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LOCAL STRONG SOLUTIONS OF THE NAVIER-STOKES EQUATIONS FOR NONHOMOGENEOUS INCOMPRESSIBLE FLUIDS IN A BOUNDED DOMAIN Ω

Tesi di Laurea Magistrale in Matematica

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Chapter 0

Introduction to the problem

0.0.1 The inhomogeneous incompressible Navier-Stokes equations

Let Ω a bounded domain in $\mathbb{R}^3$ with smooth boundary. The motion of a nonhomogeneous incompressible viscous fluid in Ω is governed by the *inhomogeneous incompressible Navier-Stokes equations* (*INSE*, briefly); solving the *partial differential equation* associated to this problem consists in finding a triple of functions (ρ, u, P) which satisfies the system of equations

$$\begin{cases} \rho_t(x,t) + \nabla \cdot (\rho u)(x,t) = 0 \\ (\rho u)_t(x,t) + \nabla \cdot (\rho u \otimes u)(x,t) - \mu \Delta u(x,t) + \nabla P(x,t) = 0 \\ \nabla \cdot u(x,t) = 0 \end{cases} \tag{1}$$

together with the standard mathematical data given by the initial value problem and the boundary value problem, that is

$$\begin{cases} \rho(x,0) = \rho_0(x) \\ u(x,0) = u_0(x) \end{cases} \qquad \forall\ x \in \Omega \tag{2}$$

$$u(x,t) = 0 \quad \forall\ (x,t) \in \partial\Omega \times (0,T) \tag{3}$$

where $\partial\Omega$ is the boundary of Ω.

Before specifying the historical process of the study of these equations, the actual aims of the present thesis, and, consequently, the hypothesis we will require on the initial data, we briefly point out the physical interpretation of the functions involved in the INSE.

Physical interpretation of the problem. In a physical interpretation the function ρ that appears in (1) denotes the density of the fluid that we are considering, and it is a scalar function $\rho : \Omega \times I \to \mathbb{R}^{\geq 0}$, where $I \subseteq \mathbb{R}$ is an interval of time; in the case of a global solution, it coincides with $\mathbb{R}$. On the other hand, u is the velocity field, and assumes values in $\mathbb{R}^3$, $u : \Omega \times I \to \mathbb{R}^3$. Finally $P : \Omega \times I \to \mathbb{R}$ denotes the pressure of the fluid, it is a scalar function, to which is associated a gradient pressure term $\nabla P : \Omega \times I \to \mathbb{R}^3$. Finally $\mu > 0$ is the positive constant of *viscosity*.

The first equation in (1) is called *mass equation* (and it is a transport equation). This equation has a simple physical interpretation. We explain this with a formal argument. Consider in fact the density function $\rho(x,t)$ over a domain Ω. The amount of fluid at time t in a subdomain V of Ω is given by

$$\int_V \rho(x,t)\ dx$$

However, if we introduce a function $F(x,t)$ that describes the flux of the fluid, we can consider this flux through the surface of V, that is given by

$$\int_{\partial V} F(x,t)\ d\sigma(x)$$

It is well known in physics that *the variation in time of the amount of fluid equals the opposite of the flux through the surface*, that is

$$\frac{d}{dt}\int_V \rho(x,t)\ dx = -\int_{\partial V} F(x,t)\ d\sigma(x) \tag{4}$$

If the functions involved in (4) are regular enough, we can rewrite

$$\int_V \rho_t(x,t)\ dx = -\int_V \nabla\cdot F(x,t)\ dx$$

By the arbitrariness of the subdomain, where this physical interpretation holds, we can deduce that

$$\rho_t(x,t) = -\nabla\cdot F(x,t) \quad \text{over} \ \ \Omega$$

It is physically reasonable to define the flux as $F(x,t) = \rho(x,t)u(x,t)$, a vectorial function with module given by the product of density and velocity, that has the same direction and verse of the velocity (since $\rho \geq 0$). The equation assumes the form

$$\rho_t = -\nabla\cdot(\rho u) = -\rho\,(\nabla\cdot u) - u\cdot\nabla\rho$$

If we assume the *incompressibility condition* $\nabla\cdot u = 0$ for the velocity, we have

$$\rho_t + u\cdot\nabla\rho = 0$$

that is the so called *transport equation.* It is clear that the mass equation is a scalar equation. The second equation of the system is named *momentum equation*, while the latter is the *solenoidal condition* or *incompressibility equation.* The momentum equation is a vectorial equation, while the solenoidal condition is clearly scalar.

Consider for a moment the two equations together, to outline a part the story of the Navier Stokes equations (whose nature is located in the momentum equation, that expands the physical description of the motion of a fluid merely given by the *law of conservation*).

What today is known as *Navier-Stokes* System was proposed for the first time by the French engineer *C.L.M.H. Navier* in 1822, [21, p. 414], in the form

$$\begin{cases} \rho\left(\frac{\partial}{\partial t}v + v\cdot\nabla v\right) = \mu\Delta v - \nabla\pi - \rho f(x,t) \\ \nabla\cdot v = 0 \end{cases} \tag{5}$$

on the basis of a suitable molecular model. Nevertheless, the problem was known, in a very different stamp, before the formalization actualized by Navier. It is well represented by the following comment of Truesdell [30, p. 455]:

"*Such models were not new, having occurred in philosophical or qualitative speculations for millennia past. Navier's magnificent achievement was to put these notions into sufficiently concrete form that he could derive equations of motion for them.*"

Notice that here the pressure is not a thermodynamic variable; rather it represents the "reaction force" that must act on the fluid in order to leave any material volume unchanged, in contrast to the compressible scheme, where the pressure is a thermodynamic variable.

Even though the problem has been known before Navier, it was only later, by the efforts of Poisson (1831), de Saint Venant (1843), and mainly by the clarifying work of Stokes (1845), that equations (5) found a completely satisfactory justification on the basis of the continuum mechanics approach. Nowadays, equations (5) are usually referred to as *Navier–Stokes equations*.

Today is accepted to relate the nature of the Navier-Stokes equations to the Newtonian nature of the fluid. A *Newtonian fluid* is, in the language of modern rational mechanics, a fluid that respect the dynamical equation

$$T = -\pi I + 2\mu D \tag{6}$$

where T is the so called *Cauchy stress tensor*, that define the state of stress at a point inside a material (in this case, a fluid) in the deformed configuration; the tensor I is the identity; $D = \{D_{ij}\}$ is defined by

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{7}$$

and π is the pressure introduced above.

In words, the relation (6) states that the stress in a viscous liquid produces a gradient of velocity that is proportional to the stress. In Newton's words: "*The resistance, arising from the want of lubricity in the parts of a fluid is, cæteris paribus, proportional to the velocity with which the parts of the fluid are separated from each other*"; see [22, Book 2, Sect. IX, p. 373]. However, the deduction of the Navier-Stokes equations starting from their rational mechanics model is not one of the aims of this thesis.
These brief historical references are a summary of [12, Chap. 1].

Finally, we focus on the latter equation, that is the incompressibility condition. The third equation is strictly related to the transport equation: the incompressibility condition, as we will see in section 8.1.1, will ensure that the volume occupied by the fluid (i.e., the mass associated to the density-solution of the first equation[1]) will not

[1] If $\rho(x,t)$ is the density at time t and position $x \in \Omega$, the mass of the fluid in Ω at time t is

$$M(t) = \int_{\Omega} \rho(x,t)\; dx$$

change, remarking the *incompressible* nature of the fluid.

The notions of solution that will be involved here are various and to be clarified. In particular, we are going to introduce new functional spaces to solve the problem.

0.1 Results presented in this thesis

As typical in PDEs, solving the Navier-Stokes equations is not a well posed problem if we do not clarify what "kind of solution" we are searching for. Fixed the class of functions in which we want to find some solutions, the problem requires suitable hypothesis on the initial data to be resolved. The variety of possible situations splits the present problem into several subproblems: in particular, the existence of smooth solutions to the Navier Stokes equations is a problem that has never been proved or disproved. This particular version of the Navier Stokes problem is, from May 2000, one of the so called *Millennium problems*, since the study of these equations in applied sciences has revealed to be fundamental, in particular in applied physics and engineering. So, *NSEs* deserved four points (existence and breakdown questions) in the Clay Institute's list of prize. The matter is perfectly exposed in Fefferman's [11]. Another unsolved problem related to the Navier Stokes equations is the Euler equation ($\mu = 0$), although this particular problem is not on the Clay Institute's list.

Apart from this general introduction, in the present thesis we focus our attention on *local strong* solutions to the *Inhomogeneous Incompressible Navier Stokes Equations*. While the adjective *local* has a clear meaning (i.e. we are looking for a local time $T \in (0, \infty]$ of existence of the solutions), the word *strong* is less explicit. In fact the meaning of the terms *strong* and *weak*, especially dealing with PDE problems, have different facets. We will clarify this point in chapter 10. Other words involved in the heading of the problem are *incompressible* and *inhomogeneous*: the *incompressible* nature of the fluid has already been introduced in section 0.0.1, while the *inhomogeneity* of the problem is a consequence of the presence of the non linear term $\nabla \cdot (\rho u \otimes u)$.

To understand the hypothesis we will require later and the aims of this thesis, it is necessary to do a little history of the *strong theory*[2] for the Navier Stokes equations, overlooking the results that deal with the problem from a different point of view.

The following brief *excursus* is taken from [4]. The existence of strong solutions[3] has been proved until the *'80s* for initial densities ρ_0 with a positive lower bound. In particular, in three dimensions, Ladyhzenskaya and Solonnikov proved that for initial data satisfying

$$\rho_0 \in C^1(\overline{\Omega}), \quad \inf_{\Omega} \rho_0 > 0, \quad u_0 \in W^{2-2/r,r}(\Omega), \quad \nabla \cdot u_0 = 0 \tag{8}$$

and, eventually, the presence of an external force $f \in L^r\big(\Omega \times (0,T)\big)$ for some $r > 3$, there exists a time $T_* \in (0,T)$ and a *unique* solution (ρ, u, p) to the initial boundary value problem, such that

$$u \in L^r(0, T_*; W^{2,r}(\Omega)), \quad u_t \in L^r(\Omega \times (0, T_*)) \tag{9}$$

[2] In the sense of strong solutions.

[3] As before, chapter 10 will clarify what kind of strong solutions we are searching for.

$$\rho \in C^1(\overline{\Omega} \times [0, T_*]), \quad p \in L^r(0, T_*; W^{1,r}(\Omega)) \tag{10}$$

Clearly here we are requiring very restrictive conditions on the initial data, from a physical point view.

In 1987, Kim improved the regularity of the unique solution, without weakening the requests on the initial data; however, the physical validity of the model could not be confirmed since so far the class of admissible initial data, that guarantee the solvability of the problem, remains unchanged.

In 1990 Padula proved the existence of a unique strong solution for initial densities $\rho_0 \in L^1(\Omega) \cap W^{1,\infty}(\Omega)$ satisfying the additional property

$$\int_{\Omega'} \rho_0 \, dx > 0 \quad \text{for any } \Omega' \subset \Omega \text{ with positive measure} \tag{11}$$

(in which case, it is obvious that ρ_0 can only vanish on sets of measure zero).
Hypothesis (11) is again a limitation for a correct physical interpretation of the natural model: we also have to consider the case in which the fluid does not occupy the whole (accessible) space, i.e. cases in which the density is zero in some regions of the space. Indeed, this is a valid request observing physical problems suggested by our reality.
The main difficulty of achieving the existence and the higher regularity of solutions in the case of vacuum is that it seems difficult to derive a priori estimates for u_t in appropriate norms, since u_t in the momentum equation is multiplied by ρ, possibly vanishing in some regions.

In the present thesis, following [4], we will overcome this difficulty by estimating ∇u_t in L^2 norm first, then applying Sobolev inequality and finally avoiding the restrictive hypothesis on the initial density: this is the key point of the work of Choe and Kim [4], and also *one of the key points of this thesis*, as we will see below. This method requires rather higher regularity assumption and a compatibility condition on the initial data: given a bounded domain Ω, we will consider initial densities $0 \leq \rho_0 \in L^\infty(\Omega)$ and initial velocity fields $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ satisfying the compatibility condition $\mu \Delta u_0 - \nabla p_0 = \sqrt{\rho_0} g$, for some $(p_0, g) \in H^1(\Omega) \times L^2(\Omega)$, and the natural incompressibility condition on the initial velocity field $\nabla \cdot u_0 = 0$ in Ω.

Through the work of Choe and Kim, assuming these hypothesis, we will prove the existence of a positive time $T_* > 0$ and of a weak solution (ρ, u) to the initial boundary value problem (1), together with some estimates.

Moreover, improving the regularity of the initial density, i.e. assuming also $\rho_0 \in H^1(\Omega)$, we will prove the existence of a strong solution (ρ, u, p) to the initial boundary value problem (1), together with other regularity properties, as we will specify in the statements of the main theorems [See 0.1.1].

Now we list all the key points of the present thesis.

- The first goal is to revise and reformulate the resolution of the local problem for system (1) as presented in [4], collecting in a single place the theorems spread in literature[4].

- In their beautiful but short paper [4], Choe and Kim claim that, after using techniques and results based on the fundamental paper by DiPerna and Lions

[4]See, in particular, the beautiful paper [16] by Kim and the references therein.

[8] on transport theory, published in *Inventiones* in 1989, one can deduce the existence of a weak solution to the original problem (1), which satisfies some regularity estimates. Quoting Choe and Kim:

$$\left\{ \begin{array}{l} \textit{"Therefore, adapting the arguments in Lions (1996), we can easily deduce that} \\ \textit{the limit } (\rho, u) \textit{ is a weak solution of the original Eqs. (1)-(3) with the initial} \\ \textit{data } (\overline{\rho}_0, u_0) \textit{ and satisfies } [...] \ [\textit{some}] \textit{ regularity estimates } [...]" \end{array} \right. \tag{12}$$

The reference "Lions (1996)" is the paper [20] in the bibliography. The "adaptation" cited above occupies in the present thesis the sections 11.6.4 - 11.14 (about 50 pages). In fact in [4] essentially are omitted the technical details hidden in the assertion (12). Lions' paper [20] provides the devices and the strategies to deal with the problem; however these tools are spread along the pages, and employed with different aims.

- Another purpose consists in collecting, in a unique chapter, the development over the centuries of the Stokes theory, concering the *Steady State Stokes equation*, together with the related Stokes operator. Such a theory is used in order to study the regularity of a basis of functions, i.e. the sequence $\{\phi_m\}_m$, that is the starting point of the *Galerkin method*, widely employed in chapter 11. Chapter 9 gathers results from [12], [19] and [27].

- In chapter 11 a critical reading of Simon's [26] provides the devices to understand the non trivial estimates deduced in Kim's [16].

- Chapter 8 summarizes the main results concerning transport theory. This chapter is in particular focused on the fundamental *Inventiones* paper by Di Perna and Lions [8], that provides a complete (and arduous) inspection of the problem. We propose a critical approach to the problem, considering the case of bounded domain assuming the typical hypothesis of the transport equation on a bounded domain (i.e. incompressibility and zero boundary conditions). Moreover we provide a rigorous approach to the formal proofs produced in the paper.

0.1.1 Structure of the thesis

The structure of this thesis is composed by four blocks:

(i) The *introductive block*, to which belongs this chapter, contains the presentation of the matter (Chapter 0) and a brief chapter where are collected well-known (and useful) devices of mathematical analysis (Chapter 1);

(ii) The part I collects the basis of fluid mechanics, with definition of fundamental spaces of functional analysis and some tools typical of PDE theory.
Chapter 2 is a brief summary of the main results about Banach and Hilbert spaces. The approach becomes more specific in chapters 3, 4 and 5, that concern, respectively, L^p spaces, Sobolev spaces and L^p spaces involving time.

Chapter 6 is dedicated to the Helmoltz decomposition, fundamental in fluid mechanics and in particular in the present thesis. On the other hand, chapter 7 deals with weak and strong compactness of $L^p(0,T;X)$, where X is a Banach space. In chapter 10 we finally specify what *weak* and *strong* solutions are in the present context. Chapter 8 takes on the transport theory and the compactness results of DiPerna and Lions [8], that play an important role in the Navier-Stokes theory (as revealed by the presence of a transport equation in the Navier-Stokes system). Finally chapter 9 summarizes almost a century of fluid mechanincs theory, from the results of Lorentz concerning the whole space Stokes problem (with the introduction of a concrete fundamental solution, analogous to the work of Laplace in the case of the Laplacian equation) [12, Ch. IV], to the work of Ladyženskaja regarding the Stokes equation [19].
The results of this compilative part will be stated and proved (the most of them) in details, even if the proof of some of them will only be sketched, with references from inside[5] and outside the present thesis.

(iii) The core of the thesis is placed in part II: this part of the work has the purpose of proving the existence of a local time solution to the INSEs, following the article by Kim and Choe [4]. This section also takes inspiration from Simon's [26] to obtain some useful estimates, and from Kim's [16] to acquire some useful propositions and an ODE approach to the problem, in order to build the approximate solutions.

In this part we will prove the following main theorems.

Theorem 0.1. *Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary, and assume the data ρ_0, u_0 satisfy the regularity*

$$0 \leq \rho_0 \in L^\infty(\Omega), \quad u_0 \in H_0^1(\Omega) \cap H^2(\Omega) \tag{13}$$

and the compatibility condition

$$\mu \Delta u_0 - \nabla p_0 = \sqrt{\rho_0} g, \qquad \nabla \cdot u_0 = 0 \ \ in \ \ \Omega \tag{14}$$

for some $(p_0, g) \in H^1(\Omega) \times L^2(\Omega)$. Let $T > 0$ a fixed local time. Then, there exists a time $T_* \in (0,T)$ and a weak solution $(u,\rho) \in L^\infty(0,T_*;H^2(\Omega)) \times L^\infty(0,T_*;L^\infty(\Omega))$ to the initial boundary value problem

$$\begin{cases} (\rho u)_t + \nabla \cdot (\rho u \otimes u) - \mu \Delta u + \nabla p = 0 \\ \rho_t + \nabla \cdot (\rho u) = 0, \ \rho \geq 0 \qquad (x,t) \in \Omega \times (0,T_*) \\ \nabla \cdot u = 0 \end{cases} \qquad \begin{cases} \rho(x,0) = \rho_0(x) & x \in \Omega \\ u(x,0) = u_0(x) & x \in \Omega \\ u(x,t) = 0 & (x,t) \in \partial\Omega \times (0,T_*) \end{cases} \tag{15}$$

such that for a.e. $t \in (0,T_*)$ we have the estimates

$$\|\nabla u(t)\|_2^2 \leq C, \qquad \|\rho(t)\|_q = \|\rho_0\|_q \tag{16}$$

[5] In example, if a result follows from a well known functional analysis result referred in the text, a referenced note will connect the two statements, together with a few lines comment.

$$\sup_{0<s\le t}\left(\|\nabla u\|_{H^1}^2+\|\sqrt{\rho}u_t\|_2^2\right)+\int_0^t\left(\|\nabla u\|_{W^{1,6}}^2+\|u_t\|_{D_0^{1,2}}^2\right)\,ds\le C\exp\left(C\int_0^t\|\nabla u\|_2^4\,ds\right) \tag{17}$$

and

$$\sup_{(0,t]}\left(\|\nabla u\|_{H^1}^2+\|\sqrt{\rho}u_t\|_2^2\right)+\int_0^t\|\nabla u_t\|_2^2\,ds+\int_0^t\|\nabla u\|_{W^{1,6}}^2\,ds\le$$

$$\le Q\exp\left(Q\int_0^t\|\nabla u\|_2^4\,ds\right)+Q\mathcal{C}(\rho_0,u_0,p_0) \tag{18}$$

where

$$\mathcal{C}(\rho_0,u_0,p_0)\equiv\|g\|_2^2 \tag{19}$$

Here the local existence time T_* and the positive constant C,Q depend only on $\|\rho_0\|_{L^\infty}$, $\|\nabla u_0\|_2$, $\|g\|_2$ and the time T; but it is independent of the lower bound of ρ_0.

We now state a theorem that assures us, under stronger hypothesis on the initial density, the existence of strong solutions.

Theorem 0.2. *Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary, and assume the data ρ_0, u_0 satisfy the regularity*

$$0\le\rho_0\in H^1(\Omega),\qquad u_0\in H_0^1(\Omega)\cap H^2(\Omega) \tag{13}$$

and the compatibility condition

$$\mu\Delta u_0-\nabla p_0=\sqrt{\rho_0}g,\qquad \nabla\cdot u_0=0\quad in\ \Omega \tag{14}$$

for some $(p_0,g)\in H^1(\Omega)\times L^2(\Omega)$. Let $T>0$ a fixed local time. Then, there exists a time $T_*\in(0,T)$ and a strong solution (ρ,u,p) that satisfies (15) *in the sense of section 10.2. Moreover, the solutions satisfy*

$$\rho\in L^\infty(0,T_*;H^1(\Omega)),\qquad \rho_t\in L^\infty(0,T_*;L^2(\Omega)) \tag{20}$$

$$\nabla p\in L^\infty(0,T_*;L^2(\Omega))\cap L^2(0,T_*;L^6(\Omega)) \tag{21}$$

Remark 0.1. *Weak* and *strong solutions* have to be meant in a sense that will be precised in chapter 10. □ ▮

Chapter 1

Classical analysis prerequisites

1.1 Notations

The whole *PDE* theory is based on which kind of domain we are considering. We start giving the definition of *domain* that we will adopt in the present thesis.

Definition 1.1. A *domain* of $\mathbb{R}^n$ is a subset Ω of $\mathbb{R}^n$ that is open and connected. If this subset is also bounded, we will call it *bounded domain*.

Remark 1.1. The symbol $C_c^\infty(\Omega)$ represents the smooth functions with compact support in the domain Ω. Sometimes, the symbol is replaced by $C_0^\infty(\Omega)$, especially when we use this set combined with the divergence-free condition, i.e. $C_{0,\sigma}^\infty(\Omega)$. □

1.1.1 Vectors, matrices and tensors

Writing the name of a vector, we will always mean the *column representation* of the vector. The row representation will be represented with u^T. So, the juxtaposition represents the matricial product.

Definition 1.2. Let $u, v \in \mathbb{R}^n$ and $A \in M_n$ a $n \times n$-matrix. Then we define the *dot product*

$$u \cdot v := u^T v$$

The dot product is also called *inner product*. Moreover, we use the symbol $\cdot$ also to represent the application between matrices and vectors, that is

$$u \cdot A \cdot v := u^T A v, \qquad A \cdot v \equiv Av$$

Definition 1.3. It is also defined an outer product

$$u \otimes v := uv^T$$

This product is a *matrix*.

Remark 1.2. We will frequently use

$$\rho u \otimes u = (\rho u)\, u^T \tag{1.1}$$

i.e. the product with the transpose. □

Remark 1.3. The canonical euclidean norm, or 2- norm, of vectors, matrices and tensors will be represented with the simple symbol $|\cdot|$: the lack a the number 2 as a pedex has the purpose of avoiding to get confused with the L^2 norm, $\|\cdot\|_2$, that will be used massively in the future. Clearly the meaning of the symbol $|\cdot|$ is dued to the context: if v is a vector, $|v|$ is the euclidean norm of a vector; if A is a matrix, $|A|$ is the 2-norm of a matrix. This matricial norm is the norm induced by the *Frobenious inner product* for matrices[1]

$$\langle A, B\rangle = \mathrm{Tr}(A^T B)$$

For semplicity, we will use the notation

$$A \cdot B := \mathrm{Tr}(A^T B) \tag{1.2}$$

The $\cdot$ distinguishes this scalar product from the matrix product AB.

Remark 1.4. Sometimes this scalar product is represented with the notation $A : B$. However, despite the ambiguity, we prefer to maintain the usual notation of the scalar product. □

So,

$$|A| \equiv \sqrt{A \cdot A} \equiv \sqrt{\mathrm{Tr}(A^T A)} \equiv \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}^2}$$

With these devices, the *Cauchy–Schwarz inequality* assumes the form

$$|u \cdot v| \leq |u||v| \qquad \forall\, u, v \in \mathbb{R}^n$$

where at the first member $|\cdot|$ represent the absolute value.

It is useful to recall also that, if $A \in M_{n,m}$ and $B \in M_{m,l}$, it holds

$$|AB| \leq |A||B| \tag{1.3}$$

An application of (1.3) concerns the outer product. In fact, if $u, v \in \mathbb{R}^n$, we have

$$|u \otimes v| \equiv |uv^T| \leq |u||v^T| = |u||v| \tag{1.4}$$

since $u \in M_{n,1}$ and $v^T \in M_{1,n}$.

However, when $p \neq 2$, the vectorial (or, more in general, tensorial) p-norm will be represented with $|\cdot|_p$. So, for example, if v is a vector in $\mathbb{R}^n$ we have

$$|v|_p := \left(\sum_{i=1}^{n} v_i^p\right)^{\frac{1}{p}}$$

The definition is generalized to all the components in the case of tensors of bigger dimension.

[1] Remember that

$$\mathrm{Tr}(A^T B) = \mathrm{Tr}(BA^T) = \mathrm{Tr}(B^T A) = \mathrm{Tr}(AB^T)$$

In future pages, tensors will play an important role: in fact, since INSE involve some vectorial quantities, their second derivatives will be represented by tensors. Differently from a vector or a matrix, it is difficult, if not impossible, to represent a tensor graphically on a paper. However, the representation of a tensor is not useful: we will, mainly, deal with its norms. So, if T is a $n \times n \times n$ tensor, we will write

$$|T|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} T_{ijk}^2$$

In the same way, we have

$$|T|_p^p = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} |T_{ijk}|^p$$

The definitions can be adapted in the case of tensors of bigger dimension. □

1.1.2 Remarks on the vectorial nature of the INSE

Definition 1.4. The divergence of a matrix is a vector, in this case a three-dimensional vector, defined as

$$\big(\nabla \cdot (\rho u \otimes u)\big)_j := \sum_{i=1}^{3} \frac{\partial[(\rho u \otimes u)_{ij}]}{\partial x_i} \qquad j \in \{1, 2, 3\}$$

Definition 1.5. Another useful vectorial definition is the *Laplacian of a vector* that is

$$\Delta u := \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \\ \Delta u_3 \end{pmatrix} \tag{1.5}$$

Remark 1.5. For future computations, we recall here other definitions. Being u (or in general, another velocity field) a vector, we define ∇u as the *Jacobian matrix* of u. The same symbol is also used for the *gradient of a scalar function*, in example the pressure P. □

Remark 1.6. As it will be proved in (1.10), the *momentum equation* can be rewritten in a slightly different way. In particular

$$\nabla \cdot (\rho u \otimes u) = \big(\nabla \cdot (\rho u)\big)u + \rho\,(\nabla u) \cdot u$$

So, using the mass equation, we have

$$\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) - \mu \Delta u + \nabla P = \rho u_t + \rho\,(\nabla u) \cdot u - \mu \Delta u + \nabla P$$

that is an alternative way to write Navier-Stokes equations. This formulation will be very useful in order to find approximate solutions; the limit of this sequence of solutions will satisfy the weak formulation of the original momentum equation $(\rho u)_t + \nabla \cdot (\rho u \otimes u) - \mu \Delta u + \nabla P = 0$. □

1.2 Topological prerequisites

Definition 1.6. Let (X, d) a metric space, with the *topology* τ_d induced by the distance. Let $Y \subseteq X$ a subset. We will write $\overline{Y}$ to mean the *closure* of Y in X.

Definition 1.7. Let (X, d_X) and (Y, d_Y) metric spaces. A function $f : (X, d_X) \to (Y, d_Y)$ is called *Lipschitz* (or *Lipschitz continuous*) if there exists $L > 0$ such that

$$d_Y(f(x_1), f(x_2)) \leq L\ d_X(x_1, x_2) \quad \forall\ x_1, x_2 \in X$$

$L = L_f$ is called *Lipschitz constant.*

Definition 1.8. A bounded domain Ω, with boundary $\partial\Omega$, is called a *Lipschitz domain* if for every $x_0 \in \partial\Omega$ there exist an hyperplane $H^{n-1} \ni x_0$, a *Lipschitz* function f, and two numbers $r, \delta > 0$ such that

$$\Omega \cap A = \{x + y\nu |\ x \in B_r(x_0) \cap H^{n-1},\ -\delta < y < f(x)\}$$

and

$$\partial\Omega \cap A = \{x + y\nu |\ x \in B_r(x_0) \cap H^{n-1},\ y = f(x)\}$$

where ν is a unitaty normal to H^{n-1} and

$$A := \{x + y\nu |\ x \in B_r(x_0) \cap H^{n-1},\ |y| < \delta\}$$

Definition 1.9. A bounded domain Ω, with boundary $\partial\Omega$, is called a C^k *domain* if for each $x_0 \in \partial\Omega$ there exist $r > 0$ and a C^k function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$U \cap B_r(x_0) = \{x \in B_r(x_0) |\ x_n > f(x_1, ..., x_{n-1})\}$$

Moreover, it is called a *smooth domain* if $f \in C^\infty$.

We list now some lemmas that will be useful in future proofs.

Lemma 1.1. *Let Ω a bounded domain. Then it is defined*

$$d(\Omega) := \sup_{x,y \in \Omega} |x - y|$$

It is called diameter of the set.

Definition 1.10. If A, B are two subset of a domain Ω, with $A \cap B = \emptyset$, we define

$$dist(A, B) := \inf_{x \in A, y \in B} |x - y|$$

Lemma 1.2. *Let $\Omega \subseteq \mathbb{R}^n$ a domain with $n \geq 2$. Then there exist a sequence $\{\Omega_j\}_{j \in \mathbb{N}}$ of bounded Lipschitz subdomain of Ω and a sequence $\{\varepsilon_j\}_{j \in \mathbb{N}}$, with $\varepsilon_j > 0$, such that*

- $\overline{\Omega}_j \subseteq \overline{\Omega}_{j+1} \quad \forall\ j \in \mathbb{N}$;
- $dist(\partial\Omega_{j+1}, \Omega_j) \geq \varepsilon_{j+1} \quad \forall\ j \in \mathbb{N}$;

- $\lim_{j\to+\infty} \varepsilon_j = 0$;
- $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$.

Definition 1.11. Let (X,τ) a topological space and Y a subspace of X. We say that Y is relatively compact in X if its closure $\overline{Y}$ in X is a compact subset of X.

We state here also the Ascoli-Arzelà theorem, that will be very useful in a compactness argument.

Theorem 1.1 (Ascoli-Arzelà). *Let $a_n \in C([0,T])$ such that exists $C > 0$ and $K > 0$ such that*

$$|a_n(t)| \leq C \quad \forall n \in \mathbb{N},\ t \in [0,T] \tag{1.6}$$

$$|a_n(t) - a_n(\tau)| \leq K|t-\tau| \quad \forall\ t,\tau \in [0,T],\ \forall\ n \in \mathbb{N} \tag{1.7}$$

Then exists a subsequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ and $a \in C([0,T])$ such that

$$\lim_{k\to+\infty} \max_{[0,T]} |a_{n_k}(t) - a(t)| = 0$$

1.3 Useful vectorial and matricial calculus identities

We collect here some useful estimates that, going on, we will use in the calculations in the present thesis.

(i) If F is a C^1 vector field and φ is a C^1 scalar function, we have the *divergence rule*

$$\boxed{\nabla \cdot (\varphi F) = \varphi\ \nabla \cdot F + F \cdot \nabla\varphi} \tag{1.8}$$

(ii) If $w = w(x,t)$ is a C^2 vector field we have (using Schwarz theorem for partial derivatives interchainging)

$$\boxed{\frac{1}{2}\frac{d}{dt}|\nabla w|^2} = \frac{1}{2}\frac{d}{dt}\nabla w \cdot \nabla w = \frac{1}{2}\frac{d}{dt}\left\{|\nabla w_1|^2 + |\nabla w_2|^2 + |\nabla w_3|^2\right\} =$$

$$= \sum_{j=1}^{3} \nabla w_j \cdot \nabla(\partial_t w_j) = \boxed{\sum_{j=1}^{3} \nabla \cdot (\partial_t w_j \nabla w_j) - \Delta w \cdot \partial_t w} \tag{1.9}$$

where in the last equality has been used the previous point.

(iii) If a, b are two sufficiently regular vectorial fields, then

$$\boxed{\nabla \cdot (a \otimes b) = (\nabla \cdot a)\ b + \nabla b \cdot a} \tag{1.10}$$

This is a formula for the divergence of outer product. By definition, we have

$$(\nabla \cdot (a \otimes b))_j := \sum_{i=1}^{3} \frac{\partial[(a\otimes b)_{ij}]}{\partial x_i} = \sum_{i=1}^{3} \frac{\partial\,(a_i b_j)}{\partial x_i} = \sum_{i=1}^{3} (\partial_{x_i} a_i b_j + a_i \partial_{x_i} b_j) =$$

$$= (\nabla \cdot a) b_j + \nabla b_j \cdot a$$

Bracketing these elements in a column vector we get

$$\nabla \cdot (a \otimes b) = (\nabla \cdot a)\ b + \nabla b \cdot a$$

that is our assertion.

(iv) It holds

$$\boxed{\Delta u \cdot u = \sum_{i=1}^{3} \nabla \cdot (u_i \nabla u_i) - |\nabla u|^2} \tag{1.11}$$

In fact, by definition,

$$(\Delta u) \cdot u = \sum_{i=1}^{3} (\Delta u_i) u_i = \sum_{i=1}^{3} \nabla \cdot (\nabla u_i) u_i$$

But, according to the first point, we have

$$\nabla \cdot (u_i \nabla u_i) = u_i \nabla \cdot (\nabla u_i) + \nabla u_i \cdot \nabla u_i = u_i \nabla \cdot (\nabla u_i) + |\nabla u_i|^2$$

Otherwise[2]

$$|\nabla u|^2 = |\nabla u^T|^2 = |(\nabla u_1, \nabla u_2, \nabla u_3)|^2 = |\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_3|^2$$

So

$$(\Delta u) \cdot u = \sum_{i=1}^{3} \nabla \cdot (\nabla u_i) u_i = \sum_{i=1}^{3} \nabla \cdot (u_i \nabla u_i) - \sum_{i=1}^{3} |\nabla u_i|^2 = \sum_{i=1}^{3} \nabla \cdot (u_i \nabla u_i) - |\nabla u|^2$$

So, the assertion is proved.

(v) If $u \in C^3$ is a vector field, we have

$$\boxed{\Delta(\nabla \cdot u) = \nabla \cdot (\Delta u)} \tag{1.12}$$

Remember that the laplacian of a vector field is a vector field. We now prove the identity. We have, using Schwarz's theorem,

$$\Delta(\nabla \cdot u) \equiv \sum_{i=1}^{n} \partial^2_{x_i} \left(\sum_{j=1}^{n} \partial_{x_j} u_j \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_{x_j} \partial^2_{x_i} u_j = \sum_{j=1}^{n} \sum_{i=1}^{n} \partial_{x_j} \partial^2_{x_i} u_j =$$

$$= \sum_{j=1}^{n} \partial_{x_j} \left(\sum_{i=1}^{n} \partial^2_{x_i} u_j \right) = \sum_{j=1}^{n} \partial_{x_j} \Delta u_j = \nabla \cdot (\Delta u)$$

[2]The *Frobenious norm* of a matrix has the property that

$$|A|^2 = |A_1|^2 + ... + |A_m|^2$$

if A_k are the columns of the matrix.

(vi) If u is a C^1 vector field and η is a C^1 scalar function, then

$$\boxed{\nabla(u\eta) = \eta\nabla u + u \otimes \nabla\eta} \tag{1.13}$$

In fact we have

$$(\nabla(\eta u))_{ij} = \partial_{x_j}(\eta u_i) = (\partial_{x_j}\eta)\ u_i + \eta\ (\partial_{x_j}u_i) = (u \otimes \nabla\eta)_{ij} + \eta(\nabla u)_{ij}$$

(vii) If u and v are two regular verctor field we have

$$\boxed{\nabla u \cdot \nabla v = \sum_{i=1}^{3} \nabla \cdot (u_i \nabla v_i) - u \cdot \Delta v} \tag{1.14}$$

In fact we have that

$$\Big((\nabla u)(\nabla v)^T\Big)_{ii} = \sum_{j=1}^{3} \partial_{x_j} u_i \partial_{x_j} v_i$$

and so

$$\nabla u \cdot \nabla v = \mathrm{Tr}\Big((\nabla u)(\nabla v)^T\Big) = \sum_{i=1}^{3}\sum_{j=1}^{3} \partial_{x_j} u_i \partial_{x_j} v_i$$

Using that

$$\partial_{x_j}(u_i \partial_{x_j} v_i) = \partial_{x_j} u_i \partial_{x_j} v_i + u_i \partial^2_{x_j} v_i$$

it follows

$$\nabla u \cdot \nabla v = \sum_{i=1}^{3}\sum_{j=1}^{3} \partial_{x_j}(u_i \partial_{x_j} v_i) - \sum_{i=1}^{3}\sum_{j=1}^{3} u_i \partial^2_{x_j} v_i =$$

$$= \sum_{i=1}^{3} \nabla \cdot (u_i \nabla v_i) - \sum_{i=1}^{3} u_i \Delta v_i = \sum_{i=1}^{3} \nabla \cdot (u_i \nabla v_i) - u \cdot \Delta v$$

1.4 Useful well-known estimates

1. **Young's Inequality.** Let $q, p \in \mathbb{R}$ such that $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad \forall a, b \geq 0$$

In fact, remembering that $\varphi(t) := e^t$ is a convex function, we have

$$ab = \varphi\big(\ln a + \ln b\big) = \varphi\bigg(\frac{1}{p}\ln a^p + \frac{1}{q}\ln b^q\bigg) \leq \frac{1}{p}\varphi\big(\ln a^p\big) + \frac{1}{q}\varphi\big(\ln b^q\big) = \frac{a^p}{p} + \frac{b^q}{q}$$

2. **Parametric Young's Inequality.** Let $q, p \in \mathbb{R}$ such that $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ and $\varepsilon > 0$. Then

$$ab \leq \varepsilon a^p + C_\varepsilon b^q \qquad \forall\ a, b \geq 0$$

where $C_\varepsilon \equiv \frac{(\varepsilon p)^{-\frac{q}{p}}}{q}$. In fact, it is sufficient to write $ab = \Big((\varepsilon p)^{\frac{1}{p}} a\Big)\Big(\frac{b}{(\varepsilon p)^{\frac{1}{p}}}\Big)$ and apply the previous inequality.

3. **Discrete Minkowski's inequalities.** Let $a, b \in \mathbb{R}^n$ and $p \in [1, +\infty)$. Then

$$\left(\sum_{k=1}^{n} |a_k + b_k|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} \tag{1.15}$$

1.5 ODEs

Definition 1.12. A function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is *uniformly lipschitz* if exists a costant $L > 0$ such that

$$|f(x) - f(y)| \leq L|x - y| \quad \forall x, y \in D$$

Definition 1.13. A function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is *locally uniformly lipschitz* if for all $x \in D$ exists a neighborhood $U(x)$ such that f is *uniformly lipschitz* in $U(x)$.

Remark 1.7. If D is open, a function $f \in C^1(D)$ is locally uniformly lipschitz. In fact, if $x \in D$, then we can consider $\overline{B}_r(x) \subseteq D$, and

$$|f(z) - f(y)| \leq L|z - y| \qquad \forall\ z, y \in B_r(x)$$

where $\xi = \xi_{x,z} \in [x, z] := \{tx + (1-t)z|\ t \in [0, 1]\}$ and $L := \sup_{\overline{B}_r(x)} \left\| \frac{\partial f}{\partial x}(\xi) \right\|$.

Remark 1.8. We are in this situation if we choose as f the velocity field $u(x, t)$. In fact, $u \in C^2(\Omega \times [0, +\infty), \mathbb{R}^3)$, which means that $u \in C^2(A, \mathbb{R}^3)$, where A is an open set that cointains $\Omega \times [0, +\infty)$. $\square$

We recalled these definitions to recall the following proposition.

Theorem 1.2. *Let $f : D \times I \subseteq \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ a locally uniformly lipschitz function in $(x_0, t_0) \in D \times I$ with lipschitz constant $L = L(x_0, t_0)$ in the neighborhood $U(x_0, t_0)$. Let $r > 0$ and $T_0 > 0$ such that $\overline{B}_r(x_0) \times [t_0 - T_0, t_0 + T_0] \subseteq U(x_0, t_0)$. Let $\delta > 0$ such that*

$$\delta < \frac{1}{L} \quad \textit{and} \quad \delta < \min\{r, \frac{M}{r}\}$$

with $M := \max_{\overline{B}_r(x_0) \times [t_0 - T_0, t_0 + T_0]} |f(x, t)|$. Then the Cauchy problem

$$\begin{cases} \dot{x}(t) = f(x(t), t) \\ x(t_0) = x_0 \end{cases}$$

has a unique solution $x(t)$ defined for all $t \in [t_0 - \delta, t_0 + \delta]$.

Theorem 1.3. *Let $f \in C^1(D \times I; \mathbb{R}^n)$, $(x_0, t_0) \in D \times I$. Let $x(t)$ a solution of*

$$\begin{cases} \dot{x}(t) = f(x(t), t) \\ x(t_0) = x_0 \end{cases}$$

Then exists a maximal extension of $x(t)$, we say $\overline{x}(t)$, which solves the problem in an open set $J \subseteq I$.

Theorem 1.4. *Let $f \in C^1(D \times I; \mathbb{R}^n)$, $(x_0, t_0) \in D \times I$. Let $x(t)$ a solution of*

$$\begin{cases} \dot{x}(t) = f(x(t), t) \\ x(t_0) = x_0 \end{cases}$$

for $t, t_0 \in (t_1, t_2)$. Let $\{t_k\}_{k \in \mathbb{N}} \subseteq \mathbb{R}$ such that $t_k \to t_1$ as $k \to \infty$ and

$$\lim_{k \to +\infty} x(t_k) = \overline{x} \in \mathbb{R}^n$$

Then, exists $a > 0$ such that $x(t)$ is solution of the Cauchy problem for all $t \in (t_1 - a, t_2)$.

Theorem 1.5. *Let A an open set in $\mathbb{R}^n$ and let $f(\varphi, t) \in C^k(A \times [0, T])$ be a force, with $k \geq 1$. Consider $\varphi_0 \in A$ and the problem*

$$\begin{cases} \dot{\varphi}(t) = f(\varphi(t), t) \\ \varphi(0) = \varphi_0 \end{cases}$$

Then there exists a time $\tau > 0$ and a unique solution $\varphi \in C^{k+1}([0, \tau), A)$ to the problem. Moreover, we can choose $\tau > 0$ as the maximal time of existence of the solution. The maximal interval of existence of the solution is an open set.

1.5.1 Gronwall's lemmas

Lemma 1.3. *Let $f, g \in C([a, b])$, with $g \geq 0$. Suppose that $f(t) \leq f_0 + \int_a^t g(s) f(s)\, ds$ where f_0 is a constant. Then*

$$f(t) \leq f_0 \exp\left(\int_a^t g(s)\, ds \right)$$

Lemma 1.4. *Let $v : [0, \overline{T}) \to \mathbb{R}_+$ continuous such that*

$$v(t) \leq V_0 + \int_0^t \psi(s) \omega(v(s))\, ds \quad \forall t \in [0, \overline{T})$$

where $V_0 \geq 0$, $\psi : [0, \overline{T}) \to \mathbb{R}_+$ is continuous and $\omega : [0, +\infty) \to (0, +\infty)$ is continuous and monotone strictly-increasing. Then

$$v(t) \leq \phi^{-1} \left(\phi(V_0) + \int_0^t \psi(s)\, ds \right) \qquad \forall\ t \in [0, \overline{T})$$

where

$$\phi(u) := \int_{u_0}^u \frac{ds}{\omega(s)}$$

Remark 1.9. This lemma is due to Bihari, and we provide the proof in [1, p. 23, Prop. 1.1]. □

Proof. We define

$$y(t) := \int_0^t \omega(v(s))\psi(s)ds \qquad \forall\ t \in [0, \overline{T})$$

So, it follows by the hypothesis that

$$v(t) \leq V_0 + y(t)$$

Moreover

$$y'(t) = \omega(v(t))\psi(t) \leq \omega(V_0 + y(t))\psi(t) \implies \frac{y'(s)}{\omega\left(V_0 + y(s)\right)} \leq \psi(s) \tag{1.16}$$

and so, integrating over $[0, t]$, with $t \in [0, \overline{T})$, we have

$$\phi(y(t) + V_0) - \phi(V_0) = \int_0^{y(t)+V_0} \frac{d\tau}{\omega(\tau)} - \int_0^{V_0} \frac{d\tau}{\omega(\tau)} =$$

$$= \int_{V_0}^{y(t)+V_0} \frac{d\tau}{\omega(\tau)} = \int_0^t \frac{y'(s)\ ds}{\omega(V_0 + y(s))} \leq \int_0^t \psi(s)\ ds$$

So

$$\phi(v(t)) \leq \phi(y(t) + V_0) \leq \phi(V_0) + \int_0^t \psi(s)\ ds$$

By the monotony[3] of ϕ^{-1} we have $v(t) \leq \phi^{-1}\left(\phi(V_0) + \int_0^t \psi(s)\ ds\right)$, $\forall\ t \in [0, \overline{T})$. ∎

We now provide a useful Gronwall's integral lemma.

Lemma 1.5. *Let I be an interval of the real line. Let α, β, u be real valued functions defined on I. Assume that β and u are continuous and that the negative part of α is integrable on every closed and bounded subinterval of I. If, moreover, β is non negative and if u satisfies the inequality*

$$u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)\ ds \qquad \forall t \in I$$

where a is the left extreme of I, and α is non decreasing, then

$$u(t) \leq \alpha(t) \exp\left(\int_a^t \beta(s)\ ds\right) \qquad \forall t \in I$$

1.5.2 Time evolution opeator and flow of an ODE

Definition 1.14. Consider an ordinary differential equation

$$\begin{cases} \dot{x}(t) = f(x(t), t) \\ x(t_0) = x_0 \end{cases}$$

[3]In fact $\phi'(x) = \frac{1}{\omega(x)} > 0$, and so $x_1 < x_2 \iff \phi(x_1) < \phi(x_2)$.

where each term satisfies the right condition of solvability mentioned above. We define

$$\varphi(t; t_0, x_0) := x(t)$$

where $x(t)$ is the solution of the equation. Here φ is called **flow** at time t with starting data (x_0, t_0) of the ODE above.

Remark 1.10. If we fix t and t_0, $\phi(x_0) := \varphi(t; t_0, x_0)$ is a transformation $\phi : \mathbb{R}^n \to \mathbb{R}^n$. $\square$

Remark 1.11. If $t_0 \in [0, T]$, $x_0 \in \mathbb{R}^n$, $f \in C^1(\mathbb{R}^n \times [0, T], \mathbb{R}^n)$ and $|f(x, t)| \leq M$ for all $(x, t) \in \mathbb{R}^n \times [0, T]$, then the solution exists in the interval of time $[0, T]$. So we can consider $\varphi(t; t_0, x_0)$ for all $t \in [0, T]$. Because both $t, t_0 \in [0, T]$, we can also consider $\varphi(t_0; t, x_0)$. It follows that

$$\varphi(t_0; t, \varphi(t; t_0, x_0)) = x_0, \qquad \varphi(t; t_0, \varphi(t_0; t, x_0)) = x_0$$

because of unicity of solution of an ODE (level curves never cross; if they do, then the curves coincide). In this way, we have found the inverse of the trasformation. $\square$

Lemma 1.6. *Let $f \in C^k(\mathbb{R}^n \times [0, T], \mathbb{R}^n)$, with $|f| \leq M$. Let φ the flow associtated to the velocity field f. Let $t_0 \in [0, T]$. Then*

$$\varphi(t; t_0, x) \equiv g(x, t) \in C^k(\mathbb{R}^n \times [0, T], \mathbb{R}^n)$$

Theorem 1.6. *Let $f \in C^3(\mathbb{R}^n \times [0, T], \mathbb{R}^n)$ a velocity field such that $\nabla_x \cdot f = 0$ and $|f| \leq M$. Let $t_0 \in [0, T]$ and let*

$$M_t(x) := D_x \varphi(t; t_0, x)$$

with φ the flow associated to f. Then $\det(M_t(x)) \equiv \det(M_{t_0}(x)) = \det \mathbb{I} = 1$.

Remark 1.12. It is a theorem of volume conservation, a generalization of the Liouville theorem for the Hamiltonian flow. $\square$

Remark 1.13. This result is important, and justifies the name *incompressibility equation* for the divergence free equation of a flow, $\nabla \cdot f = 0$. As we will see later, this theorem says that in a change of varibles, where the change is a time evolution operator solution of a divergence free equation, the Jacobian term is unimportant, because it is costanly unitary. $\square$

1.6 Other classical results: surface integrals and divergence theorems

1.6.1 Integrals over manifolds

We first introduce some topological notions.

Definition 1.15. Let $V \subseteq \mathbb{R}^n$ a connected subset of the whole space. We say that V is a *k-manifold* if for every $x_0 \in V$ exists an open neighborhood A_{x_0} of x_0 such that $V \cap A_{x_0} = \varphi(U)$ where the pair (φ, U) satisfies the followings:

(i) U is open, connected and bounded in $\mathbb{R}^k$, so that $\overline{U}$ is a compact subspace of $\mathbb{R}^k$;

(ii) $m_k(\partial U) = 0$, where m_k is the Lebesgue measure in $\mathbb{R}^k$;

(iii) $\varphi \in C^1(\overline{U}; \mathbb{R}^n)$;

(iv) φ is an injective function;

(v) $\partial_u \varphi(u)$ has maximal rank for every $u \in \overline{U}$.

So, we have the next definitions.

Definition 1.16. Let V a k-manifold, and let (φ, U) its representation. We define the *area element* of this manifold as

$$\sigma_k(u) := \sqrt{\sum_{(i_1,...,i_k)\in I_n^k} \left| \det \frac{\partial(\varphi_{i_1}, ..., \varphi_{i_k})}{\partial(u_1, ..., u_k)}(u) \right|^2}$$

where $I_n^k := \{(i_1, ..., i_k) \in \{1, ..., n\}^k : \; i_1 \leq i_2 \leq ... \leq i_k\}$.

Finally we can introduce the most operational definition, that is the integral.

Definition 1.17. Let $f \in C(\overline{V}, \mathbb{R})$. We define

$$\int_V f \; d\sigma \equiv \int_V f \; d\sigma_k := \int_U f(\varphi(u)) \; \sigma_k(u) \; du \equiv \int_{\overline{U}} f(\varphi(u)) \; \sigma_k(u) \; du$$

Remark 1.14. The definition is well posed because ∂U has zero measure, so it holds the latter equality in the definition. Morevoer, $f \circ \varphi$ and σ_k are continuous function over the compact $\overline{U}$, so the product of the two functions is continuous over the compact and so integrable. □

Definition 1.18. If $V = \bigcup_{i=1}^{m} V_i$, where V_i is a k-manifold with representation (φ_i, U_i), we define

$$\int_V f \; d\sigma := \sum_{i=1}^{m} \int_{V_i} f \; d\sigma \equiv \sum_{i=1}^{m} \int_{\overline{U}_i} f(\varphi_i(u)) \; \sigma_{k,i}(u) \; du \tag{1.17}$$

for every[4] $f \in C(\overline{V}, \mathbb{R})$.

1.6.2 Divergence theorem for regular domains

Theorem 1.7. *Let $n \geq 2$ and $A \subseteq \mathbb{R}^n$ an open, bounded and connected set such that*

$$A = \{x \in \mathbb{R}^n : \phi(x) < 0\}, \quad \partial A = \{x \in \mathbb{R}^n : \phi(x) = 0\}, \quad \nabla\phi(x) \neq 0 \;\; \forall x \in \partial A$$

with $\phi \in C^1(\mathbb{R}^n; \mathbb{R})$. Let $F \in C^1(\overline{A}; \mathbb{R}^n)$. Then

$$\int_A (\nabla \cdot F) \; dx = \int_{\partial A} F \cdot \nu \; d\sigma_{n-1} \tag{1.18}$$

where ν is the normal vector of ∂A.

Remark 1.15. It is sufficient that A is a C^1 domain of $\mathbb{R}^n$. □

[4]And being $\overline{V}_i \subseteq \overline{V}$, $f \in C(\overline{V}_i, \mathbb{R})$.

Part I

Functional spaces in fluid mechanics and PDEs

Chapter 2

Banach and Hilbert spaces: weak and strong convergences

2.1 Hilbert spaces

Definition 2.1. A *real Hilbert space* is a vector space, equipped with a scalar product, that is complete with respect the norm induced by the inner product.

Theorem 2.1. *Let H a Hilbert space. Let $C \neq \emptyset$ a closed and convex vector subspace of H. If $v \notin C$, then $\exists\, v_0 \in C$ such that*

$$|v - v_0| = \inf_{w \in C} |v - w| \tag{2.1}$$

Definition 2.2. Let $v \in H$ and C as above. We define the *projection* of v over C as

$$p_C(v) := \begin{cases} v_0 & \text{if } v \notin C \\ v & \text{otherwise} \end{cases}$$

Definition 2.3. If W is a vector subspace of H, we define

$$W^{\perp} := \{v \in H |\ \langle v, w \rangle = 0 \ \ \forall\, w \in W\}$$

Corollary 2.1. *Let H a Hilbert space and W a closed and convex linear subspace. Let $v \in H$. Then $v_0 := p_W v \in W$ is the unique element of W such that*

$$\langle v - v_0, w \rangle = 0 \quad \forall\, w \in W$$

It follows that $v = (v - v_0) + v_0 \equiv v' + v_0$ with $v' \in W^{\perp}$.

2.2 Banach spaces

The notion of Banach space is more general then the one of Hilbert space (i.e. every Hilbert space is a Banach space). Definitions and statements are inspired by [10].

Definition 2.4. A *Banach space* is a linear normed space $(X, \|\cdot\|)$ such that is complete respect with the norm $\|\cdot\|$.

Definition 2.5. Let X, Y be Banach spaces. A bounded operator from X to Y is a function $f : X \to Y$ such that exists $C > 0$

$$\|f(x)\|_Y \leq C\|x\|_X$$

For every Banach space X we can define the dual space of X.

Definition 2.6. If X is a Banach space, then we define the *dual space of* X as

$$X^* := \{f : X \to \mathbb{R} \text{ such that } f \text{ is a bounded linear operator}\}$$

Remark 2.1. For a linear opeator between normed spaces, boundness and continuity are equivalent. □

Proposition 2.1. *The space X^* equipped with the norm*

$$\|f\| := \sup_{\|x\| \leq 1} |f(x)| \tag{2.2}$$

is a Banach space.

Definition 2.7. We say that a Banach space X is reflexive if $(X^*)^* = X$. More precisely, this means that for each $u^{**} \in (X^*)^*$, there exists $u \in X$ such that

$$\langle u^{**}, u^* \rangle = \langle u^*, u \rangle \qquad u^* \in X^*$$

where the symbol $\langle u^*, u \rangle$ denotes the real number $u^*(u)$. In other words, the symbol $\langle \cdot, \cdot \rangle$ denotes the pairing of X^* and X.

Theorem 2.2. *Every Hilbert space is reflexive; more precisely, for every $f \in H^*$ there exists a unique element $x_f \in H$ such that*

$$f(y) = \langle x, y \rangle \quad \forall y \in H$$

Moreover, the map $f \to x_f$ is a linear isomorphism of H^* onto H.

2.3 Strong and weak convergences

Definition 2.8. Let $(X, \|\cdot\|)$ a Banach space. We say that a sequence $x_k \in X$ converges to $x \in X$ (in strong sense) if

$$\lim_{k \to +\infty} \|x_k - x\| = 0$$

We use one of the following notations

$$\lim_{k \to +\infty} x_k = x, \qquad x_k \to x$$

Definition 2.9. Let $(X, \|\cdot\|)$ a Banach space and let X^* its dual space. We say that $x_k \in X$ converges weakly to $x \in X$ if

$$\lim_{k \to +\infty} f(x_k) = f(x) \quad \forall f \in X^*$$

In this case we write

$$x_k \rightharpoonup x$$

Proposition 2.2. *Let $(X, \|\cdot\|)$ be a Banach space. Let $\{x_k\}_{k\in\mathbb{N}} \subset X$ and $x \in X$. The following statements hold:*

(i) If $x_k \to x$, then $x_k \rightharpoonup x$;

(ii) If $x_k \rightharpoonup x$, then
$$\|x\| \leq \liminf_{k\to+\infty} \|x_k\| \tag{2.3}$$

(iii) If X is reflexive and $x_k \in X$ is such that $\|x_k\| \leq C$ for every $k \in \mathbb{N}$, then there exist a subsequence x_{k_j} and an element $x \in X$ such that $x_{k_j} \rightharpoonup x$.

The latter property is called weak compactness.

We can also introduce a notion of convergence in the dual space X^*.

Definition 2.10. Let $(X, \|\cdot\|)$ a Banach space, and let X^* its dual space. Consider a sequence $f_k \in X^*$. We say that f_k is weak star (or weak *) convergent to an element $f \in X^*$ if
$$\lim_{k\to+\infty} f_k(x) = f(x) \quad \forall x \in X$$
In this case we write
$$f_k \overset{*}{\rightharpoonup} f$$

Lemma 2.1. *Let $(X, \|\cdot\|)$ be a Banach space and X^* its dual. Suppose that $f_k \in X^*$ is such that $f_k \overset{*}{\rightharpoonup} f$ with $f \in X^*$. Then*
$$\|f\| \leq \liminf_{k\to+\infty} \|f_k\|$$

Remark 2.2. While the proof of (2.3) is often provided, we prove here lemma 2.1 since it is less usual. □

Proof. Define $M := \|f\| = \sup\{|f(x)| : \; x \in X, \; \|x\| \leq 1\}$. By the definition of supremum we have that for every $\varepsilon > 0$ exists $\overline{x} \in X$, $\|\overline{x}\| = 1$ such that
$$|f(\overline{x})| > M - \varepsilon$$
Since $\overline{x} \in X$, by the *weak-** convergence we have that
$$\lim_{k\to+\infty} |f_k(\overline{x})| = |f(\overline{x})|$$
So, exists a $K \in \mathbb{N}$ such that
$$|f_k(\overline{x})| > M - \varepsilon \qquad \forall\, k \geq K$$
Since $\overline{x}$ is an element in the unitaty disc of X we have that
$$\|f_k\| \geq |f_k(\overline{x})| \qquad \forall\, k \in \mathbb{N}$$
It follows that
$$\liminf_{k\to+\infty} \|f_k\| \geq M - \varepsilon$$
Since the inequality holds for every $\varepsilon > 0$ we have $\liminf_{k\to+\infty} \|f_k\| \geq M = \|f\|$. ∎

It also holds the following theorem from [18, part 4.9, problem 10, pag. 269].

Theorem 2.3. *(Banach-Alaoglu) Let Y a separable Banach space and let $M \subseteq Y^*$ a bounded subset of the dual space. Then, every sequence in M has a subsequence that converges in weak star to an element of Y^*.*

Definition 2.11. A Banach space $(X, \|\cdot\|)$ is *weakly complete* if every weak Cauchy sequence[1] is weakly convergent to some $x \in X$.

Theorem 2.4. *Let $(X, \|\cdot\|)$ a Banach space. Then*

(i) The closed unit ball $\{x \in X | \ \|x\|_X \leq 1\}$ is weakly compact if and only if X is reflexive.

(ii) If X is reflexive, then X is also weakly complete.

Remark 2.3. The theorem is [12, Th. II.1.3, pg. 32]. □

Theorem 2.5. *Let $(X, \|\cdot\|)$ a reflexive Banach space and consider a bounded sequence $\{u_k\} \subseteq X$. Then there exists a subsequence u_{k_h} and $u \in X$ such that $u_{k_h} \rightharpoonup u$. In other words, bounded sequences in a reflexive Banach space are weakly precompact.*

Theorem 2.6. *Let $(X, \|\cdot\|)$ a reflexive Banach space. Suppose that $x_n \overset{*}{\rightharpoonup} x$ in X^*. Then $x_n \rightharpoonup x$ in X.*

2.4 Compact operators on Hilbert spaces

Definition 2.12. Let X_1, X_2 be two Banach space, and let T be an operator $T : X_1 \to X_2$ linear and bounded. We say that T is compact if for every $\{x_n\} \subseteq X_1$ bounded $\Longrightarrow$ $\{Tx_n\}$ has a subsequence that converges in X_2.

Lemma 2.2. *Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be an Hilbert space and consider $T \in \mathcal{L}(\mathcal{H})$. Then there exists a unique linear bounded operator $T^* \in \mathcal{L}(\mathcal{H})$ such that*

$$\langle x, Ty \rangle = \langle T^* x, y \rangle \qquad \forall \ x, y \in \mathcal{H}$$

We say that T^* is the adjoint of T.

Lemma 2.3. *If $T \in \mathcal{L}(\mathcal{H})$ is compact, then also T^* is compact.*

Theorem 2.7. *Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ an Hilbert space. Let $B : \mathcal{H} \to \mathcal{H}$ a bounded, compact and self-adjoint operator. Then there exists a sequence $\{\lambda_k\}_{k \in \mathbb{N}}$ such that $\lambda_k \neq 0$ for every $k \in \mathbb{N}$ and*

$$|\lambda_{k+1}| \leq |\lambda_k| \qquad \forall \ k \in \mathbb{N}, \qquad \lim_{k \to +\infty} \lambda_k = 0$$

Associated to this sequence, there exists a complete orthonormal basis, $\{\varphi_k\}_{k \in \mathbb{N}} \subseteq \mathcal{H}$, such that

$$B\varphi_k = \lambda_k \varphi_k \qquad \forall \ k \in \mathbb{N}$$

[1] i.e., x_k is weak Cauchy if the following property holds, for all $l \in X^*$: given $\varepsilon > 0$, there is $\overline{n} = \overline{n}(l, \varepsilon) \in \mathbb{N}$ such that

$$|l(x_k - x_{k'})| < \varepsilon \qquad \forall k, k' \geq \overline{n}$$

Proof. The proof is provided in [23, Th. IV.16, pg. 203]. ∎

Remark 2.4. Thanks to the self-adjointness of the operator we have that the eigenvalues are real. If B is furthermore a positive operator, in the sense of $\langle B\varphi, \varphi\rangle > 0 \quad \forall \varphi \neq 0$, we have $\lambda_k > 0$ for every $k \in \mathbb{N}$. In fact, for $\varphi_k \neq 0$,

$$\lambda_k = \frac{\langle \lambda_k \varphi_k, \varphi_k\rangle}{\|\varphi_k\|^2} = \frac{\langle B\varphi_k, \varphi_k\rangle}{\|\varphi_k\|^2} > 0$$

that is, the eigenvalues are positive. □

Lemma 2.4. *Let $(H, \langle\cdot,\cdot\rangle)$ be an Hilbert space, with norm $\|\cdot\| = \langle\cdot,\cdot\rangle^{\frac{1}{2}}$. Let S a closed, positive, symmetric bilinear form with dense domain $D = D(S) \subseteq H$, equipped with the norm $\left(\|\cdot\|^2 + S(\cdot,\cdot)\right)^{\frac{1}{2}}$. Then there exists a uniquely determinded positive symmetric operator $B : D(B) \to H$ with dense domain $D(B) \subseteq D$, that satisfies*

$$\begin{cases} D(B) = \{u \in D : \; S(u, v), \forall\, v \in D, \; \textit{is continuous in } \|\cdot\|\} \\ S(u, v) = \langle Bu, v\rangle \quad \forall u \in D(B), \; v \in D \end{cases} \tag{2.4}$$

Remark 2.5. As underlined in [27, pg. 94], a proof is provided in [14, VI, Theorem 2.6] or [31, Satz 5.37]. The proof rests on the Riesz representation theorem. □

2.5 A fixed point theorem

The following theorem will help us in future chapters to find a solution to a coupled system.

Theorem 2.8 (Schauder fixed-point theorem)**.** *Let X be a Banach space and $M \subseteq X$ a closed, bounded and convex subset of X. Let $T : M \to M$ be a completely continuous operator. Then T has a fixed point in M.*

This theorem is a corollary of a well known fixed point theorem.

Theorem 2.9 (Fixed point theorem)**.** *If M is a convex, compact subset of a Banach space X, and $T : M \to M$ is continuous, then T has a fixed point in M.*

Remark 2.6. Statements and proofs of these theorems are provided in [15, pg. 10]. □

Chapter 3

L^p spaces and kernels

3.1 Lebesgue spaces

Consider the measure space $(\mathbb{R}^3, \mathcal{B}(\mathbb{R}^3), \mu)$, where $\mathcal{B}(\mathbb{R}^3)$ is the Borelian σ-algebra in $\mathbb{R}^3$ and $d\mu \equiv dx$ is the Lebesgue measure in the 3D space.

Definition 3.1. For every $\Omega \subseteq \mathbb{R}^3$ measurable set and $p \in [1, \infty)$ we define

$$L^p \equiv L^p(\Omega) := \{f|\ f \text{ is a } \mathcal{B}(\mathbb{R}^n)\text{-}\textit{measurable} \text{ function and } \int_\Omega |f(x)|^p dx < +\infty\} \tag{3.1}$$

It is well known that L^p is a vector space. Morover, it is a *normed space* with the norm

$$\|f\|_p := \left(\int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}} \tag{3.2}$$

With this norm, the space is complete, so it is a *Banach space*.

We can also consider the limit case $p = \infty$. We have

$$L^\infty \equiv L^\infty(\Omega) := \{f : \Omega \to \mathbb{R}|\ f \text{ is a } \mathcal{B}(\Omega)\text{-}\textit{measurable} \text{ function and } \sup_\Omega |f| < \infty\} \tag{3.3}$$

where $\sup_\Omega |f|$ is the *essential supremum* of $|f|$. Also $L^\infty(\Omega)$ is a *Banach space*, with norm

$$\|f\|_\infty := \sup_\Omega |f| \tag{3.4}$$

We claim now two important inequalities, that are *Hölder's generalized inequality* and the *interpolation inequality.*

Lemma 3.1. *Let Ω be a domain. Let p, q, r be such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$ and let be $f \in L^p(\Omega)$, $g \in L^q(\Omega)$ and $h \in L^r(\Omega)$. Then $fgh \in L^1(\Omega)$ and*

$$\|fgh\|_1 \leq \|f\|_p \|g\|_q \|h\|_r \tag{3.5}$$

Lemma 3.2. *Let Ω be a domain. Let $1 \le q \le \gamma \le r \le \infty$ and $\alpha \in [0,1]$ be such that $\dfrac{1}{\gamma} = \dfrac{\alpha}{q} + \dfrac{1-\alpha}{r}$. Let $f \in L^q(\Omega) \cap L^r(\Omega)$. Then $f \in L^\gamma(\Omega)$ and*

$$\|f\|_\gamma \le \|f\|_q^\alpha \|f\|_r^{1-\alpha} \tag{3.6}$$

Lemma 3.3. *Let I be an interval in $\mathbb{R}$. Let $f_n, f \in L^p(I)$. If $f_n \to f$ in $L^p(I)$, then there exists a subsequence f_{n_k} such that*

(i) $f_{n_k}(t) \to f(t)$ *a.e. in* I*;*

(ii) Exists $h \in L^p(I)$ such that $|f_{n_k}(t)| \le h(t)$ for every $k \in \mathbb{N}$ and for almost every $t \in I$.

Now we state a theorem about differentiation under integral sign for convolutions.

Theorem 3.1. *Let Ω a domain in $\mathbb{R}^n$. Let $g(x, \cdot) \in C^1(\Omega)$ and $0 \le G$ a measurable function such that*

$$|\nabla_x g(x,y)| \le G(y) \qquad \forall\, x, y \in \Omega$$

Let $f : \Omega \to \mathbb{R}$ a measurable function, and suppose that

$$F_x(y) := f(y) g(x,y) \in L^1(\Omega) \quad \forall\, x \in \Omega \quad \textit{and also} \quad |f(y)| G(y) \in L^1(\Omega)$$

Then

$$\phi(x) := \int_\Omega f(y) g(x,y)\, dy \in C^1(\Omega)$$

3.1.1 The vectorial case

If $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ we want to define the L^p for vectorial functions, say $L^p(\Omega)^n$. If $|\cdot|$ is the Euclidean norm, we define

$$\|u\|_p := \left(\int_\Omega |u|^p dx \right)^{\frac{1}{p}}$$

Another possible choice is to define the norm as

$$\|u\|'_p := \left(\sum_{i=1}^n \|u_i\|_p^p \right)^{\frac{1}{p}}$$

However the two norms are equivalent. In fact, if we look at the second one, we have

$$\|u\|'_p = \left(\sum_{i=1}^n \int_\Omega |u_i|^p \right)^{\frac{1}{p}} = \left(\int_\Omega \sum_{i=1}^n |u_i|^p \right)^{\frac{1}{p}} = \left(\int_\Omega |u|_p^p \right)^{\frac{1}{p}} \tag{3.7}$$

Since the norms $|\cdot|$ and $|\cdot|_p$ are equivalent in $\mathbb{R}^n$, we have that also the L^p norms are equivalent. A similar argument holds for the matrices.

3.2 Convolutions and mollifications

Definition 3.2. Let $u : \Omega \to \mathbb{R}$, with Ω a bounded domain, a function locally integrable and let $\Omega_{\varepsilon_0} := \{x \in \Omega | \operatorname{dist}(x, \partial\Omega) > \varepsilon_0\}$. Let $\eta \in C_c^\infty(\Omega)$ with compact support in $B(0, \varepsilon_0)$. Then, we define the *convolution*

$$(u * \eta)(x) := \int_{B(0,\varepsilon_0)} u(x-y)\eta(y)dy, \quad \forall x \in \Omega_{\varepsilon_0}$$

Theorem 3.2. *Let Ω a bounded domain. Let $u : \Omega \to \mathbb{R}$ a function such that $u \in L^2(\Omega)$ and $u' \in L^\infty(\Omega)$. Let $\eta \in C_c^\infty(\Omega)$, with compact support in $B(0, \varepsilon_0)$. Then, the convolution $u * \eta$ has the derivative $D(u * \eta) = Du * \eta$ for every $x_0 \in \Omega_{\varepsilon_0}$.*

Remark 3.1. If we prove the theorem in an interval $I \subseteq \mathbb{R}$, then the theorem holds for the partial derivatives. So, in particular, it holds in $\mathbb{R}^n$.

Proof. Consider $I \subseteq \mathbb{R}$. Let $x_0 \in I_{\varepsilon_0}$. By definition, we have

$$\lim_{h\to 0} \frac{\big(u*\eta\big)(x_0+h) - \big(u*\eta\big)(x_0)}{h} = \lim_{h\to 0} \int_{-\varepsilon_0}^{\varepsilon_0} \frac{u(x_0+h-y) - u(x_0-y)}{h}\eta(y)dy$$

Since $u \in W^{1,2}(I)$, by theorem [3, Th. VIII, pg. 122], we have that exists $\tilde{u} \in C(\overline{I})$ such that $u = \tilde{u}$ almost everywhere in I, and

$$\tilde{u}(z) - \tilde{u}(w) = \int_w^z u'(t)dt, \qquad \forall z, w \in \overline{I} \tag{3.8}$$

Since the convolution does not change if u changes on a zero measure set, and if h is small enough to have $x_0 + h - y, x_0 - y \in \overline{I}$, we have

$$\tilde{u}(x_0+h-y) - \tilde{u}(x_0-y) = \int_{x_0-y}^{x_0+h-y} u'(t)dt$$

so that

$$\frac{|\tilde{u}(x_0+h-y) - \tilde{u}(x_0-y)|}{h} \le \frac{1}{h}\int_{x_0-y}^{x_0+h-y} |u'(t)|dt \le \frac{1}{h}\int_{x_0-y}^{x_0+h-y} \|u'\|_{\infty,I}dt = \|u'\|_{\infty,I} \tag{3.9}$$

Since $\|u'\|_{\infty,I}$ is a constant and $\eta \in L^1((-\varepsilon_0, \varepsilon_0))$, we have that the incremental ratio has an integrable bound. It follow that

$$\lim_{h\to 0} \frac{\big(u*\eta\big)(x_0+h) - \big(u*\eta\big)(x_0)}{h} = \lim_{h\to 0} \frac{\big(\tilde{u}*\eta\big)(x_0+h) - \big(\tilde{u}*\eta\big)(x_0)}{h} =$$

$$= \int_{-\varepsilon_0}^{\varepsilon_0} \tilde{u}'(x_0-y)\eta(y)dy = \int_{-\varepsilon_0}^{\varepsilon_0} u'(x_0-y)\eta(y)dy = (u' * \eta)(x_0)$$

Remark 3.2. Since it holds (3.8), we have, throught the fundamental theorem of Lebesgue integral calculus, that $\tilde{u}$ has derivative $\tilde{u}'$ almost everywhere and that $\tilde{u}' = u'$ almost everywhere. □

So we have the thesis. ∎

3.2.1 Mollifiers

Consider an open set $\Omega \subset \mathbb{R}^n$. We define, for $\varepsilon > 0$,

$$\Omega_\varepsilon := \{x \in \Omega | \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$$

Moreover we define the function $\eta \in C^\infty(\mathbb{R}^n)$ by

$$\eta(x) := \begin{cases} C \exp\left(\frac{1}{|x|^2 - 1}\right) & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

The constant $C > 0$ is selected so that $\displaystyle\int_{\mathbb{R}^3} \eta(x)\ dx = 1$. Then, for every $\varepsilon > 0$, we set

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right)$$

The functions η_ε are C^∞ and satisfy

$$\int_{\mathbb{R}^n} \eta_\varepsilon\ dx = 1, \quad \operatorname{supp}(\eta_\varepsilon) \subset B(0, \varepsilon)$$

Definition 3.3. If $u : \Omega \to \mathbb{R}$ is locally integrable, we define its *mollification*

$$u^\varepsilon(x) := \int_\Omega \eta_\varepsilon(x - y) u(y)\ dy = \int_{B(0,\varepsilon)} \eta_\varepsilon(y) u(x - y)\ dy, \qquad \forall\ x \in \Omega_\varepsilon$$

We have the following theorem.

Theorem 3.3. *Let Ω be a domain. Let $\varepsilon > 0$ and consider Ω_ε. Consider u^ε the mollification of $u \in L^1_{loc}(\Omega)$. The the following properties hold.*

(i) $u^\varepsilon \in C^\infty(\Omega_\varepsilon)$;

(ii) $u^\varepsilon \to u$ almost everywhere as $\varepsilon \to 0$;

(iii) If $f \in C(\Omega)$, then $u^\varepsilon \to u$ uniformly on compact subsets of Ω;

(iv) If $1 \leq p < \infty$ and $u \in L^p_{loc}(\Omega)$, then $u^\varepsilon \to u$ in $L^p_{loc}(\Omega)$.

(v) If $1 \leq p < \infty$, $u \in L^p_{loc}(\Omega)$, Ω is bounded and V, W are open set such that $V \subset\subset W \subset\subset \Omega$, then

$$\|u^\varepsilon\|_{L^p(V)} \leq \|u\|_{L^p(W)}$$

Proof. We only prove the latter point, since it allows us to remark an aspect of the convolution. The theorem is [10, Th. 7, pg. 714].

Consider, as above, $V \subset\subset W \subset\subset \Omega$. Observe, in particular, that the closures of these three sets are compact set. In particular, the distance of V from the boundary

∂W is finite and positive. If d is this distance, we define $\varepsilon_0 := \frac{d}{2}$. Obviously, ε_0 only depends on the three sets. So, let $x \in V$, and consider that

$$|u^\varepsilon(x)| = |\int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)u(y)\ dy| \leq \int_{B(x,\varepsilon)} \eta_\varepsilon^{1-\frac{1}{p}}(x-y)\eta_\varepsilon^{\frac{1}{p}}(x-y)|u(y)|\ dy \leq$$

$$\leq \left(\int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)\ dy\right)^{1-\frac{1}{p}} \left(\int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)|u(y)|^p dy\right)^{\frac{1}{p}} = \left(\int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)|u(y)|^p dy\right)^{\frac{1}{p}}$$

since $\int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)\ dy = 1$. So, we have that

$$\|u^\varepsilon\|^p_{L^p(V)} = \int_V |u^\varepsilon(x)|^p dx \leq \int_V \left(\int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)|u(y)|^p dy\right)\ dx \leq$$

and, if $\varepsilon < \varepsilon_0$, we have that $B(x,\varepsilon) \subset W$, since $x \in V$ and the distance $d > \varepsilon_0$,

$$\leq \int_V \left(\int_W \eta_\varepsilon(x-y)|u(y)|^p dy\right)\ dx = \int_W |u(y)|^p \left(\int_V \eta_\varepsilon(x-y)dx\right)dy \leq$$

$$\leq \int_W |u(y)|^p \left(\int_{\mathbb{R}^n} \eta_\varepsilon(x-y)dx\right)dy = \int_W |u(y)|^p dy = \|u\|^p_{L^p(W)}$$

where the latter equality holds since

$$\int_{\mathbb{R}^n} \eta_\varepsilon(x-y)dx = \int_{B(y,\varepsilon)} \eta_\varepsilon(x-y)dx = 1$$

being $\eta_\varepsilon(x-y) = 0$ if $|x-y| \geq \varepsilon$. This is the thesis.

Remark 3.3. We remark that we only required $\varepsilon < \varepsilon_0$, and ε_0 is independent of u, but only depends on the domains. □ ∎

3.3 Approximation results

We list some density results about L^p spaces, that can be found in [24, Th. 3.14, pg. 69].

Theorem 3.4. *Let Ω be a domain. For $1 \leq p < \infty$, the set $C_c(\Omega)$ is dense in $L^p(\Omega)$.*

Corollary 3.1. *Let Ω be a domain. Then, for $1 \leq p < \infty$, the set $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$.*

3.4 L^p spaces as functional spaces

Theorem 3.5. *Let $f_n, f \in L^p(\Omega)$. Suppose that*

$$\lim_{n\to\infty} \|f_n\|_p = \|f\|_p, \qquad f_n \rightharpoonup f \quad \text{in } L^p(\Omega)$$

Then $f_n \to f$ in $L^p(\Omega)$.

Remark 3.4. It is an application to the L^p spaces of the theorem [3, Proposition III.30, pg. 52]. □

Theorem 3.6. *Let $f_n \in L^p(\Omega)$ and $f \in L^p(\Omega)$ such that, for every $\varphi \in C_c^\infty(\Omega)$,*

$$\lim_{n\to\infty} \int_\Omega f_n(x)\varphi(x)\ dx = \int_\Omega f(x)\varphi(x)\ dx$$

Suppose moreover that exists $C > 0$ such that $\|f_n\|_p \leq C$ for every $n \in \mathbb{N}$. Then $f_n \rightharpoonup f$, that is

$$\lim_{n\to\infty} \int_\Omega f_n(x)g(x)\ dx = \int_\Omega f(x)g(x)\ dx \qquad \forall\ g \in L^q(\Omega)$$

Proof. Let $g \in L^q(\Omega)$ and $\varepsilon > 0$. Then we fix $\varphi_\varepsilon \in C_c^\infty(\Omega)$ such that $\|g - \varphi_\varepsilon\|_q < \varepsilon$. It follows that

$$\left| \int_\Omega (f_n - f)g\ dx \right| = \left| \int_\Omega (f_n - f)\varphi_\varepsilon\ dx + \int_\Omega (f_n - f)(g - \varphi_\varepsilon)\ dx \right| \leq$$

$$\leq \left| \int_\Omega (f_n - f)\varphi_\varepsilon\ dx \right| + \|f_n - f\|_p \|g - \varphi_\varepsilon\|_q \leq \left| \int_\Omega (f_n - f)\varphi_\varepsilon\ dx \right| + \Big(C + \|f\|_p \Big)\varepsilon$$

that is small for n large enough, using the hypothesis on the test functions. ∎

Theorem 3.7. *Let $f_n \in L^p(\Omega)$ a sequence of function in $L^p(\Omega)$ such that $\sup_n \|f_n\|_p < \infty$. Then, for every $\varepsilon > 0$ exists $M_\varepsilon > 0$ such that*

$$\sup_{n\in\mathbb{N}} \left\{ \int_{\{x\in\Omega:\ |f_n(x)|>M_\varepsilon\}} |f_n(x)|\ dx \right\} < \varepsilon$$

Proof. Let $M \in (0, \infty)$. Then

$$\chi_{\{x\in\Omega:\ |f_n(x)|>M\}}(x)|f_n(x)|M^{p-1} \leq |f_n(x)|^p$$

for every $x \in \Omega$. Integrating the expression, we have that

$$M^{p-1} \int_{\{x\in\Omega:\ |f_n(x)|>M\}} |f_n(x)|\, dx \leq \|f_n\|_p^p \implies \int_{\{x\in\Omega:\ |f_n(x)|>M\}} |f_n(x)|\, dx \leq M^{1-p} \Big(\sup_n \|f_n\|_p \Big)^p$$

and the thesis follows since $p > 1$. ∎

3.5 Convergence in measure

Definition 3.4. Let $(\Omega, \mathcal{M}, \mu)$ a finite measure space. Let f_n, f measurable functions over Ω. Then we say that $f_n \to f$ *in measure* if and only if, for every $\varepsilon > 0$

$$\lim_{n\to\infty} \mu(\{x \in \Omega :\ |f_n(x) - f(x)| \geq \varepsilon\}) = 0$$

We have the following properties of convergence in measure.

Proposition 3.1. *Let $(\Omega, \mathcal{M}, \mu)$ a measure space, with $\mu(\Omega) < \infty$. Let f_n, f measurable functions over Ω. Then the following properties hold.*

(i) Let $\varepsilon > 0$. If for every $\delta > 0$ there exists $N_\delta \in \mathbb{N}$ such that

$$\mu(\{x \in \Omega : \ |f_n(x) - f_m(x)| \geq \varepsilon\}) \leq \delta \qquad \forall\, n, m \geq N_\delta \tag{3.10}$$

then exists a measurable function f such that $f_n \to f$ in measure.

Conversely, if $f_n \to f$ in measure, then equation (3.10) holds.

(ii) If $f_n \to f$ almost everywhere in Ω, then $f_n \to f$ in measure.

(iii) Suppose that $f_n \to f$ in $L^p(\Omega)$. Then $f_n \to f$ in measure.

(iv) If $f_n \to f$ in measure, then there exists a subsequence n_k such that $f_{n_k}(x) \to f(x)$ for almost every $x \in \Omega$.

(v) On the other hand, if $f_n \to f$ in measure and exists $g \in L^p(\Omega)$ such that $|f_n| \leq g$, then $f_n \to f$ in $L^p(\Omega)$.

(vi) If $f_n \to f$ in measure and β is a continuous function over $\mathbb{R}$, then $\beta(f_n) \to \beta(f)$ in measure.

(vii) Let f_n a sequence of measurable functions, such that for every β_k piecewise differentiable such that

$$\beta_k(t) := \begin{cases} \beta_k(t) = 0 & |t| \leq \frac{1}{k} \\ \beta_k'(t) > 0 & |t| > \frac{1}{k} \\ \beta_k, \ \beta_k' \text{ are bounded} \end{cases}$$

exists v_k measurable function such that

$$\beta_k(f_n) \to v_k \qquad \text{in measure as } n \to \infty$$

If moreover $f_n \in L^p(\Omega)$, with $\sup\limits_{n \in \mathbb{N}} \|f_n\|_{L^p(\Omega)} < \infty$, it follows that exists f measurable function such that

$$f_n \to f \qquad \text{in measure as } n \to \infty$$

Proof. We only prove the point *(vii)*. We consider the family of functions

$$\beta_k(t) := \begin{cases} \beta_k(t) = 0 & |t| \leq \frac{1}{k} \\ \beta_k(t) = 1 & |t| \geq \frac{1}{k} \\ \beta_k'(t) = 1 & t \in A_k \end{cases}$$

where $A_k := [-k, -\frac{1}{k}] \cup [\frac{1}{k}, k]$ and β_k is continuous. The function β_k defined above is piecewise differentiable. This is less than the hypothesis required above; however it is not a problem. We prove the proposition in this case. It is clear that

$$\mathbb{R} = (-\infty, -k] \cup A_k \cup [-\frac{1}{k}, \frac{1}{k}] \cup [k, \infty)$$

Using point *(i)*, we have that $\beta_k(f_n)$ is a Cauchy sequence in measure. So, if we consider the set

$$E_m^n(\varepsilon) := \{x \in \Omega : \ |f_n(x) - f_m(x)| > \varepsilon\}$$

it can be decomposed in a finite number of subsets, depending on the subset $[-\frac{1}{k}, \frac{1}{k}]$, A_k or $[-\infty, k] \cup [k, \infty]$ where $u_n(x)$ and $u_m(x)$ live. We also define

$$B_k := [-\frac{1}{k}, \frac{1}{k}], \qquad C_k := (-\infty, -k] \cup [k, \infty)$$

In particular, we define now the set

$$E_{m,D'}^{n,D}(\varepsilon) := \{x \in \Omega : \ |f_n(x) - f_m(x)| > \varepsilon, \ f_n(x) \in D, \ f_m(x) \in D'\}$$

where $D, D' \in \{A_k, B_k, C_k\}$. It follows that

$$E_m^n(\varepsilon) = \bigcup_{D,D' \in \{A_k, B_k, C_k\}} E_{m,D'}^{n,D}(\varepsilon)$$

We now show that, as n, m are large enough, and $k \in \mathbb{N}$ is large but fixed, the set $E_{m,D'}^{n,D}$ has small measure.

(i) We first consider $E_{m,A_k}^{n,B_k}(\varepsilon)$. In this set we have

$$\varepsilon < |f_n(x) - f_m(x)| \leq |f_n(x) - \beta_k(f_n(x))| + |\beta_k(f_n(x)) - \beta_k(f_m(x))| + |\beta_k(f_m(x)) - f_m(x)|$$

and so it follows that[1]

$$E_{m,A_k}^{n,B_k}(\varepsilon) \subseteq \{|f_n(x) - \beta_k(f_n(x))| \geq \frac{\varepsilon}{3}\} \cup \{|\beta_k(f_n(x)) - \beta_k(f_m(x))| \geq \frac{\varepsilon}{3}\} \cup \{|\beta_k(f_m(x)) - f_m(x)| \geq \frac{\varepsilon}{3}\}$$

The first set is empty if k is large enough, since $|f_n(x)| \leq \frac{1}{k}$. The second set has small measure, thanks to the convergence in measure. The latter set is empty, since $\beta_k(f_m(x)) = f_m(x)$ for $f_m(x) \in A_k$. Clearly $E_{m,B_k}^{n,A_k}(\varepsilon)$ can be studied in the same way.

(ii) On the other hand we can consider the set

$$E_{m,A_k}^{n,A_k} := \{x \in \Omega : \ |f_n(x) - f_m(x)| > \varepsilon, \ f_n(x), \ f_m(x) \in A_k\} \subseteq$$

$$\subseteq \{x \in \Omega : \ |\beta_k(f_n(x)) - \beta_k(f_m(x))| > \varepsilon\}$$

and so the measure is small.

[1] If $a, b, c \geq 0$ and $a + b + c > \varepsilon$, then a element in the set $\{a, b, c\}$ is $> \varepsilon$. In fact, if every element is $\leq \frac{\varepsilon}{3}$, we have a contradiction.

(iii) Moreover in every set of the form $E^{n,D}_{m,D'}(\varepsilon)$, with $D = C_k$ or $D' = C_k$, we have, if in example $D = C_k$,

$$E^{n,D}_{m,D'}(\varepsilon) \subseteq \{x \in \Omega : \ |f_n(x)| > k\}$$

and so

$$|E^{n,D}_{m,D'}(\varepsilon)| \leq |\{x \in \Omega : \ |f_n(x)| > k\}| \leq \frac{1}{k^p}\left(\sup_{n\in\mathbb{N}} \|f_n\|_p\right)$$

So the measure is small if k is large enough.

(iv) Finally, the term

$$E^{n,B_k}_{m,B_k} := \{x \in \Omega : \ |f_n(x) - f_m(x)| > \varepsilon, \ |f_n(x)|, \ |f_m(x)| < \frac{1}{k}\}$$

has small measure if k is large enough and n goes to infinity. In fact

$$\varepsilon < |f_n(x)-f_m(x)| \leq |f_n(x)-\beta_k(f_n(x))|+|\beta_k(f_n(x))-\beta_k(f_m(x))|+|\beta_k(f_m(x))-f_m(x)|$$

and so

$$E^{n,B_k}_{m,B_k} \subseteq \{|f_n(x)-\beta_k(f_n(x))| \geq \frac{\varepsilon}{3}\}\cup\{|\beta_k(f_n(x))-\beta_k(f_m(x))| \geq \frac{\varepsilon}{3}\}\cup\{|\beta_k(f_m(x))-f_m(x)| \geq \frac{\varepsilon}{3}\}$$

The first and the latter term are empty if k is large enough. The measure of the second term vanishes if $n \to \infty$, thanks to the hypothesis.

So we have that f_n is Cauchy convergent in measure. Then, thanks to point *(i)*, there exists a measurable function f such that $f_n \to f$ in measure. ∎

3.6 Lebesgue's Differentiation Theorem

Theorem 3.8. *Let Ω be an open subset of $\mathbb{R}^n$. Suppose that $f \in L^1_{loc}(\Omega)$. Then, for almost every $x_0 \in \Omega$ it holds*

$$\lim_{r\to 0} \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} f(x) \ dx = f(x_0) \tag{3.11}$$

A point x_0 at which (3.11) *holds is called a Lebesgue point of f.*

Corollary 3.2. *Let I be an open interval of $\mathbb{R}$. Suppose that $f \in L^1_{loc}(I)$. Let $\psi^{t_0}_n \in C^\infty_c(I)$ be a sequence of test functions over I such that $\int_I \varphi^{t_0}_n(t) \ dt \equiv 1$ for every $n \in \mathbb{N}$ and $supp(\varphi^{x_0}_n) \subset (-\frac{1}{n} + t_0, t_0 + \frac{1}{n})$. Then, for almost every $t_0 \in I$,*

$$\lim_{n\to\infty} \int_I \varphi^{t_0}_n(t) f(t) \ dt = f(t_0) \tag{3.12}$$

3.7 Further integration theory: kernels

We start this further integration theory with a fundamental definition.

Definition 3.5. A *Calderón-Zygmund kernel* is a function $K \in L^1_{loc}(\mathbb{R}^n/\{0\})$ such that exists a constant $B \geq 0$ with the following properties:

(i) $|K(x)| \leq \dfrac{B}{|x|^n} \quad \forall\, x \neq 0;$

(ii) $\displaystyle\int_{\{x\in\mathbb{R}^n:|x|>2|y|\}} |K(x-y)-K(x)|\; dx \leq B \quad \forall\, y \neq 0;$

(iii) $\displaystyle\int_{r_1<|x|<r_2} K(x)\; dx = 0 \quad \forall\, r_1, r_2 > 0.$

Remark 3.5. Keep in mind that a compact subset C of $\mathbb{R}^n/\{0\}$ is far enough from the origin 0. In fact if for every $r > 0$ there is a point $x_0 \in B(0,r)$ that is a point of C, then we can build a sequence $x_n \in C$ such that $x_n \to 0$. But also every subsequence of x_n converges to 0, so we have find a sequence in C such that it hasn't a subsequence that converges in C. This contradicts the compactness of C. So there exists a $r > 0$ such that each point of $B(0,r)$ is not a point of C. So, if $K \in C(\mathbb{R}^n/\{0\})$ and C is a compact subset of $\mathbb{R}^n/\{0\}$, we have that K, and $|K|$, are continuous on C. So, $|K|$ is summable on C. This means that $K \in L^1_{loc}(\mathbb{R}^n/\{0\})$. □

Remark 3.6. Except for the second condition, the other requests seem natural. This second condition is called *Hormander* condition. The following lemma gives us a more operational formulation. □

Lemma 3.4. *Suppose that exists $B \geq 0$ such that*

$$|K(x)| \leq \frac{B}{|x|^n} \quad \forall\, x \neq 0, \qquad \int_{r_1<|x|<r_2} K(x)dx = 0 \quad \forall\, r_1, r_2 > 0 \tag{3.13}$$

Suppose moreover that $K \in C^1(\mathbb{R}^n/\{0\})$ and it holds

$$|\nabla K(x)| \leq \frac{B}{|x|^{n+1}} \quad \forall\, x \neq 0 \tag{3.14}$$

Then it holds also the Hormander condition.

Proof. Since $\partial_t K(x-ty) = -\nabla K(x-ty)\cdot y$, by the *Fundamental Calculus theorem* we have

$$K(x) - K(x-y) = -\int_0^1 \nabla K(x-ty)\cdot y\; dt$$

Now if $2|y| < |x|$ we have[2]

$$|K(x) - K(x-y)| \leq B|x-ty|^{-n-1}|y| \leq \frac{2^{n+1}B}{|x|^{n+1}}|y|$$

[2]Observe that the previous expression makes sense if $x - ty \neq 0$. But with the last assumption, if $x = ty$ for a $t \in [0,1]$,

$$|x| = t|y| \leq |y| < \frac{|x|}{2}$$

that is an absurd.

where has been used that

$$\frac{|x|}{2} < |x| - |y| \leq |x| - t|y| \leq ||x| - t|y|| \leq |x - ty|$$

Now we integrate and

$$\int_{\{x:\ |x|>2|y|\}} |K(x) - K(x-y)|\ dx \leq 2^{n+1}B|y| \int_{\{x:\ |x|>2|y|\}} |x|^{-n-1}dx$$

Writing the latter integral in polar coordinates we have

$$\int_{\{x:\ |x|>2|y|\}} |x|^{-n-1}dx = \frac{C}{2|y|}$$

Substituting B with a costant B' (and eventually taking the maximum between B and B' to have the same constant in the two inequalities), we have the thesis. ∎

The following proposition will be very useful in a moment.

Proposition 3.2. *Let $f : \mathbb{R}^n/\{0\} \to \mathbb{R}$ a function in $C^1(\mathbb{R}^n/\{0\})$. Suppose that f is homogeneous of exponent α. Then*

(i) $\partial_{x_i} f(x)$ is homogeneous of exponent $\alpha - 1$ for every $i \in \{1, ..., n\}$;

(ii) $|f(x)| \leq C|x|^\alpha \quad \forall\ x \neq 0$, where $C = \max\limits_{|x|=1} |f(x)|$.

Proof. By definition, we have

$$\partial_{x_i} f(tx) = \lim_{h\to 0} \frac{f(tx + h\hat{e}_i) - f(tx)}{h} = t^\alpha \lim_{h\to 0} \frac{f(x + \frac{h}{t}\hat{e}_i) - f(x)}{h} = t^{\alpha-1}\partial_{x_i} f(x)$$

On the other side $|f(x)| = \left|f\left(\frac{x}{|x|}|x|\right)\right| = |x|^\alpha \left|f\left(\frac{x}{|x|}\right)\right| \leq |x|^\alpha \max\limits_{|x|=1} |f|$. ∎

Remark 3.7. If f is sufficiently regular in $\mathbb{R}^n/\{0\}$, then $|f(x)| \leq C_0|x|^\alpha$ and moreover, being $|\partial_{x_i} f(x)| \leq C_i|x|^{\alpha-1}$, we have

$$|\nabla f(x)| = \left(\sum_{i=1}^n |\partial_{x_i} f(x)|^2\right)^{\frac{1}{2}} \leq \sqrt{n}\ \overline{C}|x|^{\alpha-1}$$

where $\overline{C} := \max_i C_i$. □

The following lemma paves the way to the hypothesis of the Calderón-Zygmund theorem.

Lemma 3.5. *Let K be a Caldeòn-Zygmund kernel and let $f \in C_0^\infty(\mathbb{R}^n)$. Then the function*

$$\phi(x) := \lim_{\varepsilon\to 0} \int_{|x-y|\geq\varepsilon} K(x-y)f(y)dy \tag{3.15}$$

is defined for every $x \in \mathbb{R}^n$.

Proof. Let B the constant provided by the definition of Calderón-Zygmund kernel. With a change of coordinates (namely a translation), we have

$$\int_{|x-y|\geq\varepsilon} K(x-y)f(y)dy = \int_{|y|\geq\varepsilon} K(y)f(x-y)dy =$$

$$= \int_{1\geq|y|\geq\varepsilon} K(y)f(x-y)dy + \int_{|y|\geq 1} K(y)f(x-y)dy - \int_{1\geq|y|\geq\varepsilon} K(y)f(x)dy$$

where $\varepsilon < 1$ and it has been used a property of a Calderón-Zygmynd kernel. We can rewrite this expression as

$$\int_{|x-y|\geq\varepsilon} K(x-y)f(y)dy = \int_{1\geq|y|\geq\varepsilon} K(y)[f(x-y)-f(x)]dy \ + \int_{|y|\geq 1} K(y)f(x-y)dy$$

The first integral can be estimated as follows:

$$|K(y)[f(x-y)-f(x)]| \leq B\|\nabla f\|_\infty |y|^{1-n}$$

using the Lagrange theorem[3] and the estimate for K. So

$$\int_{1\geq|y|\geq\varepsilon} |K(y)[f(x-y)-f(x)]|dy \leq B\|\nabla f\|_\infty \int_{1\geq|y|\geq\varepsilon} \frac{1}{|y|^{n-1}} \ dy$$

But the function $|y|^{1-n}$ is integrable near the origin (since $n-1<n$) and so the limit

$$\lim_{\varepsilon\to 0} \frac{1}{|y|^{n-1}} \ dy$$

exists. So also the limit

$$\lim_{\varepsilon\to 0} \int_{1\geq|y|\geq\varepsilon} |K(y)[f(x-y)-f(x)]|dy$$

exists. This imply the existence of the limit of the integral of positive and negative part of the integrand, and so of the limite of the whole function, that is the difference of the two limites mentioned above.
So, being

$$\int_{|x-y|\geq\varepsilon} K(x-y)f(y)dy = \int_{1\geq|y|\geq\varepsilon} K(y)[f(x-y)-f(x)]dy \ + \int_{|y|\geq 1} K(y)f(x-y)dy$$

we have that the left side is equal to the sum of two integrals with existing limit. It follows that also exists the limit of the left side. ∎

[3]We have

$$|f(x+h)-f(x)| = |\int_0^1 \frac{d}{dt} f(x+th)dt| = |\int_0^1 \nabla f(x+th)\cdot h \ dt| \leq \|\nabla f\|_\infty |h|$$

and we choose $h=-y$.

Remark 3.8. If the kernel is such that the integral

$$\int_{\mathbb{R}^3} K(x-y)f(y)dy$$

is well defined, i.e. the product $|K(x-y)||f(y)|$ is summable over $\mathbb{R}^3$ for every x, we have in particular that, for a fixed x,

$$u(x) := \int_{\mathbb{R}^3} K(x-y)f(y)dy = \lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon} K(x-y)f(y)dy$$

using the Lebesgue dominated convergence theorem. The Calderón-Zygmund theorem will provide us information about the integrability of the function u. In particular, as we will see in a moment, if $|K(x-y)||f(y)| \in L^1(\mathbb{R}^3)$ for every $x \in \mathbb{R}^3$ and $f \in L^2(\mathbb{R}^3)$, we will have

$$\|u\|_2 \leq C\|f\|_2$$

i.e. also $u \in L^2$. □

3.8 The Calderón-Zygmund theorem

Theorem 3.9. *Let $K \in L^1_{loc}(\mathbb{R}^n/\{0\})$ a Calderón-Zygmund kernel. Let $p \in (1,\infty)$ and consider the operator*

$$T_\varepsilon(f) := \int_{|y|\geq\varepsilon} f(x-y)K(y)\ dy$$

for $f \in L^p(\mathbb{R}^n)$. Then there exists a constant C_p depending only on B, n, p, and independent of ε, such that

$$\|T_\varepsilon(f)\| \leq C_p\|f\|_p \tag{3.16}$$

Moreover, for every $f \in L^p(\mathbb{R}^n)$ exists the strong limit

$$T_K(f) := \lim_{\varepsilon\to 0} T_\varepsilon(f) \qquad in\ \ L^p(\mathbb{R}^n)$$

The operator T_K is bounded in $L^p(\mathbb{R}^n)$ and obeys to the same bound (3.16).

The rest of this section is committed to prove this Calderón-Zygmund theorem.

3.8.1 The Calderón-Zygmund decomposition

Definition 3.6. In the following claims, a *cube Q with sides parallel to the axes* in $\mathbb{R}^n$ will be the closed cube

$$Q := \{x \in \mathbb{R}^n :\ x_i \in [a_i, b_i],\ |a_i - b_i| \equiv l \quad \forall\ i = 1, ..., n\}$$

with measure $|Q| = l^n$. By $\overset{\circ}{Q}$ we will mean the *interior* of the cube Q.

A *daughter* of the cube Q is a cube Q' of side $\frac{l}{2}$ obtained dividing Q into 2^n sub-cubes, i.e. dividing $I_i \equiv [a_i, b_i] = \left[a_i, a_i + \frac{l}{2}\right] \cup \left[a_i + \frac{l}{2}, b_i\right] \equiv I_i^1 \cup I_i^2$,

$$Q'_{(k_1,\dots,k_n)} := \left\{x \in \mathbb{R}^n : \; x_i \in I_i^{k_i}, \quad k_i \in \{1,2\}, \quad \forall\, i = 1, \dots, n\right\}$$

The cube Q is said to be the *father* of all the cubes $Q'_{(k_1,\dots,k_n)}$.

Theorem 3.10. *Let $f \in L^1(\mathbb{R}^n)$ and $\alpha > 0$. Then there exists a countable collection of cubes with sides parallel to the axes, say $\{Q_j\}_{j\in\mathbb{N}}$, with disjoint interiors such that*

$$\alpha < \frac{1}{|Q_j|}\int_{Q_j} |f(x)|\, dx \le 2^n \alpha$$

Moreover we can decompose the function f. In particular, consider

$$\Omega := \bigcup_{j\in\mathbb{N}} Q_j, \qquad F := \mathbb{R}^n/\Omega$$

Then, we have the following:

(i) $|\Omega| \le \alpha^{-1}\|f\|_{L^1(\mathbb{R}^n)}$;

(ii) $\exists\, E \subseteq \mathbb{R}^n$, $|E| = 0$ *such that* $|f(x)| \le \alpha$ *for every* $x \in F/E$;

(iii) There exists two functions g and b such that

$$f(x) = g(x) + b(x)$$

such that $|g(x)| \le 2^n\alpha$ almost everywhere and for every $p \in [1, \infty]$

$$\|g\|_{L^p(\mathbb{R}^n)} \le \alpha^{\frac{p-1}{p}}(1 + 2^{np})^{\frac{1}{p}}\|f\|_{L^1(\mathbb{R}^n)}^{\frac{1}{p}}$$

and $b(x) = 0$ for all $x \in F$. Finally

$$\int_{Q_j} b(x)\, dx = 0 \qquad \forall\, j \in \mathbb{N}, \qquad \|b\|_{L^1(\mathbb{R}^n)} \le (1 + 2^n)\|f\|_{L^1(\mathbb{R}^n)} \tag{3.17}$$

Proof. First of all, we can write $\mathbb{R}^n$ as the union of a countable collection of cubes with disjoint interiors such that

$$\|f\|_{L^1(\mathbb{R}^n)} \le \alpha|Q|$$

for every cube Q of the family. Each cube can be divided into 2^n daughters, with sides parallel to the axes. For each daughter Q' of a cube Q, we can compute the number $\dfrac{1}{|Q'|}\displaystyle\int_{Q'} |f(x)|\, dx$. At this point we ask if this number is larger or smaller that α. We have two possibilities.

- If $\frac{1}{|Q'|}\int_{Q'} |f(x)|\, dx \le \alpha$, we consider the daughters of the cube Q', now considered as a father;
- If $\frac{1}{|Q'|}\int_{Q'} |f(x)|\, dx > \alpha$ then we retain to this cube a special role.

This "sketched" algorithm allows us to construct a countable sequence of cubes.

Let $\{Q_j\}_j$ a first countable family of cubes, with the property

$$\|f\|_{L^1(\mathbb{R}^n)} \le \alpha |Q_j| \tag{3.18}$$

Dividing each cube Q_j we obtain a new countable sequence of cubes. In particular, to each father cube Q_j they are related

$$Q_j \longrightarrow \begin{cases} Q_j^{(1)} \\ \dots \\ Q_j^{(k)} \\ \dots \\ Q_j^{(2^n)} \end{cases}$$

The relation with the father is important in view of the next step. If now

$$\frac{1}{|Q_j^{(k)}|}\int_{Q_j^{(k)}} |f(x)|\, dx \le \alpha$$

we subdivide $Q_j^{(k)}$ into 2^n daughters. On the orher hand. if

$$\frac{1}{|Q_j^{(k)}|}\int_{Q_j^{(k)}} |f(x)|\, dx > \alpha$$

we retain the cube as one of the $\{C_h\}_h$.

Remark 3.9. Observe that

$$\|f\|_{L^1(\mathbb{R}^n)} \le \alpha |Q_j| \implies \frac{1}{|Q_j|}\int_{Q_j} |f(x)|\, dx \le \alpha$$

So the original cubes Q_j have to be divided into daughters. □

So, we have described a first pass of the construction. Starting from the original cubes Q_j we have obtained some daughters that have been retained in the $\{C_h\}_h$, and some daughters that have been subdivided into other sub-cubes. At the end of this step we have a countable number of cubes retained and a countable number of cubes to which we will apply again this process. Since the number of steps is "scanned" by the separation of the cubes (that is a countable process), and each steps produces a countable number of cubes to retain, we have that the family $\{C_h\}_h$ of set to retain is countable, since countable union of countable sets.

Define now $\Omega := \bigcup_{h\in\mathbb{N}} C_h$. Observe that the interior $\overset{\circ}{C_h}$ are disjoint by construction, since we have started with disjoint interiors and the construction mantained the property. In particular, we have that, if P_h is the father of C_h, then $|P_h| = 2^n|C_h|$ and

$$\frac{1}{|P_h|}\int_{P_h} |f(x)|\ dx \leq \alpha$$

since otherwise we would have already stopped the process at P_h. So

$$\alpha < \frac{1}{|C_h|}\int_{C_h} |f(x)|\ dx \leq 2^n\alpha \tag{3.19}$$

being $\frac{1}{2^n|C_h|}\int_{C_h} |f(x)|\ dx \leq \frac{1}{|P_h|}\int_{P_h} |f(x)|\ dx \leq \alpha$. Furthermore observe that

$$|\Omega| \leq \sum_{h\in\mathbb{N}} |C_h| \leq \sum_{h\in\mathbb{N}} \alpha^{-1}\int_{C_h} |f(x)|\, dx = \alpha^{-1}\left(\sum_{h\in\mathbb{N}}\int_{\overset{\circ}{C_h}} |f(x)|\, dx\right) = \alpha^{-1}\int_{\bigcup_{h\in\mathbb{N}}\overset{\circ}{C_h}} |f(x)|\, dx \leq$$

$$\leq \alpha^{-1}\|f\|_{L^1(\mathbb{R}^n)} \tag{3.20}$$

Now, by the Lebesgue differentiation theorem we have that exists $E \subseteq \mathbb{R}^n$, $|E| = 0$ such that

$$f(x) = \lim_{Q\to x}\frac{1}{|Q|}\int_Q f(y)\ dy \qquad \forall\ x \in \mathbb{R}^n/E$$

where Q is a family of cubes that contain x and $Q \to x$ means that their diamters converge to zero.

Let now $x \in F/E$. We have that $x \notin \Omega$, so that $x \notin C_h$ for every $h \in \mathbb{N}$. So, there exists a subsequence of cubes Q'_m containing x whose diameters converge to zero and which are not elements of the family $\{C_h\}_h$, thanks to what we have just said. So, in particular,

$$\frac{1}{|Q'_m|}\int_{Q'_m} |f(x)|\ dx \leq \alpha$$

It follows that $|f(x)| \leq \alpha$. We now define

$$g(x) := \begin{cases} f(x) & x \in F \\ \dfrac{1}{|C_h|}\displaystyle\int_{C_h} f(x)\ dx & x \in \overset{\circ}{C_h} \end{cases}$$

This defines g almost everywhere. Moreover, using that $|f(x)| \leq \alpha$ over F, equation (3.19) and that $2^n \geq 1$, we have

$$|g(x)| \leq 2^n\alpha \quad \text{almost everywhere}$$

Moreover, we have that

$$\int_F |g(x)|^p\ dx = \int_F |f(x)|^p\ dx \leq \alpha^{p-1}\|f\|_{L^1(\mathbb{R}^n)}$$

and, using (3.20),

$$\int_\Omega |g(x)|^p\ dx \le 2^{np}\alpha^p|\Omega| \le \alpha^{p-1}2^{np}\|f\|_{L^1(\mathbb{R}^n)}$$

We now consider the function $b(x) := f(x) - g(x)$, that is defined almost everywhere and, by definition, it vanishes on F. Moreover

$$\int_{C_h} b(x)\ dx = \int_{C_h} (f(x) - g(x))\ \ dx = 0$$

Finally, being $|b| \le |f| + |g|$, we have

$$\int_\Omega |b(x)|\ dx \le \|f\|_{L^1(\mathbb{R}^n)} + \|g\|_{L^1(\Omega)} \le \|f\|_{L^1(\mathbb{R}^n)} + 2^n\alpha|\Omega| \le \|f\|_{L^1(\mathbb{R}^n)}(1 + 2^n)$$

and this completes the proof. ∎

3.8.2 The Marcinkiewicz Interpolation

Definition 3.7. An real operator T over a vector space V is said to be *sublinear* if

(i) $T(\gamma v) = \gamma T(v)$ for every $\gamma > 0$;

(ii) $T(v + w) \le T(v) + T(w)$ for every $v, w \in V$.

Definition 3.8. A sublinear operator is said to be weak type (p, q), with $q < \infty$, if for every $f \in L^p(\mathbb{R}^n)$

$$|\{x \in \mathbb{R}^n|\ |Tf(x)| > \alpha\}| \le \left(\frac{C\|f\|_{L^p(\mathbb{R}^n)}}{\alpha}\right)^q \tag{3.21}$$

for every $\alpha > 0$, with C independent of α and f.

Remark 3.10. If $\|Tf\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}$ holds, then by the Chebyshev's inequality we have that it is weak type (p, q). □

Remark 3.11. If $p \in (p_1, p_2)$, then

$$L^p(\mathbb{R}^n) \subset L^{p_1}(\mathbb{R}^n) + L^{p_2}(\mathbb{R}^n)$$

In fact, let $f \in L^p(\mathbb{R}^n)$ and choose $\gamma > 0$. Define

$$f_\gamma(x) := \begin{cases} f(x) & |f(x)| > \gamma \\ 0 & |f(x)| \le \gamma \end{cases}$$

and

$$f^\gamma(x) := \begin{cases} 0 & |f(x)| > \gamma \\ f(x) & |f(x)| \le \gamma \end{cases}$$

Obviously $f = f_\gamma + f^\gamma$. Moreover

$$\|f_\gamma\|_{L^{p_1}(\mathbb{R}^n)}^{p_1} \le \gamma^{p_1-p}\|f\|_{L^p(\mathbb{R}^n)}^p, \qquad \|f^\gamma\|_{L^{p_2}(\mathbb{R}^n)}^{p_2} \le \gamma^{p_2-p}\|f\|_{L^p(\mathbb{R}^n)}^p$$

In fact, observe that

$$\|f_\gamma\|_{L^{p_1}(\mathbb{R}^n)}^{p_1} = \int_{\{x\in\mathbb{R}^n:\ |f(x)|>\gamma\}} |f(x)|^{p_1}\ dx = \int_{\{x\in\mathbb{R}^n:\ |f(x)|>\gamma\}} |f(x)|^{p_1-p}|f(x)|^p\ dx =$$

$$= \int_{\{x\in\mathbb{R}^n:\ |f(x)|>\gamma\}} |f(x)|^p \frac{1}{|f(x)|^{p-p_1}}\ dx \le \gamma^{p_1-p}\int_{\mathbb{R}^n} |f(x)|^p\ dx$$

and analogously we have

$$\|f^\gamma\|_{L^{p_2}(\mathbb{R}^n)}^{p_2} = \int_{\{x\in\mathbb{R}^n:\ |f(x)|\le\gamma\}} |f(x)|^{p_2}\ dx = \int_{\{x\in\mathbb{R}^n:\ |f(x)|\le\gamma\}} |f(x)|^p|f(x)|^{p_2-p}\ dx \le \gamma^{p_2-p}\|f\|_{L^p(\mathbb{R}^n)}^p$$

and these are the desired estimates. □

Theorem 3.11. *Let $r \in (1,\infty]$. Assume that T is subadditive and weak type $(1,1)$ and weak type (r,r). Then, for every $p \in (1,r)$, there exists a constant C_p such that*

$$\|Tf\|_{L^p(\mathbb{R}^n)} \le C_p\|f\|_{L^p(\mathbb{R}^n)} \tag{3.22}$$

Proof. Fist consider $r < \infty$. We take $\alpha > 0$ and denote

$$\lambda(\alpha) := |\{x \in \mathbb{R}^n :\ |Tf(x)| > \alpha\}|$$

Decompose now $f = f_\alpha + f^\alpha$. Observe moreover that

$$\{x \in \mathbb{R}^n :\ |Tf(x)| > \alpha\} \subseteq \{x \in \mathbb{R}^n :\ |Tf_\alpha(x)| > \frac{\alpha}{2}\} \cup \{x \in \mathbb{R}^n :\ |Tf^\alpha(x)| > \frac{\alpha}{2}\}$$

since $|T(f)| = |T(f_\alpha + f^\alpha)| \le |T(f_\alpha)| + |T(f^\alpha)|$ by sublinearity. It follows that

$$\lambda(\alpha) \le |\{x \in \mathbb{R}^n :\ |Tf_\alpha(x)| > \frac{\alpha}{2}\}| + |\{x \in \mathbb{R}^n :\ |Tf^\alpha(x)| > \frac{\alpha}{2}\}| \le$$

$$\le \frac{C_1\|f_\alpha\|_{L^1(\mathbb{R}^n)}}{\alpha} + \left(\frac{C_2\|f^\alpha\|_{L^r(\mathbb{R}^n)}}{\alpha}\right)^r = \frac{C_1}{\alpha}\int_{\mathbb{R}^n} |f_\alpha(x)|\ dx + \frac{C_2^r}{\alpha^r}\int_{\mathbb{R}^n} |f^\alpha(x)|^r\ dx =$$

$$= \frac{C_1}{\alpha}\int_{\{x\in\mathbb{R}^n:\ |f(x)|>\alpha\}} |f(x)|\ dx + \frac{C_2^r}{\alpha^r}\int_{\{x\in\mathbb{R}^n:\ |f(x)|\le\alpha\}} |f(x)|^r\ dx \tag{3.23}$$

using the hypothesis on T as a weak operator. Now, we multiply by $p\alpha^{p-1}$ and integrate for $\alpha \in (0,\infty)$. The first term becomes

$$\int_0^\infty \alpha^{p-1}\alpha^{-1}\left(\int_{|f|>\alpha} |f|\, dx\right) d\alpha = \int_{\mathbb{R}^n} |f(x)|\left(\int_0^{|f(x)|} \alpha^{p-2}\, d\alpha\right) dx = \frac{1}{p-1}\int_{\mathbb{R}^n} |f(x)|^p\, dx$$

and the second

$$\int_0^\infty \alpha^{p-1}\alpha^{-r}\left(\int_{|f|\le\alpha} |f(x)|^r\, dx\right) d\alpha = \int_{\mathbb{R}^n} |f(x)|^r\left(\int_{|f(x)|}^\infty \alpha^{p-1-r} d\alpha\right) dx = \frac{1}{r-p}\int_{\mathbb{R}^n} |f(x)|^p\, dx$$

where Fubini's theorem can be employed since the second iterated integral is finite, being $f \in L^p(\mathbb{R}^n)$. Moreover

$$\int_0^\infty \alpha^{p-1}|\{x \in \mathbb{R}^n : \ |Tf(x)| > \alpha\}| d\alpha = \int_0^\infty \alpha^{p-1} \left(\int_{\{x\in\mathbb{R}^n: \ |Tf(x)|>\alpha\}} dx \right) d\alpha =$$

and so, if $f \in L^p(\mathbb{R}^n)$, we have, looking at the inequality (3.23) that the latter integral is finite. So, using Fubini's Theorem we have that the integrals can be interchanged, and thus, continuing the chain of equality,

$$= \int_{\mathbb{R}^n} \left(\int_0^{|Tf(x)|} \alpha^{p-1} \ d\alpha \right) dx = \frac{1}{p} \int_{\mathbb{R}^n} |Tf(x)|^p \ dx = \frac{1}{p} \|Tf\|^p_{L^p(\mathbb{R}^n)}$$

It follows that $Tf \in L^p(\mathbb{R}^n)$. So, we have the thesis in this case. If $r = \infty$ the proof can be adapted. ∎

3.8.3 Fourier transform

Definition 3.9. Let $u \in L^1(\mathbb{R}^n)$. We define the *Fourier transform* of u as

$$\hat{u}(y) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix\cdot y} u(x) \ dx$$

On the other hand, the *inverse Fourier transform* of u is

$$\check{u}(y) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\cdot y} u(x) \ dx$$

Remark 3.12. Since $|e^{\pm ix\cdot y}| = 1$ and $u \in L^1(\mathbb{R}^n)$, then the integrals converge.

The following theorems list some important theorems.

Theorem 3.12. *Let $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then $\hat{u}$, $\check{u} \in L^2(\mathbb{R}^n)$ and*

$$\|\hat{u}\|_{L^2(\mathbb{R}^n)} = \|\check{u}\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)} \tag{3.24}$$

Definition 3.10. Fourier transform in $L^2(\mathbb{R}^n)$. Let $u \in L^2(\mathbb{R}^n)$. Then there exists $u_k \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ such that

$$\lim_{k\to\infty} \|u_k - u\|_{L^2(\mathbb{R}^n)} = 0$$

Using equation (3.24), we have that

$$\|\hat{u}_k - \hat{u}_j\|_{L^2(\mathbb{R}^n)} = \|\widehat{u_k - u_j}\|_{L^2(\mathbb{R}^n)} = \|u_k - u_j\|_{L^2(\mathbb{R}^n)}$$

so that $\hat{u}_k$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$. We define $\hat{u}$ has its limit in $L^2(\mathbb{R}^n)$. Similarly we can define $\check{u}$.

Remark 3.13. If $u \in L^2(\mathbb{R}^n)$, we have that $\hat{u}$ is defined and

$$\|u\|_{L^2(\mathbb{R}^n)} = \lim_{k\to\infty} \|u_k\|_{L^2(\mathbb{R}^n)} = \lim_{k\to\infty} \|\hat{u}_k\|_{L^2(\mathbb{R}^n)} = \|\hat{u}\|_{L^2(\mathbb{R}^n)}$$

where $u_k \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. So, the equality (3.24) holds also in $L^2(\mathbb{R}^n)$. □

Theorem 3.13. *Assume $u, v \in L^2(\mathbb{R}^n)$. Then the following properties hold:*

- $\displaystyle\int_{\mathbb{R}^n} u\overline{v}\; dx = \int_{\mathbb{R}^n} \hat{u}\overline{\hat{v}}\; dx$;
- $\hat{D^\alpha u} = (iy)^\alpha \hat{u}$ *for each multiindex α such that $D^\alpha u \in L^2(\mathbb{R}^n)$;*
- $\widehat{(u * v)} = (2\pi)^{\frac{n}{2}} \hat{u}\hat{v}$;
- $u = \check{\hat{u}}$.

The following theorem let us finally introduce a generalization of Sobolev spaces.

Theorem 3.14. *Let $k \geq 0$ be an integer. Then a function $u \in L^2(\mathbb{R}^n)$ belongs to $H^k(\mathbb{R}^n)$ if and only if*

$$(1 + |y|^k)\hat{u} \in L^2(\mathbb{R}^n)$$

In addition, there exists a positive constant C such that

$$\frac{1}{C}\|u\|_{H^k(\mathbb{R}^n)} \leq \|(1 + |y|^k)\hat{u}\|_{L^2(\mathbb{R}^n)} \leq C\|u\|_{H^k(\mathbb{R}^n)}$$

for each $u \in H^k(\mathbb{R}^n)$.

3.8.4 Properties of singular kernels

We now list some properties of the Calderón-Zygmund kernels introduced at the beginning of section 3.7.

Definition 3.11. Let K be a Calderón-Zygmund kernel. For every $\varepsilon > 0$ we define

$$(C_\varepsilon(K))\,(x) := \begin{cases} K(x) & |x| \geq \varepsilon \\ 0 & |x| < \varepsilon \end{cases}$$

Moreover, we consider

$$(\tau_\varepsilon K)(x) := \varepsilon^{-n} K\left(\frac{x}{\varepsilon}\right)$$

and for $f \in L^p(\mathbb{R}^n)$ we set $(\delta_\varepsilon f)(x) := f(\varepsilon x)$.

Definition 3.12. Moreover, we define the convolution operator

$$T_K(f)(x) = \int_{\mathbb{R}^n} K(x - y)f(y)\; dy$$

Proposition 3.3. *The following properties hold.*

(i) *If K is a Calderón-Zygmund kernel, then also $\tau_\varepsilon K$ is a Calderón-Zygmund kernel with the same constant B, for every $\varepsilon > 0$;*

(ii) *For any $\varepsilon > 0$*

$$C_\varepsilon(K) = \tau_\varepsilon(C_1(\tau_{\frac{1}{\varepsilon}}(K)))$$

(iii) *If K is a Calderón-Zygmund kernel, then $C_1(K)$ is also a Calderón-Zygmund kernel, with a constant $B_1 > 0$ depending on B and the dimension of the space only;*

(iv) *Suppose that f, K and ε are such that the image $T_{\tau_\varepsilon K} f$ has sense. Then*

$$\left(\delta_{\frac{1}{\varepsilon}} T_K \delta_\varepsilon\right) f = T_{\tau_\varepsilon K} f$$

(v) *If $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a bounded linear operator, then the family $\delta_{\frac{1}{\varepsilon}} T \delta_\varepsilon$ satisfies the uniform bound*

$$\sup_{\varepsilon>0} \|\delta_{\frac{1}{\varepsilon}} T \delta_\varepsilon\|_{\mathcal{L}(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n))} \leq C$$

(vi) *It holds $\mathcal{F}\left(C_\varepsilon(K)\right)(\xi) = \mathcal{F}\left(C_1(\tau_{\frac{1}{\varepsilon}} K)\right)(\varepsilon\xi)$, where $\mathcal{F}g$ is the Fourier transform of g.*

Remark 3.14. Since $C_\varepsilon(K) \equiv \tau_\varepsilon(C_1(\tau_{\frac{1}{\varepsilon}}(K)))$, we have, thanks to *(i)* that $\tau_{\frac{1}{\varepsilon}} K$ is a Calderón-Zygmund kernel, so that, thanks to *(iii)*, $C_1(\tau_{\frac{1}{\varepsilon}} K)$ is a Calderón-Zygmund kernel. Finally, using again *(i)*, we have that $C_\varepsilon(K)$ is a Calderón-Zygmund kernel. □

Proof. We first prove that $\tau_\varepsilon K$ is a Calderón-Zygmund kernel with constant B. In fact,

$$|(\tau_\varepsilon K)(x)| := |\varepsilon^{-n} K\left(\frac{x}{\varepsilon}\right)| \leq \varepsilon^{-n} B \left|\frac{x}{\varepsilon}\right|^{-n} = B|x|^{-n}$$

Moreover

$$\int_{|x|>2|y|} |(\tau_\varepsilon K)(x-y) - (\tau_\varepsilon K)(x)| \; dx = \varepsilon^{-n} \int_{|x|>2|y|} \left| K\left(\frac{x-y}{\varepsilon}\right) - K\left(\frac{x}{\varepsilon}\right)\right| \; dx =$$

$$\overset{z=\frac{x}{\varepsilon}}{=} \varepsilon^{-n} \int_{|z|>\frac{2|y|}{\varepsilon}} \left| K\left(z - \frac{y}{\varepsilon}\right) - K(z)\right| \; \varepsilon^n \; dz \leq B$$

Finally

$$\int_{r_1<|x|<r_2} (\tau_\varepsilon K)(x) \; dx = \varepsilon^{-n} \int_{r_1<|x|<r_2} K\left(\frac{x}{\varepsilon}\right) \; dx \overset{z=\frac{x}{\varepsilon}}{=} \varepsilon^{-n} \int_{\frac{r_1}{\varepsilon}<|z|<\frac{r_2}{\varepsilon}} K(z) \; \varepsilon^n \; dz = 0$$

We now compute, in order to prove the second point, the composition $\tau_\varepsilon(C_1(\tau_{\frac{1}{\varepsilon}}(K)))$. First of all

$$(\tau_{\frac{1}{\varepsilon}} K)(x) = \varepsilon^n K(\varepsilon x)$$

So, it follows that

$$\left(C_1\left(\tau_{\frac{1}{\varepsilon}} K\right)\right)(x) = \begin{cases} \varepsilon^n K(\varepsilon x) & |x| \geq 1 \\ 0 & |x| < 1 \end{cases}$$

Finally

$$\left(\tau_\varepsilon\left(C_1\left(\tau_{\frac{1}{\varepsilon}} K\right)\right)\right)(x) = \begin{cases} K(x) & |x| \geq \varepsilon \\ 0 & |x| < \varepsilon \end{cases} =: \left(C_\varepsilon(K)\right)(x)$$

We now prove the third point. Consider a Calderón-Zygmund kernel K. By definition, for every $x \neq 0$,

$$|C_1(K)(x)| := \begin{cases} |K(x)| & |x| \geq 1 \\ 0 & |x| < 1 \end{cases} \leq B|x|^{-n}$$

Fix now $\lambda > 1$ and $\rho > 0$. Then

$$\int_{\rho \leq |x| \leq \lambda\rho} |K(x)| \, dx \leq B \int_{\rho \leq |x| \leq \lambda\rho} |x|^{-n} \, dx = \omega_n B \log \lambda \tag{3.25}$$

where ω_n is the area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. Using this estimate, we have

$$\int_{|x|>2|y|,\ |x-y|>1,\ |x|<1} |K(x-y)| \, dx \leq \omega_n B \log \frac{3}{2}$$

choosing $\rho = 1$ and $\lambda = \frac{3}{2}$, since $1 < |x-y| \leq |x| + \frac{|x|}{2} < \frac{3}{2}$. On the other hand

$$\int_{|x|>2|y|,\ |x-y|<1,\ |x|>1} |K(x)| \, dx \leq \omega_n B \log 2$$

since we can choose $\rho = 1$ and $\lambda = 2$, being $1 < |x| \leq |x-y| + |y| < 1 + \frac{|x|}{2}$, that is $|x| < 2$. So, the constant B_1 can be choosen as $B_1 := (1 + \omega_n \log 3)B$.

We now prove the fourth point. Remember that

$$(\delta_\varepsilon f)(x) := f(\varepsilon x)$$

and thus we have

$$T_K(\delta_\varepsilon f)(x) := \int_{\mathbb{R}^n} K(x-y) f(\varepsilon y) \, dy$$

Then it follows that

$$\delta_{\frac{1}{\varepsilon}} \left(T_K(\delta_\varepsilon f)\right)(x) = \int_{\mathbb{R}^n} K\left(\frac{x}{\varepsilon} - y\right) f(\varepsilon y) \, dy = \int_{\mathbb{R}^n} K\left(\frac{1}{\varepsilon}(x - \varepsilon y)\right) f(\varepsilon y) \, dy \overset{z=\varepsilon y}{=}$$

$$= \int_{\mathbb{R}^n} K\left(\frac{x-z}{\varepsilon}\right) f(z) \frac{dz}{\varepsilon^n} = \left(T_{\tau_\varepsilon K} f\right)(x)$$

Let now $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ be a bounded linear operator. In particular

$$\|T\|_{L^p(\mathbb{R}^n), L^p(\mathbb{R}^n)} := \sup_{0 \neq f \in L^p(\mathbb{R}^n)} \frac{\|Tf\|_{L^p(\mathbb{R}^n)}}{\|f\|_{L^p(\mathbb{R}^n)}} \equiv C < \infty$$

Consider now the operator

$$f(x) \xrightarrow{\delta_\varepsilon} f(\varepsilon x) \xrightarrow{T} T(f(\varepsilon \cdot)) = g(\cdot) \xrightarrow{\delta_{\frac{1}{\varepsilon}}} g(\frac{x}{\varepsilon})$$

So, we have

$$\| \left(\delta_{\frac{1}{\varepsilon}} T \delta_\varepsilon\right) f\|_{L^p(\mathbb{R}^n)} = \|\delta_{\frac{1}{\varepsilon}} g\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |g\left(\frac{x}{\varepsilon}\right)|^p \, dx\right)^{\frac{1}{p}} \overset{y=\frac{x}{\varepsilon}}{=} \left(\int_{\mathbb{R}^n} |g(y)|^p \varepsilon^n \, dy\right)^{\frac{1}{p}} =$$

$$= \varepsilon^{\frac{n}{p}} \|T(\delta_\varepsilon f)\|_{L^p(\mathbb{R}^n)} \leq \varepsilon^{\frac{n}{p}} C \|\delta_\varepsilon f\|_{L^p(\mathbb{R}^n)} = \varepsilon^{\frac{n}{p}} C \bigg(\int_{\mathbb{R}^n} |f(\varepsilon x)|^p \, dx \bigg)^{\frac{1}{p}} =$$

$$\overset{y=\varepsilon x}{=} \varepsilon^{\frac{n}{p}} C \bigg(\int_{\mathbb{R}^n} |f(y)|^p \varepsilon^{-n} \, dy \bigg)^{\frac{1}{p}} = C \|f\|_{L^p(\mathbb{R}^n)}$$

that is the conclusion of this point.

For the following final point, remember that

$$\mathcal{F}f(\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) \, dx$$

Since

$$(C_\varepsilon(K))(x) := \begin{cases} K(x) & |x| \geq \varepsilon \\ 0 & |x| < \varepsilon \end{cases}$$

So, by definition ,we have

$$\mathcal{F}\left(C_\varepsilon(K)\right)(\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} (C_\varepsilon(K))(x) \, dx = \int_{|x|\geq\varepsilon} e^{-ix\cdot\xi} K(x) \, dx \overset{y=\frac{x}{\varepsilon}}{=} \varepsilon^n \int_{|y|\geq 1} e^{-i\varepsilon y\cdot\xi} K(\varepsilon y) \, dy =$$

$$= \int_{|y|\geq 1} e^{-iy\cdot(\varepsilon\xi)} (\tau_{\frac{1}{\varepsilon}} K)(y) \, dy = \int_{\mathbb{R}^n} e^{-iy\cdot(\varepsilon\xi)} C_1(\tau_{\frac{1}{\varepsilon}})(y) \, dy = \mathcal{F}(C_1(\tau_{\frac{1}{\varepsilon}}))(\varepsilon\xi)$$

and this concludes the proof of the proposition. ∎

Concerning the Fourier transform, we also have this lemma.

Lemma 3.6. *Let K be a Calderón-Zygmund kernel with constant B. Then, there exists a constant γ depending only on dimension of space so that*

$$\sup_{\varepsilon>0} \sup_{\xi\in\mathbb{R}^n} |\mathcal{F}C_\varepsilon K(\xi)| \leq \gamma B \tag{3.26}$$

Proof. Let K be a Calderón-Zygmund kernel with constant B. Then, define

$$K_1 := C_1(K)$$

and remember that, for every $\varepsilon > 0$, $\tau_{\frac{1}{\varepsilon}} K$ is a Calderón-Zygmund kernel with constant B. Then, by proposition 3.3, we have that

$$\mathcal{F}(C_\varepsilon(K))(\xi) = \mathcal{F}\left(C_1(\tau_{\frac{1}{\varepsilon}} K))\right)(\varepsilon\xi)$$

and so it is enought to prove that $|\hat{K}_1(\xi)| \leq \gamma B$, if K is a Calderón-Zygmund kernel with constant B (and we apply this to $K \longleftrightarrow \tau_{\frac{1}{\varepsilon}} K$).

We remark first of all that $C_1(K)(x) = \chi_{|x|\geq 1}(x) K(x)$, with $|K(x)| \leq \frac{B}{|x|^n}$ for every $x \neq 0$. It follows that

$$|C_1(K)(x)| \leq \chi_{|x|\geq 1}(x) \frac{B}{|x|^{2n}}$$

that is in $L^1(\mathbb{R}^n)$, since $2n > n$, being $n \in \mathbb{N}$. By definition of Fourier transform we have

$$\hat{K}_1(\xi) := \lim_{R\to\infty} \int_{|x|\le R} e^{-ix\cdot\xi} K_1(x)\ dx$$

Fix $0 \neq \xi \in \mathbb{R}^n$, and consider $R > \frac{3\pi}{|\xi|}$. Then we can write

$$\int_{|x|\le R} e^{-ix\cdot\xi} K_1(x)\ dx = \int_{|x|\le \frac{2\pi}{|\xi|}} e^{-ix\cdot\xi} K_1(x)\ dx + \int_{\frac{2\pi}{|\xi|}\le|x|\le R} e^{-ix\cdot\xi} K_1(x)\ dx \equiv I_1(\xi) + I_2(\xi)$$

We now estimate I_1 and I_2. We first deal with I_1. Consider $r_1 < 1$. Then

$$\int_{|x|\le \frac{2\pi}{|\xi|}} K_1(x)\ dx = \int_{r_1\le|x|\le \frac{2\pi}{|\xi|}} K_1(x)\ dx = 0$$

and so

$$|I_1(\xi)| = \left| \int_{|x|\le \frac{2\pi}{|\xi|}} \left(e^{-ix\cdot\xi} - 1\right) K_1(x)\ dx \right| \le \int_{|x|\le \frac{2\pi}{|\xi|}} |x||\xi| B|x|^{-n}\ dx = 2\pi\omega_n B$$

We now deal with the other integral. Choose $\eta := \frac{\pi\xi}{|\xi|^2}$. Then it follows that $|\eta| = \frac{\pi}{|\xi|}$ and $e^{-i\xi\cdot\eta} = -1$. So, we can use the equality

$$I_2(\xi) = \int_{\frac{2\pi}{|\xi|}\le|x|\le R} e^{-ix\cdot\xi}\left(K_1(x) - K_1(x-\eta)\right)\ dx + \int_{\frac{2\pi}{|\xi|}\le|x|\le R} e^{-ix\cdot\xi} K_1(x-\eta)\ dx$$

Observe that

$$\int_{\frac{2\pi}{|\xi|}\le|x|\le R} e^{-ix\cdot\xi} K_1(x-\eta)\ dx = \int_{\frac{2\pi}{|\xi|}\le|x|\le R} e^{-i(x-\eta)\cdot\xi+\eta\cdot\xi} K_1(x-\eta)\ dx =$$

$$\overset{x'=x-\eta}{=} \int_{\frac{2\pi}{|\xi|}\le|x'+\eta|\le R} e^{-ix'\cdot\xi+\eta\cdot\xi} K_1(x')\ dx' = -\int_{\frac{2\pi}{|\xi|}\le|x'+\eta|\le R} e^{-ix'\cdot\xi} K_1(x')\ dx' =$$

$$= -\int_{\frac{2\pi}{|\xi|}\le|x'|\le R} e^{-ix'\cdot\xi} K_1(x')\ dx' + \left\{ \int_{\frac{2\pi}{|\xi|}\le|x'|\le R} e^{-ix'\cdot\xi} K_1(x')\ dx' - \int_{\frac{2\pi}{|\xi|}\le|x'+\eta|\le R} e^{-ix'\cdot\xi} K_1(x')\ dx' \right\} \equiv$$

$$\equiv -I_2(\xi) + E(\xi)$$

where

$$E(\xi) := \int_{\frac{2\pi}{|\xi|}\le|x|\le R} e^{-ix\cdot\xi} K_1(x)\ dx - \int_{\frac{2\pi}{|\xi|}\le|x+\eta|\le R} e^{-ix\cdot\xi} K_1(x)\ dx =$$

$$= \int_A e^{-ix\cdot\xi} K_1(x)\ dx - \int_B e^{-ix\cdot\xi} K_1(x)\ dx$$

where

$$A := \left\{ \frac{2\pi}{|\xi|} \le |x| \le R \right\} / \left\{ \frac{2\pi}{|\xi|} \le |x+\eta| \le R \right\}, \quad B := \left\{ \frac{2\pi}{|\xi|} \le |x+\eta| \le R \right\} / \left\{ \frac{2\pi}{|\xi|} \le |x| \le R \right\}$$

It follows that

$$2I_2(\xi) = \int_{\frac{2\pi}{|\xi|}\leq|x|\leq R} e^{-ix\cdot\xi}\left(K_1(x) - K_1(x-\eta)\right)\, dx + E(\xi)$$

Consider now $x \in A$. Then

$$|x+\eta| < \frac{2\pi}{|\xi|} \quad \text{or} \quad |x+\eta| > R$$

In the first situation we have $|x| < \frac{3\pi}{|\xi|}$, while in the second $|x| > R - \frac{\pi}{|\xi|}$. With these considerations, we have

$$A \subset \left\{\frac{2\pi}{|\xi|} \leq |x| \leq \frac{3\pi}{|\xi|}\right\} \cup \left\{R - \frac{\pi}{|\xi|} \leq |x| \leq R\right\}$$

Similarly, we have

$$B \subset \left\{\frac{\pi}{|\xi|} \leq |x| \leq \frac{2\pi}{|\xi|}\right\} \cup \left\{R \leq |x| \leq R + \frac{\pi}{|\xi|}\right\}$$

Thus, it follows that

$$|E(\xi)| \leq \int_A |K_1(x)|\, dx + \int_B |K_1(x)|\, dx \leq$$

$$\leq \int_{\frac{2\pi}{|\xi|}\leq|x|\leq\frac{3\pi}{|\xi|}} |K_1(x)|\, dx + \int_{\frac{\pi}{|\xi|}\leq|x|\leq\frac{2\pi}{|\xi|}} |K_1(x)|\, dx + \int_{R-\frac{\pi}{|\xi|}\leq|x|\leq R+\frac{\pi}{|\xi|}} |K_1(x)|\, dx$$

Using now (3.25), we have that the integral only depends on the ratio of the boundary values on the ring domain. So, we have that

$$\int_{\frac{2\pi}{|\xi|}\leq|x|\leq\frac{3\pi}{|\xi|}} |K_1(x)|\, dx \leq \omega_n B \log\frac{3}{2}, \qquad \int_{\frac{\pi}{|\xi|}\leq|x|\leq\frac{2\pi}{|\xi|}} |K_1(x)|\, dx \leq \omega_n B \log 2$$

and

$$\int_{R-\frac{\pi}{|\xi|}\leq|x|\leq R+\frac{\pi}{|\xi|}} |K_1(x)|\, dx \leq \omega_n B \log\left(\frac{R+\frac{\pi}{|\xi|}}{R-\frac{\pi}{|\xi|}}\right) \leq \omega_n B \log 2$$

since

$$\frac{R+\frac{\pi}{|\xi|}}{R-\frac{\pi}{|\xi|}} = \frac{1+\frac{\pi}{R|\xi|}}{1-\frac{\pi}{R|\xi|}} = f(\frac{\pi}{R|\xi|})$$

where $f(t) := \frac{1+t}{1-t}$ and

$$R > \frac{3\pi}{|\xi|} \Longrightarrow \frac{1}{3} > \frac{\pi}{R|\xi|} > 0$$

so that

$$\frac{1+\frac{\pi}{R|\xi|}}{1-\frac{\pi}{R|\xi|}} \leq 2 = \max_{t\in[0,\frac{1}{3}]} f(t)$$

So we have

$$|E(\xi)| \leq c\omega_n B$$

Moreover, observe that since $|x| \geq \frac{2\pi}{|\xi|} = 2|\eta|$, we have

$$\int_{\frac{2\pi}{|\xi|}\leq|x|\leq R} |K_1(x) - K_1(x-\eta)|\ dx \leq \int_{|x|\geq 2|\eta|} |K_1(x) - K_1(x-\eta)|\ dx \overset{\text{Prop. 3.3}}{\leq} (1+\omega_n \log 3)B$$

so that the proof is complete. ∎

3.8.5 The Calderón-Zygmund operator

In this conclusive section we prove two theorems concerning the Calderón-Zygmund operator.

Theorem 3.15. *Let K be a Calderón-Zygmund kernel with constant B. Then*

$$\sup_{\varepsilon>0} \sup_{\xi\in\mathbb{R}^n} |\hat{K}_\varepsilon(\xi)| \leq \gamma B \tag{3.27}$$

for some $\gamma > 0$, where $K_\varepsilon(y) := K(y)\chi_{|y|\geq\varepsilon}(y)$. Then, for each $p \in (1, \infty)$, the operator

$$T_\varepsilon(f)(x) := \int_{\mathbb{R}^n} K_\varepsilon(x-y) f(y)\ dy$$

is defined for $f \in L^p(\mathbb{R}^n)$. Moreover $T_\varepsilon(f) \in L^p(\mathbb{R}^n)$ and exists a constant $C_p = C_p(n,p,B)$ such that

$$\|T_\varepsilon\|_{\mathcal{L}(L^p(\mathbb{R}^n),L^p(\mathbb{R}^n))} \leq C_p$$

uniformly in $\varepsilon > 0$.

Proof. First of all, we underline that $T_\varepsilon(f) \equiv K_\varepsilon * f$. Moreover, since $f \in L^p(\mathbb{R}^n)$, the convolution is defined. In fact, if $q \in (1, \infty)$ is the conjugate of p, we have

$$\int_{|y|\geq\varepsilon} |f(x-y)||K(y)|\ dy = \int_{\mathbb{R}^n} |f(x-y)||K(y)|\chi_{|y|\geq\varepsilon}(y)\ dy \leq$$

$$\leq \left(\int_{\mathbb{R}^n} |f(x-y)|^p\ dy\right)^{\frac{1}{p}} \left(\int_{|y|\geq\varepsilon} |K(y)|^q\ dy\right)^{\frac{1}{q}} \leq \|f\|_p \left(\int_{|y|\geq\varepsilon} \frac{B^q}{|y|^{nq}}\ dy\right)^{\frac{1}{q}} < \infty$$

since $nq > n$. So, the definition of the operator is well-posed in every space $L^p(\mathbb{R}^n)$.

Remark 3.15. However, we want the integrability condition to be inherited by $T_\varepsilon(f)$. To do this, we have to proceed in steps. We first control the $L^2(\mathbb{R}^n)$ integrability, starting from a smaller space. □

Remark 3.16. We want to obtain a uniform bound, independent of ε. In order to do this, we have to deduce the estimate in the space $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. The definition can be extended by density. □

Definition over $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Let $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. By Young's inequality for convolutions, we have that

$$\|T_\varepsilon(f)\|_2 = \|K_\varepsilon * f\|_2 \leq \|K_\varepsilon\|_2 \|f\|_1 < \infty$$

So, if $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then $T_\varepsilon(f) \in L^2(\mathbb{R}^n)$. In this case, we have, moreover,

$$\|T_\varepsilon(f)\|_2 = \|\mathcal{F}(T_\varepsilon(f))\|_2 = (2\pi)^{\frac{n}{2}} \|\hat{K}_\varepsilon \hat{f}\|_2 \overset{(3.27)}{\leq} (2\pi)^{\frac{n}{2}} \gamma B \|\hat{f}\|_2 = \gamma B \|f\|_2 \tag{3.28}$$

where the bound (3.27) is assured by theorem 3.6. So, the operator is defined in $L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Since this subset of $L^2(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, we can extend the operator to the whole $L^2(\mathbb{R}^n)$: if $f \in L^2(\mathbb{R}^n)$, we can find $f_k \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ such that

$$\lim_{k \to \infty} \|f_k - f\|_2 = 0$$

So, since $T_\varepsilon(af + bg) = aT_\varepsilon(f) + bT_\varepsilon(g)$, we have

$$\|T_\varepsilon(f_m) - T_\varepsilon(f_h)\|_2 = \|T_\varepsilon(f_m - f_h)\|_2 \leq \gamma B \|f_m - f_h\|_2 \to 0$$

as $m, h \to \infty$. So the sequence $T_\varepsilon(f_m)$ converges to a function v in $L^2(\mathbb{R}^n)$. So, we define

$$T_\varepsilon(f) := \lim_{m \to \infty} T_\varepsilon(f_m)$$

where the limit has to be meant in $L^2(\mathbb{R}^n)$. On the other hand, if $f \in L^2(\mathbb{R}^n)$, we have already seen that it is defined

$$\int_{\mathbb{R}^n} K_\varepsilon(x-y) f(y)\, dy =: \Lambda f(x)$$

It is now clear that if $f_m \to f$ in $L^2(\mathbb{R}^n)$, we have

$$\left| \int_{\mathbb{R}^n} K_\varepsilon(x-y) f_m(y)\, dy - \int_{\mathbb{R}^n} K_\varepsilon(x-y) f(y)\, dy \right| \leq \int_{\mathbb{R}^n} |K_\varepsilon(x-y)| |f_m(y) - f(y)|\, dy \leq$$

$$\leq \left(\int_{\mathbb{R}^n} |K_\varepsilon(x-y)|^2\, dy \right)^{\frac{1}{2}} \|f_m - f\|_{L^2(\mathbb{R}^n)} \to 0 \quad \text{as } m \to \infty$$

This means that

$$\lim_{m \to \infty} \int_{\mathbb{R}^n} K_\varepsilon(x-y) f_m(y)\, dy = \int_{\mathbb{R}^n} K_\varepsilon(x-y) f(y)\, dy \tag{3.29}$$

pointwise. Since $T_\varepsilon(f_m)$ converges to $T_\varepsilon(f)$ in $L^2(\mathbb{R}^n)$ by definition, there exists a subsequence $T_\varepsilon(f_{m_k})$ such that $T_\varepsilon(f_{m_k})(x) \to T_\varepsilon(f)(x)$ for almost every $x \in \mathbb{R}^n$. But $T_\varepsilon(f_{m_k})(x) \to \Lambda f(x)$, thanks to (3.29), and so $\Lambda f = T_\varepsilon(f)$ almost everywhere in $\mathbb{R}^n$. So the extention of $T_\varepsilon\big|_{L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)}$ to the space $L^2(\mathbb{R}^n)$ coincides with the convolution of the space $L^2(\mathbb{R}^n)$ with the kernel K_ε. That is, if $f \in L^2(\mathbb{R}^n)$,

$$T_\varepsilon(f) = \int_{\mathbb{R}^n} K_\varepsilon(x-y) f(y)\, dy$$

It follows that

$$\|T_\varepsilon(f)\|_{L^2(\mathbb{R}^n)} = \lim_{m\to\infty} \|T_\varepsilon(f_m)\|_{L^2(\mathbb{R}^n)} \overset{(3.28)}{\leq} \lim_{m\to\infty} \left(\gamma B \|f_m\|_{L^2(\mathbb{R}^n)}\right) = \gamma B \|f\|_{L^2(\mathbb{R}^n)} \quad (3.30)$$

So the thesis holds in the case $p = 2$, that is T_ε is a bounded operator in $\mathcal{L}(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$.

We now want to prove that T_ε is also a weak type $(1,1)$ operator, so that, using theorem 3.11, we can conclude that T_ε is bounded in $L^p(\mathbb{R}^n)$ for $p \in (1,2)$.

Let $\alpha > 0$ and $f \in L^1(\mathbb{R}^n)$. We consider the Calderón-Zygmund decomposition at height α. We have

$$T_\varepsilon(f) = T_\varepsilon(g) + T_\varepsilon(b)$$

It follows that

$$|\{x \in \mathbb{R}^n : |T_\varepsilon(f)(x)| > \alpha\}| \leq |\{x \in \mathbb{R}^n : |T_\varepsilon(g)(x)| > \frac{\alpha}{2}\}| + |\{x \in \mathbb{R}^n : |T_\varepsilon(b)(x)| > \frac{\alpha}{2}\}|$$

We will estimate these two pieces separately. From theorem 3.10 we know that

$$\|g\|^2_{L^2(\mathbb{R}^n)} \leq (1 + 2^{2n})\alpha \|f\|_{L^1(\mathbb{R}^n)}$$

Therefore we have

$$|\{x \in \mathbb{R}^n : |T_\varepsilon(g)(x)| > \frac{\alpha}{2}\}| \leq \frac{4}{\alpha^2}\|T_\varepsilon(g)\|^2_{L^2(\mathbb{R}^n)} \overset{(3.30)}{\leq} cB^2\alpha^{-2}\|g\|^2_{L^2(\mathbb{R}^n)} \leq c'B\alpha^{-1}\|f\|_{L^1(\mathbb{R}^n)}$$

It remains to estimate $T_\varepsilon(b)$. In order to do so, remember the cubes Q_j in the Calderón-Zygmund decomposition, that is theorem 3.10. To each cube Q_j we associate a larger cube Q_j^*, concentric with Q_j, with diameter $2\sqrt{n}$ times larger. Thus, we define

$$\Omega^* := \bigcup_{j\in\mathbb{N}} Q_j^*$$

and $F^* = \mathbb{R}^n / \Omega^*$. Observe now that, since $|Q_j^*| = \lambda |Q_j|$, with λ independent of j and dipending only on n,

$$|\Omega^*| \leq \sum_{j\in\mathbb{N}} |Q_j^*| = \lambda \sum_{j\in\mathbb{N}} |Q_j| = \lambda|\Omega| \overset{(3.20)}{\leq} \lambda\alpha^{-1}\|f\|_{L^1(\mathbb{R}^n)}$$

Moreover, denote with y_j the common center of Q_j and Q_j^*. Then, if $x \notin Q_j^*$, we have that $|x - y_j| \geq 2|y - y_j|$ for all $y \in Q_j$. In fact,

$$|x - y_j| \geq R_j = 2r_j \geq 2|y - y_j|$$

where R_j is the radius of the ball inscribed in Q_j^*, while r_j is the radius of the ball circumscribing Q_j. Observe that $2r_j = R_j$ by elementary geometry considerations. So, we define

$$b_j(x) := \begin{cases} b(x) & x \in Q_j \\ 0 & \text{otherwise} \end{cases}$$

Observe that $b(x) = \sum_{j\in\mathbb{N}} b_j(x)$. Moreover, since Q_j have mutually disjoint intersiors nad $b(x) = 0$ on F, we have that the sum reduces to one term only for almost every $x \in \mathbb{R}^n$. Observe moreover that

$$T_\varepsilon(b_j)(x) := \int_{Q_j} K_\varepsilon(x-y)b(y)\ dy = \int_{Q_j} \left(K_\varepsilon(x-y) - K_\varepsilon(x-y_j)\right) b(y)\ dy$$

since $\int_{Q_j} b(y)\ dy = 0$. Remember now that

$$F^* = \mathbb{R}^n/\Omega^* = \bigcap_{j\in\mathbb{N}} (\mathbb{R}^n/Q_j^*)$$

and so

$$\|T_\varepsilon(b)\|_{L^1(F^*)} := \int_{F^*} |T_\varepsilon b(x)|\ dx \le \sum_{j\in\mathbb{N}} \int_{F^*} |T_\varepsilon b_j(x)|\ dx \le$$

$$\le \sum_{j\in\mathbb{N}} \int_{\{x\notin Q_j^*\}} \left(\int_{Q_j} |K_\varepsilon(x-y) - K_\varepsilon(x-y_j)||b(y)|\ dy \right) dx =$$

$$= \sum_{j\in\mathbb{N}} \int_{Q_j} |b(y)| \left(\int_{\{x\notin Q_j^*\}} |K_\varepsilon(x-y) - K_\varepsilon(x-y_j)|\ dx \right) dy \le$$

$$\le \sum_{j\in\mathbb{N}} \int_{Q_j} |b(y)| \left(\int_{|x-y_j|\ge 2|y-y_j|} |K_\varepsilon(x-y_j-(y-y_j)) - K_\varepsilon(x-y_j)| \right) dy \le$$

using the properties of the Calderón-Zygmund kernel and remark 3.14,

$$\le B_1 \sum_{j\in\mathbb{N}} \int_{Q_j} |b(y)|\ dy = B_1 \|b\|_{L^1(\mathbb{R}^n)} \overset{(3.17)}{\le} B_1(1+2^n)\|f\|_{L^1(\mathbb{R}^n)}$$

By Chebyshev we have

$$|\{x \in F^* |\ |T_\varepsilon b(x)| > \frac{\alpha}{2}\}| \le 2\alpha^{-1} \int_{F^*} |T_\varepsilon b(x)|\ dx \le 2\alpha^{-1} B_1(1+2^n)\|f\|_{L^1(\mathbb{R}^n)}$$

It follows that

$$|\{x \in \mathbb{R}^n |\ |T_\varepsilon b(x)| > \frac{\alpha}{2}\}| \le |\{x \in F^* |\ |T_\varepsilon b(x)| > \frac{\alpha}{2}\}| + |\Omega^*| \le$$

$$\le 2\alpha^{-1} B_1(1+2^n)\|f\|_{L^1(\mathbb{R}^n)} + \lambda\alpha^{-1}\|f\|_{L^1(\mathbb{R}^n)}$$

This means that T_ε is a weak type $(1,1)$. The Marcinkiewicz interpolation theorem implies the result for $p \in (1,2)$, as explained above.

We want now to deduce the thesis for $p > 2$. Let $p \in (2,\infty)$. Its conjugate exponent is $p' \in (1,2)$. The adjoint operator of T_ε is computed convolving with $K_\varepsilon(-x)$ instead of $K_\varepsilon(x)$. Also the reflexed kernel $\overline{K}_\varepsilon(x) := K_\varepsilon(-x)$ satisfies the properties of the theorem. Since the convolution with $\overline{K}_\varepsilon$ is the adjoint operator of T_ε, we have that the thesis also holds for $p \in (2,\infty)$. ∎

Theorem 3.16. *Let K be a Calderón-Zygmund kernel, with constant B. Fix $p \in (1,\infty)$. Consider the operator*

$$T_\varepsilon(f)(x) := \int_{|y|\geq\varepsilon} f(x-y)K(y)\ dy$$

There exists a constant $C_p = C_p(p,n,B)$ such that

$$\|T_\varepsilon f\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}$$

holds uniformly for all $\varepsilon > 0$. Moreover, for each $f \in L^p(\mathbb{R}^n)$, the strong limit

$$T_K(f) := \lim_{\varepsilon\to 0} T_\varepsilon(f)$$

exists in the norm $L^p(\mathbb{R}^n)$. The operator T_K is bounded in $L^p(\mathbb{R}^n)$ with operator norm bounded by C_p.

Proof. We first of all do some well-posedness considerations.

Remark 3.17. T_ε **is defined for every** $f \in L^p(\mathbb{R}^n)$**.** If $p \in (1,\infty)$ and $q \in (1,\infty)$ is its conjugate, we have that, for every $f \in L^p(\mathbb{R}^n)$,

$$\int_{|y|\geq\varepsilon} |f(x-y)||K(y)|\ dy = \int_{\mathbb{R}^n} |f(x-y)||K(y)|\chi_{|y|\geq\varepsilon}(y)\ dy \leq$$

$$\leq \left(\int_{\mathbb{R}^n} |f(x-y)|^p\ dy\right)^{\frac{1}{p}} \left(\int_{|y|\geq\varepsilon} |K(y)|^q\ dy\right)^{\frac{1}{q}} \leq \|f\|_p \left(\int_{|y|\geq\varepsilon} \frac{B^q}{|y|^{nq}}\ dy\right)^{\frac{1}{q}} < \infty$$

since $f \in L^p(\mathbb{R}^n)$ and $nq > n$. $\square$

The operators T_ε are nothing but convolution operators with kernels $C_\varepsilon(K)$. Thanks to proposition 3.3, it is a Calderón-Zygmund kernel with constant B. So, using lemma 3.6, we are in the hypothesis of theorem 3.15, since moreover[4] $C_\varepsilon(K)$ is in $L^2(\mathbb{R}^n)$. So we have

$$\|T_\varepsilon(f)\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)} \tag{3.31}$$

We now prove the convergence. We first focus our attention on a dense subset of $L^p(\mathbb{R}^n)$.

Definition and convergence over $C_c^\infty(\mathbb{R}^n)$. Consider the dense subset $C_c^\infty(\mathbb{R}^n)$ of $L^p(\mathbb{R}^n)$. Let $f \in C_c^\infty(\mathbb{R}^n)$. Then

$$(T_\varepsilon f)(x) = \int_{|y|\geq 1} K(y)f(x-y)\ dy + \int_{\varepsilon\leq|y|\leq 1} K(y)\left(f(x-y) - f(x)\right)\ dy$$

The first integral is a fixed function in $L^p(\mathbb{R}^n)$ since it is the convolution $(K\chi_{|y|\geq 1}) * f$ and by Young's convolution inequality

$$\|(K\chi_{|y|\geq 1}) * f\|_{L^p(\mathbb{R}^n)} \leq \|K\chi_{|y|\geq 1}\|_{L^p(\mathbb{R}^n)} \|f\|_{L^1(\mathbb{R}^n)} \tag{3.32}$$

[4] In fact,

$$\int_{|y|\geq\varepsilon} |K(y)|^2\ dy \leq \int_{|y|\geq\varepsilon} \frac{B^2}{|y|^{2n}}\ dy < \infty$$

since $K\chi_{|y|\geq 1} \in L^p(\mathbb{R}^n)$ for every $p \in (1, \infty)$. Looking at the second integral, we have that the convergence is even uniform. In fact, let $C_f := \text{supp}(f)$, a compact set in $\mathbb{R}^n$. Remember first of all that

$$|f(x-y) - f(x)| = \left| \left(\int_0^1 \nabla f(x - ty)\ dt \right) \cdot (-y) \right| \leq \sup_{\xi \in \mathbb{R}^n} |\nabla f(\xi)||y| \tag{3.33}$$

Define $m(\varepsilon, \eta) := \min\{\varepsilon, \eta\}$ and $M(\varepsilon, \eta) := \max\{\varepsilon, \eta\}$. So we have

$$\left| \int_{\varepsilon \leq |y| \leq 1} K(y) \left(f(x-y) - f(x) \right)\ dy - \int_{\eta \leq |y| \leq 1} K(y) \left(f(x-y) - f(x) \right)\ dy \right| =$$

$$= \left| \int_{m(\varepsilon,\eta) \leq |y| \leq M(\varepsilon,\eta)} K(y) \left(f(x-y) - f(x) \right)\ dy \right| \leq \int_{m(\varepsilon,\eta) \leq |y| \leq M(\varepsilon,\eta)} |K(y)||f(x-y) - f(x)|\ dy \leq$$

using the property of Calderón-Zygmynd kernels with constant B and (3.33)

$$\leq B \sup_{\xi \in \mathbb{R}^n} |\nabla f| \int_{m(\varepsilon,\eta) \leq |y| \leq M(\varepsilon,\eta)} \frac{|y|}{|y|^n}\ dy = B \sup_{\xi \in \mathbb{R}^n} |\nabla f| \int_{m(\varepsilon,\eta) \leq |y| \leq M(\varepsilon,\eta)} \frac{dy}{|y|^{n-1}} \leq$$

$$\leq B \sup_{\xi \in \mathbb{R}^n} |\nabla f| \int_{m(\varepsilon,\eta)}^{M(\varepsilon,\eta)} \frac{\rho^{n-1}}{\rho^{n-1}}\ d\rho = B \sup_{\xi \in \mathbb{R}^n} |\nabla f| \left(M(\varepsilon,\eta) - m(\varepsilon,\eta) \right) \to 0$$

as $\varepsilon, \eta \to 0$, since also $0 < m(\varepsilon, \eta) \leq M(\varepsilon, \eta) \to 0$.

Since the Cauchy convergence is uniform in $x \in \mathbb{R}^n$, we have that the sequence $\displaystyle\int_{\varepsilon \leq |y| \leq 1} K(y) \left(f(x-y) - f(x) \right)\ dy$ converges uniformly to a continuous function in $\mathbb{R}^n$. This means that exists $F \in \mathbb{R}^n$ such that

$$\lim_{\varepsilon \to 0} \sup_{x \in \mathbb{R}^n} \left| \int_{\varepsilon \leq |y| \leq 1} K(y) \left(f(x-y) - f(x) \right)\ dy - F(x) \right| = 0$$

Remark 3.18. Let $R > 0$ such that $C_f \subseteq B(0, R)$. Let $x \in \mathbb{R}^n$ such that $|x| > R + 1$. It follows that $|x| > R$ and, since $|y| \leq 1$,

$$|x - y| \geq |x| - |y| \geq |x| - 1 > R$$

It follows that $f(x) = f(x - y) = 0$, so that, if $|x| > R + 1$

$$\int_{\varepsilon \leq |y| \leq 1} K(y) \left(f(x-y) - f(x) \right)\ dy = 0$$

for every $\varepsilon > 0$. Then, this function is compactly supported, with support contained in $B(0, R + 1)$. Since the uniform limit of functions with compact support in a fixed compact is supported in the same compact, we have that F is supported in $B(0, R+1)$. So $F \in L^p(\mathbb{R}^n)$ for every $p \in (1, \infty)$. $\square$

Obviously the uniform convergence implies that

$$\lim_{\varepsilon \to 0} \left\| T_\varepsilon f - \overline{F} \right\|_{L^p(\mathbb{R}^n)} = 0$$

where now $\overline{F} \in L^p(\mathbb{R}^n)$ also consider the integral over $|y| \geq 1$. We use the estimate (3.32) to say that the sum belongs to $L^p(\mathbb{R}^n)$. So we can define

$$T_K f := \lim_{\varepsilon \to 0} T_\varepsilon f \quad \text{in the sense of } L^p(\mathbb{R}^n) \tag{3.34}$$

Remember also that $\|T_\varepsilon f\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}$, as proved above.

Definition and convergence over $L^p(\mathbb{R}^n)$. Let $f \in L^p(\mathbb{R}^n)$. Then we can consider a sequence $\{f^m\}_{m\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^n)$ such that $\|f^m - f\|_{L^p(\mathbb{R}^n)} \to 0$ as $m \to \infty$. By the argument above it is defined $T_K f^m$ for every $m \in \mathbb{N}$, in the sense that

$$\lim_{\varepsilon\to 0} \|T_K f^m - T_\varepsilon f^m\|_{L^p(\mathbb{R}^n)} = 0$$

So we have

$$\|T_K f^m\|_{L^p(\mathbb{R}^n)} = \lim_{\varepsilon\to 0} \|T_\varepsilon f^m\|_{L^p(\mathbb{R}^n)} \leq C_p \|f^m\|_{L^p(\mathbb{R}^n)} \tag{3.35}$$

Obsiously the operator T_K is linear, since if $a, b \in \mathbb{R}$ and $g_1, g_2 \in C_c^\infty(\mathbb{R}^n)$

$$T_K(ag_1 + bg_2) := \lim_{\varepsilon\to 0} T_\varepsilon(ag_1 + bg_2) = aT_K g_1 + bT_K g_2$$

It follows that

$$\|T_K(f^m) - T_K(f^h)\|_{L^p(\mathbb{R}^n)} = \|T_K(f^m - f^h)\|_{L^p(\mathbb{R}^n)} \leq C_p \|f^m - f^h\|_{L^p(\mathbb{R}^n)} \to 0$$

where $f^m - f^h \in C_c^\infty(\mathbb{R}^n)$ and $f^m, f^h \to f$ in $L^p(\mathbb{R}^n)$. So $T_K(f^m)$ is a Cauchy sequence in $L^p(\mathbb{R}^n)$. It follows that we can define

$$T_K(f) := \lim_{m\to\infty} T_K(f^m) \quad \text{in the sense of } L^p(\mathbb{R}^n)$$

Now they hold two limits. At first

$$\|T_K(f)\|_{L^p(\mathbb{R}^n)} = \lim_{m\to\infty} \|T_K(f^m)\|_{L^p(\mathbb{R}^n)} \overset{(3.35)}{\leq} C_p \lim_{m\to\infty} \|f^m\|_{L^p(\mathbb{R}^n)} = C_p \|f\|_{L^p(\mathbb{R}^n)} \tag{3.36}$$

Moreover

$$T_K(f) \equiv \lim_{\varepsilon\to 0} T_\varepsilon(f) \quad \text{in the sense of } L^p(\mathbb{R}^n) \tag{3.37}$$

In fact, let $\delta > 0$. Fix $N = N(\delta) \in \mathbb{N}$ such that $\|f^N - f\|_{L^p(\mathbb{R}^n)} < \frac{\delta}{C_p}$. Then, we can find $\varepsilon_0 = \varepsilon_0(f^N) = \varepsilon_0(N(\delta)) = \varepsilon_0(\delta)$, such that if $\varepsilon < \varepsilon_0$

$$\|T_K(f^N) - T_\varepsilon(f^N)\|_{L^p(\mathbb{R}^n)} < \delta$$

since, being $f^N \in C_c^\infty(\mathbb{R}^n)$, (3.34) holds. So we have[5]

$$\begin{aligned}
\|T_K(f) - T_\varepsilon(f)\|_{L^p(\mathbb{R}^n)} &= \|T_K(f) - T_K(f^N) + T_K(f^N) - T_\varepsilon(f^N) + T_\varepsilon(f^N) - T_\varepsilon(f)\|_{L^p(\mathbb{R}^n)} \leq \\
&\leq \|T_K(f) - T_K(f^N)\|_{L^p(\mathbb{R}^n)} + \|T_K(f^N) - T_\varepsilon(f^N)\|_{L^p(\mathbb{R}^n)} + \|T_\varepsilon(f^N) - T_\varepsilon(f)\|_{L^p(\mathbb{R}^n)} \leq \\
&\leq C_p\|f - f^N\|_{L^p(\mathbb{R}^n)} + \|T_K(f^N) - T_\varepsilon(f^N)\|_{L^p(\mathbb{R}^n)} + C_p\|f^N - f\|_{L^p(\mathbb{R}^n)} \leq \\
&\leq \delta + \|T_K(f^N) - T_\varepsilon(f^N)\|_{L^p(\mathbb{R}^n)} + \delta < 3\delta
\end{aligned}$$

if $\varepsilon < \varepsilon_0(\delta)$. This says that (3.37) holds. Thus we have

$$\|T_K(f)\|_{L^p(\mathbb{R}^n)} \equiv \left\|\lim_{\varepsilon\to 0} T_\varepsilon(f)\right\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}$$

that is the thesis of the Calderón-Zygmund theorem. ∎

[5]Since $f^N, f \in L^p(\mathbb{R}^n)$, we have

$$\|T_K(f) - T_K(f^N)\|_{L^p(\mathbb{R}^n)} = \|T_K(f - f^N)\|_{L^p(\mathbb{R}^n)} \leq C_p\|f - f_N\|_{L^p(\mathbb{R}^n)}$$

using linearity (since T_K is linear on smooth function and the limit conserves the linearity) and using (3.36). Similarly, we have that

$$\|T_\varepsilon(f^N) - T_\varepsilon(f)\|_{L^p(\mathbb{R}^n)} = \|T_\varepsilon(f^N - f)\|_{L^p(\mathbb{R}^n)} \leq C_p\|f - f^N\|_{L^p(\mathbb{R}^n)}$$

using the linearity of T_ε and the bound (3.31).

Chapter 4

Sobolev spaces

Definition 4.1. Let $\Omega \subseteq \mathbb{R}^n$ an open subset. We define

$$L^1_{loc}(\Omega) := \left\{ f : \Omega \to \mathbb{R} \;\middle|\; \int_K |f(x)| \, dx < +\infty \;\; \forall \; K \text{ compact subset of } \Omega \right\}$$

Definition 4.2. A *multiindex* of lenght k is a vector

$$\alpha = (\alpha_1, ..., \alpha_m)$$

such that $\alpha_i \in \mathbb{N}$ and $|\alpha| := \alpha_1 + ... + \alpha_m = k$.

Definition 4.3. Let U an open subset of $\mathbb{R}^n$. A *test function* on U is an element of the space

$$C^\infty_c(U) := \{\phi : U \to \mathbb{R} | \; \phi \in C^\infty(U) \text{ and } \operatorname{supp}(\phi) \text{ is a compact subset of } U\}$$

Definition 4.4. Let U an open subset of $\mathbb{R}^n$ and $u \in L^1_{loc}(U)$. We say that u admits the α^{th}-weak partial derivative if exists a function $v \in L^1_{loc}(U)$ such that

$$\int_U u D^\alpha \phi \; dx = (-1)^{|\alpha|} \int_U v \phi \; dx \qquad \forall \phi \in C^\infty_c(U)$$

In this situation, we write

$$D^\alpha u = v \quad \text{or} \quad \nabla^\alpha u = v$$

If $k \in \mathbb{N}$, it is often used the notation

$$\nabla^k u := \{D^\alpha u | \; |\alpha| = k\}$$

where in this situation the symbol means a whole class of derivatives.

Remark 4.1. The two integral in the definition are well-posed, because a test function can be dominated by its maximum and the characteristical function of its (compact) support. □

Lemma 4.1. *If a function $u \in L^1_{loc}(U)$ admists two α^{th}-weak partial derivatives, say $v, \tilde{v}$, then*

$$v = \tilde{v} \quad a.e.$$

Remark 4.2. If v is the α^{th}-weak derivative of u on the open set U, then it is the α^{th}-weak derivative of u in all the open subsets $V \subseteq U$. In fact, obviously $u, v \in L^1_{\text{loc}}(V)$ and for every $\phi \in C_c^\infty(V) \subseteq C_c^\infty(U)$, we have

$$\int_V u D^\alpha \phi \, dx = \int_U u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_U v\phi \, dx = (-1)^{|\alpha|} \int_V v\phi \, dx$$

where has been used that ϕ and its deriviatives vanish outside V. The central equality follows from the fact that v is the α^{th}-weak derivative of u on U. □

4.0.1 Definition of Sobolev spaces

Definition 4.5. Let $p \in [1, \infty]$ and $k \geq 0$ an integer. We define the Sobolev space as

$$W^{k,p}(U) := \{u \in L^1_{loc}(U) |\ D^\alpha u \text{ exists in the weak sense and } D^\alpha u \in L^p(U)\ \forall \alpha :\ |\alpha| \leq k\}$$

Finally we define

$$H^k(U) := W^{k,2}(U)$$

Sobolev norms:

Remark 4.3. Sobolev spaces $W^{k,p}$ are normed spaces. In fact, for $u \in W^{k,p}(U)$, we can define

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \left(\sum_{|\alpha| \leq k} \int_U |D^\alpha u|^p dx \right)^{\frac{1}{p}} = \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_p^p \right)^{\frac{1}{p}} & \text{if } 1 \leq p < +\infty \\ \sum_{|\alpha| \leq k} \sup_U |D^\alpha u| & \text{if } p = \infty \end{cases} \tag{4.1}$$

These naturally induce a distance and so a topology. □

Morover, it is called *homogeneous Sobolev space* the set

$$D^{k,p}(U) := \{u \in L^1_{loc}(U) |\ D^\alpha u \text{ exists in the weak sense and } D^\alpha u \in L^p(U)\ \forall \alpha :\ |\alpha| = k\}$$

It can be equipped with the *seminorm*

$$|u|_{D^{k,p}(U)} := \left(\sum_{|\alpha| = k} \int_U |D^\alpha u|^p dx \right)^{\frac{1}{p}}$$

When $k = 1$ it actually coincides with the p-norm of the gradient. Let also be

$$D_0^{k,p}(U) := \{u \in D^{k,p}(U) |\ \exists\ \{u_m\} \subseteq C_c^\infty(U) \text{ such that } \lim_{m \to +\infty} |u_m - u|_{D^{k,p}(U)} = 0\}$$

Definition 4.6. We also define the so called *Sobolev space with zero boundary values*[1]

$$W_0^{k,p}(U) := \{u \in W^{k,p}(U) |\ \exists\ \{u_m\} \subseteq C_c^\infty(U) \text{ such that } \lim_{m \to +\infty} \|u_m - u\|_{W^{k,p}(U)} = 0\}$$

Remark 4.4. We can equip the space with the norm

$$\|u\|_{W_0^{k,p}(U)} := \lim_{n \to +\infty} \|u_n\|_{W^{k,p}(U)} = \|u\|_{W^{k,p}(U)}$$

where the last identity holds because $|\|u_n\|_{W^{k,p}(U)} - \|u\|_{W^{k,p}(U)}| \leq \|u_n - u\|_{W^{k,p}(U)} \to 0$ as $n \to +\infty$. □

[1]In a sense that will be specified in subsection 4.3.

Equivalent norms and vector valued functions. In the following chaptes, we will use mainly vectorial functions; moreover, it will be very useful to introduce an equivalent Sobolev norm that helps in inequalities and estimates.

Let $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$. We say that u is in $W^{k,p}(\Omega)^n$ if it is true for every u_i, $i \in \{1, ..., n\}$. In this case we can consider the p- norm of each derivative.
In the future we will represent

$$\|u\|_{W^{k,p}} := \Big(\int_\Omega |u|^p dx + \int_\Omega |\nabla u|^p dx + ... + \int_\Omega |\nabla^k u|^p dx \Big)^{\frac{1}{p}} \equiv \Big(\int_\Omega \sum_{j=0}^k |\nabla^j u|^p dx \Big)^{\frac{1}{p}}$$

where $\nabla^j u$ is the tensor of the j-th derivatives.
Since a tensor space is a finite dimensional normed space, we have the equivalence of all the norms. So

$$C_{1,1}^p |u|_p^p \leq |u|^p \leq C_{2,1}^p |u|_p^p, \quad ..., \quad C_{1,k}^p |\nabla^k u|_p^p \leq |\nabla^k u|^p \leq C_{2,k}^p |\nabla^k u|_p^p$$

and for an h-dimensional tensor

$$|T|_p^p = \sum |T_{i_1,...,i_h}|^p$$

we have the equivalence of the norms:

$$\|u\|_{W^{k,p}} \leq \Big(\int_\Omega \sum_{j=0}^k C_{2,j}^p |\nabla^j u|_p^p \Big)^{\frac{1}{p}} \leq \Big(\int_\Omega \sum_{j=0}^k C_{2,j^*}^p |\nabla^j u|_p^p \Big)^{\frac{1}{p}} = C_{2,j^*} \Big(\int_\Omega \sum_{j=0}^k |\nabla^j u|_p^p \Big)^{\frac{1}{p}}$$

where $C_{2,j^*} := \max\{C_{2,1}, ..., C_{2,k}\}$. Since $|\nabla^j u|_p^p = \sum_{i=1}^3 \sum_{|\alpha|=j} |D^\alpha u_i|^p$, we have

$$\|u\|_{W^{k,p}} \leq C_{2,j^*} \Big(\int_\Omega \sum_{i=1}^3 \sum_{|\alpha| \leq k} |D^\alpha u_i|^p dx \Big)^{\frac{1}{p}} = C_{2,j^*} \Big(\sum_{i=1}^3 \sum_{|\alpha| \leq k} \|D^\alpha u_i\|_p^p \Big)^{\frac{1}{p}}$$

and the latter one is the vectorial version of the norm introduced in (4.1). Since the other inequality is similar, we have the equivalence of the norms.

From now on, we will consider

$$\|u\|_{W^{k,p}} := \Big(\int_\Omega \sum_{j=0}^k |\nabla^j u|^p dx \Big)^{\frac{1}{p}}$$

It is of course a norm. From Minkowsky's inequality it follows the triangular inequality. In fact

$$\|u+v\|_{W^{k,p}} = \Big(\sum_{j=0}^k \|\nabla^j (u+v)\|_p^p \Big)^{\frac{1}{p}} \leq \Big(\sum_{j=0}^k \big(\|\nabla^j u\|_p + \|\nabla^j v\|_p \big)^p \Big)^{\frac{1}{p}} \leq$$

$$\stackrel{(1.15)}{\leq} \Big(\sum_{j=0}^k \|\nabla^j u\|_p^p \Big)^{\frac{1}{p}} + \Big(\sum_{j=0}^k \|\nabla^j v\|_p^p \Big)^{\frac{1}{p}} = \|u\|_{W^{k,p}} + \|v\|_{W^{k,p}}$$

4.1 Sobolev spaces in the case $\Omega = \mathbb{R}^n$

Important properties of Sobolev spaces dipend on the nature of the domain where the space is defined.

Definition 4.7. If $n \in \mathbb{N}$ and $1 \leq p < n$ we define the Sobolev conjugate of p as

$$p^* := \frac{np}{n-p}$$

Remark 4.5. We have

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$$

and $p^* > p$. □

Theorem 4.1. *Let $n \in \mathbb{N}$ and $1 \leq p < n$. Let p^* the Sobolev conjugate of p. Then*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C|u|_{D^{1,p}(\mathbb{R}^n)} \equiv C\|\nabla u\|_{L^p(\mathbb{R}^n)} \quad \forall u \in C_c^\infty(\mathbb{R}^n)^n$$

with C only depending on n, p. Here

$$C_c^\infty(\mathbb{R}^n)^n := \{u : \mathbb{R}^n \to \mathbb{R}^n |\ u \in C_c^\infty(\mathbb{R}^n)\}$$

Remark 4.6. The proof that we are going to show follows the steps of the proof of the same theorem in [10]. However, we repeat the steps since we are in the case of u vectorial function and there are some differences. □

Proof. We first assume $p = 1$. By the regularity of u, applying the fundamental theorem of calculus component by component, we have

$$u(x) = \int_{-\infty}^{x_i} \partial_{x_i} u(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n)\ dy_i$$

Remember $\partial_{x_i} u$ is a vector. Using integral inequality for vectors we have

$$|u(x)| \leq \int_{-\infty}^{x_i} |\partial_{x_i} u(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n)|\ dy_i \leq$$

$$\leq \int_{-\infty}^{+\infty} |\partial_{x_i} u(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n)|\ dy_i$$

Remembering now that

$$|\nabla u| = \sqrt{|\partial_{x_1} u|^2 + ... + |\partial_{x_n} u|^2} \geq |\partial_{x_i} u| \quad \forall i = 1, ..., n$$

So

$$|u(x)| \leq \int_{-\infty}^{+\infty} |\nabla u(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n)|\ dy_i$$

If we multiply the inequality for $i = 1, ..., n$, after raising both sides to $\frac{1}{n-1}$, we get

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |\nabla u(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n)|\ dy_i \right)^{\frac{1}{n-1}}$$

Hence

$$\int_{-\infty}^{+\infty} |u(x)|^{\frac{n}{n-1}} dx_1 \leq \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |\nabla u(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n)| dy_i \right)^{\frac{1}{n-1}} dx_1 =$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |\nabla u(y_1, ..., x_n)| dy_1 \right)^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |\nabla u(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n)| dy_i \right)^{\frac{1}{n-1}} dx_1$$

$$= \left(\int_{-\infty}^{\infty} |\nabla u(y_1, ..., x_n)| dy_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |\nabla u(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n)| dy_i \right)^{\frac{1}{n-1}} dx_1$$

Now, by Hölder inequality

$$\int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |\nabla u(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n)| dy_i \right)^{\frac{1}{n-1}} dx_1 \leq$$

$$\leq \left(\prod_{i=2}^{n} \int_{-\infty}^{+\infty} \int_{-\infty}^{\infty} |\nabla u(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n)| dy_i dx_1 \right)^{\frac{1}{n-1}}$$

So we can define

$$I_1 := \int_{-\infty}^{\infty} |\nabla u(y_1, ..., x_n)| dy_1$$

$$I_i := \int_{-\infty}^{+\infty} \int_{-\infty}^{\infty} |\nabla u(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n)| dy_i dx_1$$

So

$$\int_{-\infty}^{+\infty} |u(x)|^{\frac{n}{n-1}} dx_1 \leq$$

$$\leq \left(\int_{-\infty}^{\infty} |\nabla u(y_1, ..., x_n)| dy_1 \right)^{\frac{1}{n-1}} \left(\prod_{i=2}^{n} \int_{-\infty}^{+\infty} \int_{-\infty}^{\infty} |\nabla u(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n)| dy_i dx_1 \right)^{\frac{1}{n-1}} =$$

$$= I_1^{\frac{1}{n-1}} \prod_{i=2}^{n} I_i^{\frac{1}{n-1}} = \prod_{i=1}^{n} I_i^{\frac{1}{n-1}}$$

Integrating this expression with respect x_2 we get

$$\int_{-\infty}^{\infty} \int_{-\infty}^{+\infty} |u(x)|^{\frac{n}{n-1}} dx_1 dx_2 \leq \int_{-\infty}^{\infty} I_2^{\frac{1}{n-1}} \prod_{i=1, i\neq 2}^{n} I_i^{\frac{1}{n-1}} dx_2 = I_2^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=1, i\neq 2}^{n} I_i^{\frac{1}{n-1}} dx_2 =$$

$$= \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{\infty} |\nabla u(x_1, y_2, ..., x_n)| dy_2 dx_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=1, i\neq 2}^{n} I_i^{\frac{1}{n-1}} dx_2$$

At this point we can apply Hölder to the latter term, so that

$$\int_{-\infty}^{\infty} \prod_{i=1, i\neq 2}^{n} I_i^{\frac{1}{n-1}} dx_2 \leq \prod_{i=1, i\neq 2}^{n} \left(\int_{-\infty}^{\infty} I_i \; dx_2 \right)^{\frac{1}{n-1}} =$$

$$= \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\nabla u(y_1, ..., x_n)| dy_1 dx_2 \right)^{\frac{1}{n-1}} \prod_{i=3}^{n} \left(\int_{-\infty}^{\infty} I_i \; dx_2 \right)^{\frac{1}{n-1}}$$

So we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{+\infty}|u(x)|^{\frac{n}{n-1}}dx_1dx_2 \leq$$

$$\leq \left(\int_{-\infty}^{+\infty}\int_{-\infty}^{\infty}|\nabla u(x_1,y_2,...,x_n)|dy_2dx_1\right)^{\frac{1}{n-1}}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\nabla u(y_1,...,x_n)|dy_1dx_2\right)^{\frac{1}{n-1}}\prod_{i=3}^{n}\left(\int_{-\infty}^{\infty}I_i\,dx_2\right)^{\frac{1}{n-1}}$$

Iterating the process, we get

$$\int_{\mathbb{R}^n}|u(x)|^{\frac{n}{n-1}}dx \leq \prod_{i=1}^{n}\left(\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty}|\nabla u(x_1,...,x_{i-1},y_i,x_{i+1},...,x_n)|dx_1...dy_i...dx_n\right)^{\frac{1}{n-1}} =$$

$$= \left(\int_{\mathbb{R}^n}|\nabla u(x)|dx\right)^{\frac{n}{n-1}}$$

This is the thesis if $p=1$. Let now $1<p<n$. Let

$$\gamma := \frac{p(n-1)}{n-p} > 1$$

Applying the previous result to $v := |u|^\gamma$ we have

$$\int_{\mathbb{R}^n}|u|^{\frac{\gamma n}{n-1}}dx \leq \left(\int_{\mathbb{R}^n}|\nabla |u|^\gamma|dx\right)^{\frac{n}{n-1}}$$

But

$$\nabla|u|^\gamma = \gamma|u|^{\gamma-1}\frac{u}{|u|}\nabla u$$

using the homogeneity of $y \to |y|^\gamma$ with $\gamma > 1$ to extend the derivative to the whole space. With the norm, we have

$$|\nabla|u|^\gamma| \leq \gamma|u|^{\gamma-1}|\nabla u|$$

So

$$\left(\int_{\mathbb{R}^n}|u|^{\frac{\gamma n}{n-1}}dx\right)^{\frac{n-1}{n}} \leq \gamma\int_{\mathbb{R}^n}|u|^{\gamma-1}|\nabla u|dx \leq \gamma\left(\int_{\mathbb{R}^n}|u|^{(\gamma-1)\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^n}|\nabla u|^p dx\right)^{\frac{1}{p}}$$

where Hölder inequality has been used again.
By the definition of γ we have

$$(\gamma-1)\frac{p}{p-1} = \frac{\gamma n}{n-1} = p^*$$

It follows that

$$\left(\int_{\mathbb{R}^n}|u|^{p^*}dx\right)^{\frac{n-1}{n}-\frac{p-1}{p}} \leq \gamma\|\nabla u\|_{L^p(\mathbb{R}^n)}$$

But

$$\frac{n-1}{n}-\frac{p-1}{p} = \frac{1}{p^*}$$

and so

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq \gamma\|\nabla u\|_{L^p(\mathbb{R}^n)}$$

where $\gamma = \frac{p(n-1)}{n-p} =: C(n,p) \equiv C$. ∎

Remark 4.7. It is sufficient to ask $u \in C_0^1(\mathbb{R}^n)^n$, since only the first derivative is used in the proof. □

Theorem 4.2. [2] *Let* $1 \leq p < n$ *and let* $p^* := \dfrac{pn}{n-p}$. *Then,* $W_0^{1,p}(\mathbb{R}^n) \subseteq L^{p^*}(\mathbb{R}^n)$ *and*

[2] From "Analyse Fonctionnelle"- H. Brezis, Th. IX.9.

exists $C = C(n,p)$ such that

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^n)} \quad \forall\ u \in W_0^{1,p}(\mathbb{R}^n)$$

Proof. We know by previous theorem that

$$\|\phi\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|\nabla \phi\|_{L^p(\mathbb{R}^n)} \quad \forall \phi \in C_c^\infty(\mathbb{R}^n)$$

Now, being $u \in W_0^{1,p}(\mathbb{R}^n)$, we have $\{u_m\} \subseteq C_c^\infty(\mathbb{R}^n)$ such that

$$\lim_{m\to+\infty} |u_m - u|_{D^{1,p}(\mathbb{R}^n)} = 0$$

and

$$\lim_{m\to+\infty} \|u_m - u\|_{L^p(\mathbb{R}^n)} = 0$$

So, even if currently we don't know that the L^{p^*} norm is well-posed, we have

$$\|u\|_{L^{p^*}} \leq \|u-u_m\|_{L^{p^*}}+\|u_m\|_{L^{p^*}} \leq \|u-u_m\|_{L^{p^*}}+C\|\nabla u_m\|_{L^p} = \|u-u_m\|_{L^{p^*}}+C\|\nabla u_m\|_{L^p} \leq$$
$$\leq \|u-u_m\|_{L^{p^*}} + C\big(\|\nabla u_m - \nabla u\|_{L^p} + \|\nabla u\|_{L^p}\big)$$

Remembering that C is independent by u_m, note that

$$\|\nabla u_m - \nabla u\|_{L^p} = \|\nabla(u_m - u)\|_{L^p} = |u_m - u|_{D^{1,p}} \to 0 \ \text{ as } m \to +\infty$$

Morover, since $\|u_m - u\|_{L^p} \to 0$, we have that the convergence $u_m \to u$ is in L^p and hence pointwise a.e. along a subsequence. Note that $u_i - u_j$ is a sequence in $C_0^\infty(\mathbb{R}^n)$ and

$$\|u_i - u_j\|_{L^{p^*}} \leq \overline{C}\|\nabla u_i - \nabla u_j\|_{L^p} \leq \overline{C}\big(\|\nabla u_i - \nabla u\|_{L^p} + \|\nabla u_j - \nabla u\|_{L^p}\big)$$

using Gagliardo-Nirenberg-Sobolev inequality for smooth functions with compact support. So we have that $\{u_m\}$ is a Cauchy sequence in L^{p^*}. Being this space complete, $\exists v \in L^{p^*}$ such that $u_m \to v$ in L^{p^*}. So, there exists a subsequence of u_m converging to v pointwise a.e. Being a subsequence of a sequence converging a.e. to u, we have that $v = u$ a.e. So

$$\|u - u_m\|_{L^{p^*}} = \|v - u_m\|_{L^{p^*}} \to 0 \ \text{ as } m \to +\infty$$

So the thesis holds. ∎

Moreover, thanks to the following theorem, the above lemma hols also for $u \in W^{1,p}(\mathbb{R}^n)$.

Theorem 4.3. *Let $p \in [1,\infty)$. Then $W_0^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$.*

Remark 4.8. We will prove it for $u : \mathbb{R}^n \to \mathbb{R}^n$. The scalar case is analogous. □

Proof. Obviously $W_0^{1,p}(\mathbb{R}^n) \subseteq W^{1,p}(\mathbb{R}^n)$. Let now $u \in W^{1,p}(\mathbb{R}^n)$ and let $\eta_k \in C_c^\infty(B_{k+1}(0))$ such that

$$\eta_k|_{B_k(0)} \equiv 1, \qquad \eta_k \in [0,1], \qquad \partial_{x_i}\eta_k \leq 2$$

Notice that it is sufficient to construct a such function in $\mathbb{R}$, then generalize it to a n-dimensional function via radial extension.

Then we have that $u\eta_k \in W_0^{1,p}(\mathbb{R}^n)$, as it will be proved in the following lemma. Moreover

$$u\eta_k \to u \ \text{ in } W^{1,p}(\mathbb{R}^n) \text{ as } k \to +\infty$$

In fact

$$\|u\eta_k - u\|_{W^{1,p}(\mathbb{R}^n)} \leq \|u\eta_k - u\|_{L^p(\mathbb{R}^n)} + \|\nabla(u\eta_k - u)\|_{L^p(\mathbb{R}^n)} =$$
$$= \left(\int_{\mathbb{R}^n} |u\eta_k - u|^p dx\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^n} |(1-\eta_k)\nabla u - u \otimes D\eta_k|^p dx\right)^{\frac{1}{p}} \leq$$
$$\leq \left(\int_{\mathbb{R}^n} |u\eta_k - u|^p dx\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^n} |(1-\eta_k)\nabla u|^p dx\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^n} |u \otimes D\eta_k|^p dx\right)^{\frac{1}{p}}$$

Clearly $u\eta_k \to \eta$ almost everywhere when $k \to +\infty$. Morover

$$|u(\eta_k - 1)|^p \leq |u|^p$$

is a integrable majorant, being $u \in L^p(\mathbb{R}^n)$. So by Lebesgue theorem

$$\int_{\mathbb{R}^n} |u(\eta_k - 1)|^p dx \to 0 \ \text{ as } k \to +\infty$$

Moreover $\nabla u \in L^p(\mathbb{R}^n)$, so also

$$\int_{\mathbb{R}^n} |\nabla u(\eta_k - 1)|^p dx \to 0 \ \text{ as } k \to +\infty$$

Finally

$$\left(\int_{\mathbb{R}^n} |u \otimes D\eta_k|^p dx\right)^{\frac{1}{p}} = \left(\int_{\mathbb{R}^n} |\sqrt{\sum_{i,j} u_j^2 (\partial_{x_i}\eta_k)^2}|^p dx\right)^{\frac{1}{p}} \leq 2\left(\int_{B_{k+1}(0)/B_k(0)} |u|^p dx\right)^{\frac{1}{p}}$$

Using again Lebesgue theorem, we have that also this term goes to zero as $k \to +\infty$.

This implies that $u\eta_k \in W_0^{1,p}(\mathbb{R}^n)$ is a Cauchy sequence in the $W_0^{1,p}(\mathbb{R}^n)$ norm. Being this space complete, we have that exists $v \in W_0^{1,p}(\mathbb{R}^n)$ such that

$$\|u\eta_k - v\|_{W_0^{1,p}(\mathbb{R}^n)} \to 0 \ \text{ as } k \to +\infty$$

But $W_0^{1,p}(\mathbb{R}^n)$ is equipped with the same norm of $W^{1,p}(\mathbb{R}^n)$. So

$$\|u - v\|_{W^{1,p}(\mathbb{R}^n)} \leq \|u - u\eta_k\|_{W^{1,p}(\mathbb{R}^n)} + \|u\eta_k - v\|_{W_0^{1,p}(\mathbb{R}^n)} \to 0$$

So $u = v$ up to a null measure set. It follows that $\|u - v_k\|_{W^{1,p}(\mathbb{R}^n)} = \|v - v_k\|_{W^{1,p}(\mathbb{R}^n)} \to 0$ if $v_k \in C_c^\infty(\mathbb{R}^n)$ is a sequence that approaches v in the norm $\|\cdot\|_{W^{1,p}(\mathbb{R}^n)}$. So $u \in W_0^{1,p}(\mathbb{R}^n)$. ∎

This lemma, used in the previous proof, holds for all the domains.

Lemma 4.2. *Let Ω open and let $w \in W^{1,p}(\Omega)$ such that supp(w) is a compact subset of Ω. Then $w \in W_0^{1,p}(\Omega)$.*

Remark 4.9. If $\Omega = \mathbb{R}^n$ open set, and $w = u\eta_k$ with $u \in W^{1,p}(\mathbb{R}^n)$, we have that $\text{supp}(w) \subseteq B_{k+1}(0)$ and $w \in W^{1,p}(\mathbb{R}^n)$ because $u \in W^{1,p}(\mathbb{R}^n)$ and η_k is bounded. This is exactly what we used in the previous proof. □

Proof. See [17, Lemma 1.23, pg. 19]. ∎

4.2 Sobolev spaces in the case of bounded domains

The main inequality of this case is the following.

Lemma 4.3. *Let $\Omega \subseteq \mathbb{R}^n$ a bounded domain. Let $q \in (1, +\infty)$ and*

$$d(\Omega) := \sup_{x,y\in\Omega} |x-y|$$

Then

$$\|u\|_{L^q(\Omega)^n} \leq C(q,d)\|\nabla u\|_{L^q(\Omega)^{n^2}} \quad \forall\ u \in W_0^{1,q}(\Omega)$$

This is called Poincaré inequality.

We have seen that $W_0^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$. However, the equality does not hold for a bounded domain Ω. In fact, if we consider $\Omega = B_R(0)$ with $R > 0$ and $u \equiv 1 \in W^{1,p}(\Omega)$ a scalar function[3], it follows from previous lemma that if, for absurd, $u \in W_0^{1,p}(\Omega)$, then, taking in example $q = 2$ and being $d(\Omega) = 2R$,

$$\pi R^2 = \mu(\Omega) = \int_\Omega dx \leq C(2, 2R)\int_\Omega 0\ dx = 0$$

That's obviously absurd[4].

This tells us that $W_0^{1,p}(U) \neq W^{1,p}(U)$. However, we will deduce a characterization of the set $W_0^{1,p}(U)$: this coincides with the set of the functions of $W^{1,p}(U)$ those "vanish on the boundary", in a sense that will be formalized by the so called *trace operator*. This operator, roughly speaking, map a function to its values at the boundary: for $C(\overline{U})$ functions, it will coincide with the restriction to the boundary.

Sobolev Inequalities in bounded domain. We have two main theorems from [10].

Theorem 4.4. *Let Ω a bounded and open subset of $\mathbb{R}^n$, with $\partial\Omega \in C^1$. Assume $1 \leq p < n$, and let $p^* := \frac{np}{n-p}$. Assume $u \in W^{1,p}(\Omega)$ a real valued function. Then $u \in L^{p^*}(\Omega)$ and*

$$\|u\|_{L^{p^*}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$$

where C depends only on p, n and Ω.

Theorem 4.5. *Let Ω a bounded open subset of $\mathbb{R}^n$. Suppose $u \in W_0^{1,p}(\Omega)$ a real valued function, with $1 \leq p < n$. Then we have the estimate*

$$\|u\|_{L^q(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)} \quad \forall q \in [1, p^*]$$

where the constant C depends only on p, q, n, Ω.

[3] The inequality also holds for scalar functions.

[4] The function u is here actually a regular function on an open set. So, the weak derivative coincides with the classical derivative.

Remark 4.10. We will be particulary intrerested in functions $u \in W^{1,p}(\Omega)^n$ or $W_0^{1,p}(\Omega)^n$, i.e. $\mathbb{R}^n$- vector valued functions. The estimates above continue to hold. In fact, if we write $u = (u_1, ..., u_n)$, we have[5]

$$\|u_i\|_{L^{p^*}(\Omega)} \leq C\|u_i\|_{W^{1,p}(\Omega)} \qquad \|u_i\|_{L^q(\Omega)} \leq C\|\nabla u_i\|_{L^p(\Omega)}$$

So, using the equivalence of the norms as in (3.7),

$$\|u\|^q_{L^q(\Omega)} := \int_\Omega |u|^q dx \leq K \int_\Omega \sum_{i=1}^n |u_i|^q dx = K \sum_{i=1}^n \|u_i\|^q_{L^q(\Omega)} \leq KC^q \sum_{i=1}^n \|\nabla u_i\|^q_{L^p(\Omega)} =$$

$$= KC^q \sum_{i=1}^n \left(\int_\Omega |\nabla u_i|^p \; dx \right)^{\frac{q}{p}} \leq KC^q (K')^q \left(\sum_{i=1}^n \int_\Omega |\nabla u_i|^p \; dx \right)^{\frac{q}{p}}$$

using that, if $y = (y_1, ..., y_n)$, $\|y\|^q_q = \sum_{i=1}^n |y_i|^q \leq (K')^q \|y\|^q_p = (K')^q \left(\sum_{i=1}^n |y_i|^p \right)^{\frac{q}{p}}$, since on vectors q-norm and p-norm are equivalent. We have used $y_i = \left(\int_\Omega |\nabla u_i|^p dx \right)^{\frac{1}{p}}$.
Remember that $|\cdot|$ is the vectorial Euclidean norm, while $|\cdot|_p$ is the vectorial p-norm. Since, using again norm equivalence, we have $|\nabla u_i|^p \leq C_1^p |\nabla u_i|_p^p$ it follows

$$\|u\|^q_{L^q(\Omega)} \leq KC^q (K')^q \left(\int_\Omega \sum_{i=1}^n C_1^p |\nabla u_i|_p^p \; dx \right)^{\frac{q}{p}} = KC^q (K')^q C_1^q \left(\int_\Omega |\nabla u|_p^p \; dx \right)^{\frac{q}{p}} \leq$$

$$\leq KC^q (K')^q C_1^q \left(\int_\Omega C_2^p |\nabla u|^p dx \right)^{\frac{q}{p}}$$

where C_2 is such that $|\nabla u|_p \leq C_2 |\nabla u|$ for the equivalence of matricial norms.
We have proved that

$$\|u\|_{L^q} \leq K'' \|\nabla u\|_{L^p}$$

where K'' depends on the same parameters as before.

A similar argument holds when u is a matrix. It will be proved in the special case $p = 2$ in (11.40). □

---

[5]If $u \in W_0^{1,p}(\Omega)^n$ we have exists $\varphi_m \in C_c^\infty(\Omega)^n$ such that

$$0 = \lim_{m \to +\infty} \|u - \varphi_m\|_{W^{1,p}} = \lim_{m \to +\infty} \left(\sum_{|\alpha| \leq 1} \int_\Omega |D^\alpha (u - \varphi_m)|^p dx \right)^{\frac{1}{p}} \geq$$

$$\geq \lim_{m \to +\infty} \left(\int_\Omega |u - \varphi_m|^p \; dx \right)^{\frac{1}{p}} \geq \lim_{m \to +\infty} \left(\int_\Omega |u_i - (\varphi_m)_i|^p \; dx \right)^{\frac{1}{p}}$$

so that each component is in $W_0^{1,p}(\Omega)$.

4.2.1 General Sobolev inequalities

Definition 4.8. We say u^* is a *version* of a given function u provided $u = u^*$ almost everywhere.

Theorem 4.6. *Let U be a bounded domain, open subset of $\mathbb{R}^n$, and suppose that ∂U is C^1. Assume that $n < p \leq \infty$ and $u \in W^{1,p}(U)$. Then u has a version $u^* \in C^{0,\gamma}(\overline{U})$, for $\gamma = 1 - \frac{n}{p}$, with the estimate*

$$\|u^*\|_{C^{0,\gamma}(\overline{U})} \leq C\|u\|_{W^{1,p}(U)}$$

The constant C depends only on p, n and U.

Theorem 4.7. *Let U ba a bounded open subset of $\mathbb{R}^n$, with a C^1 boundary. Assume $u \in W^{k,p}(U)$. If $k > \frac{n}{p}$, then $u \in C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\overline{U})$, where*

$$\gamma := \begin{cases} \left[\frac{n}{p}\right] + 1 - \frac{n}{p}, & \frac{n}{p} \text{ is not an integer} \\ \text{any positive number} < 1, & \frac{n}{p} \text{ is an integer} \end{cases}$$

In addition, it holds the estimate

$$\|u\|_{C^{k-[\frac{n}{p}]-1,\gamma}(\overline{U})} \leq C\|u\|_{W^{k,p}(U)}$$

the constant C depending only on k, p, n, γ and U.

Theorem 4.8. *Let Ω a bounded domain with C^1 boundary. Let $u \in W^{k,p}(\Omega)$. If $k > \frac{n}{p}$ then $u \in C^{k-[\frac{n}{p}]-1,\gamma}(\overline{\Omega})$, where*

$$\gamma := \begin{cases} [\frac{n}{p}] + 1 - \frac{n}{p}, & \text{if } \frac{n}{p} \text{ is not an integer} \\ \text{any positive number } < 1, & \text{if } \frac{n}{p} \text{ is an integer} \end{cases}$$

4.3 Trace operator

Traces concern the possibility of assigning boundary balues along $\partial\Omega$ to a function $u \in W^{1,p}(\Omega)$, assuming $\partial\Omega \in C^1$. We have the following (trace) Theorem.

Theorem 4.9. *Assume Ω is bounded and $\partial\Omega$ is C^1. Then, there exists a bounded linear operator*

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

such that

(i) $Tu = u|_{\partial\Omega}$ if $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$;

(ii) $\|Tu\|_{L^p(\partial\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$ for each $u \in W^{1,p}(\Omega)$ with the constant C depending only on p and Ω.

Definition 4.9. We call Tu the trace of u on $\partial\Omega$.

We examine now what it means for a function to have zero trace.

Theorem 4.10. *Assume Ω is bounded and $\partial\Omega$ is C^1. Suppose furthermore that $u \in W^{1,p}(\Omega)$. Then*

$$u \in W_0^{1,p}(\Omega) \iff Tu = 0 \quad \text{on } \partial\Omega$$

4.4 Compactness in Sobolev Spaces

Definition 4.10. Let X and Y be Banach spaces, with $X \subset Y$. We say that X is *compactly embedded* into Y, written $X \subset\subset Y$ if

(i) exists a constant $C > 0$ such that $\|x\|_Y \leq C\|x\|_X$ for every $x \in X$;

(ii) for every $\{x_k\}_{k\in\mathbb{N}}$ bounded sequence in X, i.e. $\|x_k\|_X \leq C'$ for every $k \in \mathbb{N}$, exist a subsequence x_{k_h} and $y \in Y$ such that $\lim_{h\to\infty} \|x_{k_h} - y\|_Y = 0$.

Theorem 4.11. *Assume U is bounded open subset of $\mathbb{R}^n$, and ∂U is C^1. Suppose $1 \leq p < n$. Then*

$$W^{1,p}(U) \subset\subset L^q(U)$$

for every $1 \leq q < p^* := \frac{np}{n-p}$.

Remark 4.11. A Banach space Z embeds continuously into another Banach space X if holds only the first condition in definition 4.10. We write $Z \to X$. So, if we have the chain

$$Z \to X \subset\subset Y$$

then $Z \subset\subset Y$. In fact, let z_k a bounded sequence in Z. We have that

$$\|z_k\|_X \leq C\|z_k\|_Z \leq CC'$$

So, there exists a subsequence z_{k_h} and $y \in Y$ such that

$$\lim_{h\to\infty} \|z_{k_h} - y\|_Y = 0$$

So $Z \subset\subset Y$. $\square$

Corollary 4.1. *Let U be a bounded open subset of $\mathbb{R}^n$. Then, for every $q \in (1, \infty)$, we have*

$$W^{1,q}(U) \subset\subset L^q(U)$$

Proof. If $q < n$, then we can use $p = q$ in theorem 4.11, observing that $p^* = \frac{nq}{n-q} > q$. If $q \geq n$, since $\lim_{p\to n^-} \dfrac{np}{n-p} = +\infty$, we can find $p < n$ such that

$$\frac{np}{n-p} > q \geq n$$

So theorem 4.11 says that

$$W^{1,p}(U) \subset\subset L^q(U)$$

But since U is bounded, and $q > p$, we have that $W^{1,q}(\Omega) \to W^{1,p}(\Omega)$. In fact, if $u \in W^{1,q}(\Omega)$,

$$\|u\|_{W^{1,q}(U)} = \left(\|u\|_q^q + \|\nabla u\|_q^q\right)^{\frac{1}{q}} \leq \|u\|_q + \|\nabla u\|_q \leq |U|^{\frac{1}{r}}\|u\|_p + |U|^{\frac{1}{r}}\|\nabla u\|_p$$

with r such that $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$. So

$$\|u\|_{W^{1,q}(U)} \leq 2|U|^{\frac{1}{r}}\|u\|_{W^{1,p}(U)}$$

So, by remark 4.11, we have

$$W^{1,q}(U) \subset\subset L^q(U)$$

that is the thesis. ∎

4.5 The Homogeneous Sobolev Spaces $D^{m,q}$

For $m \in \mathbb{N}$ and $1 \leq q < \infty$ we define

$$D^{m,q}(\Omega) := \{u \in L^1_{loc}(\Omega) |\ D^l u \in L^q(\Omega),\ |l| = m\}$$

In $D^{m,q}$ we introduce the seminorm

$$|u|_{m,q} := \left(\sum_{|l|=m} \int_\Omega |D^l u|^q \right)^{\frac{1}{q}} \tag{4.2}$$

We can also define other spaces, starting from the homogeneous sobolev space.

Let P_m the class of all the polynomials of degree $\leq m-1$. For $u \in D^{m,q}$ we set

$$[u]_m := \{w \in D^{m,q} |\ w = u + p,\ \text{for some } p \in P_m\}$$

We set

$$\dot{D}^{m,q} := \{[u]_m |\ u \in D^{m,q}\}, \qquad |[u]_m|_{m,q} := |u|_{m,q}$$

Lemma 4.4. *Let Ω an arbitrary domain of $\mathbb{R}^n$, $n \geq 2$. Then $\dot{D}^{m,q}(\Omega)$ is a Banach space. In particular, if $q = 2$, then it is an Hilbert space with scalar product*

$$\langle [u]_m, [v]_m \rangle_m := \sum_{|l|=m} \int_\Omega D^l u D^l v$$

Remark 4.12. The proof of this classical result can be found in [12, Lemma II.6.2, pg. 83]. □

Remark 4.13. The functional (4.2) induces a norm in the space $C_0^\infty(\Omega)$. We define $D_0^{m,q}$ as the completion of the space $C_0^\infty(\Omega)$ with the norm $|\cdot|_{m,q}$. □

We state now a weak-compactness result, that is propesed in [12] as Exercise II.6.2, pg. 85.

Proposition 4.1. *The space $\dot{D}^{m,q}$ is separable for $1 \leq q < \infty$ and reflexive for $q \in (1, \infty)$. Thus, for $q \in (1, \infty)$ the space is weakly complete and the unit ball is weakly compact.*

Proof. Once one has proved separability and reflexivity, the weakly compactness follows from theorem 2.4. We focus our argument on the case $m = 1$, since it is simplier and equivalent to the others. We have that the set

$$W := \{w \in \left(L^q\right)^n :\ w = \left(\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n} \right)\ \exists\ u \in \dot{D}^{1,q}\}$$

It is clearly isomorphic to $\dot{D}^{1,q}$. Moreover, being $\dot{D}^{1,q}$ complete, it is a closed subset of $\left(L^q\right)^n$ thus that is separable if $q \geq 1$ and reflexive if $q > 1$. This gives the properties for $\dot{D}^{1,q}$. ∎

We also have the following lemma, that is [12, Lemma II.6.1].

Lemma 4.5. *Let Ω an arbitrary domain of $\mathbb{R}^n$, $n \geq 2$, and let $u \in D^{m,q}(\Omega)$, with $m \geq 0$ and $q \in (1, \infty)$. Then $u \in W^{m,q}_{loc}(\Omega)$ and the following inequality holds*

$$\|u\|_{m,q,\omega} \leq c \left(\sum_{|l|=m} \|D^l u\|_{q,\omega} + \|u\|_{1,\omega} \right) \tag{4.3}$$

where ω is an arbitrary bounded locally lipschitz domain with $\overline{\omega} \subset \Omega$.

Sketch. The inequality (4.3) follows from the Gagliardo-Nirenberg interpolation inequality on bounded domain, together with the Young's inequality: in fact, the Gagliardo-Nirenberg inequality allows to control the inequalities of smaller degree in terms of the norm L^1 and the norm L^q of the m-th derivative. The Young's inequality convert the product into a sum. ∎

4.6 Sobolev spaces with negative degree

Definition 4.11. Let $q \in (1, +\infty)$ an exponent and $k \in \mathbb{N}$ a positive integer degree. Let $q' := \frac{q}{q-1}$. Let $\Omega \subseteq \mathbb{R}^n$ a domain. Then we define

$$W^{-k,q}(\Omega) := \left(W^{k,q'}_0(\Omega) \right)^* \equiv \{F : W^{k,q'}_0(\Omega) \to \mathbb{R} : \text{ F is a linear and continuous operator}\}$$

i.e. the dual space of $W^{k,q'}_0(\Omega)$. This space can be equipped with the following norm:

$$\|F\|_{W^{-k,q}(\Omega)} := \sup_{0 \neq \varphi \in W^{k,q'}_0} \frac{|F(\varphi)|}{\|\varphi\|_{W^{k,q'}_0(\Omega)}}$$

Remark 4.14. The definitions can be adapted to vectorial functions. □

Remark 4.15. If $q = 2$ (and so $q' = 2$), every $f \in L^2(\Omega)$ can be seen as an element of this space:

$$F_f(\varphi) := \int_\Omega f \cdot \varphi \, dx$$

In fact, linearity is obvious, while

$$|F_f(\varphi)| \leq \int_\Omega |f||\varphi| dx \leq \|f\|_2 \|\varphi\|_2$$

This inequality says that F_f is a well-posed operator and it is continuous[6]. So $F_f \in W^{-k,2}(\Omega)$. □

Definition 4.12. Let $q \in (1, +\infty)$ an exponent and $k \in \mathbb{N}$ a positive integer degree. Let $q' := \frac{q}{q-1}$. Let $\Omega \subseteq \mathbb{R}^n$ a domain. Then we say that

$$F \in W^{-k,q}_{loc}(\Omega) \iff F|_{\Omega_0} \in W^{-k,q}(\Omega_0) \quad \forall\, \Omega_0 \text{ bounded subdomain of } \Omega \text{ such that } \overline{\Omega_0} \subseteq \Omega$$

[6] If $\varphi_n \to \varphi$ in $W^{k,2}_0$, then $\|\varphi_n - \varphi\|_2 \to 0$

Remark 4.16. If $F \in W^{-k,q}_{loc}(\Omega)$, then for every Ω_0 as above the functional F has a norm defined locally as above. □

Lemma 4.6. *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain and let $q \in (1, \infty)$. Then*

$$\|u\|_{L^q(\Omega)} \leq C \left(\|\nabla u\|_{W^{-1,q}(\Omega)} + \|u\|_{W^{-1,q}(\Omega)} \right)$$

for every $u \in L^q(\Omega)$ where $C = C(q, \Omega) > 0$ is a constant.

Remark 4.17. The proof is provided in [27, Lemma 1.1.3, pg. 45]. □

Theorem 4.12. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, with $n \geq 2$. Let $\Omega_0 \neq \emptyset$ any subdomain, and let $q \in (1, \infty)$. Then exists $C = C(q, \Omega, \Omega_0) > 0$ such that*

$$\|u\|_{L^q(\Omega)} \leq C \|\nabla u\|_{W^{-1,q}(\Omega)}$$

for all $u \in L^q(\Omega)$ such that $\displaystyle\int_{\Omega_0} u \; dx = 0$.

Remark 4.18. The proof is provided in [27, Lemma 1.5.4, pg. 58]. □

Proof. We define, first of all,

$$F_u(v) := -\int_{\Omega} u \left(\nabla \cdot v \right) \; dx$$

where $u \in L^q(\Omega)$, $v \in W^{1,q'}_0(\Omega)$ and $q' = \frac{q}{q-1}$.

We now prove the estimate. Suppose, by contrary, that $\nexists C > 0$ such that

$$\|u\|_{L^q(\Omega)} \leq C \|\nabla u\|_{-1,q}$$

for every $u \in L^q(\Omega)$ such that $\displaystyle\int_{\Omega_0} u \; dx = 0$. This means that for every $j \in \mathbb{N}$, exists $u_j \in L^q(\Omega)$ such that $\displaystyle\int_{\Omega_0} u_j \; dx = 0$ and

$$\|u_j\|_{L^q(\Omega)} > j \|\nabla u_j\|_{-1,q}$$

We define $\tilde{u}_j := \|u_j\|_q^{-1} u_j$ and consider the sequence $\{\tilde{u}_j\}_{j \in \mathbb{N}}$. Then

$$\|\tilde{u}_j\|_{L^q(\Omega)} = 1, \quad \int_{\Omega_0} \tilde{u}_j \; dx = 0, \quad \|\nabla \tilde{u}_j\|_{-1,q} < \frac{1}{j}$$

Since $L^q(\Omega)$ is reflexive, and $\{\tilde{u}_j\}_{j \in \mathbb{N}}$ is bounded, we have that exists $\tilde{u}_{j_k}$ such that $\tilde{u}_{j_k} \rightharpoonup u \in L^q(\Omega)$. We rename $\tilde{u}_j$ this subsequence. In particular it holds that $\|\nabla \tilde{u}_j\|_{-1,q} \to 0$, as $j \to \infty$. Moreover

$$|F_u(v)| = |\langle u, \nabla \cdot v \rangle| = \lim_{j \to \infty} |\langle \tilde{u}_j, \nabla \cdot v \rangle| = 0$$

for every $v \in W_0^{1,q'}(\Omega)$. This implies that $\|\nabla u\|_{-1,q} = 0$. Observe that furthermore

$$|\langle \tilde{u}_j, \nabla \cdot v\rangle| \leq \|\nabla \tilde{u}_j\|_{-1,q}\|v\|_{1,q'} \to 0 \quad \text{as } j \to \infty$$

Thus, in the sense of distributions, we have $\nabla u = 0$. This implies $u \equiv c$, and being $\int_{\Omega_0} u \, dx = 0$, we have $c = 0$.

On the other hand, lemma 4.6 says that

$$\|\tilde{u}_j\|_q = 1 \leq C \left(\|\nabla \tilde{u}_j\|_{W^{-1,q}(\Omega)} + \|\tilde{u}_j\|_{W^{-1,q}(\Omega)}\right) \tag{4.4}$$

Since $\tilde{u}_j$ is bounded in $L^q(\Omega)$ and the embedding $L^q(\Omega) \subseteq W^{-1,q}(\Omega)$ is compact, we have that exists a subsequence $\tilde{u}_{j_k}$ that converges to some $\tilde{u} \in L^q(\Omega)$. Since the convergence is in particular a weak convergence, we have that $\tilde{u} = u = 0$, since the limit is unique. This means that

$$\lim_{j\to\infty} \|\tilde{u}_j\|_{-1,q} = 0$$

So equation (4.4) becomes $1 \leq 0$, that is an absurd. So the thesis in proved. ∎

4.7 A generalized divergence theorem

The following theorem is a generalization of the divergence theorem [32, Th. 6.3.4, pg. 125]

Theorem 4.13. *Let $\Omega \subseteq \mathbb{R}^n$ an open set of class C^1 with bounded boundary Γ. Let $v \in W^{1,1}(\Omega)$. Then*

$$\int_\Omega \nabla \cdot v \, dx = \int_\Gamma (Tv) \cdot \eta \, d\sigma \tag{4.5}$$

where T is the trace operator and η the outward normal vector.

4.7.1 An application of the trace operator

This application is useful in the integration by parts: this tell us when we can get rid of the boundary piece. In future, we will often use this device to simplify our calculations.

Remark 4.19. Let $w, v \in H^2(\Omega)$ scalar functions, with $Tv = 0$ on $\partial\Omega$. Moreover suppose $v \in C^1(\overline{\Omega})$. Then we have

$$\int_{\partial\Omega} T(v\nabla w) \cdot \eta \, d\sigma = 0 \tag{4.6}$$

In fact, being $w \in H^2(\Omega) \equiv W^{2,2}(\Omega)$, in particular $\nabla w \in W^{1,2}(\Omega)$. Being Ω bounded with regular boundary, there exists a sequence $\{\varphi^n\} \subseteq C^\infty(\overline{\Omega})$ such that

$$\lim_{n\to+\infty} \|\nabla w - \varphi^n\|_{W^{1,2}(\Omega)} = 0$$

Consider now the sequence $\{v\varphi^n\}_n \subseteq C^1(\overline{\Omega})$. We have that

$$\|v\nabla w - v\varphi^n\|^2_{W^{1,2}(\Omega)} = \|v\nabla w - v\varphi^n\|_2^2 + \|\nabla(v\nabla w - v\varphi^n)\|_2^2$$

In particular

$$\nabla[v(\nabla w-\varphi^n)]=(\nabla v)(\nabla w-\varphi^n)+v\nabla(\nabla w-\varphi^n)$$

So

$$\begin{aligned}\|\nabla[v(\nabla w-\varphi^n)]\|_2^2 &\le \big(\|(\nabla v)(\nabla w-\varphi^n)\|_2+\|v\nabla(\nabla w-\varphi^n)\|_2\big)^2 \le \\ &\le 2\big(\|(\nabla v)(\nabla w-\varphi^n)\|_2^2+\|v\nabla(\nabla w-\varphi^n)\|_2^2\big)\end{aligned}$$

using that $(a+b)^2\le 2(a^2+b^2)$. Finally

$$\|v\nabla w-v\varphi^n\|_{W^{1,2}(\Omega)}^2\le\|v\nabla w-v\varphi^n\|_2^2+2\|(\nabla v)(\nabla w-\varphi^n)\|_2^2+2\|v\nabla(\nabla w-\varphi^n)\|_2^2$$

Observe now that each piece vanishes as $n\to+\infty$. In fact

$$\int_\Omega |v\nabla w-v\varphi^n|^2\,dx\le\max_{\overline{\Omega}}|v|^2\|\nabla w-\varphi^n\|_2^2 \tag{4.7}$$

$$\int_\Omega |(\nabla v)(\nabla w-\varphi^n)|^2\,dx\le\max_{\overline{\Omega}}|\nabla v|^2\|\nabla w-\varphi^n\|_2^2 \tag{4.8}$$

and

$$\int_\Omega |v\nabla(\nabla w-\varphi^n)|^2\,dx\le\max_{\overline{\Omega}}|v|^2\|\nabla(\nabla w-\varphi^n)\|_2^2 \tag{4.9}$$

since $v\in C^1(\overline{\Omega})$. Using that

$$\|\nabla w-\varphi^n\|_2^2,\|\nabla(\nabla w-\varphi^n)\|_2^2\le\|\nabla w-\varphi^n\|_{W^{1,2}(\Omega)}^2\to 0\ \text{ as } n\to+\infty$$

we have that $v\varphi^n$ approaches $v\nabla w$ in norm $W^{1,2}(\Omega)$. We have moreover that $v\varphi^n\in C^1(\overline{\Omega})$ and

$$v(x)\varphi^n(x)=0\ \ \forall x\in\partial\Omega$$

since $v\equiv 0$ in $\partial\Omega$. So

$$T(v\nabla w):=\lim_{n\to+\infty}T(v\varphi^n)$$

where the limit is taken in $L^2(\partial\Omega)$. Since the trace $T(v\varphi^n)$ is constanly zero, also the limit in $L^2(\partial\Omega)$ is zero. So the trace of $v\nabla w$ is zero and the integral above is zero too. □

4.8 Weak derivatives and mollifications

The following theorem is a different version of theorem 3.2, that generalizes the thesis in the case of weak derivatives.

Theorem 4.14. *Let $u\in W^{k,p}(\Omega)$, with Ω domain in $\mathbb{R}^n$. Consider u_ε the mollification of u. Then*

$$D^\alpha u_\varepsilon=(D^\alpha u)*\eta_\varepsilon \qquad \textit{over}\ \ \Omega_\varepsilon$$

for every α multi-index such that $|\alpha|\le k$.

Proof. The derivative of u_ε over Ω has to be intended in classical sense, being $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$. So, let $\varepsilon > 0$ fixed and $\varphi \in C_c^\infty(\Omega_\varepsilon)$. First of all observe that, if α is a multi-index, for every $x \in \Omega_\varepsilon$ we have

$$D^\alpha u_\varepsilon(x) = \int_\Omega u(y)\left(D^\alpha \eta_\varepsilon\right)(x-y)\ dy = (-1)^{|\alpha|}\int_\Omega (-1)^{|\alpha|} u(y)\left(D^\alpha \eta_\varepsilon\right)(x-y)\ dy =$$

$$= (-1)^{|\alpha|}\int_\Omega u(y) D_y^\alpha\left(\eta_\varepsilon(x-y)\right)\ dy = \int_\Omega D^\alpha u(y)\eta_\varepsilon(x-y)\ dy$$

since if $x \in \Omega_\varepsilon$ is fixed, then $\eta_\varepsilon(x-y)$ is smooth with compact support in Ω. So we have that

$$\int_{\Omega_\varepsilon} \big((D^\alpha u) * \eta_\varepsilon\big)(x)\varphi(x)\ dx = \int_{\Omega_\varepsilon}\left(\int_\Omega D^\alpha u(y)\eta_\varepsilon(x-y)\ dy\right)\varphi(x)\ dx =$$

$$= \int_{\Omega_\varepsilon} D^\alpha u_\varepsilon(x)\varphi(x)\ dx = (-1)^{|\alpha|}\int_{\Omega_\varepsilon} u_\varepsilon(x) D^\alpha\varphi(x)\ dx$$

since φ has compact support in Ω_ε. This is the thesis. ∎

Chapter 5

Spaces involving time

5.1 Bochner Integral

Often we deal with function of the form $u = u(x,t)$ with $(x,t) \in \Omega \times I$ and $\Omega \subseteq \mathbb{R}^3$, $I = [a,b] \subseteq \mathbb{R}$. This function, fixed the time, is a function of the only x-variable. In particular, it can happen that $u(x,t_0)$ is $\mathcal{B}(\mathbb{R}^3)$-misurable for a.e. $t_0 \in I$ and that for those $t_0 \in I$ we have

$$\int_\Omega |u(x,t_0)|^p \; dx < +\infty$$

In other words, $u(\cdot,t_0) \in L^p$ for a.e. $t_0 \in I$, i.e. u maps the interval $[a,b]$ to element of the functional space L^p. This justify the following definition.

Definition 5.1. A *Banach space valued function* is a function

$$u : [a,b] \to X$$

where $(X, \|\cdot\|)$ is a Banach space.

Example 5.1. The simplest example is that of a function $f \in C([a,b] \times [c,d], \mathbb{R})$.
For every $x_0 \in [a,b]$, $f(x_0,y)$ is a continuous function for $y \in [c,d]$. So it belongs to the space $(C([c,d],\mathbb{R}), \|\cdot\|_\infty)$, where $\|\cdot\|_\infty$ is the usual maximum in an interval. □

We can also introduce a notion of measurability for Banach valued functions. First of all

Definition 5.2. A Banach valued function $s : [a,b] \to X$ is called *simple* if $\exists\ m \in \mathbb{N}$ and $\{E_i\}_{i=1}^m \subseteq \mathcal{B}([a,b])$ and $s_i \in X$ for $i = 1,...,m$ such that

$$s(t) = \sum_{i=1}^m s_i \chi_{E_i}(t)$$

The *integral* of a simple function is soon defined. We define

$$\int_a^b s(t)dt := \sum_{i=1}^m s_i |E_i|$$

So we have

Definition 5.3. A Banach valued function $u : [a, b] \to X$ is *measurable* if there exists a sequence $\{s_k(t)\}_{k\in\mathbb{N}}$ of simple functions such that

$$s_k(t) \to u(t) \quad \text{for a.e. } t \in [a, b]$$

(i.e. for a.e. $t \in [a, b]$ the limit $\|s_k(t) - u(t)\| \to 0$ for $k \to +\infty$).

Definition 5.4. We say that a Banach valued function is *summable*, or *Bochner integrable*, if

(i) $\lim\limits_{k\to+\infty} \|s_k(t) - u(t)\| = 0$ for a.e. $t \in [a, b]$;

(ii) $\lim\limits_{k\to+\infty} \displaystyle\int_a^b \|s_k(t) - u(t)\| \; dt = 0.$

Here $\|\cdot\|$ is always the norm in the Banach space X. In this case we define

$$\int_a^b u(t) \; dt := \lim_{k\to+\infty} \int_a^b s_k(t) \; dt$$

Remark 5.1. The definition is well-posed. First of all, being u and s_k strong measurable[1] (the first can be approximated with simple functions; the latter is a simple function, so constantly approximated with simple functions), we have that $\|s_k(t)-u(t)\|$ is measurable[2] in[3] $([a, b], \mathcal{B}([a, b]), dt)$. Morover, we have

$$\left\| \int_a^b s_k(t) \; dt - \int_a^b s_h(t) \; dt \right\| = \left\| \int_a^b \big(s_k(t) - s_h(t)\big) \; dt \right\| \leq \int_a^b \|s_k(t) - s_h(t)\| \; dt \to 0$$

for $k, h \to +\infty$, since

$$\int_a^b \|\big(s_k(t) - u(t)\big) - \big(s_h(t) - u(t)\big)\| \; dt \leq \int_a^b \|s_k(t) - u(t)\| \; dt + \int_a^b \|s_h(t) - u(t)\| \; dt$$

Being X a Banach space (hence complete), we have that the limit exists in X. $\square$

Remark 5.2. Oftern we deal with Banach spaces of the form X^n, that is the cartesian product of n identical Banach spaces. This space can be equipped with the natural norm $\|(x_1, ..., x_n)\|_{X^n} = \sqrt{\|x_1\|^2 + ... + \|x_n\|^2}$, with $\|\cdot\| = \|\cdot\|_X$.

So, if $f : [0, T] \to X^n$, and it is a measurable function, we have that, if $s_k(t) = \sum\limits_{i=1}^{m_k} s_i^k \chi_{E_i^k}$,

$$\|s_k(t) - f(t)\|_{X^n} \to 0 \text{ a.e. } t \in [0, T]$$

Moreover, s_i^k and f have n components in X. It follows that each component of $s_k(t) = \sum\limits_{i=1}^{m_k} s_i^k \chi_{E_i^k}$ converges, in X to the respective component of f. The same holds for the

[1] Here is required the property of strong measurability to define (Bochner) integrability.

[2] This is proved, in example, in [33, Proof of Pettis Theorem, 6th edition, pg. 132].

[3] Here $\mathcal{B}([a, b])$ is the Borel-Algebra on $[a, b]$.

summability condition. So, it is defined the Bochner integral of each component.
In particular, we have

$$\lim_{k\to\infty}\left(\int_0^T s_k(t)\ dt\right)_j = \int_0^T \big(f(t)\big)_j\ dt$$

in the sense of the space X. So, we can consider the collection of the n integrals as an element of X^n.
By the definition of the norm, also $\displaystyle\int_0^T s_k(t)$ converges to this vector of X^n, since, if I is a vector of X^n such that $I_j = \displaystyle\int_0^T \big(f(t)\big)_j\ dt$, for $j = 1, ..., n$,

$$\|\int_0^T s_k(t)\ dt - I\|_{X^n} \equiv \sqrt{\sum_{j=1}^n \left\|\left(\int_0^T s_k(t)\ dt\right)_j - I_j\right\|^2} \to 0$$

as $k \to \infty$.
By the uniquess of the limit in the Banach space X^n, we have that $\displaystyle\int_0^T f(t)\ dt$ is the collection I of the Bochner integral of each component.

Observe that, if $X = L^2(\Omega)$, with Ω a domain, and if $f \in L^2(\Omega)^n$, so that $f = (f_1, ..., f_n)$, with $f_i \in L^2(\Omega)$, it holds that

$$\|f\|_{X^n} = \sqrt{\|f_1\|_2^2 + ... + \|f_n\|_2^2}$$

using that $\|\cdot\|_X = \|\cdot\|_2$. Since $\|f_i\|_2^2 = \displaystyle\int_\Omega |f_i|^2 dx$, we have

$$\|f\|_{X^n} = \left(\int_\Omega \sum_{i=1}^n |f_i|^2 dx\right)^{\frac{1}{2}} \equiv \left(\int_\Omega |f|^2 dx\right)^{\frac{1}{2}}$$

since $|f|^2 = \sum_{i=1}^n |f_i|^2$ is the Euclidean norm in $\mathbb{R}^n$. So it is the usual norm in $L^2(\Omega)^n$.
If $p \neq 2$, then we have an equivalence of the norms, since the Euclidean norm in $\mathbb{R}^n$ is equivalence to the $p-$norm in $\mathbb{R}^n$. $\square$

The definition can be generalized, replacing the Lebesgue time-measure with a general measure. But we are interested to very particular examples of this spaces, i.e. the L^p spaces involving time.

5.2 L^p spaces involving time

Definition 5.5. For $T > 0$ and $p \geq 1$ we define the space

$$L^p(0, T; X)$$

as the set of all strongly measurable functions

$$u : [0,T] \to X$$

such that

$$\int_0^T \|u(t)\|^p dt < +\infty$$

In this case, we define

$$\|u\|_{L^p(0,T;X)} := \left(\int_0^T \|u(t)\|^p \; dt \right)^{\frac{1}{p}}$$

Remark 5.3. The strongly measurability of u implies, as underlined above, that $\|u(t)\|$ is measurable in the Lebesgue sense in $[0,T]$. So, the integral is a well-posed integral of a non-negative measurable function. □

If $p = \infty$ we define

$$\|u\|_{L^\infty(0,T;X)} := \sup_{t\in[0,T]} \|u(t)\| < +\infty$$

Talking about Navier-Stokes equations, it is foundamental the case $X = L^q$. So

$$\|u\|_{L^p(0,T;L^q)} := \left(\int_0^T \|u(t)\|_{L^q}^p \; dt \right)^{\frac{1}{p}}$$

Remark 5.4. Moreover it holds the following proposition.

Proposition 5.1. *Let $u : [a,b] \to X$ a summable function, with $u \in L^p(a,b;X)$ and $p \geq 1$. Then exists $s_k(t)$ such that*

$$\lim_{k\to+\infty} \int_a^b \|s_k(t) - u(t)\|_X^p \; dt = 0$$

Proof. We know, by definition, that exists a sequence $s_k(t)$ of simple functions such that

$$\lim_{k\to+\infty} \int_a^b \|s_k(t) - u(t)\| dt = 0$$

and that this sequence also has limit $u(t)$ almost everywhere. Here we use, for sake of simplicity, $\|\cdot\|$ for $\|\cdot\|_X$. We can change slightly the structure of s_k to pass the limit under the integral sign. We define

$$\hat{s}_k(t) := \begin{cases} \frac{s_k(t)}{\|s_k(t)\|} \min\left\{ \frac{i(k)}{k}, \|s_k(t)\| \right\} & t \in \bigcup_{i=0}^{k^2} \Omega_{k,i} \\ 0 & \text{otherwise} \end{cases}$$

where $\Omega_{k,i} := \{t \in [a,b] : \frac{i}{k} \leq \|u(t)\| \leq \frac{i+1}{k}\}$, for $i \in \{0, ..., k^2\}$.
Remember that the s_k are simple functions, so they take a finite number of values.

So it is also $\hat{s}_k$. These values are taken over measurable sets, since these domains are intersections of domain of s_k and $\Omega_{k,i}$. Observe moreover that

$$\|\hat{s}_k(t)\| \leq \left\| \frac{s_k(t)}{\|s_k(t)\|} \min\left\{\frac{i(k)}{k}, \|s_k(t)\|\right\}\right\| = \frac{\|s_k(t)\|}{\|s_k(t)\|} \left|\min\left\{\frac{i(k)}{k}, \|s_k(t)\|\right\}\right| \leq \frac{i(k)}{k} \leq \|u(t)\|$$

Finally we show that $\|s_k(t)\| \to \|u(t)\|$ as $k \to +\infty$. In fact

$$t \in \Omega_{k,i} \iff \|u(t)\| \in \left[\frac{i}{k}, \frac{i+1}{k}\right] \iff \|u(t)\|k \in [i, i+1] \iff \|u(t)\|k-1 \leq i \leq \|u(t)\|k \iff$$

$$\iff \|u(t)\| - \frac{1}{k} \leq \frac{i(k)}{k} \leq \|u(t)\|$$

So we have

$$\frac{1}{\|s_k(t)\|}\min\left\{\frac{i(k)}{k}, \|s_k(t)\|\right\} := \begin{cases} 1 & \min\{\frac{i(k)}{k}, \|s_k(t)\|\} = \|s_k(t)\| \\ \frac{\frac{i(k)}{k}}{\|s_k(t)\|} \in \left[\frac{\|u(t)\|}{\|s_k(t)\|} - \frac{1}{k\|s_k(t)\|}, \frac{\|u(t)\|}{\|s_k(t)\|}\right] & \min\{\frac{i(k)}{k}, \|s_k(t)\|\} = \frac{i(k)}{k} \end{cases}$$

Since $\lim_{k\to+\infty} \|s_k(t)\| = \|u(t)\|$, we have that, for every $\varepsilon > 0$, exists $K_1 = K_1(t)$ such that

$$1 - \varepsilon \leq \frac{\|u(t)\|}{\|s_k(t)\|} \leq 1 + \varepsilon \quad \forall k \geq K_1(t)$$

It also exists $K_2 = K_2(t)$ such that[4]

$$-\varepsilon \leq \frac{1}{k\|s_k(t)\|} \leq \varepsilon \quad \forall k \geq K_2(t)$$

So, if $k \geq \max\{K_1, K_2\}$, we have

$$1 - 2\varepsilon \leq \frac{\|u(t)\|}{\|s_k(t)\|} - \frac{1}{k\|s_k(t)\|} \leq \frac{\|u(t)\|}{\|s_k(t)\|} \leq 1 + \varepsilon$$

For those k such that it holds the condition $\min\{\frac{i(k)}{k}, \|s_k(t)\|\} = \|s_k(t)\|$ we have that the function is simply 1. So, for k large enough, we have that

$$\frac{1}{\|s_k(t)\|}\min\left\{\frac{i(k)}{k}, \|s_k(t)\|\right\} - 1 \to 0$$

So[5]

$$\lim_{k\to+\infty} \|\hat{s}_k(t) - s_k(t)\| = \lim_{k\to+\infty} \left\| s_k(t)\left(\frac{1}{\|s_k(t)\|}\min\left\{\frac{i(k)}{k}, \|s_k(t)\|\right\} - 1\right)\right\| =$$

$$= \|s_k(t)\| \left| \frac{1}{\|s_k(t)\|}\min\left\{\frac{i(k)}{k}, \|s_k(t)\|\right\} - 1\right| \to \|u(t)\| \cdot 0 = 0$$

[4]Where the step of s_k is zero, so it is also the step of $\hat{s}_k$.

[5]Since for k large enough t is in $\Omega_{i,k}$ for some i.

Obviously

$$\|\hat{s}_k(t) - u(t)\| \leq \|\hat{s}_k(t) - s_k(t)\| + \|s_k(t) - u(t)\| \to 0 \quad \text{as } k \to +\infty$$

Finally

$$\|\hat{s}_k(t) - u(t)\|^p \leq \big\{\|\hat{s}_k(t)\| + \|u(t)\|\big\}^p \leq 2^p \|u(t)\|^p$$

Since $u \in L^p(a, b; X)$, we have that

$$\lim_{k\to+\infty} \int_a^b \|\hat{s}_k(t) - u(t)\|^p dt = \int_a^b \left(\lim_{k\to+\infty} \|\hat{s}_k(t) - u(t)\|^p \right) dt = 0$$

This means that $\hat{s}_k(t)$ is a sequence of step functions which limit is $u(t)$ in $L^p(a, b; X)$, that is the thesis. ∎

Theorem 5.1. *Let X be a Banach space and let $p \in [1, \infty)$. Then the collection of the functions*

$$f_n(t) := \sum_{k=1}^{n} c_k \phi_k(t) \qquad c_k \in X,\ \phi_k(t) \in C_c^\infty(0, T)$$

is dense in $L^p(0, T; X)$; that is, for every $f \in L^p(0, T; X)$ and $\varepsilon > 0$ exists $K \in \mathbb{N}$ such that

$$\|f - f_k\|_{L^p(0,T;X)} < \varepsilon \qquad \forall k \geq K$$

Observe that $f_k(t) \in C_c^\infty(0, T; X)$.

Proof. Let $f \in L^p(0, T; X)$. By proposition 5.1 we know that exists a $s_k(t) = \sum_{i=1}^{m_k} s_i^k \chi_{E_i^k}(t)$, with E_i^k measurable in $[0, T]$ and $s_i^k \in X$, such that

$$\lim_{k\to+\infty} \left(\int_0^T \|s_k(t) - f(t)\|_X^p \, dt \right)^{\frac{1}{p}} = 0$$

Now, for every $i, k \in \mathbb{N}$, we can find a function $\varphi_i^k \in C_c^\infty(0, T)$ such that

$$\|\varphi_i^k - \chi_{E_i^k}\|_p < \frac{\varepsilon}{2^i \|s_i^k\|_X}$$

since if $\|s_i^k\|_X = 0$, then $s_i^k = 0$ and we can rename the sequence. Then we can define

$$f_k(t) := \sum_{i=1}^{m_k} s_i^k \varphi_i^k(t)$$

Then we have

$$\|f - f_k\|_{L^p(0,T;X)} \equiv \left(\int_0^T \|f(t) - f_k(t)\|_X^p \, dt \right)^{\frac{1}{p}} \leq \|f - s_k\|_{L^p(0,T;X)} + \|s_k - f_k\|_{L^p(0,T;X)} =$$

$$= \left(\int_0^T \|f(t) - s_k(t)\|_X^p \, dt \right)^{\frac{1}{p}} + \Big\| \sum_{i=1}^{m_k} s_i^k \big(\chi_{E_i^k}(t) - \varphi_i^k(t)\big) \Big\|_{L^p(0,T;X)} \leq$$

$$\leq \left(\int_0^T \|f(t) - s_k(t)\|_X^p \, dt \right)^{\frac{1}{p}} + \sum_{i=1}^{m_k} \left\| s_i^k \big(\chi_{E_i^k}(t) - \varphi_i^k(t) \big) \right\|_{L^p(0,T;X)} =$$

$$= \left(\int_0^T \|f(t) - s_k(t)\|_X^p \, dt \right)^{\frac{1}{p}} + \sum_{i=1}^{m_k} \left(\int_0^T \|s_i^k \big(\chi_{E_i^k}(t) - \varphi_i^k(t) \big)\|_X^p \, dt \right)^{\frac{1}{p}} =$$

$$= \left(\int_0^T \|f(t) - s_k(t)\|_X^p \, dt \right)^{\frac{1}{p}} + \sum_{i=1}^{m_k} \|s_i^k\|_X \left(\int_0^T |\chi_{E_i^k}(t) - \varphi_i^k(t)|^p \, dt \right)^{\frac{1}{p}} =$$

$$= \left(\int_0^T \|f(t) - s_k(t)\|_X^p \, dt \right)^{\frac{1}{p}} + \sum_{i=1}^{m_k} \|s_i^k\|_X \|\chi_{E_i^k} - \varphi_i^k\|_p \leq$$

$$\leq \left(\int_0^T \|f(t) - s_k(t)\|_X^p \, dt \right)^{\frac{1}{p}} + \sum_{i=1}^{m_k} \frac{\varepsilon}{2^i} \leq \left(\int_0^T \|f(t) - s_k(t)\|_X^p \, dt \right)^{\frac{1}{p}} + \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} =$$

$$= \left(\int_0^T \|f(t) - s_k(t)\|_X^p \, dt \right)^{\frac{1}{p}} + \sum_{i=1}^{m_k} \frac{\varepsilon}{2^i} \leq \left(\int_0^T \|f(t) - s_k(t)\|_X^p \, dt \right)^{\frac{1}{p}} + \varepsilon$$

If K is such that $\left(\int_0^T \|s_k(t) - f(t)\|_X^p \, dt \right)^{\frac{1}{p}} < \varepsilon$ for every $k \geq K$, then we have

$$\|f - f_k\|_{L^p(0,T;X)} \leq 2\varepsilon \qquad \forall k \geq K$$

that is the thesis. ∎

We now focus our attention to the case $X = L^q(\Omega)$, with $q \in [1, \infty)$.

Corollary 5.1. *Let $p, q \in [1, \infty)$ and $f \in L^p(0, T; L^q(\Omega))$. Then, for every $\varepsilon > 0$, there exists $f_\varepsilon \in C_c^\infty((0, T) \times \Omega)$ such that*

$$\|f - f_\varepsilon\|_{L^p(0,T;L^q(\Omega))} < \varepsilon$$

Proof. Let be $\varepsilon > 0$. By theorem 5.1 there exists $g_\varepsilon(t) = \sum_{k=1}^{N_\varepsilon} \phi_k^\varepsilon(t) c_k^\varepsilon$ with $c_k^\varepsilon \in L^q(\Omega)$ and $\phi_k \in C_c^\infty(0, T)$ such that

$$\|f - g_\varepsilon\|_{L^p(0,T;L^q(\Omega))} < \varepsilon$$

By density of $C_c^\infty(\Omega)$ in $L^q(\Omega)$, we can find $\varphi_k^\varepsilon \in C_c^\infty(\Omega)$ such that

$$\|c_k^\varepsilon - \varphi_k^\varepsilon\|_{L^q(\Omega)} < \frac{\varepsilon}{T C_\varepsilon N_\varepsilon}$$

where

$$C_\varepsilon := \max_{k=1,\dots,N_\varepsilon} \left(\sup_{(0,T)} |\phi_k^\varepsilon| \right)$$

Define $f_\varepsilon(x,t) := \sum_{k=1}^{N_\varepsilon} \phi_k^\varepsilon(t)\varphi_k^\varepsilon(x) \in C_c^\infty((0,T)\times\Omega)$. Then

$$\|f - f_\varepsilon\|_{L^p(0,T;L^q(\Omega))} \le \|f - g_\varepsilon\|_{L^p(0,T;L^q(\Omega))} + \|g_\varepsilon - f_\varepsilon\|_{L^p(0,T;L^q(\Omega))} \le$$

$$\le \varepsilon + \left\| \sum_{k=1}^{N_\varepsilon} \phi_k\big(c_k^\varepsilon - \varphi_k^\varepsilon\big) \right\|_{L^p(0,T;L^q(\Omega))} \le \varepsilon + \sum_{k=1}^{N_\varepsilon} \big\|\phi_k\big(c_k^\varepsilon - \varphi_k^\varepsilon\big)\big\|_{L^p(0,T;L^q(\Omega))}$$

On the other hand

$$\big\|\phi_k\big(c_k^\varepsilon - \varphi_k^\varepsilon\big)\big\|_{L^p(0,T;L^q(\Omega))} = \left(\int_0^T \|\phi_k^\varepsilon(t)(c_k^\varepsilon - \varphi_k^\varepsilon)\|_{L^q(\Omega)}^p \, dt\right)^{\frac{1}{p}} = \left(\int_0^T |\phi_k^\varepsilon(t)|^p \|c_k^\varepsilon - \varphi_k^\varepsilon\|_{L^q(\Omega)}^p \, dt\right)^{\frac{1}{p}} \le$$

$$\le T\left(\sup_{(0,T)} |\phi_k^\varepsilon|\right)\|c_k^\varepsilon - \varphi_k^\varepsilon\|_{L^q(\Omega)} \le C_\varepsilon T \|c_k^\varepsilon - \varphi_k^\varepsilon\|_{L^q(\Omega)} \le \frac{\varepsilon}{N_\varepsilon}$$

It follows that $\|f - f_\varepsilon\|_{L^p(0,T;L^q(\Omega))} \le 2\varepsilon$, that is the thesis. ∎

5.3 Important functional spaces results

In fluid dynamics the temporal variable t plays a very important role. It is very common that the regularity of a certain function, or field, is different with respect to the temporal variable and the other variables.

We start with a basic proposition, inspired by [6, p. 248].

Proposition 5.2. *Let X a Banach space. If X is separable and $p \in [1, +\infty)$, then $L^p(0,T;X)$ is separable. Moreover if $p \in [1,+\infty)$ and p' is its cojugate, then*

$$(L^p(0,T;X))^* = L^{p'}(0,T;X^*)$$

The dual pairing is explicetely given by

$$\langle u, v\rangle_{L^{p'}(0,T;X'),L^p(0,T;X)} = \int_0^T \langle u(t), v(t)\rangle_{X',X} dt \quad \forall u \in L^{p'}(0,T;X'),\ v \in L^p(0,T;X)$$

In particular, if $p = 2$ and $(X, \langle,\rangle_X)$ is an Hilbert space, then $L^2(0,T;X)$ is an Hilbert space with scalar product given by

$$\langle u, v\rangle_{L^2(0,T;X)} := \int_0^T \langle u, v\rangle_X dt \quad \forall u, v \in L^2(0,T;X)$$

If $p \in (1,+\infty)$ and X is reflexive, then $L^p(0,T;X)$ is reflexive.

Remark 5.5. The proof follows from the real case $X = \mathbb{R}$. □

Moreover, the inclusion in L^p spaces involving times is the following.

Lemma 5.1. *Let $X \subseteq Y$ continuously embedded. Then also*

$$L^p(0,T;X) \subseteq L^p(0,T;Y)$$

is a continuous embedding.

Proof. Thanks to the continuity of the injection $X \to Y$, we have that exists $C > 0$ such that $\|x\|_Y \leq C\|x\|_X$ for every $x \in X$. It follows that if $f \in L^p(0,T;X)$, we have

$$\|f\|_{L^p(0,T;Y)} := \left(\int_0^T \|f\|_Y^p \, dt \right)^{\frac{1}{p}} \leq C\left(\int_0^T \|f\|_X^p \, dt \right)^{\frac{1}{p}} =: C\|f\|_{L^p(0,T;X)}$$

that is the cointinuity of the embedding. ∎

We focus now our attention to the Banach spaces X that are spaces of functions. In particular Sobolev spaces (including L^p spaces). So a function u in $L^p(0,T;X)$, with $X = W^{k,p}(\Omega)$, is a function in X at every time t, i.e.

$$u : [0,T] \to W^{k,p}(\Omega)$$

In this way, we can look at u as a function

$$u = u(t)(x)$$

More precisely, for those pairs (x,t) where u is defined, we can write

$$u = u(x,t)$$

We have now a question: since in particular $u(t) \in L^p(\Omega)$, can we think to u as a function in $L^2(\Omega \times (0,T))$, that is a function defined almost everywhere in the *4D* set $\Omega \times (0,T)$? The answer is "yes" in a very special case. It is explained by the following proposition.

Proposition 5.3. *Let $p \in [1,\infty)$. Let Ω a bounded domain and $T > 0$. Then there exists an identification*

$$L^p(0,T;L^p(\Omega)) \simeq L^p(\Omega \times (0,T))$$

Proof. An inclusion is easy. In fact, if $u(x,t) \in L^p(\Omega \times (0,T))$, we can consider, for almost every $t_0 \in (0,T)$, $u(t_0)(x) := u(x,t_0) \in L^p(\Omega)$. Moreover, since in particular $|u|^p \in L^1(\Omega \times (0,T))$, by the Fubini theorem we have

$$\int_{\Omega \times (0,T)} |u(x,t)|^p \, d(x,t) = \int_0^T \left(\int_\Omega |u(x,t)|^p \, dx \right) dt$$

Conversely, let $u \in L^p(0,T;L^p(\Omega))$.

Remark 5.6. The interval $[0,T)$ can be subdivided into a finite number of sub-intervals of constant lenght $\frac{1}{n}$. In particular, for every $t \in [0,T)$ and $n \in \mathbb{N}$, exists a unique $j = j(n,t)$ such that $t \in [\frac{j}{n}, \frac{j+1}{n})$. So, by Lebesgue theorem, for almost every $t \in [0,T]$, if $f \in L^p_{loc}(0,T)$,

$$n \int_{\frac{j(t,n)}{n}}^{\frac{j(t,n)+1}{n}} f(s)\, ds \to f(t)$$

This will be useful in a moment. $\square$

We now consider the sequence

$$L^p(\Omega) \ni u_n(t,\cdot) := n \int_{\frac{j}{n}}^{\frac{j+1}{n}} u(s,\cdot)\, ds \qquad \text{if } t \in [0,T] \cap [\frac{j}{n}, \frac{j+1}{n})$$

Then $u_n \in L^p(\Omega \times [0,T]) \subseteq L^p(0,T; L^p(\Omega))$. If we show that

$$\lim_{n\to\infty} \int_0^T \left(\int_\Omega |u_n - u|^p\, dx \right) dt = 0 \tag{5.1}$$

then

$$\lim_{m,n\to\infty} \int_0^T \left(\int_\Omega |u_n - u_m|^p\, dx \right) dt = 0 \tag{5.2}$$

In other words, $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega \times [0,T])$. By the completeness of the space $L^p(\Omega \times [0,T])$, we have that exists $\overline{u} \in L^p(\Omega \times [0,T])$ such that

$$u_n \to \overline{u} \quad \text{in } L^p(\Omega \times [0,T])$$

Moreover, (5.1) and (5.2) imply that $\|\overline{u}\|_{L^p(\Omega\times[0,T])} = \|u\|_{L^p(0,T;L^p(\Omega))}$.

So, we have only to show that (5.1) holds. First of all, observe that

$$\|u_n(t,\cdot) - u(t,\cdot)\|_p = \left\| n \int_{\frac{j}{n}}^{\frac{j+1}{n}} \big(u(s,x) - u(t,x)\big)\, ds \right\|_p \leq n \int_{\frac{j}{n}}^{\frac{j+1}{n}} \|u(s,\cdot) - u(t,\cdot)\|_p\, ds \tag{5.3}$$

This estimate, together with remark 5.6, implies that $u_n(t) \to u(t)$ in $L^p(\Omega)$ for almost every $t \in [0,T]$. So, the thesis follows. ∎

5.3.1 The case of $L^p(0,T; L^q(\Omega))$

Let $p,q \geq 1$ and Ω bounded. When $X = L^q(\Omega)$, we have, as said above, a function of two variables. Moreover, by the definition of Bochner integral we know that the integral of a function that takes values in X is an element of X. The definiton of summability in this case leads to

$$\lim_{k\to+\infty} \|s_k(t) - u(t)\| = 0 \quad \text{for a.e. } t \in [a,b]$$

Clearly, at almost every $t \in (a,b)$, is defined

$$f(t) := \int_\Omega u(x,t) dx$$

since $u(t) \in L^q(\Omega) \subseteq L^1(\Omega)$. But is the function $f(t)$ measurable over (a, b)? Observe that

$$\left|\int_\Omega u(x,t)dx - \int_\Omega s_k(x,t)dx\right| \leq \int_\Omega |u(x,t)-s_k(x,t)|dx \leq |\Omega|^{\frac{1}{q'}} \|u(t)-s_k(t)\|_q \to 0 \quad \text{as } k \to +\infty$$

So for almost every $t \in (a, b)$ we have that $c_k(t) := \displaystyle\int_\Omega s_k(x,t)dx$ is a simple function

$$c_k(t) = \int_\Omega s_k(x,t)dx = \sum_{i=1}^{m_k} \int_\Omega d_i^k(x)\chi_{E_i^k}(t)dx = \sum_{i=1}^{m_k} d_{k,i}\chi_{E_i^k}(t)$$

where $d_i^k(x) \in L^q(\Omega)$ e $d_{k,i} := \displaystyle\int_\Omega d_i^k(x)dx$.
So f is the pointwise limit of simple functions c_k, that are measurable. So, f is measurable. This means that we can consider the integral of the absolute value (also the absolute value is measurable), i.e.

$$\int_a^b |f(t)|\ dt = \int_a^b \left|\int_\Omega u(x,t)\ dx\right|\ dt$$

Since, at almost every fixed $t \in (a, b)$, $u(x,t) \in L^q(\Omega)$, we have that $\left|\displaystyle\int_\Omega u(x,t)\ dx\right| \leq \displaystyle\int_\Omega |u(x,t)|\ dx$. It follows that, since also[6] $|u(x,t)| \in L^q(\Omega)$,

$$\int_a^b |f(t)|dt \leq \int_a^b \int_\Omega |u(x,t)|dx \leq |\Omega|^{\frac{1}{q'}} \int_a^b \|u(t)\|_q dt \leq (b-a)^{\frac{1}{p'}} |\Omega|^{\frac{1}{q'}} \left(\int_a^b \|u(t)\|_q^p dt\right)^{\frac{1}{p}} < +\infty$$

So we can consider the integral with sign

$$\int_a^b f(t)dt = \int_a^b \int_\Omega u(x,t)\ dx\ dt$$

On the other hand, since we have, by definition, $\displaystyle\int_a^b u(x,t)dt \in L^q(\Omega)$, we can consider

$$\int_\Omega \int_a^b u(x,t)\ dt\ dx$$

and

$$\left|\int_\Omega \int_a^b u(x,t)\ dt\ dx\right| \leq \int_\Omega \left|\int_a^b u(x,t)\ dt\right|\ dx = \left\|\int_a^b u(t)\ dt\right\|_1 \leq$$
$$\leq \int_a^b \|u(t)\|_1\ dt = \int_a^b \left(\int_\Omega |u(x,t)|\ dx\right)\ dt$$

since $u : [0, T] \to L^q(\Omega) \subseteq L^1(\Omega)$.

---

[6] And

$$|\int_\Omega |u(x,t)|dx - \int_\Omega |s_k(x,t)|dx| \leq \int_\Omega ||u(x,t)| - |s_k(x,t)||dx \leq \int_\Omega |u(x,t) - s_k(x,t)|dx$$

so that the measurability holds also for $\displaystyle\int_\Omega |u(x,t)|dx$.

An integral interchange. We may ask if in this case is possible to interchange the integrals. Again, the definition of summability leads to

$$\lim_{k\to+\infty} \|s_k(t)-u(t)\|_q = 0 \quad \text{for a.e. } t\in[a,b]$$

As above, we have that

$$\lim_{k\to\infty} \int_\Omega s_k(x,t)dx = \int_\Omega u(x,t)dx$$

for almost every $t\in(a,b)$. As in proposition 5.1, we can redefine the sequence so that $\|s_k(t)\|_q \le \|u(t)\|_q$. In particular, we have

$$\boxed{\lim_{k\to+\infty} \int_a^b \Big(\int_\Omega s_k(x,t)dx\Big)dt = \int_a^b \Big(\int_\Omega u(x,t)dx\Big)dt}$$

since

$$\left|\int_\Omega s_k(x,t)\ dx\right| \le \|s_k(t)\|_1 \le |\Omega|^{1-\frac{1}{q}}\|s_k(t)\|_q \le |\Omega|^{1-\frac{1}{q}}\|u(t)\|_q \in L^1(a,b)$$

we can use the Lebesgue dominated convergence. On the other hand we have that

$$\int_a^b \Big(\int_\Omega s_k(x,t)dx\Big)dt = \int_a^b \Big(\int_\Omega \sum_{i=1}^{m_k} d_i^k(x)\chi_{E_i^k}(t)dx\Big)dt =$$

$$= \int_\Omega \Big(\int_a^b \sum_{i=1}^{m_k} d_i^k(x)\chi_{E_i^k}(t)dt\Big)dx = \int_\Omega \Big(\int_a^b s_k(x,t)dt\Big)dx$$

and

$$\left|\int_\Omega \Big(\int_a^b \big(s_k(x,t)-u(x,t)\big)\ dt\Big)\ dx\right| \le \left\|\int_a^b \big(s_k(x,t)-u(x,t)\big)\ dt\right\|_1 \le$$

$$\le |\Omega|^{1-\frac{1}{q}} \left\|\int_a^b \big(s_k(x,t)-u(x,t)\big)\ dt\right\|_q \le |\Omega|^{1-\frac{1}{q}} \int_a^b \|s_k(x,t)-u(x,t)\|_q\ dt \to 0$$

as $k\to\infty$, thanks to the definition 5.4. By the uniqueness of the limit, we have that

$$\boxed{\int_a^b \Big(\int_\Omega u(x,t)dx\Big)dt = \int_\Omega \Big(\int_a^b u(x,t)dt\Big)dx} \tag{5.4}$$

The case $p=q=2$. We know by the Proposition 5.3 that $L^2(0,T;L^2(\Omega)) \simeq L^2((0,T)\times\Omega)$. So in this case we have that

$$u\in L^2((0,T)\times\Omega) \implies u\in L^1((0,T)\times\Omega)$$

since the measure of $(0,T)\times\Omega$ is finite. So, by the Fubini theorem, we have[7]

$$\int_\Omega\int_a^b u(x,t)\ dt\ dx = \int_a^b\int_\Omega u(x,t)\ dx\ dt$$

[7]And since $u\in L^1((0,T)\times\Omega)$ means $|u|\in L^1((0,T)\times\Omega)$ we also have the formula

$$\int_\Omega\int_a^b |u(x,t)|\ dt\ dx = \int_a^b\int_\Omega |u(x,t)|\ dx\ dt$$

Product of functions. It is very usefull to consider $f \in L^{p_1}(0,T;L^p)$ and $g \in L^{q_1}(0,T;L^q)$, where q is the conjugate of p, and q_1 the conjugate of p_1. Since f and g are measurable, also $f \cdot g$ is measurable: in fact, if $s_k(t)$ and $v_j(t)$ are such that

$$\lim_{k\to+\infty} \|s_k(t) - f(t)\|_p = 0 \quad \text{a.e. } t, \qquad \lim_{j\to+\infty} \|v_j(t) - g(t)\|_q = 0 \quad \text{a.e. } t$$

it follows that

$$\lim_{k\to+\infty} \|s_k(t)\cdot v_k(t) - f(t)\cdot g(t)\|_1 \le \lim_{k\to+\infty} \|s_k(t)\cdot(v_k(t)-g(t))\|_1 + \|(s_k(t)-f(t))\cdot g(t)\|_1 = 0$$

and is a simple function in that takes values in $L^1(\Omega)$. The summability follows from an analogous calculation. The same argument holds for a greater number of functions, provided that these functions are in the right class in order to use the Hölder inequality.

Theorem 5.2. *Let $(H, \|\cdot\|)$ an Hilbert space. Then $L^2(0,T_*)\otimes H$ is dense in $L^2(0,T_*;H)$, where*

$$L^2(0,T_*) \otimes H := \{g = g(t) \in H : \ g(t) = \sum_{j=1}^{M} f_j(t)h_j, \ f_j \in L^2(0,T_*), \ h_j \in H\}$$

Remark 5.7. Here the symbol $\otimes$ is inappropriately used; it usually means *tensor product*, that is a more complex algebraic structure. However, the density above holds.

Proof. Let $f \in L^2(0,T_*;H)$. Then there exists a sequence of simple functions, say

$$s_k(t) := \sum_{i=1}^{m_k} \chi_{E_i}(t)h_i$$

with E_i measurable subset of $[0,T_*]$ and $h_i \in H$, such that [see section 5.2]

$$\lim_{k\to+\infty} \int_0^{T_*} \|s_k(t) - f(t)\|^2 dt = 0$$

Since $s_k(t) \in L^2(0,T_*) \otimes H$, we have in other words that, for every $\varepsilon > 0$, exists K such that

$$\|s_K - f\|_{L^2(0,T_*;H)} < \varepsilon$$

that is the required density. ∎

5.4 Sobolev spaces involving time

The basic results of this sections are inspired by the fundamental Evans' work [10].

Definition 5.6. Let X be a Banach space. The Sobolev space $W^{1,p}(0,T;X)$ consists of all the functions $u \in L^p(0,T;X)$ such that u' exists in the weak sense and belongs to $L^p(0,T;X)$. Furthermore, we set

$$\|u\|_{W^{1,p}(0,T;X)} := \begin{cases} \left(\int_0^T \|u(t)\|^p + \|u'(t)\|^p \, dt \right)^{\frac{1}{p}} & 1 \le p < \infty \\ \sup\limits_{0\le t\le T} \left(\|u(t)\| + \|u'(t)\|\right) & p = \infty \end{cases}$$

We have the following theorem.

Theorem 5.3. *Let* $u \in W^{1,p}(0,T;X)$ *for some* $p \in [1,\infty]$. *Then* $u \in C([0,T];X)$ *(after possibly being redefined on a set of measure zero). Moreover it holds*

$$u(t) = u(s) + \int_s^t u'(\tau)\ d\tau \qquad \forall 0 \le s \le t \le T$$

Furthermore, we have the estimate

$$\max_{t\in[0,T]} \|u(t)\| \le C\|u\|_{W^{1,p}(0,T;X)}$$

where C *only depends on* T.

Chapter 6

Helmholtz decomposition in L^2 spaces

Given a measure space $(\Omega, \mathcal{M}, \mu)$, the space $L^2(\mu) \equiv L^2(\Omega) \equiv L^2$ stands out other L^p spaces because of it is equipped of an inner product. Remember in fact that

$$L^2(\Omega) := \{f : \Omega \to \mathbb{R} \text{ measurable functions such that } \int_\Omega |f|^2 \, d\mu < +\infty\}$$

So, we can introduce

$$\langle f, g \rangle := \int_\Omega fg \, d\mu$$

which is well-posed because of Hölder inequality.

Definitions above can be generalized to n-dimensional vectorial field simply replacing absolute value with Euclidian norm and defining the integral component by component. The product between f and g become the Euclidian inner product. With these devices, we can continue our speech.
We define

$$C^\infty_{0,\sigma}(\Omega) := \{f \in C^\infty_c(\Omega)^n : \ \nabla \cdot f = 0\}$$

where the superscript n remembers us, at least in this chapter, that the functions take vectorial values in $\mathbb{R}^n$.

Definition 6.1. We define the closed space

$$L^2_\sigma(\Omega) := \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{L^2}}$$

that can be equipped with the standard inner product in $L^2(\Omega)$.

Remark 6.1. Remember that

$$L^2_{loc}(\Omega) := \{p : \Omega \to \mathbb{R} : \int_K |p|^2 \, d\mu < +\infty \quad \forall \, K \subseteq \Omega \text{ compact subset}\}$$

is the classical local L^2 space over Ω. □

Definition 6.2. Let Ω a domain in $\mathbb{R}^n$, $n \geq 2$. We define

$$G(\Omega) := \{f \in L^2(\Omega)^n : \; \exists p \in L^2_{loc}(\Omega) \text{ such that } f \overset{d}{=} \nabla p\}$$

where, by definition,

$$f \overset{d}{=} \nabla p \overset{def}{\iff} \int_\Omega f \cdot \varphi \; dx = -\int_\Omega p \; \nabla \cdot \varphi \; dx \quad \forall \; \varphi \in C_c^\infty(\Omega)^n$$

Remark 6.2. Note that both members make sense, the first because integral of the product of L^2 functions, the latter because $\nabla \cdot \varphi$ vanishes out of a compact, so we can use Hölder inequality inequality thanks to the fact that $p \in L^2_{loc}(\Omega)$ (and also $\nabla \cdot \varphi$ because of its regularity). $\square$

Remark 6.3. Similarly, for $\mathcal{F} \in W^{-1,q}_{loc}(\Omega)^n$ we say

$$\mathcal{F} \overset{d}{=} p \overset{def}{\iff} \langle \mathcal{F}, \varphi \rangle = -\int_\Omega p \; \nabla \cdot \varphi \; dx \quad \forall \; \varphi \in C_c^\infty(\Omega)^n$$

where $\langle \cdot, \cdot \rangle$ is the dual pairing for functionals in $W^{-1,q}_{loc}(\Omega)^n$. $\square$

6.0.1 Preliminary lemmas

The aim of this section is to prove the Helmholtz's theorem 6.1. We have first to prove some lemmas.

Lemma 6.1. *Let $\Omega \subseteq \mathbb{R}^n$ with $n \geq 2$ and let $\Omega_0 \neq \emptyset$ a bounded subdomain of Ω such that $\overline{\Omega}_0 \subseteq \Omega$. Let $f \in W^{-1,2}_{loc}(\Omega)^n$ such that*

$$f(v) = 0 \qquad \forall v \in C^\infty_{0,\sigma}(\Omega)^n$$

Then there exists a unique weak potential $p \in L^2_{loc}(\Omega)$ such that $f \overset{d}{=} \nabla p$ and

$$\int_{\Omega_0} p \; dx = 0$$

Remark 6.4. Let $q \in (1, \infty)$ and q' its conjugate. Then, every $f \in L^q(\Omega)$ induces an element in $W^{-1,q}(\Omega)^n$. In fact, consider

$$\langle \mathcal{F}_f, \varphi \rangle := \int_\Omega f \cdot \varphi \; dx$$

In fact, linearity is obvious, while

$$|\langle \mathcal{F}_f, \varphi \rangle| \leq \int_\Omega |f||\varphi| dx \leq \|f\|_q \|\varphi\|_{q'}$$

So, the inequality says that $\mathcal{F}_f$ is a well-posed operator and it is continuous, since if $\varphi_k \to \varphi$ in $W^{1,q'}_0(\Omega)^n$, then in particular $\|\varphi_k - \varphi\|_{q'} \to 0$. So $\mathcal{F}_f \in W^{-1,q}(\Omega)^n$. $\square$

Proof. Using advanced functional analysis, the proof would follows easily, using the Banach closed range theorem. However, in order to follow this way, one would prove the surjectivity of the divergence operator and theorems about unbounded operator, that will distract us from our aim. So, we follow the proof of [27, Lemma 2.2.1, pg. 73], that is less elegant but effective. Thus, we start with the proof.

The idea is to use the classical method to construct potential strarting from the work of a fixed force. First of all, keep in mind lemma 1.2. Fix $\overline{\Omega}_0 \subset \Omega$, and choose a bounded and lipschitz domain Ω_1 such that $\overline{\Omega}_0 \subseteq \Omega_1 \subseteq \overline{\Omega}_1 \subseteq \Omega$. We want to show that exists a unique $p \in L^q(\Omega)$ such that $\nabla p = f$ in distributional sense over Ω_1 and $\displaystyle\int_{\Omega_0} p\, dx = 0$. We proceed as follows. Consider a further domain such that $\overline{\Omega}_1 \subseteq \Omega_2 \subseteq \overline{\Omega}_2 \subseteq \Omega$.

Remark 6.5. Let $f \in W^{-1,2}(\Omega_2)^n$ be a functional, with Ω_2 bounded domain. Consider

$$D := \{\nabla v \in L^2(\Omega_2)^{n^2} :\ v \in W_0^{1,2}(\Omega_2)\} \subseteq L^2(\Omega_2)^{n^2}$$

Then we can define the functional $\tilde{f} : D \longrightarrow \mathbb{R}$ such that $\nabla v \mapsto \tilde{f}(\nabla v) := f(v)$. Then

$$|\tilde{f}(\nabla v)| = |f(v)| \leq \|f\|_{-1,2}\|v\|_{1,2} \leq C\|f\|_{-1,2}\|\nabla v\|_2$$

since $\|v\|_{1,2}^2 = \|v\|_2^2 + \|\nabla v\|_2^2 \leq (C^2+1)\|\nabla v\|_2^2$, using Sobolev estimates over the bounded domain Ω_2. So, the functional is continuous over $D \subseteq L^2(\Omega_2)^{n^2}$, if this set is equipped with the $\|\cdot\|_2$ norm. So, by the Hahn-Banach theorem, the functional $\tilde{f}$ can be extended to the whole $L^2(\Omega_2)^{n^2}$ with the same operator norm. In particular, exists $F \in L^2(\Omega_2)^{n^2}$ such that

$$\int_{\Omega_2} F \cdot \nabla v\, dx = f(v)$$

but F represent the whole element of the dual $\left(L^2(\Omega_2)^{n^2}\right)^*$. If $v \in C_c^\infty(\Omega_2)$, then $\nabla \cdot (-F) = f$ in the distributional sense. At this point we can redefine $F \longleftrightarrow -F$. □

So, roughly speaking, every functional of $W^{-1,2}(\Omega_2)$ on a bounded domain can be written as the divergence of a matricial functional in $L^2(\Omega_2)$.

At this point we define F^ε as the mollification of $F \in L^2(\Omega_2)^{n^2}$, where $0 < \varepsilon < \operatorname{dist}(\partial\Omega_2, \Omega_1)$. In particular $F^\varepsilon \in C^\infty(\overline{\Omega}_1)^{n^2}$. We want to prove that exists $U_\varepsilon \in C^\infty(\overline{\Omega}_1)$ such that

$$\nabla \cdot F^\varepsilon = \nabla U_\varepsilon \qquad \text{in } \overline{\Omega}_1$$

To do this, we use the following remark.

Remark 6.6. Let $w : \tau \mapsto w(\tau)$, with $\tau \in [0,1]$, a continuous function from $[0,1]$ to $\overline{\Omega}_1$. Suppose that w' exists on $[0,1]$ piecewise continuous. We say that w is the *parametrization of a curve*. It is a closed curve if $w(0) = w(1)$. Moreover, if $g \in C^\infty(\overline{\Omega}_1)^n$, we define the *work* of g along w as

$$\int_0^1 g(w(\tau)) \cdot w'(\tau)\, d\tau := \int_0^1 \left(\sum_{j=1}^n g_j(w(\tau))w_j'(\tau)\right) d\tau$$

It is a well known result of classical analysis that if for every closed curve w

$$\int_0^1 g(w)\cdot w'\, d\tau = 0$$

then exsits $u \in C^\infty(\overline{\Omega}_1)$ such that $g = \nabla U$. The scalar function U is called *scalar potential of the force g*. □

If we show that for every closed curve w

$$\int_0^1 (\nabla\cdot F^\varepsilon)\,(w(\tau))\cdot w'(\tau)\, d\tau = 0$$

then remark 6.6 provides us a potential U_ε over $\overline{\Omega}_1$. To show this, we consider the function

$$V_{w,\varepsilon}(x) := \int_0^1 \eta_\varepsilon(x - w(\tau))w'(\tau)\, d\tau$$

for every $x \in \Omega_2$, where η_ε is the usual mollificator. Clearly, $V_{w,\varepsilon} \in C_c^\infty(\Omega_2)^n$, if $w([0,1]) \subseteq \overline{\Omega}_1$, with w fixed[1]. Moreover,

$$\big(\nabla\cdot V_{w,\varepsilon}\big)(x) = \int_0^1 \sum_{j=1}^n \big(D_j\eta_\varepsilon\big)(x - w(\tau))w_j'(\tau)\, d\tau = -\int_0^1 \frac{d}{d\tau}\eta_\varepsilon(x - w(\tau))\, d\tau =$$

$$= \eta_\varepsilon(x - w(0)) - \eta_\varepsilon(x - w(1)) = 0$$

since $w(0) = w(1)$, being the curve closed. So $V_{w,\varepsilon} \in C_{0,\sigma}^\infty(\Omega_2)$. Then, by the hypothesis

$$0 = f(V_{w,\varepsilon}) = \int_{\Omega_2} F\cdot\nabla V_{w,\varepsilon}\, dx =$$

$$= \sum_{j,l=1}^n \int_{\Omega_2} F_{jl}(x)\bigg(\int_0^1 D_j\eta_\varepsilon(x - w(\tau))w_l'(\tau)\, d\tau\bigg)\, dx =$$

and so, since η_ε is smooth on the domain, w' is piecewise continuous and $F \in L^2(\Omega_2)^{n^2}$, by Fubini theorem we have

$$= \sum_{j,l=1}^n \int_0^1 w_l'(\tau)\bigg(\int_{\Omega_2} D_j\eta_\varepsilon(x - w(\tau))F_{jl}(x)\, dx\bigg)\, d\tau = -\sum_{j,l=1}^n \int_0^1 w_l'(\tau)D_j(F_{jl}^\varepsilon)(w(\tau))\, d\tau =$$

$$= -\int_0^1 (\nabla\cdot F^\varepsilon)(w(\tau))\cdot w'(\tau)\, d\tau$$

---

[1] In fact, we have choosen $\varepsilon < \mathrm{dist}(\partial\Omega_2, \Omega_1)$, so that $\varepsilon \equiv \mathrm{dist}(\partial\Omega_2, \Omega_1) - \delta$, with $\delta < \mathrm{dist}(\partial\Omega_2, \Omega_1)$. So, if x is such that $\mathrm{dist}(x, \partial\Omega_2) < \delta$, we can consider $y \in B(x, \varepsilon)$. Then $y \notin \overline{\Omega}_1$. In fact,

$$\mathrm{dist}(y, \partial\Omega_2) \le |y - x| + \mathrm{dist}(x, \partial\Omega_2) < \varepsilon + \delta \equiv \mathrm{dist}(\Omega_1, \partial\Omega_2) = \mathrm{dist}(\partial\Omega_1, \partial\Omega_2)$$

So, if $y \in \overline{\Omega}_1$, then $|x - y| \ge \varepsilon$. Then, if $w(\tau) \in \overline{\Omega}_1$ for every $\tau \in [0,1]$, we have $|x - w(\tau)| \ge \varepsilon$, and so $\eta_\varepsilon(x - w(\tau)) = 0$, if $\mathrm{dist}(x, \partial\Omega_2) < \delta$. So $V_{\varepsilon,w}$ is compactly supported.

It follows that, for every w closed curve contained in $\overline{\Omega}_1$,

$$\int_0^1 (\nabla \cdot F^\varepsilon)(w(\tau)) \cdot w'(\tau)\ d\tau = 0$$

that is what we wanted. So, exists a potential $U_\varepsilon \in C^\infty(\overline{\Omega}_1)$ such that $\nabla \cdot F^\varepsilon = \nabla U_\varepsilon$ over $\overline{\Omega}_1$, determined up to a constant. In particular, we can choose a constant $c_\varepsilon(\Omega_0)$ such that

$$\int_{\Omega_0} U_\varepsilon\ dx = 0$$

We now have to deduce some estimates. Using theorem 4.12, we have

$$\|U_\varepsilon\|_{L^2(\Omega_1)} \leq C\|\nabla U_\varepsilon\|_{W^{-1,2}(\Omega_1)} \equiv C \sup_{0\neq v\in C_c^\infty(\Omega_1)} \frac{|\langle \nabla U_\varepsilon, v\rangle_{2,2}|}{\|v\|_{W^{1,2}(\Omega_1)}} = C \sup_{0\neq v\in C_c^\infty(\Omega_1)} \frac{|\langle F_\varepsilon, \nabla v\rangle_{2,2}|}{\|v\|_{W^{1,2}(\Omega_1)}}$$

since, being $v \in C_c^\infty(\Omega_1)$,

$$\langle \nabla U_\varepsilon, v\rangle_2 = \langle \nabla \cdot F_\varepsilon, v\rangle_2 = \langle F_\varepsilon, \nabla v\rangle_2$$

Being moreover $\|v\|_{W^{1,2}(\Omega_1)} \geq \|\nabla v\|_{L^2(\Omega_1)}$, we have

$$\|U_\varepsilon\|_{L^2(\Omega_1)} \leq C\|F^\varepsilon\|_{L^2(\Omega_1)} \tag{6.1}$$

Here C is independent of ε. By the properties of mollifications, see theorem 3.3, we have

$$\lim_{\varepsilon\to 0} \|F - F^\varepsilon\|_{L^2(\Omega_1)} = 0$$

Replacing U_ε with $U_\varepsilon - U_\eta$ in (6.1), we have

$$\|U_\varepsilon - U_\eta\|_{L^2(\Omega_1)} \leq C\|F^\varepsilon - F^\eta\|_{L^q(\Omega_1)} \to 0$$

as $\varepsilon, \eta \to 0$. So, by completeness of $L^2(\Omega_1)$ we have that exists $U \in L^2(\Omega_1)$ such that

$$\lim_{\varepsilon\to 0} \|U - U_\varepsilon\|_{L^2(\Omega_1)} = 0$$

Furthermore

$$\int_{\Omega_0} U_\varepsilon\ dx = 0, \quad \overline{\Omega}_0 \subseteq \Omega_1 \Longrightarrow \int_{\Omega_0} U\ dx = 0$$

So, we have defined locally on Ω_1 a potential pressure

$$p\big|_{\Omega_1} := U$$

such that $\nabla p\big|_{\Omega_1} = \nabla U = \nabla \cdot F = f$ in the weak sense. Moreover $\displaystyle\int_{\Omega_0} p\big|_{\Omega_1}\ dx = 0$.

Let now Ω_1' be another with the same properties of Ω_1. We have that, in the intersection of the domains, $p\big|_{\Omega_1}$ and $p\big|_{\Omega_1'}$ have the same gradient. So $p\big|_{\Omega_1} - p\big|_{\Omega_1'} = c$ over $\Omega_1 \cap \Omega_1'$. Moreover, $\Omega_0 \subseteq \Omega_1 \cap \Omega_1'$ and so

$$0 = \int_{\Omega_0} \left(p\big|_{\Omega_1} - p\big|_{\Omega_1'}\right)\ dx = c|\Omega_0|$$

that is $c = 0$. So, the local pressure $p\big|_{\Omega_1}$ is well-posed with respect to the change of the local domain. Using the decomposition of the domain provided by lemma 1.2, we have that p can be defined over the whole Ω, and so we obtain a pressure[2] $p \in L^2_{loc}(\Omega)$ with the required properties. ∎

6.1 Helmholtz decomposition theorem

There is now this fondamental theorem.

Theorem 6.1. *Let $\Omega \subseteq \mathbb{R}^n$ an open subset, with $n \geq 2$. Then*

- *It holds*
$$G(\Omega) = \{f \in L^2(\Omega)^n : \langle f, g\rangle = 0 \ \forall g \in L^2_\sigma(\Omega)\}$$

- *For all $f \in L^2(\Omega)^n$ there exist unique $f_0 \in L^2_\sigma(\Omega)$ and $f_1 \in G(\Omega)$ such that*
$$f = f_0 + f_1 \qquad \langle f_0, f_1\rangle = 0$$
where there exists $p \in L^2_{loc}(\Omega)$ such that $f_1 \overset{d}{=} \nabla p$. Consequently
$$\|f\|^2_{L^2} = \|f_0\|^2_{L^2} + \|f_1\|^2_{L^2}$$

- *The operator*
$$P : L^2(\Omega)^n \to L^2_\sigma(\Omega)$$
$$f \to Pf := f_0$$
is well-defined, is linear, bounded with $\|P\| \leq 1$. Moreover, the following properties hold: if $f, g \in L^2(\Omega)^n$ then
  (i) $P(f_1) = 0$;
  (ii) $(I - P)f = f_1$;
  (iii) $P^2 f = Pf$;
  (iv) $(I - P)^2 f = (I - P)f$;
  (v) $\langle Pf, g\rangle = \langle f, Pg\rangle$;
  (vi) $\|f\|^2_{L^2} = \|Pf\|^2_{L^2} + \|(I - P)f\|^2_{L^2}$.

Remark 6.7. Observe that the property $\|P\| \leq 1$ tell us that
$$\|Pf\|_{L^2} \leq \|f\|_{L^2} \quad \forall f \in L^2(\Omega)^n$$
This would be fundamental for future estimates. □

Remark 6.8. Thanks to the theorem, we can write $G(\Omega) = (L^2_\sigma(\Omega))^\perp$. □

Remark 6.9. The property *(v)* says that P is a *self-adjoint operator*. □

[2] Since the sequence of domains is "swarming", every compact in Ω is containded in an element of the sequence.

Proof. First of all, we prove that $G(\Omega) = L^2_\sigma(\Omega)^\perp$. Let $f \in L^2_\sigma(\Omega)^\perp$. In particular, by definition, $f \in L^2(\Omega)^n$. Hence, we can consider the functional, defined in remark 6.4,

$$F_f : v \to F_f(v)$$

for every $v \in W^{1,2}_0(\Omega)^n \subseteq L^2(\Omega)^n$. In particular, if Ω_0 is a bounded subdomain of Ω, we have

$$\left| \int_{\Omega_0} f \cdot v \; dx \right| \leq \|f\|_{L^2(\Omega_0)^n} \|v\|_{L^2(\Omega_0)^n}$$

But, if $v_h, v \in W^{1,2}_0(\Omega)^n$ are such that $\lim_{h\to\infty} \|v_h - v\|_{W^{1,2}_0(\Omega)^n} = 0$, then

$$\|v_h - v\|_{L^2(\Omega_0)^n} \leq \|v_h - v\|_{L^2(\Omega)^n} \leq \|v_h - v\|_{W^{1,2}(\Omega)^n} \equiv \|v_h - v\|_{W^{1,2}_0(\Omega)^n} \to 0 \;\; \text{as } h \to +\infty$$

So, the functional

$$F_f^{\Omega_0}(v) := \int_{\Omega_0} f \cdot v \; dx$$

is a well-defined, linear and continuous operator. It follows that $F_f \in W^{-1,2}_{loc}(\Omega)^n$. Moreover, if $v \in C^\infty_{0,\sigma}(\Omega)^n$,

$$\int_\Omega f \cdot v \; dx = 0$$

because $f \in L^2_\sigma(\Omega)^\perp \equiv \{f \in L^2(\Omega)^n : \; \langle f, g\rangle = 0 \; \forall g \in L^2_\sigma(\Omega)^n\}$ and $v \in C^\infty_{0,\sigma}(\Omega)^n \subseteq L^2_\sigma(\Omega)^n$. But now we are in the hypothesis of lemma 6.1. So, there exists a $p \in L^2_{loc}(\Omega)$ such that $f \overset{d}{=} \nabla p$. This means that $f \in G(\Omega)$.

Viceversa, let $f \in G(\Omega)$. Than there exists a locally L^2 distributional potential, say p. If $v \in C^\infty_{0,\sigma}(\Omega)^n$, we have

$$\int_\Omega f \cdot v \; dx = -\int_\Omega p \; \nabla \cdot v \; dx = 0$$

using the definition of $f \overset{d}{=} \nabla p$. Let now $v \in L^2_\sigma(\Omega)^n$. This means that $v \in L^2(\Omega)^n$ and exists $v_h \in C^\infty_{0,\sigma}(\Omega)$ such that

$$\|v - v_h\|_{L^2(\Omega)^n} \to 0 \quad \text{as } h \to 0$$

Then

$$\int_\Omega |f \cdot (v - v_h)| dx \leq \int_\Omega |f||v - v_h| dx \leq \|f\|_{L^2(\Omega)^n} \|v - v_h\|_{L^2(\Omega)^n}$$

So, being $f, v \in L^2(\Omega)^n$,

$$\left| \int_\Omega f \cdot v \; dx \right| = \left| \int_\Omega f \cdot (v - v_h) \; dx \right| \leq \|f\|_{L^2(\Omega)^n} \|v - v_h\|_{L^2(\Omega)^n} \to 0 \;\; \text{as } h \to +\infty$$

Finally, for every $v \in L^2_\sigma(\Omega)^n$,

$$\int_\Omega f \cdot v \; dx = 0$$

This means that $f \in L^2_\sigma(\Omega)^\perp$. So the equality of sets holds.

We now have to use Hilbert theory to deduce the **existence of the decomposition**. If we show that $L^2_\sigma(\Omega)^n$ is a closed and convex subspace of $L^2(\Omega)^n$, then from Hilbert's spaces main theorem 2.1, we have that for every $f \in L^2(\Omega)^n$ there exist unique $f_0 \in L^2_\sigma(\Omega)$ and $f_1 \in L^2_\sigma(\Omega)^\perp \equiv G(\Omega)$ such that

$$f = f_0 + f_1, \qquad \langle f_0, f_1 \rangle = 0 \tag{6.2}$$

Convexity: let $f, g \in L^2_\sigma(\Omega)^n$. We want to show that $tf + (1-t)g \in L^2_\sigma(\Omega)^n$ for every $t \in [0,1]$. Clearly the linear interpolation $tf + (1-t)g$ is in L^2. Moreover, both f, g are approximed in L^2 by $f_h, g_h \in C_{0,\sigma}(\Omega)^n$. Also[3] $tf_h + (1-t)g_h \in C_{0,\sigma}(\Omega)^n$. Finally

$$\|t(f - f_h) + (1-t)(g - g_h)\|_{L^2} \to 0 \quad \text{as } h \to +\infty$$

So $tf + (1-t)g \in L^2_\sigma(\Omega)$.
Closure: the $L^2(\Omega)^n$ space is equipped with the distance $d_{\|\cdot\|_2}$ induced by the norm $\|\cdot\|_2$. The space $C^\infty_{0,\sigma}(\Omega)$ is a subspace of $L^2(\Omega)^n$. By definition $L^2_\sigma(\Omega)^n$ is the closure of $C^\infty_{0,\sigma}(\Omega)$ in the metric space $(L^2(\Omega)^n, d_{\|\cdot\|_2})$, so it is a closed subset of $L^2(\Omega)^n$.

So, by the theorem about Hilbert spaces mentioned above, we have that it is possible to decompose f in the sum of two orthogonal functions, as in (6.2). So we define

$$Pf := f_0 \qquad \forall\ f \in L^2(\Omega)^n$$

This decomposition immediately tell us that the operator P defined above is well posed. In fact, the decomposition is unique, and the following properties hold.

- **Linearity:** It is an immediate consequence of the uniqueness, since, if $f, g \in L^2(\Omega)^n$ and $a, b \in \mathbb{R}$, then $af_0 + bg_0$ satisfies (6.2).
- **Boundedness:** Observe that

$$\|f\|^2_{L^2(\Omega)^n} = \|f_0 + f_1\|^2_{L^2(\Omega)^n} = \|f_0\|^2_{L^2(\Omega)^n} + \|f_1\|^2_{L^2(\Omega)^n} + 2\langle f_0, f_1\rangle$$
$$= \|f_0\|^2_{L^2(\Omega)^n} + \|f_1\|^2_{L^2(\Omega)^n} \geq \|f_0\|^2_{L^2(\Omega)^n}$$

  and so

$$\|Pf\|_{L^2_\sigma(\Omega)} = \|f_0\|_{L^2(\Omega)^n} \leq \|f\|_{L^2(\Omega)^n}$$

  It follows that the operator P is bounded and $\|P\| \leq 1$.
- **$P(f_1) = 0$:** The unique decomposition of $f_1 \in G(\Omega)$ is $f_1 = 0 + f_1$, so the property follows by uniqueness.

[3] Smoothness and divergence free property are clear. Moreover, if v, w has compact support,

$$\{x \in \Omega :\ (v+w)(x) \neq 0\} \subseteq \{x \in \Omega :\ v(x) \neq 0\} \cup \{x \in \Omega :\ w(x) \neq 0\}$$

It follows that $\mathrm{supp}(v+w) \subseteq \mathrm{supp}(u) \cup \mathrm{supp}(v)$. Because the union of two compacts is a compact and a closed subset of a compact set is a compact set, also $v + w$ has compact support.

- $(I-P)f = f_1$**:** We have that

$$(I-P)f = f - Pf = f_0 + f_1 - f_0 = f_1$$

- $P^2 f = Pf$**:** We have
$$P^2 f = P(Pf) = Pf_0 = f_0$$
since the unique decomposition of f_0 is $f_0 = f_0 + 0$.

- $(I-P)^2 f = (I-P)f$**:** Calculating, we have

$$(I-P)^2 f = \big(I-P\big)((I-P)f) = \big(I-P\big)(f_1) = f_1 - P(f_1) = f_1 = (I-P)f$$

- **Self-adjointness:** We have

$$\langle Pf, g\rangle = \langle f_0, g\rangle = \langle f_0, g_0 + g_1\rangle = \langle f_0, g_0\rangle + \langle f_0, g_1\rangle =$$

since $f_1, g_1 \in G(\Omega)$,

$$= \langle f_0, g_0\rangle + \langle f_1, g_0\rangle = \langle f, g_0\rangle = \langle f, Pg\rangle$$

- L^2 **norm decomposition:** it follows from the fact that $Pf = f_0$ and $(I-P)f = f_1$, together with (6.2).

So, the proof of the theorem is complete. ∎

Chapter 7

Weak and strong compactness of $L^p(0,T;X)$ spaces

We start with some fundamental issues in weak topology theory. In these pages, a space X will always be a Banach space.

Definition 7.1. A sequence $\{u_k\}_k \subseteq X$ converges weakly to $u \in X$, written $u_k \rightharpoonup u$ is

$$\lim_{k\to+\infty} \Lambda u_k = \Lambda u \quad \forall \Lambda \in X^*$$

An important issues is also what means for a space to be compact in another.

Definition 7.2. Let X, Y Banach spaces, with $X \subseteq Y$. We say that X is *compactly embedded* in Y, and we write

$$X \subset\subset Y$$

if

$$\|u\|_Y \leq C\|u\|_X \quad \forall u \in X$$

and for every bounded sequence $\{u_k\}_k \subseteq X$ there exists a subsequence u_{k_h} and $u \in Y$ such that

$$\lim_{j\to+\infty} \|u_{k_h} - u\|_Y = 0$$

Remark 7.1. The inclusion can be substituted with another embedding. In particular, if $j(X) \subseteq Y$, where j is an embedding, the two properties of the definition become $\|u\|_Y \leq C\|j(u)\|_X$ and $\lim_{h\to+\infty} \|j(u_{k_h}) - u\|_Y = 0$.

The following theorem about weak compactness holds for every reflexive Banach space, in particular for Hilbert spaces.

Theorem 7.1. *Let X a reflexive Banach space and consider a bounded sequence $\{u_k\} \subseteq X$. Then there exists a subsequence u_{k_h} and $u \in X$ such that $u_{k_h} \rightharpoonup u$.*
In other words, bounded sequences in a reflexive Banach space are weakly precompact.

Definition 7.3. Let X a Banach space and consider the dual space X^*. A sequence $\{f_n\}_n \subseteq X^*$ in the dual space is said to be *weak-* convergent* in X^* if there exists $f \in X^*$ such that

$$\lim_{n\to+\infty} f_n(x) = f(x) \quad \forall x \in X$$

In this case we write

$$f_n \xrightharpoonup{*} f$$

Definition 7.4. A space X is *weakly compactly embedded* in Y if the embedding of the first in the latter satisfies the two properties above with the weak convergence.

The following propositions summarises some well known facts about compactness, weak compactness and compact embeddings.

Proposition 7.1. *Let U a bounded open set in $\mathbb{R}^n$ with $\partial U \in C^1$. Let $p \in [1, n)$. Then*

$$W^{1,p}(U) \subset\subset L^q(U)$$

for every $q \in [1, p^*)$.

Proposition 7.2. *Let X, Y Banach spaces and $T : X \to Y$ a linear operator. Then*

$$T \text{ is compact } \iff T^* \text{ is compact}$$

Moreover, the equivalence holds with weak compactness.

The following is taken from Evans, [10, p. 466].

Theorem 7.2. *Let $q \in (1, +\infty)$. Let $\{u_h\}$ a bounded sequence in $W^{k,p}(\Omega)$. Then there exists a subsequence $\{u_{h_j}\}$ and $u \in W^{k,p}(\Omega)$ such that $u_{h_j} \xrightarrow{*} u$ in $W^{k,p}(\Omega)$.*
Moreover if $\{u_h\} \subseteq W_0^{k,p}(\Omega)$, then $u \in W_0^{k,p}(\Omega)$.

7.0.1 Further well-known compactness results

By a theorem above, we know that $H^1(U) \subset\subset L^2(U)$, if ∂U is C^1. Without assuming ∂U to be C^1, we have $H_0^1(U) \subset\subset L^2(U)$. The inclusion is the canonical one. So we have the linear inclusion operator

$$j : H_0^1(U) \hookrightarrow L^2(U)$$

$$u \to j(u) = u$$

that is a compact operator, in particular bounded and linear. Remember that

Theorem 7.3. *Let $T : H_1 \to H_2$ a linear bounded operator. So there exists a unique bounded adjoint operator*

$$T^* : H_2 \to H_1$$

such that

$$\langle Th_1, h_2 \rangle_{h_2} = \langle h_1, T^* h_2 \rangle_{h_1} \quad \forall h_1 \in H_1, \ h_2 \in H_2$$

So, the adjoint operator

$$j^* : (L^2(U))^* \hookrightarrow (H_0^1(U))^*$$

exists and it is also compact. Since $L^2(U)$ is reflexive and $H^{-1}(U) := (H_0^1(U))^*$, we have that the embedding[1]

$$j^* : L^2(U) \hookrightarrow H^{-1}(U)$$

is compact. Moreover we have weak compacness of Sobolev spaces. Consider infact a bounded sequence $\{u_k\} \subseteq H_0^2(\Omega)$. Then in particular we have that $\{u_k\} \subseteq H_0^1(\Omega)$. So there esists a subsequence $\{u_{k_j}\}$ and $u \in H_0^1(\Omega)$ such that $u_{k_j} \xrightarrow{*} u$. Moreover $\|u\|_{H^1} \leq \|u\|_{H^2}$ for all the $u \in H_0^2(\Omega)$. So the embedding $H_0^2(\Omega) \hookrightarrow H_0^1(\Omega)$ is weakly compact. So, also the embedding $H^{-1}(\Omega) \hookrightarrow H^{-2}(\Omega)$ is compact. The injectivity is provided from the fact that $H_0^2(\Omega)$ is dense[2] in $H_0^1(\Omega)$.

7.0.2 Covergence in $C([a, b]; X)$

Before inspecting the compactness of L^p spaces involving time, we focus our attention to the following generalization of a real analysis result.

Lemma 7.1. *Let X a Banach space, and let $-\infty < a < b < \infty$. Let $f_n \in C([a, b]; X)$ a sequence such that, for every $t_0 \in [a, b]$ and for every $[a, b] \ni t_n \to t_0$*

$$\lim_{n\to\infty} \|f_n(t_n) - f(t_0)\|_X = 0 \tag{7.1}$$

with $f \in C([a, b]; X)$. Then $f_n \to f$ in $C([a, b]; X)$.

Proof. The thesis can be rewritten as

$$\lim_{n\to\infty} \sup_{t\in[a,b]} \|f_n(t) - f(t)\|_X = 0$$

By contrary, suppose that the thesis does not hold. Then, there exists $\varepsilon > 0$ such that, for every $N \in N$ exists $\overline{n} \geq N$ such that

$$\sup_{t\in[a,b]} \|f_{\overline{n}}(t) - f(t)\|_X > \varepsilon$$

Then, we can find a subsequence n_k such that

$$\sup_{t\in[a,b]} \|f_{n_k}(t) - f(t)\|_X > \varepsilon \qquad \forall k \in \mathbb{N}$$

[1]Also j^* is injective (and this means that the image of $j^*(L^2(U))$ can be seen as a subset of $H^{-1}(U)$). In fact, suppose that $j^*(\overline{u}) = 0$. Then

$$\langle j(v), \overline{u}\rangle_2 = 0 \quad \forall v \in H_0^1,\ u \in L^2$$

Then if $R(j)$ is dense in $L^2(U)$ we have that $\overline{u}$, so that j^* is injective.
But $R(j) = H_0^1(U)$ that is dense in $L^2(U)$ since

$$C_0^\infty(U) \subseteq H_0^1(U) \subseteq L^2(U)$$

and $C_0^\infty(U)$ is dense in $L^2(U)$.

[2]Quickly $C_0^\infty(\Omega) \subseteq H_0^2(\Omega) \subseteq H_0^1(\Omega)$, so taking the closure in $\|\cdot\|_{H^1}$ we have the thesis.
In other words, we can take $\phi_n \in C_0^\infty(\Omega) \subseteq H_0^2(\Omega)$ that approaches a function $u \in H_0^1(\Omega)$ in the norm $\|\cdot\|_{H^1}$, since $H_0^1(\Omega) \equiv \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^1}}$.

This means, in particular, that there exists a real $\bar{t} \in [a, b]$ such that

$$\|f_{n_k}(\bar{t}) - f(\bar{t})\|_X > \varepsilon$$

In order to remark the dependence on n_k, we define $t_{n_k} \equiv \bar{t}$. It follows that, for every $k \in \mathbb{N}$,

$$\|f_{n_k}(t_{n_k}) - f(t_{n_k})\|_X > \varepsilon$$

The sequence $t_{n_k} \in [a, b]$ is bounded, so, there exists a subsequence $t_{n_{k_h}}$ and $t_0 \in [a, b]$ such that

$$\lim_{h \to \infty} t_{n_{k_h}} = t_0$$

So, by the fact that $f \in C([a, b]; X)$, we can find H is such that

$$\|f(t_{n_{k_h}}) - f(t_0)\|_X < \frac{\varepsilon}{2} \qquad \forall h \geq H$$

We have that

$$\|f_{n_{k_h}}(t_{n_{k_h}}) - f(t_0)\| \geq \|f_{n_{k_h}}(t_{n_{k_h}}) - f(t_{n_{k_h}})\| - \|f(t_{n_{k_h}}) - f(t_0)\| > \frac{\varepsilon}{2}$$

for every $h \geq H$. But this is in contradiction with (7.1). Thus, this is the thesis. ∎

7.1 Compactness in Banach spaces involving time $L^p(0, T; X)$

In [26] it is stated the following lemma.

Lemma 7.2. *Let $X \subseteq E \subseteq Y$ Banach spaces, such that the embedding $X \hookrightarrow E$ is compact. Then the embedding*

$$L^2(0, T; X) \cap \{\varphi : \partial_t \varphi \in L^1(0, T; Y)\} \hookrightarrow L^2(0, T; E)$$

is also compact.

Nevertheless, we state and prove the following more general theorem, that is furnished in [25].

Theorem 7.4. *Let $X \subset B \subset Y$ Banach spaces, and suppose that the embedding $X \to B$ is compact, and let $p \in (1, \infty)$. Let $\mathcal{F}$ a family bounded in $L^p(0, T; X)$ and suppose that*

$$\frac{\partial \mathcal{F}}{\partial t} := \{\frac{\partial f}{\partial t} : \ f \in \mathcal{F}\}$$

is bounded in $L^1(0, T; Y)$. Then $\mathcal{F}$ is relatively compact in $L^p(0, T; B)$.

The proof is long and it is exposed in the following subsections.

7.1.1 Relative compactness

We have introduced in definition 1.11 the relative compactness. The following definition is equivalent.

Proposition 7.3. *Let $X \subset Y$ two Banach spaces. Then X is relatively compact in Y if and only if*

$$\forall \varepsilon > 0 \; \exists \; \{x_i : \; i = 1, ..., n\} \subset X : \; \forall x \in X \; \exists \; x_i \text{ such that } \|x - x_i\|_Y \leq \varepsilon \tag{7.2}$$

Proof. Suppose that (7.2) holds. Let x_k a sequence in X. If we show that x_k has a subsequence converging to a point $x \in Y$, then we have the thesis. Let $h \in \mathbb{N} \cup \{0\}$ and, with $\varepsilon = \frac{1}{2^h}$ in (7.2), consider

$$P_k := \{x_1, ..., x_{n_k}\}$$

We start with $h = 0$. There exists $y_0 \in P_0$ such that the ball $B(y_0, 1)$ of Y contains infinite points of $\{x_k\}$, being the sequence infinite. So, we define

$$I_0 := \{k \in \mathbb{N} : \; x_k \in B(y_0, 1)\}$$

Now, if $h = 1$, there exists $y_1 \in P_1$ such that $B(y_1, \frac{1}{2})$ contains infinite points of I_0, being this set infinite. So we can consider

$$I_1 := \{k \in I_0 : \; x_k \in B(y_1, \frac{1}{2})\}$$

Iterating the process, we have that y_h is such that $B(y_h, \frac{1}{2^h})$ contains infinite points of the sequence A_{h-1}. We define

$$I_h := \{k \in I_{h-1} : \; x_k \in B(y_h, \frac{1}{2^h})\}$$

We choose now a sequence k_h, strictly increasing, such that $k_h \in I_h$ for every h.
Then x_{k_h} is a Cauchy sequence. In fact, $k_h \in I_l$ for every $h \geq l$. This means that, if $h, m \geq l$, we have $k_h, k_m \in I_l$, and so

$$\|x_{k_h} - x_{k_m}\| \leq \|x_{k_h} - y_l\| + \|y_l - x_{k_m}\| \leq \frac{1}{2^{l-1}}$$

If $l \to \infty$, we have that x_{k_h} is a Cauchy sequence. So, being Y complete, we have that exists $x \in Y$ such that

$$\lim_{h \to \infty} \|x_{k_h} - x\|_Y = 0$$

This is the thesis.

Conversely, suppose that (7.2) doesn't hold. So, exists $\overline{\varepsilon} > 0$ such that X can not be covered with a finite number of balls (of Y) of radius $\overline{\varepsilon}$. So let $x_1 \in X$ arbitrary. We can find x_2 such that $\|x_1 - x_2\|_Y > \overline{\varepsilon}$ (otherwise $X \subset B(x_1, \overline{\varepsilon})$).
So, given some points $\{x_1, ..., x_k\}$, we can find $x_{k+1} \notin \{x_1, ..., x_k\}$ such that $\|x_h - x_{k+1}\|_Y > \overline{\varepsilon}$ for every $h \in \{1, ..., k\}$ (otherwise for every $x \in X/\{x_1, ..., x_k\}$ it would

exist $\overline{h}(x) \in \{1, ..., k\}$ such that $\|x - x_{\overline{h}(x)}\|_Y \leq \overline{\varepsilon}$ and so $X \subset \cup_{i=1}^k B(x_i, \overline{\varepsilon})$ that is a contradiction.). So we have a sequence $\{x_k\}$ such that

$$\|x_k - x_m\|_Y > \overline{\varepsilon}$$

for every $k \neq m$ (since for sure $k > m$ or $k < m$). So, it is impossible for the sequence x_k to have a Cauchy subsequence. So, x_k can not have a subsequence converging in Y. But x_k is a sequence in $X \subset \overline{X}$ and $\overline{X}$ is compact in Y. This is a contradiction. ∎

7.1.2 Statement of the main theorem

Definition 7.5. Given a function f defined over $[0, T]$ we define, for every $h > 0$,

$$(\tau_h f)(t) := f(t + h) \quad \text{on } [-h, T - h]$$

We first prove the following theorem.

Theorem 7.5. *Let $X \subset B \subset Y$ be Banach spaces, withe the embedding $X \to B$ compact*[3]*. Let $p \in (1, \infty)$. Suppose that*

$$F \textit{ is bounded in } L^p(0, T; X) \tag{7.3}$$

$$\|\tau_h f - f\|_{L^p(0,T-h;Y)} \to 0 \textit{ as } h \to 0, \textit{ uniformly in } f \in F \tag{7.4}$$

Then F is relatively compact in $L^p(0, T; B)$.

Remark 7.2. Theorem 7.5 implies theorem 7.4. In fact, observe that for every $g \in L^1(0, T; Y)$ we have

$$g(t) = \frac{d}{dt} \int_0^t g(s)\, ds$$

So, if we choose $g = \frac{\partial f}{\partial t}$ we get

$$\frac{d}{dt}\left(f(t) - \int_0^t \frac{\partial f}{\partial t}(s)\, ds\right) = 0$$

and so $f - \displaystyle\int_0^t \frac{\partial f}{\partial t}(s)\, ds \equiv c$, that is $f \in C(0, T; Y)$. Moreover

$$f(t + h) - f(t) = \int_t^{t+h} \frac{\partial f}{\partial t}(s)\, ds \quad \forall t \in [0, T - h]$$

Then

$$\|\tau_h f - f\|_{L^p(0,T-h;Y)} = \left\| \int_t^{t+h} \frac{\partial f}{\partial t}(s)\, ds \right\|_{L^p(0,T-h;Y)}$$

Moreover, Young's convolution inequality says that, if $g \in L^1(0, T; Y)$ and $\varphi \in L^p(0, a)$, then

$$\left\| \int_0^a g(t + \lambda)\varphi(\lambda)\, d\lambda \right\|_{L^p(0,T-a;Y)} \leq \|g\|_{L^1(0,T;Y)} \|\varphi\|_{L^p(0,a)}$$

[3]Moreover, $B \subset Y$ will always be considered a continuous embedding.

So, choosing $a = h$, $\varphi \equiv 1$ and $g(s) = \frac{\partial f}{\partial t}(s)$, we have

$$\left\| \int_t^{t+h} \frac{\partial f}{\partial t}(s)\ ds \right\|_{L^p(0,T-h;Y)} \leq h^{\frac{1}{p}} \left\| \frac{\partial f}{\partial t} \right\|_{L^1(0,T;Y)} \leq Ch^{\frac{1}{p}}$$

If $\left\| \frac{\partial f}{\partial t} \right\|_{L^1(0,T;Y)} \leq C$ uniformly in f. So we obtain the second hypothesis of theorem 7.5. ∎

Now we have to prove theorem 7.5. To do this, we need to prove some lemmas.

7.1.3 Lemmas used in the proofs

Lemma 7.3. *Let $X \subset B \subset Y$ be Banach spaces, with the embedding $X \to B$ compact. Then, for every $\eta > 0$, esists N such that*

$$\forall v \in X, \qquad \|v\|_B \leq \eta \|v\|_X + N\|v\|_Y \tag{7.5}$$

Proof. Let $\eta > 0$. Define the set

$$V_n := \{v \in B : \ \|v\|_B < \eta + n\|v\|_Y\}$$

First of all, observe that V_n is open. In fact, if $v_0 \in V_n$, then the function $\|\cdot\|_B - n\|\cdot\|_Y$ is a continuous function[4]. So, by continuity, we can find a neighbourhood of v_0 such that it holds $\|\cdot\|_B - n\|\cdot\|_Y < \eta$.

Moreover, obviously $V_n \subset V_{n+1}$ and[5]$B \subset \bigcup_{n \in \mathbb{N}} V_n$. So, we can consider $S := \{x \in X : \ \|x\|_X = 1\}$. So we have $S \subset B \subset \bigcup_{n \in \mathbb{N}} V_n$. Moreover

$$\overline{S} \subset \overline{X} \subset B$$

where $\overline{X}$ is compact in B, by the hypothesis. Since $\overline{S}$ is closed in the compact $\overline{X}$ of B, also $\overline{S}$ is compact in B. Since $\{V_n\}$ is a cover of B, we have that there exist $n_1, ..., n_m$ such that

$$S \subset \overline{S} \subset V_{n_1} \cup ... \cup V_{n_m} \equiv V_{\max\{n_1,...,n_m\}}$$

Let $N := \max\{n_1, ..., n_m\}$. Then, for every $v \in X$, with $\|v\|_X = 1$, we have

$$\|v\|_B < \eta\|v\|_X + N\|v\|_Y$$

Normalizing $v \in X$ we have the thesis. ∎

Moreover, we have the following lemma.

[4] In fact, we have

$$|(\|v\|_B - n\|v\|_Y) - (\|v_0\|_B - n\|v_0\|_Y)| \leq |\|v\|_B - \|v_0\|_B| + n|\|v\|_Y - \|v_0\|_Y| \leq \|v - v_0\|_B + n\|v - v_0\|_Y \leq$$
$$\leq \|v - v_0\|_B + nC\|v - v_0\|_B$$

since B continuously embeds into Y, that is $\|v\|_Y \leq C\|v\|_B$.

[5] If $v \in B$, then, if $v \neq 0$, we can choose n such that $n\|v\|_Y > \|v\|_B$, so that $v \in V_n$.

Lemma 7.4. *Let $X \subset B \subset Y$ be Banach spaces, with the embedding $X \to B$ compact. Let F bounded in $L^p(0,T;X)$ and relatively compact in $L^p(0,T;Y)$. Then F is relatively compact in $L^p(0,T;B)$.*

Proof. Let $\varepsilon > 0$ and M such that $\|f\|_{L^p(0,T;X)} \leq M$ for every $f \in F$. By the hypothesis, we have F relatively compact in $L^p(0,T;Y)$. So, there exists $\{f_i : \ i = 1, ..., n\} \subset F$ such that

$$\forall f \in F, \ \exists \ f_i \in F : \ \|f_i - f\|_{L^p(0,T;Y)} \leq \varepsilon$$

So, by lemma 7.3, we have that, if $f \in F$, exists $f_i \in F$ such that for every η, with $N = N(\eta)$,

$$\|f - f_i\|_{L^p(0,T;B)} \leq \eta \|f - f_i\|_{L^p(0,T;X)} + N\|f - f_i\|_{L^p(0,T;Y)} \leq C\eta + N\varepsilon$$

where $C := 2M$.
So, if $\varepsilon' > 0$, and we set $\eta := \frac{\varepsilon'}{2C}$ and $\varepsilon = \frac{\varepsilon'}{2N}$, with $N = N(\eta) = N(\varepsilon')$, we have

$$\|f - f_i\|_{L^p(0,T;B)} \leq \varepsilon'$$

This is the equivalent definition of relatively compact. So F is relatively compact also in $L^p(0,T;B)$. ∎

The following lemma is an important theorem by Ascoli and Arzelà.

Lemma 7.5. *Let Y be a Banach space. A subset F of $C(0,T;Y)$ is relatively compact if and only if*

- *$F(t) = \{f(t) : \ f \in F\}$ is relatively compact in Y, for every $0 < t < T$;*
- *F is uniformly continuous, that is $\forall \varepsilon > 0$, $\exists \eta > 0$ such that*

$$\forall 0 \leq t_1 \leq t_2 \leq T : \ |t_1 - t_2| \leq \eta \implies \|f(t_2) - f(t_1)\|_Y \leq \varepsilon \tag{7.6}$$

7.1.4 Proof of the theorem

We finally prove theorem 7.5. If we show that, with the hypothesis of theorem 7.5, F is relatively compact in $L^p(0,T;Y)$, then, since F is bounded in $L^p(0,T;X)$ we have, through lemma 7.4 that F is also relatively compact in $L^p(0,T;B)$.

We first show that, in this context,

$$I_F := \left\{ \int_{t_1}^{t_2} f(t)\ dt : \ f \in F \right\} \text{ is relatively compact in } Y, \forall\ 0 < t_1 < t_2 < T \tag{7.7}$$

To see this, consider $\displaystyle\int_{t_1}^{t_2} f(t)\ dt \in X$. We have, if M is such that $\|f\|_{L^p(0,T;X)} \leq M$ for every $f \in F$,

$$\left\| \int_{t_1}^{t_2} f(t)\ dt \right\|_X \leq \int_{t_1}^{t_2} \|f(t)\|_X\ dt \leq (t_2 - t_1)^{\frac{1}{p}} \|f\|_{L^p(0,T;X)} \leq M(t_2 - t_1)^{\frac{1}{p}}$$

So, every sequence in $I_k \in I_F$, with

$$I_k = \int_{t_1}^{t_2} f_k(t)\ dt$$

is a bounded sequence in X. Being X compactly embedded into B, we have that exists a subsequence k_h and an element $I \in B$ such that

$$\lim_{h\to\infty} \|I_{k_h} - I\|_B = 0$$

But B continuously embeds into Y, so that $I \in Y$ and

$$\|I_{k_h} - I\|_Y \le C\|I_{k_h} - I\|_B \to 0$$

as $h \to \infty$.

We define now

$$(M_a f)(t) := \frac{1}{a}\int_t^{t+a} f(s)\ ds$$

So, clearly, $M_a f \in C(0, T-a; Y)$ and, for every $0 \le t_1 \le t_2 \le T-a$ we have

$$\big\|(M_a f)(t_2) - (M_a f)(t_1)\big\|_Y = \big\|\frac{1}{a}\int_{t_1}^{t_1+a} \big(\tau_{t_2-t_1}f - f\big)(s)\ ds\big\|_Y \le \frac{1}{a}\big\|\tau_{t_2-t_1}f - f\big\|_{L^1(0,T-(t_2-t_1);Y)}$$

since $t_1 + a \le T - (t_2 - t_1) \iff a \le T - t_2$. Observe that, thanks to condition (7.4), for every $\varepsilon > 0$ we can find η such that

$$\|\tau_{t_2-t_1}f - f\big\|_{L^1(0,T-(t_2-t_1);Y)} \le a\varepsilon$$

for every $t_1 \le t_2$ such that $|t_2 - t_1| \le \eta$, and for every $f \in F$. By definition (7.6) this means that the set

$$M_a F := \{M_a f :\ f \in F\}$$

in $C(0, T-a; Y)$. Moreover we have already proved that

$$(M_a F)(t) := \left\{\frac{1}{a}\int_t^{t+a} f(s)\ ds :\ f \in F\right\}$$

is relatively compact in Y, thanks to (7.7). So, using lemma 7.5, we have that $M_a F$ is relatively compact in $C(0, T-a; Y)$.

Moreover, consider the function $\tau_h f : [0,a] \to L^p(0, T-a; Y)$ that maps $h \mapsto \tau_h f$. Thanks to (7.4) we have that this function is continuous in h.
So, using that

$$(M_a f)(t) = \frac{1}{a}\int_t^{t+a} f(s)\ ds = \frac{1}{a}\int_0^a f(t+h)\ dh = \frac{1}{a}\int_0^a \tau_h f(t)\ dh$$

we have

$$\big(M_a f - f\big)(t) = \frac{1}{a}\int_0^a (\tau_h f(t) - f(t))\ dh$$

so that

$$\big\|M_a f - f\big\|_{L^p(0,T-a;Y)} \leq \frac{1}{a}\int_0^a \|\tau_h f - f\|_{L^p(0,T-a;Y)}\, dh \leq \sup_{h\in[0,a]} \|\tau_h f - f\|_{L^p(0,T-a;Y)}$$

We remark that condition (7.4) can be rewritten as

$$\forall \varepsilon > 0\ \exists\ \eta > 0 \text{ such that } \|\tau_h f - f\|_{L^p(0,T-h;Y)} \leq \varepsilon$$

for every $h \leq \eta$ and $f \in F$; if $\delta \leq h \leq \eta$ we have $\|\tau_\delta f - f\|_{L^p(0,T-h;Y)} \leq \varepsilon$. So

$$\sup_{\delta\in[0,h]} \|\tau_\delta f - f\|_{L^p(0,T-h;Y)} \leq \varepsilon$$

This means that

$$\lim_{a\to 0} \sup_{h\in[0,a]} \|\tau_h f - f\|_{L^p(0,T-a;Y)} = 0$$

uniformly in $f \in F$. If $a < T - T_1$, that is $T_1 < T - a$, we have

$$\big\|M_a f - f\big\|_{L^p(0,T_1;Y)} \leq \big\|M_a f - f\big\|_{L^p(0,T-a;Y)} \leq \sup_{h\in[0,a]} \|\tau_h f - f\|_{L^p(0,T-a;Y)} \tag{7.8}$$

So $M_a f$ converges to f in $L^p(0,T_1;Y)$ if $a \to 0$, uniformly in $f \in F$.
Moreover, $M_a F$ is relatively compact in $C(0,T-a;Y)$. So in particular it is relatively compact in $L^p(0,T_1;Y)$. In fact, if $M_a f_k$ is a sequence in $M_a F$, we have that exists a subsequence $M_a f_{k_h}$ and $g \in C(0,T-a;Y)$ such that

$$\lim_{h\to\infty} \|M_a f_{k_h} - g\|_{C(0,T-a;Y)} = 0$$

Clearly in particular $g \in L^p(0,T_1;Y)$. Moreover

$$\|M_a f_{k_h} - g\|_{L^p(0,T_1;Y)} \equiv \left(\int_0^{T_1} \|M_a f_{k_h}(s) - g(s)\|_Y^p\, ds\right)^{\frac{1}{p}} \leq T_1^{\frac{1}{p}} \max_{s\in[0,T-a]} \|M_a f_{k_h}(s) - g(s)\|_Y = 0$$

So $M_a F$ is relatively compact in $L^p(0,T_1;Y)$. This implies that also F is relatively compact in $L^p(0,T_1;Y)$. In fact, using (7.8), for every $\varepsilon > 0$ exists $\eta > 0$ such that

$$\|M_a f - f\|_{L^p(0,T_1;Y)} \leq \varepsilon$$

for every $a \leq \eta$ and $f \in F$. Since $M_a F$ is relatively compact in $L^p(0,T_1;Y)$, there exists $\{M_a f_i :\ i = 1, ..., n\}$ such that, for every $M_a f \in M_a F$ exists $i \in \{1, ..., n\}$ such that

$$\|M_a f - M_a f_i\|_{L^p(0,T_1;Y)} \leq \varepsilon$$

Then

$$\|f - f_i\|_{L^p(0,T_1;Y)} \leq \|f - M_a f\|_{L^p(0,T_1;Y)} + \|M_a f - M_a f_i\|_{L^p(0,T_1;Y)} + \|M_a f_i - f_i\|_{L^p(0,T_1;Y)} \leq 3\varepsilon$$

So, by the correspondence $F \longleftrightarrow M_a F$ we have that also F is relatively compact in $L^p(0,T_1;Y)$.

If now we consider $\tilde{f}(t) := f(T-t)$ and define

$$\tilde{F} := \{\tilde{f} :\ f \in F\}$$

we have that the same discussion above continues to hold. So, $\tilde{F}$ is relatively compact in $L^p(0,T_1;Y)$. Looking at the definition of $\tilde{F}$ this means that F is relatively compact in $L^p(T-T_1;T;Y)$. So, if we choose $T_1 = \frac{T}{2}$ we have that F is relatively compact over the whole $L^p(0,T;Y)$. This implies the thesis, as explained above. ∎

Chapter 8

The transport equation

8.1 Classical transport theory

Definition 8.1. By *transport equation* we mean the following problem. Let $\Omega \subseteq \mathbb{R}^n$ a domain and let $I \subseteq \mathbb{R}$ bounded. Let $u(x,t) \in C(\overline{I}; C^1(\overline{\Omega}))$. The *transport equation* associated to the velocity u is the Cauchy problem

$$\begin{cases} \rho_t(x,t) + u(x,t) \cdot \nabla \rho(x,t) = 0 \\ \rho(x,0) = \rho_0(x) \end{cases} \tag{8.1}$$

where $\rho_0 \in C^1(\overline{\Omega})$ and we search for $\rho \in C^1([0,T] \times \overline{\Omega})$.

Remark 8.1. We have the following theorem, from [16], that summarizes the theory of the regular transport equation. □

Theorem 8.1 (Classical transport equation)**.** *Let Ω a bounded domain. Let $u(x,t) \in C([0,T]; C^1(\overline{\Omega}))$ with $\nabla \cdot u = 0$ and $u = 0$ for all $(x,t) \in \partial\Omega \times [0,T]$. Let $\rho_0 \in C^1(\overline{\Omega})$. Then the problem* (8.1) *has a unique solution $\rho \in C^1([0,T] \times \overline{\Omega})$. Furthermore, we have:*

- *if exist $\alpha, \beta \in \mathbb{R}$ such that $\alpha \le \rho_0(x) \le \beta$ for every $x \in \overline{\Omega}$, then*

$$\rho(x,t) \in [\alpha, \beta] \quad \forall (x,t) \in [0,T] \times \overline{\Omega}$$

- *thanks to the condition $\nabla \cdot u = 0$, the density solution of the transport equation satisfies a property of mass incompressibility, that is*

$$\|\rho(t)\|_q = \|\rho_0\|_q$$

  for every $q > 0$.

Proof. The fact that $u \in C([0,T]; C^1(\overline{\Omega}))$ means that there exists an open set E such that $u(x,t) \in C([0,T]; C^1(E))$. We can consider the ODE associated to the velocity u, that is,

$$\dot{x}(t) = u(x(t),t) \qquad x(0) = y \in \overline{\Omega}$$

with y fixed. Then, by locally existence of ODE we have a unique solution

$$x(t,y) \in C^1([0,\tilde{T}]; C^1(\overline{\Omega}))$$

The function x is nothing but the flow $x(t,y) = \varphi(t;0,y)$. It is a regular function, as specified above. Moreover, $x(t,y) \in C^1([0,\tilde{T}] \times \overline{\Omega})$. In fact, $x = x(t,y)$ is C^1 in both the variables separately, and this regularity is also uniform in x, so that it follows that the function is regular also looking at x as function of two variables.[1] Furthermore, the time $\tilde{T}$ can be replaced with T. In fact

$$|u(x,t)| \le \|u(\cdot,t)\|_{C^1(\overline{\Omega})} \le \max_{t\in[0,T]} \|u(\cdot,t)\|_{C^1(\overline{\Omega})} < +\infty$$

where $\|u(\cdot,t)\|_{C^1(\overline{\Omega})}$ is continuous because so it is $v(\cdot,t)$. So the velocity is bounded and we have global existence in $[0,T]$ of the solution to the ODE.
Moreover, from $\nabla \cdot u = 0$, it follows, as previously seen, that the Jacobian determinant of the transformation

$$S_t : \overline{\Omega} \to \overline{\Omega}$$
$$y \to x(t,y)$$

is constantly 1 for every fixed t. Notice that the codomain is $\overline{\Omega}$. In fact, if we consider the flow φ with the velocity v and we take an initial data$y \in \partial\Omega$, then $x(t,y) \equiv y$, since

$$0 = \dot{x}(t,y) = u(x(t,y),t) = 0$$

is solution.
If otherwise $y \in \Omega$, the solution can't cross the boundary, since, by unicity of the solution, it might remain on the boundary for all the times, including past times. So the solution remains in $\overline{\Omega}$. Clearly $y \to x(t,y)$ admits an inverse, that is

$$x \to \varphi(0;t,x) := y(x,t)$$

by the unicity of the solution, where φ is always the flow associated to the velocity v. As above, for initial data $x \in \overline{\Omega}$, the solution y remains in $\overline{\Omega}$. So the inverse is global. Then, since the Jacobian determinant is non zero and S_t is injective (since the inverse has been found), then S_t is invertible with inverse in C^1. That is, S_t is a C^1-diffeomorfism of $\overline{\Omega}$ onto itself. We call S_t^{-1} its inverse, and define

$$\rho(x,t) := \rho_0(S_t^{-1}(x)) \tag{8.2}$$

that is in $C^1([0,T]\times\overline{\Omega})$, since[2] $y(x,t) \in C^1([0,T]\times\overline{\Omega})$. The first point in the statement follows obviously. The second point follows from the arguments explained in section 1.5.2, as already remarked. ∎

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[1] In fact, if $(t_0,y_0) \in [0,\tilde{T}] \times \overline{\Omega}$, then

$$|x(t,y)-x(t_0,y_0)| \le |x(t,y)-x(t_0,y)|+|x(t_0,y)-x(t_0,y_0)| \le \max_{y\in\overline{\Omega}} |x(t,y)-x(t_0,y)|+|x(t_0,y)-x(t_0,y_0)|$$

and the latter is small if $|y-y_0|$ is small, thanks to the continuity respect with the initial data, while the first is small if $|t-t_0|$ is small because, by definition of $C^1([0,\tilde{T}];C^1(\overline{\Omega}))$,

$$\max_{y\in\overline{\Omega}} |x(t,y)-x(t_0,y)| \le \|x(t,\cdot)-x(t_0\cdot)\|_{C^1(\overline{\Omega})} \to 0 \quad \text{as } t \to t_0$$

[2] More precisely, we can define

$$S : (y,t) \to (x(t,y),t)$$

8.1.1 Temporal invariant property of the q-norm of the density ρ

We now consider solutions of the transport equation to deduce important properties of the density appearing in the INSE.

Theorem 8.2 (q-norm conservation)**.** *Let Ω a bounded domain in $\mathbb{R}^3$. Moreover consider a velocity field $u \in C^2([0,T];C^1(\overline{\Omega}))$ such that $\nabla \cdot u = 0$ in Ω and such that the varibles (x,t) appear in the expression of u as separated variables. Let $\rho \in C^1([0,T],C^1(\overline{\Omega}))$, for some $T > 0$, a solution of the transport equation*

$$\begin{cases} \rho_t + u \cdot \nabla \rho = 0 \\ \rho(x,0) = \overline{\rho}_0(x) \end{cases} \tag{8.3}$$

with $\overline{\rho}_0 \in C^1(\overline{\Omega})$. Then, for every $q > 0$, we have

$$\|\rho(t)\|_q = \|\overline{\rho}_0\|_q \qquad \forall\, t \in [0,T]$$

where $\rho(t) \equiv \rho(x,t)$.

Proof. We want to use the Theorem 1.6. In the hypothesis of this theorem we have supposed that the force term f is in C^3. However, looking critically at the proof of this theorem, one can notice that our u satisfies the hypotesis since it has separated variables. As above, we know that

$$\rho_t + u \cdot \nabla \rho = 0$$

So we can consider the solutions of the system

$$\begin{cases} \dot{x}(t) = u(x(t),t) \\ x(t_0) = x_0 \end{cases}$$

These give us the vectorial transformation

$$y = \varphi(t; t_0, x)$$

i.e. a vectorial function that reaches the values of the solution of the system at time t, with starting point (x,t_0). The flow φ is defined in the whole interval $[0,T]$ provided

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and since $x(t,y) \in C^1([0,T] \times \overline{\Omega})$, S is a C^1-diffeomorfism, since the inverse is

$$S^{-1} : (x,t) \to (y(x,t),t)$$

and so it is injective, and moreover

$$\partial_{(y,t)} S = \begin{pmatrix} \partial_y x(t,y) & \partial_t x(t,y) \\ 0 & 1 \end{pmatrix}$$

and its determinant is equal to the determinant of $\partial_y S_t$ with t fixed. So it is 1 for every t and every y. So, we are in the situation above again, and it follows that S is a C^1 diffeomorfism. So $y(x,t) \in C^1([0,T] \times \overline{\Omega})$.

that u is bounded and regular. But $u \in C^2([0,T], C^1(\overline{\Omega}))$, with separated variables, and so regularity and boundness in $[0,T] \times \overline{\Omega}$ are immediate.

Fixed t and t_0, we can change x. The dependence of φ on the variable x is C^1, since u is in the class C^1. Moreover φ is invertible in x and its inverse is obtained simply interchanging the position of t_0 and t; so, being the inverse a solution with different data, it is also C^1. So we can use $\phi(x) := \varphi(t; t_0, x)$ as a change of coordinates.
Thus

$$\int_\Omega |\rho(y,t)|^q \, dy = \int_\Omega |\rho(\varphi(t;t_0,x),t)|^q |\det(D\phi(x))| \, dx$$

But $\rho(\varphi(t;t_0,x),t) = \rho(x,t_0)$ as previously remarked, and

$$\det(D\phi(x)) = \det(\partial_x \varphi(t;t_0,x)) = \det(\partial_x \varphi(t_0;t_0,x)) = \det(\partial_x x) = \det(\mathbb{I}) = 1$$

using Theorem 1.6 as outlined above. Observe that φ and its inverse map Ω in itself.
So

$$\|\rho(t)\|_q^q = \int_\Omega |\rho(y,t)|^q \, dy = \int_\Omega |\rho(x,t_0)|^q \, dx = \|\rho(t_0)\|_q^q$$

This in true for every $q \in (0, +\infty)$. But

$$\|\rho(t)\|_\infty = \lim_{q \to +\infty} \|\rho(t)\|_q = \lim_{q \to +\infty} \|\rho(t_0)\|_q = \|\rho(t_0)\|_\infty$$

that is exactly what we wanted to prove. ∎

Remark 8.2. The above result holds for every $q > 0$ in this regular case. □

8.2 Weak transport theory (aprés DiPerna-Lions)

In this section we will follow the work [8] by DiPerna and Lions to prove existence and uniqueness of weak solution to the transort equation, together a fundamental *stability result* that will help us in future considerations.

Remark 8.3. The work [8] by DiPerna and Lions studies the transport equation in the whole space $\mathbb{R}^n$. In this article there is no trace of the bounded domain case. We consider here only this "new" case, that is fundamental for future arguments. We also require the velocity field to be divergence-free. Who write did not manage to find a similar discussion in literature. All the statements and proofs are written using the weak formulations, avoiding the formal notations of the enlightening paper by DiPerna-Lions.
□

8.2.1 Linear transport equation

Definition 8.2. Let Ω be a bounded domain in $\mathbb{R}^n$ and let $T > 0$. Let $p \in [1, \infty]$ be an exponent, and q such that $\frac{1}{p} + \frac{1}{q} = 1$, its conjugate exponent. Let $u \in L^1(0,T; W_0^{1,q}(\Omega))$ be a velocity field over $(0,T) \times \Omega$, with $\nabla \cdot u = 0$, i.e. the divergence-free property. Let

$\rho^0 \in L^p(\Omega)$ be the initial density. We say that the density $\rho \in L^\infty(0,T;L^p(\Omega))$ satisfies the equation

$$\begin{cases} \partial_t \rho - u \cdot \nabla \rho = 0 & \text{in } (0,T) \times \Omega \\ \rho(0) = \rho^0 \end{cases} \tag{8.4}$$

if it is a solution of (8.4) in distributional sense, that is

$$-\int_0^T \left(\int_\Omega \rho \, \partial_t \phi \, dx \right) dt - \int_\Omega \rho^0(x) \phi(0,x) \, dx + \int_0^T \left(\int_\Omega \rho \, (u \cdot \nabla \phi) \, dx \right) dt = 0 \tag{8.5}$$

for all test functions $\phi \in C^\infty([0,T] \times \Omega)$ with compact support in $[0,T) \times \Omega$. This space can also be denoted by $\mathcal{D}([0,T) \times \Omega)$.

Remark 8.4. On a bounded domain, as in the case of classical transport theory, it is necessary to assume $\nabla \cdot u = 0$ in the weak sense. □

So, we have a first existence theorem.

Theorem 8.3. *Let $p \in (1,\infty]$, $\rho_0 \in L^p(\Omega)$. Let q be its conjugate exponent. Assume*

$$u \in L^1(0,T;W_0^{1,q}(\Omega)) \tag{8.6}$$

with $\nabla \cdot u = 0$, where q is the conjugate of p. Then there exists a solution of (8.4) *in $L^\infty(0,T;L^p(\Omega))$ corresponding to the initial condition ρ_0.*

Proof. The proof is based over a classical regularization argument, as in section 11.14.3. Consider the Banach space[3]

$$Y := \{v \in W_0^{1,q}(\Omega) : \ \nabla \cdot v = 0\}$$

equipped with the norm $\| \cdot \|_Y := \| \cdot \|_{W^{1,q}(\Omega)}$. So, we can find a sequence $u^n \in C_c^\infty(0,T_*;Y)$ such that

$$\lim_{n\to\infty} \|u - u^n\|_{L^1(0,T_*;Y)} = 0 \tag{8.7}$$

Since $u^n(t) \in W_0^{1,q}(\Omega)$, each element of the sequence can be extended to be zero outside Ω. Moreover the initial density ρ^0 can be approached in $L^p(\Omega)$ with a sequence $\rho_n^0 \in C_c^\infty(\Omega)$ (by the density results in $L^p(\Omega)$). We now set

$$A_m := \{x \in \Omega_c : \ \mathrm{dist}(x, \partial\Omega) > \frac{1}{m}\}$$

and $\Omega_m := A_m^c$. We define

$$u^{m,n}(x,t) := \int_{\Omega_m} \eta_m(x-y) u^n(y,t) \, dy$$

[3] It is clearly a Banach space. In fact, give a Cauchy sequence $v^k \in Y$, by the completeness of $W_0^{1,q}(\Omega)$, we have that exists $v \in W_0^{1,q}(\Omega)$. Moreover, for every $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega v \cdot \nabla \varphi \, dx = \lim_{k\to\infty} \int_\Omega v_k \cdot \nabla \varphi \, dx = 0$$

that is $\nabla \cdot v = 0$ in the weak sense.

This convolution is smooth in x at $t \in [0,T]$ fixed. Moreover, it is continuous as a function of two variables, In fact, if $(x_0,t_0) \in \Omega_m \times [0,T]$ we have that

$$|u^{m,n}(x,t) - u^{m,n}(x_0,t_0)| \le |u^{m,n}(x,t) - u^{m,n}(x_0,t)| + |u^{m,n}(x_0,t) - u^{m,n}(x_0,t_0)| \le$$

$$\le \left| \int_{\Omega_m} \big(\eta_m(x-y) - \eta_m(x_0-y)\big)\ u^n(t,y)\ dy \right| + \left| \int_{\Omega_m} \eta_m(x_0-y)\big(u^n(t,y) - u^n(t_0,y)\big)\ dy \right| \le$$

$$\le \left| \int_{\Omega_m} \big(\eta_m(x-y) - \eta_m(x_0-y)\big)\ u^n(t,y)\ dy \right| + \|\eta_m(x_0-\cdot)\|_{p,\Omega_m} \|u^n(t,\cdot) - u^n(t_0,\cdot)\|_q$$

Since $u^n \in C^\infty([0,T];X)$, we can find $\delta_1 > 0$ such that $\|u^n(t,\cdot) - u^n(t_0,\cdot)\|_q < \frac{\varepsilon}{2}$. On the other hand, since $\eta_m(r)$ is uniformly continuos on $\mathbb{R}$, there exists $\delta_2 > 0$ such that

$$|x - x_0| = |(x-y) - (x_0-y)| < \delta_2 \implies |\eta_m(x-y) - \eta_m(x_0-y)| < \frac{\varepsilon}{2}$$

it follows that

$$|u^{m,n}(x,t) - u^{m,n}(x_0,t_0)| \le \frac{\varepsilon}{2}\|u^n(t,\cdot)\|_q + \frac{\varepsilon}{2}\|\eta_m(x_0-\cdot)\|_{p,\Omega_m} \le \frac{\varepsilon}{2}\left(\max_{t\in[0,T]} \|u^n(t,\cdot)\|_q + \|\eta_m(x_0-\cdot)\|_{p,\Omega_m} \right)$$

Moreover, thanks to the convolution properties, the x-derivative is continuos over $\overline{\Omega}_m$, and, thanks to the theorem 3.2,

$$|\nabla u^{m,n}(x,t)| = \left| \int_{\Omega_m} \eta_m(x-y) \nabla u^n(y,t)\ dy \right| \le \left(\int_{\Omega_m} |\eta_m(x-y)|^p\ dy \right)^{\frac{1}{p}} \|\nabla u^n(\cdot,t)\|_q \le$$

$$\le \left(\int_{\mathbb{R}^3} |\eta_m(x-y)|^p\ dy \right)^{\frac{1}{p}} \max_{t\in[0,T]} \|\nabla u^n(\cdot,t)\|_q \equiv \left(\int_{\mathbb{R}^3} |\eta_m(z)|^p\ dz \right)^{\frac{1}{p}} \max_{t\in[0,T]} \|\nabla u^n(\cdot,t)\|_q$$

so that

$$\sup_{t\in[0,T]} \|\nabla u^{m,n}(t)\|_\infty \le \left(\int_{\mathbb{R}^3} |\eta_m(z)|^p\ dz \right)^{\frac{1}{p}} \max_{t\in[0,T]} \|\nabla u^n(\cdot,t)\|_q$$

so $u^{m,n} \in C([0,T_*];C^1(\overline{\Omega}_m))$ and the continuity of $\nabla u^{m,n}$ in $(x_0,t_0) \in \overline{\Omega}_m \times [0,T]$ follows from the same argument above.
Finally we underline other two properties of the field $u^{m,n}$. In particular, if $x \in \partial\Omega_m$, we have

$$u^{m,n}(x,t) = \int_{\Omega_m} \eta_m(x-y) u^n(y,t)\ dy = 0$$

since $u^n(y,t) = 0$ if $y \in B(x, \frac{1}{m})$. Moreover,

$$\nabla \cdot u^{m,n}(x,t) = \int_{\Omega_m} \eta_m(x-y) \nabla \cdot u^n(y,t)\ dy = 0$$

since $\nabla \cdot u^n(y,t) = 0$ by the definition of u^n. So, we can use this velocity field to solve the transport problem

$$\begin{cases} \partial_t \rho - u^{m,n} \cdot \nabla \rho = 0 \quad \text{in } [0,T] \times \overline{\Omega}_m \\ \rho(0,x) = \rho_0^n \end{cases}$$

We can name $\rho^{m,n}$ the solution of this classical transport equation. We know, according to the classical theory studied above, that

$$\|\rho^{m,n}(t)\|_p = \|\rho_0^n\|_p \leq \|\rho_0\|_p + 1 \equiv C_0 \quad p \in [1, \infty]$$

It follows that $\|\rho^{m,n}\|_{L^\infty(0,T;L^p(\Omega))} \leq C_0$. Suppose now $p \in (1, \infty]$. Observe that $L^\infty(0,T;L^p(\Omega)) = (L^1(0,T;L^q(\Omega))^*$, where q is such that $\frac{1}{p} + \frac{1}{q} = 1$, thanks to proposition 5.2. Moreover $L^q(\Omega)$ is separable, since $q \in [1, \infty)$. So, always thanks to proposition 5.2, $L^1(0,T;L^q(\Omega))$ is separable. Then, thanks to theorem 2.3, we have that exists a subsequence weak-star converging to some $\rho \in L^\infty(0,T;L^p(\Omega))$, that is

$$\rho^{m_k,n} \overset{*}{\rightharpoonup} \rho^n$$

in $L^\infty(0,T;L^p(\Omega)) = (L^1(0,T;L^q(\Omega))^*$. In particular, the sequence satisfies, for every $\varphi \in C_c^\infty(\Omega \times [0,T))$,

$$-\int_\Omega (\rho^{m_k,n}\varphi)(0)\ dx - \int_0^T \int_\Omega \rho^{m_k,n}\varphi_t\ dx\ dt = \int_0^T \int_\Omega \rho^{m_k,n} u^{m_k,n} \cdot \nabla\varphi\ dx\ dt$$

Observe that

$$\int_\Omega \big(\rho^{m_k,n}\varphi\big)(0)\ dx \equiv \int_\Omega \rho_0^n(x)\varphi(x,0)\ dx$$

and

$$\int_0^T \int_\Omega \rho^{m_k,n}\varphi_t\ dx\ dt \to \int_0^T \int_\Omega \rho^n \varphi_t\ dx\ dt$$

as $k \to \infty$, thanks to the weak convergence. Furthermore

$$\left| \int_0^T \int_\Omega \rho^{m_k,n} u^{m_k,n} \cdot \nabla\varphi\ dx\ dt - \int_0^T \int_\Omega \rho^n u^n \cdot \nabla\varphi\ dx\ dt \right| =$$

$$= \left| \int_0^T \int_\Omega (\rho^{m_k,n} - \rho^n) u^n \cdot \nabla\varphi\ dx\ dt - \int_0^T \int_\Omega \rho^{m_k,n}(u^n - u^{m_k,n}) \cdot \nabla\varphi\ dx\ dt \right| \leq$$

$$\leq \left| \int_0^T \int_\Omega (\rho^{m_k,n} - \rho^n) u^n \cdot \nabla\varphi\ dx\ dt \right| + C\left(\int_0^T \|\rho^{m_k,n}\|_p \|u^n - u^{m_k,n}\|_q\ dt \right) \leq$$

$$\leq \left| \int_0^T \int_\Omega (\rho^{m_k,n} - \rho^n) u^n \cdot \nabla\varphi\ dx\ dt \right| + C\left(\sup_{(0,T)} \|\rho^{m_k,n}\|_p \right)\left(\int_0^T \|u^n - u^{m_k,n}\|_q\ dt \right) \leq$$

$$\leq \left| \int_0^T \int_\Omega (\rho^{m_k,n} - \rho^n) u^n \cdot \nabla\varphi\ dx\ dt \right| + CC_0 \|u^n - u^{m_k,n}\|_{L^1(0,T;L^q(\Omega))}$$

Observe now that $\|u^n - u^{m_k,n}\|_{L^1(0,T;L^q(\Omega))} \to 0$ as $k \to \infty$, thanks to (8.7), and, moreover

$$\int_0^T \|u^n \cdot \nabla\varphi\|_q\ dt \leq C \int_0^T \|u^n\|_q\ dt < \infty$$

that is $u^n \cdot \nabla\varphi \in L^1(0,T;L^q(\Omega))$ and so the weak star convergence of $\rho^{m_k,n}$ implies that

$$\int_0^T \int_\Omega (\rho^{m_k,n} - \rho^n) u^n \cdot \nabla\varphi\ dx\ dt \to 0$$

as $k \to \infty$. It follows that

$$-\int_\Omega \rho_0^n(x)\varphi(x,0)\ dx - \int_0^T \int_\Omega \rho^n \varphi_t\ dx\ dt = \int_0^T \int_\Omega \rho^n u^n \cdot \nabla\varphi\ dx\ dt \tag{8.8}$$

Moreover, by the weak convergence property, we have

$$\|\rho^n\|_{L^\infty(0,T;L^p(\Omega))} \leq \liminf_{k\to\infty} \|\rho^{m_k,n}\|_{L^\infty(0,T;L^p(\Omega))} \leq C_0 \tag{8.9}$$

We let now $n \to \infty$ in (8.8). Clearly

$$\left| \int_\Omega (\rho_0^n(x) - \rho^0(x))\varphi(x,0)\ dx \right| \leq C\|\rho_0^n - \rho^0\|_p \to 0$$

By the bound (8.9), we have that there exists a subsequence n_h and $\rho \in L^\infty(0,T;L^p(\Omega))$ such that, as $h \to \infty$,

$$\rho^{n_h} \overset{*}{\rightharpoonup} \rho$$

It follows that

$$\left| \int_0^T \int_\Omega \rho^{n_h} u^{n_h} \cdot \nabla\varphi\ dx\ dt - \int_0^T \int_\Omega \rho u \cdot \nabla\varphi\ dx\ dt \right| =$$

$$= \left| \int_0^T \int_\Omega (\rho - \rho^{n_h}) u \cdot \nabla\varphi\ dx\ dt - \int_0^T \int_\Omega \rho^{n_h}(u^{n_h} - u) \cdot \nabla\varphi\ dx\ dt \right| \leq$$

$$\leq \left| \int_0^T \int_\Omega (\rho - \rho^{n_h}) u \cdot \nabla\varphi\ dx\ dt \right| + C \int_0^T \|\rho^{n_h}\|_p \|u^{n_h} - u\|_q\ dt \leq$$

$$\leq \left| \int_0^T \int_\Omega (\rho - \rho^{n_h}) u \cdot \nabla\varphi\ dx\ dt \right| + C\left(\sup_{(0,T)} \|\rho^{n_h}\|_p \right) \int_0^T \|u^{n_h} - u\|_q\ dt \leq$$

$$\leq \left| \int_0^T \int_\Omega (\rho - \rho^{n_h}) u \cdot \nabla\varphi\ dx\ dt \right| + CC_0 \|u^{n_h} - u\|_{L^1(0,T;L^q(\Omega))}$$

Since

$$\int_0^T \|u \cdot \nabla\varphi\|_q\ dt \leq C \int_0^T \|u\|_q\ dt < \infty \Longrightarrow u \cdot \nabla\varphi \in L^1(0,T;L^q(\Omega))$$

and so, since ρ^{n_h} converges weakly star to ρ,

$$\int_0^T \int_\Omega (\rho - \rho^{n_h}) u \cdot \nabla\varphi\ dx\ dt \to 0$$

It follows that

$$-\int_\Omega \rho^0(x)\varphi(x,0)\ dx - \int_0^T \int_\Omega \rho\varphi_t\ dx\ dt = \int_0^T \int_\Omega \rho u \cdot \nabla\varphi\ dx\ dt$$

So we have found $\rho \in L^\infty(0,T;L^p(\Omega))$ such that is a weak solution to the trasport equation with velocity u and initial density ρ^0. ∎

Another important result of this section is the following: under appropriate conditions on u, weak solutions of (8.4) can be approached by smooth solution of (8.4) with small error terms. In particular, we have the following approximation theorem.

Theorem 8.4. *Let $p \in (1, \infty]$, and let $\rho \in L^\infty(0,T;L^p)$ be a solution of (8.4) with intial density $\rho_0 \in L^p(\Omega)$ and assume that $u \in L^1(0,T;W^{1,\alpha}(\Omega))$ for some $\alpha \geq q$, $\nabla \cdot u = 0$. Let $\eta_\varepsilon = \eta_\varepsilon(x)$ a regularizer kernel over Ω. In particular, if $\Omega_\varepsilon := \{x \in \Omega : dist(x, \partial\Omega) > \varepsilon\}$, we set*

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right)$$

with $C_c^\infty(\mathbb{R}^n) \ni \eta \geq 0$, $supp(\eta) \subset B(0,1)$. Let $\rho_\varepsilon(x,t) := \big(\rho(\cdot,t) * \eta_\varepsilon\big)(x,t)$. Let $\phi \in C_c^\infty([0,T) \times \Omega)$ and suppose that $\phi(x,\cdot) = 0$ for every $x \in \Omega_0^c$, with Ω_0 compact. Then, if $\varepsilon < dist(\Omega_0, \partial\Omega)$,

$$-\int_0^T \left(\int_{\Omega_\varepsilon} \rho_\varepsilon \frac{\partial \phi}{\partial t}\, dx\right) dt - \int_{\Omega_\varepsilon} \rho_\varepsilon^0\, \phi(0,x)\, dx + \int_0^T \left(\int_{\Omega_\varepsilon} \rho_\varepsilon u \cdot \nabla\phi\, dx\right) dt = \int_0^T \left(\int_\Omega r_\varepsilon \phi\, dx\right) dt \tag{8.10}$$

where

$$r_\varepsilon(x,t) = \int_\Omega \rho(y,t)(u(y,t) - u(x,t)) \cdot \nabla\eta_\varepsilon(y-x)\, dy$$

Moreover, r_ε converges to zero in $L^1(0,T;L^\beta_{loc}(\Omega))$ as $\varepsilon \to 0$, where β is such that

$$\frac{1}{\beta} = \frac{1}{\alpha} + \frac{1}{p}$$

Finally $\rho_\varepsilon^0(x) := (\rho_0 * \eta_\varepsilon)(x)$.

Remark 8.5. The convergence to zero of r_ε in $L^1(0,T;L^\beta_{loc}(\Omega))$ assures that

$$\left|\int_0^T \left(\int_\Omega r_\varepsilon \phi\, dx\right) dt\right| = \left|\int_0^T \left(\int_{\Omega_0} r_\varepsilon \phi\, dx\right) dt\right| \leq |\Omega|^{\frac{\beta-1}{\beta}} \left(\sup_{[0,T]\times\Omega} |\phi|\right) \int_0^T \|r_\varepsilon\|_{L^\beta(\Omega_0)}\, dt \to 0$$

as $\varepsilon \to 0$. $\square$

Proof. First of all, consider the integral

$$\int_0^T \left(\int_{\Omega_\varepsilon} \rho_\varepsilon(x,t) \frac{\partial\phi}{\partial t}(x,t)\, dx\right) dt = \int_0^T \left\{\int_{\Omega_\varepsilon} \left(\int_\Omega \rho(y,t)\eta_\varepsilon(x-y)\, dy\right) \frac{\partial\phi}{\partial t}(x,t)\, dx\right\} dt =$$

$$= \int_0^T \left\{\int_\Omega \left(\int_{\Omega_\varepsilon} \eta_\varepsilon(x-y) \frac{\partial\phi}{\partial t}(x,t)\, dx\right) \rho(y,t)\, dy\right\} dt = \int_0^T \left\{\int_\Omega \frac{\partial}{\partial t}\phi_\varepsilon(y,t)\, \rho(y,t)\, dy\right\} dt$$

since $\eta_\varepsilon(x-y) = \eta_\varepsilon(y-x)$ by definition, and, being $\varepsilon < \mathrm{dist}(\Omega_0, \partial\Omega)$, we have $\phi(x,t) \equiv 0$ in $\Omega/\Omega_\varepsilon$, so that

$$\int_{\Omega_\varepsilon} \eta_\varepsilon(x-y) \frac{\partial\phi}{\partial t}(x,t)\, dx = \int_\Omega \eta_\varepsilon(x-y) \frac{\partial\phi}{\partial t}(x,t)\, dx$$

In the same way, we have

$$\int_{\Omega_\varepsilon} \rho_\varepsilon^0(x)\phi(0,x)\, dx = \int_\Omega \phi_\varepsilon(0,y)\rho^0(y)\, dy$$

So, we can rewrite equation (8.10) as

$$0 = -\int_0^T \left(\int_\Omega \rho \frac{\partial \phi_\varepsilon}{\partial t}\ dx \right) dt - \int_\Omega \rho^0\ \phi_\varepsilon(0,x)\ dx + \int_0^T \left(\int_\Omega \rho u \cdot \nabla \phi_\varepsilon\ dx \right) dt =$$

$$= -\int_0^T \left(\int_{\Omega_\varepsilon} \rho_\varepsilon \frac{\partial \phi}{\partial t}\ dx \right) dt - \int_{\Omega_\varepsilon} \rho_\varepsilon^0 \phi(0,x)\ dx + \int_0^T \left(\int_{\Omega_\varepsilon} \rho_\varepsilon u \cdot \nabla \phi\ dx \right) dt +$$

$$+ \left\{ \int_0^T \left(\int_\Omega \rho u \cdot \nabla \phi_\varepsilon\ dx \right) dt - \int_0^T \left(\int_{\Omega_\varepsilon} \rho_\varepsilon u \cdot \nabla \phi\ dx \right) dt \right\}$$

If we define

$$I_\varepsilon := \left\{ \int_0^T \left(\int_\Omega \rho u \cdot \nabla \phi_\varepsilon\ dx \right) dt - \int_0^T \left(\int_{\Omega_\varepsilon} \rho_\varepsilon u \cdot \nabla \phi\ dx \right) dt \right\}$$

we have

$$I_\varepsilon = \int_0^T \left(\int_\Omega \left\{ \rho(x,t)\ u(x,t) \cdot \left(\int_\Omega \phi(y,t) \nabla \eta_\varepsilon(x-y)\ dy \right) \right\} dx \right) dt -$$

$$- \int_0^T \left(\int_{\Omega_\varepsilon} u(x,t) \cdot \nabla \phi(x,t) \left(\int_\Omega \rho(y) \eta_\varepsilon(x-y)\ dy \right) dx \right) dt$$

Remark 8.6. Notice that $\phi \in C_c^\infty(\Omega)$ is defined on the whole $\mathbb{R}^n$; so also its convolution is defined in the whole space. □

We now remark that

$$\int_0^T \left(\int_{\Omega_\varepsilon} \rho_\varepsilon(x,t) u(x,t) \cdot \nabla \phi(x,t)\ dx \right) dt =$$

$$= \int_0^T \left(\int_{\Omega_\varepsilon} \left(\int_\Omega \rho(y,t) \eta_\varepsilon(x-y)\ dy \right) u(x,t) \cdot \nabla \phi(x,t)\ dx \right) dt =$$

$$= \int_0^T \left\{ \int_\Omega \rho(y,t) \left(\int_{\Omega_\varepsilon} \eta_\varepsilon(x-y) u(x,t) \cdot \nabla \phi(x,t)\ dx \right) dy \right\} dt =$$

and being $\phi(x,t) \equiv 0$ on Ω_ε^c,

$$= \int_0^T \left\{ \int_\Omega \rho(y,t) \left(\int_\Omega \eta_\varepsilon(x-y) u(x,t) \cdot \nabla \phi(x,t)\ dx \right) dy \right\} dt =$$

$$= \int_0^T \left\{ \int_\Omega \rho(y,t) \left(\int_\Omega \eta_\varepsilon(x-y) \nabla \cdot \big(u(x,t) \cdot \phi(x,t) \big)\ dx \right) dy \right\} dt =$$

$$= -\int_0^T \left\{ \rho(y,t) \left(\int_\Omega \phi(x,t) u(x,t) \cdot \nabla \eta_\varepsilon(x-y)\ dx \right) dy \right\} dt =$$

since $\nabla \eta_\varepsilon(x-y) = -\nabla \eta_\varepsilon(y-x)$

$$= \int_0^T \left\{ \rho(y,t) \left(\int_\Omega \phi(x,t) u(x,t) \cdot \nabla \eta_\varepsilon(y-x)\ dx \right) dy \right\} dt$$

So we have, changing the name of the variables in the first block of the integral,

$$I_\varepsilon = \int_0^T \Big(\int_\Omega \Big\{ \int_\Omega \rho(y,t)\phi(x,t)u(y,t) \cdot \nabla\eta_\varepsilon(y-x)\ dx \Big\}\ dy \Big)\ dt -$$

$$- \int_0^T \Big(\int_\Omega \Big\{ \int_\Omega \rho(y,t)\phi(x,t)u(x,t) \cdot \nabla\eta_\varepsilon(y-x)\ dx \Big\}\ dy \Big)\ dt$$

It is clear that this term has the structure of the integral

$$\int_\Omega w(y)\{(B(y)-B(x)) \cdot \nabla\eta_\varepsilon(y-x)\}\ dy \tag{8.11}$$

integrated over Ω and over $(0,T)$. So we finally have

$$r_\varepsilon(x,t) = \int_\Omega \rho(y,t)(u(y,t)-u(x,t)) \cdot \nabla\eta_\varepsilon(y-x)\ dy$$

If we show that (8.11) goes to zero in the right norm as $\varepsilon \to 0$, then we have the thesis.

Remark 8.7. Let $B \in L^1(0,T;W^{1,\alpha}_{loc})$ and $w \in L^\infty(0,T;L^p_{loc})$. Then

$$(B \cdot \nabla w) * \eta_\varepsilon - B \cdot \nabla(w * \eta_\varepsilon) \to 0 \quad \text{in } L^1(0,T;L^\beta_{loc})$$

We now prove this fact. Consider a compact set $\Omega_0 \subset \Omega$. Then, from now, we take $0 < \varepsilon < \mathrm{dist}(\Omega,\Omega_0)$. We set

$$|r^1_\varepsilon(x)| = \Big| \int_\Omega w(y)\{(B(y)-B(x)) \cdot \nabla\eta_\varepsilon(x-y)\}\ dy \Big| \leq$$

$$\leq \int_\Omega |w(y)||B(y)-B(x)|\frac{1}{\varepsilon}\big|\nabla\eta\big(\frac{x-y}{\varepsilon}\big)\big|\frac{1}{\varepsilon^n}\ dy \leq$$

$$\leq C\int_\Omega |w(y)|\frac{\chi_{B(x,\varepsilon)(y)}}{\varepsilon^n}\Big\{\frac{|B(y)-B(x)|}{\varepsilon}\Big\}\chi_{B(x,\varepsilon)(y)}\ dy$$

If $1 \leq s,t$, such that $\frac{1}{s}+\frac{1}{t}=1$, then

$$|r^1_\varepsilon(x)| \leq C\Big(\int_{B(x,\varepsilon)}\Big\{\frac{|B(y)-B(x)|}{\varepsilon}\Big\}^s\ dy\Big)^{\frac{1}{s}}\Big(\int_{B(x,\varepsilon)}|w(y)|^t\frac{dy}{\varepsilon^n}\Big)^{\frac{1}{t}} =$$

$$= C\Big(\int_{B(x,\varepsilon)}\Big\{\frac{|B(y)-B(x)|}{\varepsilon}\Big\}^s\ dy\Big)^{\frac{1}{s}}\Big(\int_{\mathbb{R}^n}|w(y)|^t\frac{\chi_{B(0,\varepsilon)}(x-y)}{\varepsilon^n}\ dy\Big)^{\frac{1}{t}} =$$

$$= C\Big(\int_{B(x,\varepsilon)}\Big\{\frac{|B(y)-B(x)|}{\varepsilon}\Big\}^s\ dy\Big)^{\frac{1}{s}}\big(|w|^t * \chi_\varepsilon\big)^{\frac{1}{t}}$$

where $\chi_\varepsilon(z) := \frac{\chi_{B(0,\varepsilon)}(z)}{\varepsilon^n}$. Observe that we can consider $s=q$, so that $t=p$. Thus, we have

$$|r^1_\varepsilon(x)| \leq C\Big(\int_{B(x,\varepsilon)}\Big\{\frac{|B(y)-B(x)|}{\varepsilon}\Big\}^q\ dy\Big)^{\frac{1}{q}}\big(|w|^p * \chi_\varepsilon\big)^{\frac{1}{p}}$$

Since $\alpha \geq q$, we have that

$$|r_\varepsilon^1(x)| \leq C\biggl(\int_{B(x,\varepsilon)} \left\{\frac{|B(y)-B(x)|}{\varepsilon}\right\}^\alpha dy\biggr)^{\frac{1}{\alpha}} \bigl(|w|^p * \chi_\varepsilon\bigr)^{\frac{1}{p}}$$

So, at the power of β, we have

$$|r_\varepsilon^1(x)|^\beta \leq C^\beta\biggl(\int_{B(x,\varepsilon)} \left\{\frac{|B(y)-B(x)|}{\varepsilon}\right\}^\alpha dy\biggr)^{\frac{\beta}{\alpha}} \bigl(|w|^p * \chi_\varepsilon\bigr)^{\frac{\beta}{p}}$$

So, integrating over Ω_0, we have

$$\int_{\Omega_0} |r_\varepsilon^1(x)|^\beta \, dx \leq C^\beta \int_{\Omega_0} \left\{\biggl(\int_{B(x,\varepsilon)} \left\{\frac{|B(y)-B(x)|}{\varepsilon}\right\}^\alpha dy\biggr)^{\frac{\beta}{\alpha}} \bigl(|w|^p * \chi_\varepsilon\bigr)^{\frac{\beta}{p}}(x)\right\} dx$$

Observe now that $\beta\left(\frac{1}{\alpha}+\frac{1}{p}\right) = 1$, and so, using Hölder's inequality again, with exponents $\frac{\alpha}{\beta}$ and $\frac{p}{\beta}$, we have

$$\int_{\Omega_0} |r_\varepsilon^1(x)|^\beta \, dx \leq C^\beta\biggl(\int_{\Omega_0}\left\{\int_{B(x,\varepsilon)} \left\{\frac{|B(y)-B(x)|}{\varepsilon}\right\}^\alpha dy\right\} dx\biggr)^{\frac{\beta}{\alpha}} \biggl(\int_{\Omega_0} (|w|^p * \chi_\varepsilon) \, dx\biggr)^{\frac{\beta}{p}}$$

So, we have

$$\|r_\varepsilon^1\|_{L^\beta(\Omega_0)} \leq C\biggl(\int_{\Omega_0}\left\{\int_{B(x,\varepsilon)} \left\{\frac{|B(y)-B(x)|}{\varepsilon}\right\}^\alpha dy\right\} dx\biggr)^{\frac{1}{\alpha}} \biggl(\int_{\Omega_0} (|w|^p * \chi_\varepsilon) \, dx\biggr)^{\frac{1}{p}}$$

Using Young's convolution inequality, we have

$$\int_{\Omega_0} (|w|^p * \chi_\varepsilon) \, dx \leq \|\chi_\varepsilon\|_{L^1(\Omega_0)} \int_{\Omega_0} |w|^p \, dx = \|w\|^p_{L^p(\Omega_0)}$$

So, we have finally

$$\|r_\varepsilon^1\|_{L^\beta(\Omega_0)} \leq C\biggl(\int_{\Omega_0}\left\{\int_{B(x,\varepsilon)} \left\{\frac{|B(y)-B(x)|}{\varepsilon}\right\}^\alpha dy\right\} dx\biggr)^{\frac{1}{\alpha}} \|w\|_{L^p(\Omega_0)}$$

On the other hand

$$\biggl(\int_{\Omega_0}\left\{\int_{B(x,\varepsilon)} \left\{\frac{|B(y)-B(x)|}{\varepsilon}\right\}^\alpha dy\right\} dx\biggr)^{\frac{1}{\alpha}} = \biggl(\int_{\Omega_0}\int_{|z|\leq 1}\biggl(\int_0^1 |\nabla B(x+t\varepsilon z)| \, dt\biggr)^\alpha dz \, dx\biggr)^{\frac{1}{\alpha}} \leq$$

$$\leq C\|\nabla B\|_{L^\alpha(\Omega)}$$

So, if we integrate over $(0,T)$,

$$\int_0^T \|r_\varepsilon^1\|_{L^\beta(\Omega_0)} \, dt \leq C \sup_{t\in(0,T)} \|w\|_{L^p(\Omega_0)} \int_0^T \|\nabla B\|_{L^\alpha(\Omega_0)}$$

This density property allows us to prove the theorem in the only case of smooth functions. In a smooth scenario, we have that

$$\int_\Omega w(y)\{(B(y)-B(x))\cdot\nabla\eta_\varepsilon(y-x)\}\,dy=\int_\Omega w(y)B(y)\cdot\nabla\eta_\varepsilon(y-x)\,dy-B(x)\cdot\int_\Omega w(y)\nabla\eta_\varepsilon(y-x)\,dy\to$$

$$\to -\nabla\cdot(wB)+B\cdot\nabla w \qquad \text{in } L^\beta_{loc}(\Omega)$$

thanks to the properties of convolutions. But, in this regular case $\nabla\cdot(wB)=B\cdot\nabla w+w\nabla\cdot B$, and so, if $\nabla\cdot B=0$, exactly as $\nabla\cdot u=0$ in the hypothesis, we have the convergence to zero. This is the thesis. □ ∎

8.2.2 Uniqueness of the solution

The reformulation provided by theorem 8.4 allows us to prove some important results, concerning the weak transport equation. In particular we discuss now unieuqness of the solutions. However, before obtaining the uniqueness main result, we have an important lemma, that will introduce us to the *renormalized solutions.*

Lemma 8.1. *Let $p\in(1,\infty]$, and let $\rho\in L^\infty(0,T;L^p)$ be a solution of (8.4) with intial density $\rho_0\in L^p(\Omega)$ and assume that $u\in L^1(0,T;W^{1,\alpha}(\Omega))$ for some $\alpha\geq q$, $\nabla\cdot u=0$. Let $\eta_\varepsilon=\eta_\varepsilon(x)$ a regularizer kernel over Ω. In particular, if $\Omega_\varepsilon:=\{x\in\Omega:dist(x,\partial\Omega)>\varepsilon\}$, we set*

$$\eta_\varepsilon(x):=\frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right)$$

with $C^\infty_c(\mathbb{R}^n)\ni\eta\geq0$, $supp(\eta)\subset B(0,1)$. Let $\rho_\varepsilon(x,t):=\big(\rho(\cdot,t)*\eta_\varepsilon\big)(x,t)$. Let $\phi\in C^\infty_c([0,T)\times\Omega)$ and suppose that $\phi(x,\cdot)=0$ for every $x\in\Omega_0^c$, with Ω_0 compact. Let $\beta\in C^1(\mathbb{R})$ a function, with β' bounded and such that β vanishes near the origin. Then, if $\varepsilon<dist(\Omega_0,\partial\Omega)$, equation (8.10) holds, and implies that

$$-\int_0^T\left(\int_{\Omega_\varepsilon}\beta(\rho_\varepsilon)\frac{\partial\phi}{\partial t}\,dx\right)dt-\int_{\Omega_\varepsilon}\beta(\rho_\varepsilon^0)\;\phi(0,x)\,dx+\int_0^T\left(\int_{\Omega_\varepsilon}\beta(\rho_\varepsilon)u\cdot\nabla\phi\,dx\right)dt=$$

$$=\int_0^T\left(\int_{\Omega_\varepsilon}r_\varepsilon\beta'(\rho_\varepsilon)\phi\,dx\right)dt$$

where, as above,

$$r_\varepsilon(x,t)=\int_\Omega\rho(y,t)(u(y,t)-u(x,t))\cdot\nabla\eta_\varepsilon(y-x)\,dy$$

Proof. Consider (8.10), that is, for every $\phi\in C^\infty_c([0,T)\times\Omega)$ such that $\phi(x,\cdot)=0$ for every $x\in\Omega_0^c$, with Ω_0 compact, if $\varepsilon<\text{dist}(\Omega_0,\partial\Omega)$,

$$-\int_0^T\left(\int_{\Omega_\varepsilon}\rho_\varepsilon\frac{\partial\phi}{\partial t}\,dx\right)dt-\int_{\Omega_\varepsilon}\rho_\varepsilon^0\,\phi(0,x)\,dx+\int_0^T\left(\int_{\Omega_\varepsilon}\rho_\varepsilon u\cdot\nabla\phi\,dx\right)dt=\int_0^T\left(\int_\Omega r_\varepsilon\phi\,dx\right)dt \tag{8.12}$$

In particular, being $\rho_\varepsilon u\phi \equiv 0$ on $\partial\Omega_\varepsilon$, it follows that

$$\int_{\Omega_\varepsilon} \rho_\varepsilon \ \nabla \cdot (u\phi) \ dx = -\int_{\Omega_\varepsilon} \phi u \cdot \nabla \rho_\varepsilon \ dx + \int_{\Omega_\varepsilon} \nabla \cdot (\rho_\varepsilon u \phi) \ dx$$

so that, using the weak divergence theorem,

$$\int_{\Omega_\varepsilon} \rho_\varepsilon \ \nabla \cdot (u\phi) \ dx = -\int_{\Omega_\varepsilon} \phi u \cdot \nabla \rho_\varepsilon \ dx \tag{8.13}$$

We choose now $\phi(x,t) := \varphi(x)\psi(t)$, with $\psi \in C_c^\infty(0,T)$ and φ to be fixed. Then we have

$$-\int_0^T \Big(\int_{\Omega_\varepsilon} \rho_\varepsilon \psi'(t)\varphi(x) \ dx \Big) \ dt = \int_0^T \psi(t) \Big(\int_{\Omega_\varepsilon} (u \cdot \nabla \rho_\varepsilon + r_\varepsilon) \ \varphi(x) \Big) \ dt$$

If we choose as φ the unitary mass sequence

$$\varphi_y^n(x) := \eta_{\frac{1}{n}}(y-x)$$

it follows that

$$-\int_0^T \rho_\varepsilon(y) \ \psi'(t) \ dt = \int_0^T \psi(t)(u \cdot \nabla \rho_\varepsilon + r_\varepsilon)(y) \ dt$$

that is, in the sense of weak derivatives,

$$\frac{\partial}{\partial t}\rho_\varepsilon(y,t) = u(y,t) \cdot \nabla \rho_\varepsilon(y,t) + r_\varepsilon(y,t)$$

In particular, the equation is true for every $y \in \Omega_\varepsilon$. In particular, we have $\rho_\varepsilon(y,\cdot) \in W^{1,1}(0,T)$. On the other hand choosing, instead of ψ, the function

$$\eta_\delta^{t_0}(t) := \begin{cases} 1 & t \in [0,t_0] \\ 0 & t \in [t_0+\delta, T) \end{cases} \tag{8.14}$$

such that

$$\int_0^T (\eta_\delta^{t_0})'(t) \ dt = -1 \tag{8.15}$$

using the Lebesgue differentiation theorem, we have that, from (8.12) and (8.13) it follows that, for almost every $t_0 \in (0,T)$,

$$\rho_\varepsilon(t_0) = \rho_\varepsilon^0 + \int_0^{t_0} \big(u \cdot \nabla \rho_\varepsilon + r_\varepsilon \big) \ dt \tag{8.16}$$

This means that $\rho_\varepsilon(y,\cdot)$ is absolutely continuous and its continuos version is the right side of (8.16).

Consider now $\beta \in C^1(\mathbb{R})$ with β' bounded. The weak chain rule says that

$$\frac{\partial}{\partial t}\beta(\rho_\varepsilon) = \beta'(\rho_\varepsilon)\frac{\partial \rho_\varepsilon}{\partial t} = \beta'(\rho_\varepsilon)(u \cdot \nabla \rho_\varepsilon + r_\varepsilon) = u \cdot \nabla(\beta(\rho_\varepsilon)) + \beta'(\rho_\varepsilon) r_\varepsilon$$

since ρ_ε has classical regularity in space. So, in particular, being β' bounded, $\beta(\rho_\varepsilon) \in W^{1,1}(0,T)$ and so, moreover,

$$\beta(\rho_\varepsilon)(t) = \beta(\rho_\varepsilon^0) + \int_0^t \big(u \cdot \nabla(\beta(\rho_\varepsilon)) + \beta'(\rho_\varepsilon) r_\varepsilon\big)\ d\tau$$

In terms of weak derivatives we have

$$\int_0^T \beta(\rho_\varepsilon)\psi'(t)\ dt = -\int_0^T \psi(t)\big(u \cdot \nabla(\beta(\rho_\varepsilon)) + \beta'(\rho_\varepsilon) r_\varepsilon\big)\ dt$$

for every $\psi \in C_c^\infty(0,T)$.

Consider now $\phi \in C_c^\infty([0,T) \times \Omega)$, so that $\phi(T,x) = 0$. We know that

$$\partial_t(\beta(\rho_\varepsilon)\phi) = \partial_t(\beta(\rho_\varepsilon))\phi + \beta(\rho_\varepsilon)\partial_t\phi$$

so that $\beta(\rho_\varepsilon)\phi \in W^{1,1}(0,T)$. Moreover we have

$$0 = \beta(\rho_\varepsilon(T))\phi(T) = \beta(\rho_\varepsilon^0)\phi(0) + \int_0^T \partial_t(\beta(\rho_\varepsilon))\phi\ dt + \int_0^T \beta(\rho_\varepsilon)\partial_t\phi\ dt$$

Then we have

$$\int_0^T \left(\int_{\Omega_\varepsilon} \beta(\rho_\varepsilon)\partial_t\phi\ dx\right) dt = \int_{\Omega_\varepsilon} \left(\int_0^T \beta(\rho_\varepsilon)\partial_t\phi\ dt\right) dx =$$

$$= -\int_{\Omega_\varepsilon} \beta(\rho_\varepsilon^0)\phi(0)\ dx - \int_{\Omega_\varepsilon} \left(\int_0^T (u \cdot \nabla(\beta(\rho_\varepsilon)) + \beta'(\rho_\varepsilon) r_\varepsilon)\phi\ dt\right) dx =$$

$$= -\int_{\Omega_\varepsilon} \beta(\rho_\varepsilon^0)\phi(0)\ dx + \int_0^T \left(\int_{\Omega_\varepsilon} \beta(\rho_\varepsilon) u \cdot \nabla\phi\ dx\right) dt - \int_0^T \left(\int_{\Omega_\varepsilon} \beta'(r_\varepsilon) r_\varepsilon \phi\ dx\right) dt$$

that is the thesis. ∎

Theorem 8.5. *Let $p \in (1,\infty]$, and let $\rho \in L^\infty(0,T;L^p(\Omega))$ be a solution of* (8.4) *for the initial condition $\rho^0 \equiv 0$, with $u \in L^1(0,T;W_0^{1,q}(\Omega))$ and $\nabla \cdot u = 0$, being q the conjugate of p. Then, $\rho \equiv 0$.*

Proof. Letting $\varepsilon \to 0$ in the statement of lemma 8.1, with $\beta \in C^1(\mathbb{R})$ bounded, vanishing at the origin, and with β' bounded, we have that

$$-\int_0^T \left(\int_\Omega \beta(\rho)\frac{\partial\phi}{\partial t}\ dx\right) dt - \int_\Omega \beta(\rho_0)\phi(x,0)\ dx + \int_0^T \left(\int_\Omega \beta(\rho) u \cdot \nabla\phi\ dx\right) dt = 0 \quad (8.17)$$

where we used that $\rho^0 \equiv 0$ and $r_\varepsilon \to 0$ in $L^1(0,T;L^\beta_{loc}(\Omega))$.

Let now $M \in (0,\infty)$. We would choose $\beta(t) := (|t|^p \wedge M)$, where $a \wedge b := \min\{a,b\}$. The function is clearly bounded, but it is not in $C^1(\mathbb{R})$. However, it is possible to choose $\beta_k(t)$ a sequence such that $\beta_k \in C^1(\mathbb{R})$ for every k, $\beta_k(t) \leq \beta(t)$ for every $k \in \mathbb{N}$ and

$t \in \mathbb{R}$ and finally, for every $t \in \mathbb{R}$, $\beta_k(t) \le \beta_{k+1}(t)$, with $\beta_k(t) \to \beta(t)$ as $k \to \infty$, for almost every $t \in \mathbb{R}$. So (8.17) implies that

$$-\int_0^T \left(\int_\Omega \beta_k(\rho) \frac{\partial \phi}{\partial t}\ dx \right) dt - \int_\Omega \beta_k(\rho_0)\phi(x,0)\ dx + \int_0^T \left(\int_\Omega \beta_k(\rho) u \cdot \nabla \phi\ dx \right) dt = 0 \tag{8.18}$$

for every $k \in \mathbb{N}$. It is clear that $\beta_k(t) \le \beta(t) \le M$. By the hypothesis, we have that $u \in L^1(0,T;L^q(\Omega))$. So, using corollary 5.1, we have that if $\delta > 0$, there exists $N_\delta \in \mathbb{N}$ such that

$$\|u - u^{N_\delta}\|_{L^1(0,T;L^q(\Omega))} < \delta \tag{8.19}$$

We now choose ϕ in a precise way. In particular, we choose a sequence $\varphi_h \in C_c^\infty(\Omega)$ such that $\varphi_h \equiv 1$ over $\Omega_{\frac{1}{h}}$ (so that $|\nabla \varphi_h| \equiv 0$ over $\Omega_{\frac{1}{h}}$) and such that

$$\int_\Omega |\nabla \varphi_h|^{q'}\ dx = 1 \qquad \forall h \in \mathbb{N}$$

where q' is such that $\frac{1}{q} + \frac{1}{q'} = 1$. So, if $\delta > 0$ is fixed, and N_δ is such in (8.19), we have that $u^{N_\delta}(x,\cdot) \equiv 0$ if $x \in K_\delta^c$, with K_δ a compact set in Ω, and exists $H = H_\delta$, depending on δ such that $\Omega_{\frac{1}{h}} \supset K_\delta$ for every $h \ge H_\delta$. Then

$$\int_\Omega |\nabla \varphi_h| \left(\int_0^T |u^{N_\delta}|\ dt \right) dx = 0$$

for every $h \ge H_\delta$. Let $\phi_h(x,t) = \psi(t)\varphi_h(x)$, with $\psi \in C_c^\infty([0,T))$. It follows that

$$\begin{aligned}
\left| \int_0^T \psi(t) \left(\int_\Omega \beta_k(\rho)\ u \cdot \nabla \varphi_h\ dx \right) dt \right| &\le M \left(\max_{[0,T)} |\psi| \right) \left| \int_0^T \int_\Omega |u||\nabla \varphi_h|\ dx\ dt \right| = \\
&= M \left(\max_{[0,T)} |\psi| \right) \left| \int_\Omega \int_0^T |u||\nabla \varphi_h|\ dx\ dt - \int_\Omega |\nabla \varphi_h| \int_0^T |u^{N_\delta}|\ dt\ dx \right| = \\
&= M \left(\max_{[0,T)} |\psi| \right) \left| \int_\Omega \int_0^T \left(|u| - |u^{N_\delta}|\right) |\nabla \varphi_h|\ dx\ dt \right| \le \\
&\le M \left(\max_{[0,T)} |\psi| \right) \left(\int_\Omega \int_0^T \big||u| - |u^{N_\delta}|\big| |\nabla \varphi_h|\ dx\ dt \right) \le \\
&\le M \left(\max_{[0,T)} |\psi| \right) \left(\int_0^T \int_\Omega |u - u^{N_\delta}| |\nabla \varphi_h|\ dx\ dt \right) \le \\
&\le M \left(\max_{[0,T)} |\psi| \right) \left(\int_0^T \|u - u^{N_\delta}\|_q \|\nabla \varphi_h\|_{q'}\ dt \right) = M \left(\max_{[0,T)} |\psi| \right) \left(\int_0^T \|u - u^{N_\delta}\|_q\ dt \right)
\end{aligned} \tag{8.20}$$

since $\|\nabla \varphi_h\|_2 = 1$. Finally

$$\left| \int_0^T \psi(t) \left(\int_\Omega \beta_k(\rho)\ u \cdot \nabla \varphi_h\ dx \right) dt \right| \le M \left(\max_{[0,T)} |\psi| \right) \delta$$

for every $h \geq H_\delta$. This means that, for every $k \in \mathbb{N}$ and $\psi \in C_c^\infty([0,T))$

$$\lim_{h\to\infty} \int_0^T \psi(t)\Big(\int_\Omega \beta_k(\rho)\ u \cdot \nabla\varphi_h\ dx\Big)\ dt = 0$$

So, equation (8.18) becomes

$$-\int_0^T \psi'(t)\Big(\int_\Omega \beta_k(\rho)\varphi_h\ dx\Big)\ dt - \int_\Omega \beta(\rho_0)\psi(0)\ dx + \int_0^T \psi(t)\Big(\int_\Omega \beta_k(\rho) u \cdot \nabla\varphi_h\ dx\Big)\ dt = 0 \tag{8.21}$$

Since ψ' and β_k are bounded, and $\psi_h \to \chi_\Omega$ as $h \to \infty$, letting $h \to \infty$ we have

$$-\int_0^T \psi'(t)\Big(\int_\Omega \beta_k(\rho)\ dx\Big)\ dt - \int_\Omega \beta_k(\rho_0)\psi(0)\ dx = 0$$

We suppose $\psi(0) = 1$. Using again the boundedness of ψ' and the fact that β_k has been taken increasing, letting $k \to \infty$ we have, choosing $M = n \in \mathbb{N}$ fixed

$$-\int_0^T \psi'(t)\Big(\int_\Omega |\rho|^p \wedge n\ dx\Big)\ dt - \int_\Omega |\rho_0|^p \wedge n\ dx = 0$$

Choosing now ψ as in (8.14), (8.15), we have that for every $t_0 \in (0,T)/E_n$, with $|E_n| = 0$,

$$\int_\Omega |\rho(t_0)|^p \wedge n\ dx \equiv \Big(\int_\Omega |\rho|^p \wedge n\ dx\Big)(t_0) = \int_\Omega |\rho_0|^p \wedge n\ dx \tag{8.22}$$

Since the sequence $|\rho|^p \wedge n$ is increasing in n, and $|\rho|^p \wedge n \to |\rho|^p$ when $n \to \infty$, and (8.22) is defined for every $t_0 \in (0,T)/\bigcup_n E_n$, we have that for almost every $t_0 \in (0,T)$

$$\|\rho(t_0)\|_p = \|\rho_0\|_p \tag{8.23}$$

Since, by the hypothesis $\rho_0 \equiv 0$, this means that for almost every $t_0 \in (0,T)$, $\rho(t_0) = 0$ almost every $x \in \Omega$. This means that ρ is zero in $L^\infty(0,T;L^p(\Omega))$, that is the thesis. ∎

Remark 8.8. If now $p = \infty$, observe that in particular $\rho \in L^\infty(0,T;L^p(\Omega))$, with $p < \infty$. So $\rho \equiv 0$ by the case above. □

The next corollary follows from the proof of theorem 8.5.

Corollary 8.1. *Let $u \in L^1(0,T;L^1(\Omega))$ and $\rho_0 \in L^p(\Omega)$. Let ρ be a measurable function over $\Omega \times (0,T)$ such that, for every β admissible function,*

$$-\int_0^T \Big(\int_\Omega \beta(\rho)\frac{\partial\phi}{\partial t}\ dx\Big)\ dt - \int_\Omega \beta(\rho_0)\phi(x,0)\ dx + \int_0^T \Big(\int_\Omega \beta(\rho) u \cdot \nabla\phi\ dx\Big)\ dt = 0$$

for every $\phi \in C_c^\infty(\Omega \times [0,T))$. Then, for almost every $t_0 \in (0,T)$ we have

$$\|\rho(t_0)\|_p = \|\rho_0\|_p$$

8.2.3 Renormalized solutions

Definition 8.3. Let Ω be a bounded domain in $\mathbb{R}^n$, and $T > 0$ a time. Let $p \in (1, \infty]$, q its conjugate and $\rho^0 \in L^p(\Omega)$ an initial density. Let $u \in L^1(0, T; W_0^{1,q}(\Omega))$, $\nabla \cdot u = 0$ be a velocity field. We say that $\rho \in L^\infty(0, T; L^p(\Omega))$ is a *renormalized solution* of

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0 & \text{in } (0,T) \times \Omega \\ \rho(0) = \rho^0 \end{cases} \tag{8.24}$$

if, for every $\beta \in C^1(\mathbb{R})$, with β and $\frac{\beta'}{1+|t|}$ bounded and β such that vanishes near 0, it holds

$$-\int_0^T \left(\int_\Omega \beta(\rho) \frac{\partial \phi}{\partial t} \, dx \right) dt - \int_\Omega \beta(\rho_0(x)) \phi(x, 0) \, dx + \int_0^T \left(\int_\Omega \beta(\rho) u \cdot \nabla \phi \, dx \right) dt = 0 \tag{8.25}$$

for every $\phi \in C_c^\infty([0, T) \times \Omega)$.

Lemma 8.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ and $T > 0$ a positive time. Let $p \in (1, \infty]$ and $\rho \in L^\infty(0, T; L^p(\Omega))$ a solution of (8.4) with intial density $\rho_0 \in L^p(\Omega)$ and assume that $u \in L^1(0, T; W_0^{1,q}(\Omega))$, $\nabla \cdot u = 0$. The $\rho \in L^\infty(0, T; L^p(\Omega))$ is a renormalized solution to the problem for admissible function β with β' bounded.*

Proof. By theorem 8.1 we know that

$$-\int_0^T \left(\int_{\Omega_\varepsilon} \beta(\rho_\varepsilon) \frac{\partial \phi}{\partial t} \, dx \right) dt - \int_{\Omega_\varepsilon} \beta(\rho_\varepsilon^0) \; \phi(0, x) \, dx + \int_0^T \left(\int_{\Omega_\varepsilon} \beta(\rho_\varepsilon) u \cdot \nabla \phi \, dx \right) dt =$$

$$= \int_0^T \left(\int_{\Omega_\varepsilon} r_\varepsilon \beta'(\rho_\varepsilon) \phi \, dx \right) dt$$

with $r_\varepsilon \to 0$ in $L^1(0, T; L^\gamma_{loc}(\Omega))$, with $\dfrac{1}{\gamma} = \dfrac{1}{q} + \dfrac{1}{p} = 1 \Longrightarrow \gamma = 1$. So, letting $\varepsilon \to 0$, being β bounded and $|\beta'(\rho_\varepsilon)| \leq C_\beta$, we have that the thesis follows. ∎

8.2.4 Classical regularity of the solution

Before starting the conclusive section of the chapter about *stability*, we focus our attention on the regularity of the solutions to the weak transport equation.

Lemma 8.3. *Let $p \in (1, \infty)$ and $\rho^0 \in L^p(\Omega)$. Assume that $u \in L^1(0, T; W_0^{1,q}(\Omega))$ with $\nabla \cdot u = 0$. Then $\rho \in C([0, T]; L^p(\Omega))$.*

Remark 8.9. This theorems are Theorem II.3, Theorem II.4 and Corollary II.2 of paper [8]. □

Proof. By equation (8.23) we have that $\|\rho(t)\|_p$ has a continuous version $\|\rho(t)\|_p = \|\rho_0\|_p \in C([0, T])$. If we show that, moreover, for every $[0, T] \ni t_n \to t_0 \in [0, T]$ it holds

$$\lim_{n \to \infty} \int_\Omega \big(\rho(x, t_n) - \rho(x, t_0) \big) \cdot \varphi(x) \, dx = 0 \qquad \forall \varphi \in L^q(\Omega) \tag{8.26}$$

this means that $\rho(t_n) \rightharpoonup \rho(t_0)$ in $L^p(\Omega)$, that is $\rho(t_n)$ converges weakly to $\rho(t_0)$ in $L^p(\Omega)$. Since moreover $\|\rho(t_n)\|_p \to \|\rho(t_0)\|_p$ by continuity of the norm, we have, by theorem 3.5, that

$$\lim_{n\to\infty} \|\rho(t_n) - \rho(t_0)\|_p = 0$$

that is the continuity in $C([0,T]; L^p(\Omega))$. So, we only have to prove (8.26). We proceed as follows. If in equation (8.5) we choose $\phi(x,t) = \psi(t)\varphi(x)$ it follows that

$$-\int_0^T \psi(t)\Big(\int_\Omega \rho(x,t)\varphi(x)\ dx\Big)\ dt - \int_\Omega \rho^0(x)\psi(0)\varphi(x)\ dx +$$

$$+\int_0^T \psi(t)\Big(\int_\Omega \rho(x,t)\ \big(u(x,t)\cdot\nabla\varphi(x)\big)\ dx\Big)\ dt = 0$$

So, if $\psi(t)$ is choosen as in (8.14), (8.15), we have, for almost every $t_0 \in [0,T]$,

$$\int_\Omega \rho(x,t_0)\varphi(x)\ dx = \int_\Omega \rho^0(x)\varphi(x)\ dx - \int_0^{t_0} \Big(\int_\Omega \rho(x,t)\big(u(x,t)\cdot\nabla\varphi(x)\big)\ dx\Big)\ dt \quad (8.27)$$

The continuity of the right side implies that $\int_\Omega \rho(x,t_0)\varphi(x)\ dx$ can be defined in the whole $[0,T]$.

Consider now $h > 0$. Then, for every $\varphi \in C_c^\infty(\Omega)$ we have

$$\Big|\int_\Omega \rho(x,t_0+h)\varphi(x)\ dx - \int_\Omega \rho(x,t_0)\varphi(x)\ dx\Big| = \Big|\int_{t_0}^{t_0+h} \Big(\int_\Omega \rho(x,t)\big(u(x,t)\cdot\nabla\varphi(x)\big)\ dx\Big)\ dt\Big| \le$$

$$\le M_\varphi \int_{t_0}^{t_0+h} \|\rho(t)\|_p \|u(t)\|_q\ dt \le M_\varphi \|\rho\|_{L^\infty(0,T;L^p(\Omega))} \|u\|_{L^1(0,T;L^q(\Omega))} h$$

where $M_\varphi := \max_\Omega |\nabla\varphi|$. It follows that for every $\varphi \in C_c^\infty(\Omega)$

$$\lim_{h\to 0} \int_\Omega \rho(x,t_0+h)\varphi(x)\ dx = \int_\Omega \rho(x,t_0)\varphi(x)\ dx$$

But moreover $\|\rho(t_0+h)\|_p \le \max_{t\in[0,T]} \|\rho(t)\|_p$. So, by theorem 3.6 we have that

$$\lim_{h\to 0} \int_\Omega \rho(x,t_0+h)g(x)\ dx = \int_\Omega \rho(x,t_0)g(x)\ dx$$

for every $g \in L^q(\Omega)$. This implies (8.26) and thus the thesis. ∎

8.2.5 Stability

We now prove some consistency and stability results.

Consistency.

Lemma 8.4. *Let $\rho \in L^\infty(0,T;L^p(\Omega))$ and $u \in L^1(0,T;L^q(\Omega))$ with $p \in (1,\infty]$. If ρ is a renormalized solution, then ρ is a solution. Moreover, if ρ is a solution and $u \in L^1(0,T;W^{1,q}(\Omega))$, with $\nabla \cdot u = 0$, then ρ is a renormalized solution.*

Proof. We already know that if $u \in L^1(0,T;W^{1,q}(\Omega))$ and $\nabla \cdot u = 0$, then $\rho \in L^\infty(0,T;L^p(\Omega))$ is a renormalized solution, thanks to 8.2. We have to prove the converse implication. Suppose that $\rho \in L^\infty(0,T;L^p(\Omega))$ is a weak solution to the problem. We want to prove that it is a renormalized solution. We can consider a sequence β_n of admissible solution such that

$$|\beta_k(t)| \leq |t|, \qquad \beta_k(t) \to t \quad \text{uniformly on compacts of } \mathbb{R}$$

In particular, one can consider at first $\beta_k(t) := |t| \wedge k$, and then a C^1 approximation of this function from above, with bounded derivative. So we have

$$-\int_0^T \left(\int_\Omega \beta_k(\rho) \frac{\partial \phi}{\partial t} \, dx \right) dt - \int_\Omega \beta_k(\rho_0)\phi(x,0) \, dx + \int_0^T \left(\int_\Omega \beta_k(\rho) u \cdot \nabla \phi \, dx \right) dt = 0$$

We have now the bounds

$$\int_0^T \left(\int_\Omega |\beta_k(\rho)| |\frac{\partial \phi}{\partial t}| \, dx \right) dt \leq \int_0^T \left(\int_\Omega |\rho| |\frac{\partial \phi}{\partial t}| \, dx \right) dt < \infty$$

since $\rho \in L^\infty(0,T;L^p(\Omega))$. Similarly, we have

$$\int_\Omega |\beta_k(\rho_0)||\phi(x,0)| \, dx \leq \int_\Omega |\rho_0||\phi(x,0)| \, dx < \infty$$

$$\int_0^T \left(\int_\Omega |\beta_k(\rho)||u||\nabla \phi| \, dx \right) dt \leq \int_0^T \left(\int_\Omega |\rho||u||\nabla \phi| \, dx \right) dt < \infty$$

Since $\beta_k(t) \to t$ as $k \to \infty$ for every $t \in \mathbb{R}$, letting $k \to \infty$, we have equation 8.5 that is the weak formulation. ∎

Stability. The following theorem is the main result of this section, that in turn is one of the fundamental results of the thesis. As usual, provided a uniqueness result, as that of theorem 8.5, one expects a stability result of the solutions. We will use this fact in proposition 11.20.

Theorem 8.6. *Let $p \in (1,\infty)$. Let $u^n \in L^1(0,T;L^1(\Omega))$ be such that converges to u in $L^1(0,T;L^1(\Omega))$. Let ρ^n a bounded sequence in $L^\infty(0,T;L^p(\Omega))$, i.e. $\sup_{n\in\mathbb{N}} \|\rho_n\|_{L^\infty(0,T;L^p(\Omega))} < \infty$, such that ρ^n is a renormalized solution of the transport equation with velocity field u^n, corresponding to an initial condition $\rho_n^0 \in L^p(\Omega)$. Assume that ρ_n^0 converges in $L^p(\Omega)$ to some $\rho^0 \in L^p(\Omega)$. Suppose moreover that for every β admissible function $\beta(\rho_n^0) \to \beta(\rho^0)$ in $L^1(\Omega)$. Then ρ^n converges to $\rho \in L^\infty(0,T;L^p(\Omega))$, renormalized solution of the transport equation with velocity field u and inital density ρ^0, in $L^\infty(0,T;L^p(\Omega))$.*

Remark 8.10. To prove these theorems, we will use the following lemma by basic real analysis. See lemma 7.1. □

Proof. Now we want to prove the stability. We start with pointwise stability. Let β an admissible function, and define $v_n := \beta(\rho_n)$, where ρ^n is renormalized solution to the transport equation with velocity filed u^n and initial density ρ_0^n. Then, since β is bounded, we have that $v_n \in L^\infty(0,T;L^\infty(\Omega))$. Moreover, observe that, since ρ^n is a renormalized solution,

$$-\int_0^T \left(\int_\Omega \beta(\rho^n) \frac{\partial \phi}{\partial t} \, dx \right) dt - \int_\Omega \beta(\rho_n^0)\phi(x,0) \, dx + \int_0^T \left(\int_\Omega \beta(\rho^n) u^n \cdot \nabla \phi \, dx \right) dt = 0$$

and this can be rewritten as

$$-\int_0^T \left(\int_\Omega v_n \frac{\partial \phi}{\partial t} \, dx \right) dt - \int_\Omega v_n^0 \phi(x,0) \, dx + \int_0^T \left(\int_\Omega v_n \big(u^n \cdot \nabla \phi\big) \, dx \right) dt = 0$$

where $\beta(\rho_n^0) =: v_n^0$. On the other hand, the function β^2 is admissible yet, and, as above, $w_n := v_n^2 \in L^\infty(0,T;L^\infty(\Omega))$. Moreover, as above,

$$-\int_0^T \left(\int_\Omega w_n \frac{\partial \phi}{\partial t} \, dx \right) dt - \int_\Omega w_n^0 \phi(x,0) \, dx + \int_0^T \left(\int_\Omega w_n \big(u^n \cdot \nabla \phi\big) \, dx \right) dt = 0$$

and $w_n^0 := (v_n^0)^2$. Since the sequences are bounded in $L^\infty(0,T;L^\infty(\Omega))$, we have that exist $v, w \in L^\infty(0,T;L^\infty(\Omega))$ such that, up to extract a subsequence,

$$v_n \overset{*}{\rightharpoonup} v, \quad w_n \overset{*}{\rightharpoonup} w \quad \text{in } L^\infty(0,T;L^\infty(\Omega))$$

In particular

$$\|v\|_{L^\infty(0,T;L^\infty(\Omega))} \le \liminf_{n\to\infty} \|v_n\|_{L^\infty(0,T;L^\infty(\Omega))} \le C_\beta$$

where $\beta(s) \le C_\beta$ for every $s \in \mathbb{R}$. Observe that, up to extract a subsequence, we can suppose that ρ_n^0 converges to ρ^0 almost everywhere in Ω.

Since $u^n \to u$ in $L^1(0,T;L^1(\Omega))$, we have that, considering in example the case of v_n (that of w_n is analogous),

$$\int_0^T \left(\int_\Omega v_n \frac{\partial \phi}{\partial t} \, dx \right) dt \to \int_0^T \left(\int_\Omega v_n \frac{\partial \phi}{\partial t} \, dx \right) dt, \qquad \int_\Omega v_n^0 \phi(0,x) \, dx \to \int_\Omega v^0 \phi(0,x) \, dx$$

since $\partial_t \phi \in L^1(0,T;L^1(\Omega))$ and $\phi(0,x) \in L^1(\Omega)$. Moreover

$$\left| \int_0^T \left(\int_\Omega v_n \big(u^n \cdot \nabla \phi\big) \, dx \right) dt - \int_0^T \left(\int_\Omega v \big(u \cdot \nabla \phi\big) \, dx \right) dt \right| \le$$

$$\le \left| \int_0^T \left(\int_\Omega v_n \big((u^n - u) \cdot \nabla \phi\big) \, dx \right) dt - \int_0^T \left(\int_\Omega (v - v_n) \big(u \cdot \nabla \phi\big) \, dx \right) dt \right| \le$$

$$\le \int_0^T \int_\Omega |v_n||u^n - u||\nabla \phi| \, dx \, dt + \left| \int_0^T \left(\int_\Omega (v - v_n) \big(u \cdot \nabla \phi\big) \, dx \right) dt \right| \le$$

$$\leq M\|v_n\|_{L^\infty(0,T;L^\infty(\Omega))}\|u^n - u\|_{L^1(0,T;L^1(\Omega))} + \left|\int_0^T \left(\int_\Omega (v - v_n)\big(u \cdot \nabla\phi\big)\ dx\right)\ dt\right|$$

where M is such that $|\nabla\phi| \leq M$. Since $\|v_n\|_{L^\infty(0,T;L^\infty(\Omega))}$ is bounded and $u^n \to u$ in $L^1(0,T;L^1(\Omega))$, we only have to prove that also the other term vanishes. But

$$\int_0^T \int_\Omega |u||\nabla\phi|\ dx\ dt \leq M\|u\|_{L^1(0,T;L^1(\Omega))} < \infty$$

that is $u \cdot \nabla\phi \in L^1(0,T;L^1(\Omega))$, and since $v_n \overset{*}{\rightharpoonup} v$ in $L^\infty(0,T;L^\infty(\Omega))$, we have that also this term vanishes. So finally

$$-\int_0^T \left(\int_\Omega v\frac{\partial\phi}{\partial t}\ dx\right)\ dt - \int_\Omega v^0(x)\ \phi(x,0)\ dx + \int_0^T \left(\int_\Omega v\ \big(u \cdot \nabla\phi\big)\ dx\right)\ dt = 0 \quad (8.28)$$

and, in the same way,

$$-\int_0^T \left(\int_\Omega w\frac{\partial\phi}{\partial t}\ dx\right)\ dt - \int_\Omega w^0(x)\ \phi(x,0)\ dx + \int_0^T \left(\int_\Omega w\ \big(u \cdot \nabla\phi\big)\ dx\right)\ dt = 0$$

Remark 8.11. Observe that, since $\rho_0^n \to \rho_0$ almost everywhere and β is bounded, we have $\beta^\alpha(\rho_0^n) \to \beta^\alpha(\rho_0)$ in $L^1(\Omega)$, with $\alpha \in \{1,2\}$. □

Equation (8.28) says that v is a weak solution, with initial condition v^0; by the previous lemma it is a renormalized solution.

Choosing $\alpha(t) = t^2$, approaching this function with admissible $\alpha_k(t)$ such that $\alpha_k(t) \leq t^2$ and $\alpha_k(t) \to t^2$ as $k \to \infty$, for every $t \in \mathbb{R}$. So we have that

$$-\int_0^T \left(\int_\Omega \alpha_k(v)\frac{\partial\phi}{\partial t}\ dx\right)\ dt - \int_\Omega \alpha_k(v^0(x))\ \phi(x,0)\ dx + \int_0^T \left(\int_\Omega \alpha_k(v)\ \big(u \cdot \nabla\phi\big)\ dx\right)\ dt = 0$$

implies, letting $k \to \infty$,

$$-\int_0^T \left(\int_\Omega v^2\frac{\partial\phi}{\partial t}\ dx\right)\ dt - \int_\Omega (v^0)^2(x)\ \phi(x,0)\ dx + \int_0^T \left(\int_\Omega v^2\ \big(u \cdot \nabla\phi\big)\ dx\right)\ dt = 0$$

since $|v(x,t)|^2 \leq \|v\|^2_{L^\infty(0,T;L^\infty(\Omega))} \leq C_\beta^2$ and $v^0 = \beta(\rho^0) \leq C_\beta$, so that the integrals are well-posed.

So, v^2 is a weak solution to the transport equation with initial condition $(v^0)^2$. But also w is a weak solution to the same transport equation with initial condition $(v^0)^2$. By uniqueness theorem 8.5, we have $v^2 \equiv w$.

This means that

$$v_n^2 \overset{*}{\rightharpoonup} v^2 \quad \text{in } L^\infty(0,T;L^\infty(\Omega))$$

Moreover, notice that

$$\|v_n - v\|^2_{L^2(0,T;L^2(\Omega))} = \int_0^T \left(\int_\Omega |v_n - v|^2\ dx\right)\ dt = \langle v_n - v, v_n - v\rangle_{L^2(0,T;L^2(\Omega))} =$$

$$= \langle v_n, v_n\rangle_{L^2(0,T;L^2(\Omega))} - 2\langle v_n, v\rangle_{L^2(0,T;L^2(\Omega))} + \langle v, v\rangle_{L^2(0,T;L^2(\Omega))}$$

Observe that

$$\langle v_n, v\rangle_{L^2(0,T;L^2(\Omega))} \to \langle v, v\rangle_{L^2(0,T;L^2(\Omega))}$$

since $v \in L^\infty(0,T;L^\infty(\Omega)) \subset L^1(0,T;L^1(\Omega))$ and $v_n \stackrel{*}{\rightharpoonup} v$. Moreover, if we choose the function $\phi \equiv 1$ on $(0,T)\times\Omega$, that is in $L^1(0,T;L^1(\Omega))$, we have

$$\|v_n\|^2_{L^2(0,T;L^2(\Omega))} = \langle v_n, v_n\rangle_{L^2(0,T;L^2(\Omega))} = \int_0^T \left(\int_\Omega |v_n|^2\,dx\right)dt \equiv \int_0^T \left(\int_\Omega v_n^2\,\phi\,dx\right)dt \to$$

$$\to \int_0^T \left(\int_\Omega v^2\,\phi\,dx\right)dt = \int_0^T \left(\int_\Omega v^2\,dx\right)dt = \|v\|^2_{L^2(0,T;L^2(\Omega))} \tag{8.29}$$

as $n\to\infty$, since $\phi \in L^1(0,T;L^1(\Omega))$ and $v_n^2 \stackrel{*}{\rightharpoonup} v^2$ in $L^\infty(0,T;L^\infty(\Omega))$. This means that $v_n \to v$ in $L^2(0,T;L^2(\Omega))$.

Remark 8.12. We can choose $\alpha(t) = |t|^p$, with $p \in (1,\infty)$, and obtain the same result in $L^p(0,T;L^p(\Omega))$. In fact, this choice implies that $|v_n|^p \stackrel{*}{\rightharpoonup} |v|^p$ in $L^\infty(0,T;L^\infty(\Omega))$. Moreover, $L^p(0,T;L^p(\Omega))$ is the dual of $L^q(0,T;L^q(\Omega))$, with q and q conjugate exponents. So, for every $\nu \in L^q(0,T;L^q(\Omega))$, we have $\langle v_n,\nu\rangle_{p,q} \to \langle v,\nu\rangle_{p,q}$, as $n\to\infty$, where $\langle\cdot,\cdot\rangle_{p,q} \equiv \langle\cdot,\cdot\rangle_{L^p(0,T;L^p(\Omega)),L^q(0,T;L^q(\Omega))}$ is the dual pairing between $L^p(0,T;L^p(\Omega))$ and $L^q(0,T;L^q(\Omega))$. In fact

$$\langle v_n,\nu\rangle_{p,q} = \int_0^T \left(\int_\Omega v_n\cdot\nu\,dx\right)dt \to \int_0^T \left(\int_\Omega v\cdot\nu\,dx\right)dt = \langle v,\nu\rangle_{p,q}$$

as $n\to\infty$, since $v_n \stackrel{*}{\rightharpoonup} v$ in $L^\infty(0,T;L^\infty(\Omega))$ and $\nu \in L^q(0,T;L^q(\Omega)) \subset L^1(0,T;L^1(\Omega))$, being $q>1$. This means that $v_n \stackrel{*}{\rightharpoonup} v$ in $L^p(0,T;L^p(\Omega))$. Since, choosing $\phi\equiv 1$ as in (8.29), $|v_n|^p \stackrel{*}{\rightharpoonup} |v|^p$ in $L^\infty(0,T;L^\infty(\Omega))$ implies $\|v_n\|_{L^p(0,T;L^p(\Omega))} \to \|v\|_{L^p(0,T;L^p(\Omega))}$ as $n\to\infty$, the generalized version of theorem 3.5 (see remark 3.4) implies that $v_n \to v$ in $L^p(0,T;L^p(\Omega))$ in the strong sense. □

We now want to show that $v = \beta(\rho)$, for some $\rho \in L^\infty(0,T;L^p(\Omega))$, so that we have the convergence (in $L^p(0,T;L^p(\Omega))$) of $\beta(\rho_n)$ to v; this implies (since v satisfies the weak transport equation) that ρ is a renormalized solution, and so, by theorem 8.4, a (unique) solution.

We know that $v_n = \beta(\rho_n)$ converges to $v \in L^2(0,T;L^2(\Omega))$ in $L^2(0,T;L^2(\Omega))$. This implies that v_n converges to v in measure, that is $\beta(\rho_n)$ converges in measure to v. Since $|\Omega\times(0,T)| < \infty$ and

$$\|\rho_n\|_{L^p(0,T;L^p(\Omega)} \le C\|\rho_n\|_{L^\infty(0,T;L^p(\Omega))} \le C\left(\sup_{n\in\mathbb{N}}\|\rho_n\|_{L^\infty(0,T;L^p(\Omega))}\right) < \infty$$

using propistion 3.1, we have that exists ρ, measurable function on $\Omega\times(0,T)$, such that $\rho_n \to \rho$ in measure[4]. But, if $\beta \in C^1(\mathbb{R})$ is an admissible function, we have, by

[4]The convergence of $\beta(\rho_n)$ holds for every $\beta \in C^1(\mathbb{R})$. However, proposition 3.1 holds in this case.

proposition 3.1, that $v_n \equiv \beta(\rho_n) \to \beta(\rho)$ in measure. It follows that $v = \beta(\rho)$. In fact, we have

$$\|\beta(\rho) - v\|_{L^2(0,T;L^2(\Omega))} \leq \|\beta(\rho) - \beta(\rho_n)\|_{L^2(0,T;L^2(\Omega))} + \|\beta(\rho_n) - v\|_{L^2(0,T;L^2(\Omega))}$$

We know from above that $\|\beta(\rho_n) - v\|_{L^2(0,T;L^2(\Omega))} \to 0$ as $n \to \infty$. On the other hand, $\beta(\rho_n)$ converges to $\beta(\rho)$ in measure and $|\beta(\rho_n)| \leq C_\beta$ implies that $\beta(\rho_n)$ has a uniform integrable bound in $L^2(0,T;L^2(\Omega))$. So, again by proposition 3.1, $\|\beta(\rho_n) - \beta(\rho)\|_{L^2(0,T;L^2(\Omega))} \to 0$ as $n \to \infty$. So, $\beta(\rho) = v \in L^2(0,T;L^2(\Omega))$.

Remark 8.13. The same argument holds with $L^2(0,T;L^2(\Omega))$ replaced by $L^p(0,T;L^p(\Omega))$. □

So, the measurable function ρ is a renormalized solution of the weak transport equation, since $v = \beta(\rho)$ is a solution. Now, using the arguments in the proof of theorem 8.5 (see corollary 8.1), with $q = 1$ (where only the measurability of ρ is used), we deduce that $\|\rho(t_0)\|_p = \|\rho_0\|_p$ for almost every $t_0 \in (0,T)$. So $\rho \in L^\infty(0,T;L^p(\Omega))$, and this implies that ρ is a solution to the weak transport equation with initial density ρ_0.

Remark 8.14. The aim of the theorem is to prove that $\rho_n \to \rho$ in $C([0,T];L^p(\Omega))$, where ρ is a renormalized solution of the weak transport equation with velocity field u and initial density ρ^0. If we know *a priori* that $\rho_n \overset{*}{\rightharpoonup} \overline{\rho}$ in $L^\infty(0,T;L^\infty(\Omega))$ to some $\overline{\rho} \in L^\infty(0,T;L^p(\Omega))$, with $\overline{\rho}$ weak solution to the transport equation with field u and initial density ρ^0, then by uniqueness theorem $\rho \equiv \overline{\rho}$, and so $\rho_n \to \overline{\rho}$ in $C([0,T];L^p(\Omega))$. In this spirit we will use this stability theorem. □

We have finally that $\rho \in L^\infty(0,T;L^p(\Omega))$ is a renormalized solution, that is, if β is an admissible function, with $M > 0$ such that $|\beta(s)| \leq M$ for every $s \in \mathbb{R}$, we have

$$-\int_0^T \left(\int_\Omega \beta(\rho)\ \partial_t \phi\ dx \right) dt - \int_\Omega \beta(\rho^0(x))\phi(0,x)\ dx + \int_0^T \left(\int_\Omega \beta(\rho)\ (u \cdot \nabla \phi)\ dx \right) dt = 0$$

Choosing $\phi \in C_c^\infty([0,T) \times \Omega)$ as in (8.27), we have, for every $t_0 \in [0,T]$ (eventually redefining the function out of a zero measure set)

$$\int_\Omega \beta(\rho(t_0,x))\varphi(x)\ dx = \int_\Omega \beta(\rho^0(x))\varphi(x) - \int_0^{t_0} \left(\int_\Omega \beta(\rho(x,t))u(x,t) \cdot \nabla\varphi(x)\ dx \right) dt$$

Moreover, by the hypothesis, ρ_n is renormalized solution to the transport equation with velocity field u^n and initial density ρ_n^0. It follows that, if $t_0 \in [0,T]$, we have

$$\int_\Omega \beta(\rho_n(t_0,x))\varphi(x)\ dx = \int_\Omega \beta(\rho_n^0(x))\varphi(x) - \int_0^{t_0} \left(\int_\Omega \beta(\rho_n(x,t))u_n(x,t) \cdot \nabla\varphi(x)\ dx \right) dt$$

Let now $[0,T] \ni t_n \to t_0 \in [0,T]$ and consider that

$$\int_\Omega \beta(\rho_n(t_n,x))\varphi(x)\ dx = \int_\Omega \beta(\rho_n^0(x))\varphi(x) - \int_0^{t_n} \left(\int_\Omega \beta(\rho_n(x,t))u_n(x,t) \cdot \nabla\varphi(x)\ dx \right) dt$$

We want to show that

$$\lim_{n\to\infty} \int_\Omega \beta(\rho_n(t_n,x))\varphi(x)\ dx = \int_\Omega \beta(\rho(t_0,x))\varphi(x)\ dx \tag{8.30}$$

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for every $\varphi \in C_c^\infty(\Omega)$. But this is true. In fact

$$\int_\Omega \big(\beta(\rho_n(t_n,x)) - \beta(\rho(t_0,x))\big)\varphi(x)\ dx = \int_\Omega \big(\beta(\rho_n^0(x)) - \beta(\rho^0(x))\big)\varphi(x)\ dx -$$

$$-\Big\{\int_0^{t_n}\Big(\int_\Omega \beta(\rho_n(x,t))u_n(x,t)\cdot\nabla\varphi(x)\ dx\Big)\ dt - \int_0^{t_0}\Big(\int_\Omega \beta(\rho(x,t))u(x,t)\cdot\nabla\varphi(x)\ dx\Big)\ dt\Big\} =$$

$$= \int_\Omega \big(\beta(\rho_n^0(x)) - \beta(\rho^0(x))\big)\varphi(x)\ dx - \int_0^{t_0}\Big(\int_\Omega \beta(\rho_n)u_n\cdot\nabla\varphi\ dx - \int_\Omega \beta(\rho)u\cdot\nabla\varphi\ dx\Big)\ dt -$$

$$-\int_{t_0}^{t_n}\int_\Omega \beta(\rho_n)u_n\cdot\nabla\varphi\ dx\ dt$$

Observe, first of all, that

$$\Big|\int_\Omega \big(\beta(\rho_n^0(x)) - \beta(\rho^0(x))\big)\varphi(x)\ dx\Big| \le \|\varphi\|_\infty \|\beta(\rho_n^0) - \beta(\rho^0)\|_1 \to 0$$

as $n \to \infty$. Furthermore

$$\Big|\int_0^{t_0}\Big(\int_\Omega \beta(\rho_n)u_n\cdot\nabla\varphi\ dx - \int_\Omega \beta(\rho)u\cdot\nabla\varphi\ dx\Big)\ dt\Big| = \Big|\int_0^{t_0}\Big(\int_\Omega \big(\beta(\rho_n)u_n - \beta(\rho)u\big)\cdot\nabla\varphi\ dx\Big)\ dt\Big| =$$

$$= \Big|\int_0^{t_0}\Big(\int_\Omega \beta(\rho_n)\big(u_n - u\big)\cdot\nabla\varphi\ dx\Big)\ dt + \int_0^{t_0}\Big(\int_\Omega \big(\beta(\rho_n) - \beta(\rho)\big)u\cdot\nabla\varphi\ dx\Big)\ dt\Big| \le$$

$$\le M\|\nabla\varphi\|_\infty \int_0^T\int_\Omega |u_n - u|\ dx\ dt + \Big|\int_0^{t_0}\Big(\int_\Omega \big(\beta(\rho_n) - \beta(\rho)\big)u\cdot\nabla\varphi\ dx\Big)\ dt\Big| \to 0$$

as $n \to \infty$, since $u^n \to u$ in $L^1(0,T;L^1(\Omega))$ and $\beta(\rho_n) \overset{*}{\rightharpoonup} v = \beta(\rho)$ in $L^\infty(0,T;L^\infty(\Omega))$ and $\chi_{(0,t_0)}u\cdot\nabla\varphi \in L^1(0,T;L^1(\Omega))$.

Moreover, we have

$$\Big|\int_{t_0}^{t_n}\int_\Omega \beta(\rho_n)u_n\cdot\nabla\varphi\ dx\ dt\Big| \le \Big|\int_{t_0}^{t_n}\Big(\int_\Omega \beta(\rho_n)u_n\cdot\nabla\varphi\ dx - \int_\Omega \beta(\rho)u\cdot\nabla\varphi\ dx\Big)\ dt\Big| +$$

$$+\Big|\int_{t_0}^{t_n}\Big(\int_\Omega \beta(\rho)u\cdot\nabla\varphi\Big)\ dt\Big| \le \Big|\int_0^T \chi_{(t_0,t_n)}(t)\Big(\int_\Omega \beta(\rho_n)u_n\cdot\nabla\varphi\ dx - \int_\Omega \beta(\rho)u\cdot\nabla\varphi\ dx\Big)\ dt\Big| +$$

$$+M\|\nabla\varphi\|_\infty \int_{t_0}^{t_n}\|u\|_1\ dt \le \Big|\int_0^T \chi_{(t_0,t_n)}(t)\Big(\int_\Omega \beta(\rho_n)(u_n - u)\cdot\nabla\varphi\ dx + \int_\Omega (\beta(\rho_n) - \beta(\rho))u\cdot\nabla\varphi\ dx\Big)\ dt\Big| +$$

$$+M\|\nabla\varphi\|_\infty \int_{t_0}^{t_n}\|u\|_1\ dt \le M\|\nabla\varphi\|_\infty \|u_n - u\|_{L^1(0,T;L^1(\Omega))} + 3M\|\nabla\varphi\|_\infty \int_{t_0}^{t_n}\|u\|_1\ dt \to 0$$

as $n \to \infty$, since $t_n \to t_0$, $u^n \to u$ in $L^1(0,T;L^1(\Omega)$ as $n \to \infty$. So we have proved (8.30). Starting from this point, we want to show that also

$$\lim_{n\to\infty}\int_\Omega \rho_n(t_n,x)\varphi(x)\ dx = \int_\Omega \rho(t_0,x)\varphi(x)\ dx \tag{8.31}$$

for every $\varphi \in C_c^\infty(\Omega)$ and $t_n \to t_0$.

Remark 8.15. If (8.31) holds, then it is true for every $\varphi \in L^q(\Omega)$. Moreover, we have that

$$\|\rho_n(t_n)\|_p = \|\rho_0^n\|_p \to \|\rho_0\|_p = \|\rho_0(t_0)\|_p$$

thanks to the convergence of $\rho_0^n \to \rho_0$ in $L^p(\Omega)$ and using corollary 8.1, since $u_n, u \in L^1(0,T;L^1(\Omega))$ and ρ, ρ_n are renormalized solutions. So, it follows that $\rho_n(t_n) \to \rho(t_0)$ in $L^p(\Omega)$. $\square$

So we have to prove (8.31). Given $M \in (0, \infty)$, consider the function

$$\beta_M(s) := \begin{cases} s & |s| \leq M \\ M & |s| > M \end{cases}$$

We have to fix this M in a precise way. Let $t_n \to t_0 \in [0,T]$ and a consider the sequence $\{\rho_n(t_n)\}_{n\in\mathbb{N}} \cup \{\rho(t_0)\} \subset L^p(\Omega)$. Moreover, this sequence is bounded in $L^p(\Omega)$, since

$$\|\rho_n(t_n)\|_p, \|\rho(t_0)\|_p \leq \|\rho(t_0)\| + \sup_n \|\rho_n\|_{L^\infty(0,T;L^p(\Omega))}$$

since $\|\rho_n(t_n)\|_p \leq \sup_{(0,T)} \|\rho_n(t)\|_p \equiv \|\rho_n\|_{L^\infty(0,T;L^p(\Omega))} \leq C$ by the hypothesis. So, using theorem 3.7, we have that for every $\varepsilon > 0$ exists $M_\varepsilon > 0$ such that

$$\int_{\{x\in\Omega:\ |\rho(t_0,x)|>M_\varepsilon\}} |\rho(t_0,x)|\,dx, \int_{\{x\in\Omega:\ |\rho_n(t_n,x)|>M_\varepsilon\}} |\rho_n(t_n,x)|\,dx < \varepsilon \qquad \forall\, n \in \mathbb{N} \tag{8.32}$$

Remark 8.16. Notice that (8.32) implies that

$$M_\varepsilon|\{x \in \Omega :\ |\rho(t_0,x)| > M_\varepsilon\}|,\ M_\varepsilon|\{x \in \Omega :\ |\rho_n(t_n,x)| > M_\varepsilon\}| < \varepsilon \qquad \forall\, n \in \mathbb{N}$$

that will be useful in the future. $\square$

Fix $\varepsilon > 0$ and choose $M_\varepsilon > 0$ as above. Then we can consider β_{M_ε}. Moreover, let $\beta^k_{M_\varepsilon}$ an admissible functions that coincides with β_{M_ε} outside a neighbourhood of M_ε, and such that

$$|\beta^{k_\varepsilon}_{M_\varepsilon}(s)| \leq |\beta_{M_\varepsilon}(s)| \leq M_\varepsilon, \qquad \sup_{s\in\mathbb{R}} |\beta^k_{M_\varepsilon}(s) - \beta_{M_\varepsilon}| < \frac{1}{k}$$

If we now consider the admissible function $|\beta^{k_\varepsilon}_{M_\varepsilon}| \leq M_\varepsilon$, from We can choose $k_\varepsilon \in \mathbb{N}$ such that $\dfrac{|\Omega|\|\varphi\|_\infty}{k_\varepsilon} < \varepsilon$. So, we can write

$$\int_\Omega \rho_n(t_n,x)\varphi(x)\,dx = \int_\Omega \beta_{M_\varepsilon}(\rho_n(t_n,x))\varphi(x)\,dx + \int_\Omega \{\rho_n(t_n,x) - \beta_{M_\varepsilon}(\rho_n(t_n,x))\}\varphi(x)\,dx \tag{8.33}$$

We, at first, focus our attention to the second addend. We have

$$\left|\int_\Omega \{\rho_n(t_n,x) - \beta_{M_\varepsilon}(\rho_n(t_n,x))\}\varphi(x)\,dx\right| = \left|\int_{\{x\in\Omega:\ |\rho_n(t_n,x)|\leq M_\varepsilon\}} \{\rho_n(t_n,x) - \beta_{M_\varepsilon}(\rho_n(t_n,x))\}\varphi(x)\,dx + \right.$$

$$\left. + \int_{\{x\in\Omega:\ |\rho_n(t_n,x)|>M_\varepsilon\}} \{\rho_n(t_n,x) - \beta_{M_\varepsilon}(\rho_n(t_n,x))\}\varphi(x)\,dx\right| =$$

$$= \left| \int_{\{x\in\Omega:\ |\rho_n(t_n,x)|>M_\varepsilon\}} \{\rho_n(t_n,x) - M_\varepsilon\}\varphi(x)\ dx \right| \le$$

$$\le \|\varphi\|_\infty \left(\int_{\{x\in\Omega:\ |\rho_n(t_n,x)|>M_\varepsilon\}} |\rho_n(t_n,x)|\ dx + M_\varepsilon |\{x\in\Omega:\ |\rho_n(t_n,x)|>M_\varepsilon\}| \right) < 2\varepsilon \|\varphi\|_\infty$$

If in equation (8.33) we subtract the term $\int_\Omega \rho(t_0,x)\varphi(x)\ dx$, we have also to consider

$$\left| \int_\Omega \beta_{M_\varepsilon}(\rho_n(t_n,x))\varphi(x)\ dx - \int_\Omega \rho(t_0,x)\varphi(x)\ dx \right| \le$$

$$\le \left| \int_\Omega \big(\beta_{M_\varepsilon}(\rho_n(t_n,x)) - \beta_{M_\varepsilon}(\rho(t_0,x))\big)\varphi(x)\ dx \right| +$$

$$+ \left| \int_\Omega \big(\beta_{M_\varepsilon}(\rho(t_0,x)) - \rho(t_0,x)\big)\varphi(x)\ dx \right|$$

We deal at first with the second addend. Following the steps above, we have again

$$\left| \int_\Omega \big(\beta_{M_\varepsilon}(\rho(t_0,x)) - \rho(t_0,x)\big)\varphi(x)\ dx \right| \le$$

$$\le \left| \int_{\{x\in\Omega:\ |\rho(t_0,x)|\le M_\varepsilon\}} \big(\beta_{M_\varepsilon}(\rho(t_0,x)) - \rho(t_0,x)\big)\varphi(x)\ dx \right| +$$

$$+ \left| \int_{\{x\in\Omega:\ |\rho(t_0,x)|>M_\varepsilon\}} \big(\beta_{M_\varepsilon}(\rho(t_0,x)) - \rho(t_0,x)\big)\varphi(x)\ dx \right| =$$

$$= \left| \int_{\{x\in\Omega:\ |\rho(t_0,x)|>M_\varepsilon\}} \big(M_\varepsilon - \rho(t_0,x)\big)\varphi(x)\ dx \right| \le$$

$$\le \|\varphi\|_\infty \left(\int_{\{x\in\Omega:\ |\rho(t_0,x)|>M_\varepsilon\}} |\rho(t_0,x)|\ dx + M_\varepsilon |\{x\in\Omega:\ |\rho(t_0,x)|>M_\varepsilon\}| \right) \le 2\varepsilon \|\varphi\|_\infty$$

The other term can be written as

$$\left| \int_\Omega \big(\beta_{M_\varepsilon}(\rho_n(t_n,x)) - \beta_{M_\varepsilon}(\rho(t_0,x))\big)\varphi(x)\ dx \right| =$$

$$= \left| \int_\Omega \big(\beta_{M_\varepsilon}(\rho_n(t_n,x)) - \beta^{k_\varepsilon}_{M_\varepsilon}(\rho_n(t_n,x))\big)\varphi(x)\ dx + \int_\Omega \big(\beta^{k_\varepsilon}_{M_\varepsilon}(\rho_n(t_n,x)) - \beta^{k_\varepsilon}_{M_\varepsilon}(\rho(t_0,x))\big)\varphi(x)\ dx + \right.$$

$$\left. + \int_\Omega \big(\beta^{k_\varepsilon}_{M_\varepsilon}(\rho(t_0,x)) - \beta_{M_\varepsilon}(\rho(t_0,x))\big)\varphi(x)\ dx \right| \le$$

$$\le \|\varphi\|_\infty \int_\Omega |\beta_{M_\varepsilon}(\rho_n(t_n,x)) - \beta_{M_\varepsilon}(\rho(t_0,x))|\ dx + \left| \int_\Omega \big(\beta^{k_\varepsilon}_{M_\varepsilon}(\rho_n(t_n,x)) - \beta^{k_\varepsilon}_{M_\varepsilon}(\rho(t_0,x))\big)\varphi(x)\ dx \right| +$$

$$+ \|\varphi\|_\infty \int_\Omega |\beta_{M_\varepsilon}(\rho(t_0,x)) - \beta_{M_\varepsilon}(\rho(t_0,x))|\ dx \le$$

$$\le \frac{2\|\varphi\|_\infty |\Omega|}{k_\varepsilon} + \left| \int_\Omega \big(\beta^{k_\varepsilon}_{M_\varepsilon}(\rho_n(t_n,x)) - \beta^{k_\varepsilon}_{M_\varepsilon}(\rho(t_0,x))\big)\varphi(x)\ dx \right|$$

We have that, for every admissible function, (8.30), there exists $N = N(\beta^{k_\varepsilon}_{M_\varepsilon}) \equiv N_\varepsilon$ such that, for every $n \geq N_\varepsilon$,

$$\left| \int_\Omega \rho_n(t_n, x)\varphi(x)\ dx - \int_\Omega \rho(t_0, x)\varphi(x)\ dx \right| \leq 4\varepsilon\|\varphi\|_\infty + \frac{2\|\varphi\|_\infty |\Omega|}{k_\varepsilon} + \varepsilon \leq 4\varepsilon\|\varphi\|_\infty + 3\varepsilon$$

that is

$$\lim_{n\to\infty} \int_\Omega \rho_n(t_n, x)\varphi(x)\ dx = \int_\Omega \rho(t_0, x)\varphi(x)\ dx$$

Using remark 8.15, we have that $\rho_n(t_n) \to \rho(t_0)$ in $L^p(\Omega)$. From theorem 7.1 it follows that

$$\rho_n \to \rho \qquad \text{in } C([0, T]; L^p(\Omega))$$

that is the thesis. ∎

Chapter 9

Stationary Stokes System and the Stokes operator

The *Stokes equation* is a stationary PDE, i.e. the equation does not involve the temporal variable t. The study of this system of equations will be useful in future discussions. We will deal with the equation in the whole space and the Stationary stokes system on a domain Ω.
The theory of the Stokes equation in the whole space will concern both homogeneous and inhomogeneous Stokes equation: in the latter case we will deal with an external force in the class C_c^∞, following the ideas of Galdi's *An introduction to the Mathematical Theory of the Navier-Stokes equations* [12], that is, introducing a fundamental solution to the equations and obtaining solutions to the Stokes equations by convolution. A uniqueness theorem will be prove in the case of homogeneous equation. The whole Stokes theory is developed in the classical paper [19]. However, as mentioned before, this *compilative chapter* is based on Galdi's work [12].
To this purpose, it is fundamental to introduce the main tools of classical harmonic analysis, which will be useful in the future dissertation.
On the other hand, we will prove fundamental existence and regularity results in the case of the Stationary Stokes System in a bounded domain Ω.

9.1 Stokes equation: solution and regularization

Definition 9.1. In a bounded domain Ω, the Stokes equation is the stystem

$$\begin{cases} \Delta v = \nabla p + f \\ \nabla \cdot v = 0 \end{cases} \qquad \text{in } \Omega \tag{9.1}$$

with the adherence condition $v = v_*$ over $\partial\Omega$.

Remark 9.1. The system (9.1) can obviously be understood in classical sense, that is $v \in C^2(\Omega) \cap C(\overline{\Omega})$, $p \in C^1(\Omega) \cap C(\overline{\Omega})$, $f \in C(\overline{\Omega})$ and $v_* \in C(\partial\Omega)$. However, in a little while we will propose a weak interpretation of the equation. □

Remark 9.2. If $\Omega = \mathbb{R}^n$, the space is not bounded and the adherence is useless. We will find solution with a certain decay at the infinity. □

Remark 9.3. Observe that, if Ω is bounded and regular enough, we have, formally,

$$0 = \int_\Omega \nabla \cdot v \; dx = \int_{\partial\Omega} v_* \cdot \nu \; d\sigma$$

where ν is the outer normal of Ω. According to this, we have the following definition. $\square$

Definition 9.2. An adherence condition is *compatible* if

$$\int_{\partial\Omega} v_* \cdot \nu \; d\sigma = 0$$

Remark 9.4. In example, $v_* \equiv 0$ over $\partial\Omega$ is compatible.

We also define the following space, that will be fundamental in the next sections.

Definition 9.3. We set the space of *divergence-free test functions* as

$$\mathcal{D}(\Omega) := \{u \in C_0^\infty(\Omega) : \; \nabla \cdot u = 0 \text{ in } \Omega\}$$

The next definition introduce the weak version of the problem disclosed in remark 9.1.

Definition 9.4 (Weak solution)**.** A field $v : \Omega \to \mathbb{R}^n$ is called a *q-weak* (or *q-generalized*) solution of (9.1) if and only if

(i) $v \in D^{1,q}(\Omega)$;

(ii) v is weakly divergence free[1] in Ω;

(iii) $v = v_*$ in trace sense, or, if $v_* \equiv 0$, $v \in D_0^{1,q}(\Omega)$;

(iv) v satisfies

$$\langle \nabla v, \nabla \varphi \rangle = -\langle f, \varphi \rangle \qquad \varphi \in \mathcal{D}(\Omega) \tag{9.2}$$

Remark 9.5. In the practice, that is in the application to the problem at the core of the present thesis, we will obtain $v \in W^{1,q}(\Omega)$ or $W_0^{1,q}(\Omega)$. $\square$

Apparently, in this definition it doesn't appear the pressure term. However, it holds the following lemma.

Lemma 9.1 (Pair of weak solutions)**.** *Let Ω be a domain in $\mathbb{R}^n$, with $n \geq 2$ and let*[2] *$f \in W^{-1,q}(\Omega')$, with $q \in (1, \infty)$, for any Ω' bounded and $\overline{\Omega}' \subset \Omega$. A vector field $v \in W_{loc}^{1,q}(\Omega)$ satisfies*

$$\langle \nabla v, \nabla \varphi \rangle = -\langle f, \varphi \rangle \qquad \varphi \in \mathcal{D}(\Omega) \tag{9.3}$$

[1] i.e. divergence free in the sense of distributions, that is

$$\int_\Omega v \cdot \nabla \phi \; dx = 0 \qquad \forall \phi \in C_c^\infty(\Omega)$$

[2] This lemma is [12, Lemma IV.1.1 pg. 235]. In this book, the author asks that $f \in W_0^{-1,q}(\Omega')$. However, in the same book, page 60, Theorem II.3.5, the author consider $W_0^{-1,q}(\Omega')$ as the dual space of $W_0^{1,q'}(\Omega')$, that in our notation is, differently, defined with $W^{-1,q}(\Omega')$, as we have written in definition 4.11.

if and only if exits $p \in L^q_{loc}(\Omega)$ *such that*

$$\langle \nabla v, \nabla \psi \rangle = -\langle f, \psi \rangle + \langle p, \nabla \cdot \psi \rangle \qquad \psi \in C_0^\infty(\Omega) \tag{9.4}$$

The pair (u, p) *is often called weak solution pair.*

Proof. If we suppose that (9.4) holds for some $p \in L^q_{loc}(\Omega)$, for every $\psi \in C_0^\infty(\Omega)$. So, if $\psi \in C_{0,\sigma}^\infty(\Omega)$, clearly, being $\nabla \cdot \psi = 0$, the pressure therm vanishes and we have (9.3).

Now we have to show the other implication. We define the the functional

$$\mathcal{F}(\psi) := \langle \nabla v, \nabla \psi \rangle + \langle f, \psi \rangle$$

We now show two properties of this functional: it belongs to $W^{-1,q}(\Omega')$ and $\mathcal{F}(v) = 0$ for every $v \in C_{0,\sigma}^\infty(\Omega')$. Obviously, being $W^{-1,q}(\Omega') \equiv (W_0^{1,q'}(\Omega)')^*$, if $\psi_k \to \psi$ in $W_0^{1,q'}(\Omega')$ we have

$$|\mathcal{F}(\psi_k - \psi)| \leq \|\nabla v\|_q \|\nabla \psi_k - \nabla \psi\|_{q'} + \|f\|_q \|\psi_k - \psi\|_{q'} \to 0 \quad \text{as } k \to \infty$$

where $\|\cdot\|_p \equiv \|\cdot\|_{p,\Omega'}$. So $\mathcal{F} \in W^{-1,q}_{loc}(\Omega)$. Moreover, if $v \in C_{0,\sigma}^\infty(\Omega)$, (9.3) implies that $\mathcal{F}(v) = 0$.
So, lemma 6.1, we have that exists $p \in L^q_{loc}(\Omega)$ such that $\mathcal{F} \overset{d}{=} \nabla p$, that is

$$\mathcal{F}(\varphi) = -\int_\Omega p \; \nabla \cdot \varphi \; dx \quad \forall \varphi \in C_0^\infty(\Omega)$$

that is the thesis. ∎

Moreover, it can be proved an existence and uniquess theorem for a weak solution $q = 2$.

Theorem 9.1. *Let* $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, *be a bounded and locally Lipschitz domain. For any* $f \in D_0^{-1,2}(\Omega)$ *and* $v_* \in W^{1/2,1}(\partial\Omega)$ *that satisfies the compatibility condition, there exists one and only one weak solution* v *to* (9.1)*. Moreover, if* p *is the pressure filed associated to* v *by Lemma 9.1,*

$$\|v\|_{1,2} + \|p\|_2 \leq c\big(\|f\|_{-1,2} + \|v_*\|_{\frac{1}{2},2(\partial\Omega)}\big)$$

with $c = c(n, \Omega)$.

Remark 9.6. This existence theorem in the case $q = 2$ is fundamental to start considering the problem, but it is different from our aims: we want to *regularize solution that we already know that exist.* So we omit the proof. □

Weak and strong solutions to the Stokes problem. In the lines above we have defined *generalized weak solution* (see definition 9.4) and *pairs of weak solution* (see lemma 9.1). Moreover, we have a strong definition of solution for the Stokes system.

Definition 9.5 (Strong solutions)**.** Let Ω a domain in $\mathbb{R}^n$, with $n \geq 2$. A *strong solution* of the Stokes system (9.1) with $f \in L^2(\Omega)$ and $v_* \in L^2(\partial\Omega)$ is a pair of functions $(u, f_1) \in W^{2,2}(\Omega) \times G(\Omega)$ sucht that the equalities

$$-\mu\Delta u + f_1 = f, \qquad \nabla \cdot u = 0, \qquad Tu = v_*$$

hold a.e. in Ω.

Remark 9.7. If $v_* \equiv 0$ on $\partial\Omega$, the zero boundary conditions is satisfied if we require $u \in H_0^1(\Omega)$. □

The following proposition shows the duality between strong and weak solution. It also simplifies the implication "regular weak solution" $\implies$ "strong solution for an associtated pressure term".

Proposition 9.1. *Let $\Omega \subseteq \mathbb{R}^n$ a bounded domain, with $n \geq 2$. Consider the Stokes problem* (9.1) *over Ω. Then:*

- *A strong solution (u, f_1) of the Stokes problem is also a weak solution in the sense of definition 9.4, with $q = 2$.*
- *Moreover, for a 2-weak solution $u \in W^{2,2}(\Omega)$ of definition 9.4, with $f \in L^2(\Omega)$, there exists a pressure term $f_1 \in G(\Omega)$ such that the pair (u, f_1) is a strong solution in the sense of definition 9.5.*
- *Finally, if (u, p) is a weak solution pair in the sense of lemma 9.1, with $f \in L^2(\Omega)$, and, in addiction, $u \in H^2(\Omega)$ and $\nabla p \in L^2(\Omega)$ then $(u, \nabla p)$ is a strong solution in the sense of definition 9.5.*

Proof. Let (u, f_1) a strong solution, so that $u \in H^2(\Omega)$ and $f_1 \in G(\Omega)$. Then, if $v \in C_{0,\sigma}^\infty(\Omega)$ we have

$$-\mu \int_\Omega \Delta u \cdot v \, dx = \int_\Omega (-\mu\Delta u + f_1) \cdot v \, dx = \int_\Omega f \cdot v$$

since $f_1 \in G(\Omega)$. It also holds, using integration by parts[3] and a result about traces,

$$-\mu \int_\Omega \Delta u \cdot v \, dx = \mu \int_\Omega \nabla u \cdot \nabla v \, dx$$

that is equation (9.2). Moreover, $u \in H^2(\Omega) \subseteq D^{1,2}(\Omega)$, $u = v_*$ in trace sense and, for every $\varphi \in C_0^\infty(\Omega)$, we have

$$\int_\Omega u \cdot \nabla\varphi \, dx = -\int_\Omega (\nabla \cdot u)\varphi \, dx = 0$$

So u is a 2-weak solution.

[3]Thanks to the fact that $v \in C^1(\overline{\Omega})$ and $v = 0$ at the boundary.

Conversely, let u a weak solution in the sense of definition 9.4, that also belongs to $H^2(\Omega)$. By equation (9.2) we have

$$\mu \int_\Omega \nabla u \cdot \nabla v \; dx = \int_\Omega f \cdot v \; dx \quad \forall v \in C^\infty_{0,\sigma}(\Omega)$$

Again, as in the note above, we have the equality

$$-\mu \int_\Omega \Delta u \cdot v \; dx = \mu \int_\Omega \nabla u \cdot \nabla v \; dx$$

and so

$$\int_\Omega (-\mu \Delta u - f) \cdot v \; dx = 0 \quad \forall v \in C^\infty_{0,\sigma}(\Omega)$$

Thanks to lemma 6.1, being $f \in L^2(\Omega)$, we have

$$-\mu \Delta u - f = f_1$$

with $f_1 \in G(\Omega)$. Moreover, $Tu = v_*$ in the hypothesis and for every $\varphi \in C^\infty_0(\Omega)$,

$$0 = \int_\Omega u \cdot \nabla \varphi \; dx = -\int_\Omega \big(\nabla \cdot u)\varphi \; dx$$

so tha, by a classical measure theory result, $\nabla \cdot u = 0$ almost everywhere in Ω.

Finally, let $(u, \nabla p)$ as in the hypothesis. Then, using (9.4), $\forall \varphi \in C^\infty_0(\Omega)$ we have

$$\int_\Omega (\mu \nabla u \cdot \nabla \varphi - f \cdot \varphi) \; dx = \int_\Omega p \; \nabla \cdot \varphi \; dx = -\int_\Omega \nabla p \cdot \varphi \; dx$$

So, being also $u \in H^2(\Omega)$, then

$$-\mu \int_\Omega \Delta u \cdot \varphi \; dx - \int_\Omega f \cdot \varphi \; dx = -\int_\Omega \nabla p \cdot \varphi \; dx$$

So for every $\varphi \in C^\infty_0(\Omega)$, we have

$$\int_\Omega (-\mu \Delta u - f + \nabla p) \cdot \varphi \; dx = 0$$

where $-\mu \Delta u - f + \nabla p \in L^2(\Omega)$. But, for classical measure theory results, we have that the integrand is zero a.e., since it is zero the integral against any test function over Ω, so that[4]

$$-\mu \Delta u + \nabla p = f \quad \text{a.e. in} \quad \Omega$$

Moreover, as above, since $u \in H^2(\Omega)$ we have $\nabla \cdot u = 0$ almost everywhere. Furthermore, $Tu = v_*$ in trace sense by definition. ∎

[4] Here all the terms have three components. One can so apply the measure theory result for real valued functions considering $\varphi = (\varphi_1, 0, 0)$ and so on. In this way one gets the result for the three components separately, and then we put the pieces togheter.

9.2 Existence, uniquess and estimates in $\Omega = \mathbb{R}^n$

In this section we focus our attention to the Stokes problem on the whole space, with particular interest to the *inhomogeneous* problem.

Definition 9.6. Let $f, g \in C_c^\infty(\mathbb{R}^n)$ a vector and a scalar field, respectively. The Stokes problem in the whole space associated to these fields is the system

$$\begin{cases} \Delta v = \nabla p + f \\ \nabla \cdot v = g \end{cases} \quad \text{in } \mathbb{R}^n \tag{9.5}$$

and we search for $v \in C^2(\mathbb{R}^n)$ and $p \in C^1(\mathbb{R}^n)$ with suitable decay properties at infinity.

9.2.1 Solution to the problem

We simplify the problem. Consider a field $F \in C_c^\infty(\mathbb{R}^n)$. Then, we define

$$u(x) := \int_{\mathbb{R}^n} U(x-y)F(y)\ dy, \qquad \pi(x) := -\int_{\mathbb{R}^n} q(x-y)F(y)\ dy \tag{9.6}$$

where U and π have to be fixed. In the following subsections we fix the kernels U, q. In order to do this, we collect some basic results.

Harmonic functions. Here we briefly review the main definitions and result in harmonic theory.
Let $\Omega \subseteq \mathbb{R}^n$.

Definition 9.7. A function $u \in C^2(\Omega)$ is called *harmonic* in Ω if

$$\Delta u(x) = 0 \quad \forall x \in \Omega$$

Definition 9.8. A function $u \in C^4(\Omega)$ is called *biharmonic* in Ω if

$$\Delta^2 u(x) \equiv \nabla^4 u(x) := \sum_{i=1}^n \sum_{j=1}^n \partial_{x_i}^2 \partial_{x_j}^2 u(x) = 0 \quad \forall x \in \Omega$$

The operator ∇^4 is called *biharmonic* or *bilaplacian operator.*

Theorem 9.2. *Let $\Omega \subseteq \mathbb{R}^n$ a domain and let $u \in C^2(\Omega)$ an harmonic function. Then, for every $B_R(y) \subset\subset \Omega$ it holds*

$$u(y) = \frac{1}{\omega_n R^n} \int_{B_R(y)} u(x) dx$$

where ω_n is the area of the unit sphere in $\mathbb{R}^n$.

Theorem 9.3. *Let $\Omega \subseteq \mathbb{R}^n$ a connected domain and let $u \in C^2(\Omega)$ an harmonic function. If exists $y_0 \in \Omega$ such that $u(y_0) = \sup_\Omega u$, then $u \equiv const.$*

Corollary 9.1. *Let $\Omega \subseteq \mathbb{R}^n$ a bounded and connected domain. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Then*

$$\max_\Omega u = \max_{\partial\Omega} u$$

Fundamental solution of the biharmonic equation. By *biharmonic equation* we mean the PDE

$$\nabla^4 u = 0$$

Remark 9.8. We see immediately that applying the biharmonic operator to a C^4 function is equivalent to applying the laplacian operator twice. In fact, if u is C^4 in a neighborhood of x,

$$\nabla^4 u(x) = \sum_{i=1}^{n}\sum_{j=1}^{n} \partial^2_{x_i}\partial^2_{x_j} u(x) = \sum_{i=1}^{n} \partial^2_{x_i} \sum_{j=1}^{n} \partial^2_{x_j} u(x) = \sum_{i=1}^{n} \partial^2_{x_i}(\Delta u)(x) = \Delta(\Delta u)(x)$$

Definition 9.9. We call *fundamental solution of the biharmonic equation* the function

$$\Gamma(x) \equiv \phi(|x|) := \frac{|x|}{8\pi} \tag{9.7}$$

Lemma 9.2. *For every $x \neq 0$ we have $\nabla^4\Gamma(x) = 0$.*

Proof. It is an easy computation. Let $x \neq 0$. We have

$$\partial_{x_i}|x| = \frac{x_i}{|x|}, \qquad \partial_{x_i}\frac{x_i}{|x|} = \frac{1}{|x|} - \frac{x_i^2}{|x|^3}$$

so that

$$\Delta|x| = \sum_{i=1}^{3}\left(\frac{1}{|x|} - \frac{x_i^2}{|x|^3}\right) = \frac{2}{|x|}$$

using the previous remark, thanks to the fact that $|x|$ is smooth in $\mathbb{R}^n/\{0\}$. Going on we have

$$\partial_{x_i}\frac{2}{|x|} = -\frac{2x_i}{|x|^3}, \qquad \partial_{x_i}\left(-\frac{2x_i}{|x|^3}\right) = -2[\frac{1}{|x|^3} - 3\frac{x_i^2}{|x|^5}]$$

Thus

$$\Delta\frac{2}{|x|} = -2\left(\frac{3}{|x|^3} - 3\frac{|x|^2}{|x|^5}\right) = 0$$

that is the thesis. ∎

9.2.2 Lorentz's fundamental solutions

To the purpose of solving the Stokes equation, it will be useful to introduce other fundamental solutions, with the fundamental solution of Laplace's equation Γ as a model. In particular, thinking to the Stokes equation, it will be useful to find function (or, as we will say, *kernels*), say U, q, such that

$$-\Delta U(z) + \partial_{z_i} q(z) = 0$$

To this aim, we can define, given real variable function $\phi(t)$ smooth for $t \neq 0$,

$$U_{ij}(z) := (\delta_{ij}\Delta - \partial_{z_i}\partial_{z_j})\phi(|z|)$$

$$q_j(z) := -\partial_{z_j}\Delta[\phi(|z|)]$$

for every $z \neq 0$, we have

$$-\Delta U_{ij}(z) + \partial_{z_i} q_j(z) = -\Delta(\delta_{ij}\Delta[\phi(|z|)] - \partial_{z_i}\partial_{z_j}[\phi(|z|)]) - \partial_{z_i}\partial_{z_j}\Delta[\phi(|z|)] =$$

using that $\phi(|z|)$ is smooth far from the origin and so using the Schwarz lemma

$$= -\delta_{ij}\nabla^4[\phi(|z|)] + \partial_{z_i}\partial_{z_j}\Delta[\phi(|z|)] - \partial_{z_i}\partial_{z_j}\Delta[\phi(|z|)] = -\delta_{ij}\nabla^4[\phi(|z|)]$$

So if

$$\phi(t) := \frac{t}{8\pi} \implies \phi(|z|) = \frac{|z|}{8\pi} \equiv \Gamma(z)$$

It follows that, for every $z \neq 0$,

$$-\Delta U_{ij}(z) + \partial_{z_i} q_j(z) = 0$$

We now can write explicitly the expression for the kernels: using the equalities of the previous proof, we have

$$U_{ij}(z) = \frac{\delta_{ij}}{4\pi|z|} - \frac{1}{8\pi}\frac{\delta_{ij}}{|z|} + \frac{1}{8\pi}\frac{z_i z_j}{|z|^3} = \frac{\delta_{ij}}{8\pi|z|} + \frac{1}{8\pi}\frac{z_i z_j}{|z|^3}$$

$$q_j(z) = \frac{1}{4\pi}\frac{z_j}{|z|^3}$$

Remark 9.9. Notice first of all that $\sum_{i=1}^{3} \partial_{x_i} U_{ij}(x-y) = 0$. In fact, if $i = j$

$$\partial_{x_i} U_{ij}(x-y) = \frac{1}{8\pi}\left(-\frac{x_j - y_j}{|x-y|^3} + 2\frac{x_j - y_j}{|x-y|^3} - 3\frac{(x_j - y_j)^3}{|x-y|^5}\right)$$

and, if $i \neq j$,

$$\partial_{x_i} U_{ij}(x-y) = \frac{1}{8\pi}\left((x_j - y_j)(\frac{1}{|x-y|^3} - 3\frac{(x_i - y_i)^2}{|x-y|^5})\right)$$

So

$$\sum_{i=1}^{3} \partial_{x_i} U_{ij}(x-y) = \frac{1}{8\pi}\left(-3(x_j - y_j)\frac{|x-y|^2}{|x-y|^5} + 3\frac{(x_j - y_j)}{|x-y|^3}\right) = 0$$

9.2.3 Classical results about Stokes equation

Definition 9.10. Thanks to the arguments in the previous subsection, we define the so called *Lorentz's fundamental solutions*, for every $x \neq y$, as

$$U_{ij}(x-y) := \frac{1}{8\pi}\left(\frac{\delta_{ij}}{|x-y|} + \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^3}\right) \tag{9.8}$$

$$q_j(x-y) := \frac{1}{4\pi}\frac{x_j - y_j}{|x-y|^3} \tag{9.9}$$

Remark 9.10. We will now build solutions to the Stokes problem by convolution of the external force against the kernels defined above. As we will see, the convolution has no problems of definition: however, we have to do some work to derive the function we're going to define. □

Remark 9.11. If we consider the kernels U and q, these are homogenous function of degree, respectively, -1 and -2. This means that the origin is a summable singularity.
However, deriving once the kernels, we get homoegenous functions of degrees -2 and -3. The kernel obtained deriving q is thus no more summable in the origin. The same thing happens if we derive another time U. This suggests that we have to deal with these functions carefully.
□

So, we have the following theorem.

Theorem 9.4 (Classical Stokes problem in $\mathbb{R}^3$)**.** *Let $f \in C_c^\infty(\mathbb{R}^3)$ be a test external force. We define*

$$V(x) := \frac{1}{\mu}\int_{\mathbb{R}^3} U(x-y)\ f(y)\ dy, \qquad \Pi(x) := \int_{\mathbb{R}^3} q(x-y)\cdot f(y)\ dy \tag{9.10}$$

The pair (V,Π) is a solution of the Stokes equation

$$\begin{cases} -\mu\Delta V + \nabla\Pi = f \\ \nabla\cdot V = 0 \end{cases} \tag{9.11}$$

in the class $C^\infty(\mathbb{R}^3)$. Moreover this pair of solutions (V,Π) satisfies the estimates

$$|V(x)| \le \frac{W}{|x|-R} \qquad \forall\ x:\ |x|>R$$

and

$$|\Pi(x)| \le \frac{Q}{(|x|-R)^2} \qquad \forall\ x:\ |x|>R$$

where $R>0$ is such that supp$(f)\subseteq B(0,R)$.

Proof. We proceed proving the theorem by steps.

1. Well-posedness and smoothness. Well-posedness and regularity follow from a simple consideration: we can consider the integral

$$\frac{1}{\mu}\int_{\mathbb{R}^3} U(z)\ f(x-z)\ dz$$

where U represent one of the U_{ij} and f a component f_l of f. This integral is well-posed for every x. In fact, let $x_0\in\mathbb{R}^3$ and consider $B(x_0,\delta)$, for $\delta>0$, i.e. we choose an arbitrary open ball in the space. If $R>0$ is such that supp$(f)\subseteq B(0,R)$ and if $C_0 := |x_0|+\delta$, we have, for $x\in B(x_0,\delta)$,

$$|z|\ge R+C_0 \implies |x-z| \ge ||x|-|z|| \ge |z|-|x| \ge R+C_0-|x| \ge R$$

since $|x| = |x-x_0|+|x_0| \le \delta+|x_0| \equiv C_0$. So, if $|z|\ge R+C_0$, then $f(x-z)=0$. It follows that we can rewrite

$$\int_{\mathbb{R}^3} U(z)\ f(x-z)\ dz = \int_{B(0,R+C_0)} U(z)\ f(x-z)\ dz$$

So, being

$$|U(z)\ f(x-z)\chi_{B(0,R+C_0)}(z)| \le \|f\|_\infty |U(z)|\chi_{B(0,R+C_0)}(z)$$

and

$$\int_{B(0,R+C_0)} |U(z)|dz < +\infty$$

since U has an integrable singularity in the origin, being U homoegenous of exponent $\alpha=-1$. So, the integral is well posed and

$$V(x) \equiv \frac{1}{\mu}\int_{\mathbb{R}^3} U(x-y)f(y)dy = \frac{1}{\mu}\int_{\mathbb{R}^3} U(z)f(x-z)dz$$

thanks to a change of variable. Moreover, since we have also the estimate[5]

$$|U(z)\ \partial_x f(x-z)\chi_{B(0,R+C_0)}(z)| \le \|\partial_x f\|_\infty |U(z)|\chi_{B(0,R+C_0)}(z)$$

we can pass the derivative under the integral sign, and get

$$\partial_x V(x) = \frac{1}{\mu}\int_{B(0,R+C_0)} U(z)\ \partial_x f(x-z)\ dz =$$

$$= \frac{1}{\mu}\int_{\mathbb{R}^3} U(z)\ (\partial_x f)(x-z)\ dz = \frac{1}{\mu}\int_{\mathbb{R}^3} U(x-y)\ \partial_x f(y)\ dy$$

using that, being $f \equiv 0$ if $|z| \ge R + C_0$, so it is also $\partial_x f$.

The continuity follows from the same argument and the fact that $f \in C^\infty$. In fact, the same estimate, with $\|\partial_x f\|_\infty$ gives a summable bound, so we have the continuity in x_0. From the arbitrariness of x_0 and $\delta > 0$, we have that $V \in C^\infty(\mathbb{R}^3)$. The same method says $\Pi \in C^\infty(\mathbb{R}^3)$.

2. Estimates and asymptotic behaviour. Now we deduce the estimates. We start with Π. Let $R > 0$ such that $\mathrm{supp}(f) \subseteq B(0,R)$. Then

$$|\Pi(x)| = \left|\int_{\mathbb{R}^3} q(x-y)\cdot f(y)\ dy\right| = \left|\int_{B(0,R)} q(x-y)\cdot f(y)\ dy\right| \le \int_{B(0,R)} |q(x-y)\cdot f(y)|\ dy \le$$

$$\le \int_{B(0,R)} |q(x-y)||f(y)|dy$$

Remember now that each component of q, i.e. q_i, is an homogeneous function of exponent $\alpha = -2$. It follows that

$$|q(x-y)| \le \frac{1}{|x-y|^2}\max_{|z|=1}|q(z)| \equiv \frac{M}{|x-y|^2}$$

If $|x| > R$ we have

$$|x-y| \ge ||x|-|y|| \ge |x|-|y| > |x|-R$$

for every $y \in B(0,R)$. So

$$\frac{1}{|x-y|^2} \le \frac{1}{(|x|-R)^2}$$

and hence for every x such that $|x| > R$

$$|\Pi(x)| \le \int_{B(0,R)} \frac{M}{(|x|-R)^2}|f(y)|dy = \frac{Q}{(|x|-R)^2}$$

where $Q := M\displaystyle\int_{B(0,R)} |f(y)|dy$. The estimate for V is similar. We have

$$|V(x)| = \left|\frac{1}{\mu}\int_{\mathbb{R}^3} U(x-y)f(y)\ dy\right| = \left|\frac{1}{\mu}\int_{B(0,R)} U(x-y)f(y)\ dy\right| \le \frac{1}{\mu}\int_{B(0,R)} |U(x-y)||f(y)|dy$$

[5]Since also $\mathrm{supp}(\partial_x f) \subseteq B(0,R)$.

The kernel U is such that each component is an homogeneous function of exponent $\alpha = -1$. So

$$|U(x-y)| = \left|U\left(\frac{x-y}{|x-y|}|x-y|\right)\right| = \frac{1}{|x-y|}\left|U\left(\frac{x-y}{|x-y|}\right)\right| \le \frac{1}{|x-y|}\max_{|z|=1}|U(z)| \equiv \frac{M'}{|x-y|}$$

Again, for $|x| > R$, we have

$$|x-y| \ge |x| - R$$

and

$$\frac{1}{|x-y|} \le \frac{1}{|x|-R}$$

for every $y \in B(0,R)$. So

$$|V(x)| \le \frac{1}{\mu}\int_{B(0,R)} \frac{M'}{|x-y|}|f(y)|dy \le \frac{1}{\mu}\int_{B(0,R)} \frac{M'}{|x|-R}|f(y)|dy = \frac{W}{|x|-R}$$

where $W := M' \int_{B(0,R)} |f(y)|dy$. These are the estimates that we expected.

3. Further derivatives and check that are solutions. Finally we find the derivatives of V and Π and we prove that the functions solve the equation of Stokes. Even if we have proved that V, Π are smooth, we can't pass the derivative under the integral sign, because of the remarks done before the theorem. The proof of smoothness itself use the smootheness of the force f, so that's what we will use again. In order to verify that

$$-\mu\Delta V + \nabla\Pi = f$$

we have to remember that this is a vector equality. So we have to prove it component by component. For the sake of simplycity, we will prove the equality of the first component: the others are similar. So, we want to prove

$$-\mu\Delta V_1 + \partial_{x_1}\Pi = f_1$$

Remember that

$$V_1(x) = \frac{1}{\mu}\int_{\mathbb{R}^3}\sum_{l=1}^{3} U_{1l}(x-y)f_l(y)dy$$

We start immediately deriving the expression twice. Let $x \in B(x_0, \delta)$ fixed, for $x_0 \in \mathbb{R}^3$ and $\delta > 0$. We have

$$\partial^2_{x_i} V_1(x) = \frac{1}{\mu}\sum_{l=1}^{3}\partial^2_{x_i}\int_{\mathbb{R}^3} U_{1l}(z)f_l(x-z)dz = \frac{1}{\mu}\sum_{l=1}^{3}\int_{\mathbb{R}^3} U_{1l}(z)\partial^2_{x_i}[f_l(x-z)]dz$$

Using that $\partial^2_{x_i}[f_l(x-z)] = \partial^2_{z_i}[f_l(x-z)]$, we have

$$\partial^2_{x_i} V_1(x) = \frac{1}{\mu}\sum_{l=1}^{3}\int_{\mathbb{R}^3} U_{1l}(z)\partial^2_{z_i}[f_l(x-z)]dz$$

We now fix $\varepsilon > 0$. So, we can split

$$\partial^2_{x_i} V_1(x) = \frac{1}{\mu}\sum_{l=1}^{3}\int_{|z|<\varepsilon} U_{1l}(z)\partial^2_{z_i}[f_l(x-z)]dz + \frac{1}{\mu}\sum_{l=1}^{3}\int_{|z|\ge\varepsilon} U_{1l}(z)\partial^2_{z_i}[f_l(x-z)]dz$$

Notice at once that the first piece is $o_\varepsilon(1) \equiv o(1)$, since the integrand is summable in the whole space thanks to the singularity of U_{1l} and the compactness of the support of f_l (and the fact that x is banished in a fixed neighborhood). Moreover, observe that

$$\partial_{z_i}[U_{1l}(z)\partial_{z_i}(f_l(x-z))] = \partial_{z_i}U_{1l}(z)\partial_{z_i}(f_l(x-z)) + U_{1l}(z)\partial^2_{z_i}(f_l(x-z))$$

So we have

$$\partial^2_{x_i}V_1(x) = \frac{1}{\mu}\sum_{l=1}^{3}\int_{|z|\geq\varepsilon}\partial_{z_i}[U_{1l}(z)\partial_{z_i}(f_l(x-z))]dz - \frac{1}{\mu}\sum_{l=1}^{3}\int_{|z|\geq\varepsilon}\partial_{z_i}U_{1l}(z)\partial_{z_i}(f_l(x-z))dz + o(1)$$

The first addend can be reduced to a surface integral. We prove this. We have

$$\int_{|z|\geq\varepsilon}\partial_{z_i}[U_{1l}(z)\partial_{z_i}(f_l(x-z))]dz = \lim_{R\to+\infty}\int_{R\geq|z|\geq\varepsilon}\partial_{z_i}[U_{1l}(z)\partial_{z_i}(f_l(x-z))]dz$$

thanks to the fact that the integrand, as we can see looking at the equality above regarding this term, is the sum of two integrable addend on $|z| \geq \varepsilon$ (thanks to the compactness of the support of f). By the divergence theorem for annulus we have

$$\int_{R\geq|z|\geq\varepsilon}\partial_{z_i}[U_{1l}(z)\partial_{z_i}(f_l(x-z))]dz =$$

$$= \int_{|z|=R}U_{1l}(z)\partial_{z_i}(f_l(x-z))\nu_i(z)d\sigma(z) - \int_{|z|=\varepsilon}U_{1l}(z)\partial_{z_i}(f_l(x-z))\nu_i(z)d\sigma(z)$$

If R is large enough, we have already seen that $f_l(x-z) \equiv 0$, for x fixed in its neighbourhood. So it is its partial derivative. Hence, this piece vanishes as $R \to +\infty$. It follows that

$$\partial^2_{x_i}V_1(x) =$$

$$= -\frac{1}{\mu}\sum_{l=1}^{3}\int_{|z|=\varepsilon}U_{1l}(z)\partial_{z_i}(f_l(x-z))\nu_i(z)d\sigma(z) - \frac{1}{\mu}\sum_{l=1}^{3}\int_{|z|\geq\varepsilon}\partial_{z_i}U_{1l}(z)\partial_{z_i}(f_l(x-z))dz + o(1) =$$

$$= -\frac{1}{\mu}\sum_{l=1}^{3}\int_{|z|\geq\varepsilon}\partial_{z_i}U_{1l}(z)\partial_{z_i}(f_l(x-z))dz + o(1)$$

Remark 9.12. We have included one more piece in the $o(1)$ (in particular the first of the second line) since changing the variable $z \longleftrightarrow \varepsilon y$ we get

$$\int_{|z|=\varepsilon}U_{1l}(z)\partial_{z_i}(f_l(x-z))\nu_i(z)d\sigma(z) = \int_{|z|=1}U_{1l}(\varepsilon z)\partial_{z_i}(f_l(x-\varepsilon z))\nu_i(\varepsilon z)\varepsilon^2 d\sigma(z) =$$

and U_{1l} is homogeneous of degree -1

$$= \varepsilon\int_{|z|=1}U_{1l}(z)\partial_{z_i}(f_l(x-\varepsilon z))\nu_i(z)d\sigma(z) \to 0 \quad \text{as } \varepsilon \to 0$$

since we can pass the limit under the integral if the integrand is continuous on the compat set over which the integral is done. $\square$

Now, as before, we have

$$\partial_{z_i}[\partial_{z_i}U_{1l}(z)f_l(x-z)] = \partial^2_{z_i}U_{1l}(z)f_l(x-z) + \partial_{z_i}U_{1l}(z)\partial_{z_i}[f_l(x-z)]$$

and so

$$\int_{|z|\geq\varepsilon} \partial_{z_i}U_{1l}(z)\partial_{z_i}(f_l(x-z))dz =$$

$$= \int_{|z|\geq\varepsilon} \partial_{z_i}[\partial_{z_i}U_{1l}(z)f_l(x-z)]dz - \int_{|z|\geq\varepsilon} \partial^2_{z_i}U_{1l}(z)f_l(x-z)dz$$

As above, the first addend can be reduced to a surface integral, that is

$$\int_{|z|\geq\varepsilon} \partial_{z_i}[\partial_{z_i}U_{1l}(z)f_l(x-z)]dz = -\int_{|z|=\varepsilon} \partial_{z_i}U_{1l}(z)f_l(x-z)\nu_i(z)d\sigma(z)$$

We get

$$\partial^2_{x_i}V_1(x) = -\frac{1}{\mu}\sum_{l=1}^{3}\left(-\int_{|z|=\varepsilon} \partial_{z_i}U_{1l}(z)f_l(x-z)\nu_i(z)d\sigma(z) - \int_{|z|\geq\varepsilon} \partial^2_{z_i}U_{1l}(z)f_l(x-z)dz\right) + o(1) =$$

$$= \frac{1}{\mu}\sum_{l=1}^{3}\int_{|z|=\varepsilon} \partial_{z_i}U_{1l}(z)f_l(x-z)\nu_i(z)d\sigma(z) + \frac{1}{\mu}\sum_{l=1}^{3}\int_{|z|\geq\varepsilon} \partial^2_{z_i}U_{1l}(z)f_l(x-z)dz + o(1)$$

This gives an expression for the second derivatives of V_1 in x. We can now sum over i and get

$$\Delta V_1(x) = \frac{1}{\mu}\sum_{l=1}^{3}\int_{|z|=\varepsilon} \nabla U_{1l}(z)\cdot\nu(z)f_l(x-z)d\sigma(z) + \frac{1}{\mu}\sum_{l=1}^{3}\int_{|z|\geq\varepsilon} \Delta U_{1l}(z)f_l(x-z)dz + o(1)$$

We can now pass to $\partial_{x_1}\Pi(x)$. With the same x and ε, we have

$$\partial_{x_1}\Pi(x) = \partial_{x_1}\int_{\mathbb{R}^3}\sum_{l=1}^{3} q_l(x-y)f_l(y)dy = \partial_{x_1}\int_{\mathbb{R}^3}\sum_{l=1}^{3} q_l(z)f_l(x-z)dz = \sum_{l=1}^{3}\int_{\mathbb{R}^3} q_l(z)\partial_{x_1}[f_l(x-z)]dz =$$

and using that $\partial_{x_1}[f_l(x-z)] = -\partial_{z_1}[f_l(x-z)]$ we have

$$= -\sum_{l=1}^{3}\int_{\mathbb{R}^3} q_l(z)\partial_{z_1}[f_l(x-z)]dz$$

So if we use $\partial_{z_1}[q_l(z)f_l(x-z)] = \partial_{z_1}q_l(z)f_l(x-z) + q_l(z)\partial_{z_1}[f_l(x-z)]$, we get

$$\partial_{x_1}\Pi(x) = -\sum_{l=1}^{3}\int_{|z|<\varepsilon} q_l(z)\partial_{z_1}[f_l(x-z)]dz - \sum_{l=1}^{3}\int_{|z|\geq\varepsilon} q_l(z)\partial_{z_1}[f_l(x-z)]dz =$$

$$= -\sum_{l=1}^{3}\int_{|z|\geq\varepsilon} \partial_{z_1}[q_l(z)f_l(x-z)]dz + \sum_{l=1}^{3}\int_{|z|\geq\varepsilon} \partial_{z_1}q_l(z)f_l(x-z)dz + o(1)$$

where again the term over $|z|<\varepsilon$ vanishes as $\varepsilon\to 0$, thanks to the integrability of $q_l(z)$ in the origin (being homogeneous of degree -2 and being compact the support of f). Now, again, we want to replace the first term with a surface integral. In particular

$$\int_{|z|\geq\varepsilon} \partial_{z_1}[q_l(z)f_l(x-z)]dz = -\int_{|z|=\varepsilon} q_l(z)f_l(x-z)\nu_1(z)d\sigma(z)$$

So

$$\partial_{x_1}\Pi(x) = \sum_{l=1}^{3}\int_{|z|=\varepsilon} q_l(z) f_l(x-z)\nu_1(z)d\sigma(z) + \sum_{l=1}^{3}\int_{|z|\geq\varepsilon} \partial_{z_1} q_l(z) f_l(x-z)dz + o(1)$$

We now can sum the two expression, checking if they solve the equation. So, for x and ε as above, we have

$$-\mu\Delta V_1(x) + \partial_{x_1}\Pi(x) =$$

$$= -\sum_{l=1}^{3}\int_{|z|=\varepsilon} \nabla U_{1l}(z)\cdot\nu(z) f_l(x-z)d\sigma(z) - \sum_{l=1}^{3}\int_{|z|\geq\varepsilon} \Delta U_{1l}(z) f_l(x-z)dz+$$

$$+\sum_{l=1}^{3}\int_{|z|=\varepsilon} q_l(z) f_l(x-z)\nu_1(z)d\sigma(z) + \sum_{l=1}^{3}\int_{|z|\geq\varepsilon} \partial_{z_1} q_l(z) f_l(x-z)dz + o(1) =$$

$$= \sum_{l=1}^{3}\int_{|z|\geq\varepsilon} \left(-\Delta U_{1l}(z) + \partial_{z_1} q_l(z)\right) f_l(x-z)dz + \sum_{l=1}^{3}\int_{|z|=\varepsilon} \left(-\nabla U_{1l}(z)\cdot\nu(z) + q_l(z)\nu_1(z)\right) f_l(x-z)d\sigma(z) + o(1)$$

Observe that $\varepsilon > 0$, so if $|z| \geq \varepsilon$ we are far away from the origin. But

$$-\Delta U_{1l}(z) + \partial_{z_1} q_l(z) = 0 \qquad \forall z \neq 0$$

so the first integral is zero. We have to consider

$$\int_{|z|=\varepsilon} \left(-\nabla U_{1l}(z)\cdot\nu(z) + q_l(z)\nu_1(z)\right) f_l(x-z)d\sigma(z) =$$

$$= \int_{|z|=1} \left(-\nabla U_{1l}(\varepsilon z)\cdot\nu(\varepsilon z) + q_l(\varepsilon z)\nu_1(\varepsilon z)\right) f_l(x-\varepsilon z)\varepsilon^2 d\sigma(z) =$$

$$= \int_{|z|=1} \left(-\nabla U_{1l}(z)\cdot\nu(z) + q_l(z)\nu_1(z)\right) f_l(x-\varepsilon z)d\sigma(z)$$

using that ν and ν_1 are homogeneous of degree 0, while ∇U_{1l} and q_l are homogeneous of degree -2. It will be useful in a moment to know which value has the integral

$$\int_{|z|=1} [-\nabla U_{1l}(z)\cdot\nu(z) + q_l(z)\nu_1(z)]d\sigma(z)$$

for $l = 1, 2, 3$. Let first $l = 1$. Then

$$U_{11}(z) = \frac{1}{8\pi}\frac{1}{|z|} + \frac{1}{8\pi}\frac{z_1^2}{|z|^3} \qquad\qquad q_1(z) = \frac{1}{4\pi}\frac{z_1}{|z|^3}$$

It follows that, for $|z| = 1$,

$$\partial_{z_1} U_{11}(z) = \frac{1}{8\pi}\left(z_1 - 3z_1^3\right)$$

and, for $j \neq 1$,

$$\partial_{z_j} U_{11}(z) = \frac{1}{8\pi}\left(-z_j - 3z_j z_1^2\right)$$

So, if $|z| = 1$,

$$-\nabla U_{11}(z)\cdot\nu(z) + q_1(z)\nu_1(z) = -\frac{1}{8\pi}\left(z_1^2 - 3z_1^4 - z_2^2 - 3z_2^2 z_1^2 - z_3^2 - 3z_3^3 z_1^2\right) + \frac{1}{4\pi} z_1^2$$

It is straightforward to calculate the following integrals

$$\int_{|z|=1} z_1^2 \, d\sigma(z) = \int_{|z|=1} z_2^2 \, d\sigma(z) = \int_{|z|=1} z_3^2 \, d\sigma(z) = \frac{4\pi}{3}$$

$$\int_{|z|=1} z_1^4 \, d\sigma(z) = \frac{4\pi}{5}, \qquad \int_{|z|=1} z_i^2 z_j^2 \, d\sigma(z) = \frac{4\pi}{15} \quad i \neq j$$

It follows that

$$\int_{|z|=1} \left(-\nabla U_{11}(z) \cdot \nu(z) + q_1(z)\nu_1(z)\right) d\sigma(z) = 1$$

Let now $l \neq 1$. We have

$$U_{1l}(z) = \frac{1}{8\pi} \frac{z_1 z_l}{|z|^3}$$

It follows that, for $|z| = 1$,

$$\partial_{z_1} U_{1l}(z) = \frac{1}{8\pi}\left(z_l - 3z_1^2 z_l\right), \qquad \partial_{z_j} U_{1l}(z) = \frac{1}{8\pi}\left(-3z_1 z_l z_j\right) \quad j \neq l, 1, \qquad \partial_{z_l} U_{1l}(z) = \frac{1}{8\pi}\left(z_1 - 3z_1 z_l^2\right)$$

So, if $|z| = 1$,

$$-\nabla U_{1l}(z) \cdot \nu(z) + q_l(z)\nu_1(z) = -\frac{1}{8\pi}\left(z_l z_1 - 3z_1^3 z_l - 3z_1 z_l z_j^2 + z_1 z_l - 3z_1 z_l^3\right) + \frac{1}{4\pi} z_l z_1$$

Again, it is straightforward that

$$\int_{|z|=1} z_i z_k \, d\sigma(z) = \int_{|z|=1} z_i^3 z_k \, d\sigma(z) = \int_{|z|=1} z_i z_k z_h^2 \, d\sigma(z) = 0 \quad i, j, k \text{ distinct}$$

So

$$\int_{|z|=1} \left(-\nabla U_{1l}(z) \cdot \nu(z) + q_l(z)\nu_1(z)\right) \, d\sigma(z) = 0$$

Now, we consider the term

$$\int_{|z|=1} \left(-\nabla U_{1l}(z) \cdot \nu(z) + q_l(z)\nu_1(z)\right) f_l(x - \varepsilon z) \, d\sigma(z)$$

The integrand is continuous on the compact sphere $|z| = 1$, and so sending $\varepsilon \to 0$ we have

$$\int_{|z|=1} \left(-\nabla U_{1l}(z) \cdot \nu(z) + q_l(z)\nu_1(z)\right) f_l(x - \varepsilon z) d\sigma(z) \to f_l(x) \int_{|z|=1} \left(-\nabla U_{1l}(z) \cdot \nu(z) + q_l(z)\nu_1(z)\right) d\sigma(z) =$$

$$= \begin{cases} f_1(x) & \text{if } l = 1 \\ 0 & \text{otherwise} \end{cases}$$

so that

$$\sum_{l=1}^{3} \int_{|z|=\varepsilon} \left(-\nabla U_{1l}(z) \cdot \nu(z) + q_l(z)\nu_1(z)\right) f_l(x - z) d\sigma(z) \to f_1(x) \quad \text{as } \varepsilon \to 0$$

Now, if x is fixed as above, for every $\varepsilon > 0$ we have

$$-\mu \Delta V_1(x) + \partial_{x_1} \Pi(x) = \sum_{l=1}^{3} \int_{|z|=\varepsilon} \left(-\nabla U_{1l}(z) \cdot \nu(z) + q_l(z)\nu_1(z)\right) f_l(x - z) d\sigma(z) + o_\varepsilon(1)$$

Since the equality holds for every $\varepsilon > 0$ and the left side is indipendent by ε we have

$$-\mu \Delta V_1(x) + \partial_{x_1} \Pi(x) = f_1(x)$$

that is what we want.

4. Incompressibility condition. It misses to prove that V satisfies the incompressibility equation $\nabla \cdot V = 0$. The method is similar to the previous. Let $x \in \mathbb{R}^3$ in an open neighbourhood and $\varepsilon > 0$. We have

$$\partial_{x_i} V_i(x) = \partial_{x_i} \int_{\mathbb{R}^3} \sum_{l=1}^{3} U_{il}(x-y) f_l(y) dy = \sum_{l=1}^{3} \partial_{x_i} \int_{\mathbb{R}^3} U_{il}(z) f_l(x-z) dz =$$

$$= \sum_{l=1}^{3} \int_{\mathbb{R}^3} U_{il}(z) \partial_{x_i}[f_l(x-z)] dz = - \sum_{l=1}^{3} \int_{\mathbb{R}^3} U_{il}(z) \partial_{z_i}[f_l(x-z)] dz$$

As above

$$\partial_{z_i}[U_{il}(z) f_l(x-z)] = \partial_{z_i} U_{il}(z) f_l(x-z) + U_{il}(z) \partial_{z_i}[f_l(x-z)]$$

So

$$\partial_{x_i} V_i(x) = - \sum_{l=1}^{3} \int_{|z|<\varepsilon} U_{il}(z) \partial_{z_i}[f_l(x-z)] dz - \sum_{l=1}^{3} \int_{|z|\geq\varepsilon} U_{il}(z) \partial_{z_i}[f_l(x-z)] dz =$$

$$= - \sum_{l=1}^{3} \int_{|z|\geq\varepsilon} \partial_{z_i}[U_{il}(z) f_l(x-z)] dz + \sum_{l=1}^{3} \int_{|z|\geq\varepsilon} \partial_{z_i} U_{il}(z) f_l(x-z) dz + o(1) =$$

$$= \sum_{l=1}^{3} \int_{|z|=\varepsilon} U_{il}(z) f_l(x-z) \nu_i(z) d\sigma(z) + \sum_{l=1}^{3} \int_{|z|\geq\varepsilon} \partial_{z_i} U_{il}(z) f_l(x-z) dz + o(1) =$$

$$= \sum_{l=1}^{3} \int_{|z|\geq\varepsilon} \partial_{z_i} U_{il}(z) f_l(x-z) dz + o(1)$$

where the surface integral has been included in the $o(1)$ since

$$\int_{|z|=\varepsilon} U_{il}(z) f_l(x-z) \nu_i(z) d\sigma(z) = \int_{|z|=1} U_{il}(\varepsilon z) f_l(x-\varepsilon z) \nu_i(\varepsilon z) \varepsilon^2 d\sigma(z) =$$

$$= \varepsilon \int_{|z|=1} U_{il}(z) f_l(x-\varepsilon z) \nu_i(z) d\sigma(z) \to 0 \quad \text{as } \varepsilon \to 0$$

where has been used that U_{il} is an homogeneous function of degree -1 and we have passed the limit under the integral sign because the integral is continuous on a compact set. It follows that

$$\nabla \cdot V(x) = \sum_{i=1}^{3} \partial_{x_i} V_i(x) = \sum_{i=1}^{3} \sum_{l=1}^{3} \int_{|z|\geq\varepsilon} \partial_{z_i} U_{il}(z) f_l(x-z) dz + o(1) =$$

$$= \sum_{l=1}^{3} \int_{|z|\geq\varepsilon} \sum_{i=1}^{3} \partial_{z_i} U_{il}(z) f_l(x-z) dz + o(1)$$

Using that, as noticed at the beginning,

$$\sum_{i=1}^{3} \partial_{z_i} U_{ij}(z) = 0 \quad \forall z \neq 0,\ j \in \{1,2,3\}$$

we have, since $\varepsilon > 0$,

$$\nabla \cdot V(x) = o(1)$$

being the other integrals zero because so it is the integrand far from the origin. Being x fixed, and $\varepsilon > 0$ arbitrary, we can send $\varepsilon \to 0$ and get

$$\nabla \cdot V(x) = 0$$

This completes the proof. ∎

Remark 9.13. We now have build a solution to the problem in the very special case of incompressible fluid. The following paragraph generalizes the problem to the case $\nabla \cdot v = g \not\equiv 0$. □

Reduction to the original problem. What we solved at this point is the incompressible Stokes problem, that is

$$\begin{cases} \mu\Delta v = \nabla p + f \\ \nabla \cdot v = 0 \end{cases} \tag{9.12}$$

We want to generalize the proof to the case

$$\begin{cases} \mu\Delta v = \nabla p + f \\ \nabla \cdot v = g \end{cases} \tag{9.13}$$

with $g \in C_c^\infty(\mathbb{R}^n)$. In order to solve (9.13), we shall look for a solution

$$\begin{cases} v = u + h \\ p = \pi \end{cases}$$

where u and π are volume potensials introduced above corresponding to $F := f - \mu\Delta h$, with

$$h = \nabla(\mathcal{E} * g)$$

where

$$\mathcal{E}(x) := \begin{cases} (2\pi)^{-1} \ln|x-y| & n = 2 \\ [n(n-2)\omega_n]^{-1}|x-y|^{2-n} & n \geq 3 \end{cases}$$

that is the fundamental solution of the Laplace's equation. By the properties of the Laplace's solution and some calculus, we have

$$\Delta h = \nabla g \in C_c^\infty(\mathbb{R}^n), \quad \nabla \cdot h = g \in C_c^\infty(\mathbb{R}^n)$$

At this point, it is clear that

$$\mu\Delta v = \mu\Delta u + \mu\Delta h = \nabla\pi + F + \mu\Delta h = \nabla p + f$$

and

$$\nabla \cdot v = \nabla \cdot u + \nabla \cdot h = g$$

9.3 Estimates of the solution on the whole space

We now need some estimates over this integral solutions. In order to do so, we will use theorems about integration of kernels and some applications. So, in the next section, keep in mind the results of section 3.7.

Remark 9.14. For sake of semplicity, we will prove the results with only two derivatives. With the same devices one can prove the results stated in susection 9.3.3: however the calculations are prohibitive and distract us from our aims. □

9.3.1 Estimates over the velocity field

We now deduce some estimates concerning the velocity field. First of all observe that

$$u_k(x) = -\frac{1}{\mu}\int_{\mathbb{R}^3}\sum_{l=1}^{3} U_{kl}(x-y)f_l(y)dy =$$

$$= -\frac{1}{\mu}\int_{|x-y|<\varepsilon}\sum_{l=1}^{3} U_{kl}(x-y)f_l(y)dy - \frac{1}{\mu}\int_{|x-y|\geq\varepsilon}\sum_{l=1}^{3} U_{kl}(x-y)f_l(y)dy \tag{9.14}$$

if $\varepsilon > 0$ is arbitrarly fixed. Deriving with respect the variable x, remembering that $D_{ij} = \partial_{x_j}\partial_{x_i}$ and using the differentiation under integral sign previously discussed, we have

$$D_{ij}u_k(x) = -\frac{1}{\mu}D_{ij}\int_{|x-y|\geq\varepsilon}\sum_{l=1}^{3} U_{kl}(x-y)f_l(y)dy \ -\frac{1}{\mu}D_{ij}\int_{|z|<\varepsilon}\sum_{l=1}^{3} U_{kl}(z)f_l(x-z)dz =$$

So, we have to deal with two pieces, that is

$$\text{(I)}\quad D_{ij}\int_{|x-y|\geq\varepsilon}\sum_{l=1}^{3} U_{kl}(x-y)f_l(y)dy, \qquad \text{(II)}\quad D_{ij}\int_{|z|<\varepsilon}\sum_{l=1}^{3} U_{kl}(z)f_l(x-z)dz$$

Considerations about the (I) piece. For the first term notice that

$$D_{ij}\int_{|x-y|\geq\varepsilon} U_{kl}(x-y)f_l(y)dy = D_{ij}\int_{|z|\geq\varepsilon} U_{kl}(z)f_l(x-z)dz = \int_{|z|\geq\varepsilon} U_{kl}(z)D_{ij}[f_l(x-z)]dz =$$

$$= \int_{|z|\geq\varepsilon} U_{kl}(z)D^z_{ij}[f_l(x-z)]dz$$

since $\partial_{x_j}\partial_{x_i}[f_l(x-z)] = \partial_{z_j}\partial_{z_i}[f_l(x-z)]$. But

$$D^z_j[U_{kl}(z)D^z_i(f_l(x-z))] = D^z_jU_{kl}(z)D^z_i(f_l(x-z)) + U_{kl}(z)D^z_jD^z_i(f_l(x-z))$$

and

$$D^z_i[D^z_jU_{kl}(z)f_l(x-z)] = D^z_iD^z_jU_{kl}(z)f_l(x-z) + D^z_jU_{kl}(z)D^z_i[f_l(x-z)]$$

so that

$$U_{kl}(z)D^z_{ij}[f_l(x-z)] = D^z_j[U_{kl}(z)D^z_i(f_l(x-z))] - D^z_jU_{kl}(z)D^z_i(f_l(x-z)) =$$

$$= D^z_j[U_{kl}(z)D^z_i(f_l(x-z))] - D^z_i[D^z_jU_{kl}(z)f_l(x-z)] + D^z_iD^z_jU_{kl}(z)f_l(x-z)$$

Integrating on $B(0,\varepsilon)^c$ we have

$$\int_{|z|\geq\varepsilon} U_{kl}(z)D^z_{ij}[f_l(x-z)]dz =$$

$$= \int_{|z|\geq\varepsilon} D^z_j[U_{kl}(z)D^z_i(f_l(x-z))]dz - \int_{|z|\geq\varepsilon} D^z_i[D^z_jU_{kl}(z)f_l(x-z)]dz + \int_{|z|\geq\varepsilon} D^z_iD^z_jU_{kl}(z)f_l(x-z)dz$$

The first piece is

$$\int_{|z|\geq\varepsilon} D^z_j[U_{kl}(z)D^z_i(f_l(x-z))]dz = \lim_{R\to+\infty}\int_{R\geq|z|\geq\varepsilon} D^z_j[U_{kl}(z)D^z_i(f_l(x-z))]dz$$

and by divergence theorem over an annulus (avoiding the singularity for $\varepsilon > 0$), we have

$$\int_{R\geq|z|\geq\varepsilon} D_j^z[U_{kl}(z)D_i^z(f_l(x-z))]dz =$$

$$= \int_{|z|=R} U_{kl}(z)D_i^z(f_l(x-z))\nu_j(z)d\sigma(z) \; - \int_{|z|=\varepsilon} U_{kl}(z)D_i^z(f_l(x-z))\nu_j(z)d\sigma(z)$$

We have fixed $x \in \mathbb{R}^3$, and so, if $x \in B(x_0, \delta)$, we have if $R > R_0 + C_0$, with $C_0 = |x_0| + \delta$ and R_0 such that $\mathrm{supp}(f) \subseteq B(0, R_0)$, then, as previously seen, $f \equiv 0$, togheter with its derivatives. So for such R the first piece vanishes. Concerning the second, we have

$$\int_{|z|=\varepsilon} |U_{kl}(z)D_i^z(f_l(x-z))\nu_j(z)|d\sigma(z) \leq M \int_{|z|=\varepsilon} |U_{kl}(z)|d\sigma(z)$$

and

$$\int_{|z|=\varepsilon} |U_{kl}(z)|d\sigma(z) = \int_{|y|=1} |U_{kl}(\varepsilon y)|\varepsilon^2 d\sigma(y) = \varepsilon \int_{|y|=1} |U_{kl}(y)|d\sigma(y)$$

since U_{kl} is homogeneous of degree $\alpha = -1$. So this piece vanishes as $\varepsilon \to 0$. Thus the whole first term vanishes. Now we consider

$$\int_{|z|\geq\varepsilon} D_i^z[D_j^z U_{kl}(z)f_l(x-z)]dz = \lim_{R\to+\infty} \int_{R\geq|z|\geq\varepsilon} D_i^z[D_j^z U_{kl}(z)f_l(x-z)]dz$$

where again we can write the integral as limit of integrals since $D_i^z[D_j^z U_{kl}(z)f_l(x-z)]$, is integrable over $|z| \geq \varepsilon$ thanks to the regularity of f_l and the compactness of its support. Again by divergence theorem over an annulus we have

$$\int_{R\geq|z|\geq\varepsilon} D_i^z[D_j^z U_{kl}(z)f_l(x-z)]dz =$$

$$= \int_{|z|=R} D_j^z U_{kl}(z)f_l(x-z)\nu_i(z)d\sigma(z) - \int_{|z|=\varepsilon} D_j^z U_{kl}(z)f_l(x-z)\nu_i(z)d\sigma(z)$$

For the surface integral over $\partial B(0,R)$ it holds the same argument about the support of f_l. So

$$\lim_{\varepsilon\to0} D_{ij} \int_{|x-y|\geq\varepsilon} U_{kl}(x-y)f_l(y)dy = \lim_{\varepsilon\to0} \int_{|z|\geq\varepsilon} U_{kl}(z)D_{ij}^z[f_l(x-z)]dz =$$

$$= \lim_{\varepsilon\to0} \left(\int_{|z|=\varepsilon} D_j^z U_{kl}(z)f_l(x-z)\nu_i(z)d\sigma(z) + \int_{|z|\geq\varepsilon} D_i^z D_j^z U_{kl}(z)f_l(x-z)dz \right) =$$

$$= \lim_{\varepsilon\to0} \left(\int_{|x-y|=\varepsilon} D_j U_{kl}(x-y)f_l(y)\nu_i(y)d\sigma(y) + \int_{|x-y|\geq\varepsilon} D_i D_j U_{kl}(x-y)f_l(y)dy \right) \quad (9.15)$$

where $\nu_i(y)$ is the i-th component of $\pm\frac{x-y}{|x-y|}$ but the dipendence by x is hidden (also $\nu_i(z) = \pm\frac{z_i}{|z|}$).

Consideration about the (II) piece. Notice that

$$\left| D_{ij} \int_{|z|<\varepsilon} U_{kl}(z)f_l(x-z)dz \right| = \left| \int_{|z|<\varepsilon} U_{kl}(z)D_{ij}f_l(x-z)dz \right| \leq$$

$$\leq \|D_{ij}f_l\|_\infty \int_{|z|<\varepsilon} |U_{kl}(z)|\chi_K dz \to 0 \quad \text{as } \varepsilon \to 0$$

since $U_{kl}(z)$ has an integrable singlularity at the origin, being homogenous of degree $\alpha = -1$, and K is such that if $z \notin K$ then $f(x-z) = 0$. So $|U_{kl}|\chi_K$ is summable near the origin.

Considerations about the two pieces. Remembering equation (9.14), we have to study the sum obtained in (9.15)

$$-\frac{1}{\mu}\int_{|x-y|\geq\varepsilon}\sum_{l=1}^{3} D_{ij}U_{kl}(x-y)f_l(y)dy \;-\frac{1}{\mu}\int_{|x-y|=\varepsilon}\sum_{l=1}^{3} D_iU_{kl}(x-y)f_l(y)\nu_j(y)d\sigma(y) \quad (9.16)$$

as $\varepsilon \to 0$. In fact, every other term vanishes, as proved. Remember that as $x \neq y$, U_{kl} is regular and so $D_{ij} = D_{ji}$.

We want first to estimate the second term of (9.16). For this aim, it is necessary to remark that, reading the explicit forms for D_iU_{kl}, we see that

$$|D_iU_{kl}(\alpha(x-y))| = \frac{1}{\alpha^2}|D_iU_{kl}(x-y)| \quad \forall\alpha > 0$$

So

$$\left|\sum_{l=1}^{3}\int_{|x-y|=\varepsilon} D_i[U_{kl}(x-y)]f_l(y)\nu_j(y)\; d\sigma(y)\right| \leq \sum_{l=1}^{3}\int_{|x-y|=\varepsilon}|D_i[U_{kl}(x-y)]||f_l(y)||\nu_j(y)|\; d\sigma(y) \leq$$

$$\leq \sum_{l=1}^{3}\int_{|x-y|=\varepsilon}|D_i[U_{kl}(x-y)]||f_l(y)|\; d\sigma(y)$$

where has been used that $|\nu_j(y)| \leq |\nu(y)| = 1$. Being, for every $\varepsilon > 0$, $|D_i[U_{kl}(x-y)]| \geq 0$ integrable, since the only singulatiry is when $x = y$, and being f_l cointinous because of the hypothesis about f, we have that

$$\int_{|x-y|=\varepsilon}|D_i[U_{kl}(x-y)]||f_l(y)|\; d\sigma(y) = |f_l(y_\varepsilon)|\int_{|x-y|=\varepsilon}|D_i[U_{kl}(x-y)]|\; d\sigma(y)$$

where y_ε is a point in $\partial B_\varepsilon(x) \equiv \{y \in \mathbb{R}^n :\; |x-y| = \varepsilon\}$. With a change of coordinates, we have[6]

$$\int_{|x-y|=\varepsilon}|D_i[U_{kl}(x-y)]|\; d\sigma(y) = \int_{|x-y|=1}|D_i[U_{kl}(\varepsilon(x-y))]|\varepsilon^2\; d\sigma(y) =$$

$$= \int_{|x-y|=1}\frac{1}{\varepsilon^2}|D_i[U_{kl}(x-y)]|\varepsilon^2\; d\sigma(y) = \int_{|x-y|=1}|D_i[U_{kl}(x-y)]|\; d\sigma(y)$$

The last integral is well-defined because $x \neq y$ on $|x-y| = 1$. Observe moreover that $|y_\varepsilon - x| = \varepsilon \to 0$ as $\varepsilon \to 0$. So $\lim_{\varepsilon\to 0} y_\varepsilon = x$. By the continuity of f_l we have

$$\lim_{\varepsilon\to 0}|f_l(y_\varepsilon)| = |f_l(x)|$$

Furthemore we have that

$$\int_{|x-y|=1}|D_i[U_{kl}(x-y)]|d\sigma(y) = \int_{|z|=1}|D_iU_{kl}(z)|d\sigma(z)$$

[6]

$$\int_{|x-y|=\varepsilon}|D_i[U_{kl}(x-y)]|d\sigma(y) = \int_{|z|=\varepsilon}|D_i[U_{kl}(z)]|d\sigma(z) =$$

$$= \varepsilon^2\int_{|z|=1}|D_i[U_{kl}(\varepsilon z)]|d\sigma(z) = \varepsilon^2\int_{|x-y|=1}|D_i[U_{kl}(\varepsilon(x-y))]|d\sigma(y)$$

so this is simply a number that does not depend on x or y. Hence we define

$$A_{ikl} := \int_{|z|=1} |D_i U_{kl}(z)| d\sigma(z) \tag{9.17}$$

So the limit exists, and it is

$$\lim_{\varepsilon\to 0} \int_{|x-y|=\varepsilon} |D_i[U_{kl}(x-y)]||f_l(y)|d\sigma(y) = A_{ikl}|f_l(x)|$$

Finally we get

$$\lim_{\varepsilon\to 0} \sum_{l=1}^{3} \int_{|x-y|=\varepsilon} |D_i[U_{kl}(x-y)]||f_l(y)|d\sigma(y) = \sum_{l=1}^{3} A_{ikl}|f_l(x)| \leq$$
$$\leq C_{ik}[|f_1(x)| + |f_2(x)| + |f_3(x)|] \leq 3C_{ik}|f(x)| \tag{9.18}$$

where $C_{ik} := \max_{l=1,2,3} A_{ikl}$ and $f(x) = (f_1(x), f_2(x), f_3(x))$.

We want now to say something about the other piece of (9.16), that is

$$\lim_{\varepsilon\to 0} \int_{|x-y|\geq\varepsilon} \sum_{l=1}^{3} D_{ij}[U_{kl}(x-y)]f_l(y)dy$$

Remark 9.15. We have that

$$U_{kl}(z) = -\frac{1}{8\pi}\left(\frac{\delta_{kl}}{|z|} + \frac{z_k z_l}{|z|^3}\right)$$

is an homogeneous function with exponent $\alpha = -1$. □

Remark 9.16. Being U_{kl} homogenous of order -1, we have that $\partial_{x_i} U_{kl}$ is homogeneous with order -2. So, $\partial_{x_j}\partial_{x_i} U_{kl}$ is homogeneous of order -3 and finally $\nabla(\partial_{x_j}\partial_{x_i} U_{kl})$ is homogeneous of order -4 and so the Hormander condition holds, since the proposition 3.2 says

$$|\nabla D_{ij} U_{kl}(z)| \equiv |\nabla(\partial_{x_j}\partial_{x_i} U_{kl})(z)| \leq \frac{C}{|z|^4} \quad \forall z \neq 0$$

On the other hand we know that

$$|D_{ij} U_{kl}(z)| \equiv |\partial_{x_j}\partial_{x_i} U_{kl}(z)| \leq \frac{C'}{|z|^3} \quad \forall z \neq 0$$

The maximum between C and C' satisfies both the inequalities. If $D_{ij}U_{kl}$ also is such that

$$\int_{r_1<|x|<r_2} D_{ij} U_{kl}(x)dx = 0$$

then we are in the hypothesis of the Calderón-Zygmund theorem. But by the divergence theorem in an annulus we have

$$\int_{r_1<|x|<r_2} D_{ij} U_{kl}(x)dx = \int_{|x|=r_2} \partial_{x_i} U_{kl}(x)\frac{x_j}{|x|}dx - \int_{|x|=r_1} \partial_{x_i} U_{kl}(x)\frac{x_j}{|x|}dx =$$
$$= \int_{|x|=1} \partial_{x_i} U_{kl}(r_2 x)\frac{r_2 x_j}{|r_2 x|} r_2^2 dx - \int_{|x|=1} \partial_{x_i} U_{kl}(r_1 x)\frac{r_1 x_j}{r_1|x|} r_1^2 dx =$$
$$= \int_{|x|=1} \partial_{x_i} U_{kl}(x)\frac{x_j}{|x|}dx - \int_{|x|=1} \partial_{x_i} U_{kl}(x)\frac{x_j}{|x|}dx = 0$$

using a change of variable and the fact that $\partial_{x_i} U_{kl}$ is homogeneous of degree -2. So we can apply the Calderón-Zygmund theorem. □

Application of Calderón-Zygmund theorem. Since in remark 9.16 we just proved that $D_{ij}U_{kl}$ is a Calderón-Zygmund kernel, we can apply theorem 3.16 to deduce that

$$\left\| \lim_{\varepsilon\to 0} \int_{|x-y|\geq\varepsilon} D_{ij}U_{kl}(x-y)f_l(y)dy \right\|_{L^p(\mathbb{R}^3)} \leq C_{p,l}\|f_l\|_{L^p(\mathbb{R}^3)} \tag{9.19}$$

and so

$$\left\| \lim_{\varepsilon\to 0} \sum_{l=1}^{3} \int_{|x-y|\geq\varepsilon} \frac{1}{\mu} D_{ij}[U_{kl}(x-y)]f_l(y)dy \right\|_{L^p(\mathbb{R}^3)} \leq \frac{1}{\mu}\sum_{l=1}^{3} \left\| \lim_{\varepsilon\to 0} \int_{|x-y|\geq\varepsilon} D_{ij}[U_{kl}(x-y)]f_l(y)dy \right\|_{L^p(\mathbb{R}^3)} \leq$$

$$\overset{(9.19)}{\leq} \frac{1}{\mu}\sum_{l=1}^{3} C_{p,l}\|f_l\|_{L^p(\mathbb{R}^3)} \leq \frac{1}{\mu}\|f\|_{L^p(\mathbb{R}^3)} \sum_{l=1}^{3} C_{p,l} \equiv \frac{C_p}{\mu}\|f\|_{L^p(\mathbb{R}^3)} \tag{9.20}$$

since for every $l \in \{1,2,3\}$ we have $|f_l(x)|^2 \leq |f_1(x)|^2 + |f_2(x)|^2 + |f_3(x)|^2 = |f(x)|^2$ and so

$$\|f_l\|_{L^p(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} |f_l(x)|^p \, dx \right)^{\frac{1}{p}} \leq \left(\int_{\mathbb{R}^3} |f(x)|^p \, dx \right)^{\frac{1}{p}} = \|f\|_{L^p(\mathbb{R}^3)}$$

Above we have defined $C_p \equiv \sum_{l=1}^{3} C_{p,l}$.

Estimates (9.3.1) and (9.20) will be helpful in a moment. We now underline where we were. For every $\varepsilon > 0$ we found

$$D_{ij}u_k(x) = -\frac{1}{\mu}\int_{|x-y|\geq\varepsilon} \sum_{l=1}^{3} D_{ij}[U_{kl}(x-y)]f_l(y)dy + \int_{|x-y|=\varepsilon} \sum_{l=1}^{3} D_i[U_{kl}(x-y)]f_l(y)\nu_j(y)d\sigma(y)$$

with

$$\lim_{\varepsilon\to 0} \left| \sum_{l=1}^{3} \int_{|x-y|=\varepsilon} D_i[U_{kl}(x-y)]f_l(y)\nu_j(y)d\sigma(y) \right| \leq 3C_{ik}|f(x)| \tag{9.3.1}$$

Remark 9.17. We remark that

$$\|\nabla^2 u\|_p^p = \int_{\mathbb{R}^3} |\nabla^2 u|^p \, dx = \int_{\mathbb{R}^3} \sum_{|\alpha|=2} |D^\alpha u|^p \, dx \leq C'_p \int_{\mathbb{R}^3} \sum_{|\alpha|=2} \sum_{k=1}^{3} |D^\alpha u_k|^p \, dx =$$

$$= C'_p \sum_{|\alpha|=2} \sum_{k=1}^{3} \int_{\mathbb{R}^3} |D^\alpha u_k|^p \, dx \tag{9.21}$$

Clearly, being $|\alpha| = 2$, we can write $D^\alpha = D_{ij}$ with $i, j \in \{1,2,3\}$. □

In order to simplify the notation, we set

$$D_{ij}u_k(x) \equiv A^{\varepsilon}_{ijk}(x) + \int_{|x-y|=\varepsilon} \sum_{l=1}^{3} D_i[U_{kl}(x-y)]f_l(y)\nu_j(y)d\sigma(y)$$

where obviously

$$A^{\varepsilon}_{ijk}(x) := -\frac{1}{\mu}\int_{|x-y|\geq\varepsilon} \sum_{l=1}^{3} D_{ij}[U_{kl}(x-y)]f_l(y)dy$$

Moreover we define

$$B_{ijk}^{\varepsilon}(x) := \sum_{l=1}^{3} \int_{|x-y|=\varepsilon} |D_i[U_{kl}(x-y)]f_l(y)|d\sigma(y)$$

So

$$|D_{ij}u_k(x)| \leq |A_{ijk}^{\varepsilon}(x)| + |\int_{|x-y|=\varepsilon} \sum_{l=1}^{3} D_i[U_{kl}(x-y)]f_l(y)\nu_j(y)d\sigma(y)| \leq$$

$$\leq |A_{ijk}^{\varepsilon}(x)| + \sum_{l=1}^{3} \int_{|x-y|=\varepsilon} |D_i[U_{kl}(x-y)]f_l(y)|d\sigma(y) \equiv |A_{ijk}^{\varepsilon}(x)| + |B_{ijk}^{\varepsilon}(x)|$$

Then

$$|D_{ij}u_k(x)|^p \leq \left(|A_{ijk}^{\varepsilon}(x)| + |B_{ijk}^{\varepsilon}(x)|\right)^p \leq 2^p \left(|A_{ijk}^{\varepsilon}(x)|^p + |B_{ijk}^{\varepsilon}(x)|^p\right)$$

Since the inequality holds for every $\varepsilon > 0$ we can send $\varepsilon \to 0$ and obtain

$$|D_{ij}u_k(x)|^p \leq 2^p \left(|\lim_{\varepsilon\to 0} A_{ijk}^{\varepsilon}(x)|^p + |\lim_{\varepsilon\to 0} B_{ijk}(x)|^p\right)$$

using the continuity of $|\cdot|$ and of the power. Remembering now that[7]

$$\lim_{\varepsilon\to 0} |B_{ijk}^{\varepsilon}(x)| = \lim_{\varepsilon\to 0} \sum_{l=1}^{3} \int_{|x-y|=\varepsilon} |D_i[U_{kl}(x-y)]f_l(y)|d\sigma(y) \leq 3C_{ik}|f(x)|$$

we can write

$$|D_{ij}u_k(x)|^p \leq 2^p \left(|\lim_{\varepsilon\to 0} A_{ijk}^{\varepsilon}(x)|^p + 3^p C_{ik}^p |f(x)|^p\right)$$

Integrating over $\mathbb{R}^3$ these positive functions we get

$$\int_{\mathbb{R}^3} |D_{ij}u_k(x)|^p\, dx \leq 2^p \int_{\mathbb{R}^3} |\lim_{\varepsilon\to 0} A_{ijk}^{\varepsilon}|^p\, dx + 2^p 3^p C_{ik}^p \int_{\mathbb{R}^3} |f(x)|^p\, dx = 2^p \int_{\mathbb{R}^3} |\lim_{\varepsilon\to 0} A_{ijk}^{\varepsilon}|^p\, dx + 6^p C_{ik}^p \|f\|_{L^p(\mathbb{R}^3)}^p$$

In equation (9.20) we have seen that

$$\int_{\mathbb{R}^3} |\lim_{\varepsilon\to 0} A_{ijk}^{\varepsilon}(x)|^p\, dx \equiv \left\| \lim_{\varepsilon\to 0} \frac{1}{\mu} \int_{|x-y|\geq\varepsilon} \sum_{l=1}^{3} D_{ij}U_{kl}(x-y)f_l(y)\, dy \right\|_{L^p(\mathbb{R}^3)}^p \leq \frac{C_p^p}{\mu^p} \|f\|_{L^p(\mathbb{R}^3)}^p$$

So, putting together the pieces, we have

$$\int_{\mathbb{R}^3} |D_{ij}u_k(x)|^p\, dx \leq \left(\frac{2^p C_p^p}{\mu^p} + 6^p C_{ik}^p\right) \|f\|_{L^p(\mathbb{R}^3)}^p$$

But

$$\|\nabla^2 u\|_{L^p(\mathbb{R}^n)}^p \overset{(9.21)}{=} \sum_{i,j,k=1}^{3} \int_{\mathbb{R}^3} |D_{ij}u_k(x)|^p\, dx \leq \left(\sum_{i,j,k=1}^{3} \left(\frac{2^p C_p^p}{\mu^p} + 6^p C_{ik}^p\right)\right) \|f\|_{L^p(\mathbb{R}^3)}^p$$

[7] Also remember that

$$\lim_{\varepsilon\to 0} \sum_{l=1}^{3} \int_{|x-y|=\varepsilon} |D_i[U_{kl}(x-y)]||f_l(y)|d\sigma(y) = \sum_{l=1}^{3} A_{ikl}|f_l(x)|$$

It means that the limit exists, so the expression makes sense.

We can moreover define the constant

$$C_0 := \sum_{i,j,k=1}^{3} \left(\frac{2^p C_p^p}{\mu^p} + 6^p C_{ik}^p \right)$$

and get

$$\|\nabla^2 u\|_{L^p(\mathbb{R}^3)}^p \leq C_0 \|f\|_{L^p(\mathbb{R}^3)}^p \tag{9.22}$$

that is the estimate that we want to prove.

9.3.2 Estimates over the pressure term

A similar estimate also holds for the pressure gradient ∇p. In fact, we have first of all

$$\partial_{x_i} \int_{\mathbb{R}^3} q_l(x-y) f_l(y) dy = \partial_{x_i} \int_{\mathbb{R}^3} q_l(z) f_l(x-z) dz$$

and we can split the integral as

$$\int_{\mathbb{R}^3} q_l(z) f_l(x-z) dz = \int_{|z|\geq\varepsilon} q_l(z) f_l(x-z) dz + \int_{|z|<\varepsilon} q_l(z) f_l(x-z) dz$$

Deriving we have

$$\partial_{x_i} \int_{\mathbb{R}^3} q_l(x-y) f_l(y) dy = \int_{|z|\geq\varepsilon} q_l(z) \partial_{x_i} f_l(x-z) dz + \int_{|z|<\varepsilon} q_l(z) \partial_{x_i} f_l(x-z)\ dz \tag{9.23}$$

Since

$$\left| \int_{|z|<\varepsilon} q_l(z) \partial_{x_i} f_l(x-z)\ dz \right| \leq \int_{|z|<\varepsilon} |q_l(z) \partial_{x_i} f_l(x-z)|\ dz \leq \|\nabla f\|_\infty \int_{|z|<\varepsilon} |q_l(z)| \chi_K\ dz \to 0 \quad \text{as } \varepsilon \to 0 \tag{9.24}$$

having q_l an integrable singularity in $z = 0$, we have to say something about the first addend in (9.23). We have

$$q_l(z) \partial_{x_i} f_l(x-z) = q_l(z) \left(\nabla f_l(x-z)\right)_i = -q_l(z) \partial_{z_i} \left(f_l(x-z)\right)$$

Moreover

$$\partial_{z_i} \left(q_l(z) f_l(x-z)\right) = \partial_{z_i} q_l(z) f_l(x-z) + q_l(z) \partial_{z_i} \left(f_l(x-z)\right)$$

This means that

$$\int_{|z|\geq\varepsilon} q_l(z) \partial_{x_i} f_l(x-z) dz = \int_{|z|\geq\varepsilon} \partial_{z_i} q_l(z) f_l(x-z) dz - \int_{|z|\geq\varepsilon} \partial_{z_i} \left(q_l(z) f_l(x-z)\right)\ dz$$

We'll deal with the first addend using the Calderón-Zygmund theorem. But at first we study the latter term. We have

$$\int_{|z|\geq\varepsilon} \partial_{z_i} [q_l(z) f_l(x-z)] dz = \lim_{R\to+\infty} \int_{R\geq|z|\geq\varepsilon} \partial_{z_i} [q_l(z) f_l(x-z)] dz$$

where we can write this limit since the integrand is summable on $|z| \geq \varepsilon$ thanks to regularity of f_l and the compactness of its support, for x fixed in a certain neighbourhood. So

$$\int_{R\geq|z|\geq\varepsilon} \partial_{z_i} [q_l(z) f_l(x-z)] dz = \int_{|z|=R} q_l(z) f_l(x-z) \nu_i(z) d\sigma(z)\ - \int_{|z|=\varepsilon} q_l(z) f_l(x-z) \nu_i(z) d\sigma(z)$$

The term integrated over $\partial B(0,R)$ vanishes for R large enough, in particular $R > |x_0|+\delta+R_0$, where $x \in B(x_0,\delta)$ and $\text{supp}(f) \subseteq B(0,R_0)$. So

$$\int_{|z|\geq\varepsilon} \partial_{z_i}[q_l(z)f_l(x-z)]dz = -\int_{|z|=\varepsilon} q_l(z)f_l(x-z)\nu_i(z)d\sigma(z)$$

Sending $\varepsilon \to 0$

$$\left|\int_{|z|=\varepsilon} q_l(z)f_l(x-z)\nu_i(z)\ d\sigma(z)\right| \leq \int_{|z|=\varepsilon} |q_l(z)f_l(x-z)\nu_i(z)|\ d\sigma(z) \leq$$

$$\leq \int_{|z|=\varepsilon} |q_l(z)||f_l(x-z)|\ d\sigma(z) = |f_l(x-z_\varepsilon)| \int_{|z|=\varepsilon} |q_l(z)|\ d\sigma(z)$$

where $z_\varepsilon \in \partial B(0,\varepsilon)$ and has been used the mean value property of the integral. Moreover

$$\int_{|z|=\varepsilon} |q_l(z)|d\sigma(z) = \varepsilon^2 \int_{|y|=1} |q_l(\varepsilon y)|\ d\sigma(y) = \int_{|y|=1} |q_l(y)|d\sigma(y)$$

since q_l is an homogeneous function of exponent $\alpha = -2$. Thus it follows that

$$\left|\int_{|z|\geq\varepsilon} q_l(z)\partial_{x_i} f_l(x-z)\ dz\right| = \left|\int_{|z|\geq\varepsilon} \partial_{z_i} q_l(z)f_l(x-z)\ dz + \int_{|z|=\varepsilon} q_l(z)f_l(x-z)\nu_i(z)\ d\sigma(z)\right| \leq$$

$$\leq \left|\int_{|z|\geq\varepsilon} \partial_{z_i} q_l(z)f_l(x-z)\ dz\right| + |f_l(x-z_\varepsilon)| \int_{|y|=1} |q_l(y)|\ d\sigma(y)$$

So, it follows that

$$\lim_{\varepsilon\to 0}\left|\int_{|z|\geq\varepsilon} q_l(z)\partial_{x_i} f_l(x-z)\ dz\right| \leq \lim_{\varepsilon\to 0}\left|\int_{|z|\geq\varepsilon} \partial_{z_i} q_l(z)f_l(x-z)\ dz\right| + A_l|f_l(x)|$$

thanks to the continuity of f_l, where $A_l := \displaystyle\int_{|y|=1} |q_l(y)|\ d\sigma(y)$. Moreover

$$\lim_{\varepsilon\to 0}\left|\int_{|z|\geq\varepsilon} \partial_{z_i} q_l(z)f_l(x-z)\ dz\right| = \left|\lim_{\varepsilon\to 0}\int_{|z|\geq\varepsilon} \partial_{z_i} q_l(z)f_l(x-z)\ dz\right|$$

where the limit without the absolute value exists thanks to the Calderón-Zygmund theorem[8]; in particular

$$\psi_{i,l}(x) := \lim_{\varepsilon\to 0}\int_{|z|\geq\varepsilon} \partial_{z_i} q_l(z)f_l(x-z)dz = \lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon} \partial_{x_i} q_l(x-y)f_l(y)dy$$

exists for all $x \in \mathbb{R}^3$ thanks to the C-Z theorem, since $\partial_{z_i} q_l$ is an homogeneous function of exponent $\alpha = -3$; moreover, $\nabla(\partial_{z_i} q_l)$ is homogeneous of degree $\alpha = -4$ and

$$\int_{r_1<|x|<r_2} \partial_{z_i} q_l(z)dz = \int_{|z|=r_2} q_l(z)\frac{z_i}{|z|}d\sigma(z) - \int_{|z|=r_1} q_l(z)\frac{z_i}{|z|}d\sigma(z) =$$

[8]Notice that $\partial_{x_i} q_l(x-y) = (\nabla q_l(x-y))_i = \partial_{z_i} q_l(z)|_{z=x-y}$.

$$= \int_{|z|=1} q_l(r_2 z)\frac{r_2 z_i}{|r_2 z|} r_2^2 d\sigma(z) - \int_{|z|=1} q_l(r_1 z)\frac{r_1 z_i}{|r_1 z|} r_1^2 d\sigma(z) =$$

$$= \int_{|z|=1} q_l(z)\frac{z_i}{|z|} d\sigma(z) - \int_{|z|=1} q_l(z)\frac{z_i}{|z|} d\sigma(z) = 0$$

since q_l is homoegenous of degree $\alpha = -2$. So, by Calderón-Zygmund theorem, we have

$$\|\psi_{i,l}\|_{L^p(\mathbb{R}^3)} \leq C_p \|f_l\|_{L^p(\mathbb{R}^3)}$$

Now we remember that (9.23) holds for every $\varepsilon > 0$, and so, by the arbitrariety of ε,

$$\left| \partial_{x_i} \int_{\mathbb{R}^3} \sum_{l=1}^{3} q_l(x-y) f_l(y)\ dy \right| \leq$$

$$\leq \sum_{l=1}^{3} \left(\lim_{\varepsilon\to 0} \left| \int_{|z|\geq\varepsilon} q_l(z)\partial_{x_i} f_l(x-z)\ dz \right| + \lim_{\varepsilon\to 0} \left| \int_{|z|<\varepsilon} q_l(z)\partial_{x_i} f_l(x-z)\ dz \right| \right) \overset{(9.24)}{=}$$

$$= \sum_{l=1}^{3} \lim_{\varepsilon\to 0} \left| \int_{|z|\geq\varepsilon} q_l(z)\partial_{x_i} f_l(x-z)\ dz \right| = \sum_{l=1}^{3} \left| \lim_{\varepsilon\to 0} \int_{|z|\geq\varepsilon} q_l(z)\partial_{x_i} f_l(x-z)\ dz \right| = \sum_{l=1}^{3} |\psi_{i,l}(x)|$$

Since we know that

$$\|\nabla p\|^p_{L^p(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |\nabla p(x)|^p\ dx = \int_{\mathbb{R}^3} \left(\sum_{i=1}^{3} |\partial_{x_i} p(x)|^p \right)\ dx$$

where

$$p(x) = \int_{\mathbb{R}^3} \sum_{l=1}^{3} q_l(x-y) f_l(y) dy = \sum_{l=1}^{3} \int_{\mathbb{R}^3} q_l(x-y) f_l(y) dy$$

and so

$$\partial_{x_i} p(x) = \sum_{l=1}^{3} \partial_{x_i} \int_{\mathbb{R}^3} q_l(x-y) f_l(y)\ dy, \qquad |\partial_{x_i} p(x)| \leq \sum_{l=1}^{3} |\psi_{i,l}(x)|$$

We use[9] and thus we get

$$|\partial_{x_i} p(x)|^p \leq (|\psi_{i,1}(x)| + |\psi_{i,2}(x)| + |\psi_{i,3}(x)|)^p \leq 4^p \sum_{l=1}^{3} |\psi_{i,l}(x)|^p$$

Hence we finally have

$$\|\nabla p\|^p_{L^p(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \left(\sum_{i=1}^{3} |\partial_{x_i} p(x)|^p \right)\ dx \leq \int_{\mathbb{R}^3} \sum_{i=1}^{3} \left(4^p \sum_{l=1}^{3} |\psi_{i,l}(x)|^p \right)\ dx = 4^p \sum_{i=1}^{3} \sum_{l=1}^{3} \int_{\mathbb{R}^3} |\psi_{i,l}(x)|^p\ dx =$$

[9] We can see this, in example, applying

$$(x+y)^p \leq 2^p(x^p + y^p)$$

so that, if $a, b, c \geq 0$, we have

$$(a+b+c)^p \leq 2^p\left((a+b)^p + c^p\right) = 2^p(a+b)^p + 2^p c^p \leq 4^p(a^p + b^p) + 2^p c^p$$

$$= 4^p \sum_{i=1}^{3} \sum_{l=1}^{3} \|\psi_{i,l}\|^p_{L^p(\mathbb{R}^3)} \le 4^p \sum_{i=1}^{3} \sum_{l=1}^{3} C^2_{p,i,l} \|f_l\|^p_{L^p(\mathbb{R}^3)}$$

But

$$\|f_l\|^p_{L^p(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |f_l(x)|^p \ dx \le \int_{\mathbb{R}^3} |f(x)|^p \ dx = \|f\|^p_{L^p(\mathbb{R}^3)}$$

since $|f_l(x)|^2 \le |f_1(x)|^2 + |f_2(x)|^2 + |f_3(x)|^2 = |f(x)|^2$ for $l \in \{1, 2, 3\}$. So we finally get

$$\|\nabla p\|^p_{L^p(\mathbb{R}^3)} \le 4^p \sum_{i=1}^{3} \sum_{l=1}^{3} C^p_{p,i,l} \|f\|^p_{L^p(\mathbb{R}^3)} \equiv C'_p \|f\|^p_{L^p(\mathbb{R}^3)}$$

Together with the estimate (9.22) above, we have

$$\|\nabla^2 u\|_{L^p(\mathbb{R}^3)} + \|\nabla p\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^3)} \tag{9.25}$$

Remark 9.18. In the incompressible case $f \equiv F$ with the notations introduced above ($g \equiv 0$). In the case $g \not\equiv 0$, we have $f \longleftrightarrow f - \Delta h$, and so

$$|F| \le |f| + |\Delta h| = |f| + |\nabla g|$$

that is, by the Minkowski inequality,

$$\|F\|_q \le \|f\|_q + \|\nabla g\|_q = \|f\|_q + |g|_{1,q}$$

This provide the estimate in the case $g \not\equiv 0$. □

9.3.3 Summary of the estimates

By the structur of h as integration of a kernel and the Calderón-Zygmund theorem, we have that

$$|h|_{l+1,q} \le c|g|_{l,q} \quad \forall l \ge 0 \tag{9.26}$$

with $c = c(n, q)$. Moreover, in the previous sections we have proved that

$$|u|_{2,q} \le c_1(\|f\|_q + |g|_{1,q}), \quad |\pi|_{1,q} \le c_2(\|f\|_q + |g|_{1,q})$$

for every $q > 1$. The same calculus as above, with more difficulties, tell us that

$$\begin{cases} |u|_{l+2} \le c_1 \left(|f|_{l,q} + |g|_{l+1,q}\right) \\ |\pi|_{l+1,q} \le c_2 \left(|f|_{l,q} + |g|_{l+1,q}\right) \end{cases} \quad l \ge 0, \ q > 1 \tag{9.27}$$

Since $v = u + h$ and $p = \pi$, using (9.26),

$$|v|_{l+2,q} + |p|_{l+1,q} \le c(|f|_{l,q} + |g|_{l+1,q})$$

with $c = c(n, q)$.

9.4 Stokes fundamental solution in $\mathbb{R}^n$: Existence, uniquess and estimates

The main aim of this section is to prove the following theorem, by [12, Theorem IV.2.1, pg. 243].

Theorem 9.5. *Given*

$$f \in W^{m,q}(\mathbb{R}^n), \quad g \in W^{m+1,q}(\mathbb{R}^n), \quad m \geq 0, \quad 1 < q < \infty, \quad n \geq 2$$

there exists a pair of functions v, p such that $v \in W^{m+2,q}(B_R)$, $p \in W^{m+1,q}(B_R)$ for any $R > 0$, satisying almost everywhere the equation

$$\begin{cases} \Delta v = \nabla p + f \\ \nabla \cdot v = g \end{cases} \quad in \quad \mathbb{R}^n \tag{9.28}$$

Moreover, for all $l \in [0, m]$, $|v|_{l+2,q}$ and $|p|_{l+1,q}$ are finite and we have

$$|v|_{l+2,q} + |p|_{l+1,q} \leq c(|f|_{l,q} + |g|_{l+1,q})$$

In the above inequalities, $c = c(n, q, l)$.

It also can be proved a uniqueness result.

Lemma 9.3. *In the hypothesis of theorem 9.5, if v_1, p_1 is another solution corresponding to the data f, g with $|v_1|_{l+2,q}$ finite for some $l \in [0, m]$, then $|v_1 - v|_{l+2,q} = 0$ and $|p_1 - p|_{l+1,q} = 0$.*

9.4.1 Proof of theorem 9.5

We start the proof remembering that $W^{m',q}(\mathbb{R}^n) = W_0^{m',q}(\mathbb{R}^n)$. This result is well known when $m' = 1$ and it can be generalized to $m' \in \mathbb{N}$. It follows that there exist twi sequences $\{f_k\}, \{g_k\} \subset C_0^\infty(\mathbb{R}^n)$ such that

$$\lim_{k\to\infty} \|f_k - f\|_{m,q} = \lim_{k\to\infty} \|g_k - g\|_{m+1,q} = 0 \tag{9.29}$$

So, we can consider the problem

$$\begin{cases} \Delta v_k = \nabla p_k + f_k \\ \nabla \cdot v_k = g_k \end{cases} \text{ in } \mathbb{R}^n \tag{9.30}$$

We can solve this system thanks to the previous sections. So, the pair of solutions (v_k, p_k) satisfies the inequality

$$|v_k|_{l+2,q} + |p_k|_{l+1,q} \leq c(n, q)(|f_k|_{l,q} + |g_k|_{l+1,q}) \quad \forall l \in [0, m] \tag{9.31}$$

This implies that the left side term is bounded. In fact, we have $|f_k|_{l,q} \leq \|f_k\|_{m,q} \leq C$ and $|g_k|_{l+1,q} \leq \|g_k\|_{m+1,q} \leq C'$, thanks to the convergences in (9.29). We now define the class

$$[u]_m := \{w \in D^{m,q}(\mathbb{R}^n) : \ w = u + \mathcal{P}, \ \exists \mathcal{P} \in P_m\}$$

where P_m is the set of the polynomials with degree $\leq m-1$.
So we can define

$$\dot{D}^{m,q}(\mathbb{R}^n) := \{[u]_m |\ u \in D^{m,q}(\mathbb{R}^n)\}$$

We can equip this set with the norm

$$\|[u]_m\| := |u|_{m,q}$$

It can be proved that this space is a well defined Banach space. All the properties are listed in proposition 4.1. Morover, in the proposition we have proved that the set is reflexive Banach space. So we have $[v_k]_{l+2} \in \dot{D}^{l+2,q}(\mathbb{R}^n)$, and $[p_k]_{l+2} \in \dot{D}^{l+1,q}(\mathbb{R}^n)$, for every $l \in [0,m]$. Moreover, we have seen in (9.31), that the sequences are bounded in this space.
But sequences in a reflexive Banach space converges weakly to a limit, thanks to [10, Th. 3, pg. 639]. This means that for every $T \in (\dot{D}^{l+2,q}(\mathbb{R}^n))^*$, there exists a subsequence k_j such that

$$\lim_{j\to\infty} T(v_{k_j}) = T(v)$$

So, consider $\psi \in L^{q'}(\mathbb{R}^n)$, with $\frac{1}{q} + \frac{1}{q'} = 1$. Define

$$T_\psi(u) := \langle D^\alpha u, \psi\rangle$$

with $|\alpha| = l+2 \in [2, m+2]$. So, if $u \in \dot{D}^{l+2,q}(\mathbb{R}^n)$, we have $u = w + \mathcal{P} \in D^{l+2,q}(\mathbb{R}^n)$. So,

$$|T_\psi(u)| = |\langle D^\alpha(w+\mathcal{P}), \psi\rangle| \overset{10}{=} |\langle D^\alpha w, \psi\rangle| \leq \|D^\alpha w\|_q \|\psi\|_{q'} = \|D^\alpha u\|_q \|\psi\|_{q'} \leq |u|_{l+2,q} \|\psi\|_{q'}$$

So $T_\psi \in (\dot{D}^{l+2,q}(\mathbb{R}^n))^*$. This means that

$$\lim_{j\to\infty} \langle D^\alpha v_{k_j}, \psi\rangle = \langle D^\alpha v, \psi\rangle \qquad \forall \psi \in L^{q'}(\mathbb{R}^n)$$

Analogously, one proves

$$\lim_{j\to\infty} \langle D^\beta \nabla p_{k_j}, \psi\rangle = \langle D^\beta \nabla p, \psi\rangle \qquad \forall \psi \in L^{q'}(\mathbb{R}^n)$$

for every $|\beta| \in [0,m]$. So, choosing $l = 0$ and $|\beta| = 0$, we find $v \in D^{2,q}(\mathbb{R}^n)$ and $p \in D^{1,q}(\mathbb{R}^n)$ such that

$$\langle \Delta v - \nabla p, \psi\rangle = \langle f, \psi\rangle \qquad \forall \psi \in L^{q'}(\mathbb{R}^n) \tag{9.32}$$

since $f_k \to f$ in $W^{m,q}(\mathbb{R}^n)$, so the convergence is in particular weakly.
Now we have to do some remarks. First of all, remember lemma 4.5. Then we have that $v \in W^{2,q}(B_R)$ and $p \in W^{1,q}(B_R)$ for every $R > 0$, where $B_R \equiv B_R(0)$.
Notice now that $\psi \in C_0^\infty(\mathbb{R}^n) \subset L^{q'}(\mathbb{R}^n)$. So, writing $\mathbb{R}^n = \bigcup_{k\in\mathbb{N}} B(0,k)$, for every $k \in \mathbb{N}$ we have

$$\langle \Delta v - \nabla p - f, \psi\rangle = 0 \qquad \forall \psi \in C_0^\infty(B(0,k)) \subset C_0^\infty(\mathbb{R}^n)$$

[10] We have that the degree of $\mathcal{P}$ is $\leq l+2-1 = l+1$, so that $D^\alpha \mathcal{P} = 0$.

It follows that $\Delta v = \nabla p + f$ almost everywhere in $B(0,k)$, that is in $B(0,k)/E_k$, with $|E_k| = 0$. So, if $E := \bigcup_{k\in\mathbb{N}} E_k$, we have that $|E| = 0$ and $\Delta v = \nabla p + f$ in $\mathbb{R}^n/E$, that is almost everywhere in $\mathbb{R}^n$. In the same way we have $\nabla \cdot v = g$.
Observe that, since $(X \cap Y)^* \simeq X^* + Y^*$, we have that the classes of v_k are in the intersections of the spaces $\dot{D}^{l+2,q}(\mathbb{R}^n)$, so that the weak convergence holds with the sum of operators in the dual spaces; in particular, it holds in every dual space, choosing the null operator in the other spaces. So we have that

$$|v|_{l+2,q} + |p|_{l+1,q} \leq \liminf_{k\to\infty} |v_k|_{l+2,q} + \liminf_{k\to\infty} |p_k|_{l+1,q} \tag{9.33}$$

So, since $\lim_{k\to\infty} |f_k - f|_{l,q} = \lim_{k\to\infty} |g_k - g|_{l+1,q} = 0$ for every $l \in [0,m]$, we have, using (9.31),

$$|v|_{l+2,q} + |p|_{l+1,q} \leq 2c(n,q)(|f|_{l,q} + |g|_{l+1,q})$$

that is what we wanted. ∎

9.5 Stokes theory on bounded domains

We finally start to consider bounded domains. First of all, we have the following introductive lemma that is inspired by [12, Lemma IV.4.2].

Lemma 9.4. *Let $\Omega \subseteq \mathbb{R}^n$, with $n \geq 2$ a bounded domain. Let v, p such that $v \in W^{1,1}_{loc}(\Omega)$, $\nabla \cdot v = 0$ and $p \in L^1_{loc}(\Omega)$. Let $f \in L^1_{loc}(\Omega)$ and suppose that*

$$\langle \nabla v, \nabla \psi \rangle = -\langle f, \psi \rangle + \langle p, \nabla \cdot \psi \rangle \qquad \forall \psi \in C^\infty_c(\Omega) \tag{9.34}$$

Then, if v_ε is the regularization of v, in the sense of definition 3.3, and p_ε is the regularization of p, we have

$$\begin{cases} \Delta v_\varepsilon = \nabla p_\varepsilon + f_\varepsilon \\ \nabla \cdot v_\varepsilon = 0 \end{cases} \qquad in \ \ \Omega_0 \tag{9.35}$$

for every domain Ω_0 such that $\overline{\Omega}_0 \subset \Omega$.

Proof. Let $\overline{\Omega}_0 \subset \Omega$. Let $\varphi \in C^\infty_c(\Omega_0)$. Observe that, then, $\varphi_\varepsilon \in C^\infty_c(\Omega)$ if $\varepsilon <$ dist $(\Omega_0, \partial\Omega)$. By hypothesis (9.34), we have

$$\langle \nabla v, \nabla \varphi_\varepsilon \rangle = -\langle f, \varphi_\varepsilon \rangle + \langle p, \nabla \cdot \varphi_\varepsilon \rangle$$

Remark 9.19. Let $h, g \in L^2(\Omega)$, Ω bounded and g such that $\mathrm{supp}(g) \subset \Omega_0 \subset \Omega_\varepsilon$. Then

$$\langle g_\varepsilon, h \rangle := \int_\Omega g_\varepsilon(x) h(x) \ dx \equiv \int_\Omega \left(\int_\Omega \eta_\varepsilon(x-y) g(y) \ dy \right) h(x) \ dx =$$

$$= \int_\Omega \left(\int_\Omega \eta_\varepsilon(x-y) g(y) h(x) \ dx \right) dy \overset{11}{=} \int_{\Omega_0} g(y) \left(\int_\Omega \eta_\varepsilon(y-x) h(x) \ dx \right) dy = \langle g, h_\varepsilon \rangle_{\Omega_0}$$

that will be useful in the future. □

[11] Since $\eta(x) = \eta(-x)$, then $\eta_\varepsilon(x-y) = \eta_\varepsilon(y-x)$.

Applying remark 9.19, we have

$$\langle \nabla v, \nabla \varphi_\varepsilon \rangle = \langle \nabla v, (\nabla \varphi)_\varepsilon \rangle = \langle (\nabla v)_\varepsilon, \nabla \varphi \rangle_{\Omega_0} = \langle \nabla v_\varepsilon, \nabla \varphi \rangle_{\Omega_0} = -\langle \Delta v_\varepsilon, \varphi \rangle_{\Omega_0}$$

thanks to the fact that now $v_\varepsilon \in C^\infty(\Omega_0)$ and $\mathrm{supp}(\varphi) \subset \Omega_0$. Moreover, we have $\langle f, \varphi_\varepsilon \rangle = \langle f_\varepsilon, \varphi \rangle_{\Omega_0}$, and

$$\langle p, \nabla \cdot \varphi_\varepsilon \rangle = \langle p, (\nabla \cdot \varphi)_\varepsilon \rangle = \langle p_\varepsilon, \nabla \cdot \varphi \rangle_{\Omega_0} = -\langle \nabla p_\varepsilon, \varphi \rangle_{\Omega_0}$$

So, we have that

$$\langle \Delta v_\varepsilon, \varphi \rangle_{\Omega_0} = \langle f_\varepsilon, \varphi \rangle_{\Omega_0} + \langle \nabla p_\varepsilon, \varphi \rangle_{\Omega_0}$$

This means that

$$\int_{\Omega_0} \big(\Delta v_\varepsilon - f_\varepsilon - \nabla p_\varepsilon \big) \cdot \varphi \; dx = 0$$

for every $\varphi \in C_c^\infty(\Omega_0)$. This clearly implies that $\Delta v_\varepsilon = \nabla p_\varepsilon + f_\varepsilon$ almost everywhere in Ω_0 and, being the functions involved continuous in $\Omega_\varepsilon \supset \Omega_0$, in the whole Ω_0. ∎

We can finally prove the following theorem, that is [12, Th. IV.4.1].

Theorem 9.6. *Let $\Omega \subset \mathbb{R}^n$ a bounded domain, with $n \geq 2$. Let v a velocity field such that*[12] *$\nabla v \in L^q_{loc}(\Omega)$, $q \in (1, \infty)$ and it is weakly divergence free. Moreover, suppose that*

$$\langle \nabla v, \nabla \varphi \rangle = -\langle f, \varphi \rangle \qquad \forall \varphi \in \mathcal{D}(\Omega)$$

If $f \in W^{m,q}_{loc}(\Omega)$, with $m \geq 0$, it follows that

$$v \in W^{m+2,q}_{loc}(\Omega), \qquad p \in W^{m+1,q}_{loc}(\Omega)$$

where p is the pressure filed of lemma 9.1. Moreover, if $\Omega', \Omega'' \subset \Omega$ are bounded and such that $\overline{\Omega}' \subset \Omega'' \subset \overline{\Omega}'' \subset \Omega$, we have

$$|v|_{m+2,q,\Omega'} + |p|_{m+1,q,\Omega'} \leq c\big(\|f\|_{m,q,\Omega''} + \|v\|_{1,q,\Omega''-\Omega'} + \|p\|_{q,\Omega''-\Omega'} \big) \tag{9.36}$$

where $c = c(n, q, m, \Omega', \Omega'')$.

Proof. Keep in mind lemma 9.1. Then, the proof is essentially a corollary of the previous theorem. Consider a *cut-off* functions, that is $\varphi \in C^\infty(\Omega)$ with $\varphi \equiv 1$ over $\overline{\Omega}'$ and $\varphi \equiv 0$ over $(\Omega'')^c$. Let $\Omega_0 \supset \overline{\Omega}''$ and consider the equation

$$\big(\Delta v_\varepsilon\big)\varphi = \big(\nabla p_\varepsilon\big)\varphi + f_\varepsilon \varphi$$

Now, we define $u := \varphi v_\varepsilon$ and $\pi = \varphi p_\varepsilon$. We have

$$\Delta u = v_\varepsilon \Delta \varphi + 2\big(\nabla\varphi \cdot \nabla\big) v_\varepsilon + \varphi \Delta v_\varepsilon = v_\varepsilon \Delta + 2\big(\nabla\varphi \cdot \nabla\big) v_\varepsilon - \nabla\varphi p_\varepsilon + \nabla \pi + f_\varepsilon \varphi$$

using that $\nabla \pi = \nabla \varphi p_\varepsilon + \varphi \nabla p_\varepsilon$. Defining

$$f_c := v_\varepsilon \Delta \varphi + 2\big(\nabla\varphi \cdot \nabla\big) v_\varepsilon - \nabla \varphi p_\varepsilon, \quad f_1 := f_\varepsilon \varphi$$

[12] Lemma 4.5 implies that also $v \in L^q_{loc}(\Omega)$, so that $v \in W^{1,q}_{loc}(\Omega)$ just using hypothesis.

It follows that

$$\Delta u = \nabla \pi + f_c + f_1$$

and

$$\nabla \cdot u = \nabla \cdot \big(\varphi v_\varepsilon\big) = \big(\nabla \cdot v_\varepsilon\big)\varphi + v_\varepsilon \cdot \nabla\varphi = v_\varepsilon \cdot \nabla\varphi =: g$$

where $\nabla \cdot v_\varepsilon = 0$ since v is weakly divergence free. So the pair (u, π) solves

$$\begin{cases} \Delta u = \nabla \pi + f_c + f_1 \\ \nabla \cdot u = g \end{cases}$$

in $\mathbb{R}^n$. Observe that, since $\varphi \in C_c^\infty(\Omega_0)$ and $v_\varepsilon, p_\varepsilon, f_\varepsilon \in C^\infty(\Omega_\varepsilon)$, with $\Omega_\varepsilon \supset \overline{\Omega}_0$ if ε is small enough, we have $u, \pi, g \in C_c^\infty(\mathbb{R}^n)$ if extended as zero outside. So in particular their integral norms and those of their derivative are finite.
So, using lemma 9.3, since $f_c, f_1 \in W^{m,q}(\mathbb{R}^n)$ and $g \in W^{m+1,q}(\mathbb{R}^n)$, and since $|u|_{2,q}$ and $|\pi|_{1,q}$ are finite, we have, with $l = m = 0$, that also this solutions satisfy

$$|u|_{2,q} + |\pi|_{1,q} \leq c\big(\|f_1 + f_c\|_q + |g|_{1,q}\big)$$

In other words, the inequality is

$$\|\nabla^2 u\|_q + \|\nabla \pi\|_q \leq c\big(\|f_\varepsilon \varphi\|_q + \|v_\varepsilon \Delta\varphi\|_q + 2\|(\nabla\varphi \cdot \nabla)v_\varepsilon\|_q + \|\nabla\varphi p_\varepsilon\|_q + \|\nabla(v_\varepsilon \cdot \nabla\varphi)\|_q\big) \tag{9.37}$$

Observe now that $\nabla\big(\nabla\varphi \cdot v_\varepsilon\big) = \big(\nabla^2\varphi\big)v_\varepsilon + \nabla\varphi \cdot \nabla v_\varepsilon$. So, looking at the right member of (9.37), we notice that $\nabla\varphi, \nabla^2\varphi$ are bounded in $\Omega'' - \Omega'$ and zero outside. At the same time, φ is bounded and zero outside Ω''. So we have

$$\begin{aligned} \|\nabla^2 u\|_q + \|\nabla \pi\|_q &\leq C\big(\|f_\varepsilon\|_{q,\Omega'} + \|v_\varepsilon\|_{q,\Omega''-\Omega'} + \|\nabla v_\varepsilon\|_{q,\Omega''-\Omega'} + \|p_\varepsilon\|_{q,\Omega''-\Omega'}\big) \\ &\leq C'\big(\|f_\varepsilon\|_{q,\Omega'} + \|v_\varepsilon\|_{1,q,\Omega''-\Omega'} + \|p_\varepsilon\|_{q,\Omega''-\Omega'}\big) \end{aligned} \tag{9.38}$$

On the other side, observe that

$$\partial_{x_i} u = \partial_{x_i}\varphi v_\varepsilon + \varphi \partial_{x_i} v_\varepsilon$$

and so

$$\partial_{x_i}^2 u = \partial_{x_i}^2\varphi v_\varepsilon + 2\partial_{x_i}\varphi\partial_{x_i} v_\varepsilon + \varphi\partial_{x_i}^2 v_\varepsilon$$

So, by inverse triangular inequality, we have that

$$\begin{aligned} \|\partial_{x_i}^2 u\|_q = \|\partial_{x_i}^2\varphi v_\varepsilon + 2\partial_{x_i}\varphi\partial_{x_i} v_\varepsilon + \varphi\partial_{x_i}^2 v_\varepsilon\|_q \geq \big|\|\partial_{x_i}^2\varphi v_\varepsilon + 2\partial_{x_i}\varphi\partial_{x_i} v_\varepsilon\|_q - \|\varphi\partial_{x_i}^2 v_\varepsilon\|_q\big| \geq \\ \geq \|\varphi\partial_{x_i}^2 v_\varepsilon\|_q - \|\partial_{x_i}^2\varphi v_\varepsilon + 2\partial_{x_i}\varphi\partial_{x_i} v_\varepsilon\|_q \end{aligned}$$

So, we have that

$$\|\varphi\partial_{x_i}^2 v_\varepsilon\|_q \leq \|\partial_{x_i}^2 u\|_q + \|\partial_{x_i}^2\varphi v_\varepsilon + 2\partial_{x_i}\varphi\partial_{x_i} v_\varepsilon\|_q \leq C''\big(\|f_\varepsilon\|_{q,\Omega'} + \|v_\varepsilon\|_{1,q,\Omega''-\Omega'} + \|p_\varepsilon\|_{q,\Omega''-\Omega'}\big)$$

Since $\|\varphi\partial_{x_i}^2 v_\varepsilon\|_q \geq \|\varphi\partial_{x_i}^2 v_\varepsilon\|_{q,\Omega'} \equiv \|\partial_{x_i}^2 v_\varepsilon\|_{q,\Omega'}$, being φ constantly 1 over Ω', we have that

$$\|\partial_{x_i}^2 v_\varepsilon\|_{q,\Omega'} \leq C''\big(\|f_\varepsilon\|_{q,\Omega'} + \|v_\varepsilon\|_{1,q,\Omega''-\Omega'} + \|p_\varepsilon\|_{q,\Omega''-\Omega'}\big)$$

Since the same inequality holds for $\partial_{x_i}\pi_\varepsilon$ and the other second derivatives, we have

$$\|\nabla^2 v_\varepsilon\|_{q,\Omega'} + \|\nabla p_\varepsilon\|_{q,\Omega'} \leq \overline{C}\big(\|f_\varepsilon\|_{q,\Omega'} + \|v_\varepsilon\|_{1,q,\Omega''-\Omega'} + \|p_\varepsilon\|_{q,\Omega''-\Omega'}\big)$$

Using the pointwise convergence of mollifications, we have, as $\varepsilon \to 0$,

$$\|\nabla^2 v\|_{q,\Omega'} + \|\nabla p\|_{q,\Omega'} \leq \overline{C}\big(\|f\|_{q,\Omega'} + \|v\|_{1,q,\Omega''-\Omega'} + \|p\|_{q,\Omega''-\Omega'}\big)$$

that is the thesis. ∎

9.5.1 L^q-Estimates near the boundary

The main theorem of this subsection is the following. It is theorem [12, Th. IV.5.1].

Theorem 9.7. *Let Ω be an arbitrary domain in $\mathbb{R}^n$, with $n \geq 2$, with a boundary portion σ of class C^{m+2}, $m \geq 0$. Let Ω_0 be any bounded subdomain of Ω with $\partial\Omega_0 \cap \partial\Omega = \sigma$. Further, let*

$$v \in W^{1,q}(\Omega_0), \quad p \in L^q(\Omega_0), \quad 1 < q < \infty$$

be such that

$$\langle \nabla v, \nabla \psi \rangle = -\langle f, \psi \rangle + \langle p, \nabla \cdot \psi \rangle, \quad \psi \in C_0^\infty(\Omega_0)$$
$$\langle v, \nabla \varphi \rangle = 0 \qquad \forall \psi \in C_0^\infty(\Omega_0)$$
$$v = v_* \quad \text{at } \sigma$$

Then, if

$$f \in W^{m,q}(\Omega_0), \quad v_* \in W^{m+2-1/q,q}(\sigma)$$

we have

$$v \in W^{m+2,q}(\Omega'), \quad p \in W^{m+1,q}(\Omega')$$

for any Ω' satisfying

- *$\Omega' \subset \Omega_0$;*
- *$\partial\Omega' \cap \partial\Omega$ is a strictly interior subregion of σ.*

Finally, the following estimate holds

$$\|v\|_{m+2,q,\Omega'} + \|p\|_{m+1,q,\Omega'} \leq c\big(\|f\|_{m,q,\Omega_0} + \|v_*\|_{m+2-1/q,q(\sigma)} + \|v\|_{1,q,\Omega_0} + \|p\|_{q,\Omega_0}\big)$$

where $c = c(m, n, q, \Omega', \Omega_0)$.

Remark 9.20. The proof is very technical and it is apart from our purposes. It is completely exposed in [12, Th. IV.5.1, pg. 276]. □

9.5.2 Proof of the main theorem

In order to prove the main theorem, we have to prove a uniqueness lemma.

Lemma 9.5. *Let Ω be a bounded, C^2-smooth domain on $\mathbb{R}^n$. If v is a q-weak solution to the Stokes problem, corresponding to zero data $f \equiv 0$, $v_* \equiv 0$, then $v \equiv 0$ and $p \equiv c$ a.e. in Ω, where p is the pressure field associated to v.*

Proof. We deal with the proof in two cases. First of all, suppose $q \geq 2$. Let v, v_1 two solutions, and define $u := v - v_1$. Then u satisfies, for every $\varphi \in C_{0,\sigma}^\infty(\Omega)$,

$$\langle \nabla u, \nabla \varphi \rangle = \langle \nabla v, \nabla \varphi \rangle - \langle \nabla v_1, \nabla \varphi \rangle = 0$$

since v and v_1 are solution associated to the same data. Also the trace is zero, since T is a linear operator. So, we have that, for every $\varphi \in C_{0,\sigma}^\infty(\Omega)$,

$$0 = \int_\Omega \nabla u \cdot \nabla \varphi \, dx$$

It follows, by regularizing the euqation, that $\Delta u_\varepsilon = 0$, with zero boundary conditions. It follows that $u \equiv 0$. ∎

We finally prove the main theorem.

Definition 9.11. In the following we set

$$\|w\|_{k,q/\mathbb{R}} := \inf_{c\in\mathbb{R}} \|w + c\|_{k,q}$$

Theorem 9.8. *Let v be a q-generalized solution of the Stokes problem in a bounded domain Ω of $\mathbb{R}^n$, $n \geq 2$, of class C^{m+2}, $m \geq 0$, corresponding to*

$$f \in W^{m,q}(\Omega), \qquad v_* \in W^{m+2-1/q,q}(\partial\Omega) \tag{9.39}$$

Then

$$v \in W^{m+2,q}(\Omega), \qquad p \in W^{m+1,q}(\Omega)$$

where p is the pressure field associated to v by lemma 9.1. Moreover, the following inequality holds:

$$\|v\|_{m+2,q} + \|p\|_{m+1,q/\mathbb{R}} \leq c\big(\|f\|_{m,q} + \|v_*\|_{m+2-1/q,q(\partial\Omega)}\big)$$

with $c = c(m, n, q, \Omega)$.

Proof. Being Ω a bounded domain, we can consider the closure $\overline{\Omega}$. It is a compact set. By definition, we can cover this compact with a finite number of open balls. Using theorem 9.6 and 9.7, sectioning the domain as in 1.2, we find functions $v \in W^{m+2,q}(\Omega)$ and $p \in W^{m+1,q}(\Omega)$ such that

$$|v|_{m+2,q} + |p|_{m+1,q} \leq c\big(\|f\|_{m,q} + \|v_*\|_{m+2-1/q,q(\partial\Omega)} + \|v\|_{1,q} + \|p\|_q\big)$$

So, summing the L^q norms of all the derivatives of all the orders up to $m+1$, we finally get

$$\|v\|_{m+2,q} + \|p\|_{m+1,q/\mathbb{R}} \leq c_2\big(\|f\|_{m,q} + \|v_*\|_{m+2-1/q,q(\partial\Omega)} + \|p\|_{q/\mathbb{R}} + \|v\|_q\big) \tag{9.40}$$

Moreover, we know that, given data as in (9.39), a q-weak solution of the problem is such that $v \in W^{m+2,q}(\Omega)$, $p \in W^{m+1,q}(\Omega)$.

We now show that, provided that the solution is unique, we can prove the existence of a constant $c_3 > 0$ such that, for every $f \in W^{m,q}(\Omega)$, $v_* \in W^{m+2-1/q,q}(\partial\Omega)$, a weak solution $v \in W^{m+2,q}(\Omega)$ of the Stokes system with associated pressure field $p \in W^{m+1,q}(\Omega)$ satisfies

$$\|v\|_q + \|p\|_{q/\mathbb{R}} \leq c_3\big(\|f\|_{m,q} + \|v_*\|_{m+2-1/q,q(\partial\Omega)}\big) \tag{9.41}$$

Clearly, this result, together with (9.40), gives us the the final estimate.

If (9.41) were not true, then for every $c_k > 0$ there exists $f_k \in W^{m,q}(\Omega)$, $v_*^k \in$

$W^{m+2-1/q,q}(\partial\Omega)$ and a weak solution $v_k \in W^{m+2,q}(\Omega)$ of the problem, with associated pressure field $p_k \in W^{m+1,q}(\Omega)$, such that

$$\|v_k\|_q + \|p_k\|_{q/\mathbb{R}} > c_k\big(\|f_k\|_{m,q} + \|v_*^k\|_{m+2-1/q,q(\partial\Omega)}\big) \tag{9.42}$$

We can choose $c_k = k \in \mathbb{N}$. Moreover, without less of generality, we can suppose[13]

$$\|v_k\|_q + \|p_k\|_{q/\mathbb{R}} = 1 \text{ for all } k \in \mathbb{N} \tag{9.43}$$

Since $k \to \infty$, then

$$\lim_{k\to\infty} \big(\|f_k\|_{m,q} + \|v_*^k\|_{m+2-1/q,q(\partial\Omega)}\big) = 0 \tag{9.44}$$

By equation (9.40), we have that

$$\|v_k\|_{m+2,q} + \|p_k\|_{m+1,q/\mathbb{R}} \leq c_2\big(\|f_k\|_{m,q} + \|v_*^k\|_{m+2-1/q,q(\partial\Omega)} + \|p_k\|_{q/\mathbb{R}} + \|v_k\|_q\big) \leq C$$

So the sequence is uniformly bounded. Now we use corollary 4.1. In fact, we have that $m \geq 0$, and so $m+2 \geq 2$. So in particular $\|v_k\|_{2,q} \leq C$, that implies

$$\|v_k\|_{1,q} \leq C, \qquad \|\nabla v_k\|_{1,q} \leq C$$

Since $W^{1,q}(\Omega) \subset\subset L^q(\Omega)$ for every $q \in (1,\infty)$, we have that exist k_h and $u \in L^q(\Omega)$ such that

$$\lim_{h\to\infty} \|v_{k_h} - u\|_q = 0$$

Since now $\|\nabla v_{k_h}\|_{1,q} \leq C$, being a subsequence, we have that exist h_{h_l} and $w \in L^q(\Omega)$ such that

$$\lim_{l\to\infty} \|\nabla v_{k_{h_l}} - w\|_q = 0$$

In particular $\lim\limits_{l\to\infty} \|v_{k_{h_l}} - u\|_q = 0$, being a subsequence. So, it follows that $\nabla u = w \in L^q(\Omega)$. In fact, for every $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega u\ \partial_{x_i}\varphi\ dx = \lim_{l\to\infty} \int_\Omega v_{k_{h_l}}\ \partial_{x_i}\varphi\ dx = -\lim_{l\to\infty} \int_\Omega \partial_{x_i} v_{k_{h_l}}\ \varphi\ dx = -\int_\Omega w_i\ \varphi\ dx$$

If we rename the sequence to be v_k, we have that

$$\|v_k - u\|_{1,q} = \big(\|v_k - u\|_q + \|\nabla v_k - \nabla u\|_q\big)^{\frac{1}{q}} \to 0$$

---

[13] In fact, if $v_k \in W^{m+2,q}(\Omega)$ is a q-weak solution of the problem with data $f_k \in W^{m,q}(\Omega)$, $v_*^k \in W^{m+2-1/q,q}(\partial\Omega)$, with associated pressure field $p_k \in W^{m+1,q}(\Omega)$, we have that

$$v_k' := \frac{v_k}{\|v_k\|_q + \|p_k\|_{q/\mathbb{R}}}, \qquad p_k' := \frac{p_k}{\|v_k\|_q + \|p_k\|_{q/\mathbb{R}}}$$

is a q-weak solution to the problem with data

$$f_k' := \frac{f_k}{\|v_k\|_q + \|p_k\|_{q/\mathbb{R}}}, \qquad (v_*')_k := \frac{v_*^k}{\|v_k\|_q + \|p_k\|_{q/\mathbb{R}}}$$

Moreover, we have

$$1 > k\big(\|f_k'\|_{m,q} + \|(v_*')_k\|_{m+2-1/q,q(\partial\Omega)}\big)$$

as $k \to \infty$. Similarly, we can find $\pi \in L^q(\Omega)$ such that $\lim_{k\to\infty} \|p_k - \pi\|_{q/\mathbb{R}} = 0$. So we have

$$\|u\|_q + \|\pi\|_{q/\mathbb{R}} = 1$$

passing to the limit equation (9.43). Moreover it holds that u is a q-generalized solution to the Stokes problem in Ω. In fact

$$\langle \nabla v_k, \nabla v\rangle = \langle f_k, v\rangle$$

for every $v \in C^\infty_{0,\sigma}(\Omega)$. Since $f_k \to 0$ in $W^{m,q}(\Omega)$ thanks to (9.44), we have that $\langle \nabla u, \nabla v\rangle = 0$. Moreover, u is weakly divergence free, since so it is v_k. Finally, since $v_k \to u$ in $W^{m,q}(\Omega)$, and so in particular $v_k \to v$ in $W^{1,q}(\Omega)$, $Tv_k \to Tu$ in $L^q(\partial\Omega)$, using the estimate in theorem 4.9. Since $Tv_k = v_*^k$ converges to zero, thanks to (9.44), we have $Tu = 0$. So u is a weak solution of the Stokes problem with external force $f = 0$ and boundary data $v_* = 0$. But, thanks to the uniqueness of the solution, we have $u \equiv 0$, $\pi \equiv const$. So $\|u\|_q + \|\pi\|_{q/\mathbb{R}} = 0$, that is a contraddiction. So, the theorem is proved. ∎

9.6 The Stokes operator on a bounded domain

In the application of the *Galerkin scheme* it is fundamental the study of the eigenvalues and the eigenfunctions of a linear operator, named *Stokes operator*. The properties of this operator are strictly related to the results about the Stokes equation deduced above. We start with the definition of some spaces.

Definition 9.12. Let Ω be a bounded domain of $\mathbb{R}^n$, with $n \geq 2$. In the following, we define these spaces. Remembering that $\mathcal{D}(\Omega) = C^\infty_{0,\sigma}(\Omega)$, we set

$$L^2_\sigma(\Omega) := \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_2}$$

with the scalar product

$$\langle u, v\rangle_\Omega \equiv \langle u, v\rangle := \int_\Omega u \cdot v \, dx$$

and clearly $\|u\|_2 = \langle u, u\rangle^{\frac{1}{2}}$.

On the other hand, we set $W^{1,2}_{0,\sigma}(\Omega) := \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{W^{1,2}(\Omega)}} \subseteq L^2_\sigma(\Omega)$, with the scalar product

$$\langle u, v\rangle + \langle \nabla u, \nabla v\rangle$$

and the norm $\left(\|u\|_2^2 + \|\nabla u\|_2^2\right)^{\frac{1}{2}}$.

Remark 9.21. Observe that also $\langle \nabla u, \nabla v\rangle$ is a scalar product over $W^{1,2}_{0,\sigma}(\Omega)$. In fact, linearity, positivity and simmetry are immediate. Moreover, if $\|\nabla u\|_2 = 0$, it follows that $\nabla u \equiv 0$ and so $u \equiv c$ almost everywhere on Ω, and so the constant c is the continuous version of u. Since $Tc = c|_{\partial\Omega}$, we have $c = 0$. □

Remark 9.22. It also holds that $W^{1,2}_{0,\sigma}(\Omega)$ complactly embeds into $L^2_\sigma(\Omega)$. In symbols, $W^{1,2}_{0,\sigma}(\Omega) \subset\subset L^2_\sigma(\Omega)$. In fact, by the Sobolev theorem about (compact) embeddings we have tha $H^1_0(\Omega) \subset\subset L^2(\Omega)$. This means that

(i) $\exists C > 0$ such that $\|u\|_2 \leq C\|u\|_{H^1}$ for every $u \in H_0^1(\Omega)$;

(ii) if $\{u_k\}_{k\in\mathbb{N}}$ is a bounded sequence in $H_0^1(\Omega)$, then there exists a subsequence $\{u_{k_j}\}$ and $u \in L^2(\Omega)$ such that
$$\lim_{j\to+\infty} \|u_{k_j} - u\|_2 = 0$$

Since the norms in $W^{1,2}_{0,\sigma}(\Omega)$, $L^2_\sigma(\Omega)$ are the same of $H_0^1(\Omega)$, $L^2(\Omega)$, the first property holds for sure. So, let now $\{u_k\}_{k\in\mathbb{N}}$ in $W^{1,2}_{0,\sigma}(\Omega)$ a bounded sequence. Since it in particular lives in $H_0^1(\Omega)$, we can find a subsequence $\{u_{k_j}\}_{j\in\mathbb{N}}$ that converges to some $u \in L^2(\Omega)$ in the sense
$$\|u_{k_j} - u\|_2 \to 0 \quad \text{as } j \to +\infty$$
It remains to prove that moreover $u \in L^2_\sigma(\Omega)$. Since $u_{k_j} \in W^{1,2}_{0,\sigma}(\Omega)$, then there exists a sequence $u_h^{k_j} \in C^\infty_{0,\sigma}(\Omega)$ such that $\lim\limits_{h\to\infty} \|u_h^{k_j} - u_{k_j}\|_{H^1} = 0$. We can choose, for every k_j, the index $h_j := h(k_j)$ such that $\|u_{h_j}^{k_j} - u_{k_j}\|_2 \leq \|u_{h_j}^{k_j} - u_{k_j}\|_{H^1} < \dfrac{1}{k_j}$. It follows that $\{u_{h_j}^{k_j}\}_{j\in\mathbb{N}} \subseteq C^\infty_{0,\sigma}(\Omega)$ is a sequence such that
$$\|u_{h_j}^{k_j} - u\|_2 \leq \|u_{h_j}^{k_j} - u_{k_j}\|_2 + \|u_{k_j} - u\|_2 \to 0 \quad \text{as } j \to +\infty$$
This show that $u \in L^2_\sigma(\Omega)$. This is the thesis. □

Definition 9.13 (Stokes operator)**.** We want to define an operator
$$A : D(A) \to L^2_\sigma(\Omega)$$
with domain $D(A) \subseteq L^2_\sigma(\Omega)$ and range $R(A) := \{Au : \; u \in D(A)\}$.

We define $D(A) \subseteq W^{1,2}_{0,\sigma}(\Omega)$ to be the set of the functions $u \in W^{1,2}_{0,\sigma}(\Omega)$ such that
$$\exists f \in L^2_\sigma(\Omega) : \quad \mu\langle \nabla u, \nabla v\rangle = \langle f, v\rangle \quad \forall\, v \in C^\infty_{0,\sigma}(\Omega) \tag{9.45}$$
If $\Lambda_u(v) := \mu\langle \nabla u, \nabla v\rangle$ for every $v \in C^\infty_{0,\sigma}(\Omega)$, by the Riesz representation, we have
$$D(A) := \{u \in W^{1,2}_{0,\sigma}(\Omega) : \text{ the functional } \Lambda_u(v) \text{ is continuous in } \|\cdot\|_2\} \tag{9.46}$$
For every $u \in D(A)$, the image $Au \in L^2_\sigma(\Omega)$ is defined by
$$\mu\langle \nabla u, \nabla v\rangle = \langle Au, v\rangle \qquad \forall\, v \in C^\infty_{0,\sigma}(\Omega)$$
In other words, $Au := f$, with f defined in (9.45). The operator $A = A_\Omega$ is said *Stokes operator* on the domain Ω.

Remark 9.23. Since, if $u_1, u_2 \in D(A)$ and f_i is given by (9.45) with $u = u_i$, $i \in \{1, 2\}$, we have
$$\mu\langle \nabla(u_1 + u_2), \nabla v\rangle = \langle f_1 + f_2, v\rangle \qquad \forall v \in C^\infty_{0,\sigma}(\Omega)$$
and so $A(u_1 + u_2) = f_1 + f_2$, that is A is *linear*. □

Remark 9.24. For the future, remember that $P : L^2(\Omega) \to L^2_\sigma(\Omega)$ is the Helmholtz projection. □

9.6.1 Properties of the Stokes operator

The following theorem prove the existence and collects some properties of the Stokes operator.

Theorem 9.9. *Let $\Omega \subseteq \mathbb{R}^n$, with $n \geq 2$ a bounded domain. Then there exists $A = A_\Omega$, the Stokes operator defined above, $A : D(A) \to L^2_\sigma(\Omega)$, with the following properties:*

- *A is a positive, symmetric operator, with domain $D(A) \subseteq L^2_\sigma(\Omega)$, and $C^\infty_{0,\sigma}(\Omega) \subseteq D(A) \subseteq W^{1,2}_{0,\sigma}(\Omega)$.*
  Moreover
  $$N(A) := \{u \in D(A) : \ Au = 0\} = \{0\}$$
  and the inverse operator $A^{-1} : D(A^{-1}) \to L^2_\sigma(\Omega)$ with domain $D(A^{-1}) = R(A) = L^2_\sigma(\Omega)$ is a positive, self-adjoint operator on the Hilbert space $L^2_\sigma(\Omega)$.
- *Let $u \in W^{1,2}_{0,\sigma}(\Omega)$, $f \in L^2_\sigma(\Omega)$. Then u is a weak solution to the problem*
  $$\begin{cases} -\mu\Delta u + \nabla p = f \\ \nabla \cdot u = 0 \\ u|_{\partial\Omega} \equiv 0 \end{cases} \tag{9.47}$$
  on Ω if and only if $u \in D(A)$ and $Au = f$; moreover, the latter claim holds if and only if there exists $p \in L^2_{loc}(\Omega)$ such that
  $$-\mu\Delta u + \nabla p = f$$
  in the sense of distributions.
- *The inverse operator A^{-1} is bounded; in particular, if C is the Poincaré constant on the bounded domain Ω, we have*
  $$\|A^{-1}\| \leq C^2\mu^{-1}$$
  Here $\|\cdot\|$ is the operator norm.
- *Finally, if Ω is a C^2 domain, then*
  $$D(A) = L^2_\sigma(\Omega) \cap W^{1,2}_{0,\sigma}(\Omega) \cap W^{2,2}(\Omega) \tag{9.48}$$
  and $p \in L^2(\Omega)$. Moreover
  $$\|u\|_{W^{2,2}(\Omega)} + \mu^{-1}\|\nabla p\|_2 \leq C\mu^{-1}\|Au\|_2 \tag{9.49}$$
  for every $u \in D(A)$.

Proof. The existence is based on lemma 2.4. In our context, we set $H \equiv L^2_\sigma(\Omega)$, equipped with the norm $\|\cdot\|_2$. Since

$$C^\infty_{0,\sigma}(\Omega) \subseteq W^{1,2}_{0,\sigma}(\Omega) \subseteq L^2_\sigma(\Omega)$$

then $W^{1,2}_{0,\sigma}(\Omega)$ is dense in $(H, \|\cdot\|_2)$. So, we can set $D(S) := W^{1,2}_{0,\sigma}(\Omega)$, and

$$S : D(S) \times D(S) \to \mathbb{R}$$

$$S(u, v) := \mu\langle \nabla u, \nabla v\rangle$$

Clearly, S is symmetric. Moreover $S(u, u) = \mu\|\nabla u\|_2^2 \geq 0$ and is zero in $D(S)$ if and only if $u \equiv 0$. $D = D(S)$ is here equipped with the norm

$$\left(\|u\|_2^2 + \|\nabla u\|_2^2\right)^{\frac{1}{2}} = \|u\|_{H^1}$$

So, according to lemma 2.4, there exists a uniquely determined operator with the properties (2.4). Observe that, by continuity, S satisfies these properties. In fact, if $u \in D(A)$ and $v \in D = W^{1,2}_{0,\sigma}(\Omega)$, we can approach v in norm $\|\cdot\|_{H^1}$ with a sequence $v_k \in C^\infty_{0,\sigma}(\Omega)$. So we have

$$S(u, v) = \lim_k \mu\langle \nabla u, \nabla v_k\rangle = \lim_k \mu\langle Au, v_k\rangle = \mu\langle Au, v\rangle$$

and if $\|v\|_2 < \delta$, then $\|v_k\|_2 < \delta$ for k sufficiently large, say $k \geq K$. Then, by definition (9.46) of $D(A)$, for every $\varepsilon > 0$ exists δ such that, for every $k \geq K$,

$$|S(u, v_k)| < \varepsilon$$

So we have

$$|S(u, v)| = \lim_k |S(u, v_k)| \leq \varepsilon$$

since $v_k \to v$ in H^1. So $v \to S(u, v)$, for every $v \in D$, is continuous with respect to the norm $\|\cdot\|_2$. So, the operator A satisfies the required properties, and so $B = A$.
We start remarking some properties. Let $u, v \in D(A)$. We have that

$$\langle Au, v\rangle = \frac{1}{\mu}S(u, v) = \frac{1}{\mu}S(v, u) = \langle Av, u\rangle \tag{9.50}$$

So, we have that A is symmetric. Moreover, if $u \in D(A)$,

$$\langle Au, u\rangle = \frac{1}{\mu}S(u, u) \geq 0 \tag{9.51}$$

and if $S(u, u) = 0$, then $u = 0$ in $W^{1,2}_{0,\sigma}(\Omega)$.
Moreover, if $u \in D(A)$ and $Au = 0$, we have

$$\mu\langle \nabla u, \nabla v\rangle = 0 \qquad \forall v \in W^{1,2}_{0,\sigma}(\Omega)$$

But $u \in D(A) \subseteq W^{1,2}_{0,\sigma}(\Omega)$, and so $\|\nabla u\|_2 = 0$, that implies $u \equiv 0$ in $W^{1,2}_{0,\sigma}(\Omega)$. So, we finally have

$$N(A) = \{u \in D(A) : \ Au = 0\} = \{0\}$$

This fact allows us to define an inverse operator. In fact, consider

$$R(A) = \{f \in L^2_\sigma(\Omega) : \ \exists u \in D(A) \ f = Au\}$$

So, for every $f \in L^2_\sigma(\Omega)$, it is possible to define $A^{-1}f = u$. This operator is well posed. In fact, if $f_1, f_2 \in L^2_\sigma(\Omega)$, we have $f_i = Au_i$, $\exists u_i \in D(A)$. So, if $f_1 = f_2$, we have

$$0 = f_1 - f_2 = A(u_1 - u_2)$$

and so $u_1 = u_2$ in $W^{1,2}_{0,\sigma}(\Omega)$.
So we have the operator

$$A^{-1} : R(A) \to D(A) \subseteq L^2_\sigma(\Omega)$$

with $A^{-1}(f) = u,$ if $Au = f$.

We can now prove that $\boxed{R(A) = L^2_\sigma(\Omega)}$. We now that $R(A) \subseteq L^2_\sigma(\Omega)$ by definition. So, let $f \in L^2_\sigma(\Omega)$. Then, if $v \in W^{1,2}_{0,\sigma}(\Omega)$, we have, using the Poincaré inequality $\|v\|_2 \leq C\|\nabla v\|_2$ over[14] $W^{1,2}_{0,\sigma}(\Omega)$,

$$|\mu\langle f, v\rangle| \leq \mu\|f\|_2\|v\|_2 \leq \mu C\|f\|_2\|\nabla v\|_2$$

So, if we consider the Hilbert space $(W^{1,2}_{0,\sigma}(\Omega), \langle\cdot,\cdot\rangle_1)$, where, for every $u, v \in W^{1,2}_{0,\sigma}(\Omega)$,

$$\langle u, v\rangle_1 := \langle\nabla u, \nabla v\rangle$$

with norm $\langle v, v\rangle_1^{\frac{1}{2}} = \|\nabla v\|_2$, the operator $\lambda_f : W^{1,2}_{0,\sigma}(\Omega) \to \mathbb{R}$, defined by $\lambda_f(v) := \mu\langle f, v\rangle$, is continuous over $W^{1,2}_{0,\sigma}(\Omega)$ equipped with $\langle\cdot,\cdot\rangle_1$.
Since λ_f is clearly linear, it belongs to the dual space $(W^{1,2}_{0,\sigma}(\Omega), \langle\cdot,\cdot\rangle_1)^*$. By Riesz representation theorem for Hilbert spaces, it follows that there exists a unique $\tilde{f} \in W^{1,2}_{0,\sigma}(\Omega)$ such that

$$\mu\langle f, v\rangle = \lambda_f(v) = \langle\tilde{f}, v\rangle_1 \equiv \langle\nabla\tilde{f}, \nabla v\rangle \qquad \forall v \in W^{1,2}_{0,\sigma}(\Omega) \tag{9.52}$$

So, by definition , $\tilde{f} \in D(A)$, and $A\tilde{f} = f$. This means that $f \in R(A)$, that is what we wanted to prove.

With these devices, we can deduces the claimed properties of the operator A^{-1}. For every $f, g \in R(A) = L^2_\sigma(\Omega)$, we have that exist $u, v \in D(A)$ such that $f = Au$ and $g = Av$. So,

$$\langle A^{-1}f, g\rangle = \langle u, Av\rangle \overset{(9.50)}{=} \langle Au, v\rangle = \langle f, A^{-1}g\rangle$$

since $u, v \in D(A)$. Similarly, if $f \in R(A)$, and so $f = Au$ with $u \in D(A)$, we have

$$\langle A^{-1}f, f\rangle = \langle u, Au\rangle \overset{(9.51)}{\geq} 0$$

with $u \in D(A)$. So we have proved that

$$A^{-1} : R(A) = L^2_\sigma(\Omega) \to L^2_\sigma(\Omega)$$

[14]Thanks to the fact that the domain is bounded.

is a positive and self-adjoint operator over the Hilbert space $L^2_\sigma(\Omega)$.

We now prove the second claim of the theorem. Let $u \in W^{1,2}_{0,\sigma}(\Omega)$ and $f \in L^2_\sigma(\Omega)$. Let u be a weak solution to the Stokes problem. In other words, we have

$$\mu\langle \nabla u, \nabla v\rangle = \langle f, v\rangle \qquad \forall v \in \mathcal{D}(\Omega)$$

This automatically implies that $u \in D(A)$ and $Au = f$.
Conversely, if $Au = f$, then by definition it is a weak solution of the Stokes operator.

We now have to prove the third point, that is the boundness of the operator $\|A^{-1}\|$. Let $u \in D(A)$, and $f := Au$. As in (9.52), we have that exists $F \in L^2(\Omega)$ such that

$$\langle f, v\rangle = \langle F, \nabla v\rangle \qquad \forall v \in W^{1,2}_{0,\sigma}(\Omega)$$

So we have, being $u \in D(A) \subseteq W^{1,2}_{0,\sigma}(\Omega)$,

$$\mu\|\nabla u\|_2^2 = \mu\langle \nabla u, \nabla u\rangle = \langle Au, u\rangle = \langle f, u\rangle \le C\|f\|_2\|u\|_2 \le C\|f\|_2\|\nabla u\|_2$$

This implies $\|\nabla u\|_2 \le \mu^{-1}C\|f\|_2$. This leads to

$$\|u\|_2 \le C\|\nabla u\|_2 \le \mu^{-1}C^2\|f\|_2 = \mu^{-1}C^2\|Au\|_2$$

So, if $f \in L^2_\sigma(\Omega)$, we have $f = Au$, with $u \in D(A)$, and $u = A^{-1}f$. So

$$\|A^{-1}f\|_2 \le \mu^{-1}C^2\|f\|_2 \tag{9.53}$$

This means that $\|A^{-1}\| \le \mu^{-1}C^2$.

We finally prove the four point. Let $u \in D(A)$ and define $f := Au \in L^2_\sigma(\Omega) \subseteq L^2(\Omega)$. By the previous points we have that u is a 2-generalized solution of the Stokes system, and $f \in L^2(\Omega)$. So, theorem 9.8 implies that $u \in W^{2,2}(\Omega)$ and exists $p \in W^{1,2}(\Omega)$, associated to the velocity field by lemma 9.1, such that, also thanks to proposition 9.1,

$$-\mu\Delta u + \nabla p = f$$

Moreover, it holds the inequality

$$\|u\|_{W^{2,2}} + \mu^{-1}\|\nabla p\|_2 \le c\mu^{-1}\|f\|_2 \tag{9.54}$$

Since we already know that $u \in W^{1,2}_{0,\sigma}(\Omega) \subseteq L^2_\sigma(\Omega)$, we have $u \in W^{1,2}_{0,\sigma}(\Omega) \cap H^2(\Omega) \cap L^2_\sigma(\Omega)$.
Conversely, let $u \in W^{1,2}_{0,\sigma}(\Omega) \cap W^{2,2}(\Omega)$. Then $-\mu\Delta u \in L^2(\Omega)$. So, if $v \in C^\infty_{0,\sigma}(\Omega)$,

$$\mu\langle \nabla u, \nabla v\rangle = -\mu\langle \Delta u, v\rangle = -\mu\langle \Delta u, Pv\rangle = -\mu\langle P\Delta u, v\rangle \tag{9.55}$$

Thanks to the density of $C^\infty_{0,\sigma}(\Omega)$ in $W^{1,2}_{0,\sigma}(\Omega)$, we have

$$\langle Au, v\rangle = \mu\langle \nabla u, \nabla v\rangle = -\mu\langle P\Delta u, v\rangle \qquad \forall v \in W^{1,2}_{0,\sigma}(\Omega) \tag{9.56}$$

Equation (9.55) means that $u \in D(A)$, and equation (9.56) implies $Au = -\mu P\Delta u$. Finally, equation (9.54) applied to this case says that

$$\|u\|_{W^{2,2}} + \mu^{-1}\|\nabla p\|_2 \le c\mu^{-1}\|Au\|_2 \tag{9.57}$$

This concludes the proof of the theorem. ∎

9.7 Eigenvalues problem for the Stokes operator

Given the Stokes operator defined above, it is now useful, for future arguments in the present thesis, to study the eigenvalues problem

$$Au = \lambda u$$

Theorem 9.9 says to us that, roughly speaking, a problem of the form $Au = f$ is equivalent to a Stokes problem. This will be at the core of the next remarks.
Before starting, we define a functional space that will be fundamental also in the next chapters.
Let Ω a bounded domain, with smooth boundary[15].

Definition 9.14. From now on, we will represent with X the following functional space

$$X := \{\phi \in H_0^1(\Omega) \cap H^2(\Omega) |\ \nabla \cdot \phi = 0 \text{ in } \Omega\} \tag{9.58}$$

i.e. the *weak divergence free space.*

Remark 9.25. The space X is an Hilbert space, if equipped with the inner product and the norm of H^2.

The following theorem underlines a fundamental property of the inverse Stokes operator.

Theorem 9.10. *The inverse Stokes operator $A^{-1} : L^2_\sigma(\Omega) \to L^2_\sigma(\Omega)$ that is, as we already know, positive and self-adjoint, is, furthermore, a compact operator.*

Proof. The inverse operator $A^{-1} :\ L^2_\sigma(\Omega) \to L^2_\sigma(\Omega)$ is continuous, as outlined in (9.53).
Moreover, we can remark that every image $A^{-1}f$, with $f \in L^2_\sigma(\Omega)$, is in $D(A)$, by the definition of the inverse operator A^{-1}. Since $D(A) \subseteq W^{1,2}_{0,\sigma}$ can be equipped with the norm $\| \cdot \|_{H^1}$, we want to say something about $\|A^{-1}f\|_{H^1}$.
In particular, if $f \in L^2_\sigma(\Omega)$, then $A^{-1}f = u$, for some $u \in D(A)$, with $Au = f$. The, it holds

$$\|A^{-1}f\|_{H^1} \equiv \|u\|_{H^1} \overset{(9.57)}{\leq} \mu^{-1}c\|Au\|_2 \equiv \mu^{-1}c\|f\|_2$$

So, let $\{f_k\}_{k\in\mathbb{N}} \subseteq L^2_\sigma(\Omega)$ a bounded sequence, that is $\|f_k\|_2 \leq M$ for every $k \in \mathbb{N}$. It follows that $\|A^{-1}f_k\|_{H^1} \leq \mu^{-1}cM$ for every $k \in \mathbb{N}$. This means that $\{A^{-1}f_k\}_{k\in\mathbb{N}}$ is a bounded sequence in $W^{1,2}_{0,\sigma}(\Omega)$. But remark 9.22 outlines that $W^{1,2}_{0,\sigma}(\Omega) \subset\subset L^2_\sigma(\Omega)$. This means that there exists a subsequence $\{A^{-1}f_{k_j}\}_{j\in\mathbb{N}}$ and a function $u \in L^2_\sigma(\Omega)$ such that

$$\lim_{j\to\infty} \|A^{-1}f_{k_j} - u\|_2 = 0$$

This is the definition of compactness for the operator $A^{-1} : L^2_\sigma(\Omega) \to L^2_\sigma(\Omega)$. ∎

On compact, positive, self-adjoint operators on Hilbert spaces there exists a big class of spectral theorems. The most classical is the *Hilbert-Schmidt* theorem.

[15]To all the aims, it is enough to suppose Ω with C^2 boundary, as in the hypothesis of theorem 9.9.

9.7.1 Application of the Hilbert-Schmidt theorem

In the statement of theorem 2.7, we choose $\mathcal{H} = L^2_\sigma(\Omega)$ and $B = A^{-1}$. We find, in this way, the sequences $\{\sigma_k\}$ and $\{\varphi_k\}$ such that

$$0 < ... \le \sigma_k \le \sigma_{k-1} \le ... \le \sigma_1, \quad \lim_{k\to+\infty} \sigma_k = 0, \quad A^{-1}\varphi_k = \sigma_k\varphi_k \tag{9.59}$$

where moreover $\{\varphi_k\}_{k\in\mathbb{N}}$ is a complete ortonormal basis of $L^2_\sigma(\Omega)$. This means that, for every $f \in L^2_\sigma(\Omega)$, there exists a sequence $\{c_i^k\}_{i=1}^k$ such that

$$\lim_{k\to+\infty} \left\| f - \sum_{i=1}^k c_i^k \varphi_i \right\|_2 = 0, \quad \int_\Omega \varphi_k \cdot \varphi_j \, dx = \delta_{kj} \tag{9.60}$$

Remark 9.26. In order to start to deal with the next theorem, observe that, as in (9.58),

$$X = \{\phi \in H^1_0(\Omega) |\ \nabla \cdot \phi = 0 \text{ in } \Omega\} \cap H^2(\Omega) \tag{9.61}$$

Moreover, as highlighted in [26, Remark, pg. 1096], since the boundary $\partial\Omega$ is Lipschitz, we have $\{\phi \in H^1_0(\Omega) |\ \nabla \cdot \phi = 0 \text{ in } \Omega\} = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{H^1}} \equiv W^{1,2}_{0,\sigma}(\Omega)$. But, remembering now equation (9.48), we have that $D(A) = X$. □

Remark 9.27. We give more details to remark 9.26. Consider, in fact, $\mathcal{V} := \{\varphi \in C^\infty_c(\Omega) :\ \nabla \cdot \varphi = 0\}$, and consider $V := \overline{\mathcal{V}}^{\|\cdot\|_{H^1}}$. Then $V = \{u \in H^1_0(\Omega) :\ \nabla \cdot u = 0$.

The inclusion $\boxed{\subseteq}$ is easy, in the sense that it does not involve decomposition theorems, as the other one. Infact, consider a Cauchy sequence $\{u_n\}_{n\in\mathbb{N}}$, with $u_n \in \mathcal{V}$ and $\|u_n - u_m\|_{H^1} \to 0$ as $n, m \to \infty$. Then, by the completion of $H^1_0(\Omega)$ we have that exists $u \in H^1_0(\Omega)$ such that $u_n \to u$ in H^1. Moreover, with usual arguments, $\nabla \cdot u = 0$.

The inclusion $\boxed{\supseteq}$ involves the regularity of the boundary. It is in fact necessary to use the Helmholtz decomposition. Let in fact $f \in \mathcal{V}^\perp$. By theorem 6.1 we have that exists $p \in L^2(\Omega)$ such that $f = \nabla p$ in distributional sense. So, by integration by parts, we have $f \in \{\varphi \in H^1_0(\Omega) :\ \nabla \cdot \varphi = 0\}^\perp$. This implies that $\overline{\{\varphi \in C^\infty_c(\Omega) :\ \nabla \cdot \varphi\}} \supseteq \{\varphi \in H^1_0(\Omega) :\ \nabla \cdot \varphi = 0\}$. □

Theorem 9.11. *Let Ω a bounded domain with smooth boundary, and let $A = A_\Omega$ the Stokes operator. Then, for the operator $A : D(A) = X \to L^2_\sigma(\Omega)$, there exists a sequence of pairs (λ_k, w^k), with $0 < \lambda_1 \le \lambda_2 \le ... \le \lambda_k \le ...$, and $w^k \in X \cap L^2_\sigma(\Omega)$, such that*

$$\lim_{k\to+\infty} \lambda_k = +\infty, \qquad Aw^k = \lambda_k w^k \quad \forall k \in \mathbb{N}, \qquad \int_\Omega w_k \cdot w_j \, dx = \delta_{kj} \tag{9.62}$$

and, for every $u \in D(A) = X$,

$$\lim_{N\to\infty} \left\| \sum_{k=1}^N \langle u, w^k \rangle_2 w^k - u \right\|_{H^2} = 0$$

Proof. First of all, we define $\lambda_k := \dfrac{1}{\sigma_k}$ and $w^k := \varphi_k \in L^2_\sigma(\Omega)$, where σ_k and φ_k are those in (9.59). One observes immediately that

$$\lambda_k A^{-1}\varphi_k = \frac{1}{\sigma_k} A^{-1}\varphi_k = \varphi_k \implies \lambda_k\varphi_k = A\varphi_k \tag{9.63}$$

The equality (9.63) implies that $\varphi_k \in D(A)$, since it is the pre-image of $\lambda_k \varphi \in L^2_\sigma(\Omega)$. The properties of the eigenvalues are, clearly, an immediate consequence of the definition as reciprocal of σ_k. Moreover, φ_k are eigenfunctions also for the operator A, as underlined in (9.63). Obviously, since $w_k := \varphi_k$, we have

$$\int_\Omega w_k \cdot w_j \, dx = \int_\Omega \varphi_k \cdot \varphi_j = \delta_{kj}$$

It remains to prove that this is a complete basis. Since $\{\varphi_k\}_{k\in\mathbb{N}}$ is a basis of $L^2_\sigma(\Omega)$, we know that for every[16] $f \in L^2_\sigma(\Omega)$, $\lim\limits_{N\to+\infty} \| \sum\limits_{k=1}^{N} \langle f, \varphi_k\rangle_2 \varphi_k - f\|_2 = 0$.

Now we have to show that $\{w^k\}_{k\in\mathbb{N}}$ is a basis also for the space X. Remember that an Hilbert space, equipped with the H^2-norm.

Let $u \in X = D(A)$ and let $f := Au \in L^2_\sigma(\Omega)$. Define, moreover,

$$f_N := \sum_{k=1}^{N} \langle f, w^k\rangle_2 w^k$$

where $w^k \equiv \varphi_k$. Then $\lim\limits_{N\to\infty} \|f_N - f\|_2 = 0$. Observe also that, since $u, w^k \in D(A)$,

$$A^{-1} f_N = \sum_{k=1}^{N} \langle f, w^k\rangle_2 A^{-1} w^k = \sum_{k=1}^{N} \langle Au, w^k\rangle_2 \sigma_k w^k = \sum_{k=1}^{N} \langle u, Aw^k\rangle_2 \sigma_k w^k = \sum_{k=1}^{N} \langle u, w^k\rangle_2 w^k =: u_N$$

and so $Au_N = f_N$. Moreover, thanks to (9.57),

$$\|u_N - u\|_{H^2} \le c\mu^{-1} \|A(u_N - u)\|_2 \equiv c\mu^{-1} \|f_N - f\|_2$$

So

$$\lim_{N\to\infty} \left\| \sum_{k=1}^{N} \langle u, w^k\rangle_2 w^k - u \right\|_{H^2} = 0$$

for every $u \in X = D(A)$. This completes the proof. ∎

Now that we have a basis of eigenfunctions, we can deduce some regularity properties about these eigenfunctions.

[16]Observe that $c_i^k = \langle \sum_{j=1}^{k} c_j^k \varphi_j, \varphi_i\rangle = \langle \sum_{j=1}^{k} c_j^k \varphi_j - f + f, \varphi_i\rangle = \langle \sum_{j=1}^{k} c_j^k \varphi_j - f, \varphi_i\rangle + \langle f, \varphi_i\rangle$, and so

$$\left\| \sum_{i=1}^{k} \langle f, \varphi_i\rangle \varphi_i - f \right\|_2 = \left\| \sum_{i=1}^{k} (c_i^k - d_i^k)\varphi_i - f \right\|_2 \le \left\| \sum_{i=1}^{k} c_i^k \varphi_i - f \right\|_2 + \left\| \sum_{i=1}^{k} d_i^k \varphi_i \right\|_2$$

where $d_i^k := \langle \sum_{j=1}^{k} c_j^k \varphi_j - f, \varphi_i\rangle$. Moreover, thanks to the Bessel inequality, we have

$$\left\| \sum_{i=1}^{k} d_i^k \varphi_i \right\|_2^2 = \sum_{i=1}^{k} |d_i^k|^2 = \sum_{i=1}^{k} \left| \langle \sum_{j=1}^{k} c_j^k \varphi_j - f, \varphi_i\rangle \right|^2 \le \left\| \sum_{j=1}^{k} c_j^k \varphi_j - f \right\|_2^2$$

9.7.2 Regularity properties of eigenfunctions of the Stokes operator

Regularity theory is based again on theorem 9.9. This theory, is fully developed in the Ladyzhenskaya's book [19]; see in particular [19, Sec. 5, Th. 2]. However, we completely follows the work of Galdi in [12, Chapter IV], that is section 9.5.
This approach is in particular suggested by Simon in [26, pg. 1112].

Let as above Ω be a bounded domain with smooth boundary. At this level it is very important to fix the dimension in which the problem, as we will see in a moment. So, consider $\Omega \subseteq \mathbb{R}^3$, that is $n = 3$.

Let $w^k \in X \cap L^2_\sigma(\Omega)$ an eigenfunction of theorem 9.11, with eigenvalue λ_k, that is $Aw^k = \lambda_k w^k$. Since $w^k \in D(A) = X$, by (9.49) we have

$$\|w^k\|_{W^{2,2}} \leq c\mu^{-1}\|Aw^k\|_2 \equiv c\mu^{-1}\lambda_k\|w^k\|_2$$

In other words, as we have already known, $w^k \in W^{2,2}$. Now, from the Sobolev theorem 4.8, being $2 > \frac{3}{2}$, we have that $w^k \in C^{2-[\frac{3}{2}]-1,\gamma}(\overline{\Omega}) = C^{0,\frac{1}{2}}(\overline{\Omega})$.
So $w^k \in C^0(\overline{\Omega}) \equiv C(\overline{\Omega})$. We now want to get further regularity.

Since $w^k \in C(\overline{\Omega})$, we have $w^k \in L^r(\Omega)$ for every $r > 1$, since Ω is bounded. In particular, we can choose $r = 4$, so that $w^k \in L^4(\Omega)$. Since $w^k \in D(A) \cap L^2_\sigma(\Omega)$ solves the Stokes problem with force $\lambda_k w^k$, we have, by theorem 9.9, that

$$\mu\langle \nabla w^k, \nabla v\rangle = \langle \lambda_k w^k, v\rangle \qquad \forall v \in C^\infty_{0,\sigma}(\Omega)$$

Moreover, $\|\nabla w^k\|_4 \leq C\|\nabla w^k\|_{H^1}$, so that $w^k \in W^{1,4}(\Omega)$. Since w^k is continuous over $\overline{\Omega}$, and $Tw^k \equiv 0$, we have that $w^k|_{\partial\Omega} \equiv 0$. So the trace is zero also in the sense of $W^{1,4}(\Omega)$. So $w^k \in W^{1,4}_0(\Omega)$. Moreover $\nabla \cdot w^k = 0$ in the weak sense.

So, w^k is a 4-generalized solution in the sense of definition 9.4. Then, since also the force $\lambda_k w^k \in L^4(\Omega)$, theorem 9.8 assures that $w^k \in W^{2,4}(\Omega)$. Moreover it holds

$$\|w^k\|_{W^{2,4}(\Omega)} \leq C\lambda_k\|w^k\|_{L^4(\Omega)} \tag{9.64}$$

Again, by Sobolev theorem 4.8, we have, being $2 > \frac{3}{4}$,

$$w^k \in C^{2-[\frac{3}{4}]-1,\gamma}(\overline{\Omega}) = C^{1,\frac{1}{4}}(\overline{\Omega}) \subseteq C^1(\overline{\Omega})$$

So, the basis of eigenfunctions is more regular: it is one time continuously differentiable on the compact $\overline{\Omega}$.

9.7.3 Properties of the Stokes eigenfunctions

Theorem 9.12. *Consider the orthonormal basis $\{w^k\}_{k\in\mathbb{N}}$, with related eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$. Obviously by definition it holds $\int_\Omega w^j \cdot w^k \, dx = \delta_{jk}$. Moreover, it holds*

$$\int_\Omega \nabla w^j \cdot \nabla w^k \, dx = \lambda_k\delta_{jk} \tag{9.65}$$

Proof. Being $w^j \in C^1(\overline{\Omega})$, with also $w^j \in H^2(\Omega)$, using the derivative rule of the product we have

$$\nabla w^j \cdot \nabla w^k = \sum_{i=1}^{3} \nabla \cdot (w_i^j \nabla w_i^k) - w^j \cdot \Delta w^k$$

Since $\Delta w^k \in L^2(\Omega)$, we can apply the Helmolthz decomposition and so $\Delta w^k = P\Delta w^k + f_1^k$ with $f_1^k \in G(\Omega)$. Integrating over Ω we have

$$\int_\Omega \nabla w^j \cdot \nabla w^k \; dx = \sum_{i=1}^{3} \int_\Omega \nabla \cdot (w_i^j \nabla w_i^k) \; dx - \int_\Omega w^j \cdot \Delta w^k \; dx =$$

and so using the generalized divergence theorem we have

$$= \sum_{i=1}^{3} \left(\int_{\partial\Omega} T(w_i^j \nabla w_i^k) \cdot \eta \; d\sigma \right) - \int_\Omega w^j \cdot \Delta w^k dx$$

Using the devices in subsection 4.7.1, we have that

$$\int_{\partial\Omega} T(w_i^j \nabla w_i^k) \cdot \eta \; d\sigma = 0$$

So

$$\int_\Omega \nabla w^j \cdot \nabla w^k dx = - \int_\Omega w^j \cdot \Delta w^k dx =$$

$$= - \int_\Omega w^j \cdot (P\Delta w^k + f_1^k) \; dx = -\langle w^j, P\Delta w^k + f_1^k \rangle = -\langle w^j, P\Delta w^k \rangle - \langle w^j, f_1^k \rangle$$

The eigenfunction w^j is such that $Aw^j = \lambda_j w^j$, so it is in the range of the operator A, that is $L^2_\sigma(\Omega)$. So $\langle w^j, f_1^k \rangle = 0$ since f_1^k is in $G(\Omega)$. Finally

$$\int_\Omega \nabla w^j \cdot \nabla w^k dx = -\langle w^j, P\Delta w^k \rangle = \langle w^j, -P\Delta w^k \rangle = \langle w^j, Aw^k \rangle = \langle w^j, \lambda_k w^k \rangle =$$

$$= \lambda_k \langle w^j, w^k \rangle = \lambda_k \int_\Omega w^j \cdot w^k dx = \lambda_k \delta_{jk}$$

that is the thesis. ∎

9.8 A further application of the Stokes problem

We conclude the chapter with the following theorem. It is stated and partially proved in [26], and uses the devices of the Stokes problem to deduce very useful estimates that will be fundamental in the future chapters.

Lemma 9.6. *Let Ω a bounded domain of $\mathbb{R}^n$, $n \geq 2$, with smooth boundary. There exist constants $e > 0$ and $c > 0$ such that*

$$v \in H^2(\Omega) \cap \{v \in H_0^1(\Omega) : \; \nabla \cdot v = 0\} \Longrightarrow \begin{cases} \|\Delta v\|_2 \leq e \|P\Delta v\|_2 \\ \|v\|_\infty \leq c \, (\|\Delta v\|_2)^{\frac{3}{4}} \, (\|\nabla v\|_2)^{\frac{1}{4}} \end{cases}$$

Remark 9.28. This theorem will be fundamental in the next chapters, since it allows to control the essential supremum norms in terms of the norm H^2, provided that the function is, in weak sense, divergence free and with zero boundary conditions. $\square$

Proof. We start proving the first estimate. Since $v \in H^2(\Omega)$ and

$$\langle P\Delta v, w\rangle_{L^2(\Omega)} = -\langle \nabla v, \nabla w\rangle_{L^2(\Omega)} = \langle \Delta v, w\rangle_{L^2(\Omega)}$$

for every $w \in L^2_\sigma(\Omega)$, that is $\langle \Delta v - P\Delta v, w\rangle_{L^2(\Omega)} = 0$. Then, using 9.8, we have the estimate

$$\|\Delta v\|_{L^2(\Omega)} \leq C\|P\Delta v\|_{L^2(\Omega)}$$

since v solves the Stokes equation with force $P\Delta v$.

We now prove the second estimate. Using the interpolation inequality in lemma 3.2, we have that, for every $u \in L^6(\Omega) \cap L^2(\Omega)$ it holds

$$\|u\|_4 \leq C\|u\|_6^\alpha \|u\|_2^{1-\alpha} \tag{9.66}$$

with α such that $\frac{1}{4} = \frac{\alpha}{6} + \frac{1-\alpha}{2}$. So, in this case, $\alpha = \frac{3}{4}$. Since $v \in H^2(\Omega)$ in the hypothesis, we have that in particular $v \in W^{1,6}(\Omega) \cap W^{1,2}(\Omega)$. So, in particular $v, \nabla v \in L^6(\Omega) \cap L^2(\Omega)$. So, by (9.66), we have

$$\|v\|_4 \leq C\|v\|_6^{\frac{3}{4}}\|v\|_2^{\frac{1}{4}}, \qquad \|\nabla v\|_4 \leq C\|\nabla v\|_6^{\frac{3}{4}}\|\nabla v\|_2^{\frac{1}{4}}$$

But $\|v\|_6^{\frac{3}{4}}\|v\|_2^{\frac{1}{4}}$, $\|\nabla v\|_6^{\frac{3}{4}}\|\nabla v\|_2^{\frac{1}{4}} \leq C\|v\|_{W^{1,6}}^{\frac{3}{4}}\|\nabla v\|_2^{\frac{1}{4}}$, since $v \in H^1_0(\Omega)$, and so $\|v\|_2 \leq C\|\nabla v\|_2$. Thus we have

$$\|v\|_{W^{1,4}} \equiv \left(\|v\|_4^4 + \|\nabla v\|_4^4\right)^{\frac{1}{4}} \leq \|v\|_4 + \|\nabla v\|_4 \leq 2C\|v\|_{W^{1,6}}^{\frac{3}{4}}\|\nabla v\|_2^{\frac{1}{4}}$$

Since $p = 4 > n = 3$, we have by theorem 4.6,

$$\|v\|_\infty \leq C\|v\|_{W^{1,4}}$$

Moreover, $\|v\|_{W^{1,6}} \leq C_1\|v\|_{H^2} \leq C_2\|\Delta v\|_2$. Since, if $f := -\Delta v \in L^2(\Omega)$, then v solves $\Delta v = -f$, with $v \in H^1_0(\Omega)$, we have, by the theory on elliptic operators, that

$$\|v\|_{H^2} \leq C\|f\|_2 \equiv C\|\Delta v\|_2$$

So, it is clear that $\|v\|_\infty \leq C\|\Delta v\|_2^{\frac{3}{4}}\|\nabla v\|_2^{\frac{1}{4}}$ and this concludes the lemma. $\blacksquare$

Part II

Local strong solutions in the case Ω bounded domain *(après Choe and Kim)*

Chapter 10

Navier-Stokes equations: weak, strong solutions

We are interested in different kind of solutions to the Navier-Stokes equation. We start with the definition of *local weak solution* to the problem.
We give two definitions: the first is the definition of the weak solution to the *momentum equation*, the second to the *transport equation*.

10.1 Weak solutions and weak formulations

Definition 10.1. Let Ω a bounded domain in $\mathbb{R}^3$, with smooth boundary, and $T_* > 0$ a local time. Moreover, let $\mu > 0$ be a positive real number. Consider the space $W^{1,2}_{0,\sigma}(\Omega) \equiv \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{W^{1,2}}}$. Let $u_0 \in W^{1,2}_{0,\sigma}(\Omega)$ and $\rho_0 \in L^\infty(\Omega)$ given initial data. We say that the pair $(u, \rho) \in L^2(0, T_*; W^{1,2}_{0,\sigma}(\Omega)) \times L^\infty(0, T_*; L^\infty(\Omega))$ is a *local weak solution* in the interval $(0, T_*)$ of the *momentum equation*

$$(\rho u)_t + \nabla \cdot (\rho u \otimes u) - \mu \Delta u + \nabla p = 0 \tag{10.1}$$

with initial conditions

$$u(x, 0) = u_0(x), \qquad \rho(x, 0) = \rho_0(x) \tag{10.2}$$

if for every *test function* $\varphi \in C^1([0, T_*]; W^{1,2}_{0,\sigma}(\Omega))$ such that $\varphi(x, T_*) = 0$ a.e. in Ω, it holds

$$-\int_0^{T_*}\int_\Omega \rho u \cdot \varphi_t \, dx \, dt - \int_0^{T_*}\int_\Omega \rho u \cdot \nabla\varphi \cdot u \, dx \, dt + \mu \int_0^{T_*}\int_\Omega \nabla u \cdot \nabla \varphi \, dx \, dt =$$

$$= \int_\Omega \rho_0(x) u_0(x) \cdot \varphi(x, 0) \, dx \tag{10.3}$$

Remark 10.1. It is definition 1.1 of [16]. □
Remark 10.2. The integral

$$\int_\Omega |u \cdot \nabla\varphi \cdot u| \, dx \le \int_\Omega |u||\nabla\varphi||u| \, dx \le \|\nabla\varphi\|_2 \|u\|_4^2 \le C\|\nabla\varphi\|_2\|\nabla u\|_2^2 < \infty$$

is well defined, since $u \in H^1_0(\Omega)$. □

Remark 10.3. The definition makes sense with $u \in L^2(0, T_*; W^{1,2}_{0,\sigma}(\Omega))$. However, in the present thesis, we will find a solution $u \in L^\infty(0, T_*; W^{1,2}_{0,\sigma}(\Omega))$, that is a smaller space. □

Remark 10.4. In the weak formulation the pressure gradient term does not appear. We will introduce it in a slightly stronger formulation of the solution, the so called *strong (weak) formulation.* □

Definition 10.2. Let $\overline{\rho}_0 \in L^\infty(\Omega)$ an initial data and $u \in L^2(0, T_*; H^1_0(\Omega))$ a velocity field, with $\nabla \cdot u = 0$. A *weak solution of the transport equation*

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0 & \text{in } \Omega \times (0, T_*) \\ \rho(x, 0) = \overline{\rho}_0(x) \end{cases}$$

is a function $\rho \in L^\infty(0, T_*; L^\infty(\Omega))$ such that

$$\int_0^{T_*} \int_\Omega (\rho \varphi_t + \rho u \cdot \nabla \varphi)(x, t)\ dx\ dt = -\int_\Omega \overline{\rho}_0(x) \varphi(x, 0) dx$$

for every $\varphi \in C^1([0, T_*]; H^1(\Omega))$ such that $\varphi(x, T_*) = 0$ a.e. in Ω.

Remark 10.5. It is definition 1.1 of [16]. □

Remark 10.6. As above, the definition makes sense provided that $u \in L^2(0, T_*; H^1_0(\Omega))$. However, we will find a solution $u \in L^\infty(0, T_*; H^1_0(\Omega))$. □

Remark 10.7. We will consider in future divergence-free velocity fields, i.e. $\nabla \cdot u = 0$. The transport equation, in this case, can be reformulated as $\rho_t + u \cdot \nabla \rho = 0$. □

10.1.1 Brief deduction of the weak formulation for the transport equation

The weak formulation is obtained through the following formal argument[1]. We will deduce also the *weak momentum equation*, proving the main theorem of this discussion.

In fact, in a regular scenario, consider φ as above, i.e. in $C^1([0, T_*]; H^1(\Omega))$ such that $\varphi(x, T_*) = 0$ a.e. in Ω. Then

$$\rho_t \varphi + \nabla \cdot (\rho u) \varphi = 0 \qquad \forall (x, t)$$

Remember now that

$$\nabla \cdot (\rho u) \varphi = \nabla \cdot (\varphi \rho u) - \rho u \cdot \nabla \varphi, \qquad (\rho \varphi)_t = \rho_t \varphi + \rho \varphi_t$$

So, the equality above becomes

$$(\rho \varphi)_t - \rho \varphi_t + \nabla \cdot (\varphi \rho u) - \rho u \cdot \nabla \varphi = 0$$

Integrating over $\Omega \times [0, T_*)$ we get

$$0 = \int_0^{T_*} \int_\Omega [(\rho \varphi)_t - \rho \varphi_t + \nabla \cdot (\varphi \rho u) - \rho u \cdot \nabla \varphi]\ dx\ dt =$$

[1]This argument will be regularized in the proof of the main theorem.

$$\stackrel{2}{=} \int_\Omega [(\rho\varphi)(T_*) - (\rho\varphi)(0)]\ dx - \int_0^{T_*}\int_\Omega \rho\varphi_t\ dx\ dt - \int_0^{T_*}\int_\Omega \rho u\cdot\nabla\varphi]\ dx\ dt \qquad (10.4)$$

where the integral of the divergence is a boundary integral over $\partial\Omega$ through the divergence theorem and so it vanishes since $u \in H^1_0(\Omega)$. But moreover φ vanishes as $t \to T_*$, so

$$\int_0^{T_*}\int_\Omega \rho\varphi_t\ dx\ dt + \int_0^{T_*}\int_\Omega \rho u\cdot\nabla\varphi\ dx\ dt = -\int_\Omega (\rho\varphi)(0)\ dx \qquad (10.5)$$

10.2 Strong solutions

The definitions that we are going to introduce describe the *strong (weak) solutions.* The term *weak* in brakets, often avoided, remembers us that the solutions are not *strong* in the classical sense, but in a weaker sense, however stronger than the sense in definitions 10.1 and 10.2. There are many definitions of strong solutions: in general, one expects that a strong solution is in particular a weak solution. This is true, under suitable hypothesis; however, it is not in the interest of the present thesis: every time we will search for a solution, we will construct a weak solution; then, using some devices, we will prove that this solution is also strong, in the sense we are going to introduce.

Definition 10.3. Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary and $T_* > 0$. A triple (ρ, u, p) of Banach-space valued functions defined over $(0, T_*)$ in the sense

$$\rho:\ t \mapsto \rho(t) \in L^\infty(\Omega), \qquad u:\ t \mapsto u(t) \in \mathcal{V}_0^\sigma(\Omega), \qquad p:\ t \mapsto p(t) \in H^1(\Omega)$$

where $\mathcal{V}_0^\sigma := \{v \in H^1_0(\Omega) \cap H^2(\Omega):\ \nabla\cdot v = 0 \text{ in } \Omega\}$, and such that exists

$$u_t:\ t \mapsto u_t(t) \in L^2(\Omega)$$

weak derivative of u in the sense of weak differentiation of Banach-valued functions, is a *solution of the momentum equations in the Navier-Stokes equations* if, for almost every $t \in (0, T_*)$, the pair $(u(t), p(t))$ is solution to the Stokes equation

$$\begin{cases} -\mu\Delta u(t) + \nabla p(t) = f(t) \\ \nabla\cdot u(t) = 0 \end{cases} \qquad (10.6)$$

where $f(t) := -\rho(t)u_t(t) - \rho(t)u(t)\cdot\nabla u(t) \in L^2(\Omega)$.

Remark 10.8. The sense in which the pair $(u(t), p(t))$ satisfies (10.6) has been explained in chapter 9. In particular, we will prove in 11.14 that the pair $(u(t), p(t))$ we will find in the proof of the theorems in the present thesis is solution of (10.6) with $f(t) \in L^6(\Omega)$. □

Definition 10.4. Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary and $T_* > 0$. Let $u \in L^1(0, T_*; L^2(\Omega))$ a velocity field, such that $\nabla\cdot u$ exists in the weak sense and $\nabla\cdot u = 0$ for almost every $t \in (0, T_*)$. A function $\rho \in L^\infty(0, T_*; H^1(\Omega))$ is a *strong solution of the transport equation*

$$\rho_t + \nabla\cdot(\rho u) = 0$$

[2] Using FTC in Brezis' [3, pg. 122, Th. VIII.2].

if the equation holds in the sense of spacetime distributions, that is[3]

$$\int_0^T \left(\int_\Omega \rho(x,t)\ \varphi_t(x,t)\ dx \right)\ dt = \int_0^T \left(\int_\Omega u(x,t) \cdot \nabla\rho(x,t)\ \varphi(x,t)\ dx \right)\ dt \qquad (10.7)$$

for every $\varphi \in C_c^\infty((0,T_*) \times \Omega)$.

Remark 10.9. Observe that the equation (10.7) is well posed. In fact $\varphi \in L^\infty((0,T_*)\times\Omega)$ and

$$\left| \int_0^T \left(\int_\Omega u(x,t)\cdot\nabla\rho(x,t)\ dx \right)\ dt \right| \le \int_0^T \|u\|_2 \|\nabla\rho\|_2\ dt \le \|\nabla\rho\|_{L^\infty(0,T_*;L^2(\Omega))} \|u\|_{L^1(0,T_*;L^2(\Omega))}$$

The fact that the solution we will find is a strong solution to the transport equation will be proved in section (11.14.3). □

Remark 10.10. Equation (10.7) means that $\rho_t = -u \cdot \nabla\rho$ over $\Omega \times (0,T)$ in the sense of weak derivatives. □

[3]Observe that $\nabla \cdot (\rho u) = \rho\ (\nabla \cdot u) + u \cdot \nabla\rho = u \cdot \nabla\rho$.

Chapter 11

Local strong solutions in the case of bounded domain

11.1 Statement of the main theorems

We will prove in this chapter three fundamental theorems, that are part of the core of the present thesis.

Theorem 11.1. *Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary, and assume the data ρ_0, u_0 satisfy the regularity*

$$0 \le \rho_0 \in L^\infty(\Omega), \quad u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$$

and the compatibility condition

$$\mu\Delta u_0 - \nabla p_0 = \sqrt{\rho_0} g \qquad \nabla \cdot u_0 = 0 \;\; in\ \Omega \tag{11.1}$$

for some $(p_0, g) \in H^1(\Omega) \times L^2(\Omega)$. Let $T > 0$ a fixed local time. Then, there exists a time $T_* \in (0, T)$ and a weak solution $(\rho, u) \in L^\infty(0, T_*; H^2(\Omega)) \times L^\infty(0, T_*; L^\infty(\Omega))$ to the initial boundary value problem

$$\begin{cases} (\rho u)_t + \nabla \cdot (\rho u \otimes u) - \mu\Delta u + \nabla p = 0 \\ \rho_t + \nabla \cdot (\rho u) = 0, \;\; \rho \ge 0 \qquad (x,t) \in \Omega \times (0, T_*) \\ \nabla \cdot u = 0 \end{cases} \qquad \begin{cases} \rho(x,0) = \rho_0(x) & x \in \Omega \\ u(x,0) = u_0(x) & x \in \Omega \\ u(x,t) = 0 & (x,t) \in \partial\Omega \times (0, T_*) \end{cases} \tag{11.2}$$

such that for all $t \in (0, T_*)$ we have the estimates

$$\|\nabla u(t)\|_2^2 \le C, \qquad \|\rho(t)\|_q = \|\rho_0\|_q$$

$$\sup_{0<s\le t} \left(\|\nabla u\|_{H^1}^2 + \|\sqrt{\rho} u_t\|_2^2\right) + \int_0^t \left(\|\nabla u\|_{W^{1,6}}^2 + \|u_t\|_{D_0^{1,2}}^2\right) ds \le C \exp\left(C \int_0^t \|\nabla u\|_2^4 ds\right) \tag{11.3}$$

where

$$\mathcal{C}(\rho_0, u_0, p_0) \equiv \|g\|_2^2$$

Here the local existence time T_* and the positive constant C depend only on $\|\rho_0\|_{L^\infty}$, $\|\nabla u_0\|_2$, $\|g\|_2$ and the time T; but it is independent of the lower bounds of ρ_0.

Remark 11.1. The fixed time $T > 0$ in the statement above is arbitrary but fixed. Its presence is due to derivation of the proof from a more general case, that is the case with an additional force term f in the first of the equations above. In particular, in the general case, it is required

$$f, f_t, \nabla f \in L^2(0, T; L^2)$$

In our case, we have $f \equiv 0$, so our function is in the space above for all the $T > 0$. However, from the estimates above, we can see that we can't get rid of this time sending it to infinity: it will appear in the local time T_* and C definition, as stated in the previous theorem. □

Remark 11.2. In the claim, we have $\| \cdot \|_{D_0^{1,2}} = \|\nabla \cdot \|_2$, remembering that $u_t \in H_0^1(\Omega)$. □

We now state a theorem that assures us, under stronger hypothesis on the initial density, the existence of strong solutions.

Theorem 11.2. *Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary, and assume the data ρ_0, u_0 satisfy the regularity*

$$0 \leq \rho_0 \in H^1(\Omega), \quad u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$$

and the compatibility condition

$$\mu \Delta u_0 - \nabla p_0 = \sqrt{\rho_0} g \qquad \nabla \cdot u_0 = 0 \quad \text{in } \Omega \tag{11.4}$$

for some $(p_0, g) \in H^1(\Omega) \times L^2(\Omega)$. Let $T > 0$ a fixed local time. Then, there exists a time $T_* \in (0, T)$ and a strong solution (ρ, u, p) that satisfies (11.2) *in the sense of section 10.2. Moreover, the solutions satisfy*

$$\rho \in L^\infty(0, T_*; H^1(\Omega)), \qquad \rho_t \in L^\infty(0, T_*; L^2(\Omega))$$

$$\nabla p \in L^\infty(0, T_*; L^2(\Omega)) \cap L^2(0, T_*; L^6(\Omega))$$

Remark 11.3. The regularity hypothesis on the boundary can be weakened. In fact, Sobolev theorems hold provided that the boundary is C^1, while Stokes theory holds provided that the boundary is C^2. □

11.2 Construction of the weak solution

Now we start the proof of theorem 11.1, using the so called **Galerkin scheme**: it consists in solving the problem in a sequence of finite dimensional spaces, where the problem is an ODE system, then applying functional analysis arguments to extract a limit to the sequence of solutions.

The case of $\rho_0 \in C^1(\overline{\Omega})$. Let Ω be a bounded domain. Let ρ_0, u_0 be initial data such that

$$0 \leq \rho_0 \in L^\infty(\Omega), \quad u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$$

and that satisfy the compatibility condition (11.4) for some $(p_0, g) \in H^1(\Omega) \times L^2(\Omega)$. Let $T > 0$ a fixed time and consider a function $\overline{\rho}_0$ and a $\delta > 0$ such that

$$\overline{\rho}_0 \in C^1(\overline{\Omega}), \qquad \rho_0 \leq \overline{\rho}_0, \qquad 0 < \delta \leq \overline{\rho}_0 \leq \|\rho_0\|_\infty + 1 \tag{11.5}$$

Remark 11.4. The function ρ_0 is in $L^\infty(\Omega)$, so the norm $\|\cdot\|_\infty$ is the essential supremum. This norm, as every p-norm, looks at the function out of a zero measure set. So, in a zero measure set, in example a point, ρ_0 can take values that exceed $\|\rho_0\|_\infty$. These values could eventually also overtake $\|\rho_0\|_\infty + 1$. The inequality $\rho_0 \leq \overline{\rho}_0$ is so to be meant almost everywhere, while the bounds for $\overline{\rho}_0$, that is a regular function, are to be meant in the whole $\overline{\Omega}$. □

Definition 11.1. Remember that the space X has been defined in (9.58). We consider now a finite "truncation" of this functional space. Remember that $\{w^k\}_{k\in\mathbb{N}}$ is the basis of eigenfunctions of X. We set

$$X^m := \mathcal{L}(w^1, ..., w^m) \qquad \forall\ m \in \mathbb{N}$$

We want to prove the following proposition.

Proposition 11.1. *Let $\overline{\rho}_0 \in C^1(\overline{\Omega})$ such that* (11.5) *holds. Let $T > 0$. Then there exists, for every $m \in \mathbb{N}$, $u^m \in C^1([0,T]; X^m)$ and $\rho^m \in C^1([0,T]; C^1(\overline{\Omega}))$ such that*

$$\int_\Omega \left(\rho^m u_t^m + \rho^m (\nabla u^m) u^m\right) \cdot \phi + \mu \nabla u^m \cdot \nabla \phi \ dx = 0 \ \ \forall\ \phi \in X^m \tag{11.6}$$

$$u^m(0) = \sum_{k=1}^m \langle u_0, w^k \rangle w^k, \quad \rho_t^m + u^m \cdot \nabla \rho^m = 0, \quad \rho^m(0) = \overline{\rho}_0 \tag{11.7}$$

where w^k is the k-th eigengunction of the Stokes operator A.

Definition 11.2. We will call the pair (u^m, ρ^m) *approximate solution* of the Navier-Stokes inhomogeneous incompressible problem.

11.2.1 Existence of the approximate solution

We now build functions as in proposition 11.1 so that it is proved. This construction will mainly follow the paper [16], for estimates and some lemmas, but also [4] and [26].

Separation of the variables. To prove proposition 11.1, we want to reduce the integral problem (11.6) to an ordinary differential problem (i.e. a system of ordinary differential equations). We choose $\phi = w^i$ in (11.6), supposing that[1]

$$u^m(x,t) = \sum_{j=1}^m \varphi_j(t) w^j(x) \tag{11.8}$$

with φ_j depending only on time, we have

$$\sum_{j=1}^m \dot{\varphi}_j^m(t) \int_\Omega \rho^m(t)\ w^j \cdot w^i \ dx + \sum_{j=1}^m \sum_{k=1}^m \varphi_j^m(t) \varphi_k^m(t) \int_\Omega \rho^m(t) \left((\nabla w^j) w^k\right) \cdot w^i \ dx \ +$$

[1]This method is often named *separation of the variables.* We suppose the existence of a solution written as combination of element of a specific basis, and so we deduce which properties the coefficients (depending on time) have to satisfy.

$$+\mu\sum_{j=1}^{m}\varphi_j^m(t)\int_\Omega \nabla w^j\cdot\nabla w^i\ dx=0 \tag{11.9}$$

Here it is not clear what is $\rho^m(t)=\rho^m(x,t)$. We know that this function also have to solve the transport equation. In particular, as typical in transport theory, we have $\rho^m(x,t)=\overline{\rho}_0(\gamma^m(x,t))$ with $\gamma^m(x,t)$ solution of the following problem

$$\begin{cases}\dot y(s)=u^m(y,s)\\ y(0)=x\end{cases} \tag{11.10}$$

In particular, underlining the depence on a fixed flux γ, equation (11.9) can be rewritten as

$$F(\gamma;t)\partial_t\varphi^m(t)+G(\gamma;t,\varphi^m(t))=0$$

where

$$[F(\gamma;t)]_{ij}:=\int_\Omega \overline{\rho}_0(\gamma(x,t))\ w^j(x)\cdot w^i(x)\ dx \tag{11.11}$$

and

$$[G(\gamma;t,\varphi^m)]_i:=\sum_{j=1}^{m}\sum_{k=1}^{m}\varphi_j^m\varphi_k^m\int_\Omega \overline{\rho}_0\big(\gamma(x,t)\big)\big(\nabla w^j(x)w^k(x)\big)\cdot w^i(x)\ dx+$$

$$+\mu\sum_{j=1}^{m}\varphi_j^m\int_\Omega \nabla w^j(x)\cdot\nabla w^i(x)\ dx \tag{11.12}$$

Remark 11.5. F and G are regular in space and time, provided that γ is regular (since thus are w^j and $\overline{\rho}_0$). So, if the matrix $F(\gamma)$ is invertible we can try to solve this system of ODEs. □

We have the following lemma.

Lemma 11.1. *Let $0<\delta\le\overline{\rho}_0\in C(\overline{\Omega})$, $\overline{T}>0$ a time, and $\gamma(x,t)\in C^1(\overline{\Omega}\times[0,\overline{T}])$. Let*

$$[F(\gamma;t)]_{ij}:=\int_\Omega \overline{\rho}_0(\gamma(x,t))w^i(x)\cdot w^j(x)\ dx$$

Then, for every $t\in[0,\overline{T}]$, the matrix $F(\gamma;t)$ is invertible.

Proof. Suppose, by contrary, that exists a $t_0\in[0,\overline{T}]$ such that $F(t)$ is singular. That means, e.g., that is

$$[F(\gamma;t_0)]_{1,j}=\sum_{i=2}^{m}C_i[F(\gamma;t_0)]_{i,j}\ \ \forall j=1,...,m$$

This means that

$$\int_\Omega \overline{\rho}_0(\gamma(x,t_0))w^1(x)\cdot w^j(x)\ dx=$$

$$=C_2\int_\Omega \overline{\rho}_0(\gamma(x,t))w^2(x)\cdot w^j(x)\ dx+...+C_m\int_\Omega \overline{\rho}_0(\gamma(x,t))w^m(x)\cdot w^j(x)\ dx$$

that can be rewritten as

$$\int_\Omega \overline{\rho}_0(\gamma(x,t))\eta(x)\cdot w^j(x)\ dx = 0$$

where $\eta = w^1 - C_2 w^2 - ... - C_m w^m$. This holds for every $j = 1, ..., m$.
Then, summing and multiplying for C_j, we have

$$\delta \int_\Omega |\eta|^2 dx \leq \int_\Omega \overline{\rho}_0(\gamma(x,t))|\eta|^2 dx = 0$$

It follows that $\eta \equiv 0$. But $\{w^m\}_m$ is a basis, so this is an absurdum. ∎

Coupled ODEs systems. With this information, **the key point** is trying to solve the systems

$$\begin{cases} \rho_t^m = -u^m \cdot \nabla \rho^m \\ \rho^m(x,0) = \overline{\rho}_0(x) \end{cases} \tag{11.13}$$

$$\begin{cases} F(\gamma^m; t)\partial_t \varphi^m(t) = -G(\gamma^m; t, \varphi^m(t)) \\ \varphi_j^m(0) = \displaystyle\int_\Omega u_0(x)\cdot w^j(x)\ dx \end{cases} \tag{11.14}$$

where $\gamma^m(x,t) := y_x^m(t)$ is the trajectory of (11.13) defined in (8.2).

Remark 11.6. The system (11.13)-(11.14) is not immediately solvable, because $y_x^m(t)$ in the second system is the solution of the first system, but the first is solvable only if we know $\varphi^m(t)$, solution of the second system. We follow a classical way, using a *fixed point theorem.* □

11.3 Construction of the solution to the ODEs

In this section, we use the papers [16] and [26] to deduce some fundamental estimates.

In order to solve (11.13)-(11.14) we have the following proposition.

Proposition 11.2. *Let $\tilde{A}_m \in C^1([0, \overline{T}))$ a solution of* (11.14)*, with $T \geq \overline{T} > 0$. Then, if we set*

$$\tilde{U}_m(x,t) := \sum_{k=1}^m \tilde{A}_{mk}(t) w^k(x)$$

it follows that exists $L > 0$ such that

$$\|\nabla \tilde{U}_m\|_2^2 \leq L \quad \forall t \in [0, \overline{T})$$

where L is independent of m and depends on T but is independent of $\overline{T}$.

This proposition is fundamental in the next subsection; it will be very useful in a moment.

11.3.1 Existence of a solution

Let λ_1 the minimum of the eigenvalues of the Stokes operator, as defined in theorem 9.11. Let L be as in proposition 11.2 above. Fix $R \geq (\frac{L}{\lambda_1})^{\frac{1}{2}}$. We define now

$$M := \overline{B}_R \equiv \{A \in C([0,T])^m : \ \|A\|_{C([0,T])^m} \leq R\}$$

We can fix $(A_{m1}(t), ..., A_{mm}(t)) \in \overline{B}_R$ and set[2]

$$u^m(x,t) := \sum_{k=1}^{m} A_{mk}(t) w^k(x) \in C([0,T], X^m) \subseteq C([0,T], C^1(\overline{\Omega}))$$

So, being $\overline{\rho}_0 \in C^1(\overline{\Omega})$, we can find a solution to

$$\begin{cases} \rho^m_t(x,t) = -u_m(x,t) \cdot \nabla \rho^m(x,t) \\ \rho^m(x,0) = \overline{\rho}_0(x) \end{cases}$$

with $\rho^m \in C^1([0,T] \times \overline{\Omega})$, following the classical theory of transport equation as explained in theorem 8.1. In particular, the classical method of characteristics gives us, for every $x \in \overline{\Omega}$, the trajectory of a particle under the motion of the transport equation above. We can define $\gamma^m(x,t) \in C^1(\overline{\Omega} \times [0,T])$ as the trajectory of this particle at the time t obtained with the velocity field u^m. The point x is its initial data. The solution is so given by $\rho^m(x,t) := \overline{\rho}_0(\gamma^m(x,t)) \in C^1([0,T] \times \overline{\Omega})$, as underlined in (8.2). This is, in words, what explained at the beginning of chapter 8. Observe that here u^m is zero on $\partial\Omega$ and $\nabla \cdot u^m = 0$ since these properties are satisfied by w^k for every $k \in \mathbb{N}$.

Using the flux γ^m, that is γ^m **is fixed**, we can try to find a solution to the problem

$$\begin{cases} F(\gamma^m; t) \partial_t \varphi^m(t) = -G(\gamma^m; t, \varphi^m(t)) \\ \varphi^m_j(0) = \langle u_0, w^j \rangle_2 \end{cases}$$

where F and G are as above. By the previous lemma 11.1, we can invert the matrix $F(\gamma^m)(t)$. So we can solve the equation

$$\begin{cases} \partial_t \varphi^m(t) = -(F(\gamma^m; t))^{-1} G(\gamma^m; t, \varphi^m(t)) \\ \varphi^m_j(0) = \langle u_0, w^j \rangle_2 \end{cases} \tag{11.15}$$

at least locally, provided that the coefficients are regular enough.
To this aim, we write the equation above in other terms: in particular, considering only the i-th row, we have

$$\sum_{j=1}^{m} b^m_{ij}(t) \partial_t \varphi^m_j(t) + \sum_{j=1}^{m} \sum_{k=1}^{m} C^m_{ijk}(t) \varphi^m_j(t) \varphi^m_k(t) + \mu \sum_{j=1}^{m} D_{ij} \varphi^m_j(t) = 0 \tag{11.16}$$

[2]Thanks to the estimates in section 9.7.2, since $\|\cdot\|_4 \leq C\|\cdot\|_{H^1}$, so that

$$\|w^m\|_{C^1(\overline{\Omega})} \leq \|w^m\|_{W^{2,4}(\Omega)} \leq C\lambda_m \|w^m\|_{L^4(\Omega)} \leq C^2 \lambda_m \|w^m\|_{H^1(\Omega)}$$

where

$$b^m_{ij}(t) := \int_\Omega \rho^m(x,t)\ w^j(x)\cdot w^i(x)\ dx, \quad C^m_{ijk}(t) := \int_\Omega \rho^m(x,t)\big(\nabla w^j(x)\ w^k(x)\big)\cdot w^i(x)\ dx$$

$$D_{ij} := \int_\Omega \nabla w^j(x)\cdot \nabla w^i(x)\ dx$$

where we have used the notations of Kim in [16]. The following lemma specifies the regularity of these coefficients.

Lemma 11.2. *If $\gamma^m \in C^1(\overline{\Omega}\times[0,T])$ we have that $b^m_{ij}, C^m_{ijk} \in C^1([0,T])$*

Proof. We remark first of all that the coefficients D_{ij} are constant in t, so the regularity is trivial[3]. Remember moreover that $|\rho^m(x,t)| = |\overline{\rho}_0(\gamma^m(x,t))| \le \|\rho_0\|_\infty + 1$ and

$$|\rho^m_t(x,t)| = |\nabla\overline{\rho}_0(\gamma^m(x,t))\cdot\partial_t\gamma^m(x,t)| \le \max_{\overline{\Omega}\times[0,T]} |\nabla\overline{\rho}_0(\gamma^m(x,t))\cdot\partial_t\gamma^m(x,t)| \equiv R^m_0$$

So if we define $C^m_0 \equiv \|\rho_0\|_\infty + 1 + R^m_0$, we have

$$\int_\Omega |\rho^m(x,t)||w^j(x)\cdot w^i(x)|dx \le C^m_0 \int_\Omega |w^j(x)\cdot w^i(x)|dx < +\infty$$

so that we have a summable boundary (thanks to the fact that $w^k \in C^1(\overline{\Omega})$ and Ω is bounded). Furthermore

$$\int_\Omega |\partial_t\rho^m(x,t)||w^j(x)\cdot w^i(x)|dx \le C^m_0 \int_\Omega |w^j(x)\cdot w^i(x)|dx < +\infty$$

The same bounds hold for the other coefficients. In fact

$$\int_\Omega |\rho^m(x,t)|\left|\Big(\big(\nabla w^j(x)\big)w^k(x)\Big)\cdot w^i(x)\right|dx \le C^m_0 \int_\Omega \left|\Big(\big(\nabla w^j(x)\big)w^k(x)\Big)\cdot w^i(x)\right|dx < +\infty$$

and

$$\int_\Omega |\partial_t\rho^m(x,t)|\left|\Big(\big(\nabla w^j(x)\big)w^k(x)\Big)\cdot w^i(x)\right|dx \le C^m_0 \int_\Omega \left|\Big(\big(\nabla w^j(x)\big)w^k(x)\Big)\cdot w^i(x)\right|dx < +\infty$$

So the hypothesis of the Lebesgue theorem for the interchange of derivative and integral are satisfied. It follows that $b^m_{ik}, C^m_{ijk} \in C^1([0,T])$. ∎

Solution to the ODEs. We use now a general theorem for local solutions of classical ODEs, that is theorem 1.5. We can apply the theorem to our case. In fact, consider our system of ODE. We have

$$\begin{cases} \partial_t\varphi^m(t) = -(F(\gamma^m;t))^{-1}G(\gamma^m;t,\varphi^m(t)) \\ \varphi^m_j(0) = \langle u_0, w^j\rangle_2 \end{cases}$$

[3] We will write in a moment an explicit expression for these constants.

where we have inverted the matrix at the first member because of a lemma above. The matrix is always non singular, and the coefficients are C^1 in time because of the lemma just proved. So, also the coefficients of the invers are C^1, thanks to the inverted matrix formula. Also the coefficients of G are C^1. So, we can write

$$\begin{cases} \partial_t \varphi^m(t) = H^m(\varphi^m(t), t) \\ \varphi_j^m(0) = \langle u_0, w^j \rangle_2 \end{cases}$$

where $H^m(\varphi^m, t) := -(F(\gamma^m)(t))^{-1} G(\gamma^m, \varphi^m) \in C^1(\mathbb{R}^m \times [0,T])$. The regularity in φ^m is a conseguence of the quadratic structure of the equation.
So, the theorem above says that exists a local time[4] of existence $T_m > 0$ such that there exists a unique solution

$$(\tilde{A}_{m1}(t), ..., \tilde{A}_{mm}(t)) \in C^2([0, T_m)) \tag{11.17}$$

to the problem (11.15). We name $\tilde{A}_m$ this solution, leaving the φ^m-notation, in order to be consistent with the work [16]. The local time of existence depends on m because each $m \in \mathbb{N}$ gives a different ODEs system. We can moreover suppose (as in the theorem) that $[0, T_m)$ is the maximal interval of extistence of the solution $\tilde{A}_m$.

Proof of $\tilde{A}_m \in C^1([0,T]) \cap \overline{B}_R$. We want to say that this solution is also in $C^1([0,T]) \cap \overline{B}_R$. First of all we want to replace the local time T_m with T. We use the proposition 11.2. Suppose that $T_m \leq T$. Then if we set

$$\tilde{U}_m(x,t) := \sum_{k=1}^m \tilde{A}_{mk}(t) w^k(x) \tag{11.18}$$

it follows, from proposition 11.2, that exists $L > 0$ independent of T_m such that

$$\|\nabla \tilde{U}_m\|_2^2 \leq L \quad \forall t \in [0, T_m)$$

Moreover, (11.18) gives us an estimate on $\tilde{A}_m$. In fact, using that

$$\int_\Omega \nabla w^k(x) \cdot \nabla w^j(x) \, dx = \lambda_k \delta_{kj}$$

we have

$$\sqrt{L} \geq \|\nabla \tilde{U}_m\|_2 = \left\| \sum_{k=1}^m \tilde{A}_{mk}(t) \nabla w^k \right\|_2 = \left(\int_\Omega \left| \sum_{k=1}^m \tilde{A}_{mk}(t) \nabla w^k \right|^2 dx \right)^{\frac{1}{2}} =$$

$$= \left(\sum_{k=1}^m |\tilde{A}_{mk}(t)|^2 \lambda_k \right)^{\frac{1}{2}} \geq \sqrt{\lambda_1} |\tilde{A}_m(t)|$$

where λ_1 is the minimum (positive) eigenvalue. This is true for $t \in [0, T_m)$. So

$$|\tilde{A}_m(t)| \leq R \ \ \forall \ t \in [0, T_m)$$

[4]That is the $\tau > 0$ of the definition of theorem 1.5.

Now we use a continuation argument. Let t_m^h a sequence

$$0 \le t_m^1 < t_m^2 < ... < T_m$$

such that

$$\lim_{h\to+\infty} t_m^h = T_m$$

We can also consider the sequence $\tilde{A}_m(t_m^k)$ that is a vectorial sequence. Clearly, being $0 \le t_m^h < T_m$, we have

$$|\tilde{A}_m(t_m^h)| \le R \quad \forall h \in \mathbb{N}$$

So, by the Weierstrass theorem, we have that exists a subsequence $\{\tilde{A}_m(t_m^{h_n})\}_{n\in\mathbb{N}}$ such that

$$\tilde{A}_m(t_m^{h_n}) \to A_0 \quad \text{as } n \to +\infty$$

for some A_0 such that $|A_0| \le R$. Moreover $t_m^{h_n}$ is a strictly increasing sequence converging to T_m, since it is a subsequence of t_m^h. We are thus in the hypotesis of the theorem of continuation 1.4. This means that exists a $\delta_m > 0$ such that $\tilde{A}_m \in C^2([0, T_m + \delta_m))$ is a solution to the equation. But, by the definition of T_m as maximal time of existence, we have an absurd. The absurd comes from the fact that $T_m \le T$. This means that $T_m > T$ and so

$$\tilde{A}_m \in C^2([0, T]) \tag{11.19}$$

that is what we wanted. ∎

Remark 11.7. We focus on what we have done. For every $A_m \in M$, we have found another function $\tilde{A}_m \in C^2([0, T])$ such that it solves

$$\begin{cases} \partial_t \tilde{A}_m(t) = -(F(\gamma^m; t))^{-1} G(\gamma^m; t, \tilde{A}_m(t)) \\ \tilde{A}_{m,j}(0) = \langle u_0, w^j \rangle_2 \end{cases}$$

where $\gamma^m(x,t)$ is the flux that solves

$$\begin{cases} \dot{y}(s) = u^m(y, s) \\ y(t) = x \end{cases}$$

with $u^m(x,t) := \sum_{k=1}^{m} A_{mk}(t) w^k(x)$. □

11.3.2 A fixed point argument

Moreover we have now the following theorem.

Definition 11.3 (Completely continuous operator)**.** An operator $\mathcal{T} : M \to M$ is *completely continuous* if

$$\overline{\{\mathcal{T}(A) : \ A \in \mathcal{B}\}} \text{ is compact for every bounded } \mathcal{B} \subseteq M$$

The closure above is the closure in M.

Inspired by remark 11.7, we define the following operator.

Definition 11.4 (Operator associated to the ODEs system)**.** Let A_m and $\tilde{A}_m$ as above, for a fixed $m \in \mathbb{N}$. We define

$$\mathcal{T}(A_m) := \tilde{A}_m \quad \forall A_m \in \overline{B}_R$$

We have now the following fundamental proposition.

Proposition 11.3. *If M and $\mathcal{T}$ are defined as above, we have that M is a closed, convex and bounded subspace of $C([0,T])^m$ and $\mathcal{T}$ is a completely continuous operator $\mathcal{T} : M \to M$.*

To prove this proposition, we need the following argument. We will prove this fact later.

Proposition 11.4. *Let $\tilde{A}_m \in C^1([0,T])^m$ a solution of the system* (11.15) *and let $\tilde{U}_m$ as above. Then there exists $K > 0$ such that*

$$\int_0^T \|\partial_t \tilde{U}_m\|_2^2(t)\ dt \leq K$$

Proof of proposition 11.3. Clearly $M = \overline{B}_R$ is closed, convex and bounded. Moreover the operator is such that $\mathcal{T} : \overline{B}_R \to \overline{B}_R$ and it is completely continuous. We now prove these facts.

$\mathcal{T}(\overline{B}_R) \subseteq \overline{B}_R$**:** We first show that the codomain of $\mathcal{T}$ is actually $\overline{B}_R$. We will also use that

$$\int_\Omega \nabla w^k \cdot \nabla w^j = \lambda_k \delta_{kj} \tag{11.20}$$

It is proved in section 9.7.3. By the definition of $\tilde{A}_m$ we have, as stated in proposition 11.2,

$$\|\nabla \tilde{U}_m\|_2^2 \leq L \quad \forall t \in [0,T]$$

since now the solution $\tilde{A}_m$ is defined (with regularity) in $[0,T]$ and the estimate above holds, as it will be clear in the proof, also for the time $t = T$. So, again,

$$|\tilde{A}_m(t)| \leq R \quad \forall t \in [0,T]$$

and since this holds for every $t \in [0,T]$ we have

$$\|\tilde{A}_m\|_{C([0,T])^m} := \max_{t\in[0,T]} |\tilde{A}_m(t)| \leq \sqrt{\frac{L}{\lambda_1}} \leq R$$

that is

$$\tilde{A}_m \in \overline{B}_R$$

$\mathcal{T}$ **is completely continuous:** To do this, we need another bound and a classical theorem. Our aim is to show that

$$\overline{\{\mathcal{T}(A): \ A \in \mathcal{B}\}} \text{ is compact for every bounded } \mathcal{B} \subseteq \overline{B}_R$$

where the closure is in $M := \overline{B}_R$. This means that we have to show that $\{\mathcal{T}(A): \ A \in \mathcal{B}\}$ is a relatively compact set for every $\mathcal{B}$ bounded set in $\overline{B}_R$. Thus it is sufficient to show that every sequence in $\{\mathcal{T}(A): \ A \in \mathcal{B}\}$ has a subsequence that converges to a certain point in the closure of this set. To show this, let $\mathcal{B}$ a bounded subset of $\overline{B}_R$ and $\mathcal{T}(A^n_m)$, with $A^n_m \in \mathcal{B}$, a sequence in the set we want to show that is pre-compact. We want to find a subsequence $T(A^{n_k}_m)$ and a $g \in \overline{\{\mathcal{T}(A): \ A \in \mathcal{B}\}}$ such that

$$\lim_{k \to +\infty} \max_{t \in [0,T]} |\mathcal{T}(A^{n_k}_m)(t) - g(t)| = 0$$

Notice that, by the fact that $\overline{\{\mathcal{T}(A): \ A \in \mathcal{B}\}}$ is closed, if we find a g that satisfy the limit, this automatically belongs to $\overline{\{\mathcal{T}(A): \ A \in \mathcal{B}\}}$. We proceed using the Ascoli-Arzelà theorem.

We use first proposition 11.4. Observe that now we assume directly $\tilde{A}_m \in C^1([0,T])$ thanks to the argument above. The previous propostition immediately gives us a result concering a sequence $\tilde{A}_m$ that satisfies the hypothesis. First of all, observe that, if $\tilde{U}_m$ as in the propositions,

$$\|\partial_t \tilde{U}_m\|_2 = \left(\int_\Omega \left| \sum_{k=1}^m \partial_t \tilde{A}_{mk}(t) w^k \right|^2 \ dx \right)^{\frac{1}{2}} = \left(\sum_{k=1}^m |\partial_t \tilde{A}_{mk}(t)|^2 \right)^{\frac{1}{2}} = |\partial_t \tilde{A}_m(t)|$$

We choose the sequence

$$\tilde{A}^n_m(t) := \mathcal{T}(A^n_m)(t)$$

i.e. we apply the proposition to the case we are considering.
The latter inequality, togheter with the proposition, say to us that the sequence $\tilde{A}^n_m$ is equicontinuous. In fact

$$\int_0^T \|\partial_t \tilde{U}_m\|^2_2(t) \ dt \leq K$$

and it follows that

$$\int_0^T \|\partial_t \tilde{U}_m\|_2(t) \ dt \leq T^{\frac{1}{2}} \left(\int_0^T \|\partial_t \tilde{U}_m\|^2_2(t) \ dt \right)^{\frac{1}{2}} \leq \sqrt{TK}$$

So, using the mean value integral inequality, we have, if $\tau < t$,

$$|\tilde{A}^n_m(t) - \tilde{A}^n_m(\tau)| \leq |t-\tau| \int_0^1 |(\partial_t \tilde{A}^n_m)(\tau + s(t-\tau))| \ ds = |t-\tau| \int_\tau^t |(\partial_t \tilde{A}^n_m)(u)|(t-\tau) du =$$

$$= |t-\tau|^2 \int_\tau^t |(\partial_t \tilde{A}^n_m)(u)| du \leq |t-\tau|^2 \int_0^T |(\partial_t \tilde{A}^n_m)(u)| du \leq |t-\tau|^2 \int_0^T \|\partial_t \tilde{U}^n_m\|_2(u) du \leq$$

$$\leq |t-\tau|^2 \sqrt{TK} \leq T^{\frac{3}{2}} \sqrt{K} |t-\tau|$$

if $t, \tau \in [0, T]$. Here K is independent of n, m, as we will see in the proof of the proposition; it is the same for every solution in $C^1([0, T])$, as in the statement.
So the sequence $\tilde{A}_m^n$ is equicontinuous on $[0, T]$. Moreover

$$\|\tilde{A}_m^n\|_{C([0,T])^m} \leq R$$

So, we are in the hypotesis of the Ascol-Arzelà's theorem 1.1. So exists a subsequance $\tilde{A}_m^{n_k}(t) = \mathcal{T}(A_m^{n_k})(t)$ and $g \in C([0, T])$ such that

$$\lim_{k \to +\infty} \max_{t \in [0,T]} |\mathcal{T}(A_m^{n_k})(t) - g(t)| = 0$$

Furthermore $g \in \overline{\{\mathcal{T}(A) : \ A \in \mathcal{B}\}}$ since the set is closed and g is limit (in norm) of elements of the set. So the chosen sequence $\mathcal{T}(A_m^n)$ has a subsequence that converges in norm to a function in the set $\overline{\{\mathcal{T}(A) : \ A \in \mathcal{B}\}}$. This is the definition of pre-compactness on metric spaces. Clearly $\mathcal{T}$ is also continuous, since for every $A \in \overline{B}_R$ we have

$$\|\mathcal{T}(A)\|_{C([0,T])^m} \leq R$$

and so

$$\|\mathcal{T}\| := \sup \left\{ \|\mathcal{T}(A)\|_{C([0,T])^m} : \ A \in \overline{B}_R, \ \|A\|_{C([0,T])^m} \leq 1 \right\} \leq \sup_{A \in \overline{B}_R} \|\mathcal{T}(A)\|_{C([0,T])^m} \leq R$$

So the operator $\mathcal{T}$ is completely continuous, and so, by the Schauder fixed point theorem 2.8, it has a fixed point in $M = \overline{B}_R$. That is, there exists an $A_m \in \overline{B}_R$ such that

$$\mathcal{T}(A_m) = A_m$$

So the equation has finally a solution. More precisely, exists $A_m \in C([0, T])^m$ such that A_m satisfies the equation (11.14) with γ^m solution of (11.13) with velocity field

$$u^m(x, t) := \sum_{k=1}^{m} A_{mk}(t) w^k(x) \tag{11.21}$$

Moreover, $\rho^m(x, t)$ is the solution associated to the field (11.21). So (u^m, ρ^m) is the pair of functions we were searching for. ∎

Remark 11.8. Notice that we know something more about the regularity of the solution u^m. In fact, we have proved above in (11.17) that the solution of the ODE system studied above is such that

$$\tilde{A}_m \in C^2([0, T])^m$$

But $\tilde{A}_m$ is nothing but the image of A_m through the operator T (that sends an equation in $\overline{B}_R$ to the associated solution of the system). So

$$A_m = \tilde{A}_m \in C^2([0, T])^m$$

This in particular means that $u^m(x, t)$ can be derived twice respect with the time. In this case, we obtain

$$u_{tt}^m(x, t) = \sum_{k=1}^{m} \partial_t^2 A_{mk}(t) w^k(x) \in C([0, T]; X^m)$$

This will be very useful in a moment. □

Remark 11.9. This result is very optimistic: the reason is that we have neglected an eventual force in the NSE. The articles that inspired this discussion (i.e. [4],[16],[26]) consider the equation with the presence of the force. Their result at this point is in a weaker class of regularity for the weak twice derivative of u^m. $\square$

11.3.3 Proof of propositions 11.2 and 11.4

In this subsection we prove proposition 11.2 and 11.4 claimed above.

Proof of proposition 11.2. We start proving the first proposition. Keep in mind lemma 9.6. Let $\tilde{A}_m \in C^1([0,\overline{T}))$ a solution of (11.14). Define $\tilde{U}_m$ as above. Then, using that $\tilde{A}_m$ satisfies the ODE (11.16) and summing over i multiplying for the right coefficients $\partial_t \tilde{A}_m$ we have

$$\int_\Omega \rho^m(x,t)|\partial_t \tilde{U}_m(x,t)|^2 dx + \int_\Omega \rho^m(x,t)[(\nabla \tilde{U}_m(x,t))\tilde{U}_m(x,t)] \cdot \partial_t \tilde{U}_m(x,t) dx +$$

$$+\mu \int_\Omega \nabla \tilde{U}_m(x,t) \cdot \nabla(\partial_t \tilde{U}_m)(x,t) dx = 0$$

where $\rho^m(x,t)$ is a function in $C^1([0,T]\times\overline{\Omega})$ such that $\rho^m \geq \delta > 0$ and $\|\rho^m\|_\infty = \|\overline{\rho}_0\|_\infty$. Observe that $\tilde{U}_m(x,t)$ is a function where the temporal and the spatial variable belong to different functions multiplied and summed. So the operators Δ and ∂_t operate independently, without any need of the second derivatives. Moreover

$$\partial_t |\nabla \tilde{U}_m|^2 = 2\nabla \tilde{U}_m \cdot \partial_t(\nabla \tilde{U}_m)$$

Being moreover $|\nabla \tilde{U}_m|^2 \in C^1([0,\overline{T}); C(\overline{\Omega}))$, we can differentiate under integral sign and obtain

$$\int_\Omega \rho^m(x,t)|\partial_t \tilde{U}_m(x,t)|^2 dx + \int_\Omega \rho^m(x,t)\Big(\big(\nabla \tilde{U}_m(x,t)\big)\tilde{U}_m(x,t)\Big) \cdot \partial_t \tilde{U}_m(x,t)\ dx +$$

$$+\frac{\mu}{2}\frac{d}{dt}\int_\Omega |\nabla \tilde{U}_m|^2 dx = 0$$

We now estimate the second term to get

$$\left|\int_\Omega \rho^m \big((\nabla \tilde{U}_m)\tilde{U}_m\big) \cdot \partial_t \tilde{U}_m\ dx\right| \leq \int_\Omega |\sqrt{\rho^m}\big((\nabla \tilde{U}_m)\tilde{U}_m\big)||\sqrt{\rho^m}\partial_t \tilde{U}_m| dx$$

Using the Young's inequality $ab \leq \dfrac{a^2}{4} + b^2$ we have

$$\left|\int_\Omega \rho^m \big((\nabla \tilde{U}_m)\tilde{U}_m\big) \cdot \partial_t \tilde{U}_m\ dx\right| \leq \frac{1}{4}\int_\Omega \rho^m |\partial_t \tilde{U}_m|^2 dx + \int_\Omega \rho^m |\nabla \tilde{U}_m|^2 |\tilde{U}_m|^2 dx \leq$$

$$\leq \frac{1}{4}\int_\Omega \rho^m |\partial_t \tilde{U}_m|^2 dx + \int_\Omega \|\overline{\rho}_0\|_\infty |\nabla \tilde{U}_m|^2 |\tilde{U}_m|^2 dx$$

It follows that

$$2\int_\Omega \rho^m|\partial_t\tilde U_m|^2dx+\mu\frac{d}{dt}\int_\Omega|\nabla\tilde U_m|^2dx\le\frac12\int_\Omega\rho^m|\partial_t\tilde U_m|^2dx+2\int_\Omega\|\overline{\rho}_0\|_\infty|\nabla\tilde U_m|^2|\tilde U_m|^2 \tag{11.22}$$

Now we deduce another estimate. We can apply the Stokes operator A to the function $\tilde U_m$ and

$$-P\Delta\tilde U_m(x,t)=-\sum_{j=1}^m\tilde A_{mj}(t)P\Delta w^j(x)=\sum_{j=1}^m\tilde A_{mj}(t)\lambda_j w^j(x) \tag{11.23}$$

where $w^j\in X$ and so the operator is defined. So applying this in the ODE we get

$$\mu\int_\Omega P\Delta\tilde U_m\cdot P\Delta\tilde U_m\ dx\overset{5}{=}\int_\Omega\rho^m(\partial_t\tilde U_m+(\nabla\tilde U_m)\tilde U_m)\cdot P\Delta\tilde U_m dx\le$$

$$\le\frac{\mu}{2}\int_\Omega|P\Delta\tilde U_m|^2dx+\frac{1}{2\mu}\int_\Omega 2|\rho^m|^2\big(|\partial_t\tilde U_m|^2+|(\nabla\tilde U_m)\tilde U_m|^2\big)\ dx$$

Observe moreover that

$$\int_\Omega\Delta\tilde U_m\cdot P\Delta\tilde U_m\ dx=\int_\Omega|P\Delta\tilde U_m|^2dx$$

since

$$\langle\Delta\tilde U_m,P\Delta\tilde U_m\rangle=\langle\Delta\tilde U_m,P^2\Delta\tilde U_m\rangle=\langle P\Delta\tilde U_m,P\Delta\tilde U_m\rangle$$

and so, using lemma 9.6, we have

$$\frac{\mu}{2}\frac{1}{e^2}\int_\Omega|\Delta\tilde U_m|^2dx\le\frac{\mu}{2}\int_\Omega|P\Delta\tilde U_m|^2dx\le\frac1\mu\int_\Omega|\rho^m|^2|\partial_t\tilde U_m|^2dx+\frac1\mu\int_\Omega|\rho^m|^2|\nabla\tilde U_m|^2|\tilde U_m|^2dx \tag{11.24}$$

---

[5] Using that $\tilde U_m=\sum_{j=1}^m\tilde A_{mj}w^j$ and (11.23) we can notice, in example, that

$$\sum_{j=1}^m\int_\Omega\rho^m\ w^j\cdot w^i\ \partial_t\tilde A_{mj}\ dx=\int_\Omega\rho^m\partial_t\tilde U_m\cdot w^i\ dx$$

and multiplying for $-\lambda_i\tilde A_{mi}$ and summing over i we get the equality. Moreover, doing these operations, and remembering that $D_{ij}=\lambda_i\delta_{ij}$,

$$\sum_{i=1}^m\left(\mu\sum_{j=1}^m D_{ij}\tilde A_{mj}\right)\cdot(\lambda_i\tilde A_{mi})=\mu\sum_{i=1}\lambda_i^2\tilde A_{mi}^2=\mu\int_\Omega P\Delta\tilde U_m\cdot P\Delta\tilde U_m$$

At this point

$$\mu\sum_{j=1}^m D_{ij}\tilde A_{mj}=-\sum_{j=1}^m b_{ik}^m(t)\partial_t\tilde A_{mj}-\sum_{j=1}^m\sum_{k=1}^m C_{ijk}^m(t)\tilde A_{mj}\tilde A_{mk}$$

and the minus is absorbed by the equality (11.23).

So, adding the inequalities (11.22) and (11.24) we have, multiplying the second by $d > 0$ and using that $|\rho^m| \leq \|\overline{\rho}_0\|_\infty + 1 =: b$,

$$\left(\frac{3}{2} - \frac{bd}{\mu}\right) \int_\Omega |\rho^m||\partial_t \tilde{U}_m|^2 dx + \mu \frac{d}{dt} \int_\Omega |\nabla \tilde{U}_m|^2 dx + \frac{\mu d}{2e^2} \int_\Omega |\Delta \tilde{U}_m|^2 dx \leq$$

$$\leq (2b + \frac{db^2}{\mu}) \int_\Omega |\nabla \tilde{U}_m|^2 |\tilde{U}_m|^2 dx$$

Choosing $d = \frac{\mu}{2b}$, we have

$$\int_\Omega |\rho^m||\partial_t \tilde{U}_m|^2 dx + \mu \frac{d}{dt} \int_\Omega |\nabla \tilde{U}_m|^2 dx + 2\varepsilon \int_\Omega |\Delta \tilde{U}_m|^2 dx \leq$$

$$\leq \left(2b + \frac{b}{2}\right) \int_\Omega |\nabla \tilde{U}_m|^2 |\tilde{U}_m|^2 dx \leq 3b \int_\Omega |\nabla \tilde{U}_m|^2 |\tilde{U}_m|^2 dx$$

where $\varepsilon := \frac{\mu^2}{8be^2}$. Estimating the latter piece

$$\int_\Omega |\nabla \tilde{U}_m|^2 |\tilde{U}_m|^2 dx \leq \|\tilde{U}_m\|_\infty^2 \int_\Omega |\nabla \tilde{U}_m|^2 dx \overset{\text{Lemma 9.6}}{\leq} c^2 \left(\int_\Omega |\Delta \tilde{U}_m|^2 dx\right)^{\frac{3}{4}} \left(\int_\Omega |\nabla \tilde{U}_m|^2 dx\right)^{\frac{5}{4}} \leq$$

$$\overset{6}{\leq} \frac{\varepsilon}{3b} \int_\Omega |\Delta \tilde{U}_m|^2 dx + C'_\lambda \left(\int_\Omega |\nabla \tilde{U}_m|^2 dx\right)^5$$

We finally have

$$\int_\Omega |\rho^m||\partial_t \tilde{U}_m|^2 dx + \mu \frac{d}{dt} \int_\Omega |\nabla \tilde{U}_m|^2 dx + 2\varepsilon \int_\Omega |\Delta \tilde{U}_m|^2 dx \leq \varepsilon \int_\Omega |\Delta \tilde{U}_m|^2 dx + 3bC'_\lambda \left(\int_\Omega |\nabla \tilde{U}_m|^2 dx\right)^5$$

and

$$\boxed{\int_\Omega |\rho^m||\partial_t \tilde{U}_m|^2 dx + \mu \frac{d}{dt} \int_\Omega |\nabla \tilde{U}_m|^2 dx + \varepsilon \int_\Omega |\Delta \tilde{U}_m|^2 dx \leq 3bC'_\lambda \left(\int_\Omega |\nabla \tilde{U}_m|^2 dx\right)^5} \tag{11.25}$$

If $C := \frac{3bC'_\lambda}{\mu}$,

$$\frac{d}{dt} \|\nabla \tilde{U}_m\|^2 \leq C \|\nabla \tilde{U}_m\|_2^{10}$$

We integrate now between 0 and t, with $t < \overline{T}$, and get

$$\|\nabla \tilde{U}_m\|_2^2(t) - \|\nabla \tilde{U}_m\|_2^2(0) \leq C \int_0^t \|\nabla \tilde{U}_m\|_2^{10}(s) ds$$

[6] Using

$$ab \leq \lambda a^{\frac{4}{3}} + C_\lambda b^4$$

with $\lambda = \frac{\varepsilon}{3bc^2}$.

Remark 11.10. Observe that moreover

$$\boxed{\|\nabla \tilde{U}_m\|_2(0) \leq \|\nabla u_0\|_{L^2}} \tag{11.26}$$

independently of m. In fact

$$\int_\Omega |\nabla \tilde{U}_m(0,x)|^2 dx = \sum_{k=1}^m \tilde{A}^2_{mk}(0)\lambda_k = \sum_{k=1}^m \langle u_0, w^k\rangle_2^2 \lambda_k$$

and furthermore

$$\lim_{K\to+\infty} \left\| \sum_{k=1}^K \langle u_0, w^k\rangle_2 w^k - u_0 \right\|_{H^2(\Omega)} = 0$$

This in particular means that

$$\lim_{K\to+\infty} \left\| \sum_{k=1}^K \langle u_0, w^k\rangle_2 \nabla w^k - \nabla u_0 \right\|_{L^2} = 0$$

so that

$$\left| \left\| \sum_{k=1}^K \langle u_0, w^k\rangle_2 \nabla w^k \right\|_{L^2} - \|\nabla u_0\|_{L^2} \right| \leq \left\| \sum_{k=1}^K \langle u_0, w^k\rangle_2 \nabla w^k - \nabla u_0 \right\|_{L^2} \to 0$$

and so

$$\lim_{K\to+\infty} \left(\sum_{k=1}^K \langle u_0, w^k\rangle_2^2 \lambda_k \right)^{\frac{1}{2}} = \lim_{K\to+\infty} \left(\int_\Omega \left| \sum_{k=1}^K \langle u_0, w^k\rangle_2 \nabla w^k(x) \right|^2 dx \right)^{\frac{1}{2}} = \|\nabla u_0\|_2$$

since

$$\left| \sum_{k=1}^K \langle u_0, w^k\rangle_2 \nabla w^k(x) \right|^2 dx = \sum_{k=1}^K \sum_{j=1}^K \langle u_0, w^j\rangle_2 \langle u_0, w^k\rangle_2 \nabla w^j(x) \cdot \nabla w^k(x)$$

$$\implies \int_\Omega \left| \sum_{k=1}^K \langle u_0, w^k\rangle_2 \nabla w^k(x) \right|^2 dx = \sum_{k=1}^K \sum_{j=1}^K \langle u_0, w^j\rangle_2 \langle u_0, w^k\rangle_2 \int_\Omega \nabla w^j(x) \cdot \nabla w^k(x) dx =$$

$$= \sum_{k=1}^K \sum_{j=1}^K \langle u_0, w^j\rangle \langle u_0, w^k\rangle_2 \lambda_k \delta_{jk} = \sum_{k=1}^K \langle u_0, w^k\rangle_2^2 \lambda_k$$

So

$$\left(\int_\Omega |\nabla \tilde{U}_m(0,x)|^2 dx \right)^{\frac{1}{2}} = \left(\sum_{k=1}^m \langle u_0, w^k\rangle_2^2 \lambda_k \right)^{\frac{1}{2}} \leq \lim_{K\to+\infty} \left(\sum_{k=1}^K \langle u_0, w^k\rangle_2^2 \lambda_k \right)^{\frac{1}{2}} = \|\nabla u_0\|_2$$

where has been used that the series is a series of positive terms, since we have a square and $\lambda_k > 0 \ \forall k \in \mathbb{N}$. □

So we have the inequality

$$\|\nabla \tilde{U}_m\|_2^2(t) \leq \|\nabla u_0\|_{L^2}^2 + C \int_0^t \|\nabla \tilde{U}_m\|_2^{10}(s) ds \quad \forall t < \overline{T}$$

We now use the version of the Gronwall's inequality in 1.4. In our case we have, for $t \in [0, \overline{T})$,

$$\|\nabla \tilde{U}_m\|_2^2(t) \leq \|\nabla u_0\|_{L^2}^2 + C \int_0^t \|\nabla \tilde{U}_m\|_2^{10}(s) ds \leq \|\nabla u_0\|_{L^2}^2 + C \int_0^t [\|\nabla \tilde{U}_m\|_2^{10}(s) + 1] ds$$

So, if we choose $v(t) := \|\nabla \tilde{U}_m\|_2^2(t)$, $V_0 := \|\nabla u_0\|_2^2$, $\omega(v) := v^5 + 1$ and $\psi \equiv C > 0$, we have

$$\|\nabla \tilde{U}_m\|_2^2(t) \leq \phi^{-1}(\phi(V_0) + Ct) \leq \phi^{-1}(\phi(V_0) + CT)$$

since also ϕ^{-1} is an increasing function and $t < \overline{T} \leq T$. We can define $L := \phi^{-1}(\phi(V_0) + CT)$, so we get

$$\|\nabla \tilde{U}_m\|_2^2(t) \leq L \quad \forall t \in [0, \overline{T}) \tag{11.27}$$

Notice that we have used the fact that ϕ^{-1} is also stricly increasing, being the inverse of a such function. ∎

Remark 11.11. The constant L depends on T but it is independent of $\overline{T}$. □

Proof of proposition 11.4. Now we deduce the estimate on $\partial_t \tilde{U}_m$, that is

$$\int_0^T \|\partial_t \tilde{U}_m\|_2^2(t) dt \leq K$$

We are supposing now $\tilde{A}_m \in C^1([0, T])$. From equation (11.25), that holds for every t where $\tilde{A}_m$ is defined, we have

$$\boxed{\int_\Omega |\rho^m| |\partial_t \tilde{U}_m|^2 dx + \mu \frac{d}{dt} \int_\Omega |\nabla \tilde{U}_m|^2 dx + \varepsilon \int_\Omega |\Delta \tilde{U}_m|^2 dx \leq 3bC'_\lambda \Big(\int_\Omega |\nabla \tilde{U}_m|^2 dx \Big)^5}$$

and so, integrating over $[0, T]$ we have

$$\int_0^T \int_\Omega |\rho^m| |\partial_t \tilde{U}_m|^2 dx dt + \mu \Big(\int_\Omega |\nabla \tilde{U}_m|^2(T) dx - \int_\Omega |\nabla \tilde{U}_m|^2(0) dx \Big) + \varepsilon \int_0^T \int_\Omega |\Delta \tilde{U}_m|^2 dx dt \leq$$

$$\leq 3bC'_\lambda \int_0^T \Big(\int_\Omega |\nabla \tilde{U}_m|^2 dx \Big)^5 dt$$

So, now all the terms are positive and it follows

$$\int_0^T \int_\Omega |\rho^m| |\partial_t \tilde{U}_m|^2 dx dt \leq \mu \int_\Omega |\nabla \tilde{U}_m|^2(0) dx + 3bC'_\lambda \int_0^T \Big(\int_\Omega |\nabla \tilde{U}_m|^2 dx \Big)^5 dt \leq$$

$$\overset{(11.26)}{\leq} \mu \|\nabla u_0\|_2^2 + 3bC'_\lambda \int_0^T \|\nabla \tilde{U}_m\|_2^{10} dt \overset{(11.27)}{\leq} \mu \|\nabla u_0\|_2^2 + 3bC'_\lambda \int_0^T L^5 dt =$$

$$= \mu \|\nabla u_0\|_2^2 + 3bC'_\lambda T L^5$$

Now, being $\rho^m(x, t) \geq \delta > 0$, we have

$$\delta \int_0^T \int_\Omega |\partial_t \tilde{U}_m|^2 dx dt \leq \int_0^T \int_\Omega |\rho^m| |\partial_t \tilde{U}_m|^2 dx dt \leq \mu \|\nabla u_0\|_2^2 + 3bC'_\lambda T L^5$$

Defining $K := \frac{1}{\delta}\left(\mu\|\nabla u_0\|_2^2 + 3bC'_\lambda T L^5\right)$, we have

$$\int_0^T \|\partial_t \tilde{U}_m\|_2^2(t)dt = \int_0^T \int_\Omega |\partial_t \tilde{U}_m|^2 dxdt \leq K$$

that is the thesis. ∎

Remark 11.12. The constant K depends on μ, the initial data $\overline{\rho}_0$ and μ_0, the bound δ, the time T and the constant L, togheter with other constant C'_λ, and e, c that are due to some inequalities. However, the constant is independent of the solution we are considering, and this is the only thing we need. In fact we use K only for applying the Ascoli-Arzelà theorem, that is for proving that the operator $\mathcal{T}$ is completely continuous. □

11.4 Estimates on the approximate solutions

We now deduce some estimates concerning the approximate solution built in the previous section. These estimates will allow us to use a convergence argument, extracting a right subsequence which limit is our aimed solution to the problem. We collect them in some propositions. The main estimate is summarized by the following proposition.

Proposition 11.5. *Let Ω be a bounded domain in $\mathbb{R}^3$. Consider the Navier-Stokes problem over Ω as in proposition 11.1. Let $\overline{\rho}_0 \in C^1(\overline{\Omega})$ and $T > 0$. Let $\rho^m \in C^1([0,T] \times \overline{\Omega})$ and $u^m \in C^1([0,T]; X^m)$ the approximate solutions built in proposition 11.1. Then there hold the following estimates*

- $\delta \leq \rho^m(x,t) \leq \|\rho_0\|_\infty + 1, \qquad \|\rho^m(t)\|_q = \|\overline{\rho}_0\|_q$

- $\|\nabla u^m(t)\|_2^2 \leq C + C\displaystyle\int_0^t \|\nabla u^m\|_2^6 \, ds$

- $\displaystyle\sup_{0\leq s\leq t}(\|\nabla u^m\|_{H^1}^2 + \|\sqrt{\rho^m} u_t^m\|_2^2) + \int_0^t \|u_t^m\|_{D_0^{1,2}}^2 \, ds \leq C\overline{C_0}^m + C\exp\left(C\int_0^t \|\nabla u^m\|_2^4 \, ds\right)$

for every $t \in [0,T]$. Here C is a generic positive constant depending only on $\|\rho_0\|_{L^\infty}$, $\|\nabla u_0\|_2$ and T, but it is independent of δ and m. Moreover, we define

$$\overline{C_0}^m := \int_\Omega (\overline{\rho}_0)^{-1} |\mu\Delta u^m(x,0) - \nabla p_0|^2 dx$$

Remark 11.13. The first point follows from classical considerations about the transport equation: in particular the solution assumes exactly the value that assumes $\overline{\rho}_0$; the incompressibility property and the conservation of the mass integral with exponent q has already been discussed in chapter 8. See in particular theorem 8.1. □

11.4.1 A first energy estimate

Proposition 11.6. *Let Ω be a bounded domain in $\mathbb{R}^3$. Consider the Navier-Stokes problem over Ω as in proposition 11.1. Let $\overline{\rho}_0 \in C^1(\overline{\Omega})$ and $T > 0$. Let $\rho^m \in C^1([0,T] \times \overline{\Omega})$ and $u^m \in C^1([0,T]; X^m)$ the approximate solutions built in proposition 11.1. Then there holds the following energy estimate*

$$\int_\Omega \rho^m(t)|u^m|^2(t)\ dx + \int_0^t \int_\Omega |\nabla u^m|^2 dx \le C$$

for every $t \in [0,T]$. Here C is a generic positive constant depending only on $\|\rho_0\|_{L^\infty}$, $\|\nabla u_0\|_2$ and T, but it is independent of δ and m.

Proof. We now have to deduce estimates. We know that

$$\int_\Omega \{(\rho^m u_t^m + \rho^m(\nabla u^m)u^m)\cdot\phi + \mu\nabla u^m\cdot\nabla\phi\}\ dx = 0\ \ \forall\phi\in X^m$$

If we choose $\phi = u^m \in X^m$, it follows that

$$\int_\Omega \rho^m u_t^m \cdot u^m dx + \int_\Omega \rho^m u^m \cdot \big((\nabla u^m)u^m\big)dx + \mu\int_\Omega |\nabla u^m|^2 dx = 0$$

Now, using that

$$\frac{d}{dt}|u^m|^2 = 2u_t^m\cdot u^m$$

and

$$\nabla|u^m|^2 = 2(\nabla u^m)u^m$$

we have, using integration by parts,

$$\int_\Omega \rho^m u^m\cdot\nabla|u^m|^2 dx = \int_\Omega \nabla\cdot(\rho^m|u^m|^2u^m)dx - \int_\Omega |u^m|^2\ \nabla\cdot(\rho^m u^m)dx =$$

$$= \int_{\partial\Omega} \rho^m|u^m|^2u^m\cdot\nu\ d\sigma - \int_\Omega |u^m|^2\ u^m\cdot\nabla\rho^m dx =$$

$$= \int_\Omega \rho_t^m|u^m|^2 dx$$

using that $\nabla\cdot u^m = 0$ and $u^m \equiv 0$ over $\partial\Omega$, and moreover $u^m \in C^1(\overline{\Omega})$. So we have

$$\frac{1}{2}\int_\Omega \rho^m\frac{d}{dt}|u^m|^2 dx + \frac{1}{2}\int_\Omega \rho_t^m|u^m|^2 dx + \mu\int_\Omega |\nabla u^m|^2 dx = 0$$

that is, using regularity in x and t and using the compactness[7] of $\overline{\Omega}$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho^m|u^m|^2 dx + \mu\int_\Omega |\nabla u^m|^2 dx = 0$$

[7]Remember that if $f \in C([0,T]; C(\overline{\Omega}))$, then for every $(x_0,t_0)\in\overline{\Omega}\times[0,T]$ we have

$$|f(x,t) - f(x_0,t_0)| \le |f(x,t)-f(x,t_0)| + |f(x,t_0)-f(x_0,t_0)| \le \max_{x\in\overline{\Omega}}|f(x,t)-f(x,t_0)| + \varepsilon$$

if $|x - x_0| < \delta = \delta(x_0,t_0)$. Since $\lim_{t\to t_0}\|f(\cdot,t)-f(\cdot,t_0)\|_{C(\overline{\Omega})} = 0$, we have the continuity.
If now $f \in C^1([0,T]; C^1(\overline{\Omega}))$, we have the continuity of the derivatives. So $f \in C^1(\overline{\Omega}\times[0,T])$.

So integrating over $[0,t]$, with $t \leq T$, we have

$$\frac{1}{2}\int_\Omega \rho^m(t)|u^m|^2(t)\ dx - \frac{1}{2}\int_\Omega \rho^m(0)|u^m|^2(0)\ dx + \mu\int_0^t\int_\Omega |\nabla u^m|^2 dx = 0$$

Moreover

$$\int_\Omega \rho^m(0)|u^m|^2(0)\ dx \equiv \int_\Omega \rho^m(x,0)|u^m|^2(x,0)\ dx \leq \|\overline{\rho}_0\|_\infty \|u^m(0)\|_2^2 \overset{8}{\leq} \|\overline{\rho}_0\|_\infty \|u_0\|_2^2$$

So we have

$$\frac{1}{2}\int_\Omega \rho^m(t)|u^m|^2(t)\ dx + \mu\int_0^t\int_\Omega |\nabla u^m|^2 dx \leq \frac{1}{2}\|\overline{\rho}_0\|_\infty \|u_0\|_2^2 \tag{11.28}$$

We can also get rid of μ. If $\mu \geq \frac{1}{2}$, then

$$\int_\Omega \rho^m(t)|u^m|^2(t)\ dx + \int_0^t\int_\Omega |\nabla u^m|^2 dx \leq \|\overline{\rho}_0\|_\infty \|u_0\|_2^2$$

and if $\mu < \frac{1}{2}$, we have

$$\mu\int_\Omega \rho^m(t)|u^m|^2(t)\ dx + \mu\int_0^t\int_\Omega |\nabla u^m|^2 dx \leq \frac{1}{2}\|\overline{\rho}_0\|_\infty \|u_0\|_2^2$$

and so

$$\int_\Omega \rho^m(t)|u^m|^2(t)\ dx + \int_0^t\int_\Omega |\nabla u^m|^2 dx \leq \frac{\|\overline{\rho}_0\|_\infty \|u_0\|_2^2}{2\mu} \leq \frac{(\|\rho_0\|_\infty + 1)\|u_0\|_2^2}{2\mu}$$

Finally, we obtain

$$\boxed{\int_\Omega \rho^m(t)|u^m|^2(t)\ dx + \int_0^t\int_\Omega |\nabla u^m|^2 dx \leq C} \tag{11.29}$$

that is the energy estimate. ∎

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[8] Since

$$\int_\Omega |u^m|^2(x,0)dx = \sum_{k=1}^m \langle u_0, w^k\rangle_2^2 \leq \sum_{k=1}^\infty \langle u_0, w^k\rangle_2^2 = \|u_0\|_2^2$$

being $\lim_{K\to+\infty} \|\sum_{k=1}^K \langle u_0, w^k\rangle w^k - u_0\|_{H^1(\Omega)} = 0$, and so in particular

$$\left|\left(\int_\Omega \left|\sum_{k=1}^K \langle u_0, w^k\rangle w^k\right|^2 dx\right)^{\frac{1}{2}} - \|u_0\|_2\right| \leq \|\sum_{k=1}^K \langle u_0, w^k\rangle w^k - u_0\|_2 \leq \|\sum_{k=1}^K \langle u_0, w^k\rangle w^k - u_0\|_{H^1(\Omega)} \to 0$$

as $K \to +\infty$. So the estimate follows from

$$\int_\Omega \left|\sum_{k=1}^K \langle u_0, w^k\rangle w^k\right|^2 dx = \sum_{k=1}^K \langle u_0, w^k\rangle_2^2$$

11.4.2 An estimate on the velocity field

Proposition 11.7. *Let Ω be a bounded domain in $\mathbb{R}^3$. Consider the Navier-Stokes problem over Ω as in proposition 11.1. Let $\overline{\rho}_0 \in C^1(\overline{\Omega})$ and $T > 0$. Let $\rho^m \in C^1([0,T] \times \overline{\Omega})$ and $u^m \in C^1([0,T]; X^m)$ the approximate solutions built in proposition 11.1. Then there holds the following estimate*

$$\frac{1}{2}\int_\Omega \rho^m |u_t^m|^2 dx + \mu \frac{d}{dt}\int_\Omega |\nabla u^m|^2 dx \leq \int_\Omega \rho^m |\nabla u^m|^2 |u^m|^2 dx \tag{11.30}$$

for every $t \in [0,T]$.

Proof. Now we deduce an estimate on velocity. We choose $\phi = u_t^m \in X^m$ and so

$$\int_\Omega \rho^m |u_t^m|^2 dx + \int_\Omega \rho^m \big((\nabla u^m) u^m\big) \cdot u_t^m dx + \mu \int_\Omega \nabla u^m \cdot \nabla u_t^m dx = 0 \tag{11.31}$$

We have

$$\int_\Omega \nabla u^m \cdot \nabla u_t^m dx = \int_\Omega \nabla u^m \cdot \partial_t \nabla u^m dx = \frac{1}{2}\int_\Omega \partial_t |\nabla u^m|^2 dx$$

where we have used that ∇ and ∂_t operate separately. Finally

$$\int_\Omega \nabla u^m \cdot \nabla u_t^m dx = \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u^m|^2 dx$$

using the regularity on the compact $\overline{\Omega}$. Moreover

$$|\big((\nabla u^m) u^m\big) \cdot u_t^m| \leq |\nabla u^m||u^m||u_t^m| \leq \frac{1}{2}|\nabla u^m|^2 |u^m|^2 + \frac{1}{2}|u_t^m|^2$$

It follows that

$$\left|\int_\Omega \rho^m \big((\nabla u^m) u^m\big) \cdot u_t^m \, dx\right| \leq \frac{1}{2}\int_\Omega \rho^m |\nabla u^m|^2 |u^m|^2 dx + \frac{1}{2}\int_\Omega \rho^m |u_t^m|^2 dx$$

So

$$\int_\Omega \rho^m |u_t^m|^2 dx + \frac{\mu}{2}\frac{d}{dt}\int_\Omega |\nabla u^m|^2 dx \leq \frac{1}{2}\int_\Omega \rho^m |\nabla u^m|^2 |u^m|^2 dx + \frac{1}{2}\int_\Omega \rho^m |u_t^m|^2 dx$$

Finally

$$\frac{1}{2}\int_\Omega \rho^m |u_t^m|^2 dx + \frac{\mu}{2}\frac{d}{dt}\int_\Omega |\nabla u^m|^2 dx \leq \frac{1}{2}\int_\Omega \rho^m |\nabla u^m|^2 |u^m|^2 dx$$

This inequality can also be written as

$$\frac{1}{2}\int_\Omega \rho^m |u_t^m|^2 dx + \mu \frac{d}{dt}\int_\Omega |\nabla u^m|^2 dx \leq \int_\Omega \rho^m |u_t^m|^2 dx + \mu \frac{d}{dt}\int_\Omega |\nabla u^m|^2 dx \leq \int_\Omega \rho^m |\nabla u^m|^2 |u^m|^2 dx \tag{11.32}$$

that is the desired estimate. ∎

11.4.3 A second derivatives estimate on the velocity field

Proposition 11.8. *Let Ω be a bounded domain in $\mathbb{R}^3$. Consider the Navier-Stokes problem over Ω as in proposition 11.1. Let $\overline{\rho}_0 \in C^1(\overline{\Omega})$ and $T > 0$. Let $\rho^m \in C^1([0,T] \times \overline{\Omega})$ and $u^m \in C^1([0,T]; X^m)$ the approximate solutions built in proposition 11.1. Then there holds the following estimate*

$$\|\nabla^2 u^m\|_2 \le C\left(\|\rho^m u_t^m\|_2 + \|\rho^m(\nabla u^m)u^m\|_2\right) \tag{11.33}$$

for every $t \in [0,T]$. Here C is a generic positive constant depending only on $\|\rho_0\|_{L^\infty}$, $\|\nabla u_0\|_2$ and T, but it is independent of δ and m.

Proof. We now want to use again (11.31) with $\phi = Au^m$ that is in X^m, since

$$Au^m(x,t) = -P\Delta u^m(x,t) = -\sum_{j=1}^{m} \tilde{A}_{mj}(t) P\Delta w^j(x) = \sum_{j=1}^{m} \tilde{A}_{mj}(t)\lambda_j w^j(x) \in X^m$$

So we have

$$\int_\Omega \rho^m u_t^m \cdot Au^m + \int_\Omega \rho^m\big((\nabla u^m)u^m\big) \cdot Au^m dx + \mu\int_\Omega \nabla u^m \cdot \big(\nabla(Au^m)\big)dx = 0$$

We first deal with the latter piece. Equality (1.14) says that, for functions in $H^2(\Omega)$,

$$\nabla u^m \cdot \nabla(Au^m) = \sum_{i=1}^{3} \nabla \cdot \big((Au^m)_i \nabla u_i^m\big) - \Delta u^m \cdot Au^m$$

and so

$$\int_\Omega \nabla u^m \cdot \nabla(Au^m)\ dx = \sum_{i=1}^{3}\int_{\partial\Omega} \big((Au^m)_i \nabla u_i^m\big) \cdot \nu\ d\sigma - \int_\Omega \Delta u^m \cdot Au^m\ dx$$

where the first derivatives are classical derivatives[9], since $X^m \subseteq C^1(\overline{\Omega})$. Being $Au^m = 0$ over $\partial\Omega$ (since $X^m \subseteq H_0^1(\Omega)$), we have

$$\int_\Omega \rho^m u_t^m \cdot Au^m + \int_\Omega \rho^m\big((\nabla u^m)u^m\big)\cdot Au^m dx = -\mu\int_\Omega \nabla u^m \cdot \big(\nabla(Au^m)\big)\, dx = \mu\int_\Omega \Delta u^m \cdot Au^m dx =$$

$$= \mu\langle \Delta u^m, Au^m\rangle = -\mu\langle \Delta u^m, P\Delta u^m\rangle = -\mu\langle \Delta u^m, P^2\Delta u^m\rangle =$$

$$= -\mu\langle P\Delta u^m, P\Delta u^m\rangle = -\mu\int_\Omega |P\Delta u^m|^2 dx$$

So

$$\mu\int_\Omega |P\Delta u^m|^2 dx \le \int_\Omega |\rho^m u_t^m \cdot Au^m + \rho^m\big((\nabla u^m)u^m\big) \cdot Au^m| dx \le$$

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[9] Observe moreover that

$$Au^m = A\Big(\sum_{j=1}^{m} \tilde{A}_{mj} w^j\Big) = \sum_{j=1}^{m} \tilde{A}_{mj} Aw^j = \sum_{j=1}^{m} \tilde{A}_{mj}\lambda_j w^j$$

and so $Au^m \in X^m \subseteq C^1(\overline{\Omega})$.

$$\leq \int_\Omega |Au^m||\rho^m u_t^m + \rho^m\big((\nabla u^m)u^m\big)|dx \leq$$

$$\leq \frac{\mu}{2}\int_\Omega |Au^m|^2 dx + \frac{1}{2\mu}\int_\Omega 2\big(|\rho^m u_t^m|^2 + |\rho^m(\nabla u^m)u^m|^2\big)dx$$

Finally, remembering that $A = -P\Delta$,

$$\frac{\mu}{2}\int_\Omega |P\Delta u^m|^2 dx \leq \frac{1}{\mu}\big(\|\rho^m u_t^m\|_2^2 + \|\rho^m(\nabla u^m)u^m\|_2^2\big) \leq \frac{1}{\mu}\big(\|\rho^m u_t^m\|_2 + \|\rho^m(\nabla u^m)u^m\|_2\big)^2$$

So we have

$$\|Au^m\|_2 \leq \frac{\sqrt{2}}{\mu}\big(\|\rho^m u_t^m\|_2 + \|\rho^m(\nabla u^m)u^m\|_2\big)$$

Moreover, thanks to (9.49), we know that for every $\phi \in X$ it holds

$$\|\phi\|_X \equiv \|\phi\|_{H^2(\Omega)} \leq \underline{C}\|A\phi\|_2$$

So, being $u^m \in X^m \subseteq X$ it follows

$$\|\nabla^2 u^m\|_2 \leq \|u^m\|_{H^2} \leq \underline{C}\|Au^m\|_2 \leq C\big(\|\rho^m u_t^m\|_2 + \|\rho^m(\nabla u^m)u^m\|_2\big)$$

where $C := \frac{\sqrt{2}}{\mu}\underline{C}$. ∎

11.4.4 First final estimate.

We want now to deduce an estimate that includes the previous. In other words, we dedicate the following subsection to prove the following proposition.

Proposition 11.9. *Let Ω be a bounded domain in $\mathbb{R}^3$. Consider the Navier-Stokes problem over Ω as in proposition 11.1. Let $\overline{\rho}_0 \in C^1(\overline{\Omega})$ and $T > 0$. Let $\rho^m \in C^1([0,T]\times\overline{\Omega})$ and $u^m \in C^1([0,T];X^m)$ the approximate solutions built in proposition 11.1.*
Then there holds the following estimate

$$\int_0^t \big(\|\sqrt{\rho^m}u_t^m\|_2(s) + \|\nabla u^m\|_{H^1}^2(s)\big)ds + \int_\Omega |\nabla u^m(t)|dx \leq \mathcal{K} + \mathcal{K}\int_0^t \left(\int_\Omega |\nabla u^m|^2 dx\right)^3 ds \tag{11.34}$$

for every $t \in [0,T]$. Here $\mathcal{K}$ is a generic positive constant depending only on $\|\rho_0\|_{L^\infty}$, $\|\nabla u_0\|_2$ and T, but it is independent of δ and m.

Proof. Remember that $u^m \in C^1([0,T];X^m)$ and $\rho^m \in C^1([0,T],C^1(\overline{\Omega}))$. Furthermore we have proved that

$$\|u^m\|_2(0) \leq \|u_0\|_2 \tag{11.35}$$

$$\|\nabla u^m\|_2(0) \leq \|\nabla u_0\|_2 \tag{11.36}$$

By the previous subsection we have obtained

$$\|\nabla^2 u^m\|_2 \leq C\big(\|\rho^m u_t^m\|_2 + \|\rho^m(\nabla u^m)u^m\|_2\big) \tag{11.37}$$

It follows that[10]

$$\|\nabla^2 u^m\|_{L^2}^2 \leq 2C^2\big(\|\rho^m u_t^m\|_2^2+\|\rho^m(\nabla u^m)u^m\|_2^2\big) \leq 2C^2\|\rho^m\|_\infty\big(\|\sqrt{\rho^m}u_t^m\|_2^2+\|\sqrt{\rho^m}(\nabla u^m)u^m\|_2^2\big)$$

Remember that $\|\rho^m\|_\infty = \|\overline{\rho}_0\|_\infty$ and that

$$\overline{\rho}_0 \leq \|\rho_0\|_\infty + 1 \implies \|\overline{\rho}_0\|_\infty \leq \|\rho_0\|_\infty + 1$$

Finally we have

$$\|\nabla^2 u^m\|_2^2 \leq 2C^2(\|\rho_0\|_\infty + 1)\big(\|\sqrt{\rho^m}u_t^m\|_2^2 + \|\sqrt{\rho^m}(\nabla u^m)u^m\|_2^2\big) \tag{11.38}$$

Now we choose $\varepsilon > 0$ such that $8C^2\varepsilon(\|\rho_0\|_\infty + 1) < 1$. Hence we can estimate the term

$$\int_\Omega \big(\frac{1}{4}\rho^m|u_t^m|^2 + \varepsilon|\Delta u^m|^2\big)\ dx + \mu\frac{d}{dt}\int_\Omega |\nabla u^m|^2 dx$$

In fact,

$$\int_\Omega \varepsilon|\nabla^2 u^m|^2 dx \leq \frac{1}{8C^2(\|\rho_0\|_\infty + 1)}\|\nabla^2 u^m\|_{L^2}^2 \leq \frac{1}{4}\big(\|\sqrt{\rho^m}u_t^m\|_{L^2}^2 + \|\sqrt{\rho^m}(\nabla u^m)u^m\|_{L^2}^2\big) =$$

$$= \frac{1}{4}\int_\Omega \rho^m|u_t^m|^2 dx + \frac{1}{4}\int_\Omega \rho^m|(\nabla u^m)u^m|^2 dx \leq \frac{1}{4}\int_\Omega \rho^m|u_t^m|^2 dx + \frac{1}{4}\int_\Omega \rho^m|\nabla u^m|^2|u^m|^2 dx$$

where has been used the operatorial norm property $|(\nabla u^m)u^m| \leq |\nabla u^m||u^m|$. So, we have

$$\int_\Omega \big(\frac{1}{4}\rho^m|u_t^m|^2 + \varepsilon|\nabla^2 u^m|^2\big)\ dx + \mu\frac{d}{dt}\int_\Omega |\nabla u^m|^2 dx \leq$$

$$\leq \frac{1}{2}\int_\Omega \rho^m|u_t^m|^2 dx + \frac{1}{4}\int_\Omega \rho^m|\nabla u^m|^2|u^m|^2 dx + \mu\frac{d}{dt}\int_\Omega |\nabla u^m|^2 dx$$

and using the estimate (11.32) we get

$$\int_\Omega \big(\frac{1}{4}\rho^m|u_t^m|^2 + \varepsilon|\nabla^2 u^m|^2\big)\ dx + \mu\frac{d}{dt}\int_\Omega |\nabla u^m|^2 dx \leq$$

$$\leq \frac{5}{4}\int_\Omega \rho^m|\nabla u^m|^2|u^m|^2 dx \leq \frac{5}{4}\|\overline{\rho}_0\|_\infty\int_\Omega |\nabla u^m|^2|u^m|^2 dx \leq \frac{5}{4}(\|\rho_0\|_\infty+1)\int_\Omega |\nabla u^m|^2|u^m|^2 dx$$

always using invariant property of the q-norm for the density solution.

Since $\varepsilon \in (0, \frac{1}{8C^2(\|\rho_0\|_\infty+1)})$ is arbitrary, we can send ε to the right bound of its interval and get

$$\int_\Omega \left(\frac{1}{4}\rho^m|u_t^m|^2+\frac{1}{8C^2(\|\rho_0\|_\infty + 1)}|\nabla^2 u^m|^2\right) dx+\mu\frac{d}{dt}\int_\Omega |\nabla u^m|^2 dx \leq \frac{5}{4}(\|\rho_0\|_\infty+1)\int_\Omega |\nabla u^m|^2|u^m|^2 dx \tag{11.39}$$

[10]Since $(x+y)^2 \leq 2x^2 + 2y^2$.

We want now to estimate the term

$$\int_\Omega |\nabla u^m|^2 |u^m|^2 dx$$

First of all, we simply apply the Hölder inequality to $|u^m|^2$ and $|\nabla u^m|^2$ with exponents, respectively, 3 and $\frac{3}{2}$. It follows that

$$\int_\Omega |\nabla u^m|^2 |u^m|^2 dx \leq \||u^m|^2\|_3 \||\nabla u^m|^2\|_{\frac{3}{2}} \equiv \left(\int_\Omega |u^m|^6 dx\right)^{\frac{1}{3}} \left(\int_\Omega |\nabla u^m|^3 dx\right)^{\frac{2}{3}} = \|u^m\|_6^2 \|\nabla u^m\|_3^2$$

Now, using interpolated Hölder inequality, with $\gamma = 3$, $q = 2$, $r = 6$ and $\alpha = \frac{1}{2}$, we get

$$\|\nabla u^m\|_3 \leq \|\nabla u^m\|_2^{\frac{1}{2}} \|\nabla u^m\|_6^{\frac{1}{2}}$$

So,

$$\int_\Omega |\nabla u^m|^2 |u^m|^2 dx \leq \|u^m\|_6^2 \|\nabla u^m\|_2 \|\nabla u^m\|_6$$

Now we apply Sobolev-Gagliardo-Nirenberg inequality, that gives us

$$\|u^m\|_6 \leq \Lambda \|\nabla u^m\|_2$$

In this way

$$\|u^m\|_6^2 \|\nabla u^m\|_2 \|\nabla u^m\|_6 \leq \Lambda^2 \|\nabla u^m\|_2^3 \|\nabla u^m\|_6$$

The constant Λ is independent of the domain we are considering, thanks to the fact that $u^m \in H_0^1(\Omega) = W_0^{1,2}(\Omega)$ can be approached with smooth functions having compact support contained in Ω.

Now we want to estimate the term $\|\nabla u^m\|_6$. The function ∇u^m takes value in a matricial space, so usual Sobolev-Gagliardo-Nirenberg inequality seems to not hold. However, we can observe what follows. We will not repeat the argument in similar situations.

Let u a vectorial function. So we have

$$\|\nabla u\|_6 = \left(\int_\Omega |\nabla u|^6 dx\right)^{\frac{1}{6}} = \left(\int_\Omega \left(\sqrt{|\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_3|^2}\right)^6 dx\right)^{\frac{1}{6}}$$

where has been used that if A is a matrix then $|A| = |A^T| = \sqrt{|A_1|^2 + ... + |A_n|^2}$, with A_i the i-th colomn of A.
Moreover $\nabla u_i \in H^1(\Omega)$, since $u_i \in H^2(\Omega)$. Using Sobolev estimates for $W^{1,2}(\Omega)$, we found

$$\|\nabla u_i\|_6 \leq M \|\nabla u_i\|_{W^{1,2}(\Omega)}$$

where M depends only on p, n and Ω but it is independent of the function in $W^{1,2}(\Omega)$ that we are considering, so that M does not depend on i.
It follows that

$$\|\nabla u\|_6 = \||\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_3|^2\|_3^{\frac{1}{2}} \leq \left(\||\nabla u_1|^2\|_3 + \||\nabla u_2|^2\|_3 + \||\nabla u_3|^2\|_3\right)^{\frac{1}{2}} =$$

$$= \left(\|\nabla u_1\|_6^2 + \|\nabla u_2\|_6^2 + \|\nabla u_3\|_6^2\right)^{\frac{1}{2}} \leq M\left(\|\nabla u_1\|^2_{W^{1,2}(\Omega)} + \|\nabla u_2\|^2_{W^{1,2}} + \|\nabla u_3\|^2_{W^{1,2}(\Omega)}\right)^{\frac{1}{2}}$$

Remembering now that

$$\|\nabla u_i\|^2_{W^{1,2}(\Omega)} = \int_\Omega |\nabla u_i|^2 dx + \int_\Omega |\nabla(\nabla u_i)|^2 dx$$

and that

$$\|\nabla u\|_{H^1} \equiv \|\nabla u\|_{W^{1,2}} \equiv \left(\sum_{|\alpha|\leq 1} \int_\Omega |D^\alpha(\nabla u)|^2 dx\right)^{\frac{1}{2}} = \left(\int_\Omega \sum_{|\alpha|\leq 1} |D^\alpha(\nabla u)|^2 dx\right)^{\frac{1}{2}}$$

together with $|\nabla u|^2 = |\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_3|^2$ and

$$|\nabla(\nabla u)|^2 = |\nabla(\nabla u_1)|^2 + |\nabla(\nabla u_2)|^2 + |\nabla(\nabla u_3)|^2$$

where $\nabla(\nabla u)$ is the three-dimensional tensor of second derivatives, *id est*, it is the collection of three matrix, namely $\{\nabla(\nabla u_i)\}_{i=1}^3$. If T is a three-dimensional tensor, its norm is given by

$$|T|^2 \equiv \sum_{i,j,k} T^2_{ijk}$$

It is thus clear that

$$\sum_{i=1}^3 \|\nabla u_i\|^2_{W^{1,2}(\Omega)} = \|\nabla u\|^2_{H^1}$$

Finally

$$\|\nabla u\|_6 \leq M\|\nabla u\|_{H^1} \tag{11.40}$$

Hence, retracing our steps, we have

$$\int_\Omega |\nabla u^m|^2 |u^m|^2 dx \leq \Lambda_0 \|\nabla u^m\|_2^3 \|\nabla u^m\|_{H^1} \tag{11.41}$$

where $\Lambda_0 := \Lambda^2 M$.

We turn back to our estimates. We want to use the following Young's inequality: if $a, b \geq 0$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\varepsilon > 0$, then

$$ab \leq \varepsilon a^p + C(\varepsilon) b^q$$

where $C(\varepsilon) = (\varepsilon p)^{-\frac{q}{p}} q^{-1}$. Going on, set $C \longleftrightarrow \tilde{C}$ in (11.39), and let $C > 0$ be arbitrary. So, if we choose $q = p = 2$, $a = \frac{1}{\sqrt{8C}}\|\nabla u^m\|_{H^1}$ and $b = \Lambda_0\sqrt{8C}\|\nabla u^m\|_2^3$. It follows that

$$\Lambda_0 \|\nabla u^m\|_2^3 \|\nabla u^m\|_{H^1} \leq \frac{\varepsilon}{8C}\|\nabla u^m\|^2_{H^1} + \frac{2C}{\varepsilon}\Lambda_0^2 \|\nabla u^m\|_2^6$$

for every $C > 0$ and $\varepsilon > 0$. Finally

$$\int_\Omega \left(\frac{1}{4}\rho^m |u_t^m|^2 + \frac{1}{8\tilde{C}^2(\|\rho_0\|_\infty + 1)}|\nabla^2 u^m|^2\right) dx + \mu \frac{d}{dt}\int_\Omega |\nabla u^m|^2 dx \leq$$

$$\leq \frac{5}{4}(\|\rho_0\|_\infty + 1)\int_\Omega |\nabla u^m|^2|u^m|^2 dx \leq \frac{5}{4}(\|\rho_0\|_\infty + 1)\left(\frac{\varepsilon}{8C}\|\nabla u^m\|^2_{H^1} + \frac{2C}{\varepsilon}\Lambda_0^2\|\nabla u^m\|_2^6\right)$$

We have already noticed that

$$\int_\Omega |\nabla^2 u|^2 dx \equiv \|\nabla^2 u\|_2^2 \leq \|\nabla u\|^2_{H^1}$$

since the latter Sobolev norm in $H^1 = W^{1,2}$ also includes the 0-derivative terms. Here $\nabla^2 u \equiv \nabla(\nabla u)$ is the tensor of second derivatives (that is a three dimensional tensor). Let $\mu_0 := \frac{1}{2}\min\{\frac{1}{4}, \frac{1}{8\tilde{C}^2(\|\rho_0\|_\infty+1)}, \frac{2\mu}{1+6T}\}$. So

$$\mu_0\int_\Omega \rho^m|u_t^m|^2 dx + \mu_0\|\nabla u^m\|^2_{H^1} + \mu\frac{d}{dt}\int_\Omega |\nabla u^m|^2 dx \leq$$

$$\overset{11}{\leq} \int_\Omega \left(\frac{1}{4}\rho^m|u_t^m|^2 + \frac{1}{8\tilde{C}^2(\|\rho_0\|_\infty+1)}|\nabla^2 u^m|^2\right) dx + \mu_0\|\nabla u^m\|_2^2 + \mu\frac{d}{dt}\int_\Omega |\nabla u^m|^2 dx \leq$$

$$\leq \mu_0\|\nabla u^m\|_2^2 + \frac{5}{4}(\|\rho_0\|_\infty + 1)\left(\frac{\varepsilon}{8C}\|\nabla u^m\|^2_{H^1} + \frac{2C}{\varepsilon}\Lambda_0^2\|\nabla u^m\|_2^6\right)$$

So, since $\mu_0 > 0$,

$$\int_\Omega \rho^m|u_t^m|^2 dx + \|\nabla u^m\|^2_{H^1} + \frac{\mu}{\mu_0}\frac{d}{dt}\int_\Omega |\nabla u^m|^2 dx \leq$$

$$\leq \|\nabla u^m\|_2^2 + \frac{5}{4}(\|\rho_0\|_\infty + 1)\frac{\varepsilon}{8C\mu_0}\|\nabla u^m\|^2_{H^1} + \frac{5}{4}(\|\rho_0\|_\infty + 1)\frac{2C}{\varepsilon\mu_0}\Lambda_0^2\|\nabla u^m\|_2^6$$

and it follows that

$$\|\sqrt{\rho^m}u_t^m\|_2^2 + \left(1 - \frac{5}{4}(\|\rho_0\|_\infty + 1)\frac{\varepsilon}{8C\mu_0}\right)\|\nabla u^m\|^2_{H^1} + \frac{\mu}{\mu_0}\frac{d}{dt}\int_\Omega |\nabla u^m|^2 dx \leq$$

$$\leq \|\nabla u^m\|_2^2 + \frac{5}{4}(\|\rho_0\|_\infty + 1)\frac{2C}{\varepsilon\mu_0}\Lambda_0^2\|\nabla u^m\|_2^6$$

We can integrate this expressione in $(0, t)$, with $0 < t < T$. So

$$\int_0^t \|\sqrt{\rho^m}u_t^m\|_2^2(s)ds + \left(1 - \frac{5}{4}(\|\rho_0\|_\infty + 1)\frac{\varepsilon}{8C\mu_0}\right)\int_0^t \|\nabla u^m\|^2_{H^1}(s)ds + \frac{\mu}{\mu_0}\int_0^t \alpha(s)ds \leq$$

$$\leq \int_0^t \|\nabla u^m\|_2^2(s)ds + \overline{C}\int_0^t \|\nabla u^m\|_2^6(s)ds$$

where $\overline{C} := \frac{5}{4}(\|\rho_0\|_\infty + 1)\frac{2C}{\varepsilon\mu_0}\Lambda_0^2$ and $\alpha(t) := \frac{d}{dt}\|\nabla u^m\|_2^2(t)$. Now, using that $\|\nabla u^m\|_2^2$ is the anti-derivative of α, we have that

$$\frac{\mu}{\mu_0}\int_0^t \alpha(s)ds = \frac{\mu}{\mu_0}\big(\|\nabla u^m\|_2^2(t) - \|\nabla u^m\|_2^2(0)\big)$$

[11] Since $\|\nabla u\|^2_{H^1} = \|\nabla^2 u\|_2^2 + \|\nabla u\|_2^2$.

and

$$\int_0^t \|\nabla u^m\|_2^2(\omega)d\omega \le \int_0^t \sup_{s\in(0,t)} \|\nabla u^m\|_2^2(s)d\omega \le T \sup_{s\in(0,t)} \|\nabla u^m\|_2^2(s)$$

So

$$\int_0^t \|\sqrt{\rho^m}u_t^m\|_2^2(s)ds + \left(1 - \frac{5}{4}(\|\rho_0\|_\infty + 1)\frac{\varepsilon}{8C\mu_0}\right)\int_0^t \|\nabla u^m\|_{H^1}^2(s)ds + \frac{\mu}{\mu_0}\|\nabla u^m\|_2^2(t) \le$$

$$\le \frac{\mu}{\mu_0}\|\nabla u^m\|_2^2(0) + T\sup_{s\in(0,t)} \|\nabla u^m\|_2^2(s) + \overline{C}\int_0^t \|\nabla u^m\|_2^6(s)ds$$

$$\overset{(11.36)}{\le} \frac{\mu}{\mu_0}\|\nabla u_0\|_2^2 + T\sup_{s\in(0,t)} \|\nabla u^m\|_2^2(s) + \overline{C}\int_0^t \|\nabla u^m\|_2^6(s)ds$$

This is an inequality involving functions depending on t. If $t < \tau < T$, we can pass to the supremum for $t \in (0,\tau)$. If $1 - \frac{5}{4}(\|\rho_0\|_\infty + 1)\frac{\varepsilon}{8C\mu_0} > 0$, we can continue as follows. Each piece is bounded by the right side, being every term a postive term. So

$$\int_0^t \|\sqrt{\rho^m}u_t^m\|_2^2(s)ds \le \frac{\mu}{\mu_0}\|\nabla u_0\|_2^2 + T\sup_{s\in(0,t)} \|\nabla u^m\|_2^2(s) + \overline{C}\int_0^t \|\nabla u^m\|_2^6(s)ds$$

and thus

$$\sup_{t\in(0,\tau)}\int_0^t \|\sqrt{\rho^m}u_t^m\|_2^2(s)ds \le \sup_{t\in(0,\tau)}\{\frac{\mu}{\mu_0}\|\nabla u_0\|_2^2 + T\sup_{s\in(0,t)} \|\nabla u^m\|_2^2(s) + \overline{C}\int_0^t \|\nabla u^m\|_2^6(s)ds\} \le$$

$$\le \frac{\mu}{\mu_0}\|\nabla u_0\|_2^2 + T\sup_{t\in(0,\tau)}\sup_{s\in(0,t)} \|\nabla u^m\|_2^2(s) + \overline{C}\sup_{t\in(0,\tau)}\int_0^t \|\nabla u^m\|_2^6(s)ds$$

and analogously

$$\left(1 - \frac{5}{4}(\|\rho_0\|_\infty + 1)\frac{\varepsilon}{8C\mu_0}\right)\sup_{t\in(0,\tau)}\int_0^t \|\nabla u^m\|_{H^1}^2(s)ds \le$$

$$\le \frac{\mu}{\mu_0}\|\nabla u_0\|_2^2 + T\sup_{t\in(0,\tau)}\sup_{s\in(0,t)} \|\nabla u^m\|_2^2(s) + \overline{C}\sup_{t\in(0,\tau)}\int_0^t \|\nabla u^m\|_2^6(s)ds$$

and also

$$\frac{\mu}{\mu_0}\sup_{t\in(0,\tau)}\|\nabla u^m\|_2^2(t) \le \frac{\mu}{\mu_0}\|\nabla u_0\|_2^2 + T\sup_{t\in(0,\tau)}\sup_{s\in(0,t)} \|\nabla u^m\|_2^2(s) + \overline{C}\sup_{t\in(0,\tau)}\int_0^t \|\nabla u^m\|_2^6(s)ds$$

On the other hand we have[12]

$$\sup_{t\in(0,\tau)}\left\{\int_0^t \|\sqrt{\rho^m}u_t^m\|_2^2(s)ds + \left(1 - \frac{5}{4}(\|\rho_0\|_\infty + 1)\frac{\varepsilon}{8C\mu_0}\right)\int_0^t \|\nabla u^m\|_{H^1}^2(s)ds + \frac{\mu}{\mu_0}\sup_{s\in(0,\tau)}\|\nabla u^m\|_2^2(s)\right\} \le$$

[12]Notice that

$$\sup_{s\in(0,t)}\|\nabla u\|_2^2(s) \le \sup_{s\in(0,\tau)}\|\nabla u\|_2^2(s)$$

since $(0,t) \subseteq (0,\tau)$. So

$$\sup_{t\in(0,\tau)}\sup_{s\in(0,t)}\|\nabla u\|_2^2(s) \le \sup_{s\in(0,\tau)}\|\nabla u\|_2^2(s)$$

$$\le \sup_{t\in(0,\tau)} \int_0^t \|\sqrt{\rho^m} u_t^m\|_2^2(s)ds + \left(1 - \frac{5}{4}(\|\rho_0\|_\infty + 1)\frac{\varepsilon}{8C\mu_0}\right) \sup_{t\in(0,\tau)} \int_0^t \|\nabla u^m\|_{H^1}^2(s)ds + \frac{\mu}{\mu_0} \sup_{s\in(0,\tau)} \|\nabla u^m\|_2^2(s) \le$$

$$\le 3\ \{\frac{\mu}{\mu_0}\|\nabla u_0\|_2^2 + T \sup_{t\in(0,\tau)} \sup_{s\in(0,t)} \|\nabla u^m\|_2^2(s) + \overline{C} \sup_{t\in(0,\tau)} \int_0^t \|\nabla u^m\|_2^6(s)ds\} \le$$

$$\le 3\frac{\mu}{\mu_0}\|\nabla u_0\|_2^2 + 3T \sup_{s\in(0,\tau)} \|\nabla u^m\|_2^2(s) + 3\overline{C} \sup_{t\in(0,\tau)} \int_0^t \|\nabla u^m\|_2^6(s)ds$$

The first left side is the supremum in $(0,\tau)$ of a continuous function[13] of t, thanks to the regularity of the solutions and their derivatives and to the compactness of the domain. So, the supremum equals the maximum on the compact $[0,\tau]$, and thus

$$\int_0^\tau \|\sqrt{\rho^m} u_t^m\|_2^2(s)ds + \left(1 - \frac{5}{4}(\|\rho_0\|_\infty + 1)\frac{\varepsilon}{8C\mu_0}\right) \int_0^\tau \|\nabla u^m\|_{H^1}^2(s)ds + \frac{\mu}{\mu_0} \sup_{s\in(0,\tau)} \|\nabla u^m\|_2^2(s) \le$$

$$\le \sup_{t\in(0,\tau)} \left\{ \int_0^t \|\sqrt{\rho^m} u_t^m\|_2^2(s)ds + \left(1 - \frac{5}{4}(\|\rho_0\|_\infty + 1)\frac{\varepsilon}{8C\mu_0}\right) \int_0^t \|\nabla u^m\|_{H^1}^2(s)ds + \frac{\mu}{\mu_0} \sup_{s\in(0,\tau)} \|\nabla u^m\|_2^2(s) \right\} \le$$

$$\le 3\frac{\mu}{\mu_0}\|\nabla u_0\|_2^2 + 3T \sup_{s\in(0,\tau)} \|\nabla u^m\|_2^2(s) + 3\overline{C} \sup_{t\in(0,\tau)} \int_0^t \|\nabla u^m\|_2^6(s)ds \le$$

$$\overset{14}{\le} 3\frac{\mu}{\mu_0}\|\nabla u_0\|_2^2 + 3T \sup_{s\in(0,\tau)} \|\nabla u^m\|_2^2(s) + 3\overline{C} \int_0^\tau \|\nabla u^m\|_2^6(s)ds$$

Now we can derive the estimate

$$\int_0^\tau \|\sqrt{\rho^m} u_t^m\|_2^2(s)ds + \left(1 - \frac{5}{4}(\|\rho_0\|_\infty + 1)\frac{\varepsilon}{8C\mu_0}\right) \int_0^\tau \|\nabla u^m\|_{H^1}^2(s)ds +$$

$$+ \left(\frac{\mu}{\mu_0} - 3T\right) \sup_{s\in(0,\tau)} \|\nabla u^m\|_2^2(s) \le 3\frac{\mu}{\mu_0}\|\nabla u_0\|_2^2 + 3\overline{C} \int_0^\tau \|\nabla u^m\|_2^6(s)ds$$

Remember that

$$\overline{C} := \frac{5}{4}(\|\rho_0\|_\infty + 1)\frac{2C}{\varepsilon\mu_0}\Lambda_0^2$$

Moreover, we choose $\varepsilon > 0$ such that

$$1 - \frac{5}{4}(\|\rho_0\|_\infty + 1)\frac{\varepsilon}{8C\mu_0} > \frac{1}{2} \iff \frac{16C\mu_0}{5(\|\rho_0\|_\infty + 1)} > \varepsilon \tag{11.42}$$

[13] The integrale of summable functions is continuous.

[14] Since $\tau > t$ we have

$$\int_0^t \|\nabla u\|_2^2(s)ds \le \int_0^\tau \|\nabla u\|_2^2(s)ds$$

and so

$$\sup_{t\in(0,\tau)} \int_0^t \|\nabla u\|_2^2(s)ds \le \int_0^\tau \|\nabla u\|_2^2(s)ds$$

With this choice, it follows that[15]

$$\frac{1}{2}\int_0^\tau \|\sqrt{\rho^m}u_t^m\|_2^2 ds + \frac{1}{2}\int_0^\tau \|\nabla u^m\|_{H^1}^2 ds + \frac{1}{2}\sup_{t\in(0,\tau)} \|\nabla u^m\|_2^2(t) \leq$$

$$\leq 3\frac{\mu}{\mu_0}\|\nabla u_0\|_2^2 + 3\overline{C}\int_0^\tau \|\nabla u^m\|_2^6(s)ds$$

So, doubling the inequality,

$$\int_0^\tau \|\sqrt{\rho}^m u_t^m\|_2^2 ds + \int_0^\tau \|\nabla u^m\|_{H^1}^2 ds + \sup_{t\in(0,\tau)} \|\nabla u^m\|_2^2(t) \leq 6\frac{\mu}{\mu_0}\|\nabla u_0\|_2^2 + 6\overline{C}\int_0^\tau \|\nabla u^m\|_2^6(s)ds \tag{11.43}$$

Finally we take

$$\mathcal{K} := \max\{6\frac{\mu}{\mu_0}, 6\overline{C}\}$$

and we can rewrite the inequality as

$$\boxed{\int_0^\tau \|\sqrt{\rho}^m u_t^m\|_2^2 ds + \int_0^\tau \|\nabla u^m\|_{H^1}^2 ds + \sup_{t\in(0,\tau)} \|\nabla u^m\|_2^2(t) \leq \mathcal{K}\|\nabla u_0\|_2^2 + \mathcal{K}\int_0^\tau \|\nabla u^m\|_2^6(s)ds} \tag{11.44}$$

Remark 11.14. The constant μ_0 is defined as

$$\mu_0 := \frac{1}{2}\min\left\{\frac{1}{4}, \frac{1}{8\tilde{C}^2(\|\rho_0\|_\infty + 1)}, \frac{2\mu}{1+6T}\right\}$$

depend of initial data ρ_0, the viscosity μ, the time T and the constant $\tilde{C}$. The constant $\tilde{C}$, in turn, is

$$\tilde{C} = \frac{\sqrt{2}}{\mu}\underline{C}$$

where $\underline{C}$ is such that

$$|\phi|_{2,2} \leq \underline{C}\|A\phi\|_2$$

for all $\phi \in X$. On the other hand, we have

$$\overline{C} := \frac{5}{4}(\|\rho_0\|_\infty + 1)\frac{2C}{\varepsilon\mu_0}\Lambda_0^2$$

where $C > 0$ is arbitrary and $\varepsilon > 0$ depends on μ_0 and ρ_0 as in the relation (11.42). Finally, Λ_0 is given by Sobolev inequalities.

[15]Also remembering that

$$\mu_0 := \frac{1}{2}\min\{\frac{1}{4}, \frac{1}{8\tilde{C}^2(\|\rho_0\|_\infty + 1)}, \frac{2\mu}{1+6T}\} \leq \frac{2\mu}{1+6T}$$

and so

$$\frac{\mu}{\mu_0} - 3T \geq \frac{1+6T}{2} - 3T = \frac{1}{2}$$

11.4.5 Second final estimate

Proposition 11.10. *Let Ω be a bounded domain in $\mathbb{R}^3$. Consider the Navier-Stokes problem over Ω as in proposition 11.1. Let $\overline{\rho}_0 \in C^1(\overline{\Omega})$ and $T > 0$. Let $\rho^m \in C^1([0,T] \times \overline{\Omega})$ and $u^m \in C^1([0,T]; X^m)$ the approximate solutions built in proposition 11.1. Then there holds the following estimate*

$$\int_\Omega \rho^m |u_t^m|^2(t)dx + \int_\tau^t \int_\Omega |\nabla u_t^m|^2 dxds \le C' + K' \int_\Omega \rho^m |u_t^m|^2(\tau)dx + C' \left(\int_0^t \|\nabla u^m\|_2^6(s)ds \right)^3 \tag{11.45}$$

for every $t \in [0,T]$, $\tau \in (0,t)$. Here C', K' are generic positive constants depending only on $\|\rho_0\|_{L^\infty}$, $\|\nabla u_0\|_2$ and T, but independent of δ and m.

Proof. For the goal of doing other estimates, we remember that

$$u_{tt}^m(x,t) \in C([0,T]; X^m)$$

and the continuity holds also for u_t^m. Moreover we have already said that the derivatives with respect time are classical derivatives. Furthermore $\rho^m \in C^1([0,T]; C^1(\overline{\Omega}))$ and the sequence ρ^m is uniformly bounded by $\|\rho_0\|_\infty + 1$.

Thus, we deduce a further estimate. Consider again the equation

$$\int_\Omega \{ \left(\rho^m u_t^m + \rho^m (u^m \cdot \nabla u^m) \right) \cdot \phi + \mu \nabla u^m \cdot \nabla \phi \} \, dx = 0 \quad \forall \phi \in X^m$$

We want to differentiate this relation with respect t. We know from above that u^m, ∇u^m, u_t^m, ∇u_t^m, u_{tt}^m are regular in classical sense in the temporal variable. So, if $\phi \in X^m$,

$$\int_\Omega \{ \left(\rho_t^m u_t^m + \rho^m u_{tt}^m + \rho_t^m (u^m \cdot \nabla u^m) + \rho^m (u^m \cdot \nabla u^m)_t \right) \cdot \phi + \mu \nabla u_t^m \cdot \nabla \phi \} \, dx = 0$$

since the derivatives can pass under the integral sign, having the functions (and their derivatives) integral bounds uniform in the temporal variable[16]. In other terms, it follows that

$$\int_\Omega \{ \left(\rho_t^m u_t^m + \rho^m u_{tt}^m + \rho_t^m (u^m \cdot \nabla u^m) + \rho^m (u^m \cdot \nabla u_t^m) + \rho^m (u_t^m \cdot \nabla u^m) \right) \cdot \phi + \mu \nabla u_t^m \cdot \nabla \phi \} \, dx = 0 \tag{11.46}$$

Choosing $\phi = u_t^m \in X^m$ in (11.46) and reorganizing the expression we get

$$\int_\Omega \{ \frac{\rho^m}{2} \frac{d}{dt} |u_t^m|^2 + \rho^m \left(u^m \cdot \nabla u_t^m + u_t^m \cdot \nabla u^m \right) \cdot u_t^m + \mu |\nabla u_t^m|^2 \} \, dx = - \int_\Omega \{ \rho_t^m \left(u_t^m + u^m \cdot \nabla u^m \right) \cdot u_t^m \} \, dx$$

Remembering that

$$u^m \cdot \nabla \rho^m = -\rho_t^m \tag{11.47}$$

[16] It is enough to estimate the temporal part with its maximum in $[0,T]$. The remaining function depends on x and is a linear combination of element of the basis. The integrability of this functions follow from the fact that the elements of the basis (and their first derivatives) are summable in the bounded domain Ω.

and that

$$\nabla|u_t^m|^2 = 2(\nabla u_t^m)u_t^m$$

one gets

$$\int_\Omega \{[\frac{\rho^m}{2}\frac{d}{dt}|u_t^m|^2 + \frac{\rho^m}{2}(u^m \cdot \nabla|u_t^m|^2) + \rho^m(u_t^m \cdot \nabla u^m \cdot u_t^m) + \mu|\nabla u_t^m|^2\}\ dx =$$

$$\overset{(11.47)}{=} \int_\Omega \{(u^m \cdot \nabla\rho^m)\big(u_t^m + u^m \cdot \nabla u^m\big) \cdot u_t^m\}\ dx$$

Moreover, thanks to the regularity of the first derivatives, and using the divergence theorem (or integration by parts), we get

$$\int_\Omega \frac{\rho^m}{2}(u^m \cdot \nabla|u_t^m|^2)\ dx = \int_\Omega (\rho^m u^m)\ \cdot\ \nabla(\frac{1}{2}|u_t^m|^2)dx =$$

$$= \int_\Omega \nabla\cdot(\frac{\rho^m}{2}|u_t^m|^2 u^m)dx - \int_\Omega \frac{1}{2}|u_t^m|^2\ \nabla\cdot(\rho^m u^m)dx =$$

$$= -\int_\Omega \frac{1}{2}|u_t^m|^2\ \nabla\cdot(\rho^m u^m)dx \overset{(11.47)}{=} \int_\Omega \rho_t^m \frac{1}{2}|u_t^m|^2 dx$$

where has been used the generalized divergence theorem and the fact that $u^m = 0$ on $\partial\Omega$.

Also observe that

$$\partial_t(\rho^m\frac{1}{2}|u_t^m|^2) = \rho_t^m\frac{1}{2}|u_t^m|^2 + \rho^m\partial_t(\frac{1}{2}|u_t^m|^2)$$

So, usign these relations in the main equation, we get

$$\int_\Omega \left(\partial_t(\frac{\rho^m}{2}|u_t^m|^2) + \rho^m(u_t^m \cdot \nabla u^m \cdot u_t^m) + \mu|\nabla u_t^m|^2\right)\ dx =$$

$$= \int_\Omega \{(u^m \cdot \nabla\rho^m)\big(u_t^m + u^m \cdot \nabla u^m\big) \cdot u_t^m\}\ dx$$

Notice furthermore that

$$\nabla\cdot(\rho^m u^m) = \nabla\rho^m \cdot u^m + \rho^m\nabla\cdot u^m = \nabla\rho^m \cdot u^m$$

since $\nabla \cdot u^m = 0$ by construction. So

$$\frac{d}{dt}\int_\Omega \frac{\rho^m}{2}|u_t^m|^2 dx + \mu\int_\Omega |\nabla u_t^m|^2 dx =$$

$$= \int_\Omega \nabla\cdot(\rho^m u^m)\big(u_t^m + u^m\cdot\nabla u^m\big)\cdot u_t^m dx - \int_\Omega \rho^m(u_t^m \cdot \nabla u^m \cdot u_t^m)\ dx$$

using the regularity of the derivatives and the fact that the closure of Ω is compact. Now we have to deal with the right side of the latter equation. Using again integration by parts, we have

$$\int_\Omega \nabla\cdot(\rho^m u^m)\big(u_t^m + u^m\cdot\nabla u^m\big)\cdot u_t^m dx =$$

$$= \int_\Omega \nabla \cdot \{\rho^m u^m \left((u_t^m + u^m \cdot \nabla u^m)\cdot u_t^m\right)\} \, dx - \int_\Omega (\rho^m u^m)\cdot \nabla\left((u_t^m + u^m \cdot \nabla u^m)\cdot u_t^m\right) \, dx =$$

$$= - \int_\Omega (\rho^m u^m) \cdot \nabla\left((u_t^m + u^m \cdot \nabla u^m) \cdot u_t^m\right) \, dx$$

where the whole divergence piece is the integration of a trace piece; thanks to the fact that all the functions are continuous on $\overline{\Omega}$ and that $u^m = 0$ on the boundary $\partial\Omega$, we have the last equality. We use now some derivation rules. In particular

$$\nabla\{[u_t^m + u^m \cdot \nabla u^m] \cdot u_t^m\} = \nabla|u_t^m|^2 + \nabla[u^m \cdot \nabla u^m \cdot u_t^m] =$$

$$= \nabla|u_t^m|^2 + \nabla[((\nabla u^m)^T \cdot u^m) \cdot u_t^m] =$$

$$= 2u_t^m \cdot \nabla u_t^m + u_t^m \cdot (\nabla\{(\nabla u^m)^T \cdot u^m\}) + [(\nabla u^m)^T \cdot u^m] \cdot \nabla u_t^m$$

Using that

$$\nabla(Ab) = b \; \nabla(A^T) + A\nabla b$$

we have

$$\nabla((\nabla u^m)^T \cdot u^m) = u^m \; \nabla(\nabla u^m) + (\nabla u^m)^T \cdot \nabla u^m$$

So

$$|\nabla\{[u_t^m + u^m \cdot \nabla u^m] \cdot u_t^m\}| \le 2|u_t^m||\nabla u_t^m| + |\{\nabla((\nabla u^m)^T \cdot u^m)\} \cdot u_t^m| + |\nabla u_t^m||\nabla u^m||u^m| \le$$

$$\le 2|u_t^m||\nabla u_t^m| + |u_t^m||u^m||\nabla^2 u^m| + |u_t^m||\nabla u^m|^2 + |\nabla u_t^m||\nabla u^m||u^m|$$

Hence

$$\left|\int_\Omega \nabla \cdot (\rho^m u^m)[u_t^m + u^m \cdot \nabla u^m] \cdot u_t^m \; dx\right| \le$$

$$\le 2\int_\Omega \rho^m |u_t^m||\nabla u_t^m||u^m| \; dx + \int_\Omega \rho^m |u_t^m||u^m|^2|\nabla^2 u^m| \; dx +$$

$$+ \int_\Omega \rho^m |u^m||u_t^m||\nabla u^m|^2 dx + \int_\Omega \rho^m |\nabla u_t^m||\nabla u^m||u^m|^2 dx$$

Moreover

$$\left|\int_\Omega \rho^m \left(\nabla u^m \cdot u_t^m\right) \cdot u_t^m \; dx\right| \le \int_\Omega \rho^m |\nabla u^m||u_t^m|^2 dx$$

Finally we obtain the estimate

$$\frac{d}{dt}\int_\Omega \frac{\rho^m}{2}|u_t^m|^2 dx + \mu\int_\Omega |\nabla u_t^m|^2 dx =$$

$$= \int_\Omega \nabla \cdot (\rho^m u^m)\left(u_t^m + u^m \cdot \nabla u^m\right) \cdot u_t^m dx - \int_\Omega \rho^m (u_t^m \cdot \nabla u^m \cdot u_t^m) \; dx \le$$

$$\le \left|\int_\Omega \nabla \cdot (\rho^m u^m)\left(u_t^m + u^m \cdot \nabla u^m\right) \cdot u_t^m dx - \int_\Omega \rho^m (u_t^m \cdot \nabla u^m \cdot u_t^m) \; dx\right| \le$$

$$\le \int_\Omega |\nabla \cdot (\rho^m u^m)\left(u_t^m + u^m \cdot \nabla u^m\right) \cdot u_t^m| \; dx + \int_\Omega |\rho^m (u_t^m \cdot \nabla u^m \cdot u_t^m)| \; dx \le$$

$$\leq 2\int_\Omega \rho^m |u_t^m||\nabla u_t^m||u^m|\ dx + \int_\Omega \rho^m |u_t^m||u^m|^2|\nabla^2 u^m|\ dx +$$

$$+\int_\Omega \rho^m |u^m||u_t^m||\nabla u^m|^2 dx + \int_\Omega \rho^m |\nabla u_t^m||\nabla u^m||u^m|^2 dx + \int_\Omega \rho^m |\nabla u^m||u_t^m|^2 dx \quad (11.48)$$

We now want to obtain further estimates involving some Lebesgue and Sobolev norms. For this goal, we rename the last five pieces. We define

$$I_1 := 2\int_\Omega \rho^m |u^m||u_t^m||\nabla u_t^m| dx \qquad I_2 := \int_\Omega \rho^m |u^m||u_t^m||\nabla u^m|^2 dx$$

$$I_3 := \int_\Omega \rho^m |u^m|^2 |u_t^m||\nabla^2 u^m| dx \qquad I_4 := \int_\Omega \rho^m |u^m|^2 |\nabla u_t^m||\nabla u^m| dx$$

$$I_5 := \int_\Omega \rho^m |\nabla u^m||u_t^m|^2 dx$$

We will do massive use of well-known inequalities.

For the sake of semplicity, we deduce estimates for a pair of sulutions (u, ρ), avoiding the apex m.

1. First of all, we have

$$2\int_\Omega \rho |u||u_t||\nabla u_t| dx = 2\int_\Omega \sqrt{\rho}|u|\sqrt{\rho}|u_t||\nabla u_t| dx \leq 2\|\rho\|_\infty^{\frac{1}{2}} \|u\|_6 \|\sqrt{\rho} u_t\|_3 \|\nabla u_t\|_2$$

using the generalization of the Hölder inequality. Now, using the Hölder interpolated inequality, we have

$$\|\sqrt{\rho} u_t\|_3 \leq \|\sqrt{\rho} u_t\|_2^{\frac{1}{2}} \|\sqrt{\rho} u_t\|_6^{\frac{1}{2}} \quad (11.49)$$

So

$$I_1 \leq 2\|\rho\|_\infty^{\frac{1}{2}} \|u\|_6 \|\sqrt{\rho} u_t\|_2^{\frac{1}{2}} \|\sqrt{\rho} u_t\|_6^{\frac{1}{2}} \|\nabla u_t\|_2$$

But we can do further estimates. Consider that

$$\|\sqrt{\rho} u_t\|_6^{\frac{1}{2}} = \left(\int_\Omega \rho^3 |u_t|^6 dx\right)^{\frac{1}{12}} \leq \|\rho\|_\infty^{\frac{1}{4}} \|u_t\|_6^{\frac{1}{2}}$$

On the other hand, by Sobolev inequality, we get

$$\|u_t\|_6 \leq C_1 \|\nabla u_t\|_2$$

$$\|u\|_6 \leq C_2 \|\nabla u\|_2$$

where the two constants can be choosen independentely by the domain because $u \in H_0^1(\Omega)$.

So we have

$$I_1 \leq C' \|\rho\|_\infty^{\frac{3}{4}} \|\nabla u_t\|_2^{\frac{3}{2}} \|\nabla u\|_2 \|\sqrt{\rho} u_t\|_2^{\frac{1}{2}}$$

where $C' := 2\sqrt{C_1}C_2$. Finally, we use the parametric Young's inequality, with $p = \frac{4}{3}$, $q = 4$, $a = \|\nabla u_t\|_2^{\frac{3}{2}}$ and $b = C'\|\rho\|_\infty^{\frac{3}{4}}\|\nabla u\|_2\|\sqrt{\rho}u_t\|_2^{\frac{1}{2}}$. The result is

$$\boxed{I_1 \leq \varepsilon\|\nabla u_t\|_2^2 + C_\varepsilon\|\nabla u\|_2^4\|\sqrt{\rho}u_t\|_2^2}$$

where $\left(\|\rho\|_\infty^{\frac{3}{4}}\right)^4 = \|\rho\|_\infty^3 = \|\overline{\rho}_0\|_\infty^3 \leq (\|\rho_0\|_\infty + 1)^3$ has been included in the constant C_ε.

2. Now we have, also using the generalized version of Hölder,

$$\int_\Omega \rho|u||u_t||\nabla u|^2 dx \leq \|\rho\|_\infty \int_\Omega |u||u_t||\nabla u||\nabla u| dx \leq$$

$$\leq \|\rho\|_\infty\|u\|_6\|u_t\|_6\|\nabla u\|_6\|\nabla u\|_2$$

Again, thanks to Sobolev inequality, and using (11.40),

$$\|u\|_6 \leq C_1\|\nabla u\|_2, \quad \|u_t\|_6 \leq C_2\|\nabla u_t\|_2, \quad \|\nabla u\|_6 \leq C_3\|\nabla u\|_{H^1}$$

Here C_1 and C_2 are independent of the domain since $u, u_t \in H_0^1(\Omega)$. The constant C_3 depends a priori on the domain. So

$$I_2 \leq C\|\rho\|_\infty\|\nabla u\|_2\|\nabla u_t\|_2\|\nabla u\|_{H^1}\|\nabla u\|_2$$

where $C := C_1C_2C_3$. Finally, again by the parametric Young's inequality, with $p = q = 2$ and $a = \|\nabla u_t\|_2$, $b = C\|\rho\|_\infty\|\nabla u\|_2^2\|\nabla u\|_{H^1}$, we have

$$\boxed{I_2 \leq \varepsilon\|\nabla u_t\|_2^2 + C_\varepsilon\|\nabla u\|_2^4\|\nabla u\|_{H^1}^2}$$

where also this time $\|\rho\|_\infty$ has been replaced with $\|\rho_0\|_\infty + 1$ as above.

3. This point is similar to the previous. We have

$$\int_\Omega \rho|u|^2|u_t||\nabla^2 u| dx \leq \|\rho\|_\infty \int_\Omega |u|^2|u_t||\nabla^2 u| dx = \|\overline{\rho}_0\|_\infty \int_\Omega |u||u||u_t||\nabla^2 u| dx \leq$$

$$\leq (\|\rho_0\|_\infty + 1)\|u\|_6^2\|u_t\|_6\|\nabla^2 u\|_2$$

and using

$$\|u\|_6 \leq C_1\|\nabla u\|_2 \qquad \|u_t\|_6 \leq C_2\|\nabla u_t\|_2$$

we have

$$I_3 \leq (\|\rho_0\|_\infty + 1)C_1^2\|\nabla u\|_2^2 C_2\|\nabla u_t\|_2\|\nabla^2 u\|_2$$

Again C_1 and C_2 don't depend on the domain. Again by parametric Young's inequality, with $p = q = 2$ and $a = \|\nabla u_t\|_2$, $b = C_1^2C_2(\|\rho_0\|_\infty + 1)\|\nabla u\|_2^2\|\nabla^2 u\|_2$, we have

$$\boxed{I_3 \leq \varepsilon\|\nabla u_t\|_2^2 + C_\varepsilon\|\nabla u\|_2^4\|\nabla^2 u\|_2^2}$$

4. We have, always by Hölder,

$$\int_\Omega \rho|u|^2|\nabla u_t||\nabla u|dx \leq \|\rho\|_\infty \int_\Omega |u|^2|\nabla u_t||\nabla u|dx = \|\overline{\rho}_0\|_\infty \int_\Omega |u||u||\nabla u_t||\nabla u|dx \leq$$

$$\leq (\|\rho_0\|_\infty + 1)\|u\|_6^2\|\nabla u_t\|_2\|\nabla u\|_6$$

So, being, by Sobolev inequality,

$$\|u\|_6 \leq C_1\|\nabla u\|_2$$

and again by (11.40)

$$\|\nabla u\|_6 \leq C_2\|\nabla u\|_{H^1}$$

Here the situation is similar to a point above: C_1 is independent of the domain, while C_2 depends on the domain and this dipendence can be specified, as will be done in future.
Thus

$$I_4 \leq (\|\rho_0\|_\infty + 1)C_1^2\|\nabla u\|_2^2\|\nabla u_t\|_2 C_2\|\nabla u\|_{H^1}$$

By the usual parametric Young's inequality, we have, with $p = q = 2$ and $a = \|\nabla u_t\|_2$, $b = C_1^2 C_2(\|\rho_0\|_\infty + 1)\|\nabla u\|_2^2\|\nabla u\|_{H^1}$,

$$\boxed{I_4 \leq \varepsilon\|\nabla u_t\|_2^2 + C_\varepsilon\|\nabla u\|_2^4\|\nabla u\|_{H^1}^2}$$

5. We finally deal with the last piece. We have, also by the interpolated Hölder's inequality in (11.49),

$$\int_\Omega \rho|\nabla u||u_t|^2 dx = \int_\Omega \sqrt{\rho}|\nabla u|[\sqrt{\rho}|u_t|]|u_t|dx \leq \|\overline{\rho}_0\|_\infty^{\frac{1}{2}} \int_\Omega |\nabla u|[\sqrt{\rho}|u_t|]|u_t|dx \leq$$

$$\leq \|\overline{\rho}_0\|_\infty^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_3\|\nabla u\|_2\|u_t\|_6 \leq (\|\rho_0\|_\infty + 1)^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_2^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_6^{\frac{1}{2}}\|\nabla u\|_2\|u_t\|_6$$

and being

$$\|\sqrt{\rho}u_t\|_6^{\frac{1}{2}} \leq \|\overline{\rho}_0\|_\infty^{\frac{1}{4}}\|u_t\|_6^{\frac{1}{2}} \leq (\|\rho_0\|_\infty + 1)^{\frac{1}{4}}\|u_t\|_6^{\frac{1}{2}}$$

and

$$\|u_t\|_6 \leq C_1\|\nabla u_t\|_2$$

where the constant C_1 is independent of the domain.
We have

$$I_5 \leq (\|\rho_0\|_\infty + 1)^{\frac{3}{4}}\|\sqrt{\rho}u_t\|_2^{\frac{1}{2}}\sqrt{C_1}\|\nabla u_t\|_2^{\frac{1}{2}}\|\nabla u\|_2 C_1\|\nabla u_t\|_2 =$$

$$= C_2(\|\rho_0\|_\infty + 1)^{\frac{3}{4}}\|\sqrt{\rho}u_t\|_2^{\frac{1}{2}}\|\nabla u_t\|_2^{\frac{3}{2}}\|\nabla u\|_2$$

where $C_2 = C_1^{\frac{3}{2}}$.
Finally, by Young's parametric inequality, with $p = \frac{4}{3}$, $q = 4$, $a = \|\nabla u_t\|_2^{\frac{3}{2}}$ and $b = C_2(\|\rho_0\|_\infty + 1)^{\frac{3}{4}}\|\sqrt{\rho}u_t\|_2^{\frac{1}{2}}\|\nabla u\|_2$, we have

$$\boxed{I_5 \leq \varepsilon\|\nabla u_t\|_2^2 + C_\varepsilon\|\sqrt{\rho}u_t\|_2^2\|\nabla u\|_2^4}$$

We continue for a moment to use this notation, avoiding the apex m. Equation (11.4.5) can be written as

$$\frac{d}{dt}\int_\Omega \frac{\rho}{2}|u_t|^2dx + \mu\int_\Omega |\nabla u_t|^2 dx \le \sum_{j=1}^5 I_j$$

More explicitly, using the estimates just deduced,

$$\sum_{j=1}^5 I_j \le [\varepsilon\|\nabla u_t\|_2^2 + C_{1,\varepsilon}\|\nabla u\|_2^4\|\sqrt{\rho}u_t\|_2^2] + [\varepsilon\|\nabla u_t\|_2^2 + C_{2,\varepsilon}\|\nabla u\|_2^4\|\nabla u\|_{H^1}^2]+$$

$$+[\varepsilon\|\nabla u_t\|_2^2 + C_{3,\varepsilon}\|\nabla u\|_2^4\|\nabla^2 u\|_2^2] + [\varepsilon\|\nabla u_t\|_2^2 + C_{4,\varepsilon}\|\nabla u\|_2^4\|\nabla u\|_{H^1}^2]+$$
$$+[\varepsilon\|\nabla u_t\|_2^2 + C_{5,\varepsilon}\|\sqrt{\rho}u_t\|_2^2\|\nabla u\|_2^4] =$$
$$= 5\varepsilon\|\nabla u_t\|_2^2 + \|\nabla u\|_2^4\{(C_{1,\varepsilon}+C_{5,\varepsilon})\|\sqrt{\rho}u_t\|_2^2 + (C_{2,\varepsilon}+C_{4,\varepsilon})\|\nabla u\|_{H^1}^2 + C_{3,\varepsilon}\|\nabla^2 u\|_2^2\} \le$$
$$\le 5\varepsilon\|\nabla u_t\|_2^2 + \|\nabla u\|_2^4\{(C_{1,\varepsilon}+C_{5,\varepsilon})\|\sqrt{\rho}u_t\|_2^2 + (C_{2,\varepsilon}+C_{4,\varepsilon}+C_{3,\varepsilon})\|\nabla u\|_{H^1}^2\}$$

So

$$\frac{d}{dt}\int_\Omega \frac{\rho}{2}|u_t|^2dx + \mu\|\nabla u_t\|_2^2 \le$$
$$\le 5\varepsilon\|\nabla u_t\|_2^2 + \|\nabla u\|_2^4\{(C_{1,\varepsilon}+C_{5,\varepsilon})\|\sqrt{\rho}u_t\|_2^2 + (C_{2,\varepsilon}+C_{4,\varepsilon}+C_{3,\varepsilon})\|\nabla u\|_{H^1}^2\}$$

that is

$$\frac{d}{dt}\int_\Omega \frac{\rho}{2}|u_t|^2dx + (\mu-5\varepsilon)\|\nabla u_t\|_2^2 \le$$
$$\le \|\nabla u\|_2^4\{(C_{1,\varepsilon}+C_{5,\varepsilon})\|\sqrt{\rho}u_t\|_2^2 + (C_{2,\varepsilon}+C_{4,\varepsilon}+C_{3,\varepsilon})\|\nabla u\|_{H^1}^2\}$$

Since the inequality holds for every $\varepsilon > 0$, we can choose $\varepsilon = \frac{\mu}{10}$ and we get

$$\boxed{\frac{d}{dt}\int_\Omega \rho|u_t|^2dx + \mu\|\nabla u_t\|_2^2 \le \|\nabla u\|_2^4\{C_1\|\sqrt{\rho}u_t\|_2^2 + C_2\|\nabla u\|_{H^1}^2\}}$$

Integrating over the interval (τ, t), with $\tau > 0$, we have[17]

$$\int_\Omega \rho|u_t|^2(t)dx - \int_\Omega \rho|u_t|^2(\tau)dx + \mu\int_\tau^t\int_\Omega |\nabla u_t|^2dxds \le C\int_\tau^t \|\nabla u\|_2^4\{\|\sqrt{\rho}u_t\|_2^2 + \|\nabla u\|_{H^1}^2\}ds$$

and so

$$\int_\Omega \rho|u_t|^2(t)dx + \mu\int_\tau^t\int_\Omega |\nabla u_t|^2dxds \le C\int_\tau^t \|\nabla u\|_2^4\{\|\sqrt{\rho}u_t\|_2^2 + \|\nabla u\|_{H^1}^2\}ds + \int_\Omega \rho|u_t|^2(\tau)dx \tag{11.50}$$

Now we deal with the third piece. We get

$$\int_\tau^t \|\nabla u\|_2^4\{\|\sqrt{\rho}u_t\|_2^2 + \|\nabla u\|_{H^1}^2\}ds = \int_\tau^t \left(\|\nabla u\|_2^2\right)^2\{\|\sqrt{\rho}u_t\|_2^2 + \|\nabla u\|_{H^1}^2\}\ ds \le$$
$$\le \left(\sup_{s\in(0,t)} \|\nabla u\|_2^2(s)\right)^2 \int_0^t \{\|\sqrt{\rho}u_t\|_2^2 + \|\nabla u\|_{H^1}^2\}\ ds$$

[17] $C := \max\{C_1, C_2\}$.

Remember now the inequality (11.43) deduced in the previous section, i.e.

$$\int_0^t \|\sqrt{\rho}u_t\|_2^2 ds \ + \int_0^t \|\nabla u\|_{H^1}^2 ds + \sup_{s\in(0,t)} \|\nabla u\|_2^2(s) \leq 6\frac{\mu}{\mu_0}\|\nabla u_0\|_2^2 + 6\overline{C}\int_0^t \|\nabla u\|_2^6(s)ds$$

We can read in (11.43) the following inequalities,

$$\int_0^t \|\sqrt{\rho}u_t\|_2^2 ds \ + \int_0^t \|\nabla u\|_{H^1}^2 ds \leq 6\frac{\mu}{\mu_0}\|\nabla u_0\|_2^2 + 6\overline{C}\int_0^t \|\nabla u\|_2^6(s)ds$$

$$\sup_{s\in(0,t)} \|\nabla u\|_2^2(s) \leq 6\frac{\mu}{\mu_0}\|\nabla u_0\|_2^2 + 6\overline{C}\int_0^t \|\nabla u\|_2^6(s)ds$$

and

$$6\frac{\mu}{\mu_0}\|\nabla u_0\|_2^2 + 6\overline{C}\int_0^t \|\nabla u\|_2^6(s)ds \leq M_0 + M_0\int_0^t \|\nabla u\|_2^6(s)ds$$

where M_0 is the maximum of the two constants. We have

$$\int_\tau^t \|\nabla u\|_2^4\{\|\sqrt{\rho}u_t\|_2^2 + \|\nabla u\|_{H^1}^2\}ds \leq \left\{M_0 + M_0\int_0^t \|\nabla u\|_2^6(s)ds\right\}^3 \tag{11.51}$$

Remember now that

$$\frac{(1+y)^3}{1+y^3} \leq 4 \qquad \forall y \geq 0$$

and so

$$M_0^3\left(1 + \int_0^t \|\nabla u\|_2^6(s)ds\right)^3 \leq 4M_0^3\left\{1 + \left(\int_0^t \|\nabla u\|_2^6(s)ds\right)^3\right\}$$

Finally, using (11.50) and (11.51),

$$\int_\Omega \rho|u_t|^2(t)dx + \mu\int_\tau^t\int_\Omega |\nabla u_t|^2 dxds \leq \int_\Omega \rho|u_t|^2(\tau)dx + 4CM_0^3\left\{1 + (\int_0^t \|\nabla u\|_2^6(s)ds)^3\right\} =$$

$$= 4CM_0^3 + \int_\Omega \rho|u_t|^2(\tau)dx + 4CM_0^3\left(\int_0^t \|\nabla u\|_2^6(s)ds\right)^3$$

Looking at a left piece a time, and dividing for μ, the equality can also be rewrite as

$$\int_\Omega \rho|u_t|^2(t)dx \ + \int_\tau^t\int_\Omega |\nabla u_t|^2 dxds \leq$$

$$\leq \left(4 + \frac{4}{\mu}\right)CM_0^3 + \left(1 + \frac{1}{\mu}\right)\int_\Omega \rho|u_t|^2(\tau)dx + \left(4 + \frac{4}{\mu}\right)CM_0^3\left(\int_0^t \|\nabla u\|_2^6(s)ds\right)^3 \equiv$$

$$\equiv C' + K'\int_\Omega \rho|u_t|^2(\tau)dx + C'\left(\int_0^t \|\nabla u\|_2^6(s)ds\right)^3$$

where the constants have been renamed.
We write the just deduced inequality in a line, remembering the dependence on m:

$$\int_\Omega \rho^m|u_t^m|^2(t)dx + \int_\tau^t\int_\Omega |\nabla u_t^m|^2 dxds \leq C' + K'\int_\Omega \rho^m|u_t^m|^2(\tau)dx + C'\left(\int_0^t \|\nabla u^m\|_2^6(s)ds\right)^3 \tag{11.52}$$

that is out thesis. ∎

11.4.6 A third final estimate

This proposition states a slightly different version of the previous one.

Proposition 11.11. *Let Ω be a bounded domain in $\mathbb{R}^3$. Consider the Navier-Stokes problem over Ω as in proposition 11.1. Let $\overline{\rho}_0 \in C^1(\overline{\Omega})$ and $T > 0$. Let $\rho^m \in C^1([0,T] \times \overline{\Omega})$ and $u^m \in C^1([0,T]; X^m)$ the approximate solutions built in proposition 11.1. Then there holds the following estimate*

$$\int_\Omega \rho^m |u_t^m|^2(t)\ dx + \int_0^t \int_\Omega |\nabla u_t^m|^2\ dx\ ds \leq C'' + C''\overline{\mathcal{C}_0}^m + C''\left(\int_0^t \|\nabla u\|_2^6(s)\ ds\right)^3 \tag{11.53}$$

for every $t \in [0,T]$. Here C'' is a generic positive constant depending only on $\|\rho_0\|_{L^\infty}$, $\|\nabla u_0\|_2$ and T, but independent of δ and m. On the other hand

$$\overline{\mathcal{C}_0}^m := \int_\Omega (\overline{\rho}_0)^{-1} |\mu \Delta u^m(0) - \nabla p_0|^2 dx$$

Proof. We consider again the equation

$$\int_\Omega \{(\rho u_t^m + \rho^m u^m \cdot \nabla u^m) \cdot \phi + \mu \nabla u^m \cdot \nabla \phi\}\ dx = 0$$

for $\phi \in X^m$. Choosing $\phi = u_t^m$, we get

$$\int_\Omega \{\rho^m |u_t^m|^2 + \rho^m u^m \cdot \nabla u^m \cdot u_t^m + \mu \nabla u^m \cdot \nabla u_t^m\}\ dx = 0$$

Consider now $p_0 \in H^1(\Omega)$, that is fixed as in proposition 11.1. We have, using integration by parts, that

$$\int_\Omega \nabla p_0 \cdot u_t^m\ dx = \int_\Omega \nabla \cdot (p_0 u_t^m)\ dx - \int_\Omega p_0\ \nabla \cdot u_t^m\ dx = 0$$

since the trace of u_t^m is zero at the boundary of Ω (here we are using the argument in section (4.7.1) with $u^m \in C^1(\overline{\Omega})$) and also $\nabla \cdot u_t^m = 0$ in Ω. So we can rewrite the equality above as

$$\int_\Omega \rho^m |u_t^m|^2 dx = -\int_\Omega \rho^m u^m \cdot \nabla u^m \cdot u_t^m\ dx - \mu \int_\Omega \nabla u^m \cdot \nabla u_t^m\ dx - \int_\Omega \nabla p_0 \cdot u_t^m\ dx$$

Observe that

$$\int_\Omega \nabla u^m \cdot \nabla u_t^m\ dx = -\int_\Omega u_t^m \cdot \Delta u^m\ dx$$

thanks again to the result in section (4.7.1).
So we have

$$\int_\Omega \rho^m |u_t^m|^2 dx = -\int_\Omega \rho^m u^m \cdot \nabla u^m \cdot u_t^m\ dx + \mu \int_\Omega u_t^m \cdot \Delta u^m\ dx - \int_\Omega \nabla p_0 \cdot u_t^m\ dx =$$

$$= \int_\Omega (-\rho^m u^m \cdot \nabla u^m \cdot u_t^m + \mu \Delta u^m \cdot u_t^m - \nabla p_0 \cdot u_t^m)\ dx =$$

$$= \int_\Omega (-\sqrt{\rho^m} u^m \cdot \nabla u^m + \frac{1}{\sqrt{\rho^m}}[\mu\Delta u^m - \nabla p_0]) \cdot (\sqrt{\rho^m} u^m_t)\ dx$$

thanks to the fact that $\rho^m \geq \delta > 0$. In absolute value, we have

$$\int_\Omega \rho^m |u^m_t|^2 dx \leq \int_\Omega |-\sqrt{\rho^m} u^m \cdot \nabla u^m + \frac{1}{\sqrt{\rho^m}}[\mu\Delta u^m - \nabla p_0]||\sqrt{\rho^m} u^m_t|\ dx \leq$$

$$\leq \int_\Omega |\sqrt{\rho^m} u^m \cdot \nabla u^m||\sqrt{\rho^m} u^m_t| dx + \int_\Omega |\frac{1}{\sqrt{\rho^m}}[\mu\Delta u^m - \nabla p_0]||\sqrt{\rho^m} u^m_t| dx \leq$$

$$\leq \int_\Omega |\sqrt{\rho^m}||\nabla u^m||u^m||\sqrt{\rho^m} u^m_t| dx + \int_\Omega |\frac{1}{\sqrt{\rho^m}}[\mu\Delta u^m - \nabla p_0]||\sqrt{\rho^m} u^m_t| dx \leq$$

$$\overset{18}{\leq} \int_\Omega \left(\rho^m |\nabla u^m|^2 |u^m|^2 + \frac{1}{4}\rho^m |u^m_t|^2\right)\ dx + \int_\Omega \left(\frac{1}{\rho^m}|\mu\Delta u^m - \nabla p_0|^2 + \frac{1}{4}\rho^m |u^m_t|^2\right)\ dx =$$

$$= \int_\Omega \rho^m |\nabla u^m|^2 |u^m|^2 dx + \int_\Omega (\rho^m)^{-1} |\mu\Delta u^m - \nabla p_0|^2 dx + \frac{1}{2}\int_\Omega \rho^m |u^m_t|^2 dx$$

So we get

$$\frac{1}{2}\int_\Omega \rho^m |u^m_t|^2 dx \leq \int_\Omega \rho^m |\nabla u^m|^2 |u^m|^2 dx + \int_\Omega (\rho^m)^{-1} |\mu\Delta u^m - \nabla p_0|^2 dx$$

that is

$$\int_\Omega \rho^m |u^m_t|^2 dx \leq 2\Big(\int_\Omega \rho^m |\nabla u^m|^2 |u^m|^2 dx + \int_\Omega (\rho^m)^{-1} |\mu\Delta u^m - \nabla p_0|^2 dx\Big)$$

We can rewrite this inequality at the time τ, that is

$$\int_\Omega \rho^m(\tau) |u^m_t|^2(\tau) dx \leq 2\Big(\int_\Omega \rho^m(\tau) |\nabla u^m|^2(\tau) |u^m|^2(\tau) dx + \int_\Omega (\rho^m)^{-1}(\tau) |\mu\Delta u^m(\tau) - \nabla p_0|^2 dx\Big) \tag{11.54}$$

Our aim is to take the limit for $\tau \to 0^+$.

We have to do some considerations. First of all, for every $\tau \in [0, T]$ we have

$$\rho^m(\tau) \leq \|\rho^m(\tau)\|_\infty = \|\overline{\rho}_0\|_\infty \leq \|\rho_0\|_\infty + 1$$

[18] Using

$$ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}$$

for every $\varepsilon > 0$ and $a, b \geq 0$. This is a particular case of the Young's inequality. So, if $\varepsilon = 2 > 0$,

$$|\sqrt{\rho}||\nabla u||u||\sqrt{\rho} u_t| \leq \frac{\rho |u_t|^2}{4} + \frac{2\rho |\nabla u|^2 |u|^2}{2}$$

and

$$|\frac{1}{\sqrt{\rho}}[\mu\Delta u - \nabla p_0]||\sqrt{\rho} u_t| \leq \frac{\rho |u_t|^2}{4} + \frac{2\rho^{-1} |\mu\Delta u - \nabla p_0|^2}{2}$$

On the other hand we have $(\rho^m)^{-1}(\tau) \leq \frac{1}{\delta}$. Remember now that

$$u_m(x,t) := \sum_{k=1}^{m} \tilde{A}_{mk}(t) w^k(x), \qquad |\tilde{A}_{mk}(t)| \leq |\tilde{A}_m(t)| \leq R \quad \forall t \in [0,T]$$

(here $|\cdot|$ is the usual Euclidean norm). Consequentely we have

$$\nabla u^m(x,t) = \sum_{k=1}^{m} \tilde{A}_{mk}(t) \nabla w^k(x), \qquad \Delta u^m(x,t) = \sum_{k=1}^{m} \tilde{A}_{mk}(t) \Delta w^k(x)$$

and finally

$$u_t^m(x,t) = \sum_{k=1}^{m} \partial_t \tilde{A}_{mk}(t) w^k(x)$$

Moreover we know that $\tilde{A}_m \in C^1([0,T])$. So we find the following bounds:

$$|u_m(x,t)| \leq R\left(\sum_{k=1}^{m} |w^k(x)|\right), \quad |\nabla u^m(x,t)| \leq R\left(\sum_{k=1}^{m} |\nabla w^k(x)|\right),$$

$$|\Delta u^m(x,t)| \leq R\left(\sum_{k=1}^{m} |\Delta w^k(x)|\right), \quad |u_t^m(x,t)| \leq R_0^m \sum_{k=1}^{m} |w^k(x)|$$

where $R_0^m := \max_{[0,T]} |\partial_t \tilde{A}_m|$. Observe that the bounds are uniform in t and are in $L^2(\Omega)$, since $w^k \in H^2(\Omega)$. We now have summable bounds for the integrands above: in fact

$$\rho^m(\tau)|u_t^m|^2(\tau) \leq (\|\rho_0\|_\infty + 1)(R_0^m)^2 \Bigg(\sum_{k=1}^{m} |w^k(x)|\Bigg)^2 \in L^1(\Omega) \tag{11.55}$$

$$\rho^m(\tau)|\nabla u^m|^2(\tau)|u^m|^2(\tau) \leq (\|\rho_0\|_\infty + 1)R^4 \Bigg(\sum_{k=1}^{m} |\nabla w^k(x)|\Bigg)^2 \Bigg(\sum_{k=1}^{m} |w^k(x)|\Bigg)^2 \leq$$

$$\leq (\|\rho_0\|_\infty + 1)R^4 \Bigg(\sum_{k=1}^{m} |\nabla w^k(x)|\Bigg)^2 \Bigg(\sum_{k=1}^{m} \|w^k\|_\infty\Bigg)^2 \in L^1(\Omega)$$

where $\|w^k\|_\infty = \max_{\overline{\Omega}} |w_k|$ since $w^k \in C^1(\overline{\Omega})$.
Finally

$$(\rho^m)^{-1}(\tau)|\mu \Delta u^m(\tau) - \nabla p_0|^2 \leq 2\delta^{-1}\big(\mu^2 |\Delta u^m|^2(\tau) + |\nabla p_0|^2\big) \leq$$

$$\leq 2\delta^{-1}\Bigg(\mu^2 R^2 \Bigg(\sum_{k=1}^{m} |\Delta w^k(x)|\Bigg)^2 + |\nabla p_0|^2\Bigg) \in L^1(\Omega)$$

where the summability is dued to fact that $p_0 \in H^1(\Omega)$ and $\Delta w^k \in L^2(\Omega)$. Since the following limits exist

$$\lim_{\tau \to 0^+} \rho^m(\tau)|u_t^m|^2(\tau) = \overline{\rho}_0 |u_t^m(0)|^2, \quad \lim_{\tau \to 0^+} \rho^m(\tau)|\nabla u^m|^2(\tau)|u^m|^2(\tau) = \overline{\rho}_0 |\nabla u^m|^2(0)|u^m|^2(0)$$

$$\lim_{\tau\to 0^+}(\rho^m)^{-1}(\tau)|\mu\Delta u^m(\tau)-\nabla p_0|^2=(\overline{\rho}_0)^{-1}|\mu\Delta u^m(0)-\nabla p_0|^2 \tag{11.56}$$

where $\overline{\rho}_0\geq\delta>0$. We used the continuity respect with the temporal variable: in particular, we have, in example

$$\|\Delta u^m(\tau)-\Delta u^m(0)\|_2=\|\sum_{k=1}^m\big(\tilde{A}_{mk}(\tau)-\tilde{A}_{mk}(0)\big)\Delta w^k\|_2\leq\sum_{k=1}^m|\tilde{A}_{mk}(\tau)-\tilde{A}_{mk}(0)|\|\Delta w^k\|_2\to 0$$

as $\tau\to 0^+$. This means that, provided that we understand the limit along a sequence, $\Delta u^m(\tau_n)\to\Delta u^m(0)$ almost everywhere in Ω.
So, thanks to the Lebesgue dominated convergence, we have that also exist the limits

$$\lim_{\tau\to 0^+}\int_\Omega\rho^m(\tau)|u_t^m|^2(\tau)\ dx=\int_\Omega\overline{\rho}_0|u_t^m(0)|^2\ dx$$

$$\lim_{\tau\to 0^+}\left(\int_\Omega\rho^m(\tau)|\nabla u^m|^2(\tau)|u^m|^2(\tau)dx\right)=\int_\Omega\overline{\rho}_0|\nabla u^m|^2(0)|u^m|^2(0)\ dx$$

$$\lim_{\tau\to 0^+}\left(\int_\Omega(\rho^m)^{-1}(\tau)|\mu\Delta u^m(\tau)-\nabla p_0|^2dx\right)=\int_\Omega(\overline{\rho}_0)^{-1}|\mu\Delta u^m(0)-\nabla p_0|^2\ dx$$

So, taking the limit both sides in the inequality (11.54), we have

$$\lim_{\tau\to 0^+}\int_\Omega\rho^m(\tau)|u_t^m|^2(\tau)\ dx=\int_\Omega\lim_{\tau\to 0^+}\big(\rho^m(\tau)|u_t^m|^2(\tau)\big)\ dx\leq$$

$$\leq 2\int_\Omega\overline{\rho}_0|\nabla u^m|^2(0)|u^m|^2(0)\ dx+2\int_\Omega(\overline{\rho}_0)^{-1}|\mu\Delta u^m(0)-\nabla p_0|^2\ dx \tag{11.57}$$

We have already proved that

$$\|\nabla u^m\|_2(0)\leq\|\nabla u_0\|_2\qquad\|u^m\|_2(0)\leq\|u_0\|_2$$

for every u^m approximate solution. So, we can estimate the integrals in (11.57) as

$$\int_\Omega\overline{\rho}_0|\nabla u^m|^2(0)|u^m|^2(0)\ dx\leq(\|\rho_0\|_\infty+1)\int_\Omega|\nabla u^m|^2(0)|u^m|^2(0)\ dx\leq$$

$$\leq(\|\rho_0\|+1)\|u^m\|_\infty^2(0)\int_\Omega|\nabla u^m|^2(0)\ dx=(\|\rho_0\|_\infty+1)\|u^m\|_\infty^2(0)\|\nabla u^m\|_2^2(0)\leq$$

$$\leq(\|\rho_0\|_\infty+1)\|u^m\|_\infty^2(0)\|\nabla u_0\|_2^2$$

So, using lemma 9.6, it follows that

$$\|u^m(0)\|_\infty\leq c\big(\|\Delta u^m(0)\|_2\big)^{\frac{3}{4}}\big(\|\nabla u^m(0)\|_2\big)^{\frac{1}{4}}\leq c\big(\|\Delta u^m(0)\|_2\big)^{\frac{3}{4}}\big(\|\nabla u_0\|_2\big)^{\frac{1}{4}}$$

Moreover, we know that

$$\|\Delta u^m(0)\|_2\leq\|\Delta u^m(0)-\Delta u_0\|_2+\|\Delta u_0\|_2$$

Observe now that

$$\|\Delta u^m(0) - \Delta u_0\|_2 \leq \sqrt{5}\|\nabla^2(u^m(0) - u_0)\|_2 \leq \sqrt{5}\|u^m(0) - u_0\|_{H^2} < 1$$

for some $M \in \mathbb{N}$, and every $m \geq M$. In fact we have that $u^m(0) \to u_0$ in X equipped with the H^2 norm. So

$$\|u^m(0)\|_\infty^2 \leq c^2\big(1 + \|\Delta u_0\|_2\big)^{\frac{3}{2}}\|\nabla u_0\|_2^{\frac{1}{2}}$$

for every $m \geq M$. Notice that, defining

$$E_0 := \max\{c^2\big(1 + \|\Delta u_0\|_2\big)^{\frac{3}{2}}\|\nabla u_0\|_2^{\frac{1}{2}},\ \|u^m(0)\|_\infty^2;\ m = 1, ..., M-1\}$$

it holds

$$\|u^m(0)\|_\infty^2 \leq E_0 \qquad \forall m \in \mathbb{N}$$

So E_0 does not depend on m. Thus, we have

$$\int_\Omega \overline{\rho}_0|\nabla u^m|^2(0)|u^m|^2(0)\ dx \leq (\|\rho_0\|_\infty + 1)E_0\|\nabla u_0\|_2^2 \tag{11.58}$$

Remember now the inequality (11.52)

$$\int_\Omega \rho|u_t|^2(t)dx\ + \int_\tau^t \int_\Omega |\nabla u_t|^2 dxds \leq C' + K'\int_\Omega \rho|u_t|^2(\tau)dx + C'\bigg(\int_0^t \|\nabla u\|_2^6(s)ds\bigg)^3$$

So taking a sequence $\tau_n \to 0^+$ as $n \to \infty$, we have

$$\int_\Omega \rho^m|u_t^m|^2(t)dx + \lim_{n\to\infty}\int_{\tau_n}^t\int_\Omega |\nabla u_t^m|^2 dxds \leq$$

$$\leq C' + K'\lim_{n\to\infty}\int_\Omega \rho^m|u_t^m|^2(\tau_n)dx + C'\bigg(\int_0^t \|\nabla u^m\|_2^6(s)\ ds\bigg)^3 \leq \tag{11.59}$$

$$\overset{(11.57)+(11.58)}{\leq} C' + 2K'(\|\rho_0\|_\infty + 1)E_0\|\nabla u_0\|_2^2 + 2K'\bigg(\int_\Omega (\overline{\rho}_0)^{-1}|\mu\Delta u^m(0) - \nabla p_0|^2\ dx\bigg) +$$

$$+C'\bigg(\int_0^t \|\nabla u^m\|_2^6(s)\ ds\bigg)^3 \leq C'' + C''\overline{\mathcal{C}_0}^m + C''(\int_0^t \|\nabla u^m\|_2^6(s)ds)^3$$

where C'' is the maximum between the constants that depend only on the initial data and

$$\overline{\mathcal{C}_0}^m := \int_\Omega (\overline{\rho}_0)^{-1}|\mu\Delta u^m(0) - \nabla p_0|^2 dx$$

Finally, since $\|\nabla u_t^m\|_2^2(s)$ is integrable in time[19] (since $u_t^m \in C([0,T];X^m)$), we have that

$$\int_{\tau_n}^t\int_\Omega |\nabla u_t^m|^2 dxds \to \int_0^t\int_\Omega |\nabla u_t^m|^2 dxds \ \text{ as } n \to \infty$$

[19] Observe that

$$|\|\nabla u_t^m\|_2(s) - \|\nabla u_t^m\|_2(s_0)| \leq \|\nabla u_t^m(s) - \nabla u_t^m(s_0)\|_2 \leq \sum_{k=1}^m |\partial_t\tilde{A}_{mk}(s) - \partial_t\tilde{A}_{mk}(s_0)|\|\nabla w^k\|_2 \to 0$$

as $s \to s_0$. So, the function is continuous.

We finally find

$$\int_\Omega \rho^m |u_t^m|^2(t)dx + \int_0^t \int_\Omega |\nabla u_t^m|^2 dxds \le C'' + C''\overline{\mathcal{C}_0}^m + C''\left(\int_0^t \|\nabla u\|_2^6(s)\, ds\right)^3 \quad (11.60)$$

Remark 11.15. In equation (11.59) we have considered the limit $\lim_{\tau\to 0^+} \int_\Omega \rho^m |u_t^m|^2(\tau)\, dx$ as a number, also estimating it, since we have proved above that the Lebesgue dominate convergence assures us that it is actually a number. See (11.55), (11.56). $\blacksquare$

Remark 11.16. We can also write the estimate as

$$\boxed{\|\nabla u^m\|_2^2(t) \le H + H\int_0^t \|\nabla u^m\|_2^6(s)ds} \quad (11.61)$$

for some constant $H > 0$ and for every $t \in [0, T]$. In fact, rembember equation (11.43)

$$\int_0^t \|\sqrt{\rho^m} u_t^m\|_2^2\, ds + \int_0^t \|\nabla u^m\|_{H^1}^2\, ds + \sup_{s\in(0,t)} \|\nabla u^m\|_2^2(s) \le 6\frac{\mu}{\mu_0}\|\nabla u_0\|_2^2 + 6\overline{C}\int_0^t \|\nabla u^m\|_2^6(s)\, ds$$

It follows that

$$\sup_{s\in(0,t)} \|\nabla u^m\|_2^2(s) \le 6\frac{\mu}{\mu_0}\|\nabla u_0\|_2^2 + 6\overline{C}\int_0^t \|\nabla u^m\|_2^6(s)ds \le H + H\int_0^t \|\nabla u^m\|_2^6(s)\, ds$$

taking H as the maximum of the two constants. Moreover, $\|\nabla u^m\|_2(s)$ is continuous, since

$$|\|\nabla u^m\|_2(s) - \|\nabla u^m\|_2(s_0)| \le \|\nabla u^m(s) - \nabla u^m(s_0)\|_2 \le \sum_{k=1}^m |\tilde{A}_{mk}(s) - \tilde{A}_{mk}(s_0)|\|\nabla w^k\|_2 \to 0$$

as $s \to s_0$. So, since the supremum of a continuous function on an open set is the maximum of the function on the closure of the set, we have

$$\|\nabla u^m(t)\|_2^2 \le \max_{s\in[0,t]} \|\nabla u^m(s)\|_2^2 \le H + H\int_0^t \|\nabla u^m\|_2^6(s)\, ds$$

This estimate will provide, in future arguments, a local time of existence. $\square$

11.4.7 A further regularity estimate

Proposition 11.12. *Let Ω be a bounded domain in $\mathbb{R}^3$. Consider the Navier-Stokes problem over Ω as in proposition 11.1. Let $\overline{\rho}_0 \in C^1(\overline{\Omega})$ and $T > 0$. Let $\rho^m \in C^1([0,T]\times\overline{\Omega})$ and $u^m \in C^1([0,T]; X^m)$ the approximate solutions built in proposition 11.1. Then there holds the following estimate*

$$\sup_{\tau\in(0,t)} \{\|\nabla u^m\|_{H^1}^2 + \|\sqrt{\rho^m} u_t^m\|_2^2\} + \int_0^t \|\nabla u_t^m\|_2^2\, ds \le \hat{H}\overline{\mathcal{C}_0}^m + \hat{H}\exp\left(\hat{H}\int_0^t \|\nabla u^m\|_2^4\, ds\right) \quad (11.62)$$

for every $t \in [0,T]$. Here $\hat{H}$ is a generic positive constant depending only on $\|\rho_0\|_{L^\infty}$, $\|\nabla u_0\|_2$ and T, but independent of δ and m. On the other hand

$$\overline{\mathcal{C}_0}^m := \int_\Omega (\overline{\rho}_0)^{-1}|\mu\Delta u^m(0) - \nabla p_0|^2 dx$$

Proof. We want now to prove this further regularity estimate. We use lemma 1.3. If we look at estimate (11.61) and in lemma 1.3 we choose $f(t) := \|\nabla u^m\|_2^2(t)$ and $f_0 = H$ and $g(s) := H\|\nabla u^m\|_2^4(s)$ and $a = 0$, then

$$\|\nabla u^m\|_2^2(t) \le H \exp\left(H \int_0^t \|\nabla u^m\|_2^4(s) ds\right) \tag{11.63}$$

If τ is fixed and $t < \tau$ we have

$$\|\nabla u^m\|_2^2(t) \le H \exp\left(H \int_0^t \|\nabla u^m\|_2^4(s) ds\right) \le H \exp\left(H \int_0^\tau \|\nabla u^m\|_2^4(s) ds\right)$$

and so

$$\sup_{t\in(0,\tau)} \|\nabla u^m\|_2^2(t) \le H \exp\left(H \int_0^\tau \|\nabla u^m\|_2^4(s) ds\right) \tag{11.64}$$

Remember now the following estimates previously deduced:

$$\int_0^\tau \|\sqrt{\rho^m} u_t^m\|_2^2 \, ds + \int_0^\tau \|\nabla u^m\|_{H^1}^2 ds + \sup_{t\in(0,\tau)} \|\nabla u^m\|_2^2(t) \le K_0 + K_0 \int_0^\tau \|\nabla u^m\|_2^6(s) \, ds \tag{11.43}$$

where K_0 has been taken as the maximum of the two constants;

$$\int_\Omega \rho^m |u_t^m|^2(t) dx + \int_0^t \int_\Omega |\nabla u_t^m|^2 dx ds \le C'' + C'' \overline{\mathcal{C}_0}^m + C'' \left(\int_0^t \|\nabla u\|_2^6(s) \, ds \right)^3 \tag{11.60}$$

We immediately have from (11.43) that

$$\sup_{t\in(0,\tau)} \|\nabla u^m\|_2^2(t) \le K_0 + K_0 \int_0^\tau \|\nabla u^m\|_2^6(s) \, ds$$

Moreover, by (11.60), we get

$$\sup_{t\in(0,\tau)} \|\sqrt{\rho^m} u_t^m\|_2^2 \le C'' + C'' \overline{\mathcal{C}_0}^m + C'' \left(\int_0^\tau \|\nabla u^m\|_2^6(s) \, ds \right)^3$$

using the trick above of taking a time $t < \tau$. Furthermore again by (11.43) we have

$$\int_0^\tau \|\sqrt{\rho^m} u_t^m\|_2^2 \, ds + \int_0^\tau \|\nabla u^m\|_{H^1}^2 \, ds \le K_0 + K_0 \int_0^\tau \|\nabla u^m\|_2^6(s) \, ds$$

Observe that from (11.63) it follows

$$f(t) := \int_0^t \|\nabla u^m\|_2^6 \, ds \le \exp\left(H \int_0^t \|\nabla u^m\|_2^4 \, ds\right) =: g(t) \tag{11.65}$$

In fact, deriving both sides, we have

$$f'(t) = \|\nabla u^m\|_2^6(t), \qquad g'(t) = H\|\nabla u^m\|_2^4(t) \exp\left(H \int_0^t \|\nabla u^m\|_2^4 \, ds\right)$$

So, using (11.63), $f'(t) \leq g'(t)$ since

$$\|\nabla u^m\|_2^6(t) \leq H\|\nabla u^m\|_2^4(t) \exp\left(H\int_0^t \|\nabla u^m\|_2^4\,ds\right) \iff \|\nabla u^m\|_2^2(t) \leq H \exp\left(H\int_0^t \|\nabla u^m\|_2^4\,ds\right)$$

Moreover $f(0) = 0 < 1 = g(0)$. So, we have

$$\int_0^t \|\nabla u^m\|_2^6\,ds = f(t) = f(0)+\int_0^t f'(s)\,ds \leq g(0)+\int_0^t g'(s)\,ds = g(t) = \exp\left(H\int_0^t \|\nabla u^m\|_2^4\,ds\right)$$

Using (11.60) we know for sure that

$$\int_0^t\int_\Omega |\nabla u_t^m|^2 dxds,\ \sup_{\tau\in(0,t)} \|\sqrt{\rho^m}u_t^m\|_2^2 \leq C''+C''\overline{\mathcal{C}_0}^m+C''\left(\int_0^t \|\nabla u^m\|_2^6(s)ds\right)^3 \quad (11.66)$$

Moreover we know that $\|\nabla u^m\|_{H^1}^2 = \|\nabla u^m\|_2^2 + \|\nabla^2 u^m\|_2^2$ and we can study the two pieces independently. For the first we have

$$\|\nabla u^m\|_2^2 \overset{(11.63)}{\leq} H\exp\left(H\int_0^t \|\nabla u^m\|_2^4(s)\,ds\right)$$

On the other hand, the second can be treated as follows. We have

$$\begin{aligned}\|\nabla^2 u^m\|_2^2 &\overset{(11.38)}{\leq} 2\tilde{C}^2(\|\rho_0\|_\infty+1)\big(\|\sqrt{\rho^m}u_t^m\|_2^2 + \|\sqrt{\rho^m}(\nabla u^m)u^m\|_2^2\big) \equiv \\ &\equiv \tilde{K}\big(\|\sqrt{\rho^m}u_t^m\|_2^2 + \|\sqrt{\rho^m}(\nabla u^m)u^m\|_2^2\big) \leq \\ &\overset{(11.41)}{\leq} \tilde{K}\|\sqrt{\rho^m}u_t^m\|_2^2 + \tilde{K}(\|\rho_0\|_\infty+1)\Lambda_0\|\nabla u^m\|_2^3\|\nabla u^m\|_{H^1} \leq \\ &\overset{20}{\leq} \tilde{K}\|\sqrt{\rho^m}u_t^m\|_2^2 + C'''\Lambda_0\left(\frac{1}{4\varepsilon}\|\nabla u^m\|_2^6 + \varepsilon\|\nabla u^m\|_2^2 + \varepsilon\|\nabla^2 u^m\|_2^2\right)\end{aligned}$$

where $C''' := \tilde{K}(\|\rho_0\|_\infty+1)$ and $\varepsilon > 0$. We find, in this way,

$$\big(1-\varepsilon C'''\Lambda_0\big)\|\nabla^2 u^m\|_2^2 \leq \tilde{K}\|\sqrt{\rho^m}u_t^m\|_2^2 + \frac{C'''\Lambda_0}{4\varepsilon}\|\nabla u^m\|_2^6 + \varepsilon C'''\Lambda_0\|\nabla u^m\|_2^2$$

Since $\varepsilon > 0$ is arbitrary, we can fix $\varepsilon = \frac{1}{2C'''\Lambda_0}$, then

$$\|\nabla^2 u^m\|_2^2 \leq 2\tilde{K}\|\sqrt{\rho^m}u_t^m\|_2^2 + (C'''\Lambda_0)^2\|\nabla u^m\|_2^6 + \|\nabla u^m\|_2^2$$

and so

$$\|\nabla u^m\|_{H^1}^2 = \|\nabla u^m\|_2^2 + \|\nabla^2 u^m\|_2^2 \leq 2\tilde{K}\|\sqrt{\rho^m}u_t^m\|_2^2 + (C'''\Lambda_0)^2\|\nabla u^m\|_2^6 + 2\|\nabla u^m\|_2^2$$

[20] Here we used the inequality

$$ab \leq \frac{a^2}{2\varepsilon'} + \frac{\varepsilon' b^2}{2}$$

with $\varepsilon' = 2\varepsilon$.

Passing to the supremum for $\tau \in (0,t)$ we get

$$\sup_{\tau\in(0,t)} \|\nabla u^m\|_{H^1}^2 \le 2\tilde{K} \sup_{\tau\in(0,t)} \|\sqrt{\rho^m} u_t^m\|_2^2 + (C'''\Lambda_0)^2 \sup_{\tau\in(0,t)} \|\nabla u^m\|_2^6 + 2 \sup_{\tau\in(0,t)} \|\nabla u^m\|_2^2 \le$$

$$\overset{(11.66)+(11.43)}{\le} 2\tilde{K}\left\{C''+C''\overline{\mathcal{C}_0}^m+C''\left(\int_0^t \|\nabla u^m\|_2^6(s)\,ds\right)^3\right\}+(C'''\Lambda_0)^2\left(K_0+K_0\int_0^t \|\nabla u^m\|_2^6(s)\,ds\right)^3+$$

$$+2\left(K_0 + K_0\int_0^t \|\nabla u^m\|_2^6(s)\;ds\right)$$

Since $(1+y)^3 \le 4(1+y^3)$ for every $y \ge 0$ and also holds (11.65), we have

$$\sup_{\tau\in(0,t)} \|\nabla u^m\|_{H^1}^2 \le 2\tilde{K}\left\{C''+C''\overline{\mathcal{C}_0}^m+C''\left(\int_0^t \|\nabla u^m\|_2^6(s)ds\right)^3\right\}+4(C'''\Lambda_0)^2K_0^3\left\{1+\left(\int_0^t \|\nabla u^m\|_2^6(s)ds\right)^3\right\}+$$

$$+2\left(K_0+K_0\int_0^t \|\nabla u^m\|_2^6(s)\,ds\right) \le \hat{C}+\hat{C}\overline{\mathcal{C}_0}^m+\hat{C}\left(\int_0^t \|\nabla u^m\|_2^6(s)\,ds\right)^3+\hat{C}\int_0^t \|\nabla u^m\|_2^6(s)\,ds$$

where $\hat{C}$ is the maximum of the constants. If we now consider

$$\sup_{\tau\in(0,t)} \{\|\nabla u^m\|_{H^1}^2+\|\sqrt{\rho^m}u_t^m\|_2^2\}+\int_0^t \|\nabla u_t^m\|_2^2 ds \le \sup_{\tau\in(0,t)} \|\nabla u^m\|_{H^1}^2+ \sup_{\tau\in(0,t)} \|\sqrt{\rho^m}u_t^m\|_2^2+\int_0^t \|\nabla u_t^m\|_2^2 ds \le \tag{11.67}$$

$$\overset{(11.66)}{\le} \hat{C}+\hat{C}\overline{\mathcal{C}_0}^m+\hat{C}\left(\int_0^t \|\nabla u^m\|_2^6(s)ds\right)^3+\hat{C}\int_0^t \|\nabla u^m\|_2^6(s)ds+2\left\{C''+C''\overline{\mathcal{C}_0}^m+C''\left(\int_0^t \|\nabla u^m\|_2^6(s)ds\right)^3\right\} \equiv$$

$$\equiv \hat{D} + \hat{D}\overline{\mathcal{C}_0}^m + \hat{D}\left(\int_0^t \|\nabla u^m\|_2^6(s)ds\right)^3 + \hat{C}\int_0^t \|\nabla u^m\|_2^6(s)ds$$

where $\hat{D} = \hat{C} + 2C''$. But, moreover, we know, from (11.65), that

$$\left(\int_0^t \|\nabla u^m\|_2^6 ds\right)^3 \le \exp\left(3H\int_0^t \|\nabla u^m\|_2^4 ds\right)$$

So

$$\sup_{\tau\in(0,t)} \{\|\nabla u^m\|_{H^1}^2 + \|\sqrt{\rho^m}u_t^m\|_2^2\} + \int_0^t \|\nabla u_t^m\|_2^2 ds \le$$

$$\le \hat{D} + \hat{D}\overline{\mathcal{C}_0}^m + \hat{D}\exp\left(3H\int_0^t \|\nabla u^m\|_2^4 ds\right) + \hat{C}\exp\left(H\int_0^t \|\nabla u^m\|_2^4 ds\right) \le$$

$$\le \hat{D}\overline{\mathcal{C}_0}^m + 3\hat{D}\exp\left(3H\int_0^t \|\nabla u^m\|_2^4 ds\right) \le \hat{H}\overline{\mathcal{C}_0}^m + \hat{H}\exp\left(\hat{H}\int_0^t \|\nabla u^m\|_2^4 ds\right)$$

since $\exp(\alpha) \ge 1$ if $\alpha \ge 0$. Here $\hat{H} := \max\{\hat{D}, 3\hat{D}, 3H\}$.
We have finally obtained the inequality

$$\boxed{\sup_{\tau\in(0,t)} \{\|\nabla u^m\|_{H^1}^2 + \|\sqrt{\rho^m}u_t^m\|_2^2\} + \int_0^t \|\nabla u_t^m\|_2^2\; ds \le \hat{H}\overline{\mathcal{C}_0}^m + \hat{H}\exp\left(\hat{H}\int_0^t \|\nabla u^m\|_2^4\; ds\right)} \tag{11.68}$$

Remark 11.17. The constants involved in the inequality depend on the constants introduced before. □ ∎

11.5 A local time of existence

We now use the estimates deduced in the previous subsections to find a local time of existence. We have the following proposition.

Proposition 11.13. *Let Ω be a bounded domain in $\mathbb{R}^3$. Consider the Navier-Stokes problem over Ω as in proposition 11.1. Let $\overline{\rho}_0 \in C^1(\overline{\Omega})$ and $T > 0$. Let $\rho^m \in C^1([0,T] \times \overline{\Omega})$ and $u^m \in C^1([0,T]; X^m)$ the approximate solutions built in proposition 11.1. Then there exist a time $T_* \in (0,T)$ and a constant $C > 0$ such that*

$$\sup_{t \in [0,T_*]} \|\nabla u^m\|_2(t) \le C$$

Here C and T_* are positive constants depending only on $\|\rho_0\|_{L^\infty}$, $\|\nabla u_0\|_2$ and T, but independent of δ and m.

Proof. Remember first of all the inequality

$$\|\nabla u^m\|_2^2(t) \le H + H \int_0^t \|\nabla u^m\|_2^6(s)\, ds \qquad (11.61)$$

where H does not depend of δ and m. In particular, we want to find a time T_*, independent of m, δ and eventually the size of the domain, and a bound $M > 0$, such that

$$\sup_{t \in [0,T_*]} \|\nabla u^m\|_2(t) \le M$$

We star from the inequality we have above. First of all note that we can replace H with $\max\{H, \|\nabla u_0\|_2^2 + 1\} > \|\nabla u_0\|_2^2$. We fix $T_1 \in (0,T)$ and consider $T_0 \in [0,T_1)$. We define

$$f^m(s) := \|\nabla u^m\|_2^2(s)$$

and

$$\beta(t) := \begin{cases} \sup_{s \in [0,t]} f^m(s) & t \neq 0 \\ f^m(0) & t = 0 \end{cases}$$

The function β is continuous on $[0,T_0]$, since f^m is continuous on $[0,T_0]$. In fact, let $t_0 \in [0,T_0]$. We have

$$\left| \max_{s \in [0,t]} f^m(s) - \max_{s \in [0,t_0]} f^m(s) \right| = \begin{cases} 0 & (*) \\ |f(\overline{t}) - f(\underline{t})| & (**) \end{cases}$$

where $(*)$ happens when the maximum of f^m over $[0,T_0]$ is achieved in both cases in $[0, \min\{t, t_0\}]$, and the case $(**)$ otherwise. In particular, in the second case we can use the uniformly continuity of f^m to deduce the smallness of the difference[21].
On the other hand, observe that, from the inequality above, we have for every $\boxed{T_0 \in [0,T_1]}$,

$$\sup_{t \in [0,T_0]} \|\nabla u^m\|_2^2(t) \le H + H \int_0^{T_0} \|\nabla u^m\|_2^6(s) ds \le H + HT_0 \sup_{t \in [0,T_0]} \|\nabla u^m\|_2^6(t) \le$$

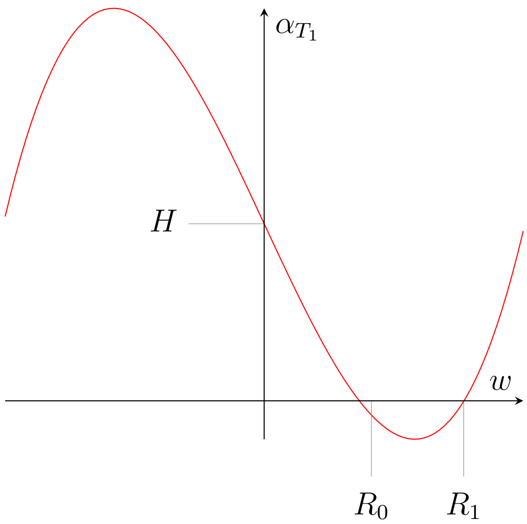
[21] If the maximum is in $[0, \min\{t, t_0\})$, then we have the smallness choosing δ_1 sufficiently small; if it is in the right boundary of $[0, \min\{t, t_0\}]$, then by uniformly continuity we can choose again δ_1 so that the difference is small.

$$\leq H + HT_1 \sup_{t\in[0,T_0]} \|\nabla u^m\|_2^6(t)$$

where the supremum is a maximum, because of the continuity of $\|\nabla u^m\|_2$. So if $w := \sup_{[0,T_0]} \|\nabla u^m\|_2^2$ we have that

$$w \leq H + HT_1 w^3$$

We consider the polynomial $\alpha_{T_1}(w) := H + HT_1 w^3 - w$ and the inequality $\alpha_{T_1}(w) \geq 0$. We plot in the figure below a qualitative graph of the function α_{T_1}.



Remark 11.18. In the graph are also reported the point $w = R_0, R_1$ that will be defined in a moment. □

So we have

$$\alpha'_{T_1}(w) = 3HT_1 w^2 - 1 = 0 \iff w = \pm\sqrt{\frac{1}{3HT_1}}$$

and

$$\alpha_{T_1}\left(\sqrt{\frac{1}{3HT_1}}\right) = H + HT_1\left(\frac{1}{3HT_1}\right)^{\frac{3}{2}} - \sqrt{\frac{1}{3HT_1}} = H + \frac{H}{(3H)^{\frac{3}{2}}}T_1^{-\frac{1}{2}} - \frac{1}{\sqrt{3H}}\frac{1}{\sqrt{T_1}} =$$

$$= H + \frac{1}{\sqrt{T_1 H}}\left(\frac{1}{3^{\frac{3}{2}}} - \frac{1}{\sqrt{3}}\right)$$

If T_1 **is very small, in a way depending on** H, we have that the minimum is negative. In particular, if $\alpha := \frac{1}{3^{\frac{3}{2}}} - \frac{1}{\sqrt{3}}$,

$$H + \frac{\alpha}{\sqrt{T_1 H}} < 0 \iff H < \frac{|\alpha|}{\sqrt{T_1 H}} \iff T_1 < \frac{|\alpha|^2}{H^3}$$

Remark 11.19. If H, that depends on the initial data, is bounded, that is $H \leq H_0$, uniformly with respect the initial data, we can choose

$$T_1 < \frac{|\alpha|^2}{H_0^3} \leq \frac{|\alpha|^2}{H^3}$$

Note also that a zero $w_0 > 0$ of α_{T_1}, for $T_1 > 0$, satisfies

$$0 < T_1 = \frac{1}{Hw_0^2}(1 - \frac{H}{w_0}) = \frac{w_0 - H}{Hw_0^3}$$

Being $w_0 > 0$ we have $w_0 > H > \|\nabla u_0\|_2^2$. So every positive zero (that exists because we imposed the minimum, reached in a positive value, to be negative and moreover $\alpha_{T_1}(0) = H > 0$) is strictly greater than $\|\nabla u_0\|_2^2$.
We define as $[0, R_0]$ an interval that intercepts only the first zero.
So, for every $T_0 \in [0, T_1)$, the supremum w, that is positive, can only live in $[0, R_0]$ or in an interval $[R_1, +\infty)$, where the polynomial is positive. Furthermore the supremum function is continuous in $[0, T_1)$. At the time $t = 0$ we have

$$\beta(0) = \|\nabla u^m\|_2^2(0) \leq \|\nabla u_0\|_2^2 < H < w_0 \leq R_0$$

By continuity, living at starting time in the first interval, the function can't jump beyond R_1. So, for every $T_0 \in [0, T_1)$ we have

$$\sup_{t \in [0, T_0]} \|\nabla u^m\|_2^2(t) \leq R_0$$

and we can that $T_* = T_0 \in (0, T_1) \subseteq (0, T)$ and we get

$$\sup_{t \in [0, T_*]} \|\nabla u^m\|_2(t) \leq \sqrt{R_0} \tag{11.69}$$

with T_* and R_0 only depending on H and T, that are independent on δ, m. ∎

Remark 11.20. As in remark 11.19, we can consider the existence of H_0. So, we can choose $T_1 = \dfrac{|\alpha|}{2H_0^3}$ and $T_* = T_0 \geq \dfrac{|\alpha|}{4H_0^3}$. □

11.5.1 A uniform upper bound for the sequence $\overline{\mathcal{C}_0}^m$

As final result of the section, we want to find a uniform upper bound for $\overline{\mathcal{C}_0}^m$. In particular, we prove the following proposition.

Proposition 11.14. *Let Ω be a bounded domain in $\mathbb{R}^3$. Consider the Navier-Stokes problem over Ω as in proposition 11.1. Let $\overline{\rho}_0 \in C^1(\overline{\Omega})$ and $T > 0$. Let $\rho^m \in C^1([0, T] \times \overline{\Omega})$ and $u^m \in C^1([0, T]; X^m)$ the approximate solutions built in proposition 11.1. Consider*

$$\overline{\mathcal{C}_0}^m := \int_\Omega (\overline{\rho}_0)^{-1} |\mu \Delta u^m(0) - \nabla p_0|^2 dx$$

Then there exists a constant $W_0 = W_0(\delta, u_0, g)$ such that

$$\overline{\mathcal{C}_0}^m \leq W_0 \qquad \forall m \in \mathbb{N}$$

Proof. We have to keep in mind the compatibility condition (11.4). So, if follows

$$\overline{\mathcal{C}_0}^m := \int_\Omega (\overline{\rho}_0)^{-1}|\mu\Delta u^m(0) - \nabla p_0|^2 dx = \int_\Omega (\overline{\rho}_0)^{-1}|\mu\Delta u^m(0) - \mu\Delta u_0 + \mu\Delta u_0 - \nabla p_0|^2 dx \leq$$

$$\leq 2\delta^{-1}\int_\Omega \left(|\mu\Delta u^m(0) - \mu\Delta u_0|^2 + |\rho_0||g|^2\right) dx \leq 2\delta^{-1}\mu^2 \int_\Omega |\Delta u^m(0) - \Delta u_0|^2 + 2\delta^{-1}(\|\rho_0\|_\infty + 1)\|g\|_2^2$$

Moreover, we know that $\lim_{N\to+\infty} \|\sum_{k=1}^N \langle u_0, w^k\rangle_2 w^k - u_0\|_{H^2} = 0$.

Remark 11.21. Observe that, if $v_n = (v_n^1, v_n^2, v_n^3)$, $\lim_{n\to+\infty} \|\nabla^2 v_n\|_2^2 = \lim_{n\to+\infty} \int_\Omega |\nabla^2 v_n|^2 dx$

$$|\nabla^2 v_n|^2 = \sum_{i=1}^3 \sum_{j=1}^3 \sum_{l=1}^3 |\partial^2_{ij} v_n^l|^2 \geq \sum_{l=1}^3 \left(|\partial^2_{11} v_n^l|^2 + |\partial^2_{22} v_n^l|^2 + |\partial^2_{33} v_n^l|^2\right)$$

Finally[22]

$$|\Delta v_n|^2 := \sum_{l=1}^3 |\Delta v_n^l|^2 \leq \sum_{l=1}^3 \left(|\partial^2_{11} v_n^l| + |\partial^2_{22} v_n^l| + |\partial^2_{33} v_n^l|\right)^2 \leq 5\sum_{l=1}^3 \left(|\partial^2_{11} v_n^l|^2 + |\partial^2_{22} v_n^l|^2 + |\partial^2_{33} v_n^l|^2\right) \leq 5|\nabla^2 v_n|^2$$

and so

$$\int_\Omega |\Delta v_n|^2 \leq 5\int_\Omega |\nabla^2 v_n|^2 dx \leq 5\|v_n\|_{H^2}^2$$

this will be useful in a moment. □

In fact, in our case,

$$\int_\Omega |\Delta u^m(0) - \Delta u_0|^2 dx \leq 5\left\|\sum_{k=1}^m \langle u_0, w^k\rangle_2 w^k - u_0\right\|_{H^2}^2$$

No, if $M \in \mathbb{N}$ is such that

$$\left\|\sum_{k=1}^m \langle u_0, w^k\rangle_2 w^k - u_0\right\|_{H^2}^2 < \frac{1}{5} \quad \forall m \geq M$$

we can bound the latter term for $m \geq M$. In particular we get

$$\overline{\mathcal{C}_0}^m \leq 2\delta^{-1}\left(\mu^2 + (\|\rho_0\|_\infty + 1)\|g\|_2^2\right) \equiv W_0 \quad \forall m \geq M \tag{11.70}$$

If we rename the sequences (ρ^m, u^m) so that they start with $m = M$, we have that $\overline{\mathcal{C}_0}^m \leq W_0$ is true for every element of the sequence $\{(\rho^m, u^m)\}_{m\in\mathbb{N}}$. ∎

Remark 11.22. There is no trace of this argument in [4]. However, the same authors take care of this point in their work [5]. □

[22]Using that, if $a, b, c \geq 0$, ordering $0 \leq a \leq b \leq c$,

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac \leq a^2 + 3b^2 + 5c^2 \leq 5(a^2 + b^2 + c^2)$$

11.6 An extraction argument: weak limits of the approximate solutions

In this section we will use the estimates deduced in the sections above to extract convergent subsequence of the approximate solution. It is clear that, once proved that this limits exist, we have to ask if they satisfy the original equation. However, the latter question will be answered in the next section, that is, together with the present section, the core of the thesis, as clarified in section 0.1.

11.6.1 Weak-star limit function for the sequence u^m

In this subsection we prove the following proposition.

Proposition 11.15. *Let Ω be a bounded domain in $\mathbb{R}^3$. Consider the Navier-Stokes problem over Ω as in proposition 11.1. Let $\overline{\rho}_0 \in C^1(\overline{\Omega})$ and $T > 0$. Let $\rho^m \in C^1([0,T] \times \overline{\Omega})$ and $u^m \in C^1([0,T]; X^m)$ the approximate solutions built in proposition 11.1. Then there exists a function $u \in L^\infty(0, T_*; H^2(\Omega))$ and a subsequence $\{u^{m_k}\}_{k\in\mathbb{N}}$ of $\{u^m\}_{m\in\mathbb{N}}$ such that*

$$u^{m_k} \overset{*}{\rightharpoonup} u$$

In other words, for every $v \in L^1(0, T_*; H^2(\Omega))$ it holds

$$\lim_{k\to+\infty} \int_0^{T_*} \langle u^{m_k}(t), v(t)\rangle_{H^2}\, dt = \int_0^{T_*} \langle u(t), v(t)\rangle_{H^2}\, dt \tag{11.71}$$

Here T_* is the local time provided by proposition 11.13.

Proof. In equation (11.68), if we choose $t = T_* < T$, we have

$$\sup_{s\in[0,T_*]} \|\nabla u^m\|_{H^1}^2 \le \hat{H}\overline{\mathcal{C}_0}^m + \hat{H} \exp\left(\hat{H}\int_0^{T_*} \|\nabla u^m\|_2^4 ds\right) \le$$

$$\overset{(11.70)+(11.69)}{\le} \hat{H}W_0 + \hat{H}\exp\left(\hat{H}T_* R_0^2\right) \le \hat{H}W_0 + \hat{H}\exp\left(\hat{H}TR_0^2\right)$$

On the other hand, since $u^m \in H_0^1$, we have $\|u^m\|_2 \le K\|\nabla u^m\|_2$. So,

$$\sup_{s\in[0,T_*]} \|u^m\|_2^2 \le K^2 \sup_{s\in[0,T_*]} \|\nabla u^m\|_2^2 \overset{(11.69)}{\le} K^2 R_0$$

Observing that $\|u^m\|_{H^2}^2 = \|u^m\|_2^2 + \|\nabla u^m\|_{H^1}^2$, we have

$$\sup_{s\in[0,T_*]} \|u^m\|_{H^2}^2 \le \sup_{s\in[0,T_*]} \|u^m\|_2^2 + \sup_{s\in[0,T_*]} \|\nabla u^m\|_{H^1}^2 \le$$

$$\le K^2R_0 + \hat{H}W_0 + \hat{H}\exp(\hat{H}TR_0^2)$$

and so

$$\|u^m\|_{L^\infty(0,T_*;H^2(\Omega))} \equiv \sup_{[0,T_*]} \|u^m\|_{H^2} \le \sqrt{K^2R_0 + \hat{H}W_0 + \hat{H}\exp(\hat{H}TR_0^2)} \equiv \hat{K} \tag{11.72}$$

Going on, keep in mind proposition 5.2. At this point we consider first of all the functions $u^m \in C^1([0,T_*],X^m)$. The space X^m is a finitely generated Banach space, equipped with the $\|\cdot\|_{H^2}$ norm (since every finitely generated subspace is closed and so it is a Banach subspace); this space is contained in $(H^2(\Omega),\|\cdot\|_{H^2})$.
So in particular $u^m \in L^\infty(0,T_*;H^2(\Omega))$. Observe now that, according to proposition 5.2,

$$L^\infty(0,T_*;H^2(\Omega)) \simeq (L^1(0,T_*;H^2(\Omega)))^*$$

where $L^1(0,T_*;H^2(\Omega))$ is a separable Banach space. We use now the version of Banach-Alaoglu theorem in Theorem 2.3, very usefull in PDE issues.
So, if $Y = L^1(0,T_*;H^2(\Omega))$, we have just proved that u^m is bounded in the dual space Y^*. So, there exists a function $u \in L^\infty(0,T_*;H^2(\Omega))$ and a subsequence u^{m_k} of u^m such that

$$u^{m_k} \overset{*}{\rightharpoonup} u$$

The weak $*$ convergence notion is the usual, i.e. the one introduced in the proposition 5.2. It can be translated as

$$\lim_{k\to+\infty}\int_0^{T_*}\langle u^{m_k}(t),v(t)\rangle_{H^2}dt = \int_0^{T_*}\langle u(t),v(t)\rangle_{H^2}dt \qquad \forall\ v\in L^1(0,T_*;H^2(\Omega)) \quad (11.73)$$

This prove the proposition. ∎

11.6.2 Weak-star limit for the sequence ρ^m

We now prove a proposition very similar to the previous one.

Proposition 11.16. *Let Ω be a bounded domain in $\mathbb{R}^3$. Consider the Navier-Stokes problem over Ω as in proposition 11.1. Let $\overline{\rho}_0 \in C^1(\overline{\Omega})$ and $T>0$. Let $\rho^m \in C^1([0,T]\times\overline{\Omega})$ and $u^m \in C^1([0,T];X^m)$ the approximate solutions built in proposition 11.1.*
Then there exists a function $\rho \in L^\infty(0,T_*;L^\infty(\Omega))$ and a subsequence $\{\rho^{m_k}\}_{k\in\mathbb{N}}$ of $\{\rho^m\}_{m\in\mathbb{N}}$ such that

$$\rho^{m_k} \overset{*}{\rightharpoonup} \rho$$

In other words, for every $v \in L^1(0,T_*;L^1(\Omega))$ it holds

$$\lim_{k\to+\infty}\int_0^{T_*}\langle\rho^{m_k}(t),v(t)\rangle_{\infty,1}\ dt = \int_0^{T_*}\langle u(t),v(t)\rangle_{\infty,1}\ dt \qquad (11.74)$$

Here T_* is the local time provided by proposition 11.13.

Proof. First of all remember that $\|\rho^m(t)\|_\infty \le \|\rho_0\|_\infty + 1$. So, we have the estimate

$$\sup_{s\in[0,T_*]}\|\rho^m\|_\infty(s) \le \|\rho_0\|_\infty + 1$$

So, as above, the sequence $\rho^{m_k} \in C^1([0,T_*],C^1(\overline{\Omega}))$ is in $L^\infty(0,T_*;L^\infty(\Omega))$ and in the latter space the function is bounded. Moreover

$$L^\infty(0,T_*;L^\infty(\Omega)) \simeq (L^1(0,T_*;L^1(\Omega)))^*$$

and so, again by the Theorem 2.3, we have that exist a subsequence $\rho^{m_{k_h}}$ and a function $\rho \in L^\infty(0, T_*; L^\infty(\Omega))$ such that

$$\rho^{m_{k_h}} \overset{*}{\rightharpoonup} \rho$$

that is what we wanted to prove. ∎

Remark 11.23. Since $u^{m_{k_h}}$ is a subsequence of u^{m_k}, it is moreover true that

$$u^{m_{k_h}} \overset{*}{\rightharpoonup} u$$

since every convergence passes to subsequences, being every notion of convergence defined through numerical sequences.
We can say for brevity, renaming the subsequences as u^m and ρ^m,

$$(\rho^m, u^m) \overset{*}{\rightharpoonup} (\rho, u) \quad \text{in} \quad L^\infty(0, T_*; L^\infty(\Omega) \times H^2(\Omega))$$

where the meaning of these symbols is that defined in propositions 11.15 and 11.16.
A similar argument about subsequences will be used again in future. □

11.6.3 Weak limit for the sequence u_t^m

We now want to know something more about the derivative u_t^m. We first prove a lemma.

Lemma 11.3. *Let Ω be a bounded domain in $\mathbb{R}^3$, and suppose, as above, that*

$$u^m \overset{*}{\rightharpoonup} u \quad \textit{in } L^\infty(0, T_*; H^2(\Omega))$$

with $u \in L^\infty(0, T_*; H^2(\Omega))$. Then, it also holds $u^m \rightharpoonup u$ in $L^2(0, T_*; H^2(\Omega))$.

Proof. We know that $u^m \overset{*}{\rightharpoonup} u$ in $L^\infty(0, T_*; H^2(\Omega))$. Since $L^\infty(0, T_*; H^2(\Omega)) \simeq (L^1(0, T_*; H^2(\Omega))^*$, keeping in mind the dual pairing, we have that this means

$$\lim_{m\to+\infty} \int_0^{T_*} \langle u^m(t), v(t)\rangle_{H^2} dt = \int_0^{T_*} \langle u(t), v(t)\rangle_{H^2} dt \quad \forall v \in L^1(0, T_*; H^2(\Omega))$$

Notice that by the Cauchy-Schwarz inequality, we have that, if $v \in L^2(0, T_*, H^2(\Omega))$,

$$\int_0^{T_*} \|v(t)\|_{H^2} \, dt \le \sqrt{T_*} \left(\int_0^{T_*} \|v(t)\|_{H^2}^2 \, dt \right)^{\frac{1}{2}} < +\infty$$

so that $v \in L^1(0, T_*; H^2(\Omega))$. We have, in this, way that

$$\lim_{m\to+\infty} \int_0^{T_*} \langle u^m(t), v(t)\rangle_{H^2} \, dt = \int_0^{T_*} \langle u(t), v(t)\rangle_{H^2} \, dt \quad \forall v \in L^2(0, T_*; H^2(\Omega))$$

that means, looking at the dual pairing[23], $u^m \overset{*}{\rightharpoonup} u$ in $L^2(0, T_*; H^2(\Omega))$. Using now theorem 2.6, since $L^2(0, T_*; H^2(\Omega))$ is reflexive, we have in particular that $u^m \rightharpoonup u$ in $L^2(0, T_*; H^2(\Omega))$. This is what we wanted to prove. ∎

Now we are ready to prove the following proposition.

[23]Notice that $u^m, u \in L^\infty(0, T_*; H^2(\Omega)) \subseteq L^2(0, T_*; H^2(\Omega)) = \big(L^2(0, T_*; H^2(\Omega))\big)^*$.

Proposition 11.17. *Let Ω be a bounded domain in $\mathbb{R}^3$. Consider the Navier-Stokes problem over Ω as in proposition 11.1. Let $\overline{\rho}_0 \in C^1(\overline{\Omega})$ and $T > 0$. Let $\rho^m \in C^1([0,T] \times \overline{\Omega})$ and $u^m \in C^1([0,T]; X^m)$ the approximate solutions built in proposition 11.1.*
Then there exists a function $u \in L^\infty(0, T_*; H^2)$ and a subsequence $\{u^{m_k}\}_{k\in\mathbb{N}}$ of $\{u^m\}_{m\in\mathbb{N}}$ such that

$$u^{m_k} \overset{*}{\rightharpoonup} u$$

Moreover, u has the weak derivative $u_t \in L^2(0, T_*; H_0^1(\Omega))$ and it holds

$$u_t^{m_k} \rightharpoonup u_t \qquad \text{in } L^2(0, T_*; H_0^1(\Omega))$$

Here T_* is the local time provided by proposition 11.13.

Proof. To this aim, consider the fact that $u_t^m \in L^2(0, T_*; H_0^1(\Omega))$, since $w^k \in H_0^1(\Omega)$ and

$$\int_0^{T_*} \|u_t^m\|_{H^1}^2 dt = \int_0^{T_*} \|u_t^m\|_2^2 \, dt + \int_0^{T_*} \|\nabla u_t^m\|_2^2 \, dt$$

and using that $u_t^m \in H_0^1(\Omega) \Longrightarrow \|u_t^m\|_2 \leq \overline{K} \|\nabla u_t^m\|_2$, with $\overline{K} = \overline{K}(\Omega)$,

$$\int_0^{T_*} \|u_t^m\|_{H^1}^2 dt \leq (1 + \overline{K}^2) \int_0^{T_*} \|\nabla u_t^m\|_2^2 \, dt \overset{(11.68)}{\leq} (1 + \overline{K}^2)[\hat{H}\overline{\mathcal{C}_0}^m + \hat{H} \exp(\hat{H} T_* M^4)] \leq$$

$$\leq (1 + \overline{K}^2)[\hat{H} W_0 + \hat{H} \exp(\hat{H} T M^4)] \tag{11.75}$$

So we have obtained that $u_t^m \in L^2(0, T_*; H_0^1(\Omega))$ and the sequence is bounded in this space, uniformly in m. Moreover, since $H_0^1(\Omega)$ is an Hilbert space, and so a reflexive Banach space, then $L^2(0, T_*; H_0^1(\Omega))$ is reflexive, thanks to proposition 5.2. So we can use theorem 2.5, and we have that there exist $v \in L^2(0, T_*; H_0^1(\Omega))$ and a subsequence $u_t^{m_k}$ such that

$$u_t^{m_k} \rightharpoonup v \quad \text{in} \quad L^2(0, T_*; H_0^1(\Omega))$$

Taking the subsequence m_k also in u^{m_k} and ρ^{m_k}, and using that convergence of subsequences is preserved, we have, renaming the subsequences as (u^m, ρ^m, u_t^m),

$$(\rho^m, u^m) \overset{*}{\rightharpoonup} (\rho, u) \in L^\infty(0, T_*; L^\infty(\Omega) \times H^2(\Omega))$$

$$u_t^m \rightharpoonup v \in L^2(0, T_*; H_0^1(\Omega))$$

Now we want to prove that u admits a weak temporal derivative and that this derivative is v, so that v can be renamed u_t.

We that $u \in L^\infty(0, T_*; H^2(\Omega))$. This means that

$$C \equiv \text{ess sup}_{t\in[0,T_*]} \|u(t)\|_{H^2} < +\infty$$

This implies that

$$\int_0^{T_*} \|u(t)\|_{H^2} \, dt \leq C T_* < +\infty$$

So $u \in L^1(0, T_*; H^2(\Omega))$. Moreover $v \in L^2(0, T_*; H^1_0(\Omega)) \subseteq L^1(0, T_*; H^1_0(\Omega))$.
In this way, v is the weak derivative of u provided that

$$\int_0^{T_*} \partial_t \phi(t) u(x,t)\ dt = -\int_0^{T_*} \phi(t) v(x,t)\ dt \quad \forall \phi \in C^\infty_c(0, T_*)$$

We now use lemma 11.3. We have that $u^m \rightharpoonup u$ in $L^2(0, T_*; H^2(\Omega))$. So we start with our argument. Let A a measurable subset of Ω, and $\phi \in C^\infty_c(0, T_*)$. We have, defining $u_i = \pi_i(u)$ and $v_i = \pi_i(v)$, where π_i is the projection on the i-th component[24],

$$\int_A \int_0^{T_*} \big(\phi_t u_i + \phi v_i\big)\ dt\ dx =$$

$$= \int_A \int_0^{T_*} \big(\phi_t u_i - \phi_t u_i^m + \phi_t u_i^m + \phi (u_i^m)_t - \phi (u_i^m)_t + \phi v_i\big)\ dt\ dx$$

Thanks to the regularity of u^m with respect time and the integration by parts, it holds the equality

$$\int_0^{T_*} \phi_t u_i^m\ dt = -\int_0^{T_*} (u_i^m)_t \phi\ dt$$

So, we have

$$\int_A \int_0^{T_*} (\phi_t u_i + \phi v_i)\ dt\ dx = \int_A \int_0^{T_*} (\phi_t u_i - \phi_t u_i^m - \phi (u_i)_t^m + \phi v_i)\ dt\ dx =$$

$$= \int_A \int_0^{T_*} \phi_t (u_i - u_i^m)\ dt\ dx + \int_A \int_0^{T_*} \phi (v_i - (u_i^m)_t)\ dt\ dx$$

The last term is the sum of two functionals. In particular we can define

$$f_i(w) := \int_A \int_0^{T_*} \phi_t w_i\ dt\ dx \quad i \in \{1, 2, 3\}$$

that maps $f_i : L^2(0, T_*; H^2(\Omega)) \to \mathbb{R}$. Morover we have

$$g_i(w) := \int_A \int_0^{T_*} \phi w_i\ dt\ dx$$

that maps $g : L^2(0, T_*; H^1_0(\Omega)) \to \mathbb{R}$. The functionals are continuous. To see this, we first remark a fact. Observe that

$$L^2(0, T_*; H^2(\Omega)), L^2(0, T_*; H^1_0(\Omega)) \subseteq L^2(0, T_*; L^2(\Omega)) \simeq L^2(\Omega \times (0, T_*)) \tag{11.76}$$

Keeping this in mind, we have

$$|f_i(w)| \leq \int_A \int_0^{T_*} |\phi_t||w_i|\ dt\ dx \leq \|\phi_t\|_{t,\infty} \int_A \int_0^{T_*} |w_i|\ dt\ dx = \|\phi_t\|_{t,\infty} \int_0^{T_*} \int_A |w_i|\ dx\ dt$$

[24]The functions are in fact vectors.

where in the last equality we used the Tonelli theorem for non-negative functions thanks to the fact that the equivalence (11.76) remarked above says to us that the function is measurable. So we have

$$|f_i(w)| \leq \|\phi_t\|_{t,\infty} \int_0^{T_*} |A|^{\frac{1}{2}} \left(\int_A |w_i|^2 dx \right)^{\frac{1}{2}} dt \leq \|\phi_t\|_{t,\infty} |A|^{\frac{1}{2}} \int_0^{T_*} \|w_i\|_2 \ dt \leq$$

$$\leq \|\phi_t\|_{t,\infty} |A|^{\frac{1}{2}} T_*^{\frac{1}{2}} \left(\int_0^{T_*} \|w_i\|_2^2 dt \right)^{\frac{1}{2}} \leq \|\phi_t\|_{t,\infty} |A|^{\frac{1}{2}} T_*^{\frac{1}{2}} \left(\int_0^{T_*} \|w\|_{H^2}^2 dt \right)^{\frac{1}{2}}$$

So the functional is continuous. Thus, since $u^m \rightharpoonup u$ in $L^2(0, T_*; H^2(\Omega))$, we have

$$\lim_{m \to +\infty} \int_A \int_0^{T_*} \phi_t u_i^m \ dt \ dx = \int_A \int_0^{T_*} \phi_t u_i \ dt \ dx$$

The other limit is very similar. In fact, g_i is continuous since

$$|g_i(w)| \leq \|\phi\|_{t,\infty} \int_A \int_0^{T_*} |w_i| \ dt \ dx \leq \|\phi\|_{t,\infty} \int_0^{T_*} \int_A |w_i| \ dt \ dx$$

always using the Tonelli theorem and the identification (11.76) above. So

$$|g_i(w)| \leq \|\phi\|_{t,\infty} |A|^{\frac{1}{2}} \int_0^{T_*} \left(\int_A |w_i|^2 dx \right)^{\frac{1}{2}} dt \leq \|\phi\|_{t,\infty} |A|^{\frac{1}{2}} \int_0^{T_*} \|w_i\|_2 \ dt \leq$$

$$\leq \|\phi\|_{t,\infty} |A|^{\frac{1}{2}} \int_0^{T_*} \|w\|_2 \, dt \leq \|\phi\|_{t,\infty} |A|^{\frac{1}{2}} T_*^{\frac{1}{2}} \left(\int_0^{T_*} \|w\|_2^2 \, dt \right)^{\frac{1}{2}} \leq \|\phi\|_{t,\infty} |A|^{\frac{1}{2}} T_*^{\frac{1}{2}} \left(\int_0^{T_*} \|w\|_{H^1}^2 dt \right)^{\frac{1}{2}}$$

So also this functional is continuous. We deduce that

$$\lim_{m \to +\infty} \int_A \int_0^{T_*} \phi (u_t^m)_i \ dt \ dx = \int_A \int_0^{T_*} \phi v_i \ dt \ dx$$

since $u_t^m \rightharpoonup v$ in $L^2(0, T_*; H_0^1(\Omega))$. This means that

$$\int_A \int_0^{T_*} (\phi_t u_i + \phi v_i) \ dt \ dx = 0$$

Since the equality is true for every A measurable subset of Ω and every $\phi \in C_c^\infty(0, T_*)$, we have, at ϕ fixed,

$$\int_0^{T_*} (\phi_t u_i + \phi v_i) \ dt = 0 \quad \text{a.e. in } \Omega$$

In other words

$$\int_0^{T_*} \phi_t u_i \ dt = - \int_0^{T_*} \phi v_i \ dt \quad \text{a.e. in } \Omega$$

Thus, if $\phi \in C_c^\infty(0, T_*)$ we have, since the two integrals belong to $L^2(\Omega)$ by the definition of Bochner integral,

$$\int_0^{T_*} \phi_t u \ dt = - \int_0^{T_*} \phi v \ dt \ \text{ in } \ L^2(\Omega)$$

Here we used the equality component by component of the integrals[25] and the fact that the equality almost everywhere is the equality in the sense of the Banach space $L^2(\Omega)$. In other words, we can write $u_t := v \in L^2(0, T_*; H_0^1(\Omega))$ in the weak sense. ∎

[25] See remark 5.2.

11.6.4 Boundary condition and incompressibility of the velocity field

Before claiming and proving the main theorem of the subsection, that will assure important properties of the velocity field, we have to prove a preliminary proposition, that will be very useful also in future arguments.

Proposition 11.18. *Let Ω be a bounded subset of $\mathbb{R}^3$. Suppose that we are in the hypothesis of proposition 11.17. Then, the sequence u^m with weak-star limit u also admits a subsequence $\{u^{m_k}\}_{k\in\mathbb{N}}$ such that*

$$u^{m_k} \to u, \quad \nabla u^{m_k} \to \nabla u \qquad \textit{in} \quad L^2(0,T_*;L^2(\Omega))$$

Proof. We consider now lemma 7.2. In this context, we can choose $X = H^1_0(\Omega) \cap H^2(\Omega)$ with the norm[26] $\|\cdot\|_{H^1_0} + \|\cdot\|_{H^2}$. Moreover we choose $B = Y = L^2(\Omega)$. The embedding $X \hookrightarrow L^2(\Omega) = B$ is compact thanks to the Rellich-Kondrachov theorem[27]. So, by lemma 7.2, the embedding

$$L^2(0,T_*;H^1_0 \cap H^2) \cap \{\varphi : \partial_t\varphi \in L^1(0,T_*;L^2)\} \hookrightarrow L^2(0,T_*;L^2)$$

is compact. In particular we follow the hypothesis of lemma 7.2. In fact, we can consider the sequence u^m. It is in $L^2(0,T_*;H^1_0 \cap H^2)$ and it is bounded, since

$$\left(\int_0^{T_*} \left(\|u^m\|_{H^1_0} + \|u^m\|_{H^2}\right)^2 dt\right)^{\frac{1}{2}} \le 2T_*^{\frac{1}{2}} \sup_{(0,T_*)} \|u^m\|_{H^2} \le 2T_*^{\frac{1}{2}}\hat{K}$$

thanks to the estimate (11.72). Moreover u^m_t is in $L^1(0,T_*;L^2)$ and

$$\int_0^{T_*} \|u^m_t\|_2 \, dt \le T_*^{\frac{1}{2}}\left(\int_0^{T_*} \|u^m_t\|_2^2 dt\right)^{\frac{1}{2}} \le T_*^{\frac{1}{2}}\left(\int_0^{T_*} \|u^m_t\|_{H^1}^2 dt\right)^{\frac{1}{2}} \le$$

$$\stackrel{(11.75)}{\le} T_*^{\frac{1}{2}}\sqrt{(1+\overline{K}^2)[\hat{H}W_0 + \hat{H}\exp(\hat{H}T_*M^4)]}$$

So, the sequence of temporal derivatives is bounded in the space. Thus, eventually passing to a subsequence, we have that

$$u^{m_k} \to w \ \text{ in } L^2(0,T_*;L^2(\Omega))$$

[26] The space $(H^1_0 \cap H^2, \|\cdot\|_{H^2})$ is a Banach space. In fact, being a subset of H^2, the norm is well defined and satisfies the properties of the definition of Banach space. Moreover, let w_n a Cauchy sequence in this space. Then it is a Cauchy sequence in H^1_0 and H^2, since

$$\|w_n - w_m\|_{H^1} \le \|w_n - w_m\|_{H^2} < \varepsilon \quad \forall m,n \ge N$$

So, being $w_n \in H^1_0, H^2$, we have that, by the completeness of the two spaces, $w^n \to w' \in H^1_0$ in H^1 and $w^n \to w'' \in H^2$ in H^2. Moreover

$$\|w' - w''\|_2 \le \|w' - w^n\|_2 + \|w^n - w''\|_2 \le \|w' - w^n\|_{H^1} + \|w^n - w''\|_{H^2} \to 0$$

so that $w' = w''$ almost everywhere. Moreover, observe that the norm used above $\|\cdot\|_{H^1_0} + \|\cdot\|_{H^2}$ is equivalent to $\|\cdot\|_{H^2}$.

[27] If u_k is a bounded sequence in X, it is in particular a bounded sequence in $H^1_0(\Omega)$ (that is injected $\hookrightarrow L^2(\Omega)$). So we can extract $u_{k_j} \to u \in L^2(\Omega)$ in $\|\cdot\|_2$.

Observe now that the inclusion $i : L^2(0,T_*;H^2(\Omega)) \to L^2(0,T_*;L^2(\Omega))$ is continuous, since

$$\|v\|_{L^2(0,T_*;L^2(\Omega))} = \left(\int_0^{T_*} \|v\|_2^2 \, dt\right)^{\frac{1}{2}} \leq \left(\int_0^{T_*} \|v\|_{H^2}^2 \, dt\right)^{\frac{1}{2}} = \|v\|_{L^2(0,T_*;H^2(\Omega))}$$

So, if $f \in (L^2(0,T_*;L^2(\Omega)))^*$, we have that

$$\lim_{m\to+\infty} f(i(u^{m_k})) = f(i(u))$$

since[28] $u^m \rightharpoonup u$ in $L^2(0,T_*;H^2(\Omega))$. This means that $u^{m_k} \rightharpoonup u$ in $L^2(0,T_*;L^2(\Omega))$. Moreover $u^{m_k} \rightharpoonup w$ in $L^2(0,T_*;L^2(\Omega))$, since strong convergence implies weak convergence. The uniqueness of the weak limit leads to

$$w = u$$

in the sense[29] of $L^2(0,T_*;L^2(\Omega))$, i.e. $\|w-u\|_2 = 0$ almost every $t \in (0,T_*)$. So,

$$\int_0^{T_*} \left\|u^{m_k} - u\right\|_2^2 \, dt = \int_0^{T_*} \left\|u^{m_k} - w + w - u\right\|_2^2 \, dt \leq$$

$$\leq 2\int_0^{T_*} \left\|u^{m_k} - w\right\|_2^2 \, dt + 2\int_0^{T_*} \left\|w-u\right\|_2^2 \, dt = 2\int_0^{T_*} \|u^{m_k} - w\|_2^2 \, dt \to 0 \;\text{ as } k \to +\infty$$

Now we want to pass to another subsequence for proving the same result for ∇u^{m_k}. Consider this time the chain $X := H^1(\Omega) \subseteq B := L^2(\Omega) =: Y$.
The inclusion $X \hookrightarrow B$ is compact thaks to the Rellich-Kondrachov theorem. So it is compact also the inclusion

$$L^2(0,T_*;H^1(\Omega)) \cap \{\varphi : \; \partial_t\varphi \in L^1(0,T_*;L^2(\Omega))\} \hookrightarrow L^2(0,T_*;L^2(\Omega))$$

More precisely, if we consider ∇u^{m_k}, it is bounded in both the spaces. In fact

$$\left(\int_0^{T_*} \|\nabla u^{m_k}\|_{H^1} dt\right)^{\frac{1}{2}} \leq \left(\int_0^{T_*} \|u^{m_k}\|_{H^2} dt\right)^{\frac{1}{2}} \leq T_*^{\frac{1}{2}}\hat{K}$$

thanks to the estimate (11.72). Moreover, since $\partial_t \nabla u^{m_k} = \nabla u_t^{m_k}$, thanks to the regularity and the fact that the varible x, t are separated, we have

$$\left(\int_0^{T_*} \|\partial_t \nabla u^{m_k}\|_2^2 dt\right)^{\frac{1}{2}} = \left(\int_0^{T_*} \|\nabla u_t^{m_k}\|_2^2 dt\right)^{\frac{1}{2}} \overset{(11.75)}{\leq} \sqrt{\hat{H}W_0 + \hat{H}\exp(\hat{H}T_* M^4)}$$

[28] We have $|f(i(u))| \leq C\|i(u)\|_{L^2(0,T_*;L^2)} \leq C\|u\|_{L^2(0,T_*;H^2)}$.
[29] In fact

$$\|w-u\|_{L^2(0,T_*;L^2)} = \langle w-u, w-u\rangle_{L^2(0,T_*;L^2)} = \langle w, w-u\rangle_{L^2(0,T_*;L^2)} - \langle u, w-u\rangle_{L^2(0,T_*;L^2)} =$$

$$= \lim_{m\to+\infty} \langle u^m, w-u\rangle_{L^2(0,T_*;L^2)} - \lim_{m\to+\infty} \langle u^m, w-u\rangle_{L^2(0,T_*;L^2)} = 0$$

So, passing again to a subsequence, we have that exists $w \in L^2(0, T_*; L^2(\Omega))$ such that

$$\nabla u^{m_{k_h}} \to w \quad \text{in } L^2(0, T_*; L^2(\Omega))$$

The strong convergence of $u^{m_{k_h}}$ to u is also true, since we only have passed to a subsequence. If we prove that $w = \nabla u$ in the sense of $L^2(0, T_*; L^2(\Omega))$, then we conclude. Remember that $u^{m_{k_h}} \rightharpoonup u$ in $L^2(0, T_*; H^2(\Omega))$, thanks to lemma 11.3. Moreover we have $\nabla u^{m_{k_h}} \rightharpoonup w$ in $L^2(0, T_*; L^2(\Omega))$. In fact, consider the gradient operator

$$\nabla : L^2(0, T_*; H^2(\Omega)) \to L^2(0, T_*; L^2(\Omega))$$

$$v \to \nabla v$$

The operator is continuous. In fact

$$\|\nabla v\|_{L^2(0,T_*;L^2(\Omega))} := \left(\int_0^{T_*} \|\nabla v\|_2^2 \, dt \right)^{\frac{1}{2}} \leq \left(\int_0^{T_*} \|v\|_{H^2}^2 \, dt \right)^{\frac{1}{2}} =: \|v\|_{L^2(0,T_*;H^2(\Omega))}$$

So, if $f \in (L^2(0, T_*; L^2(\Omega)))^*$, then for every $v \in L^2(0, T_*; H^2(\Omega))$,

$$|f(\nabla v)| \leq \|\nabla v\|_{L^2(0,T_*;L^2(\Omega))} \leq \|v\|_{L^2(0,T_*;H^2(\Omega))}$$

so that $f(\nabla \cdot) \in (L^2(0, T_*; H^2(\Omega)))^*$. Then

$$\lim_{h \to +\infty} f(\nabla u^{m_{k_h}}) = f(\nabla u)$$

thanks to the weak convergence of $u^{m_{k_h}}$ to u in $L^2(0, T_*; H^2(\Omega))$. This means that

$$\nabla u^{m_{k_h}} \rightharpoonup \nabla u \quad \text{in } L^2(0, T_*; L^2(\Omega))$$

By the uniquess of the weak limit, as above, we have that $\nabla u = w$ in $L^2(0, T_*; L^2(\Omega))$, and so

$$\int_0^{T_*} \left\|\nabla u^{m_{k_h}} - \nabla u\right\|_2^2 \, dt \leq 2 \int_0^{T_*} \left\|\nabla u^{m_{k_h}} - w\right\|_2^2 \, dt + 2 \int_0^{T_*} \left\|w - \nabla u\right\|_2^2 \, dt = \tag{11.77}$$

$$= 2 \int_0^{T_*} \left\|\nabla u^{m_{k_h}} - w\right\|_2^2 \, dt \to 0 \quad \text{as } h \to +\infty$$

So we have the thesis. ∎

We finally prove the main proposition of the subsection.

Proposition 11.19. *Let Ω be a bounded subset of $\mathbb{R}^3$. Suppose that we are in the hypothesis of proposition 11.17. Then, the weak-star limit $u \in L^\infty(0, T_*; H^2(\Omega))$ of the sequence u^m is such that, for almost every $t \in (0, T_*)$, $u(t) \in H_0^1(\Omega)$ and $\nabla \cdot u(t) = 0$. In particular, this means that for almost every $t \in (0, T_*)$, $u(t) \in X$.*

Proof. We first remember that, for almost every $t \in (0, T_*)$, we have $u(t) \in H^1(\Omega) = W^{1,2}(\Omega)$. So, we can use theorem 4.10 concerning the trace operator: thanks to this theorem, we only have to verify that $Tu = 0$ on $\partial\Omega$.
Let u^{m_k} the sequence assured by proposition 11.18. Remember, in particular, that $u^{m_k}(t) \in C^1(\overline{\Omega})$ for every $t \in (0, T_*)$. In particular, since $w^h|_{\partial\Omega} = 0$, it is zero on $\partial\Omega$. It follows that

$$Tu^{m_k} = 0 \quad \text{on } \partial\Omega$$

For almost every $t \in (0, T_*)$, sa $t \in (0, T_*)/E$, with $|E| = 0$, we can consider Tu, since $u(t) \in H^1(\Omega)$. So

$$\|Tu\|_{L^2(\partial\Omega)} = \|T(u - u^{m_k})\|_{L^2(\partial\Omega)} \leq C\|u - u^{m_k}\|_{H^1(\Omega)}$$

Squaring both sides and integrating in $(0, T_*)$ we have

$$\int_0^{T_*} \|Tu\|^2_{L^2(\partial\Omega)}\, dt \leq C^2 \int_0^{T_*} \|u - u^{m_k}\|^2_{H^1(\Omega)}\, dt = C^2 \int_0^{T_*} \|u - u^{m_k}\|_2^2\, dt + C^2 \int_0^{T_*} \|\nabla u - \nabla u^{m_k}\|_2^2\, dt$$

Since the two pieces on the right side vanish, we have that

$$\int_0^{T_*} \|Tu\|^2_{L^2(\partial\Omega)}\, dt = 0$$

This means that for $t \in (0, T_*)/A$, with $|A| = 0$, the trace is $\|Tu\|_{L^2(\partial\Omega)}(t) = 0$, i.e. $(Tu)(t) = 0$ on $\partial\Omega$. So, for $t \in (0, T_*)/(A \cup E)$, since $u(t) \in H^1(\Omega)$, we have that $u(t) \in H^1_0(\Omega)$.

It remains to prove the incompressibility condition. We know that, for almost every $t \in (0, T_*)$, $u(t) \in H^1(\Omega)$ and in particular, by the definition of weak derivative,

$$\int_\Omega u_i(t)\partial_{x_i}\varphi\, dx = -\int_\Omega \partial_{x_i}u_i(t)\varphi\, dx \quad \forall\varphi \in C_c^\infty(\Omega)$$

If $\nabla \cdot u$ is the weak divergence, we have that

$$\int_\Omega \nabla \cdot u(t)\varphi\, dx = \sum_{i=1}^3 \int_\Omega \partial_{x_i}u_i(t)\varphi\, dx = -\sum_{i=1}^3 \int_\Omega u_i(t)\partial_{x_i}\varphi\, dx = -\int_\Omega u(t) \cdot \nabla\varphi\, dx$$

Remark 11.24. If we consider the functional

$$f(w) := -\int_\Omega w \cdot \nabla\varphi\, dx \quad \forall w \in L^2(\Omega)$$

then it is linear and continuous. In fact

$$|f(w)| \leq \|w\|_2\|\nabla\varphi\|_2$$

and so the continuity is proved, as a functional $f : L^2(\Omega) \to \mathbb{R}$. $\square$

We know moreover that $u^{m_k} \to u$ in $L^2(0,T_*;L^2(\Omega))$.
In other words

$$\lim_{k\to+\infty}\int_0^{T_*}\|u^{m_k}-u\|_2^2\,dt=0$$

and so, using Theorem 3.12, pg. 68 in [24], there exists a subsequence $u^{m_{k_h}}$ such that, for almost every $t\in(0,T_*)$,

$$\lim_{h\to+\infty}\|u^{m_{k_h}}-u\|_2(t)=0$$

So, for almost every $t\in(0,T_*)$,

$$\lim_{h\to+\infty}\int_\Omega u^{m_{k_h}}(t)\cdot\nabla\varphi\,dx=\int_\Omega u(t)\cdot\nabla\varphi\,dx$$

On the other hand we have

$$\int_\Omega u^{m_{k_h}}(t)\cdot\nabla\varphi\,dx=-\int_\Omega\nabla\cdot u^{m_{k_h}}(t)\varphi\,dx=0$$

since $\nabla\cdot u^{m_{k_h}}(t)=0$ for every t in classic sense, by construction.
It follows that, for almost every $t\in(0,T_*)$,

$$\int_\Omega u(t)\cdot\nabla\varphi\,dx=0\quad\varphi\in C_c^\infty(\Omega)$$

In other words, for almost every $t\in(0,T_*)$ we have $\nabla\cdot u(t)=0$ in the weak sense. ∎

11.6.5 Integrability property of the limits

In order to proceed with the proof of the fact that the pair (u,ρ) is solution to the original Navier-Stokes equation, we need to prove the following lemma.

Lemma 11.4. *Let Ω be a bounded domain of $\mathbb{R}^3$, and suppose that (u,ρ) are the functions built in propositions 11.15-11.19. Then for almost every $t\in(0,T_*)$ we have that*

$$\rho u_t+\rho\big(u\cdot\nabla u\big)-\mu\Delta u\in L^2(\Omega)$$

Moreover, the following integrals are finite

$$\int_0^{T_*}\int_\Omega\big|\rho(t)u_t(t)\cdot\phi\big|\,dx\,dt,\quad\int_0^{T_*}\int_\Omega\big|\rho(t)\big(u(t)\cdot\nabla u(t)\big)\cdot\phi\big|\,dx\,dt,\quad\int_0^{T_*}\int_\Omega\big|\Delta u(t)\cdot\phi\big|\,dx\,dt \tag{11.78}$$

Finally

$$\int_0^{T_*}\|\rho(t)u_t(t)+\rho(t)u(t)\cdot\nabla u(t)-\mu\Delta u(t)\|_2^2\,\,dt<+\infty \tag{11.79}$$

Observe that (11.78) *allows us to write the integrals without the absolute value.*

Proof. For those t such that $u(t) \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\rho(t) \in L^\infty(\Omega)$, we have, thanks to lemma 9.6,

$$\|u(t)\|_{L^\infty(\Omega)} \leq c\big(\|\Delta u(t)\|_2\big)^{\frac{3}{4}}\big(\|\nabla u(t)\|_2\big)^{\frac{1}{4}}$$

Moreover, we have that

$$\|\rho(t)u_t(t) + \rho(t)u(t)\cdot\nabla u(t) - \mu\Delta u(t)\|_2 \leq \|\rho(t)u_t(t)\|_2 + \|\rho(t)u(t)\cdot\nabla u(t)\|_2 + \mu\|\Delta u(t)\|_2$$

We want to estimate the three addends separately. In particular we have

$$\|\rho(t)u_t(t)\|_2 = \bigg(\int_\Omega |\rho(t)|^2|u_t(t)|^2 dx\bigg)^{\frac{1}{2}} \leq \|\rho(t)\|_\infty\|u_t(t)\|_2 < +\infty$$

since $\rho(t) \in L^\infty(\Omega)$. Moreover

$$\|\rho(t)u(t)\cdot\nabla u(t)\|_2 = \bigg(\int_\Omega |\rho(t)|^2|u(t)\cdot\nabla u(t)|^2 dx\bigg)^{\frac{1}{2}} \leq \|\rho(t)\|_\infty\bigg(\int_\Omega |u(t)|^2|\nabla u(t)|^2 dx\bigg)^{\frac{1}{2}} \leq$$

$$\leq \|\rho(t)\|_\infty\|u(t)\|_\infty\|\nabla u(t)\|_2$$

that is finite. Finally $\|\Delta u(t)\|_2 < +\infty$ since $u(t) \in H^2(\Omega)$. So, for almost every $t \in (0, T_*)$, the function is in $L^2(\Omega)$.

Moreover, we have the following integrability property. In fact, if $\phi \in L^2(\Omega)$,

$$\int_0^{T_*}\int_\Omega \big|\rho(t)u_t(t)\cdot\phi\big|\, dx\, dt \leq \int_0^{T_*} \|\rho(t)\|_\infty\|u_t(t)\|_2\|\phi\|_2 \leq \sqrt{T_*}\|\rho\|_{L^\infty(0,T_*;L^\infty(\Omega))}\|\phi\|_2\|u_t\|_{L^2(0,T_*;L^2(\Omega))}$$

and

$$\int_0^{T_*}\int_\Omega \big|\Delta u(t)\cdot\phi\big|\, dx\, dt \leq \int_0^{T_*} \|\Delta u(t)\|_2\|\phi\|_2 dt \leq \|\phi\|_2\sqrt{5}\int_0^{T_*} \|\nabla^2 u(t)\|_2 dt \leq$$

$$\leq \|\phi\|_2\sqrt{5}T_*^{\frac{1}{2}}\bigg(\int_0^{T_*} \|\nabla^2 u(t)\|_2^2 dt\bigg)^{\frac{1}{2}} \leq \sqrt{5}T_*^{\frac{1}{2}}\|\phi\|_2\|u\|_{L^2(0,T_*;H^2(\Omega))}$$

and finally

$$\int_0^{T_*}\int_\Omega \big|\rho(t)\big(u(t)\cdot\nabla u(t)\big)\cdot\phi\big|\, dx\, dt \leq \int_0^{T_*} \|\rho(t)\|_\infty\int_\Omega |u(t)\cdot\nabla u(t)||\phi| dx\, dt \leq$$

$$\leq \|\rho\|_{L^\infty(0,T_*;L^\infty(\Omega))}\|\phi\|_2\int_0^{T_*} \|u(t)\cdot\nabla u(t)\|_2\, dt$$

Moreover, since $u \in H_0^1$, $\|u\|_4 \leq C_1\|\nabla u\|_2$ and moreover, using (11.40), $\|\nabla u\|_4 \leq |\Omega|^{\frac{1}{12}}\|\nabla u\|_6 \leq |\Omega|^{\frac{1}{12}}C_2\|\nabla u\|_{H^1} \leq |\Omega|^{\frac{1}{12}}C_2\|u\|_{H^2}$, so that

$$\|u(t)\cdot\nabla u(t)\|_2 \leq \bigg(\int_\Omega |u(t)|^2|\nabla u(t)|^2 dx\bigg)^{\frac{1}{2}} \leq \|u(t)\|_4\|\nabla u(t)\|_4 \leq C_1|\Omega|^{\frac{1}{12}}C_2\|u(t)\|_{H^2}^2$$

and so

$$\int_0^{T_*} \|u(t)\cdot\nabla u(t)\|_2 \, dt \le C_1|\Omega|^{\frac{1}{12}} C_2 \int_0^{T_*} \|u(t)\|^2_{H^2} \, dt < +\infty$$

since $u \in L^2(0,T_*;H^2(\Omega))$.

Finally we have to prove

$$\int_0^{T_*} \|\rho(t)u_t(t) + \rho(t)u(t)\cdot\nabla u(t) - \mu\Delta u(t)\|^2_2(t) \, dt < +\infty$$

It holds the inequality

$$\big\|\rho(t)u_t(t)+\rho(t)u(t)\cdot\nabla u(t)-\mu\Delta u(t)\big\|^2_2 \le \big(\|\rho(t)u_t(t)\|_2+\|\rho(t)u(t)\cdot\nabla u(t)\|_2+\mu\|\Delta u(t)\|_2\big)^2 \le$$

$$\le 5\big(\|\rho(t)u_t(t)\|^2_2 + \|\rho(t)u(t)\cdot\nabla u(t)\|^2_2 + \mu\|\Delta u(t)\|^2_2\big) \le$$

$$\le 5\big(\|\rho(t)\|^2_\infty\|u_t(t)\|^2_2 + c^2\|\rho(t)\|^2_\infty\|\Delta u(t)\|^{\frac{3}{2}}_2\|\nabla u(t)\|^{\frac{1}{2}}_2\|\nabla u(t)\|^2_2 + \mu\|\Delta u(t)\|^2_2\big)$$

Finally

$$\int_0^{T_*} \big\|\rho(t)u_t(t) + \rho(t)u(t)\cdot\nabla u(t) - \mu\Delta u(t)\big\|^2_2 \, dt \le$$

$$\le 5\bigg(\int_0^{T_*} \|\rho(t)\|^2_\infty\|u_t(t)\|^2_2 dt + c^2\int_0^{T_*} \|\rho(t)\|^2_\infty\|\Delta u(t)\|^{\frac{3}{2}}_2\|\nabla u(t)\|^{\frac{5}{2}}_2 \, dt + \mu\int_0^{T_*} \|\Delta u(t)\|^2_2 dt\bigg) \le$$

$$\le 5\bigg(\|\rho\|^2_{L^\infty(0,T_*;L^\infty)}\int_0^{T_*} \|u_t\|^2_2 \, dt + 5^{\frac{3}{4}}c^2\|\rho\|^2_{L^\infty(0,T_*;L^\infty)}T_*\|u\|^4_{L^\infty(0,T_*;H^2)} + 5\mu\int_0^{T_*} \|u\|^2_{H^2} \, dt\bigg)$$

that is finite since[30] $\rho \in L^\infty(0,T_*;L^\infty)$, $u_t \in L^2(0,T_*;H^1_0(\Omega))$, $u \in L^\infty(0,T_*;H^2)$. ∎

11.7 Further regularity results: the transport equation

Proposition 11.20. *Let Ω be a bounded domain in $\mathbb{R}^3$. Consider the Navier-Stokes problem over Ω as in proposition 11.1. Let $\overline{\rho}_0 \in C^1(\overline{\Omega})$ and $T > 0$. Let $\rho^m \in C^1([0,T]\times\overline{\Omega})$ and $u^m \in C^1([0,T];X^m)$ the approximate solutions built in proposition 11.1 and the function $\rho \in L^\infty(0,T_*;L^\infty(\Omega))$ such that*

$$\rho^m \overset{*}{\rightharpoonup} \rho \qquad in \;\; L^\infty(0,T_*;L^\infty(\Omega)) \tag{11.80}$$

We know that hold all the properties proved in section 11.6. Here T_* is the local time provided by proposition 11.13. Then, we have moreover that

$$\rho^m \to \rho \quad in \;\; L^\infty(0,T_*;L^p(\Omega))$$

[30]We also used that $\|\Delta u(t)\|_2 \le \sqrt{5}\|u\|_{H^2}$.

Proof. The whole proof is a rereading of theorem 8.6. In fact, remember that $u^m \in L^\infty(0,T_*;H^2(\Omega)) \subseteq L^1(0,T_*;W^{1,1}(\Omega))$ and $\nabla \cdot u^m \equiv 0$. Moreover, since $u^m \to u$ in $L^2(0,T_*;L^2(\Omega))$, we have that u^m converges to u in $L^1(0,T_*;L^1(\Omega))$. Now, ρ^m is a weak solution of

$$\begin{cases} \partial_t \rho^m + u^m \cdot \nabla \rho^m = 0 \\ \rho^m(0) = \overline{\rho}_0 \end{cases}$$

and $u^m \in L^1(0,T;W^{1,1}(\Omega))$, $\nabla \cdot u^m = 0$, so, using theorem 8.2, with $p = \infty$ and $q = 1$, ρ^m is a renormalized solution. The initial condition is $\rho^m(0) = \overline{\rho}_0 \in C^1(\overline{\Omega})$. This implies that for every β admissible function $\beta(\rho^m(0)) \to \beta(\overline{\rho}_0)$ in $L^1(\Omega)$. Moreover ρ^m is bounded in $L^\infty(0,T_*;L^\infty(\Omega))$, and so in $L^\infty(0,T_*;L^p(\Omega))$ for every $p \in [1,\infty]$. Then, using theorem 8.4, ρ^m converges to some $\overline{\rho}$, renormalized solution with initial condition $\overline{\rho}_0$, in $C([0,T_*];L^p(\Omega))$.

On the other hand, we have that ρ^m is also a weak solution with initial density $\rho^m(0)$, that is

$$\int_0^{T_*} \left(\int_\Omega \rho^m \varphi_t + \rho^m u^m \cdot \nabla \varphi \, dx \right) dt = - \int_\Omega \overline{\rho}_0(x) \varphi(x,0) \, dx$$

for every $\varphi \in C^1([0,T_*];H^1(\Omega))$ with $\varphi(x,T_*) = 0$ a.e. in Ω. Thanks to (11.80), and the fact that $\varphi_t \in L^1(0,T;L^1(\Omega))$, it follows that

$$\lim_{m\to\infty} \int_0^{T_*} \int_\Omega \rho^m \varphi_t \, dx \, dt = \int_0^{T_*} \int_\Omega \rho \, \varphi_t \, dx \, dt$$

Moreover, observe that

$$\left| \int_0^{T_*} \left(\int_\Omega (\rho^m u^m - \rho u) \cdot \nabla \varphi \, dx \right) dt \right| =$$

$$= \left| \int_0^{T_*} \left(\int_\Omega (\rho^m - \rho) u \cdot \nabla \varphi \, dx \right) dt - \int_0^{T_*} \left(\int_\Omega \rho^m (u^m - u) \cdot \nabla \varphi \, dx \right) dt \right| \leq$$

$$\leq \left| \int_0^{T_*} \left(\int_\Omega (\rho^m - \rho) u \cdot \nabla \varphi \, dx \right) dt \right| + \|\rho^m\|_{L^\infty(0,T_*;L^\infty(\Omega))} \|u^m - u\|_{L^2(0,T_*;L^2(\Omega))} \|\nabla \varphi\|_{L^2(0,T_*;L^2(\Omega))}$$

Since $\|\rho^m\|_{L^\infty(0,T_*;L^\infty(\Omega))}$ is bounded and $u^m \to u$ in $L^2(0,T_*;L^2(\Omega))$, the second addend vanishes. Moreover, we have

$$\int_0^{T_*} \left(\int_\Omega |u \cdot \nabla \varphi| \, dx \right) dt \leq \|u\|_{L^2(0,T_*;L^2(\Omega))} \|\nabla \varphi\|_{L^2(0,T_*;L^2(\Omega))}$$

so that $u \cdot \nabla \varphi \in L^1(0,T_*;L^1(\Omega))$. So, thanks again to the weak star convergence (11.80), we have

$$\lim_{m\to\infty} \int_0^{T_*} \left(\int_\Omega (\rho^m - \rho) u \cdot \nabla \varphi \, dx \right) dt = 0$$

If follows that

$$\int_0^{T_*} \left(\int_\Omega \rho \varphi_t + \rho u \cdot \nabla \varphi \, dx \right) dt = - \int_\Omega \overline{\rho}_0(x) \varphi(x,0) \, dx$$

that is, ρ is a weak solution to the transport equation with velocity field u and initial density $\overline{\rho}_0$. In particular, applying lemma 8.4 with $p=\infty$ and $q=1$, we have that ρ is a renormalized solution. Subtracting the two definitions of renormalized solution, we have that

$$\int_0^{T_*}\left(\int_\Omega (\beta(\rho)-\beta(\overline{\rho}))\varphi_t\,dx\right)dt+\int_0^{T_*}\left(\int_\Omega (\beta(\rho)-\beta(\overline{\rho}))u\cdot\nabla\varphi\,dx\right)dt=0$$

In other words $\beta(\rho)-\beta(\overline{\rho})\in L^\infty(0,T;L^\infty(\Omega))$ is weak solution of the transport equation with velocity field $u\in L^1(0,T_*;W^{1,1}(\Omega))$, and initial density $\rho_0'\equiv 0$. Using theorem 8.5 with $p=\infty$ and $q=1$, we have that $\beta(\rho)\equiv\beta(\overline{\rho})$ for every admissible function β. So, choosing β_M such that $\beta_M(s)=s$ if $|s|\le M$, and β_M bounded, $C^1(\mathbb{R})$ and admissible, we have that

$$\rho\equiv\overline{\rho}\qquad\text{over }\{|\rho|\le M,\ |\overline{\rho}|\le M\}$$

Letting $M\to\infty$, we have the equality of the functions in the whole space $\Omega\times(0,T_*)$. ∎

11.8 Weak solution to the incompressible Navier-Stokes equations

The notion of weak solution for the Navier-Stokes equation (and, clearly, also the transport equation) has been introduced in Chapter 10. In this section we will, first, provide a further integral of the momentum equation, verifying, then, that the pair of solutions (ρ,u) satisfies also the weak formulation as introduced in Chapter 10.
We collect here the properties deduced in sections 11.2-11.7.

$$u^m\overset{*}{\rightharpoonup}u\ \text{ in } L^\infty(0,T_*;H^2(\Omega)),\quad \rho^m\to\rho\ \text{ in } L^\infty(0,T_*;L^q(\Omega)) \tag{11.81}$$

$$u_t^m\rightharpoonup u_t\ \text{ in } L^2(0,T_*;H_0^1(\Omega)) \tag{11.82}$$

Remember also that $\rho^m\overset{*}{\rightharpoonup}\rho$ in $L^\infty(0,T_*;L^\infty(\Omega))$. Moreover, u^m can be choosen such that

$$u^m\rightharpoonup u\quad\text{in } L^2(0,T_*;H^2(\Omega)) \tag{11.83}$$

and

$$u^m\to u,\quad \nabla u^m\to\nabla u\qquad\text{in}\quad L^2(0,T_*;L^2(\Omega)) \tag{11.84}$$

11.8.1 Statement of the theorems

In this subsection we prove an integral version of the problem, that in particular implies the usual definition.

Proposition 11.21. *Let Ω be a bounded domain of $\mathbb{R}^3$. Consider the pair of solution (u,ρ), as introduced in sections[31] 11.2-11.7. Then, for every $\nu\in X$, exists a subset $E_\nu\subset(0,T_*)$, with $|E_\nu|=0$, such that*

$$\int_\Omega(\rho u_t+\rho\,(u\cdot\nabla u)-\mu\Delta u)\cdot\nu\,dx=0\qquad\forall t\in(0,T_*)/E_\nu$$

[31] *i.e.*, with the properties summarized above.

Corollary 11.1. *Let Ω be a bounded domain of $\mathbb{R}^3$. Consider the pair of solution (u,ρ), as introduced in sections 11.2-11.7. Then exists a subset $E \subset (0,T_*)$ with $|E| = 0$ such that, for every $\psi \in W^{1,2}_{0,\sigma}(\Omega)$,*

$$\int_\Omega (\rho u_t + \rho\,(u\cdot\nabla u) - \mu\Delta u)\cdot\psi\ dx = 0 \qquad \forall t \in (0,T_*)/E$$

Theorem 11.3. *Let Ω be a bounded domain of $\mathbb{R}^3$. Consider the pair of solution (u,ρ), as introduced in sections 11.2-11.7. Then for every $\varphi \in C^1([0,T_*];W^{1,2}_{0,\sigma}(\Omega))$ such that $\varphi(x,T_*) = 0$ a.e. in Ω, we have*

$$\int_0^{T_*}\int_\Omega (\rho u_t + \rho\,(u\cdot\nabla u) - \mu\Delta u)\cdot\varphi(x,t)\ dx\ dt = 0$$

11.8.2 Proof of proposition 11.21

Remember, from proposition 11.1, that the pair (u^m,ρ^m) is such that

$$\int_\Omega \left(\rho^m u^m_t + \rho^m(\nabla u^m)u^m\right)\cdot\phi + \mu\nabla u^m\cdot\nabla\phi\ dx = 0\ \ \forall\phi\in X^m \tag{11.85}$$

Remark 11.25. We have extracted a lot of subsequences of u^m. So, the sequence that we are considering is not indexed by the natural numbers, but by a subsequence of $\mathbb{N}$, $m \longleftrightarrow n_m$. However, the property continues to hold, provided that $\phi\in X^m$ with the right m. □

Remark 11.26. We have that, if $\phi\in X^m$,

$$\int_\Omega \nabla u^m\cdot\nabla\phi\ dx = -\int_\Omega \Delta u^m\cdot\phi\ dx$$

using the arguments in section 9.7.3. □

Let, now, E be a measurable subset of $(0,T_*)$. Then if a function g is in $L^p(0,T_*;X)$, it is in particular in $L^p(E;X)$ in the sense that

$$\|g\|_{L^p(E;X)} \le \|g\|_{L^p(0,T_*;X)}$$

since the L^p temporal norm is bigger in a larger domain (and this is true for $p=\infty$ thanks to the property of the supremum, and for $p<\infty$, thanks to the monotonicity property of the integral operator). Moreover, the integrals

$$\int_E\int_\Omega \rho u_t\cdot\phi\ dx\ dt,\quad \int_E\int_\Omega \Delta u\cdot\phi\ dx\ dt,\quad \int_E\int_\Omega (\rho u\cdot\nabla u)\cdot\phi\ dx\ dt$$

are defined, as proved in section 11.6.5, provided that $\phi\in L^2(\Omega)$. So, if we prove that

$$\lim_{m\to+\infty}\int_E\int_\Omega \rho^m u^m_t\cdot\nu^m dx\ dt = \int_E\int_\Omega \rho u_t\cdot\phi\ dx \tag{11.86}$$

$$\lim_{m\to+\infty}\int_E\int_\Omega \Delta u^m\cdot\nu^m dx\ dt = \int_E\int_\Omega \Delta u\cdot\phi\ dx\ dt \tag{11.87}$$

$$\lim_{m\to+\infty}\int_E\int_\Omega(\rho^m u^m\cdot\nabla u^m)\cdot\nu^m dx\ dt=\int_E\int_\Omega(\rho u\cdot\nabla u)\cdot\phi\ dx\ dt \tag{11.88}$$

for a suitable sequence $\{\nu^m\}_m$ such that $\nu^m\to\phi\in X$ in the H^2-norm, we have practically finished: if we fix $\phi\in X$ and let E change in the measurable subsets of $(0,T_*)$, we have that

$$\int_\Omega(\rho u_t+\rho u\cdot\nabla u-\mu\Delta u)\cdot\phi\ dx=0\quad\forall t\in(0,T_*)/E$$

with $|E|=0$, thanks to a well-known result of measure theory. Clearly, in this situation $E=E_\phi$, since the integrand function depends on ϕ. We now choose the sequence $\{\nu^m\}_m$. We know that for every $\phi\in X$ holds the limit

$$\lim_{m\to+\infty}\left\|\sum_{k=1}^m\langle\phi,\phi^k\rangle_2\phi^k-\phi\right\|_{H^2}=0$$

So, we can take

$$\nu^m:=\sum_{k=1}^m\langle\phi,\phi^k\rangle_2\phi^k\in X^m \tag{11.89}$$

Observe that $\nu^m\in X^m$, as required in (11.85).

So, (11.86), (11.87), (11.88) holds, and thanks to (11.85), we have the thesis.

Remark 11.27. If (11.86), (11.87), (11.88) does not hold for every $m\in\mathbb{N}$ but only for a subsequence, the same subsequence where the convergences (11.81)-(11.84) hold, we can consider the same subsequence in ν^m; the convergence continues to hold, since we are considering a subsequence. □

We now prove the equalities (11.86), (11.87), (11.88).

Proof of equality (11.86)**.** To prove this first limit, we can write

$$\left|\int_E\int_\Omega\rho^m u_t^m\cdot\nu^m dx\ dt-\int_E\int_\Omega\rho u_t\cdot\phi\ dx\ dt\right|=$$

$$=\left|\int_E\int_\Omega\rho^m u_t^m\cdot(\nu^m-\phi)\ dx\ dt+\int_E\int_\Omega(\rho^m u_t^m-\rho u_t)\cdot\phi\ dx\ dt\right|\le$$

$$\le\int_E\int_\Omega|\rho^m u_t^m||\nu^m-\phi|\ dx\ dt+|\int_E\int_\Omega(\rho^m u_t^m-\rho u_t)\cdot\phi\ dx\ dt|\le$$

$$\le\int_0^{T_*}\|\rho^m u_t^m\|_2\|\nu^m-\phi\|_2dt+|\int_E\int_\Omega(\rho^m u_t^m-\rho u_t^m+\rho u_t^m-\rho u_t)\cdot\phi\ dx\ dt|\le$$

$$\le M_0^*\|\nu^m-\phi\|_{H^2}\left(\int_0^{T_*}\|u_t^m\|_2^2\right)^{\frac12}+|\int_E\int_\Omega\big((\rho^m-\rho)u_t^m\big)\cdot\phi\ dx\ dt|+|\int_E\int_\Omega\big(\rho(u_t^m-u_t)\big)\cdot\phi\ dx\ dt|$$

where $M_0^*:=(\|\rho_0\|_\infty+1)\sqrt{T_*}$. The first piece is bounded by

$$M_0^*\|\nu^m-\phi\|_{H^2}\left(\int_0^{T_*}\|u_t^m\|_{H^1}^2dt\right)^{\frac12}\equiv M_0^*\|\nu^m-\phi\|_{H^2}\|u_t^m\|_{L^2(0,T_*;H_0^1(\Omega))}\to0\ \text{ as } m\to+\infty \tag{11.90}$$

since $\|u_t^m\|_{L^2(0,T_*;H_0^1(\Omega))}$ is bounded in m. In fact, see i.e. [10, p.723], any weakly convergent sequence is bounded, and we know that $u_t^m \rightharpoonup u_t$ in $L^2(0,T_*;H_0^1(\Omega))$.

In the second piece we use the convegence of ρ^m to ρ and, again, the boundness of u_t^m. We have

$$|\int_E\int_\Omega \big((\rho^m-\rho)u_t^m\big)\cdot\phi\, dx\, dt| \le \int_E\int_\Omega |u_t^m||(\rho^m-\rho)\phi| dx\, dt \le \int_0^{T_*} \|u_t^m\|_2\|(\rho^m-\rho)\phi\|_2 dt \le$$

$$\le \Big(\int_0^{T_*} \|u_t^m\|_2^2 dt\Big)^{\frac{1}{2}}\Big(\int_0^{T_*} \|(\rho^m-\rho)\phi\|_2^2 dt\Big)^{\frac{1}{2}} \le \|u_t^m\|_{L^2(0,T_*;H_0^1(\Omega))}\Big(\int_0^{T_*}\int_\Omega |(\rho^m-\rho)\phi|^2 dx\, dt\Big)^{\frac{1}{2}}$$

The last term has the first factor bounded, as above; the second term can be treated as follows:

$$\int_\Omega |(\rho^m-\rho)\phi|^2 dx = \int_\Omega |\rho^m-\rho|^2|\phi|^2 dx \le \||\rho^m-\rho|^2\|_{\frac{3}{2}}\||\phi|^2\|_3 =$$

$$= (\int_\Omega |\rho^m-\rho|^3 dx)^{\frac{2}{3}}(\int_\Omega |\phi|^6 dx)^{\frac{1}{3}} = \|\rho^m-\rho\|_3^2\|\phi\|_6^2 \le C^2\|\rho^m-\rho\|_3^2\|\nabla\phi\|_2^2$$

using that

$$\|v\|_6 \le C\|\nabla v\|_2 \tag{11.91}$$

for every $v \in W_0^{1,2}$ (and $\phi \in H_0^1 \equiv W_0^{1,2}(\Omega)$). Finally we get

$$\Big(\int_0^{T_*}\int_\Omega |(\rho^m-\rho)\phi|^2 dx\, dt\Big)^{\frac{1}{2}} \le C\|\nabla\phi\|_2\Big(\int_0^{T_*} \|\rho^m-\rho\|_3^2 dt\Big)^{\frac{1}{2}} \le$$

$$\le CT_*^{\frac{1}{2}}\|\nabla\phi\|_2 \sup_{t\in(0,T_*)} \|\rho^m-\rho\|_3 \to 0 \quad \text{as } m\to+\infty$$

since $\rho^m \to \rho$ in $L^\infty(0,T_*;L^q(\Omega))$ for every $q \ge \frac{3}{2}$. So choosing $q=3$, we have

$$\lim_{m\to+\infty} \sup_{t\in(0,T_*)} \|\rho^m-\rho\|_3 = 0$$

Now we deal with the latter piece, that is

$$\Big|\int_E\int_\Omega \big(\rho(u_t^m-u_t)\big)\cdot\phi\, dx\, dt\Big| = \Big|\int_E\int_\Omega (u_t^m-u_t)\cdot(\rho\phi)\, dx\, dt\Big|$$

We know that $u_t^m \rightharpoonup u_t$ in $L^2(0,T_*;H_0^1(\Omega))$, that means

$$\lim_{m\to+\infty} f(u_t^m) = f(u_t)$$

for every continuous functional $f : L^2(0,T_*;H_0^1(\Omega)) \to \mathbb{R}$. So if we consider the functional

$$f(w) := \int_E\int_\Omega w\cdot(\rho\phi)\, dx\, dt \quad \forall w \in L^2(0,T_*;H_0^1(\Omega))$$

it is well-posed (as we will see in a moment) and linear (thanks to the linearity of the integrals and the euclidian scalar product). Moreover we have

$$|f(w)| \leq \int_E \int_\Omega |w||\rho\phi| \; dx \; dt \leq \int_0^{T_*} \|w\|_2 \|\rho\phi\|_2 \; dt \overset{32}{\leq} C_1 \|\phi\|_{H^1} \int_0^{T_*} \|\rho\|_4 \|w\|_{H^1} \; dt \leq$$

$$\leq C_1 \|\phi\|_{H^1} \left(\int_0^{T_*} \|\rho\|_4^2 \; dt \right)^{\frac{1}{2}} \left(\int_0^{T_*} \|w\|_{H^1}^2 \; dt \right)^{\frac{1}{2}} \leq C_1 \|\phi\|_{H^1} T_*^{\frac{1}{2}} \left(\sup_{t \in (0,T_*)} \|\rho\|_4 \right) \|w\|_{L^2(0,T_*;H_0^1(\Omega))}$$

Observe that, since $\rho^m \to \rho$ in $L^\infty(0,T_*;L^q(\Omega))$, we have $\rho \in L^\infty(0,T_*;L^4)$ for $q = 4$. This shows the well-posedness of the operator and its continuity. So we have

$$\lim_{m \to +\infty} f(u_t^m) = f(u_t)$$

that means

$$\lim_{m \to +\infty} \int_E \int_\Omega u_t^m \cdot (\rho\phi) \; dx \; dt = \int_E \int_\Omega u_t \cdot (\rho\phi) \; dx \; dt$$

This is what we wanted. ∎

Proof of equality (11.87)**.** Now we deal with another limit. As above, observe that

$$\left| \int_E \int_\Omega (\Delta u^m \cdot \nu^m - \Delta u \cdot \phi) \; dx \; dt \right| =$$

$$= \left| \int_E \int_\Omega (\Delta u^m \cdot \nu^m - \Delta u^m \cdot \phi + \Delta u^m \cdot \phi - \Delta u \cdot \phi) \; dx \; dt \right| \leq$$

$$\leq \int_E \int_\Omega |\Delta u^m||\nu^m - \phi| dx \; dt + \left| \int_E \int_\Omega (\Delta u^m - \Delta u) \cdot \phi \; dx \; dt \right| \leq$$

$$\leq \int_0^{T_*} \|\Delta u^m\|_2 \|\nu^m - \phi\|_2 dt + \left| \int_E \int_\Omega (\Delta u^m - \Delta u) \cdot \phi \; dx \; dt \right|$$

We know that $u^m \rightharpoonup u$ in $L^2(0,T_*;H^2(\Omega))$. So we can choose

$$f(w) := \int_E \int_\Omega \Delta w \cdot \phi \; dx \; dt \qquad \forall w \in L^2(0,T_*;H^2(\Omega))$$

The linearity of the functional is obvious; well posedness and continuity follows from this consideration:

$$|f(w)| \leq \int_E \int_\Omega |\Delta w||\phi| \; dx \; dt \leq \int_0^{T_*} \|\Delta w\|_2 \|\phi\|_2 dt$$

[32] Since

$$\|\rho\phi\|_2^2 = \int_\Omega |\rho|^2 |\phi|^2 dx \leq \||\rho|^2\|_2 \||\phi|^2\|_2 = \|\rho\|_4^2 \|\phi\|_4^2 \leq C_1^2 \|\rho\|_4^2 \|\nabla\phi\|_2^2 \leq C_1^2 \|\rho\|_4^2 \|\phi\|_{H^1}^2$$

So

$$|f(w)| \overset{33}{\leq} \sqrt{5}\|\phi\|_2 \int_0^{T_*} \|w\|_{H^2} dt \leq \sqrt{5}\|\phi\|_2 T_*^{\frac{1}{2}} \left(\int_0^{T_*} \|w\|_{H^2}^2 dt \right)^{\frac{1}{2}} \equiv \sqrt{5}\|\phi\|_2 T_*^{\frac{1}{2}} \|w\|_{L^2(0,T_*;H^2(\Omega))}$$

So the functional f is well-posed and continuous over $L^2(0, T_*; H^2(\Omega))$, i.e. it is in $(L^2(0, T_*; H^2(\Omega)))^*$.
It follows that

$$\lim_{m\to+\infty} \int_E \int_\Omega \Delta(u^m - u) \cdot \phi \; dx \; dt = 0$$

for every $\phi \in X$. The other piece is immediate noticing that

$$\int_0^{T_*} \|\Delta u^m\|_2 \|\nu^m - \phi\|_2 dt \leq \sqrt{5} \int_0^{T_*} \|u^m\|_{H^2} \|\nu^m - \phi\|_2 dt$$

using equation (11.72), that is $\sup_{[0,T_*]} \|u^m\|_{H^2} \leq \hat{K}$, we have

$$\int_0^{T_*} \|\Delta u^m\|_2 \|\nu^m - \phi\|_2 dt \leq \sqrt{5} \int_0^{T_*} \hat{K} \|\nu^m - \phi\|_{H^2} dt = \sqrt{5}\hat{K} T_* \|\nu^m - \phi\|_{H^2} \to 0 \text{ as } m \to +\infty$$

Proof of equality (11.86)**.** We deal now with the latter limit. As usual, we write

$$\left| \int_E \int_\Omega (\rho^m u^m \cdot \nabla u^m) \cdot \nu^m dx \; dt - \int_E \int_\Omega (\rho u \cdot \nabla u) \cdot \phi \; dx \; dt \right| =$$

$$= \left| \int_E \int_\Omega (\rho^m u^m \cdot \nabla u^m) \cdot (\nu^m - \phi) \; dx \; dt + \int_E \int_\Omega (\rho^m u^m \cdot \nabla u^m - \rho u \cdot \nabla u) \cdot \phi \; dx \; dt \right| \leq$$

$$\leq \int_0^{T_*} \int_\Omega |\rho^m u^m \cdot \nabla u^m| |\nu^m - \phi| \; dx \; dt + \left| \int_E \int_\Omega (\rho^m u^m \cdot \nabla u^m - \rho u \cdot \nabla u) \cdot \phi \; dx \; dt \right|$$

The first addend can be treated as above. We have

$$\int_0^{T_*} \int_\Omega |\rho^m u^m \cdot \nabla u^m| |\nu^m - \phi| \; dx \; dt \leq (\|\rho_0\|_\infty + 1) \int_0^{T_*} \int_\Omega |u^m| |\nabla u^m| |\nu^m - \phi| \; dx \; dt \leq$$

$$\leq (\|\rho_0\|_\infty + 1) \int_0^{T_*} \| |u^m| |\nabla u^m| \|_2 \|\nu^m - \phi\|_2 dt = (\|\rho_0\|_\infty + 1) \|\nu^m - \phi\|_2 \int_0^{T_*} \| |u^m| |\nabla u^m| \|_2 dt$$

Moreover

$$\int_\Omega |u^m|^2 |\nabla u^m|^2 dx \leq \| |u^m|^2 \|_2 \| |\nabla u^m|^2 \|_2 = \left(\int_\Omega |u^m|^4 dx\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla u^m|^4 dx\right)^{\frac{1}{2}} = \|u^m\|_4^2 \|\nabla u^m\|_4^2$$

Now we have some Sobolev inequalities. In fact, being $u^m \in H_0^1(\Omega)$, we know that

$$\|u^m\|_4 \leq C_1 \|\nabla u^m\|_2 \leq C_1 \|u^m\|_{H^2} \tag{11.92}$$

[33] Using that

$$\|\Delta w\|_2 := \left(\int_\Omega |\Delta w|^2 dx \right)^{\frac{1}{2}} \leq \left(\int_\Omega 5 |\nabla^2 w|^2 dx \right)^{\frac{1}{2}} \leq \sqrt{5} \|w\|_{H^2}$$

where the constant C_1 depends on Ω. At the same time, being $\nabla u^m \in W^{1,2}$, using (11.40) we get

$$\|\nabla u^m\|_6 \le C_2 \|\nabla u^m\|_{W^{1,2}} \le C_2 \|u^m\|_{H^2}$$

Moreover

$$\|\nabla u^m\|_4 \le |\Omega|^{\frac{1}{12}} \|\nabla u^m\|_6 \le |\Omega|^{\frac{1}{12}} C_2 \|u^m\|_{H^2}$$

So

$$\||u^m||\nabla u^m|\|_2 = (\int_\Omega |u^m|^2 |\nabla u^m|^2 dx)^{\frac{1}{2}} \le \|u^m\|_4 \|\nabla u^m\|_4 \le C_1 |\Omega|^{\frac{1}{12}} C_2 \|u^m\|^2_{H^2}$$

It follows that

$$\int_0^{T_*} \||u^m||\nabla u^m|\|_2 \, dt \le C_1 |\Omega|^{\frac{1}{12}} C_2 \int_0^{T_*} \|u^m\|^2_{H^2} dt \overset{(11.72)}{\le} C_1 |\Omega|^{\frac{1}{12}} C_2 \int_0^{T_*} \hat{K}^2 dt = C_1 |\Omega|^{\frac{1}{12}} C_2 T_* \hat{K}^2$$

Thanks to this bounds, the convergence of the first piece follows from the fact that $\|\nu^n - \phi\|_{H^2} \to 0$. We deal now with the piece

$$\left| \int_E \int_\Omega (\rho^m u^m \cdot \nabla u^m - \rho u \cdot \nabla u) \cdot \phi \; dx \; dt \right| =$$

$$= \left| \int_E \int_\Omega \big((\rho^m - \rho) u^m \cdot \nabla u^m + \rho (u^m \cdot \nabla u^m - u \cdot \nabla u) \big) \cdot \phi \; dx \; dt \right| =$$

$$\le \left| \int_E \int_\Omega \Big((\rho^m - \rho) u^m \cdot \nabla u^m + \rho \big((u^m - u) \cdot \nabla u^m + u \cdot (\nabla u^m - \nabla u) \big) \Big) \cdot \phi \; dx \; dt \right| \le$$

$$\le \left| \int_E \int_\Omega (\rho^m - \rho) u^m \cdot \nabla u^m \cdot \phi \; dx \; dt \right| + \left| \int_E \int_\Omega \rho (u^m - u) \cdot \nabla u^m \cdot \phi \; dx \; dt \right| +$$

$$+ \left| \int_E \int_\Omega \rho u \cdot (\nabla u^m - \nabla u) \cdot \phi \; dx \; dt \right|$$

We start with the second addend. We have

$$| \int_E \int_\Omega \rho (u^m - u) \cdot \nabla u^m \cdot \phi \; dx \; dt | \le \int_E \int_\Omega |\rho||u^m - u||\nabla u^m||\phi| dx \; dt \le$$

$$\le \int_0^{T_*} \|u^m - u\|_2 \||\rho||\nabla u^m||\phi|\|_2 dt$$

On the other hand

$$\||\rho||\nabla u^m||\phi|\|^2_2 = \int_\Omega |\rho|^2 |\nabla u^m|^2 |\phi|^2 dx \le \||\rho|^2\|_3 \||\nabla u^m|^2\|_3 \||\phi|^2\|_3 =$$

$$= \|\rho\|^2_6 \|\nabla u^m\|^2_6 \|\phi\|^2_6 \overset{(11.91)+(11.40)}{\le} \|\rho\|^2_6 C_2^2 C_1^2 \|u^m\|^2_{H^2} \|\nabla \phi\|^2_2 \overset{(11.72)}{\le} (\|\rho\|_6 C_1 C_2)^2 \hat{K}^2 \|\nabla \phi\|^2_2$$

and so

$$| \int_E \int_\Omega \rho (u^m - u) \cdot \nabla u^m \cdot \phi \; dx \; dt | \le (C_1 C_2 \hat{K} \|\nabla \phi\|_2) \int_0^{T_*} \|\rho\|_6 \|u^m - u\|_2 dt \le$$

$$\leq (C_1 C_2 \hat{K} \|\nabla\phi\|_2)\left(\int_0^{T_*} \|\rho\|_6^2 dt\right)^{\frac{1}{2}}\left(\int_0^{T_*} \|u^m - u\|_2^2 dt\right)^{\frac{1}{2}} \leq$$

$$\leq (C_1 C_2 \hat{K} \|\nabla\phi\|_2) T_*^{\frac{1}{2}}\left(\sup_{t\in(0,T_*)} \|\rho\|_6\right)\|u^m - u\|_{L^2(0,T_*;L^2)} \to 0 \quad \text{as } m \to +\infty$$

Here again we know that $\rho \in L^\infty(0, T_*; L^6)$ since $\rho^m \to \rho$ in $L^\infty(0, T_*; L^q)$ with $q = 6$. Moreover we used (11.84).

A similar argument holds for the term

$$|\int_E \int_\Omega (\rho^m - \rho) u^m \cdot \nabla u^m \cdot \phi \; dx \; dt| \leq \int_E \int_\Omega |\rho^m - \rho||u^m||\nabla u^m||\phi| dx \; dt \leq$$

$$\leq \int_0^{T_*} \|\rho^m - \rho\|_2 \||u^m||\nabla u^m||\phi|\|_2 dt$$

We now have

$$\||u^m||\nabla u^m||\phi|\|_2^2 = \int_\Omega |u^m|^2 |\nabla u^m|^2 |\phi|^2 dx \leq \||u^m|^2\|_3 \||\nabla u^m|^2\|_3 \||\phi|^2\|_3 =$$

$$= \|u^m\|_6^2 \|\nabla u^m\|_6^2 \|\phi\|_6^2$$

and we use the inequalities

$$\|u^m\|_6 \leq C_1' \|\nabla u^m\|_2 \leq C_1' \|u^m\|_{H^2}, \quad \|\phi\|_6 \leq C_2' \|\nabla\phi\|_2 \leq C_2' \|\phi\|_{H^2}$$

since $u^m, \phi \in H_0^1(\Omega)$. Moreover, since $\nabla u^m \in W^{1,2}$, with (11.40) we have

$$\|\nabla u^m\|_6 \leq C_3' \|\nabla u^m\|_{H^1} \leq C_3' \|u^m\|_{H^2}$$

It follows that

$$\||u^m||\nabla u^m||\phi|\|_2 \leq C_1' \|u^m\|_{H^2} C_3' \|u^m\|_{H^2} C_2' \|\phi\|_{H^2} \overset{(11.72)}{\leq} C_1' C_3' C_2' \hat{K}^2 \|\phi\|_{H^2}$$

Finally

$$\left|\int_E \int_\Omega (\rho^m - \rho) u^m \cdot \nabla u^m \cdot \phi \; dx \; dt\right| \leq C_1' C_3' C_2' \hat{K}^2 \|\phi\|_{H^2} \int_0^{T_*} \|\rho^m - \rho\|_2 dt \leq$$

$$\leq C_1' C_3' C_2' \hat{K}^2 T_* \|\phi\|_{H^2} \sup_{t\in(0,T_*)} \|\rho^m - \rho\|_2 = C_1' C_3' C_2' \hat{K}^2 T_* \|\phi\|_{H^2} \|\rho^m - \rho\|_{L^\infty(0,T_*;L^2(\Omega))} \to 0$$

since $\rho^m \to \rho$ in $L^\infty(0, T_*; L^q(\Omega))$ for every $q \geq \frac{3}{2}$.

We finally deal with the remaining piece, i.e.

$$\left|\int_E \int_\Omega \rho u \cdot (\nabla u^m - \nabla u) \cdot \phi \; dx \; dt\right|$$

We define, for every $w \in L^2(0, T_*; H^2(\Omega))$, the functional

$$f(w) := \int_E \int_\Omega \rho u \cdot \nabla w \cdot \phi \; dx \; dt$$

The functional is obviously linear. It is also well posed and continuous. In fact

$$|f(w)| \leq \int_E \int_\Omega |\rho||u||\nabla w||\phi| \; dx \; dt \leq \int_0^{T_*} \|\nabla w\|_2 \||\rho||u||\phi|\|_2 dt$$

and, since $u \in H^2$ and $\phi \in H_0^1$,

$$\||\rho||u||\phi|\|_2^2 = \int_\Omega |\rho|^2 |u|^2 |\phi|^2 dx \leq \||\rho|^2\|_3 \||u|^2\|_3 \||\phi|^2\|_3 =$$

$$= \|\rho\|_6^2 \|u\|_6^2 \|\phi\|_6^2 \leq \|\rho\|_6^2 C_2^2 \|u\|_{H^1}^2 C_1^2 \|\nabla \phi\|_2^2$$

So we have

$$|f(w)| \leq C_2 \bigg(\sup_{t \in (0,T_*)} \|u\|_{H^1} \bigg) C_1 \|\nabla \phi\|_2 \int_0^{T_*} \|\rho\|_6 \|\nabla w\|_2 dt$$

Remember that $\sup\limits_{t \in (0,T_*)} \|u\|_{H^2}$ is a number since $u \in L^\infty(0, T_*; H^2(\Omega))$. Finally

$$|f(w)| \leq C_2 \|u\|_{L^\infty(0,T_*;H^2)} C_1 \|\nabla \phi\|_2 \bigg(\int_0^{T_*} \|\rho\|_6^2 dt \bigg)^{\frac{1}{2}} \bigg(\int_0^{T_*} \|\nabla w\|_2^2 dt \bigg)^{\frac{1}{2}} \leq$$

$$\leq C_1 C_2 \|u\|_{L^\infty(0,T_*;H^2)} \|\nabla \phi\|_2 T_*^{\frac{1}{2}} \|\rho\|_{L^\infty(0,T_*;L^6)} \bigg(\int_0^{T_*} \|w\|_{H^2}^2 dt \bigg)^{\frac{1}{2}}$$

Here $\|\rho\|_{L^\infty(0,T_*;L^6)}$ is a number since $\rho^m \to \rho$ in $L^\infty(0, T_*; L^q)$ with $q = 6$. So the functional is well-posed and continuous. Since $u^m \rightharpoonup u$ in $L^2(0, T_*; H^2(\Omega))$, we have that

$$f(u^m - u) \to 0 \;\; \text{as } m \to +\infty$$

In other words

$$\lim_{m \to +\infty} \int_E \int_\Omega \rho u \cdot \nabla (u^m - u) \cdot \phi \; dx \; dt = 0$$

So, we have proved the proposition. ∎

11.8.3 Proof of corollary 11.1

We have already remarked that the zero measure set E found above depends on the $\nu \in X$ that we fix. What is true is the following assertion.
Let $\nu \in X$. Then there exists a subset $E_\nu \subseteq (0, T_*)$, with $|E_\nu| = 0$, such that

$$\int_\Omega (\rho u_t + \rho u \cdot \nabla u - \mu \Delta u) \cdot \nu \; dx = 0 \quad t \in (0, T_*)/E_\nu \tag{11.93}$$

We now want to generalize the result. We can do vary particular choice of ν. For $\nu = w^m \equiv \phi^m$, element of the basis of X, we set

$$E_m := E_{w^m}$$

and

$$E := \bigcup_{m\in\mathbb{N}} E_m$$

Being countable union of zero measure set, we have that $|E| = 0$. So, for every $m \in \mathbb{N}$ we have

$$\int_\Omega (\rho u_t + \rho u \cdot \nabla u - \mu\Delta u) \cdot w^m \, dx = 0 \quad \forall t \in (0, T_*)/E \tag{11.94}$$

Moreover, possibly except over another zero measure set, say A, $|A| = 0$, the integral above is well defined, as we have seen in 11.4, since also $w^m \in L^2(\Omega)$. If now $\nu \in X$, we can find ν^m such that

$$\lim_{m\to+\infty} \|\nu^m - \nu\|_2 \leq \lim_{m\to+\infty} \|\nu^m - \nu\|_X = 0$$

where $\nu^m \in X^m$ is define in (11.89) with $\phi = \nu$. This is possible since $\{w^m\}$ is a basis. So, for those $t \in (0, T_*)/A$ such that $\rho u_t + \rho u \cdot \nabla u - \mu\Delta u \in L^2(\Omega)$, as in section 11.6.5, with $t \notin E$, we have

$$\left| \int_\Omega (\rho u_t + \rho u \cdot \nabla u - \mu\Delta u) \cdot \nu \, dx \right| = \left| \int_\Omega (\rho u_t + \rho u \cdot \nabla u - \mu\Delta u) \cdot (\nu - \nu^m) \, dx \right| \leq$$

$$\leq \|\rho(t)u_t(t) + \rho(t)u(t) \cdot \nabla u(t) - \mu\Delta u(t)\|_2 \|\nu - \nu^m\|_2$$

If we send $m \to \infty$ we find that the left-side is zero. Here we used that

$$\int_\Omega (\rho u_t + \rho u \cdot \nabla u - \mu\Delta u) \cdot \nu^m \, dx = 0$$

for every $t \in (0, T_*)/E$, since $\nu^m := \displaystyle\sum_{k=1}^m \langle \nu, \phi^k \rangle_2 \phi^k$ and so by (11.94) and linearity the integral is zero. Finally, for every $t \in (0, T_*)/(A \cup E)$ and for every $\nu \in X$ we have

$$\boxed{\int_\Omega (\rho u_t + \rho u \cdot \nabla u - \mu\Delta u) \cdot \nu \, dx = 0} \tag{11.95}$$

Observe that now A and E does not depend on ν.

Moreover, in (9.61), we have observed that $X = W^{1,2}_{0,\sigma}(\Omega) \cap H^2(\Omega)$. So, if $\psi \in W^{1,2}_{0,\sigma}(\Omega)$, we have that there exists a sequence, say $\{\nu^k\}_{k\in\mathbb{N}} \subseteq C^\infty_{0,\sigma}(\Omega)$, such that

$$\lim_{k\to\infty} \|\nu^k - \psi\|_2 \leq \lim_{k\to\infty} \|\nu^k - \psi\|_{H^1} = 0$$

Since ν^k are smooth, we have that in particular $\nu^k \in X$. So, for every $t \in (0, T_*)/(A\cup E)$, and $k \in \mathbb{N}$ we have

$$\int_\Omega (\rho u_t + \rho u \cdot \nabla u - \mu\Delta u) \cdot \nu^k \, dx = 0$$

It follows that

$$\left|\int_\Omega (\rho u_t + \rho u\cdot\nabla u - \mu\Delta u)\cdot\psi\ dx\right| = \left|\int_\Omega (\rho u_t + \rho u\cdot\nabla u - \mu\Delta u)\cdot(\psi - \nu^k)\ dx\right| \leq$$

$$\leq \|\rho u_t + \rho u\cdot\nabla u - \mu\Delta u\|_2(t)\|\psi - \nu^k\|_2 \to 0$$

as $k \to \infty$. This means that, for every $t \in (0,T_*)/(A\cup E)$, $|A\cup E| \leq |A| + |E| = 0$,

$$\int_\Omega (\rho u_t + \rho u\cdot\nabla u - \mu\Delta u)\cdot\psi\ dx = 0 \tag{11.96}$$

for every $\psi \in W^{1,2}_{0,\sigma}(\Omega)$. ∎

11.8.4 Proof of theorem 11.3

The integral property (11.96) holds for every time-independent ψ function in $W^{1,2}_{0,\sigma}(\Omega)$. We want to increase the class of functions such that the equality holds. For this purpose, consider $\varphi \in C^1([0,T_*]; W^{1,2}_{0,\sigma}(\Omega))$, where $(W^{1,2}_{0,\sigma}(\Omega), \|\cdot\|_{H^1})$ has to be meant as an Hilbert space[34]. In particular, a function in this class is in $L^2(0,T_*; W^{1,2}_{0,\sigma}(\Omega))$. So, we can use the theorem 5.2 above, from [6].

Remember now that $\displaystyle\int_0^{T_*} \|\rho u_t + \rho u\cdot\nabla u - \mu\Delta u\|_2^2(t)\ dt < +\infty$, as seen in Section 11.6.5. So for every $\varphi \in C^1([0,T_*]; W^{1,2}_{0,\sigma}(\Omega))$ we have

$$\left|\int_0^{T_*}\int_\Omega (\rho u_t + \rho u\cdot\nabla u - \mu\Delta u)\cdot\varphi(x,t)\ dx\ dt\right| \leq \int_0^{T_*} \|\rho u_t + \rho u\cdot\nabla u - \mu\Delta u\|_2\|\varphi(t)\|_2\ dt \leq$$

$$\leq \left(\int_0^{T_*} \|\rho u_t + \rho u\cdot\nabla u - \mu\Delta u\|_2^2(t)\ dt\right)^{\frac{1}{2}}\left(\int_0^{T_*}\|\varphi(t)\|_2^2\ dt\right)^{\frac{1}{2}} < +\infty$$

Moreover, using theorem 5.2, for every $\varepsilon > 0$ there exists $\varphi_\varepsilon \in L^2(0,T_*)\otimes W^{1,2}_{0,\sigma}(\Omega)$ such that

$$\|\varphi - \varphi_\varepsilon\|_{L^2(0,T_*;W^{1,2}_{0,\sigma}(\Omega))} < \varepsilon$$

So, the estimates above hold with φ substituted by φ_ε, and, moreover, we have

$$\int_0^{T_*}\int_\Omega (\rho u_t + \rho u\cdot\nabla u - \mu\Delta u)\cdot\varphi_\varepsilon(x,t)\ dx\ dt =$$

$$= \int_0^{T_*}\int_\Omega (\rho u_t + \rho u\cdot\nabla u - \mu\Delta u)\cdot\Big(\sum_{i=1}^{m_\varepsilon}\chi_{E_i^\varepsilon}(t)h_i^\varepsilon(x)\Big)\ dx\ dt =$$

$$= \sum_{i=1}^{m_\varepsilon}\int_0^{T_*}\chi_{E_i^\varepsilon}(t)\int_\Omega (\rho u_t + \rho u\cdot\nabla u - \mu\Delta u)\cdot h_i^\varepsilon(x)\ dx\ dt = 0$$

[34]It is a closed subspace of the Hilbert space $H^1(\Omega) \equiv W^{1,2}(\Omega)$

since $h_i^\varepsilon \in W^{1,2}_{0,\sigma}(\Omega)$ and so we used (11.96). It follows that

$$\left| \int_0^{T_*} \int_\Omega (\rho u_t + \rho u \cdot \nabla u - \mu \Delta u) \cdot \varphi(x,t) \; dx \; dt \right| =$$

$$= \left| \int_0^{T_*} \int_\Omega (\rho u_t + \rho u \cdot \nabla u - \mu \Delta u) \cdot (\varphi(x,t) - \varphi_\varepsilon(x,t)) \; dx \; dt \right| \le$$

$$\le \int_0^{T_*} \|\rho u_t + \rho u \cdot \nabla u - \mu \Delta u\|_2 \|\varphi(t) - \varphi_\varepsilon(t)\|_2 \; dt \le$$

$$\le \left(\int_0^{T_*} \|\rho u_t + \rho u \cdot \nabla u - \mu \Delta u\|_2^2 \; dt \right)^{\frac{1}{2}} \left(\int_0^{T_*} \|\varphi(t) - \varphi_\varepsilon(t)\|_2^2 \; dt \right)^{\frac{1}{2}} \le$$

$$\le \left(\int_0^{T_*} \|\rho u_t + \rho u \cdot \nabla u - \mu \Delta u\|_2^2 \; dt \right)^{\frac{1}{2}} \|\varphi - \varphi_\varepsilon\|_{L^2(0,T_*;W^{1,2}_{0,\sigma}(\Omega))}$$

Since the latter piece is small, we have that for every $\varphi \in C^1([0,T_*]; W^{1,2}_{0,\sigma}(\Omega))$

$$\boxed{\int_0^{T_*} \int_\Omega (\rho u_t + \rho u \cdot \nabla u - \mu \Delta u) \cdot \varphi(x,t) \; dx \; dt = 0} \tag{11.97}$$

So we have proved the theorem. ∎

11.8.5 The weak solution is a weak strong solution with a pression gradient term

Theorem 11.4. *Let Ω be a bounded domain of $\mathbb{R}^3$. Consider the pair of solution (u,ρ), as introduced in sections 11.2-11.7. Then, for almost every $t \in (0,T_*)$, there exists a function $p(t) \in L^2_{loc}(\Omega)$ such that $p(t)$ has weak derivative in Ω and*

$$\rho u_t + \rho\big(u \cdot \nabla u\big) - \mu \Delta u = \nabla p$$

Proof. Remember that

$$C^\infty_{0,\sigma}(\Omega) \subseteq \{\phi \in C^\infty_c(\Omega) : \; \nabla \cdot \phi = 0 \text{ in } \Omega\} \subseteq W^{1,2}_{0,\sigma}(\Omega)$$

Thanks to theorem 11.1, there exists a set of zero measure $A \cup E$ such that for every $\phi \in W^{1,2}_{0,\sigma}(\Omega)$ we have

$$\int_\Omega (\rho u_t + \rho u \cdot \nabla u - \mu \Delta u) \cdot \phi \; dx = 0 \quad \forall t \in (0,T_*)/(A \cup E)$$

Using Lemma 6.1, we have that, for a.e. $t \in (0,T_*)$,

$$\int_\Omega (\rho u_t + \rho u \cdot \nabla u - \mu \Delta u) \cdot \varphi \; dx = -\int_\Omega p(t) \; \nabla \cdot \varphi \; dx \qquad \forall \, \varphi \in C^\infty_c(\Omega)$$

for some $p(t) \in L^2_{loc}(\Omega)$. If ψ is a scalar test function and $\varphi := \psi\hat{e}_i$, we have that

$$\int_\Omega (\rho u_t + \rho u \cdot \nabla u - \mu\Delta u)_i \ \psi \ dx = -\int_\Omega p(t) \ \partial_i \psi \ dx \qquad \forall \ \psi \in C_c^\infty(\Omega)$$

This means that

$$(\rho u_t + \rho u \cdot \nabla u - \mu\Delta u)_i = \partial_i p(t)$$

where the derivative is a weak derivative. In other words, for almost every $t \in (0, T_*)$, we have

$$\rho u_t + \rho u \cdot \nabla u - \mu\Delta u = \nabla p(t)$$

The equality above is to be meant in the weak derivative sense, so it is true almost everywhere. ∎

Remark 11.28. This theorem immediately gives us an important property of the pair (u, ρ): it is a solution in a strong sense. □

11.9 Weak solution to the problem with regular initial density $\overline{\rho}_0$: the momentum equation

We want now to deduce now the weak formulation of the problem, as introduced in chapter 10. To do this, we need to derive the functions in classical sense. So, for a moment, we turn back to the approximate solutions in the following way. In particular, at first, we have the following theorem. In fact, in the previous paragraph we deduced an integral weak form of the momentum equation. However, this formulation doesn't involve the initial data. In this subsection, we will modify the integral equation, changing it into the real weak formulation of the problem as introduced in chapter 10.

Theorem 11.5. *Let Ω be a bounded domain of $\mathbb{R}^3$. Consider the pair of solution (u, ρ), as introduced in sections 11.2-11.7, with initial data $(u_0, \overline{\rho}_0)$, as fixed at the beginning of chapter 11. Then, for every $\varphi \in C^1([0, T_*]; W^{1,2}_{0,\sigma}(\Omega))$ such that $\varphi(x, T_*) = 0$ a.e. in Ω, we have*

$$-\int_0^{T_*}\int_\Omega \rho u \cdot \varphi_t \ dx \ dt - \int_0^{T_*}\int_\Omega \rho u \cdot (\nabla\varphi) \cdot u \ dx \ dt + \mu \int_0^{T_*}\int_\Omega \nabla u \cdot \nabla\varphi \ dx \ dt =$$

$$= \int_\Omega \overline{\rho}_0(x) u_0(x) \cdot \varphi(x, 0) \ dx$$

Proof. Let $\varphi \in C^1([0, T_*]; W^{1,2}_{0,\sigma}(\Omega))$ such that $\varphi(x, T_*) = 0$ a.e. in Ω. Consider

$$\int_0^{T_*}\int_\Omega (\rho^m u_t^m + \rho^m u^m \cdot \nabla u^m - \mu\Delta u^m) \cdot \varphi(x, t) \ dx \ dt$$

and remember that

$$\rho_t^m + u^m \cdot \nabla\rho^m = 0, \quad \nabla \cdot u^m = 0$$

By the regularity in time, we have that

$$(\rho^m u^m \cdot \varphi)_t = \rho_t^m (u^m \cdot \varphi) + \rho^m (u^m \cdot \varphi)_t = \rho_t^m u^m \cdot \varphi + \rho^m (u_t^m \cdot \varphi + u^m \cdot \varphi_t)$$

and so

$$\rho^m u_t^m \cdot \varphi = (\rho^m u^m \cdot \varphi)_t - \rho_t^m u^m \cdot \varphi - \rho^m u^m \cdot \varphi_t =$$
$$= (\rho^m u^m \cdot \varphi)_t + (u^m \cdot \nabla \rho^m)(u^m \cdot \varphi) - \rho^m u^m \cdot \varphi_t$$

using the mass equation. On the other hand we have

$$\nabla(u^m \cdot \varphi) = \varphi \cdot (\nabla u^m) + u^m \cdot (\nabla \varphi) \tag{11.98}$$

using the Leibniz rule for weak derivatives, since $u^m \in C^1(\overline{\Omega})$. So

$$\varphi \cdot \nabla u^m \cdot (\rho^m u^m) \overset{(11.98)}{=} [\nabla(u^m \cdot \varphi) - u^m \cdot (\nabla \varphi)] \cdot (\rho^m u^m) =$$
$$= \nabla(u^m \cdot \varphi) \cdot (\rho^m u^m) - \rho^m u^m \cdot (\nabla \varphi) \cdot u^m \tag{11.99}$$

Moreover we have that, if ϕ is a scalar field and A is vectorial,

$$A \cdot \nabla \phi = \nabla \cdot (\phi A) - \phi(\nabla \cdot A)$$

and so, with $A = \rho^m u^m$ and $\phi = u^m \cdot \varphi$,

$$(\rho^m u^m) \cdot \nabla(u^m \cdot \varphi) = \nabla \cdot (\rho^m u^m (u^m \cdot \varphi)) - (u^m \cdot \varphi) \nabla \cdot (\rho^m u^m) \tag{11.100}$$

using again the Leibniz rule and the fact that $\rho^m u^m$ is regular in x.
Moreover

$$\nabla \cdot (\rho^m u^m) = \rho^m \nabla \cdot u^m + u^m \cdot \nabla \rho^m = u^m \cdot \nabla \rho^m \tag{11.101}$$

since $\nabla \cdot u^m = 0$.
So we have

$$(\rho^m u^m) \cdot \nabla(u^m \cdot \varphi) \overset{(11.100)+(11.101)}{=} \nabla \cdot (\rho^m u^m (u^m \cdot \varphi)) - (u^m \cdot \varphi)(u^m \cdot \nabla \rho^m) \tag{11.102}$$

and then

$$\varphi \cdot \nabla u^m \cdot (\rho^m u^m) \overset{(11.99)+(11.102)}{=} \nabla \cdot (\rho^m u^m (u^m \cdot \varphi)) - (u^m \cdot \varphi)(u^m \cdot \nabla \rho^m) - \rho^m u^m \cdot (\nabla \varphi) \cdot u^m$$

Putting the pieces togheter we have

$$\rho^m u_t^m \cdot \varphi + \varphi \cdot \nabla u^m \cdot (\rho^m u^m) = \tag{11.103}$$
$$= (\rho^m u^m \cdot \varphi)_t - \rho^m u^m \cdot \varphi_t + \nabla \cdot (\rho^m u^m (u^m \cdot \varphi)) - \rho^m u^m \cdot (\nabla \varphi) \cdot u^m$$

The aim is now clear: integration in Ω will remove the divergence term, while integration in t will provide us an initial time term.

At the same time we have, thanks to (1.14),

$$\Delta u^m \cdot \varphi = \sum_{i=1}^{3} \nabla \cdot (\varphi_i \nabla u_i^m) - \nabla u^m \cdot \nabla \varphi$$

In the last equation there is, however, a little problem: the equality is obtained using the Leibniz rule, but ∇u_i^m or φ_i aren't in C^1 class. So, we have to use a little approximation argument. Since $\varphi(t) \in H_0^1(\Omega)$, and the boundary $\partial\Omega$ is regular, then there exists a sequence of functions $\eta^k(t) \in C_c^\infty(\Omega) \subseteq C^\infty(\overline{\Omega})$ such that

$$\lim_{k\to+\infty} \|\varphi(t) - \eta_k(t)\|_{H^1} = 0$$

It follows that

$$\int_\Omega \Delta u^m(x,t)\cdot\varphi(x,t)\;dx \overset{35}{=} \lim_{k\to+\infty}\int_\Omega \Delta u^m(x,t)\cdot\eta_k(x,t)\;dx =$$

$$\overset{(1.14)}{=} \lim_{k\to+\infty}\left(\sum_{i=1}^3 \int_\Omega \nabla\cdot\big((\eta_k(x,t))_i\nabla u_i^m(x,t)\big)\;dx - \int_\Omega \nabla u^m(x,t)\cdot\nabla\eta_k(x,t)\;dx\right)$$

since now we are in classical hypothesis. If we prove that

$$\int_\Omega \nabla\cdot\big((\eta_k(x,t))_i\nabla u_i^m(x,t)\big)\;dx = 0 \quad \forall k,m\in\mathbb{N},\ \forall i\in\{1,2,3\} \tag{11.104}$$

then

$$\int_\Omega \Delta u^m(x,t)\cdot\varphi(x,t)\;dx = -\int_\Omega \nabla u^m(x,t)\cdot\nabla\varphi(t)\;dx$$

using the same argument convergence of note 35.
To prove (11.104) we use the generalized divergence theorem. In particular, we have

$$\int_\Omega \nabla\cdot\big((\eta_k(x,t))_i\nabla u_i^m(x,t)\big)\;dx = \int_{\partial\Omega} T\big((\eta_k(x,t))_i\nabla u_i^m(x,t)\big)\cdot\nu\;d\sigma$$

where ν is the outward normal vector.

If we show that the **trace is zero** we have done. We can use the argument in (4.6), since $u_i^m(t)\in H^2(\Omega)$ and $\eta_k(t)\in C^\infty(\overline{\Omega})$. Since $T\eta_k(t)=0$ on $\partial\Omega$ (it is a continuous function over $\overline{\Omega}$ and the boundary value is zero), we have the thesis.
So, passing to the integrals, we have

$$\int_0^{T_*}\int_\Omega \Delta u^m(x,t)\cdot\varphi(x,t)\;dx\;dt = -\int_0^{T_*}\int_\Omega \nabla u^m(x,t)\cdot\nabla\varphi(x,t)\;dx\;dt$$

Measurability and summability are not a problem in the latter equality, thanks to the discussion in the chapter dedicated to the Bochner integral.
We now restart from the initial integral equality. We have

$$\int_0^{T_*}\int_\Omega (\rho^m u_t^m + \rho^m\nabla u^m\cdot u^m - \mu\Delta u^m)\cdot\varphi(x,t)\;dx\;dt =$$

[35] Observe that

$$\left|\int_\Omega \Delta u^m(x,t)\cdot\varphi(x,t)\;dx - \int_\Omega \Delta u^m(x,t)\cdot\eta_k(x,t)\;dx\right| \le \int_\Omega |\Delta u^m(x,t)||\varphi(x,t)-\eta_k(x,t)|dx \le$$

$$\le \|\Delta u^m(t)\|_2\|\varphi_2(t)-\eta_k(t)\|_2 \to 0 \quad \text{as } k\to+\infty$$

$$\overset{(11.103)}{=} \int_0^{T_*}\int_\Omega (\rho^m u^m\cdot\varphi)_t - \rho^m u^m\cdot\varphi_t + \nabla\cdot\big(\rho^m u^m(u^m\cdot\varphi)\big) - \rho^m u^m\cdot(\nabla\varphi)\cdot u^m \; dx\; dt -$$

$$-\mu\int_0^{T_*}\int_\Omega \Delta u^m\cdot\varphi \; dx\; dt =$$

$$= \int_0^{T_*}\int_\Omega (\rho^m u^m\cdot\varphi)_t - \rho^m u^m\cdot\varphi_t + \nabla\cdot(\rho^m u^m(u^m\cdot\varphi)) - \rho^m u^m\cdot(\nabla\varphi)\cdot u^m \; dx\; dt +$$

$$+\mu\int_0^{T_*}\int_\Omega \nabla u^m\cdot\nabla\varphi \; dx\; dt$$

We now have to reach two goals: **interchange the integrals** in the first term and **get rid of the divergence term**.

To interchange the integrals observe that the product $\rho^m u^m\cdot\varphi$ is regular in time; in particular we can observe that

$$(\rho^m u^m\cdot\varphi)_t = \rho^m_t u^m\cdot\varphi + \rho^m u^m_t\cdot\varphi + \rho^m u^m\cdot\varphi_t \in C([0,T_*];L^2(\Omega)) \subseteq L^2(0,T_*;L^2(\Omega))$$

So, the function is in $L^2\big((0,T_*)\times\Omega\big)$, by the isomorphism between the two spaces, and so by the Fubini theorem

$$\int_0^{T_*}\int_\Omega (\rho^m u^m\cdot\varphi)_t \; dx\; dt = \int_\Omega\int_0^{T_*}(\rho^m u^m\cdot\varphi)_t \; dt\; dx$$

Using the FCT with Bochner integrals, we have

$$\int_0^t (\rho^m u^m\cdot\varphi)_t \; dt = (\rho^m u^m\cdot\varphi)(t) - c_0$$

for almost every $t\in[0,T_*]$; here c_0 is a contant in $L^2(\Omega)$. Actually the equality holds for every $t\in[0,T_*]$, thanks to the regularity in t of the integrand function. So we have

$$c_0 = (\rho^m u^m\cdot\varphi)(x,0)$$

See [9, pg. 51-52, th. 4.9]. So, it follows that

$$\int_0^{T_*}\big(\rho^m u^m\cdot\varphi\big)_t \; dt = \big(\rho^m u^m\cdot\varphi\big)(T_*) - \big(\rho^m u^m\cdot\varphi\big)(x,0) =$$

$$= -\big(\rho^m u^m\cdot\varphi\big)(x,0) \quad \text{a.e. } x\in\Omega$$

since $\varphi(x,T_*)=0$ almost everywhere in Ω.
It follows that

$$\int_0^{T_*}\int_\Omega \big(\rho^m u^m\cdot\varphi\big)_t \; dx\; dt = -\int_\Omega \big(\rho^m u^m\cdot\varphi\big)(x,0)\; dx$$

Our aim is now to show

$$\int_\Omega \nabla\cdot\big(\rho^m u^m(u^m\cdot\varphi)\big)\; dx = 0$$

Using the generalized divergence theorem we have

$$\int_\Omega \nabla \cdot \big(\rho^m u^m (u^m \cdot \varphi)\big)\, dx = \int_{\partial\Omega} T\big(\rho^m u^m (u^m \cdot \varphi)\big) \cdot \nu \, dx$$

We now have to show that the trace is zero. Since the boudary is regular (and the domain is bounded) we can approach, for almost every $t \in (0, T_*)$, $\varphi(t) \in H^1(\Omega)$ with a sequence $\eta_k(t) \in C^\infty(\overline{\Omega})$ such that

$$\lim_{k\to+\infty} \|\varphi(t) - \eta_k(t)\|_{H^1} = 0 \tag{11.105}$$

If we show that $\rho^m u^m(u^m \cdot \eta_k) \in C^1(\overline{\Omega})$ converges to $\rho^m u^m(u^m\cdot\varphi)$ in H^1, then we have done, as will be precised later.
Consider

$$\|\rho^m u^m(u^m\cdot\eta_k) - \rho^m u^m(u^m\cdot\varphi)\|_{H^1}^2 = \|\rho^m u^m\big(u^m\cdot(\eta_k-\varphi)\big)\|_{H^1}^2 =$$

$$= \|\rho^m u^m\big(u^m\cdot(\eta_k-\varphi)\big)\|_2^2 + \|\nabla\bigg(\rho^m u^m\big(u^m\cdot(\eta_k-\varphi)\big)\bigg)\|_2^2 \overset{36}{\leq}$$

$$\overset{(11.106)}{\leq} \sup_\Omega \big(|u^m|^2|\rho^m u^m|^2\big)\|\eta_k-\varphi\|_2^2 + 2\sup_\Omega |\rho^m u^m|^2 \int_\Omega |\nabla\big(u^m\cdot(\eta_k-\varphi)\big)|^2 dx +$$

$$+2\sup_\Omega \big(|u^m|^2|\nabla(\rho^m u^m)|^2\big)\|\eta_k-\varphi\|_2^2$$

The first and the third pieces go to zero as $k\to+\infty$, thanks to (11.105). For the second term we have

$$\nabla\big(u^m\cdot(\eta_k-\varphi)\big) = (\eta_k-\varphi)\cdot(\nabla u^m) + u^m\cdot\nabla(\eta_k-\varphi)$$

and so

$$\int_\Omega |\nabla\big(u^m\cdot(\eta_k-\varphi)\big)|^2 dx \leq 2\int_\Omega |\eta_k-\varphi|^2|\nabla u^m|^2 dx + 2\int_\Omega |u^m|^2|\nabla(\eta_k-\varphi)|^2 dx \leq$$

$$\leq 2\sup_\Omega |\nabla u^m|^2 \|\eta_k-\varphi\|_2^2 + 2\sup_\Omega |u^m|^2\|\nabla(\eta_k-\varphi)\|_2^2$$

that goes to zero as $k\to+\infty$, thanks again to (11.105). So, for almost every $t\in(0,T_*)$, we have

$$\big(\rho^m u^m(u^m\cdot\eta_k)\big)(t) \to \big(\rho^m u^m(u^m\cdot\varphi)\big)(t) \quad \text{in } H^1 \text{ as } k\to+\infty$$

---

[36]

$$= \int_\Omega |u^m\cdot(\eta_k-\varphi)|^2|\rho^m u^m|^2 dx + \int_\Omega |\nabla\big(u^m\cdot(\eta_k-\varphi)\big)\otimes(\rho^m u^m) + \big(u^m\cdot(\eta_k-\varphi)\big)\nabla(\rho^m u^m)|^2 dx \leq$$

$$\leq \int_\Omega |u^m|^2|\eta_k-\varphi|^2|\rho^m u^m|^2 dx + 2\int_\Omega |\nabla\big(u^m\cdot(\eta_k-\varphi)\big)|^2|\rho^m u^m|^2 dx + 2\int_\Omega |u^m|^2|\eta_k-\varphi|^2|\nabla(\rho^m u^m)|^2 dx \leq \tag{11.106}$$

where we used that $|w_1\otimes w_2| \equiv |w_1 w_2^T| \leq |w_1||w_2|$, since it can be meant as a matrix product.

Since $\rho^m u^m(u^m\cdot\eta_k)\in C^1(\overline{\Omega})$, by definition we have for almost every $t\in(0,T_*)$,

$$T\big(\rho^m u^m(u^m\cdot\varphi)\big)(t)=\lim_{k\to+\infty}T\big(\rho^m u^m(u^m\cdot\eta_k)\big)(t)\quad\text{in } L^2(\partial\Omega)$$

But $T\big(\rho^m u^m(u^m\cdot\eta_k)\big)(t)=0$ since the function is in $C^1(\overline{\Omega})$ and at the boundary the function is zero, since $u^m=0$ on $\partial\Omega$.
Since the limit of a zero sequence in $L^2(\partial\Omega)$ is zero, we have that for almost every $t\in(0,T_*)$

$$T\big(\rho^m u^m(u^m\cdot\varphi)\big)(t)=0\quad\text{a.e. on }\partial\Omega$$

It follows that

$$\int_\Omega\nabla\cdot\big(\rho^m u^m(u^m\cdot\varphi)\big)\;dx=\int_{\partial\Omega}T\big(\rho^m u^m(u^m\cdot\varphi)\big)\cdot\nu\;dx=0$$

for almost every $t\in(0,T_*)$. Then

$$\int_0^{T_*}\int_\Omega\nabla\cdot\big(\rho^m u^m(u^m\cdot\varphi)\big)\;dx\;dt=0$$

So, our initial expression becomes as follows:

$$\int_0^{T_*}\int_\Omega(\rho^m u^m_t+\rho^m u^m\cdot\nabla u^m-\mu\Delta u^m)\cdot\varphi\;dx\;dt=$$

$$=\int_0^{T_*}\int_\Omega(\rho^m u^m\cdot\varphi)_t-\rho^m u^m\cdot\varphi_t+\nabla\cdot(\rho^m u^m(u^m\cdot\varphi))-\rho^m u^m\cdot(\nabla\varphi)\cdot u^m\;dx\;dt+$$

$$+\mu\int_0^{T_*}\int_\Omega\nabla u^m\cdot\nabla\varphi\;dx\;dt=$$

$$=-\int_\Omega(\rho^m u^m\cdot\varphi)(x,0)\;dx-\int_0^{T_*}\int_\Omega\rho^m u^m\cdot\varphi_t\;dx\;dt-\int_0^{T_*}\int_\Omega\rho^m u^m\cdot(\nabla\varphi)\cdot u^m\;dx\;dt+$$

$$+\mu\int_0^{T_*}\int_\Omega\nabla u^m\cdot\nabla\varphi\;dx\;dt$$

At this point, we want now to **take the limit** another time. The first member goes to zero, since it is the limit from which we started in this section. For the second member we go in order. First of all we show that

$$\boxed{\lim_{m\to+\infty}\int_\Omega(\rho^m u^m\cdot\varphi)(x,0)\;dx=\int_\Omega\overline{\rho}_0(x)u_0(x)\cdot\varphi(x,0)dx}\tag{11.107}$$

In fact observe that, since $\rho^m(x,0)=\overline{\rho}_0(x)$ for the choice of the initial data,

$$\left|\int_\Omega\big(\rho^m u^m\cdot\varphi\big)(x,0)\;dx-\int_\Omega\overline{\rho}_0(x)u_0(x)\cdot\varphi(x,0)dx\right|=\left|\int_\Omega\overline{\rho}_0(x)\big(u^m(x,0)-u_0(x)\big)\cdot\varphi(x,0)\;dx\right|\le$$

$$\le\int_\Omega|\overline{\rho}_0(x)||u^m(x,0)-u_0(x)||\varphi(x,0)|dx\le\sup_\Omega|\overline{\rho}_0|\|u^m(0)-u_0\|_2\|\varphi(0)\|_2\to0$$

as m goes to infinity, since $u^m(0)$ goes to u_0 in X, that is in H^2-norm.

Now we deal with the second piece. We want to show that

$$\boxed{\lim_{m\to+\infty}\int_0^{T_*}\int_\Omega \rho^m u^m\cdot\varphi_t\ dx\ dt=\int_0^{T_*}\int_\Omega \rho u\cdot\varphi_t\ dx\ dt} \tag{11.108}$$

Using well know arguments, we have

$$\left|\int_0^{T_*}\int_\Omega(\rho^m u^m-\rho u)\cdot\varphi_t\ dx\ dt\right|=\left|\int_0^{T_*}\int_\Omega(\rho^m u^m-\rho^m u+\rho^m u-\rho u)\cdot\varphi_t\ dx\ dt\right|\le$$

$$\le\int_0^{T_*}\int_\Omega|\rho^m||u^m-u||\varphi_t|\ dx\ dt+\int_0^{T_*}\int_\Omega|\rho^m-\rho||u||\varphi_t|\ dx\ dt\le$$

$$\le(\|\rho_0\|_\infty+1)\int_0^{T_*}\|u^m-u\|_2\|\varphi_t\|_2\ dt+\int_0^{T_*}\|\rho^m-\rho\|_3\|u\|_3\|\varphi_t\|_3 dt\le$$

$$\le(\|\rho_0\|_\infty+1)\Big(\int_0^{T_*}\|u^m-u\|_2^2dt\Big)^{\frac12}\Big(\int_0^{T_*}\|\varphi_t\|_2^2dt\Big)^{\frac12}+\sup_{(0,T_*)}\|\rho^m-\rho\|_3\int_0^{T_*}\|u\|_3\|\varphi_t\|dt$$

Observe that

$$\int_0^{T_*}\|u\|_3\|\varphi_t\|_3\ dt\overset{37}{\le}\overline{C}\Big(\int_0^{T_*}\|u\|_{H^1}^2dt\Big)^{\frac12}\Big(\int_0^{T_*}\|\varphi_t\|_{H^1}^2dt\Big)^{\frac12}<+\infty$$

since $u\in L^\infty(0,T_*;H^2)$ and $\varphi_t\in C([0,T_*];W^{1,2}_{0,\sigma}(\Omega))$.

So, since at the beginning we can select the sequences with the properties $u^m\to u$ in $L^2(0,T_*;L^2)$ and $\rho^m\to\rho$ in $L^\infty(0,T_*;L^q)$ with $q\ge\frac32$, we have also the convergence of this piece.

We now have to prove that

$$\boxed{\lim_{m\to+\infty}\int_0^{T_*}\int_\Omega \rho^m u^m\cdot(\nabla\varphi)\cdot u^m\ dx\ dt=\int_0^{T_*}\int_\Omega \rho u\cdot(\nabla\varphi)\cdot u\ dx\ dt} \tag{11.109}$$

We have

$$\left|\int_0^{T_*}\int_\Omega\rho^m u^m\cdot(\nabla\varphi)\cdot u^m-\rho u\cdot(\nabla\varphi)\cdot u\ dx\ dt\right|=$$

$$=\left|\int_0^{T_*}\int_\Omega(\rho^m-\rho)u^m\cdot(\nabla\varphi)\cdot u^m+\rho(u^m-u)\cdot(\nabla\varphi)\cdot u^m+\rho u\cdot(\nabla\varphi)\cdot u^m-\rho u\cdot(\nabla\varphi)\cdot u\ dx\ dt\right|\le$$

$$\le\int_0^{T_*}\int_\Omega|\rho^m-\rho||\nabla\varphi||u^m|^2dx\ dt+\int_0^{T_*}\int_\Omega|\rho||\nabla\varphi||u^m||u^m-u|\ dx\ dt+$$

[37]Using that

$$\|u\|_3\|\varphi_t\|_3\le|\Omega|^{\frac16}\|u\|_6|\Omega|^{\frac16}\|\varphi_t\|_6\le\overline{C}\|u\|_{H^1}\|\varphi_t\|_{H^1}$$

since $u\in L^\infty(0,T_*;H^2)$ and $\varphi_t\in C([0,T_*];W^{1,2}_{0,\sigma}(\Omega))$.

$$+\int_0^{T_*}\int_\Omega |\rho u||\nabla\varphi||u^m-u|\ dx\ dt \tag{11.110}$$

Remember now that, from (11.77), we have chosen a sequence such that

$$\lim_{m\to+\infty}\int_0^{T_*}\|\nabla u^m-\nabla u\|_2^2 dt=0 \tag{11.111}$$

So, we deal with the three pieces in 11.9, in order.
We start with

$$\int_0^{T_*}\int_\Omega |\rho^m-\rho||\nabla\varphi||u^m|^2dx\ dt \le \int_0^{T_*}\|\nabla\varphi\|_2\||\rho^m-\rho||u^m|^2\|_2\ dt \le$$

$$\le \sup_{(0,T_*)}\|\nabla\varphi\|_2\int_0^{T_*}\||\rho^m-\rho||u^m|^2\|_2\ dt$$

On the other side we have

$$\||\rho^m-\rho||u^m|^2\|_2^2=\int_\Omega|\rho^m-\rho|^2|u^m|^4dx\le\||\rho^m-\rho|^2\|_3\||u^m|^4\|_{\frac{3}{2}}=\|\rho^m-\rho\|_6^2\|u^m\|_6^4$$

where we have used the Hölder inequality with $p=3$ and $q=\frac{3}{2}$. Since $\|u^m\|_6\le C\|\nabla u^m\|_2\le C\|u^m\|_{H^2}$, and $u^m\in L^\infty(0,T_*;H^2)$ with $\|u^m\|_{L^\infty(0,T_*;H^2)}\le\hat{H}$, we have

$$\int_0^{T_*}\int_\Omega |\rho^m-\rho||\nabla\varphi||u^m|^2dx\ dt\le\sup_{(0,T_*)}\|\nabla\varphi\|_2T_*C\hat{K}^2\sup_{(0,T_*)}\|\rho^m-\rho\|_6$$

Since $\rho^m\to\rho$ in $L^\infty(0,T_*;L^q)$ for every $q\ge\frac{3}{2}$ we have that this piece goes to zero.

The second piece is similar. We have

$$\int_0^{T_*}\int_\Omega |\rho||\nabla\varphi||u^m||u^m-u|\ dx\ dt\le\int_0^{T_*}\|\nabla\varphi\|_2\||\rho||u^m||u^m-u|\|_2\ dt\le$$

$$\le\sup_{(0,T_*)}\|\nabla\varphi\|_2\int_0^{T_*}\||\rho||u^m||u^m-u|\|_2\ dt$$

On the other hand

$$\||\rho||u^m||u^m-u|\|_2^2=\int_\Omega|\rho|^2|u^m|^2|u^m-u|^2dx\le\||\rho|^2\|_3\||u^m|^2\|_3\||u^m-u|^2\|_3=\|\rho\|_6^2\|u^m\|_6^2\|u^m-u\|_6^2$$

Since $u^m,u\in H_0^1(\Omega)$ for almost every $t\in(0,T_*)$ (being a numberable sequence), we have $\|u^m-u\|_6\le C\|\nabla u^m-\nabla u\|_2$. Finally we get

$$\int_0^{T_*}\int_\Omega|\rho||\nabla\varphi||u^m||u^m-u|\ dx\ dt\le C\hat{K}\sup_{(0,T_*)}\|\nabla\varphi\|_2\sup_{(0,T_*)}\|\rho\|_6\int_0^{T_*}\|\nabla u^m-\nabla u\|_2dt$$

Using (11.111), we have that also this term goes to zero.

The latter term is similar. In fact

$$\int_0^{T_*}\int_\Omega |\rho u||\nabla\varphi||u^m - u|\, dx\, dt \le \int_0^{T_*} \|\nabla\varphi\|_2 \||\rho u||u^m - u|\|_2\, dt \le \sup_{(0,T_*)} \|\nabla\varphi\|_2 \int_0^{T_*} \||\rho u||u^m - u|\|_2\, dt$$

So, as above,

$$\||\rho u||u^m - u|\|_2^2 = \int_\Omega |\rho u|^2 |u^m - u|^2 dx = \int_\Omega |\rho|^2 |u|^2 |u^m - u|^2\, dx \le \||\rho|^2\|_3 \||u|^2\|_3 \||u^m - u|^2\|_3 =$$

$$= \|\rho\|_6^2 \|u\|_6^2 \|u^m - u\|_6^2 \le C^2 \|\rho\|_6^2 \|u\|_6^2 \|\nabla u^m - \nabla u\|_2^2$$

Since $u \in L^\infty(0,T_*;H^2)$, we have $\|u\|_6 \le C\|u\|_{H^1} \le C\|u\|_{H^2}$ and so $\|u\|_{L^\infty(0,T_*;L^6)} \le C\|u\|_{L^\infty(0,T_*;H^2)} < \infty$. So it follows

$$\int_0^{T_*}\int_\Omega |\rho u||\nabla\varphi||u^m - u|\, dx\, dt \le C^2 \|\rho\|_{L^\infty(0,T_*;L^6)} \|u\|_{L^\infty(0,T_*;H^2)} \int_0^{T_*} \|\nabla u^m - \nabla u\|_2 dt$$

that vanishes thanks, again, to equation (11.111).

Finally we have to deal with the easiest piece. We have to prove that

$$\lim_{m\to+\infty} \int_0^{T_*}\int_\Omega \nabla u^m \cdot \nabla\varphi\, dx\, dt = \int_0^{T_*}\int_\Omega \nabla u \cdot \nabla\varphi\, dx\, dt$$

We have

$$\left|\int_0^{T_*}\int_\Omega (\nabla u^m - \nabla u)\cdot\nabla\varphi\, dx\, dt\right| \le \int_0^{T_*}\int_\Omega |\nabla u^m - \nabla u||\nabla\varphi|\, dx\, dt \le$$

$$\le \int_0^{T_*} \|\nabla u^m - \nabla u\|_2 \|\nabla\varphi\|_2 dt \le \left(\int_0^{T_*} \|\nabla u^m - \nabla u\|_2^2 dt\right)^{\frac{1}{2}} \left(\int_0^{T_*} \|\nabla\varphi\|_2^2 dt\right)^{\frac{1}{2}} \le$$

$$\le T_*^{\frac{1}{2}} \sup_{[0,T_*]} \|\varphi\|_{H^1} \left(\int_0^{T_*} \|\nabla u^m - \nabla u\|_2^2 dt\right)^{\frac{1}{2}}$$

that goes to zero as remarked above.

Finally we have

$$0 = \lim_{m\to+\infty} \int_0^{T_*}\int_\Omega (\rho^m u_t^m + \rho^m u^m \cdot \nabla u^m - \mu\Delta u^m)\cdot\varphi(x,t)\, dx\, dt$$

$$= \lim_{m\to+\infty} \left\{ -\int_\Omega (\rho^m u^m \cdot \varphi)(x,0)\, dx - \int_0^{T_*}\int_\Omega \rho^m u^m \cdot \varphi_t\, dx\, dt - \int_0^{T_*}\int_\Omega \rho^m u^m \cdot (\nabla\varphi)\cdot u^m\, dx\, dt + \right.$$

$$\left. + \int_0^{T_*}\int_\Omega \nabla u^m \cdot \nabla\varphi\, dx\, dt \right\} =$$

$$= -\int_\Omega \overline{\rho}_0(x) u_0(x)\cdot\varphi(x,0) dx - \int_0^{T_*}\int_\Omega \rho u\cdot\varphi_t\, dx\, dt - \int_0^{T_*}\int_\Omega \rho u\cdot(\nabla\varphi)\cdot u\, dx\, dt + \mu\int_0^{T_*}\int_\Omega \nabla u\cdot\nabla\varphi\, dx\, dt$$

This means that for every $\varphi \in C^1([0,T_*]; W^{1,2}_{0,\sigma}(\Omega))$ with $\varphi(x,T_*) = 0$ almost everywhere in Ω we have the desired equality. ∎

11.10 Weak solution to the problem with regular initial density $\overline{\rho}_0$: the transport equation

We have proved the weak formulation of the transport equation. However, to be a weak solution of the Navier-Stokes equations, the pair (u, ρ) also have to solve the weak transport equation, as introduced in chapter 10. So we have the following theorem.

Theorem 11.6. *Let Ω be a bounded domain of $\mathbb{R}^3$. Consider the pair of solution (u, ρ), as introduced in sections 11.2-11.7, with initial data $(u_0, \overline{\rho}_0)$, as fixed at the beginning of chapter 11. Then, for every $\varphi \in C^1([0, T_*]; H^1(\Omega))$ such that $\varphi(x, T_*) = 0$ a.e. in Ω, we have*

$$\int_0^{T_*}\int_\Omega \rho\varphi_t \; dx \; dt + \int_0^{T_*}\int_\Omega \rho u \cdot \nabla\varphi \; dx \; dt = -\int_\Omega \overline{\rho}_0(x)\varphi(x, 0)dx$$

Proof. Let $\varphi \in C^1([0, T_*]; H^1(\Omega))$ such that $\varphi(x, T_*) = 0$ a.e. in Ω. We have that, clearly, approximate solutions (ρ^m, u^m) are classical solution of the transport equation; so, following the argument of (10.5), we have

$$\int_0^{T_*}\int_\Omega (\rho^m\varphi_t + \rho^m u^m \cdot \nabla\varphi)(x, t) \; dx \; dt = -\int_\Omega \overline{\rho}_0(x)\varphi(x, 0)dx$$

If we prove that for every test function φ

$$\lim_{m\to+\infty}\int_0^{T_*}\int_\Omega \rho^m\varphi_t \; dx \; dt = \int_0^{T_*}\int_\Omega \rho\varphi_t \; dx \; dt$$

$$\lim_{m\to+\infty}\int_0^{T_*}\int_\Omega \rho^m u^m \cdot \nabla\varphi \; dx \; dt = \int_0^{T_*}\int_\Omega \rho u \cdot \nabla\varphi \; dx \; dt$$

we have that ρ is a weak solution to the transport equation. We know that $\rho^m \to \rho$ in $L^\infty(0, T_*; L^q(\Omega))$ for every $q \in [\frac{3}{2}, \infty)$. So

$$\left|\int_0^{T_*}\int_\Omega (\rho^m - \rho)\varphi_t \; dx \; dt\right| \le \int_0^{T_*} \|\rho^m - \rho\|_2\|\varphi_t\|_2 dt \le \|\rho^m - \rho\|_{L^\infty(0,T_*;L^2(\Omega))}\int_0^{T_*} \|\varphi_t\|_2 dt \to 0$$

We deal now with the other limit. We can write

$$\int_0^{T_*}\int_\Omega (\rho^m u^m - \rho u)\cdot\nabla\varphi \, dx \, dt = \int_0^{T_*}\int_\Omega (\rho^m - \rho)u^m\cdot\nabla\varphi \, dx \, dt + \int_0^{T_*}\int_\Omega \rho(u^m - u)\cdot\nabla\varphi \, dx \, dt$$

We can deal with the first integral in a way very similar to the one discussed above. In fact

$$\left|\int_0^{T_*}\int_\Omega (\rho^m - \rho)u^m\cdot\nabla\varphi \, dx \, dt\right| \le \int_0^{T_*}\int_\Omega |\rho^m - \rho||u^m||\nabla\varphi| \, dx \, dt \le \int_0^{T_*} \||\rho^m - \rho||u^m|\|_2\|\nabla\varphi\|_2 dt$$

where

$$\||\rho^m - \rho||u^m|\|_2^2 = \int_\Omega |\rho^m - \rho|^2|u^m|^2 dx \le \|\rho^m - \rho\|_4^2\|u^m\|_4^2$$

From Sobolev inequalities, since $u^m \in H_0^1(\Omega)$, we have $\|u^m\|_4 \leq C\|\nabla u^m\|_2$. So it follows that

$$\||\rho^m - \rho||u^m|\|_2 \leq C\|\rho^m - \rho\|_4\|\nabla u^m\|_2 \leq C\|\rho^m - \rho\|_4\|u^m\|_{H^2}$$

Finally

$$\left|\int_0^{T_*}\int_\Omega (\rho^m - \rho)u^m \cdot \nabla\varphi \; dx \; dt\right| \leq \left(\int_0^{T_*} \||\rho^m - \rho||u^m|\|_2^2 dt\right)^{\frac{1}{2}} \left(\int_0^{T_*} \|\nabla\varphi\|_2^2 dt\right)^{\frac{1}{2}} \leq$$

$$\leq C\left(\int_0^{T_*} \|\rho^m - \rho\|_4^2\|u^m\|_{H^2}^2 dt\right)^{\frac{1}{2}} \left(\int_0^{T_*} \|\nabla\varphi\|_2^2 dt\right)^{\frac{1}{2}} \leq$$

$$\leq C \sup_{(0,T_*)} \|\rho^m - \rho\|_4 \left(\int_0^{T_*} \|u^m\|_{H^2}^2 dt\right)^{\frac{1}{2}} \left(\int_0^{T_*} \|\nabla\varphi\|_2^2 dt\right)^{\frac{1}{2}} \leq$$

$$\leq CT_*^{\frac{1}{2}}\hat{K}\left(\int_0^{T_*} \|\nabla\varphi\|_2^2 dt\right)^{\frac{1}{2}} \|\rho^m - \rho\|_{L^\infty(0,T_*;L^4(\Omega))}$$

since $\sup_{[0,T_*]} \|u^m\|_{H^2} \leq \hat{K}$. So this piece goes to zero as $m \to \infty$.

To deal with the other integral, we use that $u^m \rightharpoonup u$ in $L^2(0,T_*;H^2(\Omega))$. So, we can consider the functional

$$f(w) := \int_0^{T_*}\int_\Omega \rho w \cdot \nabla\varphi \; dx \; dt \quad \forall w \in L^2(0,T_*;H^2(\Omega))$$

We now show that this functional is well defined and continuous (it is obviously linear). We have

$$|f(w)| \leq \int_0^{T_*}\int_\Omega |\rho w||\nabla\varphi| \; dx \; dt \leq \int_0^{T_*} \|\rho w\|_2\|\nabla\varphi\|_2 dt \leq$$

noticing that $\|\rho w\|_2 = \left(\int_\Omega |\rho|^2|w|^2 dx\right)^{\frac{1}{2}} \leq \||\rho|^2\|_2^{\frac{1}{2}}\||w|^2\|_2^{\frac{1}{2}} = \|\rho\|_4\|w\|_4 \leq \|\rho\|_4\|w\|_{H^2}$, we have

$$\leq \|\rho\|_{L^\infty(0,T_*;L^4)}\int_0^{T_*} \|w\|_{H^2}\|\nabla\varphi\|_2 \; dt \leq \|\rho\|_{L^\infty(0,T_*;L^4)}\left(\int_0^{T_*} \|w\|_{H^2}^2 \; dt\right)^{\frac{1}{2}} \left(\int_0^{T_*} \|\nabla\varphi\|_2^2 \; dt\right)^{\frac{1}{2}} \leq$$

$$\leq \|\rho\|_{L^\infty(0,T_*;L^4)}\left(\int_0^{T_*} \|\nabla\varphi\|_2^2 \; dt\right)^{\frac{1}{2}} \|w\|_{L^2(0,T_*;H^2(\Omega))}$$

So continuity and well-posedness are proved. It follows that

$$\lim_{m\to+\infty} f(u^m - u) = 0$$

and so

$$\lim_{m\to+\infty}\int_0^{T_*}\int_\Omega \rho^m u^m \cdot \nabla\varphi \; dx \; dt = \int_0^{T_*}\int_\Omega \rho u \cdot \nabla\varphi \; dx \; dt$$

It follows that, for every φ test function $\varphi \in C^1([0,T_*];H^1)$,

$$\int_0^{T_*}\int_\Omega \rho\varphi_t \ dx \ dt + \int_0^{T_*}\int_\Omega \rho u\cdot\nabla\varphi \ dx \ dt =$$

$$= \lim_{m\to+\infty}\int_0^{T_*}\int_\Omega (\rho^m\varphi_t + \rho^m u^m\cdot\nabla\varphi)(x,t) \ dx \ dt = -\int_\Omega \overline{\rho}_0(x)\varphi(x,0)dx$$

So, we have proved that the limit solution satisfies (in weak sense) the transport equation. ∎

11.11 *A posteriori* estimates on the pair (ρ, u)

Now we have finally found the weak solution to the Navier-Stokes equation, for initial density $\overline{\rho}_0$ with certain regularity hypothesis. Now we deduce some useful estimates on the pair (u,ρ) of solutions.

11.11.1 Estimates on the density ρ

In this subsection, we have the following lemma.

Lemma 11.5. *Let $\rho(t) \in L^\infty(0,T_*;L^q)$ the limit solution above. Then it hold the following properties*

$$0 \le \rho(t) \le \|\rho_0\|_\infty + 1, \qquad \|\rho(t)\|_q = \|\overline{\rho}_0\|_q \qquad \forall t\in(0,T_*)$$

Remark 11.29. This properties hold a.e., since a function in $L^p(0,\overline{T};X)$ is defined a.e., being (respect with time) in an L^p space. □

Proof. We know that, for every $q\in[\frac{3}{2},+\infty)$, it holds

$$\lim_{m\to+\infty}\|\rho^m-\rho\|_{L^\infty(0,T_*;L^q(\Omega))} = 0$$

We can fix a point $\overline{t}\in(0,T_*)$ such that $\overline{t}\notin I_m$ for every $m\in\mathbb{N}$, where I_m is such that $\|\rho^m(t)-\rho(t)\|_q \le \|\rho^m-\rho\|_{L^\infty(0,T_*;L^q(\Omega))} < \infty$ for every $t\in(0,T_*)/I_m$. In particular $I := \bigcup_{m\in\mathbb{N}} I_m$ has measure zero. So, for $\overline{t}\in(0,T_*)/I$ we have

$$\|\rho^m(\overline{t})-\rho(\overline{t})\|_q \le \|\rho^m-\rho\|_{L^\infty(0,T_*;L^q(\Omega))} \to 0$$

as $m\to\infty$. This means that $\{\rho^m(\overline{t})\}_m$ is a Cauchy sequence in $L^q(\Omega)$. This implies that there exists a subsequence $\{\rho^{m_k}(\overline{t})\}_k$ such that

$$\lim_{k\to+\infty}\rho^{m_k}(x,\overline{t}) = \rho(x,\overline{t}) \quad \text{a.e. } x\in\Omega$$

Since $\rho^m(x,t)\ge\delta$, we have that also

$$\rho(x,\overline{t})\ge\delta \quad \text{a.e. } x\in\Omega$$

Moreover, since $\rho^m(x,t)\le\|\rho_0\|_\infty+1$, it follows that

$$\rho(x,\overline{t})\le\|\rho_0\|_\infty+1 \quad \text{a.e. } x\in\Omega$$

Remark 11.30. We have obtained the limit solutions from "astract" theorems of functional analysis that provide us the solution in a non constructive way. We know that, in example, $\rho \in L^\infty(0, T_*; L^q)$. This has to be read in terms of spaces involving time: we fix a time (almost everywhere) in $(0, T_*)$. At this time, ρ gives back a vector in $L^q(\Omega)$ that is a function defined almost everywhere. So, almost every where in time, we have information about the density almost everywhere in the space. In other words, stopping the time at a certain point, the configuration of the density at that freezed time is known almost everywhere. The same holds for the other functions, e.g. the vectorial velocity u. $\square$

So for a.e. $t \in (0, T_*)$, we have

$$\delta \leq \rho(t) \leq \|\rho_0\|_\infty + 1 \tag{11.112}$$

where the inequalities hold almost everywhere. We now deduce the second equality. For almost every $t \in (0, T_*)$ we have

$$|\|\rho^m(t)\|_q - \|\rho(t)\|_q| \leq \|\rho^m(t) - \rho(t)\|_q \leq \sup_{t\in(0,T_*)} \|\rho^m - \rho\|_q \to 0 \quad \text{as } m \to +\infty$$

So that $\lim\limits_{m\to+\infty} \|\rho^m(t)\|_q = \|\rho(t)\|_q \quad$ a.e. $t \in (0, T_*)$. Since $\|\rho^m(t)\|_q = \|\overline{\rho}_0\|_q$ for every $t \in (0, T_*)$ and $m \in \mathbb{N}$, we have $\|\rho(t)\|_q = \|\overline{\rho}_0\|_q \quad$ a.e. $t \in (0, T_*)$.

Moreover, we have that $\sup\limits_{(0,T_*)} \|\rho(t)\|_\infty \leq \|\rho_0\|_\infty + 1$. We know that $\rho \in L^\infty(0, T_*; L^\infty(\Omega))$. So, if $k \in \mathbb{N}$, we have that

$$\lim_{k\to\infty} \|\rho(t)\|_k = \|\rho(t)\|_\infty, \qquad \lim_{k\to\infty} \|\overline{\rho}_0\|_k = \|\overline{\rho}_0\|_\infty$$

provided that t is such that $\rho(t) \in L^k(\Omega)$ for every k. So, if I_k is such that $\rho(t) \in L^k(\Omega)$ and $\|\rho(t)\|_k = \|\overline{\rho}_0\|_k$ for every $t \in (0, T_*)/I_k$, we have that, for every $t \in (0, T_*)/I$, where $I := \bigcup\limits_{k\in\mathbb{N}} I_k$, we have $\rho(t) \in L^k(\Omega)$ for every k. So $\|\rho(t)\|_\infty = \|\overline{\rho}_0\|_\infty$ for almost every $t \in (0, T_*)$. ∎

11.11.2 Estimate on the velocity u

Now, we want to deduce a first estimate on the velocity u. In particular, we prove the following theorem.

Proposition 11.22. *Let Ω be a bounded domain of $\mathbb{R}^3$. Consider the pair of solution (u, ρ), as introduced in sections 11.2-11.7. Then, there exists a constant $C > 0$ such that*

$$\sup_{(0,T_*)} \|\nabla u(t)\|_2 \leq C$$

Proof. We already know, thanks to the theorem 11.13, that

$$\sup_{0\leq t\leq T_*} \|\nabla u^m(t)\|_2 \leq C \tag{11.113}$$

We have already remarked that the sequence $\{u^m\}_m$ can be chosen such that

$$\lim_{m\to+\infty}\int_0^{T_*}\|\nabla u^m-\nabla u\|_2^2\,dt=0$$

In other words, as a function of time, $\|\nabla u^m-\nabla u\|_2\to 0$ in $L^2(0,T_*)$.
This means that there exists a subsequence $\{\|\nabla u^{m_k}-\nabla u\|_2(t)\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to+\infty}\|\nabla u^{m_k}-\nabla u\|_2(t)=0\quad \text{a.e. } t\in(0,T_*)$$

Moreover we know, using estimate (11.113), that exists a measure zero set, say A, such that

$$\|\nabla u^m(t)\|_2\le C\quad \forall t\in(0,T_*)/A,\ \forall m\in\mathbb{N}$$

If B is the zero measure set such that $\|\nabla u^{m_k}-\nabla u\|_2(t)\to 0$ holds for every $t\in(0,T_*)/B$, we have that, for every $t\in(0,T_*)/(A\cup B)$,

$$\|\nabla u(t)\|_2=\lim_{k\to+\infty}\|\nabla u^{m_k}(t)\|_2\le C$$

Since the bound is true almost everywhere in $(0,T_*)$, we have that $\sup\limits_{(0,T_*)}\|\nabla u(t)\|_2\le C$, that is the thesis. ∎

11.11.3 A final *a posteriori* estimate

We finally prove in this section the following theorem.

Theorem 11.7. *Let Ω be a bounded domain of $\mathbb{R}^3$. Consider the pair of solution (u,ρ), as introduced in sections 11.2-11.7. Then, there exists a constant $C>0$ such that, for every $t\in(0,T_*)$,*

$$\sup_{\tau\in(0,t)}\left(\|\nabla u\|_{H^1}^2+\|\sqrt{\rho}u_t\|_2^2\right)+\int_0^t\|\nabla u_t\|_2^2ds\le C\mathcal{C}(\rho_0,u_0,p_0)+C\exp\left(C\int_0^t\|\nabla u\|_2^4ds\right) \tag{11.114}$$

where

$$\mathcal{C}(\rho_0,u_0,p_0):=\int_\Omega(\rho_0)^{-1}|\mu\Delta u_0-\nabla p_0|^2\,dx$$

Proof. We start with a similar estimate that we have already proved. Remember (11.67), that is

$$\sup_{\tau\in(0,t)}\|\nabla u^m\|_{H^1}^2+\sup_{\tau\in(0,t)}\|\sqrt{\rho^m}u_t^m\|_2^2+\int_0^t\|\nabla u_t^m\|_2^2ds\le\hat{H}\overline{\mathcal{C}_0}^m+\hat{H}\exp\left(\hat{H}\int_0^t\|\nabla u^m\|_2^4ds\right)$$

If we add both sides the term $\sup\limits_{\tau\in(0,t)}\|u^m\|_2^2\le\hat{K}^2$ and $\int_0^t\|u_t^m\|_2^2\,ds\le\hat{D}_0$, thanks to (11.75), we have

$$\sup_{\tau\in(0,t)}\|u^m\|_{H^2}^2+\sup_{\tau\in(0,t)}\|\sqrt{\rho^m}u_t^m\|_2^2+\int_0^t\|u_t^m\|_{H^1}^2ds\le$$

$$\leq \sup_{\tau\in(0,t)} \|u^m\|_2^2 + \sup_{\tau\in(0,t)} \|\nabla u^m\|_{H^1}^2 + \sup_{\tau\in(0,t)} \|\sqrt{\rho^m} u_t^m\|_2^2 + \int_0^t \|u_t^m\|_2^2 ds + \int_0^t \|\nabla u_t^m\|_2^2 ds \leq$$

$$\leq \hat{D}_0 + \hat{K}^2 + \sup_{\tau\in(0,t)} \|\nabla u^m\|_{H^1}^2 + \sup_{\tau\in(0,t)} \|\sqrt{\rho^m} u_t^m\|_2^2 + \int_0^t \|\nabla u_t^m\|_2^2 ds \leq$$

$$\leq \hat{D}_0 + \hat{K}^2 + \hat{H}\overline{\mathcal{C}_0}^m + \hat{H} \exp(\hat{H} \int_0^t \|\nabla u^m\|_2^4 ds)$$

Before going on, observe that the inequality can be re-written as

$$\left(\sup_{\tau\in(0,t)} \|u^m\|_{H^2}\right)^2 + \left(\sup_{\tau\in(0,t)} \|\sqrt{\rho^m} u_t^m\|_2\right)^2 + \int_0^t \|u_t^m\|_{H^1}^2 ds \leq \hat{D}_0 + \hat{K}^2 + \hat{H}\overline{\mathcal{C}_0}^m + \hat{H} \exp\left(\hat{H} \int_0^t \|\nabla u^m\|_2^4 ds\right)$$

since the functions are positive. So, taking the liminf both sides we have

$$\liminf_{m\to+\infty} \left\{ \left(\sup_{\tau\in(0,t)} \|u^m\|_{H^2}\right)^2 + \left(\sup_{\tau\in(0,t)} \|\sqrt{\rho^m} u_t^m\|_2\right)^2 + \int_0^t \|u_t^m\|_{H^1}^2 ds \right\} \leq$$

$$\leq \liminf_{m\to+\infty} \left\{ \hat{D}_0 + \hat{K}^2 + \hat{H}\overline{\mathcal{C}_0}^m + \hat{H} \exp\left(\hat{H} \int_0^t \|\nabla u^m\|_2^4 ds\right) \right\} \leq$$

$$\leq \limsup_{m\to+\infty} \left\{ \hat{D}_0 + \hat{K}^2 + \hat{H}\overline{\mathcal{C}_0}^m + \hat{H} \exp\left(\hat{H} \int_0^t \|\nabla u^m\|_2^4 ds\right) \right\}$$

So, using the properties of limsup and liminf[38] we get

$$\left(\liminf_{m\to+\infty} \sup_{\tau\in(0,t)} \|u^m\|_{H^2}\right)^2 + \left(\liminf_{m\to+\infty} \sup_{\tau\in(0,t)} \|\sqrt{\rho^m} u_t^m\|_2\right)^2 + \liminf_{m\to+\infty} \int_0^t \|u_t^m\|_{H^1}^2 ds \leq$$

$$\leq \hat{D}_0 + \hat{K}^2 + \limsup_{m\to+\infty} \hat{H}\overline{\mathcal{C}_0}^m + \limsup_{m\to+\infty} \hat{H} \exp\left(\hat{H} \int_0^t \|\nabla u^m\|_2^4 ds\right)$$

[38]We remark that, if a_n, b_n are two positive sequences, then, if $n > k \in \mathbb{N}$,

$$\big(\inf_{n>k} a_n\big)\big(\inf_{n>k} b_n\big) \leq a_n b_n \implies \big(\inf_{n>k} a_n\big)\big(\inf_{n>k} b_n\big) \leq \inf_{n>k} \big(a_n b_n\big)$$

So, sending $k \to \infty$, we have, by definition,

$$\big(\liminf_{n\to\infty} a_n\big)\big(\liminf_{n\to\infty} b_n\big) \leq \liminf_{n\to\infty} a_n b_n$$

Since $u^m \overset{*}{\rightharpoonup} u$ in $L^\infty(0, T_*; H^2(\Omega))$ we have[39]

$$\|u\|_{L^\infty(0,t;H^2(\Omega))} \le \liminf_{m\to+\infty} \|u^m\|_{L^\infty(0,t;H^2(\Omega))}$$

At the same time we can deduce the analogous inequality for $\|\sqrt{\rho^m} u_t^m\|_{L^\infty(0,t;L^2(\Omega))}$. In fact, first of all consider that, from equation (11.67),

$$\sup_{\tau\in(0,t)} \|\sqrt{\rho^m} u_t^m\|_2^2 \le \hat{H} W_0 + \hat{H} \exp(\hat{H} T_* M^4)$$

and $L^1(0, t; L^2(\Omega))$ is a reflexive Banach space, with $L^\infty(0, t; L^2(\Omega)) \simeq (L^1(0, t; L^2(\Omega)))^*$, we have that, extracting a subsequence,

$$\sqrt{\rho^m} u_t^m \overset{*}{\rightharpoonup} w \text{ in } L^\infty(0, t; L^2(\Omega))$$

for some $w \in L^\infty(0, t; L^2(\Omega))$. More precisely, we can extract the subsequence in the case $t = T_*$, then the weak star convergence is true for every t as explained in the note below. It follows that

$$\|w\|_{L^\infty(0,t;L^2(\Omega))} \le \liminf_{m\to+\infty} \|\sqrt{\rho^m} u_t^m\|_{L^\infty(0,t;L^2(\Omega))}$$

We want to prove that $\|w\|_{L^\infty(0,t;L^2(\Omega))} = \|\sqrt{\rho} u_t\|_{L^\infty(0,t;L^2(\Omega))}$. First of all observe that $\sqrt{\rho^m} u_t^m \rightharpoonup \sqrt{\rho} u_t$ in $L^2(0, t; L^2(\Omega))$. In fact, if $f \in (L^2(0, t; L^2(\Omega)))^*$, we have, with $v_f \in L^2(0, t; L^2(\Omega))$,

$$f(\sqrt{\rho^m} u_t^m - \sqrt{\rho} u_t) = \int_0^t \langle \sqrt{\rho^m} u_t^m - \sqrt{\rho} u_t, v_f \rangle_2 \, ds =$$

$$= \int_0^t \langle (\sqrt{\rho^m} - \sqrt{\rho}) u_t^m, v_f \rangle_2 \, ds + \int_0^t \langle u_t^m - u_t, \sqrt{\rho} v_f \rangle_2 \, ds$$

The second term goes to zero since $u_t^m \rightharpoonup u_t$ in $L^2(0, T_*; H_0^1(\Omega))$ and

$$\left| \int_0^t \langle g, \sqrt{\rho} v_f \rangle_2 \, ds \right| \le \int_0^t \|g\|_2 \|\sqrt{\rho} v_f\|_2 ds \le \left(\int_0^t \|\sqrt{\rho} v_f\|_2^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \|g\|_2^2 ds \right)^{\frac{1}{2}}$$

$$\le \left(\int_0^t \|\sqrt{\rho} v_f\|_2^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \|g\|_{H^1}^2 ds \right)^{\frac{1}{2}}$$

[39] We use here that a weak-star convergence in $L^\infty(0, T_*; H^2(\Omega))$ implies weak-star convergence in $L^\infty(0, t; H^2(\Omega))$ for every $t < T_*$. In fact, if $v \in L^1(0, t; H^2(\Omega))$, we can define

$$v' : (0, T_*) \to H^2(\Omega)$$

$$v'(\tau) := \begin{cases} v(\tau) & \tau \le t \\ 0 & \tau > t \end{cases}$$

and so $v' \in L^1(0, T_*; H^2(\Omega))$ and

$$\lim_m \int_0^t \langle u^m, v \rangle_{H^2} dt = \lim_m \int_0^{T_*} \langle u^m, v' \rangle_{H^2} dt = \int_0^{T_*} \langle u, v' \rangle_{H^2} dt = \int_0^t \langle u, v \rangle_{H^2} dt$$

so that the functional is continuous. The first term, on the other hand, can be treated as follows. We have

$$\left|\int_0^t \langle(\sqrt{\rho^m}-\sqrt{\rho})u_t^m, v_f\rangle_2\ ds\right| \le \int_0^t \|(\sqrt{\rho^m}-\sqrt{\rho})u_t^m\|_2\|v_f\|_2\ ds \le$$

$$\le \left(\int_0^t \|(\sqrt{\rho^m}-\sqrt{\rho})u_t^m\|_2^2\ ds\right)^{\frac{1}{2}}\left(\int_0^t \|v_f\|_2^2\ ds\right)^{\frac{1}{2}} =$$

$$= \left(\int_0^t\int_\Omega |\sqrt{\rho^m}-\sqrt{\rho}|^2|u_t^m|^2\ dx\ ds\right)^{\frac{1}{2}}\left(\int_0^t \|v_f\|_2^2\ ds\right)^{\frac{1}{2}} \le$$

$$\le \sup_{\tau\in(0,t)} \|\sqrt{\rho^m}-\sqrt{\rho}\|_4\left(\int_0^t \|u_t^m\|_4^2\ ds\right)^{\frac{1}{2}}\left(\int_0^t \|v_f\|_2^2\ ds\right)^{\frac{1}{2}}$$

using that

$$\int_\Omega |\sqrt{\rho^m}-\sqrt{\rho}|^2|u_t^m|^2dx \le \left(\int_\Omega |\sqrt{\rho^m}-\sqrt{\rho}|^4dx\right)^{\frac{1}{2}}\left(\int_\Omega |u_t^m|^4dx\right)^{\frac{1}{2}} = \|\sqrt{\rho^m}-\sqrt{\rho}\|_4^2\|u_t^m\|_4^2$$

Observe now that

$$\left(\int_0^t \|u_t^m\|_4^2ds\right)^{\frac{1}{2}} \le \overline{K}\left(\int_0^t \|\nabla u_t^m\|_2^2ds\right)^{\frac{1}{2}} \le \overline{K}\sqrt{\hat{H}W_0+\hat{H}\exp(\hat{H}T_*M^4)} \equiv \overline{F}_0$$

since $\displaystyle\int_0^{T_*} \|\nabla u_t^m\|_2^2\ dt \le [\hat{H}W_0+\hat{H}\exp(\hat{H}T_*M^4)]$. At the same time

$$\|\sqrt{\rho^m}-\sqrt{\rho}\|_4 = \left(\int_\Omega |\sqrt{\rho^m}-\sqrt{\rho}|^4dx\right)^{\frac{1}{4}} \le \left(\int_\Omega |\rho^m-\rho|^2dx\right)^{\frac{1}{4}} = \|\rho^m-\rho\|_2^{\frac{1}{2}}$$

using

$$|\sqrt{\rho^m}-\sqrt{\rho}|^2 = |\sqrt{\rho^m}-\sqrt{\rho}||\sqrt{\rho^m}-\sqrt{\rho}| \le |\sqrt{\rho^m}-\sqrt{\rho}||\sqrt{\rho^m}+\sqrt{\rho}| = |\rho^m-\rho|$$

Finally

$$\left|\int_0^t \langle(\sqrt{\rho^m}-\sqrt{\rho})u_t^m, v_f\rangle_2\ ds\right| \le \|\rho^m-\rho\|_{L^\infty(0,t;L^2(\Omega))}^{\frac{1}{2}}\overline{F}_0\left(\int_0^t \|v_f\|_2^2\ ds\right)^{\frac{1}{2}}$$

Since $\rho^m \to \rho$ in $L^\infty(0,T_*;L^2(\Omega))$ (since $2 > \frac{3}{2}$) we have that also this piece converges to zero. It follows that

$$\|w-\sqrt{\rho}u_t\|_{L^2(0,t;L^2(\Omega))}^2 = \langle w-\sqrt{\rho}u_t, w-\sqrt{\rho}u_t\rangle_{L^2(0,t;L^2(\Omega))} =$$

$$= \langle w, w-\sqrt{\rho}u_t\rangle_{L^2(0,t;L^2(\Omega))} - \langle\sqrt{\rho}u_t, w-\sqrt{\rho}u_t\rangle_{L^2(0,t;L^2(\Omega))} =$$

$$= \lim_{m\to+\infty}\langle\sqrt{\rho^m}u_t^m, w-\sqrt{\rho}u_t\rangle_{L^2(0,t;L^2(\Omega))} - \lim_{m\to+\infty}\langle\sqrt{\rho^m}u_t^m, w-\sqrt{\rho}u_t\rangle_{L^2(0,t;L^2(\Omega))} = 0$$

The result of the two limits follows from this:

$$\lim_{m\to+\infty}\langle\sqrt{\rho^m}u_t^m, w-\sqrt{\rho}u_t\rangle_{L^2(0,t;L^2(\Omega))} = \langle w, w-\sqrt{\rho}u_t\rangle_{L^2(0,t;L^2(\Omega))}$$

since $\sqrt{\rho^m}u_t^m \overset{*}{\rightharpoonup} w$ in $L^\infty(0,t;L^2(\Omega))$ and $w-\sqrt{\rho}u_t$ is in $L^1(0,t;L^2(\Omega))$. The inner product above is extacly the dual pairing between $L^1(0,t;L^2(\Omega))$ and its dual space. Moreover

$$\lim_{m\to+\infty}\langle\sqrt{\rho^m}u_t^m, w-\sqrt{\rho}u_t\rangle_{L^2(0,t;L^2(\Omega))} = \langle\sqrt{\rho}u_t, w-\sqrt{\rho}u_t\rangle_{L^2(0,t;L^2(\Omega))}$$

since $\sqrt{\rho^m}u_t^m \rightharpoonup \sqrt{\rho}u_t$ in $L^2(0,t;L^2(\Omega))$, as proved above, and

$$f(v) := \int_0^t \langle v, w-\sqrt{\rho}u_t\rangle_2 ds$$

is such that

$$|f(v)| \le \int_0^t \|v\|_2\|w-\sqrt{\rho}u_t\|_2 ds \le \left(\int_0^t \|v\|_2^2 ds\right)^{\frac{1}{2}}\left(\int_0^t \|w-\sqrt{\rho}u_t\|_2^2 ds\right)^{\frac{1}{2}} \equiv$$

$$\equiv \|v\|_{L^2(0,t;L^2(\Omega))}\left(\int_0^t \|w-\sqrt{\rho}u_t\|_2^2 ds\right)^{\frac{1}{2}}$$

Since $w-\sqrt{\rho}u_t \in L^2(0,t;L^2(\Omega))$, this means that f is continuous (and it is linear).

It follows that

$$0 = \|w-\sqrt{\rho}u_t\|_{L^2(0,t;L^2(\Omega))} = \left(\int_0^t \|w-\sqrt{\rho}u_t\|_2^2\ ds\right)^{\frac{1}{2}}$$

Thus $\|w-\sqrt{\rho}u_t\|_2 = 0$ almost everywhere in $(0,t)$. This means that

$$\|w\|_2 = \|\sqrt{\rho}u_t\|_2 \quad \text{a.e. in } (0,t)$$

It follows that

$$\sup_{\tau\in(0,t)}\|w\|_2 = \sup_{\tau\in(0,t)}\|\sqrt{\rho}u_t\|_2$$

Finally

$$\sup_{\tau\in(0,t)}\|\sqrt{\rho}u_t\|_2 = \sup_{\tau\in(0,t)}\|w\|_2 \equiv \|w\|_{L^\infty(0,t;L^2(\Omega))} \le \liminf_{m\to+\infty}\|\sqrt{\rho^m}u_t^m\|_{L^\infty(0,t;L^2(\Omega))}$$

Finally we want to say something about the term

$$\liminf_{m\to+\infty}\int_0^t \|u_t^m\|_{H^1}^2 ds$$

Remember that $u_t^m \rightharpoonup u_t$ in $L^2(0,T_*;H_0^1(\Omega))$. Since this is a reflexive Banach space, we have that also $u_t^m \overset{*}{\rightharpoonup} u_t$ in $L^2(0,T_*;H_0^1(\Omega))$. So in particular in $L^2(0,t;H_0^1(\Omega))$. It follows that

$$\left(\int_0^t \|u_t\|_{H^1}^2 ds\right)^{\frac{1}{2}} = \|u_t\|_{L^2(0,t;H_0^1(\Omega))} \leq \liminf_{m\to+\infty} \|u_t^m\|_{L^2(0,t;H_0^1(\Omega))} = \liminf_{m\to+\infty} \left(\int_0^t \|u_t^m\|_{H^1}^2 ds\right)^{\frac{1}{2}}$$

Putting all the pieces together we get

$$\sup_{\tau\in(0,t)} \|u\|_{H^2}^2 + \sup_{\tau\in(0,t)} \|\sqrt{\rho}u_t\|_2^2 + \left(\int_0^t \|u_t\|_{H^1}^2 ds\right) \leq$$

$$\leq \hat{D}_0 + \hat{K}^2 + \limsup_{m\to+\infty} \hat{H}\overline{\mathcal{C}_0}^m + \limsup_{m\to+\infty} \hat{H}\exp\left(\hat{H}\int_0^t \|\nabla u^m\|_2^4 ds\right)$$

Moreover, we have already proved that

$$\lim_{m\to+\infty} \int_0^{T_*} \|\nabla u^m - \nabla u\|_2^2 \, dt = 0$$

We can use the following lemma in [3, Th. IV.9, pg. 58], that is lemma 3.3. Using this, we can find ∇u^{m_k} such that

$$\lim_{k\to+\infty} \|\nabla u^{m_k} - \nabla u\|_2^2 = 0 \quad \text{a.e. } t\in(0,T_*)$$

Moreover, we know that

$$\sup_{[0,T_*]} \|\nabla u^m\|_2 \leq \sup_{[0,T_*]} \|u^m\|_{H^2} \leq \hat{K}$$

So, $\|\nabla u^m\|_2^4$ is bounded, and $\|\nabla u\|_2$ is the limit of $\|\nabla u^{m_k}\|$ almost everywhere. It follows that, from the Lebesgue dominated convergence theorem,

$$\lim_{k\to+\infty} \int_0^t \|\nabla u^{m_k}\|_2^4 \, ds = \int_0^t \|\nabla u\|_2^4 \, ds$$

If we think to the steps above applied to the subsequence m_k we have

$$\sup_{\tau\in(0,t)} \|u\|_{H^2}^2 + \sup_{\tau\in(0,t)} \|\sqrt{\rho}u_t\|_2^2 + \left(\int_0^t \|u_t\|_{H^1}^2 \, ds\right) \leq$$

$$\leq \hat{D}_0 + \hat{K}^2 + \limsup_{m\to+\infty} \hat{H}\overline{\mathcal{C}_0}^m + \hat{H}\exp\left(\hat{H}\int_0^t \|\nabla u\|_2^4 ds\right)$$

This can be rewritten as

$$\sup_{\tau\in(0,t)} \left(\|\nabla u\|_{H^1}^2 + \|\sqrt{\rho}u_t\|_2^2\right) + \int_0^t \|\nabla u_t\|_2^2 ds \leq \tag{11.115}$$

$$\leq \sup_{\tau\in(0,t)} \|u\|_{H^2}^2 + \sup_{\tau\in(0,t)} \|\sqrt{\rho}u_t\|_2^2 + \left(\int_0^t \|u_t\|_{H^1}^2 \, ds\right) \leq$$

$$\leq \hat{D}_0 + \hat{K}^2 + \limsup_{m\to+\infty} \hat{H}\overline{\mathcal{C}_0}^m + \hat{H}\exp\left(\hat{H}\int_0^t \|\nabla u\|_2^4\, ds\right)$$

We have to compute the limit

$$\limsup_{m\to+\infty} \hat{H}\overline{\mathcal{C}_0}^m$$

We want to prove now that[40]

$$\limsup_{m\to+\infty} \overline{\mathcal{C}_0}^m \leq \mathcal{C}(\overline{\rho}_0, u_0, p_0) \tag{11.116}$$

$$\mathcal{C}(\overline{\rho}_0, u_0, p_0) \leq \mathcal{C}(\rho_0, u_0, p_0) \tag{11.117}$$

In this case, since $\Delta u^m(x,0) \to \Delta u_0(x)$ almost every[41] $x \in \Omega$, we have, by Fatou's Lemma,

$$\mathcal{C}(\overline{\rho}_0, u_0, p_0) = \int_\Omega (\overline{\rho}_0)^{-1}|\mu\Delta u_0 - \nabla p_0|^2 dx \leq \liminf_{m\to+\infty} \int_\Omega (\overline{\rho}_0)^{-1}|\mu\Delta u^m(0) - \nabla p_0|^2 dx =$$

$$= \liminf_{m\to+\infty} \overline{\mathcal{C}_0}^m \leq \limsup_{m\to+\infty} \overline{\mathcal{C}_0}^m \leq \mathcal{C}(\overline{\rho}_0, u_0, p_0)$$

So, in this case, $\limsup_{m\to+\infty} \overline{\mathcal{C}_0}^m = \mathcal{C}(\overline{\rho}_0, u_0, p_0) \leq \mathcal{C}(\rho_0, u_0, p_0)$.

If we assume that (11.116)-(11.117) hold, we can deduce the desired estimate. We consider two cases.

If $\mathcal{C}(\rho_0, u_0, p_0) = 0$, the inequality above becomes

$$\sup_{\tau\in(0,t)} \left(\|\nabla u\|_{H^1}^2 + \|\sqrt{\rho}u_t\|_2^2\right) + \int_0^t \|\nabla u_t\|_2^2 ds \leq \hat{D}_0 + \hat{K}^2 + \hat{H}\exp\left(\hat{H}\int_0^t \|\nabla u\|_2^4\, ds\right) \leq$$

$$\leq \hat{D}_0 + \hat{K}^2 + (\hat{H} + \hat{D}_0 + \hat{K}^2)\exp\left((\hat{H} + \hat{D}_0 + \hat{K}^2)\int_0^t \|\nabla u\|_2^4\, ds\right) \leq$$

$$\leq 2(\hat{H} + \hat{D}_0 + \hat{K}^2)\exp\left((\hat{H} + \hat{D}_0 + \hat{K}^2)\int_0^t \|\nabla u\|_2^4\, ds\right)$$

since

$$(\hat{H} + \hat{D}_0 + \hat{K}^2)\exp\left((\hat{H} + \hat{D}_0 + \hat{K}^2)\int_0^t \|\nabla u\|_2^4\, ds\right) \geq \hat{H} + \hat{D}_0 + \hat{K}^2 \geq \hat{D}_0 + \hat{K}^2$$

If $\mathcal{C}(\rho_0, u_0, p_0) > 0$, we can take ε such that $\varepsilon < \mathcal{C}(\rho_0, u_0, p_0)$. Thus

$$\sup_{\tau\in(0,t)} \left(\|\nabla u\|_{H^1}^2 + \|\sqrt{\rho}u_t\|_2^2\right) + \int_0^t \|\nabla u_t\|_2^2 ds \leq$$

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[40] The second inequality is easy; observe that the latter term only depends on the initial conditions.

[41] In fact

$$\|\Delta u^m(0) - \Delta u_0\|_2 \leq \sqrt{5}\|u^m(0) - u_0\|_{H^2} \to 0$$

so $\Delta u^m(0) \to \Delta u_0$ in L^2. This implies that there is a subsequence that converges to Δu_0 almost everywhere. So we can pass to this subsequence.

$$\leq \hat{D}_0 + \hat{K}^2 + \hat{H} \limsup_{m\to+\infty} \overline{\mathcal{C}_0}^m + \hat{H} \exp\left(\hat{H}\int_0^t \|\nabla u\|_2^4 \, ds\right) \leq$$

$$\leq \hat{D}_0 + \hat{K}^2 + \hat{H}\mathcal{C}(\rho_0, u_0, p_0) + \hat{H} \exp\left(\hat{H}\int_0^t \|\nabla u\|_2^4 \, ds\right) \leq$$

$$\leq \hat{D}_0 + \hat{K}^2 + (\hat{H} + \frac{\hat{D}_0 + \hat{K}^2}{\varepsilon})\mathcal{C}(\rho_0, u_0, p_0) + \hat{H} \exp\left(\hat{H}\int_0^t \|\nabla u\|_2^4 \, ds\right) \leq$$

and so[42] we have

$$\leq 2\left(\hat{H} + \frac{\hat{D}_0 + \hat{K}^2}{\varepsilon}\right)\mathcal{C}(\rho_0, u_0, p_0) + \hat{H} \exp\left(\hat{H}\int_0^t \|\nabla u\|_2^4 \, ds\right)$$

So, in every case, we can write the inequality in the form

$$\sup_{\tau\in(0,t)} \left(\|\nabla u\|_{H^1}^2 + \|\sqrt{\rho} u_t\|_2^2\right) + \int_0^t \|\nabla u_t\|_2^2 \, ds \leq C\mathcal{C}(\rho_0, u_0, p_0) + C\exp\left(C\int_0^t \|\nabla u\|_2^4 \, ds\right)$$

provided that

$$\boxed{\limsup_{m\to+\infty} \overline{\mathcal{C}_0}^m \leq \mathcal{C}(\overline{\rho}_0, u_0, p_0) \leq \mathcal{C}(\rho_0, u_0, p_0)}$$

We now prove this fact. Actually the inequality holds with the limit. In fact, consider sequence such that $\Delta u^m(0) \to \Delta u_0$ almost everywhere. In particular we have that $\|\Delta u^m(0) - \Delta u_0\|_2 \to 0$. So there exists[43] a function $U \in L^2(\Omega)$ such that

$$|\Delta u^m(x,0) - \Delta u_0(x)| \leq U(x) \quad \forall m, \ \forall x \in \Omega$$

eventually extracting another subsequence and considering the initial inequality adapted to this subsequence (convergence properties keep to hold as long as the kind of convergence is defined through a numerical sequence). So we can estimate the function

$$(\overline{\rho}_0)^{-1}|\mu\Delta u^m(0) - \nabla p_0|^2 \leq \delta^{-1}\big(\mu|\Delta u^m(0)| + |\nabla p_0|\big)^2 \leq 2\delta^{-1}\big(\mu^2|\Delta u^m(0)|^2 + |\nabla p_0|^2\big)$$

Moreover

$$|\Delta u^m(0)| \leq |\Delta u^m(0) - \Delta u_0| + |\Delta u_0| \leq U + |\Delta u_0|$$

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[42] Using that

$$\left(\hat{H} + \frac{\hat{D}_0 + \hat{K}^2}{\varepsilon}\right)\mathcal{C}(\rho_0, u_0, p_0) > \hat{D}_0 + \hat{K}^2$$

[43] For every $k \in \mathbb{N}$ there exists m_k such that

$$\|\Delta u^{m_k}(0) - \Delta u_0\|_2 \leq \frac{1}{2^k}$$

So

$$U(x) := \sum_{k\in\mathbb{N}} |\Delta u^{m_k}(x,0) - \Delta u_0(x)| \geq |\Delta u^{m_h}(x,0) - \Delta u_0(x)| \quad \forall h \in \mathbb{N}$$

and $\|U\|_2 \leq \sum_{k\in\mathbb{N}} \|\Delta u^{m_k}(0) - \Delta u_0\|_2 \leq \sum_{k\in\mathbb{N}} \frac{1}{2^k} < +\infty$.

The function $U + |\Delta u_0|$ is in $L^2(\Omega)$ since it is sum of functions in $L^2(\Omega)$. Thus

$$(\overline{\rho}_0)^{-1}|\mu\Delta u^m(0) - \nabla p_0|^2 \leq 2\delta^{-1}\big((U + |\Delta u_0|)^2 + |\nabla p_0|^2\big) \equiv G$$

where $G \in L^1(\Omega)$.
So the sequence has a summable bound. Then, we can apply the Lebesgue dominated convergence theorem, and we get

$$\lim_{m\to+\infty} \overline{\mathcal{C}_0}^m = \lim_{m\to+\infty} \int_\Omega (\overline{\rho}_0)^{-1}|\mu\Delta u^m(0) - \nabla p_0|^2 dx = \int_\Omega (\overline{\rho}_0)^{-1}|\mu\Delta u_0 - \nabla p_0|^2 dx$$

Now we prove (11.117). In fact, in the last integral we want to replace $\overline{\rho}_0$ with the initial data ρ_0. We have supposed to hold the *compatibility condition*

$$\mu\Delta u_0 - \nabla p_0 = \sqrt{\rho_0}g$$

where $g \in L^2(\Omega)$. The functions $\overline{\rho}_0, \rho_0$ are non negative, and $\rho_0 \in L^\infty(\Omega)$, so in particulare the initial density is a non negative measurable function. We can consider

$$P_0 := \{x \in \Omega : \ \rho_0(x) = 0\} = \rho_0^{-1}(\{0\})$$

This set is measurable subset of Ω, since ρ_0 is measurable and $\{0\}$ is a Borelian set. For every $x \in P_0$, we know for sure that $|\mu\Delta u_0 - \nabla p_0| = 0$ by the equality above. Also g is a measurable function.
So we can redefine these functions as follows. We set

$$(\sqrt{\rho_0'}(x))^{-1} := \begin{cases} (\sqrt{\rho_0}(x))^{-1} & x \notin P_0 \\ \infty & x \in P_0 \end{cases}$$

and

$$g'(x) := \begin{cases} g(x) & x \notin P_0 \\ 0 & x \in P_0 \end{cases}$$

We now, for sake of semplicity, drop the apices. With this devices, we can write

$$(\sqrt{\rho_0})^{-1}|\mu\Delta u_0 - \nabla p_0| = |g|$$

Observe that now this condition is completely equivalent to the compatibility condition, keeping in mind the product in the extended positive line $0 \cdot \infty = 0$, as in example introduced by [24].
Moreover the measurability of the functions is preserved[44]. Moreover we have that

[44] In fact let g a measurable function over a measure space $(M, \mathcal{M})$ and let $g_0 \in [0, \infty]$ and G_0 a measurable set. Then

$$f(x) := \begin{cases} g(x) & x \notin G_0 \\ g_0 & x \in G_0 \end{cases}$$

is a measurable function. In fact, if A is an open set, we can discern two cases. If $g_0 \notin A$, then

$$f^{-1}(A) = g^{-1}(A) \in \mathcal{M}$$

If $g_0 \in A$ we have

$$f^{-1}(A) = (g^{-1}(A) \cap G_0^c) \cup G_0 \in \mathcal{M}$$

$\rho_0 \leq \overline{\rho}_0$. So, for every $x \notin P_0$,

$$(\overline{\rho}_0)^{-1}(x) \leq (\rho_0)^{-1}(x)$$

and if $x \in P_0$

$$(\overline{\rho}_0)^{-1}(x) \leq \delta^{-1} \leq \infty = (\rho_0)^{-1}(x)$$

So we have

$$\int_\Omega (\overline{\rho}_0)^{-1}|\mu\Delta u_0 - \nabla p_0|^2 dx \leq \int_\Omega (\rho_0)^{-1}|\mu\Delta u_0 - \nabla p_0|^2 dx = \int_\Omega |g|^2 dx < \infty$$

The latter integral is actually $\|g'\|_2^2$, but it holds $\|g'\|_2^2 \leq \|g\|_2^2$, since the functions are equal or $|g'|$ is zero while $|g| \geq 0$. The points where $\rho_0^{-1} = \infty$ give no contribution to the integral. If fact

$$\rho_0^{-1} = \infty \;\Leftrightarrow\; x \in P_0 \;\Rightarrow\; |\mu\Delta u_0 - \nabla p_0| = 0$$

so that the product between infinity and zero is zero. Since $g \in L^2(\Omega)$, $\mathcal{C}(\rho_0, u_0, p_0) = \displaystyle\int_\Omega (\rho_0)^{-1}|\mu\Delta u_0 - \nabla p_0|^2 dx$ is a number.

Finally we have, for every $t \in (0, T_*)$,

$$\boxed{\sup_{\tau\in(0,t)} \left(\|\nabla u\|_{H^1}^2 + \|\sqrt{\rho}u_t\|_2^2\right) + \int_0^t \|\nabla u_t\|_2^2 \, ds \leq C\mathcal{C}(\rho_0, u_0, p_0) + C\exp\left(C\int_0^t \|\nabla u\|_2^4 \, ds\right)} \tag{11.118}$$

Remark 11.31. Another similar estimate can be deduced without other computation. In fact, we know that

$$\limsup_{m\to+\infty} \exp\left(\hat{H}\int_0^t \|\nabla u^m\|_2^4 ds\right) \leq \exp\left(\hat{H}\int_0^t C^4 ds\right) \leq \exp\left(\hat{H}T_* C^4\right)$$

thanks to estimate (11.113).
So equation (11.115) becomes

$$\sup_{\tau\in(0,t)} \left(\|\nabla u\|_{H^1}^2 + \|\sqrt{\rho}u_t\|_2^2\right) + \int_0^t \|\nabla u_t\|_2^2 ds \leq \hat{D}_0 + \hat{K}^2 + \limsup_{m\to+\infty} \hat{H}\overline{\mathcal{C}_0}^m + \hat{H}\exp\left(\hat{H}T_* C^4\right)$$

This leads to the inequality to

$$\sup_{\tau\in(0,t)} \left(\|\nabla u\|_{H^1}^2 + \|\sqrt{\rho}u_t\|_2^2\right) + \int_0^t \|\nabla u_t\|_2^2 \, ds \leq C\mathcal{C}(\rho_0, u_0, p_0) + C$$

11.12 Regularization of the initial density

In this subsection we prove the following lemma.

Lemma 11.6. *Let $\delta \in (0,1)$ and $\rho_0 \in L^\infty(\Omega)$. Then there exists a regularized initial density $\rho_{0,\delta} \in C^1(\overline{\Omega})$ such that*

$$0 < \max\{\rho_0, \delta\} \leq \rho_{0,\delta} \leq \|\rho_0\|_\infty + \delta$$

Moreover, if we consider these functions as a sequence of functions indixed by δ, we have

$$\lim_{\delta \to 0} \rho_{0,\delta}(x) = \rho_0(x) \quad \text{for almost every } x \in \Omega$$

Remark 11.32. If we choose $\overline{\rho}_0 = \rho_{0,\delta}$ for a fixed δ, the choice satisfies the hypothesis (11.5), since $\delta \in (0,1)$. Moreover, in the following section we will consider sequence $\delta = \delta_m$. So, in the following proof, from a certain point, we will consider such a sequence. □

Proof. We have $\rho_0 \in L^\infty(\Omega)$ and Ω bounded domain. So, we can consider $B = B_R(0) \supseteq \overline{\Omega}$. We can extend now the function. In particular, we consider

$$\rho_0(x) := \begin{cases} \rho_0(x) & x \in \Omega \\ \|\rho_0\|_\infty & x \in B/\Omega \end{cases}$$

Moreover, define, for every $\varepsilon > 0$, $B_\varepsilon := \{x \in B : \operatorname{dist}(x, \partial B) > \varepsilon\}$. Then, there exists $\overline{\varepsilon} > 0$ such that $\Omega \subset B_\varepsilon$ for every $\varepsilon < \overline{\varepsilon}$.

We define

$$g_{0\delta}(x) := \max\{\rho_0(x), \delta\}$$

Clearly $g_{0\delta}(x) \leq \|g_{0\delta}\|_\infty \leq \|\rho_0\|_\infty + \delta$. On the other side, we consider

$$g'_{0\delta}(x) := -g_{0\delta}(x) + \|\rho_0\|_\infty + \delta \geq 0$$

We consider now the regularization

$$\overline{g}^\varepsilon_{0\delta}(x) := \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)\big(g'_{0\delta}(y) - \frac{\delta}{4}\big)\ dy \geq 0$$

The non-negativity follows from the fact that $-g_{0\delta}(x) + \|\rho_0\|_\infty + \delta \geq \frac{\delta}{4}$ if $\delta \leq \|\rho_0\|_\infty$. Observe now that

$$\lim_{(\varepsilon,\delta)\to(0,0)} \overline{g}^\varepsilon_{0\delta}(x) = -\rho_0(x) + \|\rho_0\|_\infty \quad \text{a.e. } x \in \Omega \tag{11.119}$$

We now prove that limit (11.119) holds. In fact

$$\big|\overline{g}^\varepsilon_{0\delta}(x) - (-\rho_0(x) + \|\rho_0\|_\infty)\big| = \left| \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)\big(g'_{0\delta}(y) - \frac{\delta}{4} + \rho_0(x) - \|\rho_0\|_\infty\big)\ dy \right| \leq$$

$$\leq \int_{B(x,\varepsilon)} \frac{1}{\varepsilon^3}\left|\eta(\frac{x-y}{\varepsilon})\right|\left|g'_{0\delta}(y) - \frac{\delta}{4} + \rho_0(x) - \|\rho_0\|_\infty\right|\ dy$$

Let $C > 0$ such that $|\eta(z)| \leq C$ for every $z \in B(0,1)$. Observe now that

$$\left| g'_{0\delta}(y) - \frac{\delta}{4} + \rho_0(x) - \|\rho_0\|_\infty \right| = |-\max\{\rho_0(y), \delta\} + \|\rho_0\|_\infty + \delta + \rho_0(y) - \rho_0(y) - \frac{\delta}{4} + \rho_0(x) - \|\rho_0\|_\infty| \leq$$

$$\leq |-\max\{\rho_0(y), \delta\} + \rho_0(y)| + \frac{3}{4}\delta + |\rho_0(y) - \rho_0(x)| \leq \delta + \frac{3}{4}\delta + |\rho_0(y) - \rho_0(x)|$$

since

$$\max\{\rho_0(y), \delta\} - \rho_0(y) = \begin{cases} 0 < \delta & \rho_0(y) \geq \delta \\ \delta - \rho_0(y) \leq \delta & \delta \geq \rho_0(y) \end{cases} \geq 0$$

So we have

$$\left|\overline{g}^{\varepsilon}_{0\delta}(x) - (-\rho_0(x) + \|\rho_0\|_\infty)\right| \leq C\left(\delta + \frac{3}{4}\delta + \frac{1}{\varepsilon^3}\int_{B(x,\varepsilon)} |\rho_0(y) - \rho_0(x)|\, dy\right) \to 0$$

as $\varepsilon, \delta \to 0$, thanks to the Lebesgue theorem.

On the other hand, we have that

$$\lim_{\varepsilon \to 0} \overline{g}^{\varepsilon}_{0\delta}(x) = g'_{0\delta}(x) - \frac{\delta}{4} \quad \text{a.e. } x \in \Omega$$

since it is the regularization of the function. So, for every $\alpha > 0$, exists $\overline{\varepsilon} = \overline{\varepsilon}(\alpha)$ such that

$$|\overline{g}^{\varepsilon}_{0\delta}(x) - g'_{0\delta}(x) + \frac{\delta}{4}| < \alpha$$

for every $\varepsilon < \overline{\varepsilon}$. So, if we choose $\alpha = \frac{\delta}{4}$, we have

$$\overline{g}^{\varepsilon}_{0\delta}(x) \leq g'_{0\delta}(x)$$

for every $\varepsilon < \overline{\varepsilon}(\delta)$. So, if $\varepsilon_0(\delta) := \frac{\overline{\varepsilon}(\delta)}{2} < \overline{\varepsilon}(\delta)$,

$$\overline{g}^{\varepsilon_0(\delta)}_{0\delta}(x) \leq g'_{0\delta}(x)$$

So, if now $\delta = \delta_k = \frac{1}{k}$, we have $\varepsilon_1 := \varepsilon_0(\delta_1)$, and $\varepsilon_2 < \min\{\varepsilon_0(\delta_2), \frac{\varepsilon_1}{2}\}$. So, in general

$$\varepsilon_k < \min\{\varepsilon_0(\delta_k), \frac{\varepsilon_{k-1}}{2}\}$$

So $\varepsilon_k \to 0$ and $\overline{g}^{\varepsilon_k}_{0\delta_k}(x) \leq g'_{0\delta_k}(x)$ and so the sequence $\overline{g}^{\varepsilon_k}_{0\delta_k}(x) \to -\rho_0(x) + \|\rho_0\|_\infty$ from below. This means that

$$-\overline{g}^{\varepsilon_k}_{0\delta_k}(x) + \|\rho_0\|_\infty + \delta_k \to \rho_0(x), \qquad -\overline{g}^{\varepsilon_k}_{0\delta_k}(x) + \|\rho_0\|_\infty + \delta_k \geq g_{0\delta_k}(x)$$

and moreover $-\overline{g}^{\varepsilon_k}_{0\delta_k}(x) + \|\rho_0\|_\infty + \delta_k \in C^1(\overline{\Omega})$, that is the thesis. ∎

11.13 Weak and strong solution to momentum equation with initial density $\rho_0 \in L^\infty(\Omega)$

11.13.1 Weak solution to the momentum equations

We now briefly repeat the argument above for another sequence.

The regularized approximation of the previous subsection gives us a sequence of initial data in the regularity class that we needed. In particular, we have $\{\rho_{0\delta}\}_{\delta\in(0,1)} \subseteq C^1(\overline{\Omega})$ such that, for every $\delta \in (0,1)$, it holds

$$0 < \max\{\rho_0(x), \delta\} \leq \rho_{0\delta}(x) \leq \|\rho_0\|_\infty + \delta$$

and moreover it holds the limit

$$\lim_{\delta\to 0} \rho_{0\delta}(x) = \rho_0(x) \quad \text{a.e. } x \in \Omega \tag{11.120}$$

The first inequality above gives us an important information. In fact, we have that

$$\rho_0(x) \leq \max\{\rho_0(x), \delta\} \leq \rho_{0\delta}(x)$$

and

$$0 < \delta \leq \max\{\rho_0(x), \delta\} \leq \rho_{0\delta}(x) \leq \|\rho_0\|_\infty + \delta < \|\rho_0\|_\infty + 1$$

So, if we choose the initial density $\overline{\rho}_0(x) = \rho_{0\delta}(x)$, we have that this density satisfies the hypothesis required in (11.5). Under these hypothesis, we have already discussed the construction of a weak solution to the problem, in the sense that we will clarify in a moment. We have that exists a pair of solutions $(\rho^\delta, u^\delta) \in L^\infty(0, T_*; L^\infty(\Omega)) \times L^\infty(0, T_*; H^2(\Omega))$ of weak solution to the INSE, with initial data[45] $(\overline{\rho}_0, u_0) = (\rho_{0\delta}, u_0)$, that is

$$\int_0^{T_*} \int_\Omega \rho^\delta(\varphi_t + \rho^\delta u^\delta \cdot \nabla\varphi)\, dx\, dt = -\int_\Omega \overline{\rho}_0(x)\varphi(x,0)\, dx = -\int_\Omega \rho_{0\delta}(x)\varphi(x,0)\, dx$$

for every $\varphi \in C^1([0, T_*]; H^1(\Omega)$ with $\varphi(x, T_*) = 0$ a.e. in Ω; and

$$-\int_0^{T_*} \int_\Omega \rho^\delta u^\delta \cdot \varphi_t\, dx\, dt - \int_0^{T_*} \int_\Omega \rho^\delta u^\delta \cdot \nabla\varphi \cdot u\, dx\, dt + \mu \int_0^{T_*} \int_\Omega \nabla u^\delta \cdot \nabla\varphi\, dx\, dt =$$

$$= \int_\Omega \overline{\rho}_0(x) u_0(x) \cdot \varphi(x,0)\, dx = \int_\Omega \rho_{0\delta}(x) u_0(x) \cdot \varphi(x,0)\, dx$$

for every $\varphi \in C^1([0, T_*]; X)$ with $\varphi(x, T_*) = 0$ almost everywhere in Ω.
Remember that T_* is independent of the lower bound δ of the initial density and also of the initial density $\overline{\rho}_0$ itself.
It also exists the weak derivative $u^\delta_t \in L^2(0, T_*; H^1_0(\Omega))$ of u^δ. Moreover, for such a solution, we have already proved the following estimates: for every $t \in (0, T_*)$ we have that

$$0 \leq \rho^\delta(t) \leq \|\rho_0\|_\infty + 1$$

[45] Looking at the initial density, it is clear the role of δ in (ρ^δ, u^δ).

$$\|\rho^\delta(t)\|_q = \|\rho_{0\delta}\|_q, \qquad \|\nabla u^\delta(t)\|_2^2 \le C$$

$$\sup_{(0,t]} \left(\|\nabla u^\delta\|_{H^1}^2 + \|\sqrt{\rho^\delta} u_t^\delta\|_2^2\right) + \int_0^t \|\nabla u_t^\delta\|_2^2 ds \le C\mathcal{C}(\rho_0, u_0, p_0) + C\exp\left(C\int_0^t \|\nabla u^\delta\|_2^4 ds\right) \tag{11.121}$$

uniformly in δ. From the family of solutions $\{(\rho^\delta, u^\delta) : \ \delta \in (0,1)\}$ we can consider a sequence. In fact, if we take $\delta = \delta_m = \frac{1}{m} \in (0,1)$ for every $m \ge 2$, we have[46] a sequence $\{(\rho^m, u^m)\}_{m\ge 2}$ with the boundaries and the properties above, where δ has to be replaced with m.
With this (more familiar) notation, we now that exists $(u, \rho) \in L^\infty(0, T_*; L^\infty) \times L^\infty(0, T_*; H^2)$ such that

$$\rho^m \overset{*}{\rightharpoonup} \rho \quad \text{in } L^\infty(0, T_*; L^\infty(\Omega)), \qquad u^m \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0, T_*; H^2(\Omega))$$

and morever exists $u_t \in L^2(0, T_*; H_0^1(\Omega))$ weak derivative of u, and

$$u_t^m \rightharpoonup u_t \quad \text{in } L^2(0, T_*; H_0^1(\Omega))$$

In fact, looking at section 11.6, we have that the limit-extraction argument is based only on the features of the functional spaces considered and the estimates above; so, although in 11.6 the sequence (ρ^m, u^m) has some regularity properties, the extraction works in the same way in this less regular case.

Finally, the pair (ρ, u) is the candidate to be out local solution. We have to show that it is a weak solution with initial data (ρ_0, u_0) and moreover that the main estimate (11.121) holds without δ. In particular, using the m-notation, we have that

$$\int_0^{T_*}\int_\Omega \rho^m(\varphi_t + \rho^m u^m \cdot \nabla\varphi)\, dx\, dt = -\int_\Omega \rho_{0\frac{1}{m}}(x)\varphi(x,0)\, dx$$

for every $\varphi \in C^1([0, T_*]; H^1(\Omega))$ with $\varphi(x, T_*) = 0$ a.e. in Ω; and

$$-\int_0^{T_*}\int_\Omega \rho^m u^m \cdot \varphi_t\, dx\, dt - \int_0^{T_*}\int_\Omega \rho^m u^m \cdot \nabla\varphi \cdot u\, dx\, dt + \mu\int_0^{T_*}\int_\Omega \nabla u^m \cdot \nabla\varphi\, dx\, dt =$$

$$= \int_\Omega \rho_{0\frac{1}{m}}(x) u_0(x) \cdot \varphi(x,0)\, dx$$

for every $\varphi \in C^1([0, T_*]; X)$ with $\varphi(x, T_*) = 0$ almost everywhere in Ω.
Remember moreover that

$$\lim_{m\to+\infty} \rho_{0\frac{1}{m}}(x) = \rho_0(x) \quad \text{a.e. } x \in \Omega, \qquad \rho_{0\frac{1}{m}}(x) \le \|\rho_0\|_\infty + 1$$

So, thanks to the Lebesgue dominated theorem, we have

$$\lim_{m\to+\infty}\int_\Omega \rho_{0\frac{1}{m}}(x)\varphi(x,0)\, dx = \int_\Omega \rho_0(x)\varphi(x,0)\, dx$$

and

$$\lim_{m\to+\infty}\int_\Omega \rho_{0\frac{1}{m}}(x) u_0(x) \cdot \varphi(x,0)\, dx = \int_\Omega \rho_0(x) u_0(x) \cdot \varphi(x,0)\, dx$$

[46] With an abuse of notation, we consider $\rho^m \equiv \rho^{\frac{1}{m}}$ and $u^m \equiv u^{\frac{1}{m}}$.

If we show that the limit in m allows us to get rid of the m in the notation, than we proved that (ρ, u) is a weak solution with initial data (ρ_0, u_0).
We start with the transport (or mass) equation. Remember that, thanks to the adaptation of the DiPerna-Lions compactness result [8], we have that

$$\rho^m \to \rho \quad \text{in } \; L^\infty(0, T_*; L^q)$$

since ρ^m is a weak solution and the properties in 11.7 hold: in particular, equation (11.120) says that for every β bounded, $\beta(\rho_{0\delta}) \to \beta(\rho_0)$ in $L^1(\Omega)$, and $\rho_0 \in L^\infty(\Omega)$. Moreover, since $|\rho_{0\delta}(x)| \leq \|\rho_0\|_\infty + 1$ and there is convergence almost everywhere, Lebesgue dominated convergence imlies that $\rho_{0\delta} \to \rho_0$ in $L^p(\Omega)$, for every $p \in [1, \infty]$. This implies, as proved in section 11.10, that

$$\lim_{m \to +\infty} \int_0^{T_*} \int_\Omega \rho^m \varphi_t \; dx \; dt = \int_0^{T_*} \int_\Omega \rho \varphi_t \; dx \; dt$$

$$\lim_{m \to +\infty} \int_0^{T_*} \int_\Omega \rho^m u^m \cdot \nabla\varphi \; dx \; dt = \int_0^{T_*} \int_\Omega \rho u \cdot \nabla\varphi \; dx \; dt$$

for every $\varphi \in C^1([0, T_*]; H^1(\Omega))$ such that $\varphi(x, T_*) = 0$ a.e. in Ω. So the limit solution satisfies the weak transport equation. Moreover in section 11.9 we have already proved that

$$\lim_{m \to +\infty} \int_0^{T_*} \int_\Omega \rho^m u^m \cdot \varphi_t \; dx \; dt = \int_0^{T_*} \int_\Omega \rho u \cdot \varphi_t \; dx \; dt$$

$$\lim_{m \to +\infty} \int_0^{T_*} \int_\Omega \rho^m u^m \cdot (\nabla\varphi) \cdot u^m \; dx \; dt = \int_0^{T_*} \int_\Omega \rho u \cdot (\nabla\varphi) \cdot u \; dx \; dt$$

$$\lim_{m \to +\infty} \int_0^{T_*} \int_\Omega \nabla u^m \cdot \nabla\varphi \; dx \; dt = \int_0^{T_*} \int_\Omega \nabla u \cdot \nabla\varphi \; dx \; dt$$

provided that, as above, $\rho^m \to \rho$ in $L^\infty(0, T_*; L^q)$. So, also the momentum equation is satisfied. This is a first result that we were aiming. Observe that all the arguments of the previous sections recalled here don't involve the fact that in those previous sections (ρ^m, u^m) are very regular; the arguments we used are only convergences and continuity of some operators between functional spaces (in particular L^p spaces involving time).
It remains to prove the main inequality for the weak solution. We have the inequality

$$\sup_{(0,t]} \left(\|\nabla u^m\|_{H^1}^2 + \|\sqrt{\rho^m} u_t^m\|_2^2\right) + \int_0^t \|\nabla u_t^m\|_2^2 ds \leq C\mathcal{C}(\rho_0, u_0, p_0) + C \exp\left(C \int_0^t \|\nabla u^m\|_2^4 \; ds\right)$$

using the m-notation. The inequality can be written in a slightly different way. In fact, if we consider the inequality in (11.115), we have that this inequality can be written as

$$\sup_{(0,t]} \|\nabla u^m\|_{H^1}^2 + \sup_{(0,t]} \|\sqrt{\rho^m} u_t^m\|_2^2 + \int_0^t \|\nabla u_t^m\|_2^2 ds \leq$$

$$\leq C\mathcal{C}(\rho_0, u_0, p_0) + C \exp\left(C \int_0^t \|\nabla u^m\|_2^4 \; ds\right) \tag{11.122}$$

We want now to take the limit on both sides. It is very similar to a calculus already done. As in section 11.11.3 we have

$$\liminf_{m\to+\infty}\left(\sup_{(0,t]}\left(\|\nabla u^m\|_{H^1}^2+\|\sqrt{\rho^m}u_t^m\|_2^2\right)+\int_0^t\|\nabla u_t^m\|_2^2 ds\right)\le$$

$$\le\liminf_{m\to+\infty}\left(C\mathcal{C}(\rho_0,u_0,p_0)+C\exp(C\int_0^t\|\nabla u^m\|_2^4 ds)\right)\le$$

$$\le\limsup_{m\to+\infty}\left(C\mathcal{C}(\rho_0,u_0,p_0)+C\exp(C\int_0^t\|\nabla u^m\|_2^4 ds)\right)$$

Always in 11.11.3 we have proved that

$$\|u\|_{L^\infty(0,t;H^2(\Omega))}\le\liminf_{m\to+\infty}\|u^m\|_{L^\infty(0,t;H^2(\Omega))}$$

and

$$\sup_{\tau\in(0,t)}\|\sqrt{\rho}u_t\|_2\le\liminf_{m\to+\infty}\|\sqrt{\rho^m}u_t^m\|_{L^\infty(0,t;L^2(\Omega))}$$

and

$$\left(\int_0^t\|u_t\|_{H^1}^2\ ds\right)^{\frac{1}{2}}\le\liminf_{m\to+\infty}\left(\int_0^t\|u_t^m\|_{H^1}^2\ ds\right)^{\frac{1}{2}}$$

It also holds, passing to a suitable subsequence and considering this subsequence at the beginning,

$$\lim_{m\to+\infty}\int_0^t\|\nabla u^m\|_2^4\ ds=\int_0^t\|\nabla u\|_2^4\ ds$$

Since $C\mathcal{C}(\rho_0,u_0,p_0)$ is indipendet of m (i.e. of δ_m), we have

$$\sup_{(0,t]}\left(\|\nabla u\|_{H^1}^2+\|\sqrt{\rho}u_t\|_2^2\right)+\int_0^t\|\nabla u_t^m\|_2^2 ds\le\sup_{(0,t]}\|\nabla u\|_{H^1}^2+\sup_{(0,t]}\|\sqrt{\rho}u_t\|_2^2+\int_0^t\|\nabla u_t^m\|_2^2 ds\le$$

$$\le\sup_{\tau\in(0,t]}\|u\|_{H^2}^2+\sup_{\tau\in(0,t]}\|\sqrt{\rho}u_t\|_2^2+\left(\int_0^t\|u_t\|_{H^1}^2 ds\right)=\|u\|_{L^\infty(0,t;H^2)}^2+\sup_{\tau\in(0,t]}\|\sqrt{\rho}u_t\|_2^2+\left(\int_0^t\|u_t\|_{H^1}^2 ds\right)\le$$

$$\le\left(\liminf_{m\to+\infty}\|u^m\|_{L^\infty(0,t;H^2(\Omega))}\right)^2+\left(\liminf_{m\to+\infty}\|\sqrt{\rho^m}u_t^m\|_{L^\infty(0,t;L^2(\Omega))}\right)^2+\left(\liminf_{m\to+\infty}\left(\int_0^t\|u_t^m\|_{H^1}^2 ds\right)^{\frac{1}{2}}\right)^2\le$$

and using the properties of liminf[47]

$$\le\liminf_{m\to+\infty}\|u^m\|_{L^\infty(0,t;H^2(\Omega))}^2+\liminf_{m\to+\infty}\|\sqrt{\rho^m}u_t^m\|_{L^\infty(0,t;L^2(\Omega))}^2+\liminf_{m\to+\infty}\int_0^t\|u_t^m\|_{H^1}^2 ds\le$$

[47]Remember that if $f:[0,+\infty)\to[0,+\infty)$ is continuous and increasing and $a_n\ge0$ we have

$$\liminf_{n\to+\infty}f(a_n)=\lim_{k\to+\infty}f(a_{n_k})=f\left(\lim_{k\to+\infty}a_{n_k}\right)\ge f\left(\liminf_{n\to+\infty}a_n\right)$$

where a_{n_k} is the sequence with limit the liminf. Moreover $\liminf_{n\to+\infty}a_n\le\lim_{k\to+\infty}a_{n_k}$.

$$\leq \liminf_{m\to+\infty}\left(\|u^m\|^2_{L^\infty(0,t;H^2(\Omega))}+\|\sqrt{\rho^m}u^m_t\|^2_{L^\infty(0,t;L^2(\Omega))}+\int_0^t\|u^m_t\|^2_{H^1}ds\right)\leq$$

Remembering now that, as deduced from the estimates above, $\sup_{\tau\in(0,t)}\|u^m\|_2^2\leq \hat{K}^2$ and $\int_0^t\|u^m_t\|_2^2ds\leq \hat{D}_0$, we have

$$\leq \liminf_{m\to+\infty}\left(\sup_{(0,t]}\left(\|u^m\|_2^2+\|\nabla u^m\|^2_{H^1}\right)+\|\sqrt{\rho^m}u^m_t\|^2_{L^\infty(0,t;L^2(\Omega))}+\int_0^t(\|u^m_t\|_2^2+\|\nabla u^m_t\|_2^2)ds\right)\leq$$

$$\leq \liminf_{m\to+\infty}\left(\sup_{(0,t]}\|u^m\|_2^2+\sup_{(0,t]}\|\nabla u^m\|^2_{H^1}+\|\sqrt{\rho^m}u^m_t\|^2_{L^\infty(0,t;L^2(\Omega))}+\int_0^t\|u^m_t\|_2^2dx+\int_0^t\|\nabla u^m_t\|_2^2ds\right)\leq$$

$$\liminf_{m\to+\infty}\left(\hat{K}^2+\hat{D}_0+\sup_{(0,t]}\|\nabla u^m\|^2_{H^1}+\|\sqrt{\rho^m}u^m_t\|^2_{L^\infty(0,t;L^2(\Omega))}+\int_0^t\|\nabla u^m_t\|_2^2ds\right)=$$

$$=\hat{K}^2+\hat{D}_0+\liminf_{m\to+\infty}\left(\sup_{(0,t]}\|\nabla u^m\|^2_{H^1}+\|\sqrt{\rho^m}u^m_t\|^2_{L^\infty(0,t;L^2(\Omega))}+\int_0^t\|\nabla u^m_t\|_2^2ds\right)\leq$$

$$\leq \hat{K}^2+\hat{D}_0+\liminf_{m\to+\infty}\left(C\mathcal{C}(\rho_0,u_0,p_0)+C\exp\left(C\int_0^t\|\nabla u^m\|_2^4ds\right)\right)=$$

$$=\hat{K}^2+\hat{D}_0+C\mathcal{C}(\rho_0,u_0,p_0)+\liminf_{m\to+\infty}\left(C\exp\left(C\int_0^t\|\nabla u^m\|_2^4ds\right)\right)=$$

$$=\hat{K}^2+\hat{D}_0+C\mathcal{C}(\rho_0,u_0,p_0)+C\exp\left(C\int_0^t\|\nabla u\|_2^4ds\right)\leq$$

$$\leq(\hat{K}^2+\hat{D}_0)\exp\left(C\int_0^t\|\nabla u\|_2^4ds\right)+C\mathcal{C}(\rho_0,u_0,p_0)+C\exp\left(C\int_0^t\|\nabla u\|_2^4ds\right)=$$

$$=(\hat{K}^2+\hat{D}_0+C)\exp\left(C\int_0^t\|\nabla u\|_2^4ds\right)+C\mathcal{C}(\rho_0,u_0,p_0)\leq$$

$$\leq Q\exp\left(Q\int_0^t\|\nabla u\|_2^4ds\right)+Q\mathcal{C}(\rho_0,u_0,p_0)$$

where $Q:=\max\{\hat{K}^2+\hat{D}_0+C,C\}$. We have finally proved the inequality

$$\sup_{(0,t]}\left(\|\nabla u\|^2_{H^1}+\|\sqrt{\rho}u_t\|_2^2\right)+\int_0^t\|\nabla u_t\|_2^2ds\leq Q\exp\left(Q\int_0^t\|\nabla u\|_2^4ds\right)+Q\mathcal{C}(\rho_0,u_0,p_0) \tag{11.123}$$

Moreover, in remark 11.31 we have deduced

$$\sup_{\tau\in(0,t)}\left(\|\nabla u^m\|^2_{H^1}+\|\sqrt{\rho^m}u^m_t\|_2^2\right)+\int_0^t\|\nabla u^m_t\|_2^2ds\leq C\mathcal{C}(\rho_0,u_0,p_0)+C$$

So, we have

$$\int_0^t\|\nabla u^m\|^2_{H^1}\leq t\Big(\sup_{(0,t)}\|\nabla u^m\|^2_{H^1}\Big)\leq T_*\Big(C\mathcal{C}(\rho_0,u_0,p_0)+C\Big)$$

and

$$\int_0^t \|\nabla u_t^m\|_2 dt \le T_*^{\frac{1}{2}} \left(\int_0^{T_*} \|\nabla u_t^m\|_2^2 dt \right)^{\frac{1}{2}} \le T_*^{\frac{1}{2}} \Big(C\mathcal{C}(\rho_0, u_0, p_0) + C \Big)$$

This means that

$$\nabla u^m \in L^2(0,T_*;H^1(\Omega)) \cap \{\varphi :\ \partial_t \varphi \in L^1(0,T_*;L^2(\Omega))\}$$

and the sequence is bounded as clear from above. Observe that[48] $\partial_{x_i}\partial_t u^m = \partial_t \partial_{x_i} u^m$. Here, however,

$$\int_{\Omega\times(0,T_*)} \partial_{x_i}\partial_t u^m \varphi \; d(x,t) = -\int_{\Omega\times(0,T_*)} \partial_t u^m \partial_{x_i}\varphi \; d(x,t) = \int_{\Omega\times(0,T_*)} u^m \partial_t \partial_{x_i}\varphi \; d(x,t) =$$

$$= \int_{\Omega\times(0,T_*)} u^m \partial_{x_i}\partial_t \varphi \; d(x,t)$$

thanks to the Schwarz Lemma for smooth function, and we proceed back to front. Since $H^1(\Omega) \subset\subset L^2$ by the Rellich-Kondrachov theorem, we have that the sequence $\{\nabla u^m\}_m$ has a subsequence $\{\nabla u^{m_k}\}_k$ and a function $w \in L^2(0,T_*;L^2(\Omega))$ such that

$$\lim_{m\to+\infty} \|\nabla u^{m_k} - w\|_{L^2(0,T_*;L^2(\Omega))} = 0$$

We call this subsequence u^m again. Passing to a subsequences, the convergences proved yet are heredited. We know moreover that $u^m \rightharpoonup u$ in $L^2(0,T_*;H^2(\Omega))$. We consider the functional

$$f(v) := \nabla v \quad \forall v \in L^2(0,T_*;H^2(\Omega))$$

Clearly

$$f : L^2(0,T_*;H^2(\Omega)) \to L^2(0,T_*;L^2(\Omega))$$

and

$$\|f(v)\|_{L^2(0,T_*;L^2(\Omega))} = \left(\int_0^{T_*} \|\nabla v\|_2^2 dt \right)^{\frac{1}{2}} \le \left(\int_0^{T_*} \|v\|_{H^2}^2 dt \right)^{\frac{1}{2}} = \|v\|_{L^2(0,T_*;H^2(\Omega))}$$

so that the functional is continuous. This means that[49]

$$\nabla u^m = f(u^m) \rightharpoonup f(u) = \nabla u \;\text{ in } L^2(0,T_*;L^2(\Omega))$$

[48] In the previous application of this theorem, the interchange of derivatives was simply guaranteed by the fact that the variable were separated.

[49] In fact, if $T \in (L^2(0,T_*;L^2(\Omega)))^*$, we can define $A(v) := T(f(v))$ for every $v \in L^2(0,T_*;H^2(\Omega))$. The operator is continuous, since

$$|A(v)| = |T(f(v))| \le C\|f(v)\|_{L^2(0,T_*;L^2(\Omega))} \le C\|v\|_{L^2(0,T_*;H^2(\Omega))}$$

It is also linear, since T and f are linear. Then $A \in (L^2(0,T_*;H^2(\Omega))^*$. This means that

$$\lim_{m\to+\infty} T(f(u^m)) = \lim_{m\to+\infty} A(u^m) = A(u) = T(f(v))$$

the weak convergence, in $L^2(0,T_*;L^2(\Omega))$, of $f(u^m)$ to $f(u)$.

The strong convergence $\|\nabla u^m - w\|_{L^2(0,T_*;L^2(\Omega))} \to 0$ implies that the convergence is also weak, so that

$$\nabla u^m \rightharpoonup w \ \text{ in } L^2(0,T_*;L^2(\Omega))$$

Since the weak limit is unique, we have that $w = \nabla u$ in $L^2(0,T_*;L^2(\Omega))$. So we have

$$\|\nabla u^m - \nabla u\|_{L^2(0,T_*;L^2(\Omega))} \leq \|\nabla u^m - w\|_{L^2(0,T_*;L^2(\Omega))} + \|w - \nabla u\|_{L^2(0,T_*;L^2(\Omega))} =$$

$$= \|\nabla u^m - w\|_{L^2(0,T_*;L^2(\Omega))} \to 0 \ \text{ as } m \to +\infty$$

In other words we have proved that

$$\lim_{m\to+\infty} \int_0^{T_*} \|\nabla u^m - \nabla u\|_2^2 dt = 0$$

In other words, as a function of time, $\|\nabla u^m - \nabla u\|_2 \to 0$ in $L^2(0,T_*)$.
This means that there exists a subsequence $\{\|\nabla u^{m_k} - \nabla u\|_2(t)\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to+\infty} \|\nabla u^{m_k} - \nabla u\|_2(t) = 0 \quad \text{a.e. } t \in (0,T_*)$$

Moreover we know, using estimate (11.113), that exists a measure zero set, say A, such that

$$\|\nabla u^m(t)\|_2 \leq C \quad \forall t \in (0,T_*)/A, \ \forall m \in \mathbb{N}$$

If B is the zero measure set such that $\|\nabla u^{m_k} - \nabla u\|_2(t) \to 0$ holds for every $t \in (0,T_*)/B$, we have that, for every $t \in (0,T_*)/(A \cup B)$,

$$\|\nabla u(t)\|_2 = \lim_{k\to+\infty} \|\nabla u^{m_k}(t)\|_2 \leq C$$

Since the bound is true almost every-where in $(0,T_*)$, we have that

$$\sup_{(0,T_*)} \|\nabla u(t)\|_2 \leq C \tag{11.124}$$

11.14 Strong solution to the nonhomoegeneous incompressible Navier-Stokes equations

In this section we prove that the pair of solutions (ρ, u) is a strong solution in the sense of the definitions of the chapter 10. While in the next subsection 11.14.1 we will show that the pair (ρ, u) is automatically strong solution of the momentum equation, subsection 11.14.2 and subsection 11.14.3 will specify, respectively, that an higher regularity of the velocity field holds, and that if we assume more regularity of the initial density, that is $\rho_0 \in H^1(\Omega)$, the pair (ρ, u) also satisfies the strong formulation of transport equation. This concludes our discussion, together with some *a posteriori estimates* that will be deduced in remark 11.39.

11.14.1 Strong solution to the momentum equations with pressure gradient term

In the sections above, in particular in (11.93), we have proved that if $\nu \in X$, there exists a measure zero subset $E_\nu \subseteq (0, T_*)$ such that

$$\int_\Omega (\rho^\delta u_t^\delta + \rho^\delta u^\delta \cdot \nabla u^\delta - \mu \Delta u^\delta) \cdot \nu = 0 \quad t \in (0, T_*)/E_\nu$$

It is clear that the set E_ν also depends on the choice of δ, that is the choice of the regularized initial density data. With the m-notation introduced above, we have that we can write

$$\int_\Omega (\rho^m u_t^m + \rho^m u^m \cdot \nabla u^m - \mu \Delta u^m) \cdot \nu = 0 \quad t \in (0, T_*)/E_{\nu,m}$$

In particular we can choice $\nu = \nu^m \in X^m \subseteq X$, where the apex here means that the function is in the approximate functional space X^m. So, if we write for brevity $E_{m,\nu^m} \equiv E_m$, we have

$$\int_\Omega (\rho^m u_t^m + \rho^m u^m \cdot \nabla u^m - \mu \Delta u^m) \cdot \nu^m = 0 \quad t \in (0, T_*)/E_m$$

On the other hand, for every $\phi \in X$, there exists a sequence $\{\nu^m\}_{m \in \mathbb{N}}$ with $\nu^m \in X^m$ and $\|\nu^m - \phi\|_{H^2} \to 0$ as $m \to +\infty$.

Since ρ^m and u^m converges to a limit in the same sense as in section 11.8.1, we can restart from the relation (11.85) that holds for every $t \in (0, T_*)/\bigcup_{m \in \mathbb{N}} E_m$ with $|\bigcup_{m \in \mathbb{N}} E_m| = 0$. Following the passeges above, we get that if E is a measurable subset of $(0, T_*)$, than

$$\int_E \int_\Omega (\rho u_t + \rho u \cdot \nabla u - \mu \Delta u) \cdot \nu \; dx = 0 \quad \forall \nu \in X$$

A simple property of measure theory says that if $\nu \in X$ there exists a subset $E_\nu \subset (0, T_*)$ with $|E_\nu| = 0$ such that

$$\int_\Omega (\rho u_t + \rho u \cdot \nabla u - \mu \Delta u) \cdot \nu \; dx = 0 \quad t \in (0, T_*)/E_\nu$$

So, following the arguments in section 11.8.5, we find $p(t) \in L^2_{loc}(\Omega)$ such that

$$\rho(t) u_t(t) + \rho(t) u(t) \cdot \nabla u(t) - \mu \Delta u(t) = \nabla p(t)$$

where ∇p is the weak gradient of the pressure term. Clearly, also the argument in section 11.6.5 are the same, and we use theorem 11.4.

11.14.2 Further estimates on the velocity field

We have found $u = u(t) \in H_0^1(\Omega) \cap H^2(\Omega)$ such that, for almost every $t \in (0, T_*)$ it holds

$$\begin{cases} -\mu \Delta u(t) + \nabla p(t) = -\rho(t) u_t(t) - \rho(t) u(t) \cdot \nabla u(t) \\ \nabla \cdot u(t) = 0 \end{cases} \quad \text{in } \Omega \tag{11.125}$$

i.e., it is solution to the Stokes solution with force $f(t) := -\rho(t)u_t(t) - \rho(t)u(t) \cdot \nabla u(t)$. Then we can observe that $u(t)$ is a 6-generalized solution.

In fact, consider $t \in (0, T_*)/E$, where $|E| = 0$. Then the following properties hold.

- $u \in H^2(\Omega) \subseteq D^{1,6}(\Omega)$.
- $\displaystyle\int_\Omega \sum_{i=1}^{3} u_i(\partial_{x_i}\varphi)\, dx = -\int_\Omega (\nabla \cdot u)\varphi\, dx = 0$
- We know that $u \in H^2(\Omega) \subseteq W^{1,6}(\Omega)$. Since $u \in X$, we have that

$$\lim_{m \to \infty} \left\| \sum_{i=1}^{m} \langle u, w^i \rangle_2 w^i - u \right\|_{H^2} = 0$$

We define $u^m := \displaystyle\sum_{i=1}^{m} \langle u, w^i \rangle_2 w^i \in X$. The function u^m is a linear combination of eigenfunctions, and we know that $w^i \in C^1(\overline{\Omega})$. Moreover, $Tw^i = u|_{\partial\Omega} \equiv 0$, and $\nabla \cdot w^i = 0$. Since also $w^i \in C^1(\overline{\Omega})$, we have that $Tw^i = 0$ also in the sense of $W^{1,6}(\Omega)$. So, $w^i \in W_0^{1,6}(\Omega)$ and also $u^m \in W_0^{1,6}(\Omega)$. So, we find

$$\|u^m - u\|_{W^{1,6}} = \left(\|u^m - u\|_6^6 + \|\nabla u^m - \nabla u\|_6^6\right)^{\frac{1}{6}} \leq \|u^m - u\|_6 + \|\nabla u^m - \nabla u\|_6$$

But $\|u^m - u\|_6 \leq C\|u^m - u\|_{H^1}$ and $\|\nabla u^m - \nabla u\|_6 \leq C'\|\nabla u^m - \nabla u\|_{H^1}$, so that

$$\|u^m - u\|_{W^{1,6}} \leq C''\|u^m - u\|_{H^2} \to 0$$

as $m \to \infty$. Being $W_0^{1,6}(\Omega)$ closed, we have that also $u \in W_0^{1,6}(\Omega)$.

- Observe that $f(t) \in L^6(\Omega)$. In fact we have

$$\|\rho u_t\|_6 \leq \|\rho\|_\infty \|u_t\|_6 \leq \|\rho\|_\infty \|\nabla u_t\|_2 < \infty \tag{11.126}$$

$$\begin{aligned} \|\rho(u \cdot \nabla u)\|_6 &\leq \|\rho\|_\infty \|u \cdot \nabla u\|_6 \leq \|\rho\|_\infty \|u\|_\infty \|\nabla u\|_6 \leq \\ &\leq cM\|\rho\|_\infty \left(\|\Delta u\|_2\right)^{\frac{3}{4}} \left(\|\nabla u\|_2\right)^{\frac{1}{4}} \|\nabla u\|_{H^1} < \infty \end{aligned} \tag{11.127}$$

thanks to (11.40) and lemma 9.6. So we can consider, if $\varphi \in C_{0,\sigma}^\infty(\Omega)$,

$$-\mu \Delta u + \nabla p = f \implies -\mu \Delta u \cdot \varphi + \nabla p \cdot \varphi - f \cdot \varphi = 0$$

so that

$$-\mu \int_\Omega \Delta u \cdot \varphi\, dx + \int_\Omega \nabla p \cdot \varphi = \int_\Omega f \cdot \varphi = \langle f, \varphi \rangle$$

Since $\varphi \in C^\infty_{0,\sigma}(\Omega)$, and, being $\nabla p \in G(\Omega)$, $\langle \nabla p, g\rangle = 0$ for every $g \in C^\infty_{0,\sigma}(\Omega) \subseteq L^2_\sigma(\Omega)$, then

$$-\mu \int_\Omega \Delta u \cdot \varphi \, dx = \langle f, \varphi\rangle$$

On the other hand

$$-\mu \int_\Omega \Delta u \cdot \varphi \, dx = -\mu \sum_{i=1}^3 \int_\Omega \Delta u_i \cdot \varphi_i \, dx = -\mu \sum_{i=1}^3 \int_\Omega \left(\sum_{j=1}^3 \partial^2_{x_j} u_i \right) \cdot \varphi_i \, dx =$$

$$= -\mu \sum_{i=1}^3 \sum_{j=1}^3 \left(- \int_\Omega \partial_{x_j} u_i \cdot \partial_{x_j} \varphi_i \right) = \mu \sum_{i=1}^3 \int_\Omega \nabla u_i \cdot \nabla \varphi_i \, dx = \mu \int_\Omega \nabla u \cdot \nabla \varphi$$

since also $\varphi \in C^\infty_0(\Omega)$.

So we proved that the function is a 6-generalized solution. We now use theorem 9.8. We deduce that

$$\|\nabla^2 u\|_6 + \inf_{c \in \mathbb{R}} \|p + c\|_{1,6} \le C\|f\|_6$$

Using that $\|p+c\|_{1,6} = \big(\|p+c\|_6 + \|\nabla p\|_6^6\big)^{\frac{1}{6}} \ge \|\nabla p\|_6$, and so that $\inf_{c\in\mathbb{R}} \|p+c\|_{1,6} \ge \|\nabla p\|_6$, we have

$$\|\nabla^2 u\|_6^2 + \|\nabla p\|_6^2 \le \big(\|\nabla^2 u\|_6 + \|\nabla p\|_6\big)^2 \le 2C\big(\|\rho u_t\|_6^2 + \|\rho\big(u \cdot \nabla u\big)\|_6^2\big)$$

So, thanks to (11.126)-(11.127), and the fact that $\|\rho(t)\|_\infty \le \|\rho_0\|_\infty + 1$,

$$\|\nabla^2 u\|_6^2 \le C'(\|\nabla u_t\|_2^2 + \|u \cdot \nabla u\|_6^2)$$

Moreover $\|\nabla u\|_6^2 \le M\|u\|_{H^2}^2$, and so, thanks to (11.127),

$$\|\nabla u\|_{W^{1,6}}^2 = \big(\|\nabla u\|_6^6 + \|\nabla^2 u\|_6^6\big)^{\frac{1}{3}} \le \|\nabla u\|_6^2 + \|\nabla^2 u\|_6^2 \le C''\big(\|\nabla u_t\|_2^2 + \|u\|_{H^2}^4 + \|u\|_{H^2}^2\big)$$

So, using estimate (11.123), we have[50]

$$\int_0^t \|\nabla u\|_{W^{1,6}}^2 \le C'' \left(\int_0^t \|\nabla u_t\|_2^2 \, dt + \int_0^t \|u\|_{H^2}^4 \, dt + \int_0^t \|u\|_{H^2}^2 \, dt \right) \le$$

$$\le C''' \left(Q \exp\left(Q \int_0^t \|\nabla u\|_2^4 ds \right) + Q\mathcal{C}(\rho_0, u_0, p_0) \right)$$

[50] Using also that that

$$\|u\|_2 \le C\|\nabla u\|_2 \le C\|\nabla u\|_{H^1} \implies \sup_{(0,t)} \|u\|_{H^2}^2 = \sup_{(0,t)} \big(\|u\|_2^2 + \|\nabla u\|_{H^1}^2\big) \le (C+1) \sup_{(0,t)} \|\nabla u\|_{H^1}^2$$

and so using (11.123) we have

$$\sup_{(0,t)} \|u\|_{H^2}^2 \le (C+1) \left(Q \exp\left(Q \int_0^t \|\nabla u\|_2^4 ds \right) + Q\mathcal{C}(\rho_0, u_0, p_0) \right)$$

So equation (11.123) can be rewritten as

$$\boxed{\sup_{(0,t]}\left(\|\nabla u\|_{H^1}^2+\|\sqrt{\rho}u_t\|_2^2\right)+\int_0^t\|\nabla u_t\|_2^2\,ds+\int_0^t\|\nabla u\|_{W^{1,6}}^2\,ds\le Q\exp\left(Q\int_0^t\|\nabla u\|_2^4\,ds\right)+Q\mathcal{C}(\rho_0,u_0,p_0)} \tag{11.128}$$

11.14.3 Strong solution to the transport equation

In the sections above we have found a weak solution to the problem; in particular we constructed a pair of weak solution (u,ρ). Here u is a weak velocity field; in this section we want to regularize the density solution ρ of the mass equation in the INSE system. We have the following theorem.

Theorem 11.8. *Let $u\in L^\infty(0,T_*;W_0^{1,2}(\Omega)\cap W^{2,2}(\Omega))$ the weak divergence-free velocity field (that is $\nabla\cdot u=0$) constructed in the sections above. Remember that*

$$\nabla^2 u\in L^2(0,T_*;L^6(\Omega)) \tag{11.129}$$

Let $\rho\in L^\infty(0,T_*;L^2(\Omega))$ a weak solution to the transport problem

$$\begin{cases}\rho_t+\nabla\cdot(\rho u)=0 & \text{in } (0,T_*)\times\Omega\\ \rho(0)=\rho_0 & \text{in } \Omega\end{cases} \tag{11.130}$$

with $\rho_0\in L^2(\Omega)$. Then, if we also suppose $\rho_0\in H^1(\Omega)$, ρ is a strong solution to the transport equation with $\rho\in L^\infty(0,T_*;H^1(\Omega))$ and it holds the estimate

$$\|\rho(t)\|_{H^1}\le\|\rho_0\|_{H^1}\exp\left(C\int_0^t\|\nabla u\|_{W^{1,6}}ds\right)\quad\forall\ t\in[0,T_*) \tag{11.131}$$

Remark 11.33. Notice that here we have added a further hypothesis on the initial density ρ_0, that now suppose an integrability property also for the weak derivative.

Remark 11.34. The weak solution obtained in the previous sections is $\rho\in L^\infty(0,T_*;L^\infty(\Omega))$. So, this density is in particular in $L^\infty(0,T_*;L^2(\Omega))$ since

$$\sup_{(0,T_*)}\|\rho(t)\|_2=\sup_{(0,T_*)}\left(\int_\Omega|\rho(x,t)|^2dx\right)^{\frac12}\le\sup_{(0,T_*)}\left(\int_\Omega\|\rho(t)\|_\infty^2dx\right)^{\frac12}=|\Omega|^{\frac12}\sup_{(0,T_*)}\|\rho(t)\|_\infty<+\infty$$

Proof of theorem 11.8. In this section $\rho\in L^\infty(0,T_*;L^2)$ will always represent the weak solution in the hypothesis of theorem 11.8, that is the weak solution built in the previous sections.
From the construction of the velocity field explained in the sections above, we have that $u\in L^\infty(0,T_*;H^2)$: the hypothesis remark that $\nabla\cdot u=0$ and $u\in L^\infty(0,T_*;D_0^{1,2}\cap D^{2,2})$. Moreover, we are supposing $\nabla^2u\in L^2(0,T_*;L^6(\Omega))$. All these hypothesis will help us to prove the statement.

We want to approach the initial density and the velocity field, so that the problem can be considered as a classical problem. First consider the density ρ_0.

Remark 11.35. We want an external approximation also for the initial density. Since the domain is bounded, we can consider an extension of the function to the whole $\mathbb{R}^3$, that is $\rho_0 \in H^1(\mathbb{R}^3)$ such that

$$\|\rho_0\|_{H^1(\mathbb{R}^3)} \leq C'' \|\rho_0\|_{H^1(\Omega)}$$

This new initial density coincides almost everywhere in Ω with the old one. Since $H^1(\mathbb{R}^3) \equiv H_0^1(\mathbb{R}^3)$, as already observed, we have that exists a sequence $\rho_n^0 \in C_c^\infty(\mathbb{R}^3) \subseteq C^\infty(\overline{\Omega})$, such that

$$\lim_{n\to\infty} \|\rho_0 - \rho_n^0\|_{H^1(\Omega')} \leq \lim_{n\to\infty} \|\rho_0 - \rho_n^0\|_{H^1(\mathbb{R}^3)} = 0$$

for every $\Omega' \subset \mathbb{R}^3$. Moreover, for every $k \in \mathbb{N}$ exists m_k such that

$$\|\rho_{n_h}^0\|_{H^1(\Omega')} - \|\rho_0\|_{H^1(\Omega')} \leq \|\rho_0 - \rho_{n_h}^0\|_{H^1(\mathbb{R}^3)} \leq \frac{1}{h}$$

So

$$\lim_{h\to\infty} \|\rho_{n_h}^0\|_{H^1(\Omega')} \leq \|\rho_0\|_{H^1(\Omega')} \tag{11.132}$$

So, the approximating sequence can be chosen with the property (11.132). Observe that the sequence does not depend on Ω'. □

We also want to regularize also the velocity field, so that equation (11.130) con be considered in classical sense. In particular, we want a (possibly) smooth field u with zero boundary values and a divergence free property. Remembering that $u \in H_0^1(\Omega)$ and $\nabla \cdot u = 0$ in the weak sense, we have that a simple convolution will assure a divergence free property for the regularized function; however, the mollification would modify the boundary values. So we have to proceed with caution.

Remark 11.36. In particular observe that the following extension of u in the whole $\mathbb{R}^3$ space maintain the properties underlined above

$$\overline{u}(x,t) := \begin{cases} u(x,t) & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$$

and

$$\nabla\overline{u}(x,t) = \begin{cases} \nabla u(x,t) & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$$

In particular $\overline{u} \in L^\infty(0, T_*; H^2(\Omega))$. Weak derivatives continue to hold. In fact the equality

$$\int_{\mathbb{R}^3} \overline{u}(x,t)\varphi_{x_i}\, dx = \int_\Omega u(x,t)\varphi_{x_i}\, dx = -\int_\Omega \partial_{x_i} u(x,t)\varphi\, dx = -\int_{\mathbb{R}^3} \partial_{x_i}\overline{u}(x,t)\varphi\, dx$$

holds for sure if $\varphi \in C_c^\infty(\Omega)$; if this is not true, but $\varphi \in C_c^\infty(\mathbb{R}^3)$, then we can approximate the intersection of the support of φ and Ω with smooth functions from inside. Thus, it follows $\nabla \cdot \overline{u}(t) = 0$. Moreover $\nabla^2\overline{u} \in L^2(0, T_*; L^6(\Omega))$ and the weak derivative argument continues to hold.

Observe moreover that $\overline{u}(t) \in H_0^1(\Omega)$, since the approximation with test functions of u can be extended to an approximation of $\overline{u}$ by defining as zero[51] the test function outside Ω. $\square$

At this point we observe that $u \in L^2(0, T_*; H^{2,6}(\Omega))$. In fact by the hypothesis on u we have, thanks to (11.40),

$$\|\nabla u\|_6 \le M\|\nabla u\|_{H^1} < \infty$$

Moreover

$$\int_0^{T_*} \|\nabla^2 u\|_6^2 \, dt < \infty, \qquad \int_0^{T_*} \|\nabla u\|_6^2 \, dt \le M^2 \int_0^{T_*} \|\nabla u\|_{H^1}^2 dt \le M^2 T_* \Big(\sup_{(0,T_*)} \|\nabla u\|_{H^1} \Big)^2 < \infty$$

since $u \in L^\infty(0, T_*; H^2)$. So, in particular,

$$C_1 := \int_0^{T_*} \|\nabla u\|_\infty^2(s) \, ds \le C^2 \int_0^{T_*} \|\nabla u\|_{W^{1,6}(\Omega)}^2 ds \le C^2 \Big(\int_0^{T_*} \|\nabla u\|_6^2 \, dt + \int_0^{T_*} \|\nabla^2 u\|_6^2 \, dt \Big) < \infty \tag{11.133}$$

since

$$\|\nabla u\|_{W^{1,6}}^2 = \big(\|\nabla u\|_6^6 + \|\nabla^2 u\|_6^6\big)^{\frac{1}{3}} \overset{52}{\le} \|\nabla u\|_6^2 + \|\nabla^2 u\|_6^2$$

Obviously, $u \in L^2(0, T_*; L^6(\Omega))$.

So, by properties of L^2-Banach valued functions space, we have that, if X is a Banach space, exists $u^n \in C^\infty([0, T_*]; X)$ such that[53]

$$\lim_{n\to\infty} \|u^n - u\|_{L^2(0,T_*;X)} = 0$$

If we choose

$$X := \{\phi \in H_0^1(\Omega) \cap W^{2,6} : \ \nabla \cdot \phi = 0 \text{ in } \Omega\}$$

equipped with the norm $\|\cdot\|_X := \|\cdot\|_{2,6}$. It clearly is a Banach space[54]. This implies that

$$\lim_{n\to\infty} \|\nabla^2 u^n - \nabla^2 u\|_{L^2(0,T_*;L^6(\Omega))} = 0 \implies \lim_{n\to\infty} \Big(\int_0^{T_*} \|\nabla^2 u^n\|_6^2 \, dt \Big)^{\frac{1}{2}} = \Big(\int_0^{T_*} \|\nabla^2 u\|_6^2 \, dt \Big)^{\frac{1}{2}} \tag{11.134}$$

[51] Clearly the smoothness is maintained.

[52] Observe that, if $a, b \ge 0$, $a^6 + b^6 \le (a^2 + b^2)^3 = a^6 + 3a^4b^2 + 3a^2b^4 + b^6$.

[53] See theorem 5.1.

[54] It is obviously a vector space and the norm is well defined. Consider now a Cauchy sequence ϕ_k, and remember that Ω is bounded. For every $\varepsilon > 0$ we have that exists K such that

$$\|\phi_k - \phi_h\|_{2,6} < \varepsilon \qquad \forall k, h \ge K$$

Since $\|\phi_k - \phi_h\|_{1,2} \le C\|\phi_k - \phi_h\|_{2,6}$, and so, being Sobolev spaces complete, $\phi_k \to \phi \in H^1 \cap W^{2,6}$. Being $\phi_k \in H_0^1$, that is closed, moreover $\phi \in H_0^1$. Finally, for every $\varphi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega \phi \cdot \nabla\varphi \, dx = \lim_{k\to\infty} \int_\Omega \phi_k \cdot \nabla\varphi \, dx = 0$$

that is $\nabla \cdot \phi = 0$ in the weak sense in Ω.

and $u^n \in H^1_0(\Omega)$, with $\nabla \cdot u^n = 0$ in Ω. Furthermore we also have

$$\lim_{n\to\infty} \|\nabla u^n - \nabla u\|_{L^2(0,T_*;L^6)} = 0$$

Moreover, we want to regularize the function also respect with the x-variable. To this aim, we define the set A_m such that

$$A_m := \{x \in \Omega^c : \ \mathrm{dist}(x, \partial\Omega) > \frac{1}{m}\}$$

So we can consider $\Omega_m := A_m^c$.

We set

$$u^{m,n}(x,t) := \int_{\Omega_m} \eta_m(x-y) u^n(y,t) \ dy$$

where the field u^n has to be understood as in remark 11.36. At fixed $t \in [0, T_*]$, this convolution is clearly smooth in x thanks to properties of convolution. Moreover, it is continuos as a function of two variables, In fact, if $(x_0, t_0) \in \Omega_m \times [0, T_*]$ we have that

$$|u^{m,n}(x,t) - u^{m,n}(x_0,t_0)| \le |u^{m,n}(x,t) - u^{m,n}(x_0,t)| + |u^{m,n}(x_0,t) - u^{m,n}(x_0,t_0)| \le$$

$$\le \left| \int_{\Omega_m} \big(\eta_m(x-y) - \eta_m(x_0-y)\big) \ u^n(t,y) \ dy \right| + \left| \int_{\Omega_m} \eta_m(x_0-y)\big(u^n(t,y) - u^n(t_0,y)\big) \ dy \right| \le$$

$$\le \left| \int_{\Omega_m} \big(\eta_m(x-y) - \eta_m(x_0-y)\big) \ u^n(t,y) \ dy \right| + \|\eta_m(x_0 - \cdot)\|_{2,\Omega_m} \|u^n(t,\cdot) - u^n(t_0,\cdot)\|_2$$

Since $u^n \in C^\infty([0,T_*]; X)$, we can find $\delta_1 > 0$ such that $\|u^n(t,\cdot) - u^n(t_0,\cdot)\|_2 < \frac{\varepsilon}{2}$. On the other hand, since $\eta_m(r)$ is uniformly continuos on $\mathbb{R}$, there exists $\delta_2 > 0$ such that

$$|x - x_0| = |(x-y) - (x_0-y)| < \delta_2 \implies |\eta_m(x-y) - \eta_m(x_0-y)| < \frac{\varepsilon}{2}$$

it follows that

$$|u^{m,n}(x,t) - u^{m,n}(x_0,t_0)| \le \frac{\varepsilon}{2}\|u^n(t,\cdot)\|_2 + \frac{\varepsilon}{2}\|\eta_m(x_0 - \cdot)\|_{2,\Omega_m} \le \frac{\varepsilon}{2}\left(\max_{t\in[0,T_*]} \|u^n(t,\cdot)\|_2 + \|\eta_m(x_0 - \cdot)\|_{2,\Omega_m} \right)$$

Moreover, thanks to the convolution properties, the x-derivative is continuos over $\overline{\Omega}_m$, and, thanks to the theorem 3.2,

$$|\nabla u^{m,n}(x,t)| = \left| \int_{\Omega_m} \eta_m(x-y) \nabla u^n(y,t) \ dy \right| \le \left(\int_{\Omega_m} |\eta_m(x-y)|^2 \ dy \right)^{\frac{1}{2}} \|\nabla u^n(\cdot,t)\|_2 \le$$

$$\le \left(\int_{\mathbb{R}^3} |\eta_m(x-y)|^2 \ dy \right)^{\frac{1}{2}} \max_{t\in[0,T_*]} \|\nabla u^n(\cdot,t)\|_2 \equiv \left(\int_{\mathbb{R}^3} |\eta_m(z)|^2 \ dz \right)^{\frac{1}{2}} \max_{t\in[0,T_*]} \|\nabla u^n(\cdot,t)\|_2$$

so that

$$\sup_{t\in[0,T_*]} \|\nabla u^{m,n}(t)\|_\infty \le \left(\int_{\mathbb{R}^3} |\eta_m(z)|^2 \ dz \right)^{\frac{1}{2}} \max_{t\in[0,T_*]} \|\nabla u^n(\cdot,t)\|_2$$

so $u^{m,n} \in C([0,T_*];C^1(\overline{\Omega}_m))$.

Finally we underline other two properties of the field $u^{m,n}$. In particular, if $x \in \partial\Omega_m$, we have

$$u^{m,n}(x,t) = \int_{\Omega_m} \eta_m(x-y)u^n(y,t)\ dy = 0$$

since $u^n(y,t) = 0$ if $y \in B(x,\frac{1}{m})$. Moreover,

$$\nabla \cdot u^{m,n}(x,t) = \int_{\Omega_m} \eta_m(x-y)\nabla \cdot u^n(y,t)\ dy = 0$$

since $\nabla \cdot u^n(y,t) = 0$ by the definition of u^n. So, we can use this velocity field to solve the transport problem

$$\begin{cases} \partial_t\rho + u^{m,n}\cdot\nabla\rho = 0 & \overline{\Omega}_m \times [0,T_*] \\ \rho(0) = \rho_0^n \end{cases} \tag{11.135}$$

By the classical theory exposed at the beginning of chapter 8, we have a solution $\rho^{m,n}$ in the classical sense. Considering this solution, we now do some classical estimates. We have

$$\partial_t\rho^{m,n} + u^m \cdot \nabla\rho^{m,n} = 0 \implies \partial_t\partial_{x_j}\rho^{m,n} + \partial_{x_j}u^{m,n}\cdot\nabla\rho^{m,n} + u^{m,n}\cdot\nabla\partial_{x_j}\rho^{m,n} = 0$$

Multiplying this equality by $\partial_{x_j}\rho^{m,n}$ we have

$$\big(\partial_t\partial_{x_j}\rho^{m,n}\big)\partial_{x_j}\rho^{m,n} + \big(\partial_{x_j}u^{m,n}\cdot\nabla\ \rho^m\big)\partial_{x_j}\rho^{m,n} + \big(u^{m,n}\cdot\nabla\partial_{x_j}\rho^{m,n}\big)\partial_{x_j}\rho^{m,n} = 0$$

and so

$$\frac{1}{2}\partial_t|\partial_{x_j}\rho^{m,n}|^2 + \big(\partial_{x_j}u^{m,n}\cdot\nabla\rho^{m,n}\big)\partial_{x_j}\rho^{m,n} + \big(u^{m,n}\cdot\nabla\partial_{x_j}\rho^{m,n}\big)\partial_{x_j}\rho^{m,n} = 0$$

Since $\nabla\cdot u^{m,n} = 0$, we have that

$$\nabla\cdot(|\rho^{m,n}_{x_j}|^2u^{m,n}) = |\rho^{m,n}_{x_j}|^2\big(\nabla\cdot u^{m,n}\big) + u^{m,n}\cdot\nabla|\rho^{m,n}_{x_j}|^2 = u^{m,n}\cdot\nabla|\rho^{m,n}_{x_j}|^2 = 2\big(u^{m,n}\cdot\nabla\rho^{m,n}_{x_j}\big)\rho^{m,n}_{x_j}$$

So, it follows

$$\frac{1}{2}\partial_t|\partial_{x_j}\rho^{m,n}|^2 + \big(\partial_{x_j}u^{m,n}\cdot\nabla\rho^{m,n}\big)\partial_{x_j}\rho^{m,n} + \frac{1}{2}\nabla\cdot(|\rho^{m,n}_{x_j}|^2\ u^{m,n}) = 0$$

Summing over j and integrating on Ω_m we have

$$\frac{1}{2}\int_{\Omega_m}\partial_t|\nabla\rho^{m,n}|^2dx + \sum_{j=1}^{3}\int_{\Omega_m}\big(\partial_{x_j}u^{m,n}\cdot\nabla\rho^{m,n}\big)\partial_{x_j}\rho^{m,n}\ dx + \frac{1}{2}\sum_{j=1}^{3}\int_{\Omega_m}\nabla\cdot(|\rho^{m,n}_{x_j}|^2\ u^{m,n})\ dx = 0$$

Since $u^{m,n}$ is zero on $\partial\Omega_m$, through the divergence theorem we have that

$$\int_{\Omega}\partial_t|\nabla\rho^{m,n}|^2dx = -2\sum_{j=1}^{3}\int_{\Omega}\big(\partial_{x_j}u^{m,n}\cdot\nabla\rho^{m,n}\big)\partial_{x_j}\rho^{m,n}\ dx$$

So, estimating, we have

$$\frac{d}{dt}\int_{\Omega_m} |\nabla\rho^{m,n}|^2 dx \le C\int_{\Omega_m} |\nabla u_{m,n}||\nabla\rho^{m,n}|^2 dx \le C\|\nabla u_{m,n}\|_{\infty,\Omega_m}\int_{\Omega_m} |\nabla\rho^{m,n}|^2 dx$$

and so by Gronwall's inequality

$$\|\nabla\rho^{m,n}(t)\|_{2,\Omega_m}^2 = \int_{\Omega_m} |\nabla\rho^{m,n}|^2(t)\,dx \le \left(\int_{\Omega_m} |\nabla\rho^{m,n}|^2(0)\,dx\right)\exp\left(\int_0^t C\|\nabla u_{m,n}\|_{\infty,\Omega_m}(s)\,ds\right) \le$$

$$\le \left\|\rho^{m,n}(0)\right\|_{H^1(\Omega_m)}^2 \exp\left(\int_0^t C\|\nabla u_{m,n}\|_{\infty,\Omega_m}(s)\,ds\right) \tag{11.136}$$

Remember that $\|\rho^{m,n}(t)\|_{2,\Omega_m} = \|\rho^{m,n}(0)\|_{2,\Omega_m}$, since this holds for the solutions of transport equation with the incompressibility condition $\nabla\cdot u^{m,n}=0$. We have

$$\|\rho^{m,n}(t)\|_{H^1(\Omega_m)}^2 = \|\rho^{m,n}(t)\|_{2,\Omega_m}^2 + \|\nabla\rho^{m,n}(t)\|_{2,\Omega_m}^2 \le$$

$$\le \left\|\rho^{m,n}(0)\right\|_{2,\Omega_m}^2 + \left\|\rho^{m,n}(0)\right\|_{H^1(\Omega_m)}^2 \exp\left(\int_0^t C\|\nabla u_{m,n}\|_{\infty,\Omega_m}(s)\,ds\right) \le$$

$$\le 2\left\|\rho^{m,n}(0)\right\|_{H^1(\Omega_m)}^2 \exp\left(\int_0^t C\|\nabla u_m\|_{\infty,\Omega_m}(s)\,ds\right)$$

using that $\exp(\alpha)\ge 1$ for every $\alpha\ge 0$. Taking the square root we have

$$\|\rho^{m,n}(t)\|_{H^1(\Omega_m)} \le \sqrt{2}\left\|\rho^{m,n}(0)\right\|_{H^1(\Omega_m)} \exp\left(\int_0^t C'\|\nabla u_{m,n}\|_{\infty,\Omega_m}(s)\,ds\right) \equiv \tag{11.137}$$

$$\equiv \sqrt{2}\left\|\rho_0^n\right\|_{H^1(\Omega_m)} \exp\left(\int_0^t C'\|\nabla u_{m,n}\|_{\infty,\Omega_m}(s)\,ds\right)$$

We consider now the term $\|\nabla u_{m,n}\|_{\infty,\Omega_m}$. We know that

$$\|\nabla u^{m,n}(t)\|_{\infty,\Omega_m} = \sup_{x\in\Omega_m}\left|\int_{\Omega_m}\eta_m(x-y)\nabla u^n(y,t)\,dy\right| \le \|\nabla u^n(t)\|_{\infty,\Omega}$$

since $0\le \int_{\Omega_m}\eta_m(x-y)\,dy \le \int_{\mathbb{R}^3}\eta_m(x-y)\,dy = 1$. If follows that

$$\|\rho^{m,n}(t)\|_{H^1(\Omega_m)} \le \sqrt{2}\left\|\rho_0^n\right\|_{H^1(\Omega_1)} \exp\left(\int_0^t C'\|\nabla u^n\|_{\infty,\Omega}(s)\,ds\right)$$

where $\Omega\subset\Omega_m\subset\Omega_1$. If follows that, for $n\in\mathbb{N}$ fixed,

$$\|\rho^{m,n}(t)\|_{H^1(\Omega)} \le \sqrt{2}\|\rho_0^n\|_{H^1(\Omega_1)} \exp\left(\int_0^{T_*} C'\|\nabla u^n\|_\infty(s)\,ds\right) \equiv \Lambda_0 \quad \forall t\in(0,T_*) \tag{11.138}$$

Taking the supremum, we have

$$\|\rho^{m,n}\|_{L^\infty(0,T_*;H^1(\Omega))} \le \Lambda_0$$

The space $L^\infty(0,T_*;L^2(\Omega))$ is the dual space of $L^1(0,T_*;L^2(\Omega))$ that is separable since $L^2(\Omega)$ is separable and the exponent is ≥ 1. So, thanks to a versione of Banach-Hanaoglu, $\rho^{m,n}$ and $\nabla\rho^{m,n}$ are bounded in its dual space $L^\infty(0,T_*;L^2(\Omega))$ and then there exists a subsequence m_k and $\overline{\rho}^n, f^n \in L^\infty(0,T_*;L^2(\Omega))$ such that

$$\rho^{m_k,n} \overset{*}{\rightharpoonup} \overline{\rho}^n, \quad \nabla\rho^{m_k,n} \overset{*}{\rightharpoonup} f^n \quad \text{in } L^\infty(0,T_*;L^2(\Omega))$$

Clearly $f^n = \nabla\overline{\rho}^n$. In fact, for every $\phi \in C_c^\infty(\Omega \times (0,T_*))$,

$$\int_{\Omega\times(0,T_*)} \overline{\rho}^n \partial_{x_i}\phi \, dx = \lim_{k\to+\infty} \int_{\Omega\times(0,T_*)} \rho^{m_k,n}\partial_{x_i}\phi \, dx =$$

$$= -\lim_{k\to+\infty} \int_{\Omega\times(0,T_*)} \partial_{x_i}\rho^{m_k,n}\phi \, dx = -\int_{\Omega\times(0,T_*)} f_i^n \phi \, dx$$

Remark 11.37. Observe that ϕ and its derivatives are bounded and $\overline{\rho}^n \in L^\infty(0,T_*;L^2(\Omega))$. So

$$\int_0^{T_*}\left(\int_\Omega |\overline{\rho}^n|^2|\partial_{x_i}\phi|^2 dx\right)dt \leq M\|\overline{\rho}^n\|^2_{L^2(0,T_*;L^2(\Omega))} < \infty$$

So $\overline{\rho}^n\partial_{x_i}\phi \in L^2(0,T_*;L^2(\Omega))$ and so, as we have already discussed, it is in $L^2((0,T_*)\times\Omega)$ and the Fubini theorem holds. The same is true for f_i^n. □

It holds moreover that

$$\|\overline{\rho}^n\|_{L^\infty(0,T_*;L^2(\Omega)} \leq \liminf_{k\to\infty}\|\rho^{m_k,n}\|_{L^\infty(0,T_*;L^2(\Omega)}, \quad \|\nabla\overline{\rho}^n\|_{L^\infty(0,T_*;L^2(\Omega)} \leq \liminf_{k\to\infty}\|\nabla\rho^{m_k,n}\|_{L^\infty(0,T_*;L^2(\Omega)}$$

It follows that

$$\|\overline{\rho}^n\|_{L^\infty(0,T_*;H^1(\Omega))} \leq \|\overline{\rho}^n\|_{L^\infty(0,T_*;L^2(\Omega))} + \|\nabla\overline{\rho}^n\|_{L^\infty(0,T_*;L^2(\Omega))} \leq$$

$$\leq \liminf_{k\to\infty}\|\rho^{m_k,n}\|_{L^\infty(0,T_*;L^2(\Omega)} + \liminf_{k\to\infty}\|\nabla\rho^{m_k,n}\|_{L^\infty(0,T_*;L^2(\Omega)} \leq$$

$$\leq \liminf_{k\to\infty}\left(\|\rho^{m_k,n}\|_{L^\infty(0,T_*;L^2(\Omega)} + \|\nabla\rho^{m_k,n}\|_{L^\infty(0,T_*;L^2(\Omega))}\right) \leq$$

$$\leq 2\sqrt{2}\|\rho_0^n\|_{H^1(\Omega_1)}\exp\left(\int_0^{T_*} C'\|\nabla u^n\|_\infty(s)\, ds\right)$$

So

$$\boxed{\|\overline{\rho}^n\|_{L^\infty(0,T_*;H^1(\Omega))} \leq 2\sqrt{2}\|\rho_0^n\|_{H^1(\Omega_1)}\exp\left(\int_0^{T_*} C'\|\nabla u^n\|_\infty(s)\, ds\right)} \tag{11.139}$$

Before going on, we want to say somthing about $\|u^{m_k,n} - u\|_{L^2(0,T_*;L^2)}$. Observe that

$$\|u^{m_k,n}(t)\|_{L^2} = \|\eta_{m_k} * u^n(t,\cdot)\|_2 \leq \|\eta_{m_k}\|_1\|u^n(t,\cdot)\|_2$$

using Young's inequality for convolutions. So, it follows that

$$\|u^{m_k,n}(t) - u^n(t)\|_{L^2} \leq C\|u^n(t,\cdot)\|_2$$

that has summable square, being $u^n \in L^2(0, T_*; L^2)$. Since moreover, $\|u^{m_k,n}(t) - u^n(t)\|_{2,\Omega} \to 0$ as $k \to \infty$, being $\Omega \subset \Omega_1$ bounded, we have

$$\int_0^{T_*} \|u^{m_k,n}(t) - u^n(t)\|_2^2 \, dt \to 0 \tag{11.140}$$

as $k \to \infty$. With the same argument, we have that

$$\|\nabla^2 u^{m_k,n}\|_6 = \|\eta_{m_k} * \nabla^2 u^n(t,\cdot)\|_6 \le \|\eta_{m_k}\|_1 \|\nabla^2 u^n(t,\cdot)\|_6$$

and, being $\nabla^2 u^n \in L^2(0, T_*; L^6)$, and being $\Omega \subset \Omega_1$ bounded $\|\nabla^2 u^{m_k,n}(t) - \nabla^2 u^n(t)\|_6 \to 0$ as $k \to \infty$. It follows that

$$\int_0^{T_*} \|\nabla^2 u^{m_k,n}(t) - \nabla^2 u^n(t)\|_6^2 \, dt \to 0 \tag{11.141}$$

as $k \to \infty$. Thus, these convergence allow us to prove that the limit $\overline{\rho}^n$ is a strong solution to the transport equation. We now prove this fact. Consider, for every $k \in \mathbb{N}$,

$$\rho_t^{m_k,n} + u^{m_k,n} \cdot \nabla \rho^{m_k,n} = 0 \quad \text{over } \Omega \times (0, T_*) \tag{11.142}$$

Remark 11.38. As in remark 11.37, φ is bounded together with its derivatives, while $\rho^{m_k,n}$ is in $L^\infty(0, T_*; L^2(\Omega))$ and also $u^{m_k,n} \cdot \nabla \rho^{m_k,n}$. In fact, we have that

$$\sup_{(0,T_*)} \|u^{m_k,n} \cdot \nabla \rho^{m_k,n}\|_{L^2(\Omega)} \le \sup_{(0,T_*)} \left(\int_\Omega |u^{m_k,n}|^2 |\nabla \rho^{m_k,n}|^2 dx \right)^{\frac{1}{2}} \le$$

$$\le \sqrt{|\Omega|} \sup_{\Omega \times (0,T_*)} |u^{m_k,n}| \sup_{(0,T_*)} \|\nabla \rho^{m_k,n}\|_{L^2(\Omega)}$$

The second factor is bounded thanks to the estimate (11.138). On the other hand we have that the other term is bounded thanks to the regularity of the velocity field. So, the integrand is in $L^2(0, T_*; L^2(\Omega))$ and so in $L^2(\Omega \times (0, T_*)) \subseteq L^1(\Omega \times (0, T_*))$. Moreover, Fubini holds. $\square$

Using the definition of weak derivative, we have

$$\int_{\Omega \times (0,T_*)} \rho^{m_k,n} \varphi_t \, d(x,t) = \int_{\Omega \times (0,T_*)} \left(u^{m_k,n} \cdot \nabla \rho^{m_k,n}\right) \varphi \, d(x,t)$$

for every $\varphi = \varphi(x,t) \in C_c^\infty(\Omega \times (0, T_*))$. Using the Fubini-Tonelli theorem, we have

$$\int_\Omega \int_0^{T_*} \rho^{m_k,n} \varphi_t \, dt \, dx = \int_\Omega \int_0^{T_*} \left(u^{m_k,n} \cdot \nabla \rho^{m_k,n}\right) \varphi \, dt \, dx \tag{11.143}$$

Our aim is to prove, strarting from (11.143), that

$$\int_\Omega \int_0^{T_*} \overline{\rho}^n \varphi_t \, dt \, dx = \int_\Omega \int_0^{T_*} \left(u^n \cdot \nabla \overline{\rho}^n\right) \varphi \, dt \, dx \tag{11.144}$$

for every $\varphi \in C_c^\infty(\Omega \times (0, T_*))$; that is

$$\overline{\rho}_t^n \equiv -u^n \cdot \nabla \overline{\rho}^n \tag{11.145}$$

in the sense of weak derivative. So we prove (11.144).

Our purpose is now to pass from (11.143) to (11.144) with a limit argument. We have quite immediately that

$$\lim_{k\to+\infty}\int_\Omega\int_0^{T_*}\rho^{m_k,n}\varphi_t\;dt\;dx=\int_\Omega\int_0^{T_*}\overline{\rho}^n\varphi_t\;dt\;dx$$

thanks to the weak star convergence of $\rho^{m_k,n}$ to $\overline{\rho}^n$ (and the integral operator above is the dual pairing). It remains to prove that

$$\lim_{k\to+\infty}\int_\Omega\int_0^{T_*}\left(u^{m_k,n}\cdot\nabla\rho^{m_k,n}\right)\varphi\;dt\;dx=\int_\Omega\int_0^{T_*}\left(u^n\cdot\nabla\overline{\rho}^n\right)\varphi\;dt\;dx$$

With the usual devices, we have

$$\left|\int_0^{T_*}\int_\Omega\left(u^{m_k,n}\cdot\nabla\rho^{m_k,n}-u^n\cdot\nabla\overline{\rho}^n\right)\varphi\;dt\;dx\right|=$$

$$=\left|\int_0^{T_*}\int_\Omega\left(u^{m_k,n}\cdot\nabla\rho^{m_k,n}-u^n\cdot\nabla\rho^{m_k,n}+u^n\cdot\nabla\rho^{m_k,n}-u^n\cdot\nabla\overline{\rho}^n\right)\varphi\;dt\;dx\right|\le$$

$$\le\left|\int_0^{T_*}\int_\Omega(u^{m_k,n}-u^n)\cdot\nabla\rho^{m_k,n}\varphi\;dx\;dt\right|+\left|\int_0^{T_*}\int_\Omega u^n\cdot(\nabla\rho^{m_k,n}-\nabla\overline{\rho}^n)\;\varphi\;dx\;dt\right|\le$$

The first limit follows from Hölder's inequality. In fact

$$\left|\int_0^{T_*}\int_\Omega(u^{m_k,n}-u)\cdot\nabla\rho^{m_k,n}\varphi\;dx\;dt\right|\le\int_0^{T_*}\|u^{m_k,n}-u^n\|_{L^2(\Omega)}\|\nabla\rho^{m_k,n}\varphi\|_{L^2(\Omega)}\le$$

$$\le\left(\int_0^{T_*}\|u^{m_k,n}-u^n\|^2_{L^2(\Omega)}dt\right)^{\frac12}\left(\int_0^{T_*}\|\nabla\rho^{m_k,n}\varphi\|^2_{L^2(\Omega)}dt\right)^{\frac12}$$

Observe that

$$\int_0^{T_*}\|\nabla\rho^{m_k,n}\varphi\|^2_{L^2(\Omega)}\;dt=\int_0^{T_*}\left(\int_\Omega|\nabla\rho^{m_k,n}|^2|\varphi|^2dx\right)dt\le\sup_{(0,T_*)\times\Omega}|\varphi|^2\int_0^{T_*}\|\nabla\rho^{m_k,n}\|^2_{L^2(\Omega)}\;dt\le$$

and using (11.138)

$$\le\sup_{(0,T_*)\times\Omega}|\varphi|^2\int_0^{T_*}\Lambda_0^2\;dt<\infty$$

Since $u^{m_k,n}\to u^n$ in $L^2(0,T_*;L^2(\Omega))$, thanks to (11.140), we have that this first limit vanishes. For the second piece, remember that $\varphi\in C_c^\infty(\Omega\times(0,T_*))$. So, in particular

$$\int_0^{T_*}\int_\Omega u^n\cdot(\nabla\rho^{m_k,n}-\nabla\overline{\rho}^n)\;\varphi\;dx\;dt\to 0\quad\text{as }k\to\infty$$

since $\nabla\rho^{m_k,n}\overset{*}{\rightharpoonup}\nabla\overline{\rho}^n$ in $L^\infty(0,T_*;L^2(\Omega))$ and being[55] $u^n\varphi$ in $L^1(0,T_*;L^2(\Omega))$. So we have the thesis

$$\int_\Omega\int_0^{T_*}\overline{\rho}^n\varphi_t\;dt\;dx=\int_\Omega\int_0^{T_*}\left(u^n\cdot\nabla\overline{\rho}^n\right)\varphi\;dt\;dx\tag{11.147}$$

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[55]

$$\int_0^{T_*}\|u^n\varphi\|_{L^2(\Omega)}=\int_0^{T_*}\left(\int_\Omega|u^n\varphi|^2dx\right)^{\frac12}dt\le MT_*\sup_{(0,T_*)}\|u^n\|_{L^2(\Omega)}<\infty\tag{11.146}$$

where M is a bound for φ and $u^n\in L^\infty(0,T_*;H^2(\Omega))$.

for every $\varphi \in C_c^\infty(\Omega\times(0,T_*))$. We now have to consider again the estimate (11.139). Moreover, thanks to remark 11.35, being $\Omega_m \subset \Omega_1$, for every $m \in \mathbb{N}$, we have that

$$\|\rho_0^n\|_{H^1(\Omega_m)} \leq \|\rho_0^n\|_{H^1(\Omega_1)}, \quad \lim_{n\to\infty} \|\rho_0^n\|_{H^1(\Omega_1)} \leq \|\rho_0\|_{H^1(\Omega_1)} \leq \|\rho_0\|_{H^1(\mathbb{R}^3)} \leq C''\|\rho_0\|_{H^1(\Omega)}$$

and so

$$\limsup_{n\to\infty} \|\rho^{m,n}(0)\|_{H^1(\Omega_m)} \leq \limsup_{n\to\infty} \|\rho_0^n\|_{H^1(\Omega_1)} \leq C''\|\rho_0\|_{H^1(\Omega)}$$

On the other hand the bound $\|\rho_0^n\|_{H^1(\Omega_1)} \leq \|\rho_0\|_{H^1} + 1$ holds for every n large enough. It follows that equation (11.139) becomes

$$\|\overline{\rho}^n\|_{L^\infty(0,T_*;H^1(\Omega))} \leq 2\sqrt{2}\big(\|\rho_0\|_{H^1} + 1\big) \exp\left(\int_0^{T_*} C'\|\nabla u^n\|_\infty(s)\ ds\right) \tag{11.148}$$

Observa that

$$\int_0^{T_*} \|\nabla u^n\|_\infty(s)\ ds \leq \sqrt{T_*}\left(\int_0^{T_*} \|\nabla u^n\|_\infty^2(s)\ ds\right)^{\frac{1}{2}} \leq C\sqrt{T_*}\left(\int_0^{T_*} \|\nabla u^n\|_{W^{1,6}}^2(s)\ ds\right)^{\frac{1}{2}}$$

Since the later integral converges, thanks to (11.141), we have that $\|\overline{\rho}^n\|_{L^\infty(0,T_*;H^1(\Omega))}$ is bounded, for n large enough. So, we have that there exists a subsequence n_h and two function $\overline{\rho}$, f such that

$$\overline{\rho}^{n_h} \overset{*}{\rightharpoonup} \overline{\rho}, \quad \nabla\overline{\rho}^{n_h} \overset{*}{\rightharpoonup} f \quad \text{in} \quad L^\infty(0,T_*;L^2(\Omega))$$

In particular, repeating the arguments above, we have

$$\|\overline{\rho}\|_{L^\infty(0,T_*;L^2(\Omega)} \leq \liminf_{h\to\infty} \|\overline{\rho}^{n_h}\|_{L^\infty(0,T_*;L^2(\Omega)}, \quad \|\nabla\overline{\rho}\|_{L^\infty(0,T_*;L^2(\Omega)} \leq \liminf_{h\to\infty} \|\nabla\rho^{n_h}\|_{L^\infty(0,T_*;L^2(\Omega)}$$

Moreover, equation (11.148) can be rewritten as

$$\|\overline{\rho}^n\|_{L^\infty(0,T_*;H^1(\Omega))} \leq 2\sqrt{2}\big(\|\rho_0\|_{H^1} + 1\big) \exp\left(\int_0^{T_*} C'\|\nabla u^n\|_{W^{1,6}}(s)\ ds\right)$$

and so

$$\|\overline{\rho}\|_{L^\infty(0,T_*;H^1(\Omega))} \leq \liminf_{h\to\infty} \big(\|\overline{\rho}^{n_h}\|_{L^\infty(0,T_*;L^2(\Omega)} + \|\nabla\overline{\rho}^{n_h}\|_{L^\infty(0,T_*;L^2(\Omega))}\big) \leq$$

$$\leq \liminf_{h\to\infty} 4\sqrt{2}\|\rho_0^{n_h}\|_{H^1(\Omega_1)} \exp\left(\int_0^{T_*} C'\|\nabla u^{n_h}\|_{W^{1,6}}(s)\ ds\right) \leq$$

$$\leq \limsup_{h\to\infty} 4\sqrt{2}\|\rho_0^{n_h}\|_{H^1(\Omega_1)} \exp\left(\int_0^{T_*} C'\|\nabla u^{n_h}\|_{W^{1,6}}(s)\ ds\right)$$

Since now $\displaystyle\int_0^{T_*} \|\nabla u^{n_h}\|_{W^{1,6}}\ ds \to \int_0^{T_*} \|\nabla u\|_{W^{1,6}}\ ds$ thanks to equation (11.134), we have

$$\|\overline{\rho}\|_{L^\infty(0,T_*;H^1(\Omega))} \leq C\|\rho_0\|_{H^1} \exp\left(C\int_0^{T_*} \|\nabla u\|_{W^{1,6}}\ ds\right)$$

Finally, $\overline{\rho}$ is a strong solution of the transport equation. In fact (11.144) can again be passed to the limit. In fact, since $u^n \to u$ in $L^2(0,T;L^2)$, the equation

$$\int_\Omega\int_0^{T_*} \overline{\rho}^{n_h}\varphi_t\ dt\ dx = \int_\Omega\int_0^{T_*} \big(u^{n_h}\cdot\nabla\overline{\rho}^{n_h}\big)\ \varphi\ dt\ dx$$

can be passet to the limit with the same devices used above. So we have

$$\overline{\rho} = -u \cdot \nabla \overline{\rho}$$

Moreover, the same argument applied to the two sequences prove that this is also a weak solution. In fact, let $\varphi \in C_c^\infty(\Omega \times [0,T_*))$. We have

$$(\rho^{m_k,n}\varphi)_t = \rho_t^{m_k,n}\varphi + \rho^{m_k,n}\varphi_t$$

$$(\rho^{m_k,n}\varphi)_t - \rho^{m_k,n}\varphi_t + \nabla \cdot (\varphi\rho^{m_k,n}u^{m_k,n}) - \rho^{m_k,n}u^{m_k,n} \cdot \nabla\varphi = 0$$

and integrating over $\Omega \times (0,T_*)$ we have

$$-\int_\Omega (\rho^{m_k,n}\varphi)(0)\;dx - \int_0^{T_*}\int_\Omega \rho^{m_k,n}\varphi_t\;dx\;dt = \int_0^{T_*}\int_\Omega \rho^{m_k,n}u^{m_k,n} \cdot \nabla\varphi\;dx\;dt$$

Thus, since

$$\int_\Omega (\rho^{m_k,n}\varphi)(x,0)\;dx \equiv \int_\Omega \rho_0^n(x)\varphi(x,0)\;dx$$

and

$$\int_0^{T_*}\int_\Omega \rho^{m_k,n}\varphi_t\;dx\;dt \to \int_0^{T_*}\int_\Omega \overline{\rho}^n\varphi_t\;dx\;dt$$

since $\varphi \in L^1(0,T_*;L^2(\Omega))$, we have finally

$$\left|\int_0^{T_*}\int_\Omega \rho^{m_k,n}u^{m_k,n} \cdot \nabla\varphi\;dx\;dt - \int_0^{T_*}\int_\Omega \overline{\rho}^n u^n \cdot \nabla\varphi\;dx\;dt\right| =$$

$$= \left|\int_0^{T_*}\int_\Omega (\rho^{m_k,n}u^{m_k,n} - \rho^{m_k,n}u^n) \cdot \nabla\varphi\;dx\;dt - \int_0^{T_*}\int_\Omega (\overline{\rho}^n u^n - \rho^{m_k,n}u^n) \cdot \nabla\varphi\;dx\;dt\right|$$

that vanishes since $u^n \cdot \nabla\varphi \in L^\infty(0,T_*;L^2(\Omega))$ and $\rho^{m_k,n}$ converges in weak star to $\overline{\rho}^n$ in $L^\infty(0,T_*;L^2(\Omega))$ as $k \to \infty$. Moreover, we have, if M is a bound for $\nabla\varphi$,

$$\left|\int_0^{T_*}\int_\Omega (\rho^{m_k,n}u^{m_k,n} - \rho^{m_k,n}u^n) \cdot \nabla\varphi\;dx\;dt\right| \le M\int_0^{T_*}\int_\Omega |\rho^{m_k,n}||u^{m_k,n} - u^n|\;dx\;dt \le$$

$$\le M\int_0^{T_*} \|\rho^{m_k,n}\|_{L^2(\Omega)}\|u^{m_k,n} - u^n\|_{L^2(\Omega)}\;dt \le$$

$$\le M\left(\sup_{(0,T_*)}\|\rho^{m_k,n}\|_{L^2(\Omega)}\right)\sqrt{T_*}\left(\int_0^{T_*}\|u^{m_k,n} - u^n\|^2_{L^2(\Omega)}\;dt\right)^{\frac{1}{2}}$$

Remember that $\sup\limits_{(0,T_*)}\|\rho^{m_k,n}\|_{L^2(\Omega)} \le \Lambda_0$ thanks to equation (11.138). So, we have that also this term vanishes, since $u^{m_k,n} \to u^n$ in $L^2(0,T_*;L^2(\Omega))$. This means that, for every $\varphi \in C_c^\infty(\Omega \times [0,T_*))$,

$$\int_\Omega \rho_0^n(x)\varphi(x,0)\;dx - \int_0^{T_*}\int_\Omega \overline{\rho}^n\varphi_t\;dx\;dt = \int_0^{T_*}\int_\Omega \overline{\rho}^n\;\big(u^n \cdot \nabla\varphi\big)\;dx\;dt \tag{11.149}$$

that is, $\overline{\rho}^n$ is also a weak solution in $\Omega \times (0,T_*)$ with velocity filed u^n. Moreover, using the convergence of $\overline{\rho}^{n_h}$, and the fact that $u^n \to u$ in $L^2(0,T_*;L^2)$, we can take the limit in

$$-\int_\Omega \rho_0^{n_h}(x)\varphi(x,0)\;dx - \int_0^{T_*}\int_\Omega \overline{\rho}^{n_h}\varphi_t\;dx\;dt = \int_0^{T_*}\int_\Omega \overline{\rho}^{n_h}\;\big(u^{n_h} \cdot \nabla\varphi\big)\;dx\;dt$$

as $h \to \infty$, observing this time that

$$\left| \int_\Omega \rho_0^{n_h}(x)\varphi(x,0)\ dx - \int_\Omega \rho_0(x)\varphi(x,0)\ dx \right| \le \|\rho_0^{n_h} - \rho_0\|_{L^2(\Omega)} \|\varphi(0)\|_{L^2(\Omega)} \to 0$$

So we have that

$$-\int_\Omega \rho_0(x)\varphi(x,0)\ dx - \int_0^{T_*}\int_\Omega \overline{\rho}\varphi_t\ dx\ dt = \int_0^{T_*}\int_\Omega \overline{\rho}\ \big(u\cdot\nabla\varphi\big)\ dx\ dt$$

that is, $\overline{\rho}$ is a weak solution to the transport equation with velocity field u and initial data ρ_0. **Strong and weak solutions coincide** ($\overline{\rho} = \rho$)**:** We know, by hypothesis, that ρ is a weak solution to the problem on the whole $\Omega \times (0, T_*)$. This means that

$$\int_0^{T_*}\int_\Omega (\rho\varphi_t + \rho u\cdot\nabla\varphi)(x,t)\ dx\ dt = -\int_\Omega \rho_0(x)\varphi(x,0)dx$$

for every $\varphi \in C_c^\infty(\Omega \times [0,T_*))$. But, according to [8], or section 8.2, using in particular uniqueness theorem 8.5, weak solutions are unique. This means that $\rho = \overline{\rho}$ over $\Omega \times [0, T_*)$. So, we have that ρ is a strong solution of the transport equation over $\Omega\times(0,T_*)$. This actually proves the theorem. ∎

Remark 11.39. This theorem proves the first part of (20). The second part follows from the fact that $\rho_t = -u\cdot\nabla\rho$ and the regularities above[56]. Finally the equation (21) is proved in section 11.14.2[57]. □

[56]In particular, we have

$$\|u\cdot\nabla\rho\|_2^2 \equiv \int_\Omega |u\cdot\nabla\rho|^2\ dx \le \|u\|_\infty \|\nabla\rho\|_2^2$$

that is uniformly bounded in $(0,T_*)$ thanks to lemma 9.6 and $u \in L^\infty(0,T_*;H^2(\Omega))$ and (11.131).

[57]Actually, in this section it is proved only the belonging to $L^2(0,T;L^6(\Omega))$. However, the fact that $\nabla p \in L^\infty(0,T_*;L^2(\Omega))$ follows from

$$\|\rho u_t + \rho u\cdot\nabla u - \mu\Delta u\|_2 \le \|\sqrt{\rho}\|_\infty \|\sqrt{\rho}u_t\|_2 + \|\rho\|_\infty \|u\cdot\nabla u\|_2 + \mu\|\Delta u\|_2$$

that is uniformly bounded over $(0,T_*)$ thanks to (11.114) and the fact that

$$\|u\cdot\nabla u\|_2^2 = \int_\Omega |u\cdot\nabla u|^2\ dx \le \|u\|_4^2\|\nabla u\|_4^2 \le C\|\nabla u\|_2^2\|\nabla u\|_{H^1}^2$$

since Ω is bounded. Moreover remember that $u \in L^\infty(0,T_*;H^2(\Omega))$.

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