}{q_m^{\tau+1}} - \frac{2\gamma}{q_m^{\tau-1}}.$$

Moreover, by Lemma 15, from $\tau > 3 > 2$, for m even big enough:

$$\mu\left(\bigcup_{\frac{p}{q}\in I_{m-2}, q\geq q_m}\left(\frac{p}{q} - \frac{\gamma}{q^{\tau+1}}, \frac{p}{q} + \frac{\gamma}{q^{\tau+1}}\right)\right) < \frac{2\gamma}{q_m^{\tau-1}}.$$

Finally, if $\frac{p}{q} > \frac{p_m}{q_m}$, by the properties of continued fractions we obtain $q > q_m$, so $\frac{p}{q} - \frac{\gamma}{q^{\tau+1}} > \frac{p_m}{q_m} - \frac{\gamma}{q_m^{\tau+1}}$. Thus:

$$\mu\left(D_{\gamma,\tau}^c \cap \left(\frac{p_{m-2}}{q_{m-2}}, \frac{p_m}{q_m} - \frac{\gamma}{q_m^{\tau+1}}\right)\right) < c_m - \frac{p_{m-2}}{q_{m-2}} + \frac{2\gamma}{q_m^{\tau-1}}$$

$$< \frac{p_m}{q_m} - \frac{p_{m-2}}{q_{m-2}} - \frac{\gamma}{q_m^{\tau+1}} = \mu(\frac{p_{m-2}}{q_{m-2}}, \frac{p_m}{q_m} - \frac{\gamma}{q_m^{\tau+1}}),$$

then

$$D_{\gamma,\tau} \cap \left(\frac{p_{m-2}}{q_{m-2}}, \frac{p_m}{q_m} - \frac{\gamma}{q_m^{\tau+1}}\right) \neq \emptyset,$$

and from the fact that this holds for infinitely many m even, then α is an accumulation point of $D_{\gamma,\tau}$. ∎

Remark 18 Let $\tau > \frac{\sqrt{17}+3}{2}$, $\gamma > 0$, $\alpha \in D_{\gamma,\tau}$, if $\alpha \in \mathcal{I}^2_{\gamma,\tau}$ or $\mathcal{I}^3_{\gamma,\tau}$, then $\tau(\alpha) = \tau$. In fact if this doesn't hold, from $\alpha \notin \mathcal{I}^1_{\gamma,\tau}$ we get that for all n even or for all n odd:

$$\left|\alpha - \frac{p_n}{q_n}\right| > \frac{\gamma}{q_n^{\tau+1}}.$$

Suppose for example that this property holds for all n even. If on the contrary $\tau(\alpha) < \tau$, by Remark 17, the hypothesis of Proposition 1 are satisfied, so $\alpha \in \mathcal{A}(D_{\gamma,\tau})$, contradiction.

Corollary 6 *If* $\tau > \frac{3+\sqrt{17}}{2}$*:*

$$\mu\left(\left\{\gamma > 0 : \mathcal{I}^2_{\gamma,\tau} \neq \emptyset\right\}\right) = 0.$$

Proof Observe that, if $\alpha \in \mathcal{I}^2_{\gamma,\tau}$, then there exists $n \in \mathbb{N}$ such that

$$\left|\alpha - \frac{p_n}{q_n}\right| = \frac{\gamma}{q_n^{\tau+1}}.$$

Suppose for example that n is even, thus:

$$\alpha = \frac{p_n}{q_n} + \frac{\gamma}{q_n^{\tau+1}}.$$

Moreover, for almost all $\gamma \in (0, \frac{1}{2})$:

$$\tau\left(\frac{p}{q} + \frac{\gamma}{q^{\tau+1}}\right) = \tau\left(\frac{\gamma}{q^{\tau+1}}\right) = 1.$$

Taking the union on all the $\frac{p}{q}$ we obtain that for almost all $\gamma \in (0, \frac{1}{2})$ and for all $\frac{p}{q} \in \mathbb{Q}$,

$$\tau\left(\frac{p}{q} + \frac{\gamma}{q^{\tau+1}}\right) = 1.$$

So Corollary 6 follows by Remark 18. ∎

It remains the last one step, in which we get the Theorem.

Lemma 22 *Let* $\tau > 3$*. For almost all* $\gamma > 0$*, if* $\alpha \in \mathcal{I}(D_{\gamma,\tau})$*, there exists* $N \in \mathbb{N}$ *such that, for all* $m > N$ *even there is some* $n < m$ *even with:*

$$\frac{p_n}{q_n} + \frac{\gamma}{q_n^{\tau+1}} \geq \frac{p_m}{q_m} - \frac{\gamma}{q_m^{\tau+1}} - \frac{2\gamma}{q_m^{\tau-1}}$$

.

Proof By Corollary 6 and Remark (j) it follow that, up to a set of measure zero, we can suppose that $\mathcal{I}^1_{\gamma,\tau} = \mathcal{I}^2_{\gamma,\tau} = \emptyset$, so observe that if the Lemma were not true, it would exist $\alpha \in \mathcal{I}^3_{\gamma,\tau}$ with the even convergents that satisfy the hypothesis of Lemma 21, that implies $\alpha \in \mathcal{A}(D_{\gamma,\tau})$, contradiction. ∎

Theorem C Let $\tau > \frac{3+\sqrt{17}}{2}$. Then, for almost all $\gamma > 0$ $D_{\gamma,\tau}$ is a Cantor set.

Proof By Corollary 6 and Remark (j) it follows that, up to a set of measure zero, we can suppose that $\mathcal{I}^1_{\gamma,\tau} = \mathcal{I}^2_{\gamma,\tau} = \emptyset$. Suppose by contradiction that the statement doesn't hold, and take $0 < C_1 < C_2$ such that:

$$\mu\left(\{C_1 < \gamma < C_2 : \mathcal{I}(D_{\gamma,\tau}) \neq \emptyset\}\right) > 0,$$

and define $A := \{C_1 < \gamma < C_2 : \mathcal{I}(D_{\gamma,\tau}) \neq \emptyset\}$. By Lemma 22, for almost all $\gamma > 0$ there exists $\alpha \in \mathcal{I}(D_{\gamma,\tau})$ and there exists $N \in \mathbb{N}$ such that for all $m > N$ even, there is some $n < m$ even, with:

$$\frac{p_n}{q_n} + \frac{\gamma}{q_n^{\tau+1}} \geq \frac{p_m}{q_m} - \frac{\gamma}{q_m^{\tau+1}} - \frac{2\gamma}{q_m^{\tau-1}}.$$

Now we want to show that, for almost all chosen of $\gamma \in A$ we have:

$$\limsup \frac{q_{2k+2}}{q_{2k+1}^{\tau}} < \frac{1}{\gamma}.$$

In fact if it doesn't hold, by Remark 17 we get that for infinitely many m even:

$$q_m \sim \frac{q_{m-1}^{\tau}}{\gamma},$$

and for $m > N$ exists $n < m$ even, with:

$$\frac{p_n}{q_n} + \frac{\gamma}{q_n^{\tau+1}} \geq \frac{p_m}{q_m} - \frac{\gamma}{q_m^{\tau+1}} - \frac{2\gamma}{q_m^{\tau-1}}$$

By Lemma 19, up to a set of measure zero in A:

$$\frac{p_n}{q_n} + \frac{\gamma}{q_n^{\tau+1}} \geq \frac{p_m}{q_m} - \frac{\gamma}{q_m^{\tau+1}} - \frac{2\gamma}{q_m^{\tau-1}} \iff \frac{p_n}{q_n} + \frac{\gamma}{q_n^{\tau+1}} \geq \frac{p_m}{q_m} - \frac{\gamma}{q_m^{\tau+1}}.$$

By the properties of convergents:

$$\alpha - \frac{p_m}{q_m} < \frac{1}{q_m^2},$$

from which we get:

$$\frac{1}{q_m^2} > \alpha - \frac{p_n}{q_n} - \frac{\gamma}{q_n^{\tau+1}} - \frac{\gamma}{q_m^{\tau+1}}.$$

Moreover:

$$\alpha - \frac{p_n}{q_n} = \frac{1}{q_n(q_{n+1} + \frac{\alpha_{n+2}}{q_n})},$$

so:

$$\frac{1}{q_m^2} > \frac{1}{q_n(q_{n+1} + \frac{\alpha_{n+2}}{q_n})} - \frac{\gamma}{q_n^{\tau+1}} - \frac{\gamma}{q_m^{\tau+1}}$$

For m big enough:

$$\frac{1}{q_m^2} + \frac{\gamma}{q_m^{\tau+1}} < \frac{2}{q_m^2},$$

so:

$$\frac{2}{q_m^2} > \frac{1}{q_n(q_{n+1} + \frac{\alpha_{n+2}}{q_n})} - \frac{\gamma}{q_n^{\tau+1}} \iff$$

$$\gamma > \frac{q_n^\tau}{q_{n+1} + \frac{\alpha_{n+2}}{q_n}} - \frac{2q_n^{\tau+1}}{q_m^2},$$

moreover:

$$\gamma \leq \frac{q_n^\tau}{q_{n+1} + \frac{\alpha_{n+2}}{q_n}}.$$

So we obtain:

$$\frac{q_n^\tau}{q_{n+1} + \frac{\alpha_{n+2}}{q_n}} - \frac{2q_n^{\tau+1}}{q_m^2} < \gamma \leq \frac{q_n^\tau}{q_{n+1} + \frac{\alpha_{n+2}}{q_n}}$$

From

$$\frac{p_n}{q_n} + \frac{\gamma}{q_n^{\tau+1}} \geq \frac{p_m}{q_m} - \frac{\gamma}{q_m^{\tau+1}} - \frac{2\gamma}{q_m^{\tau-1}},$$

we get:

$$\frac{p_n}{q_n} + \frac{\gamma}{q_n^{\tau+1}} \geq \frac{p_{n+2}}{q_{n+2}} - \frac{\gamma}{q_{n+2}^{\tau+1}},$$

moreover, from $\alpha - \frac{p_n}{q_n} > \frac{\gamma}{q_n^{\tau+1}}$ for all n even, when m increase, also n increase, and by the last inequality and Remark 17 we get that $q_{n+1} \sim \frac{q_n^\tau}{\gamma}$. So

$$q_m \sim \frac{q_{m-1}^\tau}{\gamma} \geq \frac{q_{n+1}^\tau}{\gamma} \sim \frac{q_n^{\tau^2}}{\gamma^\tau} \geq \frac{q_n^{\tau^2}}{C_2^\tau}.$$

So we obtain:

$$\frac{q_n^\tau}{q_{n+1} + \frac{\alpha_{n+2}}{q_n}} - \frac{C}{q_n^{2\tau^2-\tau-1}} < \gamma \leq \frac{q_n^\tau}{q_{n+1} + \frac{\alpha_{n+2}}{q_n}}$$

with a constant $C > 0$. By Lemma 20, up to a set of measure zero, we can suppose that there exists $\epsilon > 0$ arbitrarily small such that, for n big enough:

$$a_{n+2} < q_n^{2+\epsilon}.$$

So, up to a set of measure zero, we can suppose that for all $\gamma \in A$, there exists infinitely many $q > 0$, $\frac{q^\tau}{2C_2} < p < \frac{2}{C_1 q^\tau}$, $N < q^{2+\epsilon}$ such that:

$$\frac{q^\tau}{p + \frac{N}{q}} - \frac{C}{q^{2\tau^2-\tau-1}} < \gamma \leq \frac{q^\tau}{q + \frac{N}{q}}.$$

So, for all $M \in \mathbb{N}$:

$$A \subseteq \bigcup_{q>M} \bigcup_{\frac{q^\tau}{2C_2}<p<\frac{2q^\tau}{C_1}} \bigcup_{N<q^{2+\epsilon}} \left(\frac{q^\tau}{p + \frac{N}{q}} - \frac{C}{q^{2\tau^2-\tau-1}}, \frac{q^\tau}{q + \frac{N}{q}} \right),$$

Thus:

$$\begin{aligned} \mu(A) &< \sum_{q>M} \sum_{\frac{q^\tau}{2C_2}<p<\frac{2q^\tau}{C_1}} \sum_{N<q^{2+\epsilon}} \frac{C}{q^{2\tau^2-\tau-1}} \\ &< \beta \sum_{q>M} \frac{1}{q^{2\tau^2-2\tau-3-\epsilon}} \end{aligned}$$

with some constant $\beta > 0$. Because of $\tau > \frac{3+\sqrt{17}}{2}$, for ϵ small enough the series converge, so for M that tends to infinity we obtain $\mu(A) = 0$, contradiction. So we have proved that:

$$\limsup \frac{q_{2k+2}}{q_{2k+1}^\tau} < \frac{1}{\gamma}.$$

But, by Remark 17 and Proposition 1 (used with n odd) we have that $\alpha \in \mathcal{A}(D_{\gamma,\tau})$, contradiction. So $\mu(A) = 0$. ∎

The estimate $\tau > \frac{3+\sqrt{17}}{2}$ can be improved putting a better inequality in Lemma 5. Probably the Proposition holds also with $\tau > 3$.

3.5 Final observations and questions

We have seen that, up to an equivalent number, every Diophantine point is isolated in some Diophantine set. However, there exist Diophantine points that are always accumulation points (for example, the point defined in Remark 8). Moreover, a Diophantine number may be an isolated point for infinitely many τ. Indeed, by Corollary 3 it is reasonable to expect that the statement of Theorem B holds for almost every Diophantine number. We list here some natural questions.

- We have seen that $\mathcal{T}$ is dense in $[1, +\infty)$ and that $\mathcal{T} \cap \mathbb{Q} = \emptyset$. What are the $\tau \geq 1$ such that $\mathcal{I}_\tau \neq \emptyset$? In particular, is it true that $\mathcal{T}$ is the set of Diophantine points in $[1, +\infty)$?
- Let $N \geq 3$ and define $\Delta^N_{\gamma,\tau} := \{\omega \in \mathbb{R}^N : |\omega \cdot n| \geq \frac{\gamma}{|n|^\tau} \quad \forall n \in \mathbb{Z}^N, n \neq 0\}$. What can we say about isolated points of $\Delta^N_{\gamma,\tau} \cap \mathbb{S}^{N-1}$?

- Isolated points of type 3 exist for $\tau = 1$, in fact, for $\frac{1}{3} < \gamma < \frac{1}{2}$ $D_{\gamma,\tau}$ is a finite set and, when it is not empty in general all its points are in $\mathcal{I}^3_{\gamma,\tau}$. We ask if the following holds: if $\mathcal{I}^3_{\gamma,\tau}$ is not empty then $D_{\gamma,\tau}$ is a finite set.

We have shown that in general Diophantine sets are not Cantor sets, however we believe that the following hold:

- For all $\tau \geq 1$ there exists $\gamma_\tau \in (0, \frac{1}{2})$ such that $D_{\gamma,\tau}$ is a Cantor set for almost all $\gamma \in (0, \gamma_\tau)$.

We belive also that, for any algebraic number α with degree greater then 2, there exist sequences $\tau_n \searrow 1$, $\gamma_n \searrow 0$ such that α is an isolated point of D_{γ_n,τ_n} for all n (note that, if such sequences exist, by Roth Theorem $\tau_n \searrow 1$).

4 Appendix

4.1 C^1 conjugacy implies C^∞ conjugacy (Diophantine case)

In this section we show a simple proof that, for smooth diffeomorphisms of the circle with Diophantine rotation number, C^1 conjugacy implies C^∞ conjugacy.

We use the following notation as in [73]: q is a denominator of some convergent to $\alpha = \rho(f)$ and Q is the denominator of the subsequent convergent.

So, we want to prove the following:

Proposition 2 *Let $f \in D^\infty(\mathbb{T})$ with $\alpha = \rho(f) \in D$. Suppose that the homeomorphism h that conjugate f to a rotation is of class C^1. Then, h is smooth.*

To prove the proposition we need some lemma:

Lemma 23 *There exists $C > 0$ such that, for all $n \in \mathbb{Z}$:*

$$|f^n - id - n\alpha|_0 \leq C||n\alpha||.$$

Proof By Lagrange's Theorem:

$$|(h \circ R_{n\alpha} - h - n\alpha)|_0 \leq |Dh|_0||n\alpha||.$$

So, by the identity:

$$f^n - id - n\alpha = (h \circ R_{n\alpha} - h - n\alpha) \circ h^{-1}$$

and the preceding inequality, Lemma 7 follows. ∎

We restate also the Denjoy's inequality.

Lemma 24 (Denjoy's inequality) *Let* $C := \mathrm{Var}(\log \mathrm{Df})$. *Then:*

$$|\log Df^q|_0 \leq C. \tag{157}$$

Lemma 25 *([73], lemma 5) For* $k \geq 2$ *there exists* $C_k > 0$ *such that:*

$$|D^k \log Df^q|_0 \leq C_k Q^{\frac{k}{2}}$$

The following Lemma is the main point to prove in an easy way the smoothness of h.

Lemma 26 *For* $k \geq 0, \epsilon > 0$ *there exists* $C(k, \epsilon) > 0$ *such that:*

$$|D^k \log Df^q|_0 \leq \frac{C(k,\epsilon)}{Q^{1-\epsilon}}.$$

Proof The proof of this Lemma is analogous of the Step 3 in Theorem 1. ∎

Proof (Proposition 2) For $n \in \mathbb{N}$, write n as:

$$n = \sum_{i=0}^{s} b_i q_i, \quad n \leq q_{s+1}, \quad b_i \leq \frac{q_{i+1}}{q_i}$$

Using the Diophantine condition over α, we have for ϵ small enough:

$$|D^k \log Df^n|_0 \leq C(k,\epsilon) \sum_{i \geq 0} \frac{q_{i+1}^{\epsilon}}{q_i} < C = C(k, \epsilon, \alpha).$$

In particular, the derivatives of the iterates of f are bounded in norm C^k for all $k \geq 1$. So, we have $h \in C^\infty$ (Theorem 2). ∎

4.2 Continued fractions

We recall some basic Theorem:

Theorem 10 (Cantor, [74]) *Every subset* E *of* $\mathbb{R}$ *can be written as union of a countable set and a perfect set, moreover this decomposition is unique. So the isolated points of* E *are at most countable.*

Theorem 11 (Dirichlet box principle) *Let* $n > m \in \mathbb{N}$, *if* n *elements are contained in* m *sets, then there are two distinct elements contained in the same set.*

Theorem 12 (Dirichlet, [67]) *Let $\alpha \in \mathbb{R}, Q \in \mathbb{N}$ with $Q > 1$, then there exist $q \in \mathbb{N}, p \in \mathbb{Z}$, with $q < Q$, such that:*

$$|q\alpha - p| < \frac{1}{Q}.$$

Remark 19 If α is an irrational number, by Theorem 3 there are infinitely many solutions of:

$$0 < |q\alpha - p| < \frac{1}{q}$$

with $q > 1$.

Definition 1 We define the finite continued fractions:

$$[a_0; a_1, ..., a_n] := a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots \frac{1}{a_N}}},$$

$$[a_1, ...a_n] := \frac{1}{a_1 + \frac{1}{a_2 + \dots \frac{1}{a_N}}}.$$

as functions respectively of the variables $a_0, ...a_N$ and $a_1, ...a_N$. We call $a_0, .., a_N$ the partial quotients of the continued fraction.

Remark 20 $[a_0; a_1, ..., a_N] = [a_0; a_1, ..., a_{N-1} + \frac{1}{a_N}]$.

Definition 2 Given $\alpha = [a_0; a_1, ..., a_N]$, $0 \le n \le N$ we call $[a_0; ..., a_n]$ the n-th convergent to α.

Theorem 13 *(see [32]) Define:*

$$\begin{cases} p_0 := a_0, \quad q_0 := 1, \\ p_1 := a_0 a_1 + 1, \quad q_1 := a_1, \\ p_{n+1} := a_{n+1} p_n + p_{n-1}, \quad q_{n+1} := a_{n+1} q_n + q_{n-1} \quad \forall 1 \le n < N, \end{cases} \tag{158}$$

then $\frac{p_n}{q_n} = [a_0; a_1, ..., a_n]$ *for all* $0 \le n \le N$.

Remark 21 Observe that for $n \ge 2$:

$$p_{n+1}q_n - p_n q_{n+1} = (a_{n+1}p_n + p_{n-1})q_n - p_n(a_{n+1}q_n + q_{n-1})$$
$$= -(p_n q_{n-1} - q_n p_{n-1}),$$

so by induction we get:

$$p_{n+1}q_n - p_n q_{n+1} = (-1)^n. \tag{159}$$

Now we recall some property of continued fractions. For more details see [32], where all Theorems cited below are treated.

Notations 1 In the rest of the text we always suppose that $a_0 \in \mathbb{Z}$, $a_n \in \mathbb{N}$ for $n \geq 1$.

Definition 3 If $\alpha := [a_0; a_1, ..., a_N] = \frac{p_N}{q_N}$ we say that the rational number α is represented as continued fraction.

Remark 22 Observe that the representation of a rational number α as continued fraction is not unique. In fact, if $a_N > 1$, then:

$$\alpha := [a_0; ..., a_N] = [a_0; ..., a_N - 1, 1],$$

while for $a_N = 1$, $N \geq 1$:

$$[a_0; a_1, ...a_{N-1}, 1] = [a_0; a_1, ..., a_{N-1} + 1].$$

Remark 23 Observe that $q_1 \geq q_0$ and $q_{n+1} > q_n$ for all $n \geq 1$. Moreover, by (159), for all $n \leq N$ we have $(p_n, q_n) = 1$. So, if $n \leq N$ is even:

$$\frac{p_n}{q_n} \leq \alpha,$$

while for $n \leq N$ odd:

$$\frac{p_n}{q_n} \geq \alpha.$$

Observe also that by Remark 22 we can choose the parity of N.

Theorem 14 *For all $n \geq 2$ we get:*

$$p_{n+2}q_n - q_{n+2}p_n = (-1)^n a_{n+2} \tag{160}$$

Corollary 7 *The even convergents $\frac{p_{2n}}{q_{2n}}$ increase strictly with n, while the odd convergents $\frac{p_{2n+1}}{q_{2n+1}}$ decrease strictly with n.*

Definition 4 Given $\alpha = [a_0; a_1, ..., a_N]$, $n \leq \grave{u}N$, we define the n-th complete quotient of $[a_0; a_1, ..., a_N]$ as:

$$\alpha_n := [a_n; a_{n+1}, ..., a_N].$$

Remark 24 Given $\alpha := [a_0; a_1, ..., a_N]$, for all $n < N$:

$$\alpha = \frac{\alpha_{n+1}p_n + p_{n-1}}{\alpha_{n+1}q_n + q_{n-1}}. \tag{161}$$

Remark 25 Given $\alpha = [a_0; a_1, ..., a_N]$, then $a_n = [\alpha_n]$ for all $n \leq N$.

Infinite simple continued fractions

Definition 5 Given $a_0 \in \mathbb{Z}$, $a_n \in \mathbb{N}$ for all $n \geq 1$, we define:

$$[a_0; a_1, a_2, ...] := \lim_{n\to\infty} [a_0; ...a_n].$$

Remark 26 Observe that the limit exists, in fact:

$$|[a_0; a_1, ...a_{n+1}] - [a_0; a_1, ..., a_{n+1}]| = \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right| \overset{(159)}{=} \frac{1}{q_n q_{n+1}}.$$

Thus, because of $q_n \geq n$, we get that the limit exists.

Definition 6 Given $\alpha = [a_0; a_1, a_2, ...]$, we say that $[a_0; a_1, ..., a_n]$ is the n-th convergent to α, $\alpha_n := [a_n; a_{n+1}, ...]$ is the n-th complete quotient of α.
In the rest of the text α will always denote a number $[a_0; a_1, a_2, ...]$ and $\frac{p_n}{q_n} = [a_0; a_1, ..., a_n]$ (with $(p_n, q_n) = 1$) the convergents to α.

Remark 27 Given $\alpha = [a_0; a_1, ...]$, then for all $n \geq 0$: $a_n = [\alpha_n]$.

Corollary 8 *By Remark 27 it follows that the representation of an irrational number as continued fraction is unique. Moreover, given an irrational number, it can be represented as continued fraction. In fact, given an irrational number α, if we define:*

$$\begin{cases} a_0 := [\alpha], \quad \alpha_1 := \dfrac{1}{\{\alpha\}}, \\ \\ a_n := [\alpha_n], \quad \alpha_{n+1} := \dfrac{1}{\{\alpha_n\}} \quad \forall n \geq 1, \end{cases} \tag{162}$$

then it is easy to verify that $\alpha = [a_0; a_1, ...]$.

Remark 28 By definition of n-th complete quotient and of convergent we have:

$$\alpha = [a_0; a_1, ..., a_n, \alpha_{n+1}] = \frac{\alpha_{n+1}p_n + p_{n-1}}{\alpha_{n+1}q_n + q_{n-1}}.$$

Theorem 15 *Given $\alpha = [a_0; a_1, ...]$, then:*

$$\left|\alpha - \frac{p_n}{q_n}\right| = \frac{1}{q_n(\alpha_{n+1}q_n + q_{n-1})} \tag{163}$$

Corollary 9

$$||q_n\alpha|| < \frac{1}{q_{n+1}},$$

moreover for $n \geq 1$:

$$||q_{n+1}\alpha|| < ||q_n\alpha||.$$

Definition 7 Let $\alpha, \beta \in \mathbb{R}$. We say that α is equivalent to β if there exist $a, b, c, d \in \mathbb{N}$ with $|ad - bc| = 1$ such that:

$$\beta = \frac{a\alpha + b}{c\alpha + d}.$$

It is easy to check that this is an equivalent relation on the real numbers and that any two rational numbers are equivalent.

Theorem 16 *Two irrational numbers α, β are equivalent if and only if:*

$$\alpha = [a_0; a_1, ..., a_n, c_0, c_1, ...], \quad \beta = [b_0; b_1, ..., b_m, c_0, c_1, ...].$$

Farey sequence

Definition 8 Let n be a natural number, then the Farey sequence of order n F_n is the ordered sequence of all the rational numbers $\frac{p}{q} \geq 0$ with $q \leq n$.

Despite of the name, the proof of the following properties of this sequence are not due to the geologist John Farey. In fact he simply conjectured this property and then Chauchy proved it. Moreover before Farey's conjecture, another mathematician, Charles Haros, had published similar result.

Thus the name "Farey sequence" is unjustified, but nevertheless for convention we follow the tradition.

Theorem 17 *Let n be a natural number, if $\frac{p_1}{q_1} < \frac{p_2}{q_2}$ with $0 < q_1, q_2 \leq n$ are two subsequent terms of F_n, then:*

$$p_2 q_1 - p_1 q_2 = 1$$

Moreover, all the fractions $\frac{p}{q} \in (\frac{p_1}{q_1}, \frac{p_2}{q_2})$ are of the form:

$$\frac{p}{q} = \frac{ap_1 + bp_2}{aq_1 + bq_2}$$

for some $a, b \in \mathbb{N}$. In particular $q \geq q_n + q_{n+1}$.

Remark 29 If $\frac{p_1}{q_1} < \frac{p_2}{q_2}$ are two subsequent terms of F_n, then $q_1 < n$ or $q_2 < n$. In fact, if $q_1 = q_2 = n$, then:

$$\frac{p_1}{n} < \frac{p_1}{n-1} < \frac{p_1 + 1}{n} < \frac{p_2}{n},$$

and we get a contradiction. So, no two subsequent terms of the Farey sequence have the same denominator.

Theorem 18 *For all $n \geq 1$, given $q < q_{n+1}$ we have:*

$$||q\alpha|| \geq ||q_n\alpha||.$$

Theorem 19 (Legendre) *Given a real number α, if $\frac{p}{q}$ satisfies:*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2},$$

then $\frac{p}{q}$ is a convergent to α.

Theorem 20 (Borel)*(see [36]) Given $A(\psi) := \{[a_0; a : 1, ..., a_n, ...] : 0 < a_n < \psi(n)\}$,*

$$\sum_{n\in\mathbb{N}} \frac{1}{\psi(n)} < \infty \Rightarrow \quad m(A) > 0,$$

$$\sum_{n\in\mathbb{N}} \frac{1}{\psi(n)} = \infty \Rightarrow \quad m(A) = 0.$$

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