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Some developments in Singular KAM theory

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Abstract

KAM theory predicts that a completely integrable Hamiltonian, under perturbation of sufficiently small magnitude, preserves almost all invariant tori except for a small set estimated by the square root of the perturbation size.

Such estimate is not entirely accurate because there are no effective techniques to take into account the tori formed by the presence of the perturbation, which we refer to as "secondary." For nearly a decade, L. Biasco and L. Chierchia have been addressing this problem —the total measure of invariant tori— motivated by the unresolved 1985 conjecture by Arnol'd, Kozlov, and Neishtadt.

The conjecture anticipated that the measure would be controlled by the size of the perturbation without the square root. To analyze and solve this problem, it was necessary to develop innovative and brilliant techniques, which have been described in the *"Singular KAM Theory"* published in 2023.

Thanks to this theory, it is possible to estimate the measure of these invariant tori accurately and meticulously as it has never done before, and the final result aligns with the conjecture, namely, that the measure is proportional to the perturbation size up to corrections.

However, this work has only been carried out for quasi-integrable Hamiltonians that we call "Natural" or "Mechanical" (i.e., Hamiltonians of the form $\frac{y^2}{2} + f(x)$).

The goal of this thesis is to extend the result to more general Hamiltonians, particularly in the first part we present the generalization for those system with a generic integrable and *convex* part $h(y)$ but always with a perturbation depending only on positions $f(x)$, while in the second part for those Hamiltonian with a radius of analyticity on the angles, i.e. on the x, that is different for each component (for Biasco and Chierchia was one value for each component).

The final part of the thesis intends to head towards the application of Singular KAM theory to some models of physical interest. To get into specifics, we consider the restricted, circular and planar three-body problem and we study the Fourier coefficients of the Hamiltonian that describes the system. We provide a new expansion of these coefficients that significantly improves numerical and analytical work, and, because of this, we study crucial properties like the presence of zeros and their analyticity.

Contents

Introduction

Quasi-periodic motions in Hamiltonian systems

Perturbation theory is one of the most relevant technique used in all the branches of physics: from classical to non-relativistic quantum mechanics, as well as in nuclear and atomic physics, but also in fluid dynamics and up to quantum field theory. Therefore, developing a perturbation theory in a rigorous and formal way has always been one of the crucial ambitions for mathematicians.

It's crucial to understand the primordial motivations that led us to use this kind of perturbative approach. Precisely for that reason, we have to notice that maybe the first relevant application of this theory coincides with one of the earliest problems in nature that has amazed and intrigued man since the most ancient historical times: the *motion of celestial bodies*.

This planetarian problem comes from basics observation of "regularity" in mechanisms such as the rising and the setting of the sun or monthly change of moon phases or the seasons. So astronomy began to grow and develop more sophisticated methods: calculation started to predict future positions and regularity of the motions of other planets. In this way, due to an increased knowledge of earth's non centrality in the vast solar system, men began to think about catastrophic question, and research started focus on these more natural questions:

Will we continue orbiting around the sun preserving our stability?
Which kind of fate awaits our planet?
Is there a risk that in a small period of time we will find ourselves far from solar system?

The richness of these doubts led mathematicians to develop techniques and notions that nowadays underlie the theory of "dynamical system".
In order to make our model, the solar system can be viewed as a 10 bodies system attracting each other with gravitational force: Sun has the biggest mass M, while the

other planets has $m_1, ..., m_9$ masses. If we want to make a perturbative approach, we firstly have to consider our toy model.
For this reason we can start neglecting the attraction exerted by the other planet and consider only the interaction between the Sun and a single planet, that we can call *Earth*. This problem is very easy and studied from first years of high school; it leads us to simple differential equation that one can solve obtaining the famous *Kepler's laws*: Earth orbits around Sun with following a trajectory represented by an ellipse of which one of the foci is occupied by the Sun. This motion is eternally periodic and so completely stable for all time.

Now we want to extend this result to the motion of 10 bodies. In a first approximation, we notice that $M \gg m_i$ for all $i = 1, ..., 9$ (in fact the heaviest planet is Jupiter with $m_J \approx 1,90 \times 10^{27} kg$, but $\frac{m_J}{M} \approx 1 \times 10^{-3}$) and so we can initially neglect interaction between planets, thinking solar system as the sum of 10 "single planet"-problem. In this way we have obtained a completely solvable system where each single planet will describe his elliptic trajectory indipendently from the others bodies. If we traduce this motion from a mathematical view, indicating ω_i the revolution frequency of planet with mass m_i, the motion during the time is *quasi-periodic*, namely is represented by a trigonometric series like

$$\sum_{j=(j_1,...,j_9)\in\mathbb{Z}^9} c_j e^{2\pi i(\omega\cdot j)t} \tag{0.0.1}$$

where $\omega \cdot j = j_1\omega_1 + j_2\omega_2 + ... + j_9\omega_9$.
When the solutions to the Newtonian equations of motion are only quasi-periodic, we say that the system is *integrable*, and its Hamiltonian (i.e. its energy) can be always write as a function that depends only on momenta, namely if we indicate with q the positions and with p the momenta such that $(p, q) \in \mathcal{M} \subset \mathbb{R}^{2n}$, one has

$$\mathscr{H}(p, q) = h(p).$$

Having this kind of motions means completely stability for all time, so it is clear that the question we should pay attention to is whether such quasi-periodic motions will persist if we consider mutual gravitational attraction between planets.

This problem has occupied the minds of the world's greatest mathematicians during XVII and XIX century, without positive results, until they came to the truth: this problem cannot be completely solved. Although they did not achieve great results, the researchers of the time developed revolutionary approaches and techniques that have survived to our days.

One of these new approaches developed in those years was precisely to consider the problem as a perturbation of the planetary system without interaction. This method is based on the study of the Hamiltonian function defined by

$$\mathscr{H}(p,q) = h(p) + f(p,q)$$

where f describes the presence of interaction and is small w.r.t. the integrable part. In order to take into account this smallness, given a suitable norm $||\cdot||$ on space of the real analytic functions, we can say that

$$||f(p,q)|| = \varepsilon, \text{ for a fixed } 0 < \varepsilon \ll 1$$

or equivalently to consider

$$\mathscr{H}(p,q) = h(p) + \varepsilon f(p,q)$$

with $||f(p,q)|| = 1$.

This opened the way to the perturbation theory.

A lot of brilliant mathematicians like Lagrange, Weierstrass and Poincarè (that can be considered the father of modern theory) have worked during their life to develop a rigorous perturbation theory, but they have failed due to the presence of the famous *small divisors*: integer linear combination of unperturbed frequencies ω_i, that appear at denominator when we consider the influence between planets, and lead to divergent series.

In fact, if one were looking for solutions to the perturbed problem like 0.0.1, in the expression of coefficient c_j one would find at denominator the expression

$$\omega \cdot j$$

and if there were exists an integer vector $j \in \mathbb{Z}^9$ such that $\omega \cdot j = 0$ or $\omega \cdot j \leqslant \varepsilon \ll 1$, the coefficient would be devergent.

In general when $\omega \cdot j = 0$, we say that they are in *resonance*. These resonances can occur in various systems, such as mechanical, electrical, or acoustic systems. When these effects happen, the system absorbs energy from the external force and starts vibrating with a larger amplitude, and disasters such as the collapse of bridges or buildings may be associated with resonances.

For planetary motion we can see that resonances are not a fake complication made by mathematician, but real problems. In fact if we consider the orbits of Pluto and Neptune [1]

$$T_N \approx 60\,223 \text{ days} \approx 5,2033 \times 10^9 s \quad \Rightarrow \quad \omega_N \approx 1,2075 \times 10^{-9}\,\text{Hz}$$

[1] All the planets data are taken from https://nssdc.gsfc.nasa.gov/

$$T_P \approx 90\,560 \text{ days} \approx 7,8244 \times 10^9 s \quad \Rightarrow \quad \omega_P \approx 8,0302 \times 10^{-10}\,\text{Hz}$$

we notice that

$$2\omega_N - 3\omega_P \approx 5,94 \times 10^{-12}\,\text{Hz} = 0,00000000000594\,\text{Hz} \approx 0\,\text{Hz}$$

and in the same way also, for example, Jupiter's moons Io and Europa are in a resonance with each other:

$$\omega_{IO} \approx 4,109 \times 10^{-5}\,\text{Hz} \quad \omega_{EUR} \approx 2,047 \times 10^{-5}\,\text{Hz} \;\Rightarrow\; \omega_{IO} - 2\,\omega_{EUR} \approx 1,5 \times 10^{-7}\,\text{Hz} \approx 0$$

and many others.

This problem due to the presence of small divisors in the pertubation theory has remaind unsolved until 1954. In that year, the great russian mathematician Kolmogorov published his famous paper *"On the Conservation of Conditionally Periodic Motions under Small Perturbation of the Hamiltonian"*, in which he stated the well-known:

KAM theorem [2]**, 1954.** *if the pertubation is sufficiently small, most of the quasi-periodic motions of the unperturbed system with "non–resonant" frequencies are preserved (if the unperturbed system satisfies a non–degeneracy condition).*[3]

We do not want to focus our attention on the history that is related to this paper, even though it is very interesting and fascinating. For a complete historical and mathematical view one can see [24].

The approach on which the proof of the Kolomogorov theorem is based is rather innovative: it is a construction of an iterative algorithm that converges very fast (inspired by Newton's method of tangents) and in some way "kills" the pertubation by conjugating the Hamiltonian to an unperturbed system. This is possible because is made in several steps: in the first step one conjugates the Hamiltonian to a system in which the perturbation is quadratically smaller, and then keeps making it smaller and smaller until it disappears (doing "infinite" steps). The rate at which this scheme converges is crucial: it causes the numerators to decrease so as to absorb the divergences of the small denominators. This method was a huge breakthrough in the world of dynamical systems and opened a new way of approaching serious problems like celestial mechanics, billiards, and in recent years PDEs, etc.

[2] In addition to Kolmogorov's work, the name is due to the contributions made to this theory by Arnol'd (1963 - [11]) and Moser (1967 - [37]).

[3] The non–degeneracy condition is that the unperturbed frequency map must be a diffeomorphism. The original proof made by Kolmogorov to his theorem in his 1954 work is incomplete, for full demonstration one can see for example [23].

Reading to the statement, the first question that appear is what does *most of the quasi-periodic motions* mean in this context. To be more precise, we might ask: given a generic phase space $\mathcal{M} \subset \mathbb{R}^{2n}$, which is the measure of the quasi-periodic motions that are preserved by the perturbation?

Roughly speaking, the answer is quite easy: KAM theorem ensures that the quasi–periodic motions that are preserved are the ones that correspond to *Diopanthine frequencies*. These unperturbed frequencies satisfy

$$|\omega(y) \cdot k| \geqslant \frac{\gamma}{|k|^\tau} \text{ for all } k \in \mathbb{Z}^n \backslash \{0\}$$

for $\gamma, \tau > 0$, i.e. are "quite far" from resonance. One can easily verify that set of Diophantine frequencies, that we call $\Omega_{\gamma,\tau}$, has

$$\text{meas}\,(\Omega_{\gamma,\tau}) \leqslant \mathcal{O}(\gamma)$$

and, since the optimal condition to apply KAM theorem is that $\varepsilon\,\gamma^{-2} \lesssim 1$ where ε is the size of the perturbation, [4] the right choice of γ is $\mathcal{O}(\sqrt{\varepsilon})$ such that

$$\text{meas}\,(\Omega_{\gamma,\tau}) \leqslant \mathcal{O}(\sqrt{\varepsilon}).$$

Using the fact that the frequency map is a diffeomorphism, the measure of frequencies for which quasi-periodic motions are preserved corresponds to the measure of actions for which the same holds. So if we call $\mathcal{Q}_\varepsilon$ the quasi-periodic trajectories of the perturbed system, one can conclude that

$$\text{meas}\,(\mathcal{M} \setminus \mathcal{Q}_\varepsilon) \leqslant \mathcal{O}(\sqrt{\varepsilon}).$$

We usually call the set $\mathcal{M} \setminus \mathcal{Q}_\varepsilon$ as *"non-torus"* region.

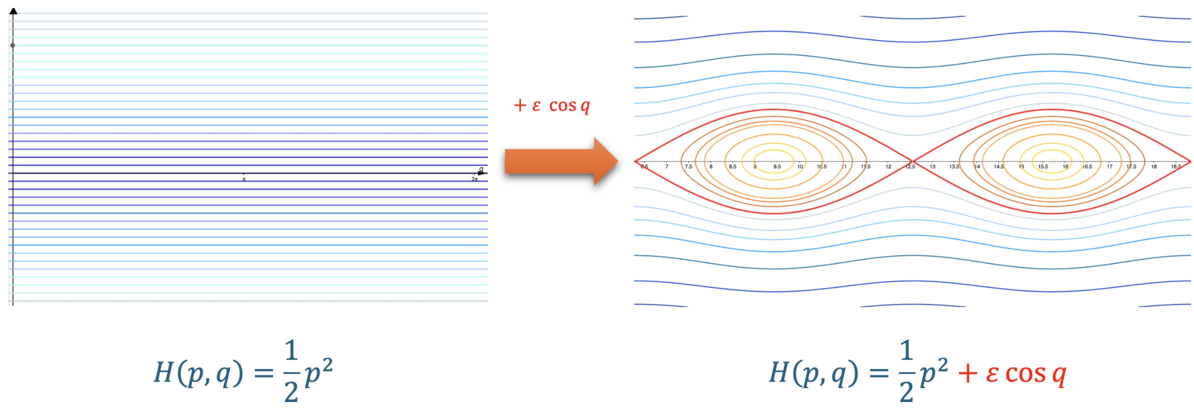
This result can be readily understood by looking at a crucial 1-dimensional example: the *pendulum* case.

Consider a point particle that moving moves under the effect of a small cosinuisodal potential (a small harmonic oscillator) with kinetic energy $\frac{1}{2}p^2$ on a 1-dimensional ring (or, for the more advanced reader, on the 1-dimensional torus), i.e. consider

$$\mathscr{H}(q,p) = \frac{1}{2}p^2 + \varepsilon \cos q, \quad (p,q) \in (B \subset \mathbb{R}^n) \times [0, 2\pi) \bmod 2\pi.$$

As we know from the basic courses of mechanics, the phase space (i.e. the space of positions and momenta) can be approximately drawn as

[4] In this heuristic introduction we leave the details for which it is necessary to impose this condition. We notice that with the Kolmogorov's 1954 proof one can only has $\gamma\,\varepsilon^{-4} \lesssim 1$, this optimality of ε^{-2} was obtained by Arnold with his new proof of KAM theorem in 1963 ([11]).



Looking at the perturbed phase space, the blue trajectories are the deformation of the unperturbed motions (that are blue straight lines), so are the effective "preserved" trajectories that are expected by KAM theorem, and are usually called "*primary tori*". Furthermore, inside these blue line, one can find the red line that describes the outline of two eyes that is called "*separatrix*".

In this case, however, we can directly compute the area that enclosed by separatrix, and so we can calculate the measure of the region in which one cannot find the primary KAM tori, i.e. the quasi-periodic motion expected by KAM theorem. Indeed, the separatrix in the phase space is expressed by the relation

$$\mathscr{H}(q,p) = \frac{1}{2}p^2 + \varepsilon\cos q = \varepsilon \;\Rightarrow\; p(q) = \pm 2\sqrt{\varepsilon(1-\cos q)}$$

so if one wants to compute the area enclosed by the separatrix in $[0, 2\pi)$ one finds

$$\text{Area} = 4\int_0^{2\pi} \sqrt{\varepsilon(1-\cos q)}dq = 8\pi\sqrt{\varepsilon} = \mathcal{O}(\sqrt{\varepsilon})$$

as we have found in the above argument. So we have checked that in this case meas $(\mathcal{M} \setminus \mathcal{Q}_\varepsilon) \leqslant \mathcal{O}(\sqrt{\varepsilon})$ where $\mathcal{Q}_\varepsilon$ represents the quasi-periodic trajectories preserved by the perturbation that were present in the $\varepsilon = 0$ case.

The most important remark that we notice from this example is that the separatrix encloses some new quasi–periodic motions that did not appear in the unperturbed case, so they were formed by the presence of the perturbation. They are called "*secondary tori*".

These secondary tori are also present directly in nature, and we can find them, for example, in some particular systems in celestial mechanics, like in the motion of two moons of Saturn.

Saturn is surrounded by a crowded family of rings and moons, and two of those moons – Epimetheus and Janus – orbit Saturn so close together that it seems as though their different orbital speeds should make them crash into each other. But due to the complex interplay of their mutual gravitational attraction and their very slightly different distances from Saturn, they never get closer than about 15,000 kilometers from each other. Instead of crashing, they exchange orbital positions in a gravitational do-si-do once every four years, in a dance that takes 100 days to play out.

Here is how the dance works. Epimetheus and Janus are small, irregularly-shaped moons with diameters of about 120 and 180 kilometers, respectively. Both are on slightly eccentric orbits around Saturn. Their orbits around Saturn differ in size by only 50 kilometers. In 2004, Epimetheus was the inner of the two satellites. Because it was closer to Saturn, Epimetheus traveled at a faster angular rate than Janus, so inner Epimetheus slowly, inexorably caught up to outer Janus. As the two approached each other in their orbits, Epimetheus tugged on Janus from behind as Janus tugged on Epimetheus with equal and opposite force.

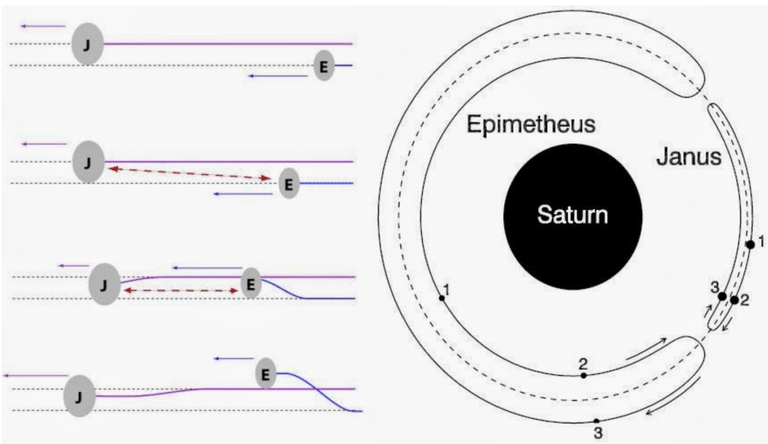


Figure 1: Foto taken from https://francis.naukas.com

The mutual tugging caused them to exchange angular momentum. Epimetheus gained momentum and rose in orbit as Janus lost an equivalent amount of momentum and fell. Because Janus is four times more massive than Epimetheus, it fell four times less than Epimetheus rose. The switch of orbital altitudes made Janus

– still ahead of Epimetheus in its orbit – the faster of the two. As a result, Janus crept ahead, and will continue to do so until catching up with Epimetheus again in 2010. The figure 1.1 below explains the behavior schematically. The closeness of the two moons is exaggerated – Epimetheus and Janus never approach closer than about 15,000 kilometers from each other, roughly 100 times their diameters.

This periodic motion of Janus and Ephimetheus would not exist if there was no perturbation (the attraction between the moons) and therefore represent what we call a secondary torus.

So, since KAM theorem is only about primary tori, a question that appears naturally is

Which is the total measure of quasi periodic motions in a small-perturbed system? (including primary and secondary tori)

Arnol'd, Kozlov and Neishtad in [9] 1985 conjectured an answer to this fundamental question.

Conjecture. (ref. [9], Remark 6.8, p.285) *It is natural to expect that in a generic system with three or more degrees of freedom the measure of the 'non-torus' set has order ε (...).*

So the presence of secondary tori causes the change of the estimate on the measure of "non-torus" zones, i.e., the complementary of quasi-periodic trajectories, from $\mathcal{O}(\sqrt{\varepsilon})$ to $\mathcal{O}(\varepsilon)$.

One can find an upper bound on the measure of the *"non–torus"* set in agreement with this conjecture for a generic class of Hamiltonian systems in the brilliant *"Singular KAM theory"* (see [4]) developed by *Luca Biasco* and *Luigi Chierchia* between 2015 and 2023.

Let us briefly review the strategy developed by Biasco and Chierchia in order to achieve this result:

- Analogously to what is done in Nekhoroshev theory (compare [25], [41]), fixed a maximal size of resonances K to be considered, one covers the action space intro three regions:
  1. Non Resonant set $\mathcal{R}^0$, (in which $|\omega \cdot k| > \alpha \ \forall \ k \in \mathbb{Z}^n \backslash \{0\}$) after high order averaging, classical KAM theory yields the existence of primary maximal KAM tori up to a set of measure $O(\sqrt{\varepsilon} e^{(-cK)})$.
  2. A Neighborhood of double and higher resonances $\mathcal{R}^2$ (in which ω is in resonance with two or more indipendent vectors of $\mathbb{Z}^n$) of measure εK^c,

where no perturbative analysis is possible because the dynamic is equivalent to a ε-free dynamic, so this is completely included in non torus set.[5]

3. a $\sqrt{\varepsilon}K^c$–neighborhood $\mathcal{R}^1$ of simple resonances, in which ω is in resonance with one vector $k \in \mathbb{Z}^n$, but $|\omega \cdot \ell| > \alpha$ for all ℓ with no component in the direction of k.

So we notice that main game has to be played on the simple–resonance neighborhood, in order to consider secondary tori and improving estimates (in classic KAM we don't consider secondary tori so that non torus set includes completely also $\mathcal{R}^1$ and this set is order $\sqrt{\varepsilon}$.)

- In neighborhoods of single resonance $k \in \mathbb{Z}^n$ one can perform high-order averaging theory developed in [1] such that the Hamiltonian is conjugated to

$$H^k(y,x) = \bar{H}^k(y,x_1) + \varepsilon f^k(y,x); f^k \approx e^{(-cK)}, \quad \bar{H}^k(y,x_1) = h^k(y) + \varepsilon g^k(y,x_1).$$

- Then the following step is to conjugate the infinitely-many 1-degree Hamiltonian $\bar{H}^k$ in action-angles variables, but this would be very difficult because one must have uniform control in all parameters as this transformation must be done for all k. In order to do this, following [2], we show that one can conjugate all $\bar{H}^k$ to a specific class of Hamiltonian call "Standard form" Hamiltonian,

$$H_{std}(p,q_1) = (1+\nu(p,q_1))p_1^2 + G(p,q).$$

For high values of k this standard form has essentially the portrait of a pendulum (i.e. is essentially $p_1^2 + \cos(q)$) so it is easy to study. For low values of k this form is quite more complicate because is only a system with finitely-many and non degenerate critical points.

In this way the problem is moved to study action-angles variables for generic standard form systems, and that's the main point of [3]. In this work action-angle transformation is fully described for this class of Hamiltonians, and the analytic proprieties of this variables are discussed. In particular in [3] Biasco and Chierchia have studied the behavior of actions function in the limit as the energy approaches the critical values that will play a main role in our intent:

$$I_1(E_{crit} \pm \delta z) = a(z) + b(z)\, z\, \log(z); \quad a, b \text{ analytic}.$$

This result will be crucial for our result.

[5]One of the most difficult open problem in dynamics is to rigorously prove that in a ε-free system the "non torus" set has $\mathcal{O}(1)$ measure.

- Now, we apply the action-angle transformation that we call ϕ_k to the original system near simple resonance
$$H_k = H^k \circ \phi_k(y, x) = h_k(I) + \varepsilon \tilde{f}^k(I, \varphi); \tilde{f}^k \approx e^{(-cK)}.$$
- The last problem we have to tackle in order to apply classical KAM theory to this Hamiltonian above, is checking the non degenerate condition on the new integrable part. Obviously this problem is not trivial only near critical points, namely with the standard form Hamiltonian in his action-angles variables. That's a very delicate question, and it becomes more serious when the distance in energy from the separatrices goes to zero, where problem becomes a singular perturbation problem with dramatic singularities. This problem is taken into account in the outstanding *"Twist Theorem"* proved in [4].
- At this point, choosing carefully the various free parameters of the game, a suitable KAM Theorem yields the existence of maximal primary and secondary KAM tori, which fill the complementary phase set of $\mathcal{R}^2 \times \mathbb{T}^n$ up to a set of measure exponentially small with K.

Choosing $K \approx |\log \varepsilon|$, it follows the result up to logarithm correction.

The main goal of this thesis is to extend Singular KAM theory to a broader class of Hamiltonian systems and to start a possible application of this theory to some relevant physical models in order to obtain some interesting new estimates on the total measure of the invariant tori for these nearly-integrable systems.

Description of the thesis

The work is divided into three chapters, each of which presents a different result. Here we will not present the rigorous results but we will do only an heuristic discussion. The formal results will be written in each chapter.

In the first section we generalize Singular KAM theory to spatial perturbation of convex Hamiltonian systems. To be more precise, the result of Biasco and Chierchia is done for natural systems described by

$$\mathscr{H}(p, q) = \frac{1}{2}p^2 + \varepsilon f(x), \qquad (p, q) \in (B \subset \mathbb{R}^n) \times \mathbb{T}^n$$

where $\mathbb{T}^n = \mathbb{R}^n \backslash (2\pi\mathbb{Z}^n)$ and with f that satisfy a precise but generic condition that we will see later in this work. In the first chapter we present the generalization of the result to systems characterized by

$$\mathscr{H}(p, q) = h(p) + \varepsilon f(x) \qquad (p, q) \in B \subset \mathbb{R}^n \times \mathbb{T}^n$$

where h is a completely generic integrable and *convex* Hamiltonian (and with the same class of potential f).

The main difference of this part w.r.t. to original result is the conjugation to *standard form* Hamiltonian near critical points. In particular, in the convex case, the surface in which the critical points lie is completely generic. Therefore, the conjugation becomes a local discourse and must be accomplished in small neighborhood of each critical point. This causes great technical difficulties because it will be necessary to cover the phase space in neighborhoods in which such conjugation is done, and all parameters have to be simultaneously uniform in k and in the choice of the critical point around which we are conjugating the system.

The second chapter generalize the result to a wider class of Hamiltonian by extending their domain of analyticity. To be more precise, in [4], the authors consider a Hamiltonian that is real analytic on a complex neighborhood of the phase space that is

$$B_r \times \mathbb{T}^n_s := \bigcup_{y \in B} \{z \in \mathbb{C}^n : |z - y| \leqslant r\} \times \{x \in \mathbb{C}^n : |\mathrm{Im}\ x| \leqslant s\} \backslash (2\pi\mathbb{Z}^n).$$

Here we generalize the result for Hamiltonians that are real analytic of a complex neighborhood of the torus that has different complex widths for each direction, namely for a positive components vector $s \in (0, +\infty) \times ... \times (0, +\infty) \subset \mathbb{R}^n$ we have considered

$$B_r \times \mathbb{T}^n_s := \bigcup_{y \in B} \{z \in \mathbb{C}^n : |z - y| \leqslant r\} \times \{x \in \mathbb{C}^n : |\mathrm{Im}\ x_i| \leqslant s_i\} \backslash (2\pi\mathbb{Z}^n) \quad \forall\ i = 1, ..., n.$$

This choice is quite more natural, especially in view of potential applications, e.g., to celestial mechanics, where typical physical potentials do have this asymmetry.

For this part the main difference w.r.t. the Biasco and Chierchia's work is about averaging, normal forms and the uniform behaviour of large Fourier modes of the potential. In particular, every argument must be made component by component, but one must be careful because every condition one wants to impose must be one for all directions.

In this way we have extended the result to a class of functions that is much more general in the integrable part (moving from the kinetic case to the convex case) and that also has a more general class of potentials (with different strips of analyticity for each direction of the angles).

In the third and last chapter we focus on a possible application of Singular KAM theory to celestial mechanics, in particular, we consider the *Restricted, planar, circular three-body problem* which is usually called RCP3BP.

Roughly speaking this model describe the bounded planar motion of a *"zero mass"* body subject to the gravitational field generated by two primary bodies revolving on circular Keplerian orbits (which are assumed to be not influenced by the small body).

When the mass ratio of the two primary bodies is small the RCP3BP is described by a nearly-integrable Hamiltonian system with two degrees of freedom; in a region of phase space corresponding to nearly elliptical motions with non small eccentricities, the system is well described by Delaunay variables.

In order to apply this new theory our main goal is to check the condition that Biasco and Chierchia impose on the generic potential, to the disturbing function of RCP3BP.

For this reason we focus on the Fourier coefficients of the perturbation. We develop a new expansion of them in terms of *Hansen coefficients* that makes the numerical work much simpler and allows us to find the asymptotic for small values of eccentricity and major axis. Furthermore, we study two main proprieties of the coefficients that will be important to control the Singular KAM condition: the presence of zeros for high Fourier modes, and their domain of analyticity.

Chapter 1

Singular KAM Theory for spatial pertubations of convex Hamiltonian systems

1.1 Set up and some standard definitions

Let $n \geqslant 2$, we consider analytic Hamiltonian systems composed of the sum of an integrable part (in the sense of Arnold-Liouville) and a small perturbation. Namely, indicating the flat n-dimensional torus by $\mathbb{T}^n := \mathbb{R}^n \backslash (2\pi\mathbb{Z}^n)$, and making use of standard action-angle coordinates $(y, x) \in B \subset \mathbb{R}^n \times \mathbb{T}^n$ where B is is the closure of a bounded connected open non empty set $\mathbb{R}^n$, associated to the symplectic two-form $\Omega := \sum_{i=1}^n dx_i \wedge dy_i$, we are interested in those systems described by

$$H(y, x) = H_\varepsilon(y, x) = h(y) + \varepsilon f(y, x) \tag{1.1.1}$$

where ε is a small parameter measuring the size of the analytic perturbation εf w.r.t. the analytic integrable part h.

Let $r, s > 0$ and $|\cdot|$ be the standard Euclidean norm, we define

$$B_r := \bigcup_{y \in B} \{z \in \mathbb{C}^n : |z - y| \leqslant r\},$$

$$\mathbb{T}^n_s := \{x \in \mathbb{C}^n : |\text{Im } x| \leqslant s\} \backslash (2\pi\mathbb{Z}^n).$$

Assume that H_ε in 1.1.1 admits an holomorphic extension for some $r, s > 0$ on the complex domain $B_r \times \mathbb{T}^n_s \subset \mathbb{C}^{2n}$.

We call the Hamiltonian flow associated to H_ε at time t $\Phi^t_{H_\varepsilon}(y_0, x_0) =: (y(t), x(t))$, representing the solution to the standard Hamiltonian equations

$$\begin{cases} \dot{y} = -\partial_x H_\varepsilon(y,x) \\ \dot{x} = \partial_y H_\varepsilon(y,x) \end{cases} \qquad \begin{cases} y(0) = y_0 \\ x(0) = x_0 \end{cases} \iff \Omega(X_{H_\varepsilon}, \cdot) = J(\nabla H_\varepsilon)^\dagger$$

where J is the standard symplectic matrix.

Remark 1.1.1. For $\varepsilon = 0$, system 1.1.1 is integrable in the sense of Arnol'd-Liouville, and its phase space is foliated by primary invariant tori carrying quasi-periodic motions.

Now we have to recall some standard definitions.

Definition 1.1.1 (MAXIMAL KAM TORI)**.** *A set $\mathcal{T} \subset \mathcal{M} = B \times \mathbb{R}^n$ is called a maximal KAM torus for a real analytic Hamiltonian $H : \mathcal{M} \longmapsto \mathbb{R}$ if there exist a real analytic embedding $\phi : \mathbb{T}^n \longmapsto \mathcal{M}$ and a Diophantine frequency vector $\omega \in \mathbb{R}^n$ such that $\mathcal{T} = \phi(\mathbb{T}^n)$, and for each $z \in \mathcal{T}$, $\Phi^t_H(z) = \phi(x + \omega t)$, where $x = \phi^{-1}(z)$.*

Definition 1.1.2 (GENERATORS OF 1D MAXIMAL LATTICES)**.** *Let $\mathbb{Z}^n_\star$ be the set of integer vectors $k \neq 0$ in $\mathbb{Z}^n$ such that the first non–null component is positive:*

$$\mathbb{Z}^n_\star := \{k \in \mathbb{Z}^n : k \neq 0 \text{ and } k_j > 0 \text{ where } j = \min\{i : k_i \neq 0\}\}$$

$\mathcal{G}^n$ denotes the set of generators of 1d *maximal lattices in $\mathbb{Z}^n$, namely, the set of vectors $k \in \mathbb{Z}^n_\star$ such that the greater common divisor (gcd) of their components is 1:*

$$\mathcal{G}^n := \{k \in \mathbb{Z}^n_\star : gcd(k_1, ..., k_n) = 1\} \tag{1.1.2}$$

for $K \geqslant 1$, we set:

$$\mathcal{G}^n_K := \mathcal{G}^n \cap \{|k|_1 \leqslant K\} \tag{1.1.3}$$

Definition 1.1.3 (1D FOURIER PROJECTORS)**.** *Given a zero–average real analytic periodic function*

$$f : x \in \mathbb{T}^n \longmapsto f(x) = \sum_{\mathbb{Z}^n \setminus \{0\}} f_k e^{ik\cdot x} \tag{1.1.4}$$

and fixed a vector $k \in \mathbb{Z}^n \setminus \{0\}$, we denote by $\pi_{\mathbb{Z}k} f$ the (real analytic) periodic function of one variable $\theta \in \mathbb{T}$ given by

$$\theta \in \mathbb{T} \longmapsto \pi_{\mathbb{Z}k} f(\theta) := \sum_{j \in \mathbb{Z}} f_{jk} e^{ij\theta} \tag{1.1.5}$$

We will also refer to the projection on the Fourier modes of the resonant maximal sublattice $\mathbb{Z}k$ *as*

$$p_{\mathbb{Z}k} f(x) := \sum_{j \in \mathbb{Z}} f_{jk} e^{ijk \cdot x}. \tag{1.1.6}$$

Notice the relation between these two functions

$$\pi_{\mathbb{Z}k} f(k \cdot x) = p_{\mathbb{Z}k} f(x) \tag{1.1.7}$$

and that one has the following (unique) decomposition:

$$f(x) = \sum_{k \in \mathcal{G}^n} \pi_{\mathbb{Z}k} f(k \cdot x) = \sum_{k \in \mathcal{G}^n} p_{\mathbb{Z}k} f(x). \tag{1.1.8}$$

Definition 1.1.4 (RESONANCES). *Given* $k \in \mathcal{G}^n$*, a resonance* $\mathcal{R}_k$ *with respect to the Hamiltonian* $h(y)$ *is the set* $\{y \in \mathbb{R}^n : \omega(y) \cdot k = 0\}$ *where* $\omega : y \in \mathbb{R}^n \to \omega(y)_i = \partial_{y_i} h(y)$*. We call* $\mathcal{R}_{k,\ell}$ *a double resonance if* $\mathcal{R}_{k,\ell} = \mathcal{R}_k \cap \mathcal{R}_\ell$ *with* k *and* ℓ *in* $\mathcal{G}^n$ *linearly indipendent; the order of a double resonance is given by* $\max\{|k|_1, |\ell|_1\}$*.*

Definition 1.1.5 (MORSE FUNCTIONS WITH DISTINCT CRITICAL VALUES). *A* C^2*-function of one variable* $\theta \longmapsto F(\theta)$ *is a Morse function if its critical points are non-degenerate, i.e.,* $F'(\theta_0) = 0 \implies F''(\theta_0) \neq 0$*; "distint critical values" means that if* $\theta_1 \neq \theta_2$ *are distinct critical points, then* $F(\theta_1) \neq F(\theta_2)$*.*

Definition 1.1.6 (β-MORSE FUNCTIONS). *A* $\mathcal{C}^2(\mathbb{T}, \mathbb{R})$ *Morse function* F *with distinct critical values is called a* β*-Morse function, with* $\beta > 0$*, if*

$$\min_{\theta \in \mathbb{T}} (|F'(\theta)| + |F''(\theta)|) \geqslant \beta, \quad \min_{i \neq j} |F(\theta_i) - F(\theta_j)| \geqslant \beta, \tag{1.1.9}$$

where $\theta_i \in \mathbb{T}$ *are the critical points of* F*.*

Definition 1.1.7 (BANACH SPACES OF REAL ANALYTIC PERIODIC FUNCTIONS AND NORMS). *For* $s > 0$ *and* $n \in \mathbb{N}$*, consider the Banach space of zero-average real analytic periodic functions on* $\mathbb{T}^n$ *with finite norm*

$$\|f\|_s := \sup_{k \in \mathbb{Z}^n} |f_k| e^{|k|_1 s}, \tag{1.1.10}$$

and denote by $\mathbb{B}^n_s$ *its closed unit ball. Besides the norm* $\|\cdot\|_s$*, we shall also use the following two (non equivalent) norms*

$$|f|_s := \sup_{\mathbb{T}^n_s} |f|, \qquad \| f \|_s := \sum_{k \in \mathbb{Z}^n} |f_k| e^{|k|_1 s}. \tag{1.1.11}$$

Note that in general $\|f\|_s \leqslant |f|_s \leqslant \|\!|f|\!\|_s$.
For functions (not necessarily holomorphic in y) $f : B_r \times \mathbb{T}^n_s \longmapsto \mathbb{C}$ *we will also use the norms*

$$|f|_{B,r,s} = |f|_{r,s} = \sup_{B_r \times \mathbb{T}^n_s} |f|, \qquad \|\!|f|\!\|_{B,r,s} = \|\!|f|\!\|_{r,s} = \sup_{y \in B_r} \sum_{k \in \mathbb{Z}^n} |f_k(y)| e^{|k|_1 s}$$
$$\|f\|_{B,r,s} = \|f\|_{r,s} := \sup_{B_r \times \mathbb{T}^n_s} |f_k(y)| e^{|k|_1 s}. \tag{1.1.12}$$

For a function depending only on $y \in B_r$ *we use* $|f|_{B,r} = |f|_r = \sup_{B_r} |f|$.

Remark 1.1.2. Those three norms are obviously not equivalent. Indeed, for any $\sigma > 0$, one has

$$\|f\|_{r,s} \leqslant |f|_{r,s} \leqslant \|\!|f|\!\|_{r,s} \leqslant \left(\coth^n(\frac{\sigma}{2}) - 1\right) \|f\|_{r,s+\sigma} \leqslant \left(\frac{2n}{\sigma}\right)^n \|f\|_{r,s+\sigma} \tag{1.1.13}$$

Definition 1.1.8 (COSINE-LIKE FUNCTION)**.** *Let* $0 \leqslant \mathsf{g} \leqslant \frac{1}{4}$. *We say that a real analytic function* $G : \mathbb{T}_1 \longmapsto \mathbb{C}$ *is* g*-cosine-like if, for some* $\eta > 0$ *and* $\theta_0 \in \mathbb{R}$ *one has*

$$|G(\theta) - \eta \cos(\theta + \theta_0)|_1 := \sup_{\mathbb{T}_1} |G(\theta) - \eta \cos(\theta + \theta_0)| \leqslant \eta \mathsf{g} \tag{1.1.14}$$

Notice that this notion is invariant by rescalings: G *is* g*-cosine-like if and only if* λG *is* g*-cosine-like for any* $\lambda > 0$.

At this point we are ready to point out our two assumptions on the Hamiltonian in 1.1.1 (one for the integrable part and one for the perturbation) and state our main result. For the issue that we are proving, we have to take the pertubation in a special class of function defined in the following

Definition 1.1.9 (THE CLASS OF POTENTIAL $\mathbb{G}^n_s$)**.** *We denote by* $\mathbb{G}^n_s$ *the subset of functions* $f \in \mathbb{B}^n_s$ *such that the following two proprietis hold:*

$$\liminf_{k \in \mathcal{G}^n : |k|_1 \to \infty} |f_k| e^{|k|_1 s} |k|_1^n > 0, \tag{1.1.15}$$

$$\forall\ k \in \mathcal{G}^n, \quad \pi_{\mathbb{Z}k} f \ \textit{is a Morse Function with distinct critical values.} \tag{1.1.16}$$

Remark 1.1.3. As it's proved in [2], this class of function is generic in $\mathbb{B}^n_s$, in the sense that $\mathbb{G}^n_s$ contains an open and dense set in $\mathbb{B}^n_s$.

Assumptions 1.1.1. *For the rest of the work we will assume two proprieties on our Hamiltonian in 1.1.1 that characterize the spatial pertubation of convex Hamiltonian systems:*
*i) The integrable part $h(y)$ is a γ-**convex** function of the action variable, i.e. for all $\xi \in \mathbb{R}^n$ and $y \in B$ the following holds:*

$$\langle \partial_y^2 h(y)\ \xi, \xi \rangle = \sum_{i,j=1}^{n} \Big(\partial_{y_i y_j} h(y) \Big) \xi_i \xi_j \geqslant \gamma |\xi|^2. \tag{1.1.17}$$

*Convex maps are in part particular **KAM Non-degenerate** function, i.e. the map*

$$y \in B_r \to \omega(y) := \partial_y h(y) \in \Omega := \omega(B_r) \subseteq \mathcal{B}(0, M) \subset \mathbb{R}^n, \quad \textit{where} \quad M := \sup_{B_r} |\omega(y)| \tag{1.1.18}$$

is a global diffeomorphism of B into Ω, where we have denoted with $\mathcal{B}(y_0, r) = \{y \in \mathbb{C}^n : |y - y_0| \leqslant r\}$. Namely we are imposing that

$$\det \left(\frac{\partial^2 h}{\partial y^2} \right) \neq 0 \tag{1.1.19}$$

and as consequence, h has a finite number of non degenerate critical points. Moreover, we define the Lipschitz constants of ω as

$$\bar{L}^{-1} |y - y_0| \leqslant |\partial_y h(y) - \partial_y h(y_0)| \leqslant L |y - y_0|, \quad \forall\ y, y_0 \in B_r. \tag{1.1.20}$$

ii) The perturbation $f \in \mathbb{G}_s^n$.

For all details concerning this new class of pertubation see [2], now we briefly discuss only the most important proprieties without proves.

As we will see in the other part of this work, our strategy depends in a crucial way on the size of the vector $k \in \mathbb{Z}^n \backslash \{0\}$ generating the resonance. In particular, we will distinguish between a finite number of "low Fourier modes" of f and an infinite number of "high Fourier modes" of the same function. In order to be able to distinguish between these two classes, we need to introduce the "cut-off" function:

Definition 1.1.10 (FOURIER CUT-OFF FUNCTION)**.** *Given $\delta \in (0, 1]$ and $n, s > 0$ define the following "Fourier cut-off function":*

$$\mathtt{N} = \mathtt{N}(\delta; s, n) := 2 \max\{1, \frac{1}{s} \log \frac{c_n}{s^n \delta}\}, \qquad \textit{where} \qquad c_n := 2^{44} (\frac{2n}{e})^n$$

Remark 1.1.4. Note that this choice of cut-off function satisfies

$$\mathtt{N} \geqslant 2\mathtt{c}_s, \quad \text{where} \quad \mathtt{c}_s := \max\{1, \frac{1}{s}\}. \tag{1.1.21}$$

We will refer to "high Fourier modes" for $|k|_1 \geqslant \mathtt{N}$, while the lower ones are for $|k|_1 < \mathtt{N}$. For low modes we can only note that since 1.1.16 holds, for all $k \in \mathcal{G}^n$ such that $|k|_1 \leqslant \mathtt{N}$ there exists $\beta > 0$ such that

$$\pi_{\mathbb{Z}k} f \text{ is } \beta\text{-Morse}. \tag{1.1.22}$$

For the high modes, instead, we have to study the

Uniform behaviour for high Fourier modes.

If f satisfies 1.1.15 it follows immediatily from basic analysis that one can find $0 < \delta \leqslant 1$ such that

$$|f_k| \geqslant \delta e^{-|k|_1 s} |k|_1^{-n} > 0, \qquad \forall\ k \in \mathcal{G}^n,\ |k|_1 \geqslant \mathtt{N} \tag{1.1.23}$$

With this simple propriety and the above choice of $\mathtt{N}$ one can show that, for $|k|_1 \geqslant \mathtt{N}$, $\pi_{\mathbb{Z}k} f$ is very close to a shifted rescaled cosine function:

Proposition 1.1.1. *Let $\delta > 0$ and $f \in \mathbb{G}^n_s$. Then, for any $k \in \mathcal{G}^n$ such that $|k|_1 \geqslant \mathtt{N}$, $\pi_{\mathbb{Z}k} f$ is 2^{-40}-cosine-like in the sense of the above Definition.*

Proof. As in [2], we shall prove something slightly stronger that will be useful for our intent, namely, that there exists $\theta_k \in [0, 2\pi)$ such that

$$\pi_{\mathbb{Z}k} f(\theta) = 2|f_k| \Big(\cos(\theta + \theta_k) + F^k_\star(\theta) \Big), \qquad F^k_\star(\theta) := \frac{1}{2|f_k|} \sum_{j \geqslant 2} f_{jk} e^{ij\theta} \tag{1.1.24}$$

with $F^k_\star(\theta) \in \mathbb{B}^1_1$ and

$$|F^k_\star|_1 \leqslant \| F^k_\star \|_1 \leqslant 2^{-40}. \tag{1.1.25}$$

Indeed, by definition of $\pi_{\mathbb{Z}k} f$,

$$\pi_{\mathbb{Z}k} f(\theta) := \sum_{j \in \mathbb{Z} \setminus \{0\}} f_{jk} e^{\mathrm{i}j\theta} = \sum_{|j|=1} f_{jk} e^{\mathrm{i}j\theta} + \sum_{|j| \geqslant 2} f_{jk} e^{\mathrm{i}j\theta},$$

and, defining $\theta_k \in [0, 2\pi)$ so that $e^{\mathrm{i}\theta_k} = f_k / |f_k|$, one has

$$\frac{1}{2|f_k|} \sum_{|j|=1} f_{jk} e^{\mathrm{i}j\theta} = \mathrm{Re} \left(\frac{f_k}{|f_k|} e^{\mathrm{i}\theta} \right) = \mathrm{Re}\, e^{\mathrm{i}(\theta + \theta_k)} = \cos(\theta + \theta_k),$$

which yields (1.1.24). Now, since $f \in \mathbb{B}^n_s$ it is $|f_k| \leqslant e^{-|k|_1 s}$ and, by (1.1.23), $|f_k| \geqslant \delta |k|_1^{-n} e^{-|k|_1 s}$. Therefore, for $|k|_1 \geqslant \mathtt{N}$, one has

$$\begin{aligned}
\|F^k_\star\|_1 \overset{(1.1.24)}{=} & \frac{1}{2|f_k|} \sum_{|j|\geqslant 2} |f_{jk}| e^{|j|} \leqslant \frac{|k|_1^n e^{|k|_1 s}}{2\delta} \sum_{|j|\geqslant 2} |f_{jk}| e^{|j|} \\
\leqslant & \frac{|k|_1^n e^{|k|_1 s}}{2\delta} \sum_{|j|\geqslant 2} e^{-|j|(|k|_1 s - 1)} \\
\leqslant & \frac{2e^2 |k|_1^n}{\delta} e^{-|k|_1 s} = \frac{2^{n+1} e^2}{s^n \delta} e^{-\frac{|k|_1 s}{2}} \left(\frac{|k|_1 s}{2}\right)^n e^{-\frac{|k|_1 s}{2}} \\
\leqslant & \left(\frac{2n}{es}\right)^n \frac{2e^2}{\delta} e^{-\frac{\mathtt{N}s}{2}} \leqslant 2^{-40}, \qquad (1.1.26)
\end{aligned}$$

where the geometric series converges since $|k|_1 s \geqslant \mathtt{N}s \geqslant 2$ (by (1.1.21)) and last inequality follows by definition of $\mathtt{N}$ in (1.1.10). □

Proposition 1.1.2. *Let $\delta > 0$ and $f \in \mathbb{G}^n_s$. Then, for any $k \in \mathcal{G}^n$ such that $|k|_1 \geqslant \mathtt{N}$, $\pi_{\mathbb{Z}k} f$ is $|f_k|$-Morse.*

Proof. As we did in the above proposition, we get

$$\left| \frac{\pi_{\mathbb{Z}k} f}{2f_k} - \cos(\theta + \theta^k) \right|_1 \overset{(2.2.5)}{=} |F^k_\star|_1 \leqslant \|F^k\|_1 \overset{(2.2.6)}{\leqslant} 2^{-40}, \qquad (1.1.27)$$

which implies that the function $F := \pi_{\mathbb{Z}k} f/(2f_k)$ is C^2–close to a (shifted) cosine: Indeed, by Cauchy estimates $\|\cdot\|_{C^2} \leqslant 2|\cdot|_1$, so that

$$\|F - \cos(\theta + \theta^k)\|_{C^2} = \max_{0\leqslant j\leqslant 2} \max_{\mathbb{T}} |\partial^j_\theta (F - \cos(\theta + \theta^k))| \leqslant 2|F^k_\star|_1 \overset{(2.2.9)}{\leqslant} 2^{-39}.$$

By Lemma 2.2.1 we see that F is $(1 - 2^{-38})$–Morse, and the claim follows by rescaling. □

Main Results

As we said before, with our system we could use classical KAM Theory that ensures the existence of a set of relative Lebesgue measure $1 - O(\sqrt{\varepsilon})$ of primary invariant tori carrying quasi-periodic motions. In particular, for fixed $\gamma > 0, \tau \geqslant n - 1$, the invariant tori of the integrable system whose associated frequencies $\omega(y)$ satisfy the Diophantine condition

$$|k \cdot \omega(y)| \geqslant \frac{\gamma}{|k|_1^\tau} \ \forall\ k \in \mathbb{Z}^n \backslash \{0\}$$

persist under any sufficiently small perturbation. One can obtain this kind of result seeing only at primary tori, that, apart from a set of order $\sqrt{\varepsilon}$ up to correction, one expects to find where there is no resonance. The aim of this work is, according to what Biasco and Chierchia have done, taking in count also tori that appears thanks to perturbation (just think to the case of the standard pendulum inside the separatrices) that we in general call *secondary tori*.

In general, region near double or higher order resonances is not expected to contain a large set of invariant tori, as its dynamics is essentially non perturbative. In fact in this domain, after the partial averaging taking into account the resonances under consideration, normalizing the deviations of the "actions" from the resonant values by the quantity ε, normalizing time, and discarding the terms of higher order, we obtain a Hamiltonian of the form $h(y) + f(x)$, which does not involve a small parameter. Returning to the original variables we obtain a "non–torus" set of measure ε.

So it remains to find and expect secondary tori to appear in region with singular resonance, as we will see in the following pages. For this reason the main result of this work is the following

Theorem 1.1.1. *Let $n \geqslant 2$, $s > 0$, B a compact set of $\mathbb{R}^n$, $b := 11n + 7$, $f \in \mathbb{G}^n_s$ with $\|f\|_s = 1$ and h a convex and integrable hamiltonian. Then, there exist a constant $\mathtt{c} > 1$, such that for all $\mathtt{K}$ and $\varepsilon > 0$ satisfying*

$$\mathtt{K} \geqslant \mathtt{c}\,, \qquad \varepsilon\, \mathtt{K}^b \leqslant 1\,, \tag{1.1.28}$$

the following holds. There exist three sets $\mathcal{R}^2 \subseteq \mathrm{B}$, $\mathcal{A} \subseteq \mathrm{B} \times \mathbb{T}^n$, $\mathcal{T} \subseteq \mathbb{R}^n \times \mathbb{T}^n$ such that:

(i) $\mathrm{B} \times \mathbb{T}^n \subseteq (\mathcal{R}^2 \times \mathbb{T}^n) \cup \mathcal{A} \cup \mathcal{T}$;

(ii) *$\mathcal{R}^2$ is a neighborhood of double resonances of order smaller than $\mathtt{K}$ satisfying the measure estimate*
$$\operatorname{meas} \mathcal{R}^2 \leqslant c_\star\, \varepsilon\, \mathtt{K}^b\,,$$
where $c_\star$ is a suitable constant depending only on n;

(iii) *$\mathcal{A}$ is exponentially small with respect to $\mathtt{K}$:*
$$\operatorname{meas} \mathcal{A} \leqslant e^{-\mathtt{K}/\mathtt{c}}\,;$$

(iv) *$\mathcal{T}$ is union of maximal KAM tori for the Hamiltonian $H(y, x; \varepsilon) := h(y) + \varepsilon f(x)$.*

Choosing $\mathtt{K} := c|\log \varepsilon|$ it follows that

Corollary 1.1.1. *Under the assumptions of the theorem above, there exists* $0 < \varepsilon_0 < 1$ *such that for* $\varepsilon < \varepsilon_0$*, all points in* $B \times \mathbb{T}^n$ *lie on a maximal KAM torus for* H_ε *in* 1.1.1*, except for a subset whose measure in bounded by* $\bar{c}\, \varepsilon |\log \varepsilon|^\gamma$ *where* $\bar{c} = 1 + (2\pi)^n c_\star c^\gamma$.

The two degrees of freedom is special: in this case the only double resonance is the origin and one can take as $\mathcal{R}^2$ a disk of measure ε^a with any $0 < a < 1$ getting a set of KAM tori of exponential density in the complementary of $\mathcal{R}^2 \times \mathbb{T}^2$.

Corollary 1.1.2. *Let the assumptions of Theorem* 1.1.1 *hold and let* $n = 2$*. Then, there exists* $0 < \varepsilon_{\rm o} < 1$*, such that for* $\varepsilon < \varepsilon_{\rm o}$ *and* $0 < a < 1$*, all points in the set* $\{y \in \mathrm{B} : |y| > \varepsilon^{a/2}\} \times \mathbb{T}^2$ *lie on a maximal KAM torus for* $\mathtt{H}$ *in* 1.1.1*, except for an exponentially small set of measure bounded by* $e^{-1/(2\mathtt{c}\varepsilon^{\hat{a}})}$*, with* $\hat{a} := (1-a)/24$.

1.2 Averaging, covering and normal forms

In this section, we discuss the high order normal forms of generic natural systems 1.1.1, especially in neighbourhoods of simple resonances. On one hand, apart from a finite (although arbitrarily large) number of simple resonances of order less than $\mathtt{N}$, the secular (averaged) Hamiltonians have a uniform normal form with a potential close to a shifted cosine. On the other hand, the secular Hamiltonians at simple resonances of order less or equal than $\mathtt{N}$ admit a simple normal form.

Normal form lemma. The first technical lemma allows to average out non-resonant Fourier modes of the perturbation f on suitable non-resonant regions, and allows for "arbitrary small" analyticity loss in the angle variables, a fact which will be crucial in our applications.

Lemma 1.2.1 (Normal form lemma with "small" analyticity loss.)**.** *Let* $r, s, \alpha > 0$, $\mathbf{K} \in \mathbb{N}$, $\mathbf{K} \geqslant 2$, $B \subset \mathbb{R}^n$ *and let* Λ *be a lattice of* $\mathbb{Z}^n$*. Let*

$$H(y, x) = h(y) + f(y, x)$$

real-analytic on $B_r \times \mathbb{T}^n_s \subset \mathbb{C}^{2n}$ *with* $|f|_{r,s} < \infty$*. Assume that* B_r *is* $(\alpha, \mathbf{K})$ *non-resonant modulo* Λ *and that*

$$\theta_\star := \frac{2^{11}\mathbf{K}^2}{\alpha r s}|f|_{r,s} < 1. \tag{1.2.1}$$

Then, there exists a real-analytic symplectic change of variable

$$\psi : (y', x') \in B_{r_\star} \times \mathbb{T}^n_{s_\star} \mapsto (y, x) \in B_r \times \mathbb{T}^n_s, \quad \text{where} \quad r_\star := \frac{r}{2},\ s_\star := s(1 - \frac{1}{\mathbf{K}}) \tag{1.2.2}$$

satisfying

$$|y-y'|_1 \leqslant \frac{\theta_\star}{2^7\mathbf{K}} r, \qquad \max_{1\leqslant i\leqslant n} |x_i - x_i'| \leqslant \frac{\theta_\star}{16\mathbf{K}^2} s, \tag{1.2.3}$$

and such that

$$H\circ\psi = h + g + f_\star; \quad \textit{where} \quad p_\Lambda g = g, \quad p_\Lambda f_\star = 0, \tag{1.2.4}$$

with

$$\| g - p_\Lambda f \|_{r_\star, s_\star} \leqslant \frac{1}{\mathbf{K}} \theta_\star \| f \|_{r,s}, \quad \| f_\star \|_{r_\star, s/2} \leqslant 2e^{-(\mathbf{K}-2)\bar{s}} \| f \|_{r,s}, \tag{1.2.5}$$

where $\bar{s} := \min\{\frac{s}{2}, \log\frac{8}{\theta_\star}\}$.

For the full demonstration see [1]

First Covering. The covering lemma is slightly different w.r.t. the one in [1]. Indeed, in order to extend our result to *generalized natural system*, we shall enlarge the zone with two or more degree of resonances.

Lemma 1.2.2 (First covering lemma)**.** *Let h be KAM non-degenerate and let ω denote its gradient. Fix $\mathtt{K} \geqslant 6\mathtt{K}_0 \geqslant 12$ and $\alpha > 0$. Then, the domain B can be covered by three sets $\mathcal{R}^i \subseteq B$,*

$$B = \mathcal{R}^0 \cup \mathcal{R}^1 \cup \mathcal{R}^2 \tag{1.2.6}$$

so that the following holds.
a) $\mathcal{R}^0$ is $(\frac{\alpha}{2}, \mathtt{K}_0)$ completely non-resonant (i.e. non-resonant modulus $\{0\}$), namely,

$$y \in \mathcal{R}^0 \Longrightarrow |\omega(y)\cdot k| \geqslant \frac{\alpha}{2}, \ \forall\ 0 < |k|_1 \leqslant \mathtt{K}_0. \tag{1.2.7}$$

b) $\mathcal{R}^1 = \bigcup_{k\in\mathcal{G}^n_{1,\mathtt{K}_0}} \mathcal{R}^{1,k}$, where, for each $k \in \mathcal{G}^n_{1,\mathtt{K}_0}$, $\mathcal{R}^{1,k}$ is a closed neighbourhood of a simple resonance $\{y \in B : \omega(y)\cdot k = 0\}$, which is $(3\alpha\mathtt{K}^{n+4}/|k|, \mathtt{K})$ non-resonant modulo $\mathbb{Z}k$, namely

$$y \in \mathcal{R}^{1,k} \Longrightarrow |\omega(y)\cdot k| < \alpha\,; \ |\omega(y)\cdot\ell| \geqslant \frac{3\alpha\mathtt{K}^{n+4}}{|k|_1}, \ \forall\ \ell \in \mathbb{Z}^n, \ \ell \notin \mathbb{Z}k, \ |\ell|_1 \leqslant \mathtt{K}. \tag{1.2.8}$$

c) $\mathcal{R}^2$ contains all the resonance of order two or more and has Lebesgue measure small with α^2: more precisely, there exists a costant $c > 0$ depending only on n and M such that

$$meas(\mathcal{R}^2) \leqslant c(n,M)\,\alpha^2\mathtt{K}^{2n+3} \tag{1.2.9}$$

The proof of this proposition is written in the section 5 of [1]. We just want to remark that the covering $\{\mathcal{R}^i\}$ is the pull back of a covering in frequency space:

$$\mathcal{R}^i := \{y \in B : \omega(y) \in \Omega^i\} \tag{1.2.10}$$

where the Ω^i's are defined as follow

$$\Omega^0 := \{\omega \in B_M(0) : \min_{k \in \mathcal{G}^n_{1,\mathtt{K}_0}} |\omega \cdot k| > \frac{\alpha}{2}\}. \tag{1.2.11}$$

Instead, to define Ω^1, denoting with $p_k^\perp$ the orthogon al projection on the subspace perpendicular to k, we use

$$\begin{aligned}
\Omega^{1,k} &:= \{\omega \in \mathbb{R}^n : |\omega \cdot k| < \alpha, |p_k^\perp \omega| < M, \text{ and } |p_k^\perp \omega \cdot \ell| > \frac{3\alpha \mathtt{K}^{n+4}}{|k|}, \ \forall\ \ell \in \mathcal{G}^n_{1,\mathtt{K}} \backslash \mathbb{Z}k\} \\
\mathcal{R}^{1,k} &:= \{y \in B : \omega(y) \in \Omega^{1,k}\} \\
\Omega^1 &:= \bigcup_{k \in \mathcal{G}^n_{1,\mathtt{K}_0}} \Omega^{1,k}.
\end{aligned} \tag{1.2.12}$$

Finally, the set Ω^2 is just the union of neighbourhoods of exact double resonances: if we call

$$R_{k,\ell} := \{\omega \cdot k = \omega \cdot \ell = 0\}, \quad k \in \mathcal{G}^n_{1,\mathtt{K}_0},\ \ell \in \mathcal{G}^n_{1,\mathtt{K}},\ \ell \notin \mathbb{Z}k, \tag{1.2.13}$$

then

$$\begin{aligned}
\Omega^2_{k,\ell} &:= \{|\omega \cdot k| < \alpha\} \cap \{|p_k^\perp \omega| < M\} \cap \{|p_k^\perp \omega \cdot \ell| \leqslant \frac{3\alpha \mathtt{K}^{n+4}}{|k|}\} \\
\Omega^2 &:= \bigcup_{k \in \mathcal{G}^n_{1,\mathtt{K}_0}} \bigcup_{\substack{\ell \in \mathcal{G}^n_{1,\mathtt{K}} \\ \ell \notin \mathbb{Z}k}} \Omega^2_{k,\ell}.
\end{aligned} \tag{1.2.14}$$

Remark 1.2.1. As we will see in the next part of this section, the right choice of α for our aim is $\alpha := \sqrt{\varepsilon}\mathtt{K}^{\frac{9}{2}n+2}$, where ε is the size of the perturbation in 1.1.1, so that

$$meas(\mathcal{R}^2) \leqslant c_\star(n, M)\, \varepsilon\, \mathtt{K}^b, \quad with \quad b := 11n + 7 \tag{1.2.15}$$

Proof. of 1.2.9. First observe that from the definitions of $\mathcal{R}^0$, $\mathcal{R}^{1,k}$ and $\mathcal{R}^2$ in (1.2.11), (1.2.12) and (1.2.13), it follows immediately that

$$\mathcal{R}^2 \subseteq \bigcup_{k \in \mathcal{G}^n_{\mathtt{K}_0}} \bigcup_{\substack{\ell \in \mathcal{G}^n_{\mathtt{K}} \\ \ell \notin \mathbb{Z}k}} \mathcal{R}^2_{k,\ell}\,, \tag{1.2.16}$$

with

$$\mathcal{R}^2_{k\ell} := \left\{ y \in \mathrm{B} : |\omega \cdot k| < \alpha;\ |\,\pi_k^\perp \omega \cdot \ell| \leqslant \tfrac{3\alpha \mathtt{K}^{n+4}}{|k|} \right\}, \qquad (k \in \mathcal{G}^n_{\mathtt{K}_o},\ \ell \in \mathcal{G}^n_{\mathtt{K}} \setminus \mathbb{Z}k)\,. \quad (1.2.17)$$

Let us, then, estimate the measure of $\mathcal{R}^2_{k,\ell}$ in (1.3.13). Denote by $v \in \mathbb{R}^n$ the projection of ω onto the plane generated by k and ℓ (recall that, by hypothesis, k and ℓ are not parallel); then,

$$|v \cdot k| = |\omega \cdot k| < \alpha\,, \qquad |\,\pi_k^\perp v \cdot \ell| = |\,\pi_k^\perp \omega \cdot \ell| \leqslant \frac{3\alpha \mathtt{K}^{n+4}}{|k|}\,. \qquad (1.2.18)$$

Set

$$h := \pi_k^\perp \ell = \ell - \tfrac{\ell \cdot k}{|k|^2} k\,. \qquad (1.2.19)$$

Then, v decomposes in a unique way as $v = ak + bh$ for suitable $a, b \in \mathbb{R}$. By (1.2.18),

$$|a| < \frac{\alpha}{|k|^2}\,, \qquad |\,\pi_k^\perp v \cdot \ell| = |bh \cdot \ell| \leqslant \frac{3\alpha \mathtt{K}^{n+4}}{|k|}\,, \qquad (1.2.20)$$

and, since $|\ell|^2|k|^2 - (\ell \cdot k)^2$ is a positive integer (recall, that k and ℓ are integer vectors not parallel),

$$|h \cdot \ell| \overset{(1.2.19)}{=} \frac{|\ell|^2|k|^2 - (\ell \cdot k)^2}{|k|^2} \geqslant \frac{1}{|k|^2}\,.$$

Hence,

$$|b| \leqslant \frac{3\alpha \mathtt{K}^{n+4}}{|k|}\,. \qquad (1.2.21)$$

Then, write $\omega \in \Omega^2_{k,\ell}$ as $\omega = v + v^\perp$ with $v^\perp$ in the orthogonal complement of the plane generated by k and ℓ. Since $|v^\perp| \leqslant |\omega| < M$ and v lies in the plane spanned by k and ℓ inside a rectangle of sizes of length $2\alpha/|k|^2$ and $6\alpha \mathtt{K}|k|$ (compare (1.2.20) and (1.2.21)), we find

$$\mathrm{meas}(\Omega^2_{k,\ell}) \leqslant \tfrac{2\alpha}{|k|^2}\,(6\alpha \mathtt{K}^{n+4}|k|)\,(2M)^{n-2} = 3 \cdot 2^n\, M^{n-2}\, \alpha^2 \tfrac{\mathtt{K}^{n+4}}{|k|}\,, \qquad \forall \begin{cases} k \in \mathcal{G}^n_{\mathtt{K}_o}\,, \\ \ell \in \mathcal{G}^n_{\mathtt{K}} \setminus \mathbb{Z}k\,. \end{cases}$$

Thus, since $\displaystyle\sum_{k \in \mathcal{G}^n_{\mathtt{K}_o}} |k|^{-1} \leqslant c\,\mathtt{K}_o^{n-1}$ for a suitable $c = c(n)$, and since $\mathtt{K}_o \leqslant \mathtt{K}/6$, (1.2.15) follows immediately taking

$$c_\star = \frac{c}{2 \cdot 3^n} M^{n-2} \bar{L}^{-1}.$$

□

Averaging. Using normal form lemma and the covering lemma we are able to state the averaging theorem for non resonant and simply resonant zones.
To perform averaging, we need to introduce a few parameters (Fourier cut–offs, a small divisor threshold, radii of analyticity) and some notation. Let $\mathtt{K}, \mathtt{K}_0, \nu$ and α such that

$$\mathtt{K} \geqslant 6\mathtt{K}_0 \geqslant 12, \quad \nu \geqslant \frac{9}{2}n + 2 \quad \alpha := \sqrt{\varepsilon}\mathtt{K}^\nu. \tag{1.2.22}$$

For a generic $k \in \mathcal{G}^n_{1,\mathtt{K}_0}$ we define

$$\begin{aligned}
&r_0 := \frac{\alpha}{4L\mathtt{K}_0} = \sqrt{\varepsilon}\frac{\mathtt{K}^\nu}{4L\mathtt{K}_0}; \quad r_0' := \frac{r_0}{2}; \qquad r_k := \frac{\alpha}{L|k|} = \sqrt{\varepsilon}\frac{\mathtt{K}^\nu}{L|k|}; \quad r_k' := \frac{r_k}{2}; \quad \check{r}_k := \frac{r_k}{2\mathtt{K}^{n+1}}\\
&s_0 := s(1-\frac{1}{\mathtt{K}_0}); \quad s_0' := s_0(1-\frac{1}{\mathtt{K}_0}); \qquad s_\star := s(1-\frac{1}{\mathtt{K}}); \quad s_\star' := s_\star(1-\frac{1}{\mathtt{K}})\\
&\bar{\theta} := 2^{14}n^{2n}\frac{L}{s^{2n+1}\mathtt{K}^{2\nu-2n-3}}; \quad \theta := 2^{2(n-2)k}\bar{\theta}; \quad s_k' := |k|_1 s_\star'
\end{aligned} \tag{1.2.23}$$

Theorem 1.2.1 (Averaging theorem). *Let H_ε as in 1.1.1 with $\|f\|_{B,r,s} = 1$ and let 1.2.22, 1.2.23 holds. There exists a costant $b_0 = b_0(n,s) > 1$ such that if $\mathtt{K}_0 \geqslant b_0$, the following holds.*
a) There exists a symplectic change of variables

$$\Psi_0 : \mathcal{R}^0_{r_0'} \times \mathbb{T}^n_{s_0'} \longmapsto \mathcal{R}^0_{r_0} \times \mathbb{T}^n_{s_0} \tag{1.2.24}$$

such that

$$H_\varepsilon \circ \Psi_0 := h(y) + \varepsilon g^0(y) + \varepsilon f^0(y,x), \qquad \langle f^0 \rangle = 0 \tag{1.2.25}$$

with g_0 and f_0 real analytic on $\mathcal{R}^0_{r_0'} \times \mathbb{T}^n_{s_0'}$ and satisfies

$$|g^0 - \langle f \rangle|_{\mathcal{R}^0, r_0'} \leqslant \bar{\theta}, \qquad \| f^0 \|_{\mathcal{R}^0, r_0', s_0'} \leqslant 2\left(\frac{2n\mathtt{K}_0}{s}\right)^n e^{-(\mathtt{K}_0-3)s/2}. \tag{1.2.26}$$

b) for any $k \in \mathcal{G}^n_{1,\mathtt{K}_0}$ there exists a symplectic change of variables

$$\Psi_k : \mathcal{R}^{1,k}_{r_k'} \times \mathbb{T}^n_{s_\star'} \longmapsto \mathcal{R}^{1,k}_{r_k} \times \mathbb{T}^n_{s_\star} \tag{1.2.27}$$

such that

$$\begin{aligned}
H_k = H_\varepsilon \circ \Psi_k &= h^k(y) + \varepsilon g^k(y, k\cdot x) + \varepsilon f^k(y,x)\\
&= h(y) + \varepsilon g_0^k(y) + \varepsilon g^k(y, k\cdot x) + \varepsilon f^k(y,x), \quad p_{\mathbb{Z}k} f^k = 0
\end{aligned} \tag{1.2.28}$$

where g_0^k is real-analytic on $\mathcal{R}^{1,k}_{r'_k}$, $g^k(y,\cdot) \in \mathbb{B}^1_{s'_k}$ for every $y \in \mathcal{R}^{1,k}_{r'_k}$, f^k is real-analytic on $\mathcal{R}^{1,k}_{r'_k} \times \mathbb{T}^n_{s'_*}$, and

$$\begin{gathered} |g_0^k|_{\mathcal{R}^{1,k},r'_k} \leqslant \theta, \qquad \| g^k(y, k\cdot x) - p_{\mathbb{Z}k} f(y,x) \|_{\mathcal{R}^{1,k},r'_k,s'_k} \leqslant \theta. \\ \| f^k \|_{\mathcal{R}^{1,k},r'_k,\frac{s_*}{2}} < 2\left(\frac{2n\mathtt{K}}{s}\right)^n e^{-(\mathtt{K}-3)\frac{s}{2}}. \end{gathered} \tag{1.2.29}$$

c) Finally,

$$\| \pi_y \Psi_0 - y \|_{r'_0,s'_0} \leqslant \frac{r_0}{2^7 \mathtt{K}_0}, \qquad \| \pi_y \Psi_k - y \|_{r'_k,s'_*} \leqslant \frac{r_k}{2^7 \mathtt{K}^{n+1}} \tag{1.2.30}$$

and, for every fixed $y \in B$, $\pi_x \Psi_0(y,\cdot)$ and $\pi_x \Psi_k(y,\cdot)$ are diffeomorphisms on $\mathbb{T}^n$.

Remark 1.2.2. i) Observe that $r_0 \leqslant r_k \leqslant \sqrt{\varepsilon}\mathtt{K}^\nu/L$, and we have to impose the condition $r_k \leqslant r$ (where r is the analyticity radius of the unperturbed Hamiltonian, which here is a free parameter). So we have to verify the smallness condition

$$\varepsilon \leqslant \frac{r^2 L^2}{\mathtt{K}^{2\nu}}$$

but one can take $r = \frac{\mathtt{K}^\nu}{L}$ so that the condition becomes simply $\varepsilon \leqslant 1$.
ii) In order to apply lemma 1.2.1, we want to check condition 1.2.1 with our parameters in 1.2.23. But since $\| f \|_{r,s(1-1/\mathtt{K})} \leqslant (\frac{2n\mathtt{K}}{s})^n ||f||_{B,r,s}$, with simple replacing of parameters and calculation (remember that, for part b we use that $|k| \leqslant \mathtt{K}$), the condition 1.2.1 becomes

$$\mathtt{K}^{2\nu-n-4} \geqslant 2^{13+n} n^n \frac{L e^{s/2}}{s^{n+1}}. \tag{1.2.31}$$

Our choice of ν and b_0 ensures that $\mathtt{K}^{2\nu-n-4} \geqslant \mathtt{K}^{8n} \geqslant b_0^{8n}$, so that by taking b_0 large enough 2.5.15 holds.
iii) if we define

$$\theta_0 = \frac{1}{\mathtt{K}^{6n+1}} \geqslant \theta \geqslant \bar{\theta} \tag{1.2.32}$$

and by taking b_0 large enough, the smallness condition 2.5.10 becomes

$$|g^0|_{r'_0} \leqslant \theta_0, \qquad \| f^0 \|_{r'_0,s'_0} \leqslant e^{-\mathtt{K}_0 s/3} \tag{1.2.33}$$

while the condition 2.5.13 becomes

$$|g_0^k|_{r'_k} \leqslant \theta_0; \qquad \| g^k - \pi_{\mathbb{Z}k} f \|_{r'_k,s'_k} \leqslant \theta_0; \qquad \| f^k \|_{r'_k,s_*/2} \leqslant e^{-\mathtt{K}s/3}. \tag{1.2.34}$$

For the rest of the work we will refer to these simplier smallness conditions.
iv) We remeber that if a set B is $(\alpha, \mathtt{K})$ non resonant modulo Λ for h, then the complex domain B_r is $(\alpha - Lr\mathtt{K}, \mathtt{K})$ non resonant modulo Λ, provided $Lr\mathtt{K} < \alpha$. In fact, if $y \in B_r$ there exists $y_0 \in B$ such that $|y - y_0| \leqslant r$ and $|\omega(y_0) \cdot k| \geqslant \alpha$ for all $k \in \mathbb{Z}^n \backslash \Lambda, |k|_1 \leqslant \mathtt{K}$. Thus, for such k's one obtains

$$|\omega(y) \cdot k| = |\omega(y_0) \cdot k - (\omega(y_0) - \omega(y) \cdot k)| \geqslant |\omega(y_0) \cdot k| - Lr\mathtt{K} \geqslant \alpha - Lr\mathtt{K}.$$

Proof. In this work we do only a brief overview of the proof, for more details see [1]
a) By the choice of r_0, the domain $\mathcal{R}^0_{r_0}$ is $(\alpha/4, \mathtt{K}_0)$ completely non-resonant because $\mathcal{R}^0$ is $(\alpha/2, \mathtt{K}_0)$ completely non-resonant, so we can apply normal form lemma 1.2.1 to H_ε in 1.1.1 with $f, B, r, \Lambda, \alpha, \mathbf{K}, s$ replaced respectively by $\varepsilon f, \mathcal{R}^0, r_0, \{0\}, \frac{\alpha}{4}, \mathtt{K}_0, s_0$. The estimates on 2.5.10 come directly from estimates on 1.2.5.
b) By the definition of r_k, the domain $\mathcal{R}^{1,k}_{r_k}$ is $(3\alpha\mathtt{K}^{n+4}/|k| - \alpha\mathtt{K}/|k|, \mathtt{K})$ non-resonant modulo $\mathbb{Z}k$, so that it is $(\alpha\mathtt{K}^{n+4}/|k|, \mathtt{K})$ non-resonant modulo $\mathbb{Z}k$. So we can use again lemma 1.2.1 with $f, B, r, \Lambda, \alpha, \mathbf{K}, s$ replaced by $\varepsilon f, \mathcal{R}^{1,k}, r_k, \mathbb{Z}k, \frac{\alpha\mathtt{K}^{n+4}}{|k|}, \mathtt{K}, s_\star$.
Part c comes directly from estimates in 1.2.3. ∎

For high Fourier modes, a more precise and uniform normal form can be achieved:

Lemma 1.2.3 (Cosine-like Normal Forms). *Let H_ε be as in 1.1.1 with $f \in \mathbb{G}^n_s$ and let 1.2.22,1.2.23 hold. There exists a constant $\mathfrak{c}_0 = \mathfrak{c}_0(n, s, \delta) \geqslant \max\{\mathtt{N}, b_0\}$ such that if $\mathtt{K}_0 \geqslant \mathfrak{c}_0$ then the following holds. For any $k \in \mathcal{G}^n_{\mathtt{K}_0}$ such that $|k|_1 \geqslant \mathtt{N}$, then the Hamiltonian H_k in 2.5.12 takes the form:*

$$H_k = h^k(y) + 2|f_k|\varepsilon[\cos(k \cdot x + \theta_k) + F^k_\star(k \cdot x) + g^k_\star(y, k \cdot x) + f^k_\star(y, x)] \qquad (1.2.35)$$

where θ_k and $F^k_\star$ are as in proposition 1.1.1 and

$$g^k_\star := \frac{1}{2|f_k|}(g^k - \pi_{\mathbb{Z}k} f), \qquad f^k_\star := \frac{1}{2|f_k|} f^k \qquad (1.2.36)$$

Moreover $g^k_\star(y, \cdot) \in \mathbb{B}^1_1$ (for every $y \in \mathcal{R}^{1,k}_{r'_k}$), $\pi_{\mathbb{Z}k} f^k_\star = 0$, and one has

$$\|g^k_\star\|_{r'_k, 1} \leqslant \theta := \frac{1}{\mathtt{K}^{5n}}, \qquad \|f^k_\star\|_{r'_k, \frac{s_\star}{2}} \leqslant e^{-\mathtt{K}s/7}. \qquad (1.2.37)$$

Remark 1.2.3. This lemma is essentialy Theorem 2.1 of [1] and it's proved for the first time in section 7 of [1], but with some different notation. So in order to avoid any problems, we're going the write the proof with our notation.

Proof. The hypotesis above implies theorem 2.5.1, so that the results of averaging theorem 2.5.1 hold. From 1.2.36 it follows that $g^k = \pi_{\mathbb{Z}k} f - 2|f_k| g_\star^k$ which togheter with 1.1.24 and the form of H_k in 2.5.12 implies 1.2.35. Now we have to prove estimates in 1.2.37. Since $|k|_1 \geqslant \mathtt{N}$, recalling 1.2.23, one have

$$s'_k = |k|_1 s(1 - \frac{1}{\mathtt{K}})^2 > \mathtt{N}s\frac{4}{5} > 1 \tag{1.2.38}$$

in this way $g_\star^k(y,\cdot)$ is bounded on a 'large' angle–domain of size larger than 1 and has zero average since $g_\star^k(y,\cdot) \in \mathbb{B}^1_{|k|_1, s'_\star}$. Now we have to use an elementary proprety of our norm: if $s' \leqslant s$, then for any $\mathtt{N} \geqslant 1$

$$f(y,x) = \sum_{|k|_1 \geqslant \mathtt{N}} f_k(y) e^{ik\cdot x} \Longrightarrow \| f \|_{r,s'} \leqslant e^{-(s-s')\mathtt{N}} \| f \|_{r,s}. \tag{1.2.39}$$

So, if we remember that $K \geqslant 6K_0$ and if we take $\mathfrak{c}_0$ large enough we obtain

$$\begin{aligned} \| g_\star^k \|_{r'_k,1} &:= \frac{1}{2|f_k|} \| g^k - \pi_{\mathbb{Z}k} f) \|_{r'_k,1} \leqslant \frac{|k|_1^n e^{|k|_1 s}}{2\delta} \| g^k - \pi_{\mathbb{Z}k} f) \|_{r'_k,1} \\ &\leqslant \frac{|k|_1^n e^{|k|_1 s}}{2\delta} e^{-(s'_k - 1)} \| g^k - \pi_{\mathbb{Z}k} f) \|_{r'_k, s'_k} \leqslant \frac{|k|_1^n e\theta_0}{2\delta} e^{|k|_1^n (s - s'_\star)} \\ &\leqslant \frac{\mathtt{K}_0^n e}{2\delta \mathtt{K}^{6n+1}} e^{2s\mathtt{K}_0/\mathtt{K}} \leqslant \frac{1}{\mathtt{K}^{5n}} = \theta \end{aligned} \tag{1.2.40}$$

using also 1.1.23, 2.5.20 and 1.2.23.
With similar argument, possbly increasing $\mathfrak{c}_0$ one has

$$\begin{aligned} \| f_\star^k \|_{r'_k, s_\star/2} &\leqslant \frac{|k|_1^n e^{|k|_1 s}}{2\delta} \| f^k \|_{r'_k, s_\star/2} \leqslant \frac{|k|_1^n e^{|k|_1 s}}{2\delta} e^{-Ks/3} \\ &\leqslant \frac{\mathtt{K}_0^n}{2\delta} e^{-\mathtt{K}s(\frac{1}{3} - \frac{\mathtt{K}_0}{\mathtt{K}})} \leqslant \frac{\mathtt{K}^n}{2\delta 6^n} e^{-\mathtt{K}s/6} \leqslant e^{-\mathtt{K}s/7}. \end{aligned} \tag{1.2.41}$$

■

Second Covering. The averaging symplectic maps Ψ_0, Ψ_k of theorem 2.5.1 move boundaries by $\sqrt{\varepsilon}\mathtt{K}^\nu$, so one cannot use the secular Hamiltonians to describe the dynamics all the way up to the boundary of $B \times \mathbb{T}^n$. Such a problem may be overcome introducing a second covering that, thanks to the differences in the averaging process, is slightly different from the one in [2]. In fact, regarding single–resonance zone, the real domains where one applies the map Ψ_k are quite smaller w.r.t. to the ones in [2].

Definition 1.2.1.

$$\tilde{\mathcal{R}}^0 = Re\ (\mathcal{R}^0_{r'_0/2}) \qquad \tilde{\mathcal{R}}^{1,k} = Re\ (\mathcal{R}^{1,k}_{\check{r}_k}), \qquad k \in \mathcal{G}^n_{\mathtt{K}_0} \tag{1.2.42}$$

Lemma 1.2.4 (Second covering lemma)**.**

$$\begin{aligned} &i)\ \mathcal{R}^0\times\mathbb{T}^n\subseteq\Psi_0(\tilde{\mathcal{R}}^0\times\mathbb{T}^n)\\ &ii)\ \mathcal{R}^{1,k}\times\mathbb{T}^n\subseteq\Psi_k(\tilde{\mathcal{R}}^{1,k}\times\mathbb{T}^n),\qquad\forall\ k\in\mathcal{G}^n_{\mathrm{K}_0}\\ &iii)\ \mathcal{R}^2:=D\backslash(\mathcal{R}^0\cup\mathcal{R}^1)\subseteq\bigcup_{k\in\mathcal{G}^n_{\mathrm{K}_0}}\ \bigcup_{\substack{\ell\in\mathcal{G}^n_{\mathrm{K}}\\ \ell\notin\mathbb{Z}k}}\mathcal{R}^2_{k,\ell}\end{aligned}\tag{1.2.43}$$

where $\mathcal{R}^2_{k,\ell}$ is the pull back of the following set in frequency space

$$\Omega^2_{k,\ell}:=\{|\omega\cdot k|<\alpha\}\cap\{|p_k^\perp\omega|<M\}\cap\{|p_k^\perp\omega\cdot\ell|\leqslant\frac{3\alpha\mathrm{K}^{n+4}}{|k|}\}.\tag{1.2.44}$$

Remark 1.2.4. (i) Notice that from the definition of of $\widetilde{\mathcal{R}}^{1,k}$ in (2.6.1), one has that

$$\widetilde{\mathcal{R}}^{1,k}_{r_k'/2}\subseteq\mathcal{R}^{1,k}_{r_k'}\,.\tag{1.2.45}$$

(ii) Relations in (2.6.1) allow to map back the dynamics of the averaged Hamiltonians in 2.5.1 so as to describe the dynamics also *arbitrarily close to the boundary of the starting phase space*.

For the proof of the Covering Lemma we shall use the following immediate consequence of the Contraction Lemma[1]:

Lemma 1.2.5. *Fix $y_0\in\mathbb{R}^n$, $r>0$ and let $\phi:D_{2r}(y_0)\to\mathbb{C}^n$ be a real analytic map satisfying*

$$\sup_{D_{2r}(y_0)}|\phi(y)-y|\leqslant W\tag{1.2.46}$$

for some $0<W<r$. Then, $y_0\in\phi(\overline{B_r(y_0)})$.

Proof. Let $V_0:=\overline{B_r(0)}$. Solving the equation $\phi(y)=y_0$ for some $y\in\overline{B_r(y_0)}$ is equivalent to solve the fixed point equation $w=\psi_0(w):=-\psi(y_0+w)$ for $w\in V_0$ having set $\psi(y):=\phi(y)-y$. By (1.2.46) it follows that $\psi_0:V_0\to V_0$ and by the mean value theorem and Cauchy estimates we get that, for every $w,w'\in V_0$,

$$|\psi_0(w)-\psi_0(w')|=|\psi(y_0+w)-\psi(y_0+w')|\leqslant\frac{M}{r}|w-w'|\,,$$

showing that ψ_0 is a contraction on V_0 (since $M/r<1$) and the claim follows by the standard Contraction Lemma. □

[1] As usual $\overline{D}$ denotes the closure of the set D.

Proof. of Lemma 2.6.1 For $i)$ we start by proving that

$$\forall\ (y_0, x) \in \mathcal{R}^0 \times \mathbb{T}^n\,,\ \exists!\ (y, x_0) \in \widetilde{\mathcal{R}}^0 \times \mathbb{T}^n:\quad \Psi_{\rm o}(y, x) = (y_0, x_0)\,. \tag{1.2.47}$$

Define

$$W := \frac{r_{\rm o}}{2^7 \mathsf{K}_{\rm o}} \overset{(1.2.23)}{=} \frac{\alpha}{2^{11}\mathsf{K}_{\rm o}^2} < \frac{\alpha}{2^{10}\mathsf{K}_{\rm o}^2} =: r < \frac{\alpha}{2^7 \mathsf{K}_{\rm o}} \overset{(1.2.23)}{=} \frac{r'_{\rm o}}{4}\,. \tag{1.2.48}$$

Fix $(y_0, x) \in \mathcal{R}^0 \times \mathbb{T}^n$ and let $\phi(y) := \pi_y \Psi_{\rm o}(y, x)$. Then, by (1.2.48),

$$\sup_{D_{2r}(y_0)} |\phi(y) - y| \leqslant \sup_{D_{r'_{\rm o}}(y_0)} |\phi(y) - y| \leqslant |\pi_y \Psi_{\rm o} - y|_{r'_{\rm o}, s'_{\rm o}} \overset{(1.2.30)}{\leqslant} W\,.$$

Thus, by Lemma 1.2.5, since by (1.2.48) $2r < r'_{\rm o}/2$, by definition of $\widetilde{\mathcal{R}}^0$, we have that

$$y_0 \in \pi_y \Psi_{\rm o}\big(\overline{B_r(y_0)} \times \{x\}\big) \subseteq \pi_y \Psi_{\rm o}\big(\widetilde{\mathcal{R}}^0 \times \{x\}\big)\,,$$

which implies that $\Psi_{\rm o}(y, x) = (y_0, x_0)$ with $x_0 \in \mathbb{T}^n$ proving (1.2.47). Now, observe that the map $(y_0, x) \in \mathcal{R}^0 \times \mathbb{T}^n \mapsto (y, x_0) \in \widetilde{\mathcal{R}}^0 \times \mathbb{T}^n$ in (1.2.47) is nothing else than the diffeomorphism associated to the near–to–identity generating function $y_0 \cdot x + \Psi_0(y_0, x)$ of the near–to–identity symplectomorphism $\Psi_{\rm o}$. Thus, for each $y_0 \in \mathcal{R}^0$, the map $x \in \mathbb{T}^n \mapsto x_0 = x + \partial_{y_0}\Psi_0(y_0, x)$ is a diffeomorphism of $\mathbb{T}^n$ with inverse given by $x_0 \in \mathbb{T}^n \mapsto x = x_0 + \chi(y_0, x_0)$ for a suitable (small) real analytic map χ. Therefore, given $(y_0, x_0) \in \mathcal{R}^0 \times \mathbb{T}^n$, if we take $x = x_0 + \chi(y_0, x_0)$ in (1.2.47) we obtain that there exist $(y, x) \in \widetilde{\mathcal{R}}^0 \times \mathbb{T}^n$ such that $(y_0, x_0) = \Psi_0(y, x)$, proving point $i)$.

For point $ii)$ the strategy is the same. We start by proving that

$$\forall\ k \in \mathcal{G}^n_{\mathsf{K}_{\rm o}}\,,\ \forall\ (y_0, x) \in \mathcal{R}^{1,k} \times \mathbb{T}^n\,,\ \exists!\ (y, x_0) \in \widetilde{\mathcal{R}}^{1,k} \times \mathbb{T}^n:\quad \Psi_k(y, x) = (y_0, x_0)\,. \tag{1.2.49}$$

Fix $k \in \mathcal{G}^n_{\mathsf{K}_{\rm o}}$ and define

$$W := \frac{r_k}{2^7 \mathsf{K}^{n+1}} \overset{(1.2.23)}{=} \frac{\alpha}{2^7 L\,|k|\,\mathsf{K}^{n+1}} < \frac{\alpha}{2^6 L\,|k|\,\mathsf{K}^{n+1}} =: r < \frac{\check{r}_k}{2} \overset{(1.2.23)}{=} \frac{\alpha}{4L\,|k|\,\mathsf{K}^{n+1}}\,. \tag{1.2.50}$$

Fix $(y_0, x) \in \mathcal{R}^{1,k} \times \mathbb{T}^n$, and let $\phi(y) := \pi_y \Psi_k(y, x)$. By (1.2.50) and using that $4\check{r}_k < r'_k$ one has

$$\sup_{D_{2r}(y_0)} |\phi(y) - y| \leqslant \sup_{D_{2\check{r}_k}(y_0)} |\phi(y) - y| \leqslant |\pi_y \Psi_k - y|_{2\check{r}_k, s_*} \leqslant |\pi_y \Psi_k - y|_{r'_k, s_*} \overset{(1.2.30)}{\leqslant} M\,.$$

Thus, by Lemma 1.2.5 we have

$$y_0 \in \pi_y \Psi_k\big(\overline{B_r(y_0)} \times \{x\}\big) \subseteq \pi_y \Psi_k\big(\widetilde{\mathcal{R}}^{1,k} \times \{x\}\big)\,,$$

which implies that $\Psi_k(y,x) = (y_0, x_0)$ for some $x_0 \in \mathbb{T}^n$ proving (1.2.49). Now, observe that the map $(y_0, x) \in \mathcal{R}^{1,k} \times \mathbb{T}^n \mapsto (y, x_0) \in \widetilde{\mathcal{R}}^{1,k} \times \mathbb{T}^n$ in (1.2.49) is nothing else than the diffeomorphism associated to the near–to–identity generating function $y_0 \cdot x + \Psi_k(y_0, x)$ of the near–to–identity symplectomorphism Ψ_k. Thus, for each $y_0 \in \mathcal{R}^{1,k}$, the map $x \in \mathbb{T}^n \mapsto x_0 = x + \partial_{y_0}\Psi_k(y_0, x)$ is a diffeomorphism of $\mathbb{T}^n$ with inverse given by $x_0 \in \mathbb{T}^n \mapsto x = x_0 + \chi(y_0, x_0)$ for a suitable (small) real analytic map χ. Therefore, given $(y_0, x_0) \in \mathcal{R}^{1,k} \times \mathbb{T}^n$, if we take $x = x_0 + \chi(y_0, x_0)$ in (1.2.49) we obtain that there exist $(y, x) \in \widetilde{\mathcal{R}}^{1,k} \times \mathbb{T}^n$ such that $(y_0, x_0) = \Psi_k(y, x)$, proving the second point of the lemma.

For the last point if $y \in \mathcal{R}^2$ then, since $y \notin \mathcal{R}^0$, there exists $k \in \mathcal{G}^n_{\mathtt{K}_o}$ such that $|\omega \cdot k| < \alpha$, in which case, since $y \notin \mathcal{R}^1$, there exists $\ell \in \mathcal{G}^n_{\mathtt{K}} \setminus \mathbb{Z}k$ such that $|\pi_k^\perp \omega \cdot \ell| \leqslant \frac{3\alpha \mathtt{K}^{n+4}}{|k|}$, hence $y \in \mathcal{R}^2_{k,\ell}$ for some $k \in \mathcal{G}^n_{\mathtt{K}_o}$ and $\ell \in \mathcal{G}^n_{\mathtt{K}} \setminus \mathbb{Z}k$.

□

Normal form Theorem In the normal form around simple resonances the 'averaged Hamiltonian' g^k in 2.5.12 depends on angles through the linear combination $k \cdot x$, which, since $k \in \mathcal{G}^n$, defines a new well–defined angle $x_1 \in \mathbb{T}$. This fact calls for a linear symplectic change of variables, that can be possible thanks to the following conseguence of Bezout's lemma:

Lemma 1.2.6. *Let the hypotheses of theorem 2.5.1 hold.*
i) For any $k \in \mathcal{G}^n$ there exists a matrix $\widehat{A} \in \mathbb{Z}^{n-1\times n}$ such that

$$\begin{gathered} A := \begin{pmatrix} k \\ \widehat{A} \end{pmatrix} = \begin{pmatrix} k_1 \dots k_n \\ \widehat{A} \end{pmatrix} \in SL(n, \mathbb{Z}), \\ |\widehat{A}|_\infty \leqslant |k|_\infty, \quad |A|_\infty = |k|_\infty, \quad |A^{-1}|_\infty \leqslant (n-1)^{\frac{n-1}{2}} |k|_\infty^{n-1}. \end{gathered} \tag{1.2.51}$$

ii) Let Φ_0 be the linear, symplectic map on $\mathbb{R}^n \times \mathbb{T}^n$ onto itself befined by

$$\Phi_0 : (\mathtt{y}, \mathtt{x}) \longmapsto (y, x) = (A^T \mathtt{y}, A^{-1}\mathtt{x}). \tag{1.2.52}$$

Then,

$$\begin{cases} \mathtt{x}_1 = k \cdot x \\ \mathtt{x}_i = \sum_j \widehat{A}_{ij} x_j \ \forall i = 2, \dots, n \end{cases} \qquad \begin{cases} y = \mathtt{y}_1 k + \widehat{A}^T \hat{\mathtt{y}} \\ \hat{\mathtt{y}} := (\mathtt{y}_2, \dots, \mathtt{y}_n) \end{cases} \tag{1.2.53}$$

moreover, letting

$$\mathscr{D}^k := A^{-T} \tilde{\mathcal{R}}^{1,k}, \qquad \begin{cases} \tilde{r}_k := \frac{r_k}{c_1 |k|} \\ \tilde{s}_k := \frac{s}{c_1 |k|^{n-1}} \end{cases}, \qquad c_1 := 5n(n-1)^{\frac{n-1}{2}} \tag{1.2.54}$$

we find

$$\Phi_0 : \mathscr{D}^k_{\tilde r_k} \times \mathbb{T}^n_{\tilde s_k} \longmapsto \tilde{\mathcal{R}}^{1,k}_{r'_k/2} \times \mathbb{T}^n_{s_\star/2}, \qquad \Phi_0(\mathscr{D}^k \times \mathbb{T}^n) = \tilde{\mathcal{R}}^{1,k} \times \mathbb{T}^n. \tag{1.2.55}$$

Proof. *i)* From Bezout lemma it follows that: *given* $k \in \mathbb{Z}^n$, $k \neq 0$ *there exists a matrix* $A = (A_{ij})_{1\leqslant i,j\leqslant n}$ *with integer entries such that* $A_{nj} = k_j$ $\forall$ $1 \leqslant j \leqslant n$, $\det A = \gcd(k_1, ...k_1) = 1$ $(k \in \mathcal{G}^n)$, *and* $|A|_\infty = |k|_\infty$.

ii) Φ_0 is symplectic since it is generated by the generating function $\mathtt{y} \cdot Ax$. Furthermore, $\mathtt{y} \in \mathscr{D}^k_{\tilde r_k}$ if and only if $\mathtt{y} = \mathtt{y}_0 + z$ with $\mathtt{y}_0 \in \mathscr{D}^k$ and $|z| < \tilde r_k$. Thus,

$$|A^T z| \leqslant n|k||z| < n|k|\tilde r_k < \frac{r'_k}{2}. \tag{1.2.56}$$

Since $\mathtt{y}_0 \in \mathscr{D}^k \Longrightarrow A^T\mathtt{y}_0 \in \tilde{\mathcal{R}}^{1,k}$, we have that $A^T\mathtt{y} \in \tilde{\mathcal{R}}^{1,k}_{r'_k/2}$.

Regarding the $\mathtt{x}$ variable, we notice that, for any $1 \leqslant j \leqslant n$, since $\mathtt{x} \in \mathbb{T}^n_{\tilde s_k}$

$$|Im(A^{-1}\mathtt{x})_j| = |\sum_i Im(A^{-1})_{ji}\mathtt{x}_i| < n(n-1)^{\frac{n-1}{2}}|k|^{n-1}\tilde s_k \leqslant \frac{s_\star}{2} < s'_\star. \tag{1.2.57}$$

■

Finally we are ready to state the most important theorem of this section that will allow us to work with Hamiltonian in simpler form (especially near semple risonance).

Theorem 1.2.2 (Normal form Theorem)**.** *Let* H_ε *be as in* 1.1.1 *with* $f \in \mathbb{G}^n_s$ *with the cutoff function* $\mathtt{N}$ *as in definition* 1.1.10, *and let* 1.2.23, 1.2.22 *hold. There exists a constant* $c_0 = c_0(n, s, \delta) \geqslant \max\{\mathtt{N}, b_0\}$ *such that, if* $\mathtt{K}_0 \geqslant c_0$, $k \in \mathcal{G}^n_{\mathtt{K}_0}$, *and* $\mathscr{D}^k$, $\tilde r_k, \tilde s_k$ *as in* 1.2.54, *then there exist real-analytic symplectic maps*

$$\Psi_0 : \mathcal{R}^0_{r'_0} \times \mathbb{T}^n_{s'_0} \mapsto \mathcal{R}^0_{r_0} \times \mathbb{T}^n_{s_0}, \qquad \Psi^k := \Psi_k \circ \Phi_0 : \mathscr{D}^k_{\tilde r_k} \times \mathbb{T}^n_{\tilde s_k} \mapsto \mathcal{R}^{1,k}_{r_k} \times \mathbb{T}^n_{s_\star} \tag{1.2.58}$$

where Ψ_k *is defined in* 2.5.11 *and* Φ_0 *in* 1.2.52, *having the following proprieties.*
i)

$$H_0(y, x) = H_\varepsilon \circ \Psi_0(y, x) := h(y) + \varepsilon g^0(y) + \varepsilon f^0(y, x), \tag{1.2.59}$$

with g^0 *and* f^0 *satisfying* 2.5.19 *and* $\langle f^0\rangle = 0$.
ii) H_k *in* 2.5.12, *in the symplectic variables* $(\mathtt{y}, \mathtt{x})$ *takes the form*

$$\begin{aligned} &\mathcal{H}_k(\mathtt{y}, \mathtt{x}) := H_k \circ \Phi_0(\mathtt{y}, \mathtt{x}) = \bar{\mathtt{H}}_k(\mathtt{y}, \mathtt{x}_1) + \varepsilon \mathtt{f}^k(\mathtt{y}, \mathtt{x}), \qquad (\mathtt{y}, \mathtt{x}) \in \mathscr{D}^k_{\tilde r_k} \times \mathbb{T}^n_{\tilde s_k} \\ &\mathtt{f}^k(\mathtt{y}, \mathtt{x}) := f^k(A^T\mathtt{y}, A^{-1}\mathtt{x}). \end{aligned} \tag{1.2.60}$$

where the "secular Hamiltonian"

$$\bar{\mathtt{H}}_k(\mathtt{y}, \mathtt{x}_1) := h^k(\mathtt{y}) + \varepsilon \mathtt{g}^k(\mathtt{y}, \mathtt{x}_1), \qquad h^k(\mathtt{y}) := h(A^T\mathtt{y}) + \varepsilon \mathtt{g}^k_0(\mathtt{y}); \tag{1.2.61}$$

is a real-analytic function for $\mathtt{y} \in \mathscr{D}^k_{\tilde{r}_k}$ *and* $\mathtt{x}_1 \in \mathbb{T}^1_{s'_k}$, *futhermore* $\mathtt{g}^k(\mathtt{y}, \cdot) \in \mathbb{B}^1_{s'_k}$ *and the following estimates hold:*

$$|\mathtt{g}^k_0|_{\tilde{r}_k} \leqslant \theta_0; \qquad \|\mathtt{g}^k - \pi_{\mathbb{Z}k} f\|_{\tilde{r}_k, s'_k} \leqslant \theta_0; \qquad \|\mathtt{f}^k\|_{\tilde{r}_k, \tilde{s}_k} \leqslant e^{-\mathtt{K}s/3}. \tag{1.2.62}$$

iii) if $|k|_1 \geqslant \mathtt{N}$, *there exists* $\theta_k \in [0, 2\pi)$ *such that*

$$\mathcal{H}_k = h(A^T \mathtt{y}) + \varepsilon \mathtt{g}^k_0(\mathtt{y}) + 2|f_k|\varepsilon[\cos(\mathtt{x}_1 + \theta_k) + F^k_\star(\mathtt{x}_1) + \mathtt{g}^k_\star(\mathtt{y}, \mathtt{x}_1) + \mathtt{f}^k_\star(\mathtt{y}, \mathtt{x})] \tag{1.2.63}$$

where $F^k_\star$ *is as in 1.1.24 and satisfies* $F^k_\star \in \mathbb{B}^1_1$ *and* $|F^k_\star|_1 \leqslant 2^{-40}$. *Moreover* $\mathtt{g}^k_\star(\mathtt{y}, \cdot) \in \mathbb{B}^1_1$ *for every* $\mathtt{y} \in \mathscr{D}^k_{\tilde{r}_k}$, $\pi_{\mathbb{Z}k} \mathtt{f}^k_\star = 0$ *and one has*

$$\|\mathtt{g}^k_\star\|_{\tilde{r}_k, 1} \leqslant \theta, \quad \|\mathtt{f}^k_\star\|_{\tilde{r}_k, \tilde{s}_k} \leqslant e^{-\mathtt{K}s/7}. \tag{1.2.64}$$

Proof. First relation in 1.2.58 is 2.5.8, for the second one we only have to remember that $\frac{s_\star}{2} \leqslant s'_\star$ and the thesis comes from the definition of Ψ_0 and Ψ_k in the previous results.

i) Follows from part *a)* of theorem 2.5.1.

ii) Equations 1.2.60,1.2.61 follow immediately from the definition of the symplectic map Φ_0 in the prevoius lemma that acts on 2.5.12. Estimates in 1.3.29 comes from 2.5.20 calling

$$\mathtt{g}^k_0(\mathtt{y}) := g^k_0(\mathrm{A}^T \mathtt{y}), \qquad \mathtt{g}^k(\mathtt{y}, \mathtt{x}_1) := g^k(\mathrm{A}^T \mathtt{y}, \mathtt{x}_1). \tag{1.2.65}$$

iii) follows directly by proposition1.1.1 and lemma 1.2.3 with the definition of $\mathtt{y}$ and $\mathtt{x}$ and the notation

$$\mathtt{g}^k_\star := \frac{1}{2|f_k|}\left(\mathtt{g}^k - \pi_{\mathbb{Z}k} f\right), \qquad \mathtt{f}^k_\star := \frac{1}{2|f_k|}\bar{f}^k \tag{1.2.66}$$

noting that, w.r.t. lemma 1.2.3 $\mathtt{g}^k_\star(\mathtt{y}, \mathtt{x}_1) = g^k_\star(\mathrm{A}^T \mathtt{y}, \mathtt{x}_1)$ and that $\mathtt{f}^k_\star(\mathtt{y}, \mathtt{x}) = f^k_\star(\mathrm{A}^T \mathtt{y}, \mathrm{A}^{-1}\mathtt{x})$.
■

Remark 1.2.5. (i) Beware that, while $\Psi_{\rm o}$ is a map close to the identity, Ψ^k is not, as it is the composition of a linear transformation[2] with a near–to–identity map.
(ii) Point (iii) in Theorem 1.2.2 shows that the secular Hamiltonian $\overline{\mathtt{H}}_k$ (obtained disregarding the exponentially small perturbation $f^k_\star$) has a potential, which is $O(1/\mathtt{K}^{5n})$–perturbation of the 'cosine–like function'

$$\cos(\mathtt{x}_1 + \theta_k) + F^k_\star(\mathtt{x}_1), \qquad \text{where} \quad |F^k_\star|_1 \leqslant 2^{-40}.$$

[2]Namely, the symplectic transformation, the generating function of which is given by $\mathtt{y} \cdot \mathrm{A}x$, and which maps the resonant combination $k \cdot x$ to the 'resonant' angle $\mathtt{x}_1$.

This means that for $|k|_1 \geqslant \mathtt{N}$, *the secular Hamiltonians at simple resonances all look the same*, allowing, in particular, for a uniform analysis in terms of action–angle variables (compare § 1.4 below).

Notice also that the perturbation $f^k_\star$, which is bounded by $e^{-\mathtt{K}s/7}$ has a factor $|f_k|$ in front of it and that such a factor, in turn, may be exponentially small (since $|f_k| \sim e^{-|k|_1 s}$ for large $|k|_1$).

(iii) For later use we observe that[3]

$$\mathtt{K}_{\rm o} \geqslant \mathtt{N} \geqslant 2\mathtt{c}_s\,, \qquad \text{where} \qquad \mathtt{c}_s := \max\{1, 1/s\}\,. \tag{1.2.67}$$

In this way near non-resonant zone we average out non-resonant Fourier modes of f, in order to apply KAM theory with exponentially small perturbation. Near simple resonance, instead, we are now ready to study our secular hamiltonian, in particular we would like to analyze the action-angles variables of these functions and their analytic structure.

1.3 Conjugations near simple resonance

We want firstly to analyze integrable part of the secular Hamiltonian that is

$$h^k(\mathtt{y}) = h(y) + \varepsilon \mathtt{g}^k_0(\mathtt{y}) \qquad \text{real analytic on } \mathscr{D}^k_{\tilde r_k}. \tag{1.3.1}$$

Lemma 1.3.1. *h^k in (1.3.1) satisfies the following twist property*

$$\inf_{\mathtt{y}\in\mathscr{D}^k} \left|\partial^2_{\mathtt{y}_1}[h^k(\mathtt{y})]\right| \geqslant \gamma_k := \frac{\gamma |k|_1^2}{2n} > 0. \tag{1.3.2}$$

Proof. The proof is based on the fact that h is a convex function and using Cauchy estimates. Indeed by construction

$$\partial^2_{\mathtt{y}_1}[h^k(\mathtt{y})] = \sum_{i,j}\Big((A^T)_{i,1}\ (A^T)_{j,1}\ \partial_{\mathtt{y}_i\mathtt{y}_j} h(y) + \varepsilon (A^T)_{i,1}\ (A^T)_{j,1}\ \partial_{\mathtt{y}_i\mathtt{y}_j} g^k_0(y)\Big). \tag{1.3.3}$$

For the second part we use Cauchy estimates to obtain

$$\begin{aligned} \sup_{\mathtt{y}\in\mathscr{D}^k} |\sum_{i,j}(A^T)_{i,1}\ (A^T)_{j,1}\ \partial_{\mathtt{y}_i\mathtt{y}_j} g^k_0(y)| &\leqslant |A^T|^2_\infty |\partial_{\mathtt{y}_i\mathtt{y}_j} g^k_0|_{\tilde r_k/2} \leqslant |k|^2_\infty 2 |g^k_0|_{\tilde r_k} (\frac{\tilde r_k}{2})^{-2} \leqslant \\ &\leqslant |k|^2_\infty 8\theta_0 \frac{c_1^2 L^2 |k|^4}{\varepsilon \mathtt{K}^{2\nu}} \leqslant \frac{8\ c(n) c_1^2 L^2}{\varepsilon \mathtt{K}^{15n-1}} \end{aligned} \tag{1.3.4}$$

[3] If $s \geqslant 1$ then $\mathtt{N} \geqslant 2 \geqslant 2/s$, while if $s < 1$ then the logarithm in (1.1.10) is larger than one, so that $\mathtt{N} \geqslant 2/s$ also in this case.

while for the first part we use $\gamma-$convexity of h and the form of the matrix A in 1.2.51

$$\inf_{y\in\mathscr{D}^k} |\sum_{i,j} A_{1,i}\ A_{1,j}\ \partial_{\mathtt{y}_i\mathtt{y}_j} h(y)| = \inf_{y\in\mathscr{D}^k} |\sum_{i,j} k_i\ k_j\ \partial_{\mathtt{y}_i\mathtt{y}_j} h(y)| \geqslant \gamma|k|_2^2 \geqslant \frac{\gamma|k|_1^2}{n}. \tag{1.3.5}$$

where we have used the standard norm equivalence $|k|_1 \leqslant \sqrt{n}|k|_2$. In this way we have obtained, with an inverse triangular inequality:

$$\begin{aligned}
\inf_{\mathtt{y}\in\mathscr{D}^k} |\partial^2_{\mathtt{y}_1} h^k| &= \sum_{i,j} \left|(A^T)_{i,1}\ (A^T)_{j,1}\ \partial_{\mathtt{y}_i\mathtt{y}_j} h(y) + \varepsilon (A^T)_{i,1}\ (A^T)_{j,1}\ \partial_{\mathtt{y}_i\mathtt{y}_j} g_0^k(y)\right| \\
&\geqslant \inf_{\mathscr{D}^k} \left|\sum_{i,j} (A^T)_{i,1}\ (A^T)_{j,1}\ \partial_{\mathtt{y}_i\mathtt{y}_j} h(y)\right| - \sup_{\mathscr{D}^k} \left|\sum_{i,j} \varepsilon (A^T)_{i,1}\ (A^T)_{j,1}\ \partial_{\mathtt{y}_i\mathtt{y}_j} g_0^k(y)\right| \\
&\geqslant \frac{\gamma|k|^2}{n} - \frac{8\ c(n) c_1^2 L^2}{\mathtt{K}^{15n-1}} \geqslant \frac{\gamma|k|^2}{2n} =: \gamma_k
\end{aligned} \tag{1.3.6}$$

choosing $\mathtt{K}$ sufficiently big such that $\frac{8\ c(n) n c_1^2 L^2}{\gamma|k|^2 \mathtt{K}^{15n-1}} \leqslant \frac{1}{2}$. ■

Definition 1.3.1. *We call "critical points" of h^k the $n-1$ dimensional elements of the following set*

$$\mathtt{Z}_k := \left\{\mathtt{y} \in \mathscr{D}^k :\ \partial_{\mathtt{y}_1}[h^k(\mathtt{y})] = \sum_{i=1}^{n} \left(A_{1,i}\ \partial_{\mathtt{y}_i} h(y) + \varepsilon A_{1,i}\ \partial_{\mathtt{y}_i} g_0^k(y)\right) = 0\right\}. \tag{1.3.7}$$

Remark 1.3.1. This set $\mathtt{Z}_k$ is a closed set because it is the preimage of a closed set $(\{0\})$ via a continuous function. Moreover, by definition $\mathcal{R}^{1,k} \subset B$ is compact, so also $\mathscr{D}^k_{\tilde{r}_k}$ is compact, and $\mathtt{Z}_k \subset \mathscr{D}^k_{\tilde{r}_k}$ is a compact set.

Remark 1.3.2. In order to stress the notation that we are going to use for the rest of the work, in accordance with lemma 1.2.6 we are using

$$y = A^T \mathtt{y},\ \text{such that } y \in \widetilde{\mathcal{R}}^{1,k} \Longleftrightarrow \mathtt{y} \in \mathscr{D}^k\,. \tag{1.3.8}$$

Furthermore, we are going to use the following convention: given a vector $y \in \mathbb{R}^n$, we will denote by $\hat{y} = (y_2, ..., y_n)$ the vector of the last $n-1$ coordinates such that $y = (y_1, \hat{y})$.

Remark 1.3.3. Using the Implicit Function Theorem, for every $\bar{\mathtt{y}} \in \mathtt{Z}_k$, one can find two neighborhoods of radii $0 < \rho_1, \rho_2 \leqslant \tilde{r}_k/8$ and a function η_k real analytic on $|\hat{y} - \hat{\bar{y}}| \leqslant$

ρ_2 such that for every $|y_1 - \bar{y}_1| \leqslant \rho_1$ one has $y_1 = \eta_k(\hat{y})$. Moreover, the function η_k representing the first component of the surface is unique in this neighborhood. Quantitative details regarding analyticity radii of the implicit function will be given in the proof of Proposition 1.3.1.

Lemma 1.3.2. *In $\mathscr{D}^k_{\tilde{r}_k}$ there exists a finte number of η_k.*

Proof. It is a simple application of implicite function theorem. In fact, if we assume that this cardinality is infinite, one can take a sequence $\{\eta^j_k\}_{j\in\mathbb{N}} \subset \mathtt{W}_k$, that would admit a convergent subsequence in $[-R, R] \ni y_1$ for the topological propriety of compact sets. This implies a contradiction with respect to implicit function theorem, which state that the local solution of $\partial_{\mathtt{y}_1} h^k = 0$ is unique in a small neighborhood of each point in $\mathtt{Z}_k$. So it is impossible to have accumulation of critical functions. ∎

Furthermore, h^k is convex, so there is only a single critical function η_k that is globally defined on $\widehat{\mathscr{D}^k}$. For Implicit function Theorem, one can fix a neighborhood where η_k represent the unique solutions to the equation $\partial_{\mathtt{y}_1} h^k = 0$. In this way one can associate a critical point $\bar{\mathtt{y}}$ to the couple $(\eta_k, \widehat{\bar{y}})$.

Instead of considering a critical point, we consider the graph of its associated critical function:

Definition 1.3.2. *We call "critical surface" associated to critical function η_k of h^k the following set:*

$$\mathcal{S}_k = \left\{ \mathtt{y} \in \mathtt{Z}_k : y = (\eta_k(\widehat{y}), \widehat{y});\ \textit{i.e. the graph of } \eta_k \right\}. \tag{1.3.9}$$

Remark 1.3.4. In $\mathscr{D}^k$ there is obviously only one critical surface $\mathcal{S}^k$.

Now we have to enlarge the non–resonant zone (and restrict the single–resonance region) introducin a scaling constant $\mathtt{C}$ on the bound on small divisors.

Definition 1.3.3. *Let h be KAM non–degenerate and let $\omega(y)$ denote its gradient. For $k \in \mathcal{G}^n_{1,\mathtt{K}_0}$ We denote by*

$$\begin{aligned}
\mathfrak{R}^0 &:= \{y \in B : |\omega(y) \cdot k| \geqslant \frac{\alpha}{2\mathtt{C}} \forall\ 0 < |k|_1 \leqslant \mathtt{K}_0\,.\} \\
\mathfrak{R}^{1,k} &:= \{y \in B : |\omega(y) \cdot k| \leqslant \frac{\alpha}{\mathtt{C}};\ |\omega(y) \cdot \ell| \geqslant \frac{3\alpha \mathtt{K}^{(n+4)}}{|k|_1},\ \forall\ \ell \in \mathbb{Z}^n,\ \ell \notin \mathbb{Z}k,\ |\ell|_1 \leqslant \mathtt{K}.\} \\
\mathfrak{R}^2 &:= B \backslash \big(\mathfrak{R}^0 \cup \bigcup_{k \in \mathcal{G}^n_{\mathtt{K}_0}} \mathfrak{R}^{1,k}\big)
\end{aligned} \tag{1.3.10}$$

where $\mathtt{C} = \mathtt{C}(n, L, \gamma) = \frac{12 c_1 n L}{\gamma} \geqslant 1$ *is a constant, and consequently we define*

$$\tilde{\mathfrak{R}}^{1,k} = Re\,(\mathfrak{R}^{1,k}_{\check{r}_k})\,; \qquad \check{\mathscr{D}}^k := A^{-T}\tilde{\mathfrak{R}}^{1,k}\,; \qquad \check{r}_k = \frac{r_k}{2\,\mathtt{K}^{n+1}}\,. \tag{1.3.11}$$

The choice of $\mathtt{C}$ *will be clarified in lemma 1.3.4.*

Remark 1.3.5. As we have done in the previous section, it is straightforward to see that

$$\mathfrak{R}^2 \subseteq \bigcup_{k \in \mathcal{G}^n_{\mathtt{K}_{\mathrm{o}}}} \bigcup_{\substack{\ell \in \mathcal{G}^n_{\mathtt{K}} \\ \ell \notin \mathbb{Z}k}} \mathfrak{R}^2_{k,\ell}\,, \tag{1.3.12}$$

with

$$\mathfrak{R}^2_{k\ell} := \left\{ y \in \mathrm{B} : |\omega \cdot k| < \tfrac{\alpha}{\mathtt{C}};\ |\pi_k^{\perp}\omega \cdot \ell| \leqslant \tfrac{3\alpha \mathtt{K}^{n+4}}{|k|} \right\}, \qquad (k \in \mathcal{G}^n_{\mathtt{K}_{\mathrm{o}}}\,,\ \ell \in \mathcal{G}^n_{\mathtt{K}} \setminus \mathbb{Z}k)\,. \tag{1.3.13}$$

Remark 1.3.6. Averaging procedure is essentially the same. For the non resonant region $\mathfrak{R}^0$, it is $\alpha/2\mathtt{C}$ completely non–resonant, so the paramaters taken in (1.2.23) are the same except

$$r_0 \to \mathfrak{r}_0 := \frac{\alpha}{4\,L\,\mathtt{C}\,\mathtt{K}_{\mathrm{o}}}; \qquad r_0' \to \mathfrak{r}_0' := \frac{\mathfrak{r}_0}{2} \tag{1.3.14}$$

so that $\mathfrak{R}^0_{\bar{r}_0}$ is $\alpha/4\mathtt{C}$ completely non–resonant, and one can apply 1.2.1 with $f, B, r, \Lambda, \alpha, \mathbf{K}, s$ replaced respectively by $\varepsilon f, \mathfrak{R}^0, \mathfrak{r}_0, \{0\}, \frac{\alpha}{4\mathtt{C}}, \mathtt{K}_0, s_0$ obtaining

$$\begin{aligned} &\Psi_0 : \mathfrak{R}^0_{\mathfrak{r}_0'} \times \mathbb{T}^n_{s_0'} \mapsto \mathfrak{R}^0_{\mathfrak{r}_0} \times \mathbb{T}^n_{s_0} \qquad \text{such that} \\ H_0(y,x) &= H_\varepsilon \circ \Psi_0(y,x) := h(y) + \varepsilon g^0(y) + \varepsilon f^0(y,x) \quad \text{r.a. on } \mathfrak{R}^0_{\mathfrak{r}_0'} \times \mathbb{T}^n_{s_0'} \\ &|g^0|_{\mathfrak{r}_0'} \leqslant \theta_0, \qquad \|f^0\|_{\mathfrak{r}_0',s_0'} \leqslant e^{-\mathtt{K}_0 s/3}. \end{aligned} \tag{1.3.15}$$

Regarding the single–resonant zone, instead, second covering lemma 2.6.1 and normal form theorem 1.2.2 holds in the same way with same parameters because they depend on how the single–resonant zone is non–resonant modulo $\mathbb{Z}k$, and not on how the small divisors are near zero (that is the term with $\mathtt{C}$ that is different from the old one), so that there exists

$$\Psi^k := \Psi_k \circ \Phi_0 : \check{\mathscr{D}}^k_{\check{r}_k} \times \mathbb{T}^n_{\check{s}_k} \mapsto \mathfrak{R}^{1,k}_{r_k} \times \mathbb{T}^n_{s_*}; \quad \text{with} \quad \mathfrak{R}^{1,k} \times \mathbb{T}^n \subseteq \Psi_k(\tilde{\mathfrak{R}}^{1,k} \times \mathbb{T}^n) \tag{1.3.16}$$

such that, if H_k is the one in (2.5.12), in the symplectic variables $(\mathtt{y}, \mathtt{x})$ it takes the form

$$\begin{aligned} &\mathcal{H}_k(\mathtt{y},\mathtt{x}) := H_k \circ \Phi_0(\mathtt{y},\mathtt{x}) = \bar{\mathtt{H}}_k(\mathtt{y},\mathtt{x}_1) + \varepsilon \mathtt{f}^k(\mathtt{y},\mathtt{x}), \qquad \text{r.a. on } \check{\mathscr{D}}^k_{\check{r}_k} \times \mathbb{T}^n_{\check{s}_k} \\ &\bar{\mathtt{H}}_k(\mathtt{y},\mathtt{x}_1) := h^k(\mathtt{y}) + \varepsilon \mathtt{g}^k(\mathtt{y},\mathtt{x}_1) \end{aligned} \tag{1.3.17}$$

and (1.2.62) equivalently holds with $\mathscr{D}^k \to \check{\mathscr{D}}^k$.

Now the strategy is the following: thanks to the strong proprety of convexity, we can prove that the set $\check{\mathscr{D}}^k$ is a tubular neighborhood of the critical surface $\mathcal{S}^k$, that is the content of the following:

Lemma 1.3.3. *$\check{\mathscr{D}}^k$ is contained in a proper tubular neighborhood around $\mathcal{S}_k$, i.e. $\check{\mathscr{D}}^k \subseteq \mathscr{D}^k_\star$ where*

$$\mathscr{D}^k_\star := \{\mathrm{y} \in \mathbb{R}^n : \exists\ \bar{\mathrm{y}} \in \mathcal{S}_k :\ |\hat{\mathrm{y}} - \hat{\bar{\mathrm{y}}}| < \frac{\tilde{r}_k}{16};\ |\mathrm{y}_1 - \bar{\mathrm{y}}_1| < \frac{6^n\, c_1\, r_k}{2\,\mathtt{K}\,|k|} + \frac{3}{2}\frac{\alpha}{\mathtt{C}\,\gamma_k}\}$$

Proof. By definition, we know that if $y \in \mathfrak{R}^{1,k}$ thanks to the form of the matrix A in (1.2.6)

$$|\partial_y h(y)\cdot k| = |\partial_{\mathrm{y}_1} h(y)| < \frac{\alpha}{\mathtt{C}} \Longrightarrow |\partial_{\mathrm{y}_1} h^k(\mathrm{y})| \leqslant |\partial_{\mathrm{y}_1} h(y)| + \varepsilon |\partial_{\mathrm{y}_1} g_0^k(y)| < \frac{3}{2}\frac{\alpha}{\mathtt{C}}. \tag{1.3.18}$$

taking $\mathtt{K}$ big enough. But thanks to Lagrange theorem and twist of h^k (see (1.3.6)) we also notice that for a generic point $\bar{\mathrm{y}} \in \mathcal{S}_k$

$$\begin{aligned} \partial_{\mathrm{y}_1} h^k(\mathrm{y}) = \partial_{\mathrm{y}_1} h^k(\bar{\mathrm{y}}) + \partial^2_{\mathrm{y}_1} h^k(\tilde{\mathrm{y}})\cdot|\mathrm{y}_1 - \bar{\mathrm{y}}_1|, \quad \text{for } \tilde{\mathrm{y}} : \tilde{\mathrm{y}}_1 \in (\mathrm{y}_1, \bar{\mathrm{y}}_1),\ \text{such that} \\ |\partial_{\mathrm{y}_1} h^k(\mathrm{y})| = |\partial^2_{\mathrm{y}_1} h^k(\tilde{\mathrm{y}})\cdot|\mathrm{y}_1 - \bar{\mathrm{y}}_1|| \overset{(1.3.6)}{\geqslant} \gamma_k |\mathrm{y}_1 - \bar{\mathrm{y}}_1|\,, \end{aligned} \tag{1.3.19}$$

so that by (1.3.18)

$$A^T\mathrm{y} \in \mathfrak{R}^{1,k} \Longleftrightarrow \exists\ \bar{\mathrm{y}} \in \mathcal{S}_k : |\mathrm{y}_1 - \bar{\mathrm{y}}_1| < \frac{3}{2}\frac{\alpha}{\mathtt{C}\,\gamma_k}\,. \tag{1.3.20}$$

In this way we can characterise also our set of interest: if $y \in \tilde{\mathfrak{R}}^{1,k}$, or equivalently $\mathrm{y} \in \check{\mathscr{D}}^k$, for (1.3.11) there exists a point $\hat{y} \in \mathfrak{R}^{1,k}$ such that

$$|y - \hat{y}| \leqslant \check{r}_k \Rightarrow |\mathrm{y}_1 - \hat{\mathrm{y}}_1| \overset{(1.2.6)}{\leqslant} |A^{-1}|_\infty |y - \hat{y}| \overset{(1.2.6)}{\leqslant} c_1 |k|_1^{n-1} \check{r}_k \overset{(1.2.23)}{=} c_1 |k|_1^{n-1} \frac{r_k}{2\,\mathtt{K}^{n+1}} \overset{(1.2.22)}{\leqslant} \frac{6^n\, c_1\, r_k}{2\,\mathtt{K}\,|k|} \tag{1.3.21}$$

and so, using a triangular inequality we can say that

$$\mathrm{y} \in \check{\mathscr{D}}^k \Longleftrightarrow \exists\ \bar{\mathrm{y}} \in \mathcal{S}_k : |\mathrm{y}_1 - \bar{\mathrm{y}}_1| \leqslant |\mathrm{y}_1 - \hat{\mathrm{y}}_1| + |\hat{\mathrm{y}}_1 - \bar{\mathrm{y}}_1| < \frac{6^n\, c_1\, r_k}{2\,\mathtt{K}\,|k|} + \frac{3}{2}\frac{\alpha}{\mathtt{C}\,\gamma_k}\,. \tag{1.3.22}$$

For this reason we can define

$$\mathscr{D}^k_\star := \{\mathrm{y} \in \mathbb{R}^n : \exists\ \bar{\mathrm{y}} \in \mathcal{S}_k :\ |\hat{\mathrm{y}} - \hat{\bar{\mathrm{y}}}| < \frac{\tilde{r}_k}{16};\ |\mathrm{y}_1 - \bar{\mathrm{y}}_1| < \frac{6^n\, c_1\, r_k}{2\,\mathtt{K}\,|k|} + \frac{3}{2}\frac{\alpha}{\mathtt{C}\,\gamma_k}\} \text{ such that } \check{\mathscr{D}}^k \subseteq \mathscr{D}^k_\star. \quad \square \tag{1.3.23}$$

Our goal is to perform the classical action-angles variable transformation in which $h^k(\mathtt{y})+\varepsilon \mathtt{g}^k(\mathtt{y},\mathtt{x}_1)$ become integrable. But since we are near critical surface the situation is quite complicate because the first derivative of h^k is near zero. In order to overcome to this problem, one can conjugate the hamiltonian to a natural system with generic morse potential called *standard form* system.

In few words, a standard 1D–Hamiltonian (which depends on $(n-1)$ external parameters) is a one degree–of–freedom Hamiltonian system close to a natural system with a generic potential, which may be controlled essentially by *only one parameter*, namely, the parameter κ appearing in Eq. (1.3.27) below; here, 'essentially' means, roughly speaking, that κ governs the main scaling properties of the Hamiltonian $\overline{\mathtt{H}}_k$. What is particularly relevant is that the κ parameter of the secular Hamiltonians $\overline{\mathtt{H}}_k$ is shown to be *independent of* k. The uniformity in k of the scaling proprieties of the standard form Hamiltonians allows to analyze global analytic properties: for example, the action–angle map for standard Hamiltonians, as discussed in [3], depends only on the parameter κ and therefore can be used *simultaneously* for all the secular Hamiltonians $\overline{\mathtt{H}}_k$, allowing for a nearly–integrable description of H on $\mathcal{R}^{1,k}\times\mathbb{T}^n$ with uniformly exponentially small perturbations

1.3.1 Standard form near critical surfaces

Definition 1.3.4 (STANDARD FORM HAMILTONIANS). *Let $\widehat{D}\subseteq\mathbb{R}^{n-1}$ be a bounded domain, $\mathtt{R}>0$, and $D:=(-\mathtt{R},\mathtt{R})\times\widehat{D}$. We say that a real analytic Hamiltonian $\mathtt{H}_\flat$ is in "standard form" with respect to standard symplectic variables $(p_1,q_1)\in(-\mathtt{R},\mathtt{R})\times\mathbb{T}$ and "external actions" $\widehat{p}=(p_2,...,p_n)\in\widehat{D}$ if $\mathtt{H}_\flat$ has the form*

$$\mathtt{H}_\flat(p,q_1)=(1+\nu(p,q_1))p_1^2+G(\widehat{p},q_1), \tag{1.3.24}$$

where

- *ν and G are real analytic functions defined on, respectively, $D_r\times\mathbb{T}_\mathtt{s}$ and $\widehat{D}_\mathtt{r}\times\mathbb{T}_\mathtt{s}$ for some $0<\mathtt{r}\leqslant\mathtt{R}$ and $\mathtt{s}>0$;*
- *G has zero average and there exists a function $\bar{G}$ (the "reference potential") depending only on q_1 such that, for some $\beta>0$,*

$$\bar{G}\quad \textit{is}\quad \beta-\textit{Morse},\qquad \langle\bar{G}\rangle=0; \tag{1.3.25}$$

- *the following estimates hold:*

$$\begin{cases} \sup_{\mathbb{T}^1_{\mathtt{s}}} |\bar{G}| \leqslant \epsilon, \\ \sup_{\hat{D}_r \times \mathbb{T}^1_{\mathtt{s}}} |G - \bar{G}| \leqslant \epsilon\mu, \\ \sup_{\hat{D}_r \times \mathbb{T}^1_{\mathtt{s}}} |\nu| \leqslant \mu, \end{cases} \qquad \textit{for some} \quad 0 < \epsilon \leqslant \frac{\mathtt{r}^2}{2^{16}}, \quad 0 \leqslant \mu < 1. \quad (1.3.26)$$

We shall call $(\widehat{D}, \mathtt{R}, \mathtt{r}, \mathtt{s}, \beta, \epsilon, \mu)$ the analyticity characteristics of H with respect to the unpertubed potential $\bar{G}$.

Remark 1.3.7. (i) If $\mathtt{H}_\flat$ is in standard form, then β and ϵ satisfy the relation[4] $\epsilon/\beta \geqslant 1/2$. Furthermore, one can always fix a number $\kappa \geqslant 4$ so that:

$$1/\kappa \leqslant \mathtt{s} \leqslant 1\,, \qquad 1 \leqslant \mathtt{R}/\mathtt{r} \leqslant \kappa\,, \qquad 1/2 \leqslant \epsilon/\beta \leqslant \kappa\,. \qquad (1.3.27)$$

Such a parameter κ rules the main scaling properties of these Hamiltonians.

(ii) A Hamiltonian in standard form $\mathtt{H}_\flat$ has the analytic features of its reference natural Hamiltonian

$$\bar{\mathtt{H}}_\flat := p_1^2 + \bar{G}(q_1)\,.$$

In particular, for μ small with respect to $1/\kappa$, $\mathtt{H}_\flat$ has the same finite (because of analyticity) number of equilibria (which lie on the q_1 axis) of $\bar{G}$ and in the same relative order, which is also preserved by the corresponding critical energies; compare Lemma 1.4.2 below.

(iii) Hamiltonians in standard form are particularly suited for the analytic theory of action–angle variables (in neighborhoods of separatrices) as developed in [3], where the notion of Generic Standard Form has been introduced. Such action–angle variables will be reviewed below.

(iv) The smallness of the 'adimensional ratio' $\epsilon/\mathtt{r}^2$ in (1.3.26) is needed in the analytic theory of action-angle variables for Hamiltonians in standard form developed in [2], however the factor $1/2^{16}$ is rather arbitrary and not optimal.

In this part we want to study the symplectic transformation that we need to conjugate *secular hamiltonian* in 1.2.61 with a standard form hamiltonian according to definition 1.3.4, so we consider

$$\overline{\mathtt{H}}_k(\mathtt{y}, \mathtt{x}_1) := h^k(\mathtt{y}) + \varepsilon \mathtt{g}^k(\mathtt{y}, \mathtt{x}_1), \qquad h^k(\mathtt{y}) := h(y) + \varepsilon \mathtt{g}_0^k(\mathtt{y}); \qquad (1.3.28)$$

[4]By 1.3.26, $\beta \leqslant |\bar{G}(\theta_i) - \bar{G}(\theta_i)| \leqslant 2\max_{\mathbb{T}} |\bar{G}| \leqslant 2\epsilon$.

that is a real-analytic function on $(\mathtt{y}, \mathtt{x}_1) \in \mathscr{D}^k_{\tilde{r}_k} \times \mathbb{T}^1_{s'_k}$, futhermore $\mathtt{g}^k(\mathtt{y}, \cdot) \in \mathbb{B}^1_{s'_k}$ and the following estimates hold:

$$|\mathtt{g}^k_0|_{\tilde{r}_k} \leqslant \theta_0; \qquad \|\mathtt{g}^k - \pi_{\mathbb{Z}k} f\|_{\tilde{r}_k, s'_k} \leqslant \theta_0; \qquad \|\mathtt{f}^k\|_{\tilde{r}_k, \tilde{s}_k} \leqslant e^{-\mathtt{K}s/3}. \tag{1.3.29}$$

We want to perform this conjugation to a standard form in a properly neighborhood of critical surface of h^k. So let η_k be a critical function and let $\mathcal{S}_k$ its critical surface associated. Consider a critical point is this surface $\bar{\mathtt{y}} \in \mathcal{S}_k$. We need to be very careful on the dependence upon the resonant vector k. In order to control this dependence, we introduce new parameters depending on k:

$$\begin{aligned}
&\check{\mathtt{s}} := \begin{cases} s'_k, & \text{if } |k| < \mathtt{N} \\ 1, & \text{if } |k| \geqslant \mathtt{N} \end{cases}, \qquad \boldsymbol{\varrho} = \frac{\tilde{r}_k}{8}, \quad \mu = \frac{1}{\mathtt{K}^{6n+1}} \\
&\upbeta := \begin{cases} \frac{2\varepsilon\beta}{|k|^2}, & \text{if } |k|_1 < \mathtt{N} \\ \frac{2\varepsilon|f_k|}{|k|^2}, & \text{if } |k|_1 \geqslant \mathtt{N} \end{cases}, \qquad \mathtt{s} := \begin{cases} \min\{\frac{s}{2}, 1\}, & \text{if } |k| < \mathtt{N} \\ 1, & \text{if } |k| \geqslant \mathtt{N} \end{cases}, \\
&\chi_k := \begin{cases} 1, & \text{if } |k|_1 < \mathtt{N} \\ |f_k|, & \text{if } |k|_1 \geqslant \mathtt{N} \end{cases}, \qquad \text{such that} \qquad \epsilon = \mathtt{c}_s \frac{8\varepsilon}{|k|^2}\chi_k =: \mathtt{c}_s \varepsilon_k \chi_k\,; \\
&\mathtt{R} := \frac{6^n\, c_1\, r_k}{2\,\mathtt{K}\,|k|} + \frac{3}{2}\frac{\alpha}{\mathtt{C}\,\gamma_k} + \frac{\boldsymbol{\varrho}}{8} \qquad \mathtt{D}(\bar{\mathtt{y}}_1, \hat{\bar{\mathtt{y}}}) := \{\mathtt{y} \in \mathbb{R}^n : \ |\hat{\mathtt{y}} - \hat{\bar{\mathtt{y}}}| \leqslant \boldsymbol{\varrho}; \quad |\mathtt{y}_1 - \bar{\mathtt{y}}_1| \leqslant \mathtt{R}\}\,.
\end{aligned} \tag{1.3.30}$$

Remark 1.3.8. (i) Since $|f_k| \leqslant 1$ one has:

$$|\chi_k| \leqslant 1\,. \tag{1.3.31}$$

(ii) Since $(1 - \frac{1}{\mathtt{K}})^{-2} < 2$, by definition of s'_k in 1.2.23, one has

$$\mathtt{s} \leqslant 2\check{\mathtt{s}}\,. \tag{1.3.32}$$

With the following result, we cover the entire critical surface with a finite and controlled number of neighborhood $\mathtt{D}$ where we can perform our *standard form* conjugation. The main point is that in the real space this union of $\mathtt{D}$ cover the surface, while in the complex space it is contained in the complex neighborhood of $\check{\mathscr{D}}_k$ where $\overline{\mathtt{H}}_k$ is analytic.

Lemma 1.3.4. *One can find an integer value J and $\bar{\mathtt{y}}_1, \ldots \bar{\mathtt{y}}_J \in \mathtt{Z}_k$ such that the following hold*

$$i)\ \bigcup_{j=1}^{J} \mathtt{D}(\bar{\mathtt{y}}_{j,1}, \hat{\bar{\mathtt{y}}}_j) \supseteq \mathscr{D}^k_\star \supseteq \check{\mathscr{D}}^k \qquad ii)\ \bigcup_{j=1}^{J} \mathtt{D}_{3\boldsymbol{\varrho}}(\bar{\mathtt{y}}_{j,1}, \hat{\bar{\mathtt{y}}}_j) \subseteq \check{\mathscr{D}}^k{}_{\tilde{r}_k} \tag{1.3.33}$$

Proof. Let $\ell = \frac{\boldsymbol{\varrho}}{64}$, and we consider the grid formed by square $\mathcal{Q}_i$ of side ℓ with vertices in $k\ell, k \in \mathbb{Z}^n$. Now, since the $\boldsymbol{\varrho} > \ell$, we have just moved the estimates from the number of neighborhood to the number of square with non-empty intersection with $\mathcal{S}_k$. Moreover for implicit function theorem we know that in a right neighborhood of a critical point

$$\partial_{\mathtt{y}_1} h^k(\eta(\widehat{y}), \widehat{y}) = 0$$

and for fundamental calculus theorem

$$|\partial_{\mathtt{y}_1} h(y_1, \widehat{y})| = |\int_0^{y_1} \partial_t^2 h(t, \widehat{y}) dt|| \frac{\partial y_1}{\partial \mathtt{y}_1}| \geqslant \gamma |y_1||k_1| \tag{1.3.34}$$

so that thanks to convexity of h^k, and Cauchy estimates we have that

$$\gamma\, |\eta_k(\widehat{y})| \overset{(1.3.6)}{\leqslant} |\partial_{\mathtt{y}_1} h(y)| = \varepsilon |\partial_{\mathtt{y}_1} \mathtt{g}_0^k(\mathtt{y})| \Longrightarrow \sup_{\mathtt{y} \in \mathscr{D}^k_{\tilde{r}_k}} |\eta_k| \leqslant \frac{2\varepsilon_k \chi_k \mu}{\gamma\, \tilde{r}_k} \tag{1.3.35}$$

so we can finally estimates the squares with the product between the maximum number of squares crossed by $\mathcal{S}_k$ in $\mathtt{y}_1$ direction (which is estimated by the bound on the derivative of η_k times the diameter of the set in which it is defined) and the number of squares into which we have divided the $\mathtt{y}_2, \ldots \mathtt{y}_n$ (which is estimated by the ratio between diameter of the set and side of squares) direction:

$$J \leqslant 2 \left(\frac{\varepsilon_k \chi_k m}{\gamma\, \tilde{r}_k^2\, \mathtt{K}^{6n+1}} + 1 \right) \left(\frac{m}{\ell} \right)^{n-1} \tag{1.3.36}$$

where $m = \operatorname{diam} \widehat{\mathscr{D}^k_\star}$. In this way, using that $\frac{6^n\, c_1\, r_k}{2\,\mathtt{K}\,|k|} + \frac{3}{2} \frac{\alpha}{\mathtt{C}\,\gamma_k} + \frac{\boldsymbol{\varrho}}{8} > \frac{6^n\, c_1\, r_k}{2\,\mathtt{K}\,|k|} + \frac{3}{2} \frac{\alpha}{\mathtt{C}\,\gamma_k}$, by the definition of $\mathscr{D}^k_\star$ one can easily obtain

$$\mathscr{D}^k_\star \subseteq \bigcup_{j=1,\ldots,J} \mathtt{D}(\bar{\mathtt{y}}_{j,1}, \widehat{\bar{\mathtt{y}}}_j) \tag{1.3.37}$$

For the complex extensions in *ii*), on the $n-1$ "dumb" directions it comes directly from the fact that $4\boldsymbol{\varrho} = \tilde{r}_k/2 \leqslant \tilde{r}_k$, while for the first direction we have to be careful. For the imaginary part of this first action there are no problems because $3\boldsymbol{\varrho} \leqslant \tilde{r}_k = 8\boldsymbol{\varrho}$, while for the real part we need

$$\mathtt{R} + 3\boldsymbol{\varrho} \leqslant 8\boldsymbol{\varrho} \overset{(1.3.30)}{\Longleftrightarrow} \frac{6^n\, c_1\, r_k}{2\,\mathtt{K}\,|k|} + \frac{3}{2} \frac{\alpha}{\mathtt{C}\,\gamma_k} + \frac{\boldsymbol{\varrho}}{8} + 3\boldsymbol{\varrho} \leqslant 8\boldsymbol{\varrho} \tag{1.3.38}$$

that is equivalent to ask that

$$\frac{3}{2} \frac{\alpha}{\mathtt{C}\,\gamma_k} \leqslant \frac{39}{8} \boldsymbol{\varrho} - \frac{6^n\, c_1\, r_k}{2\,\mathtt{K}\,|k|_1} \overset{(1.3.30)}{=} \frac{39}{64} \frac{r_k}{c_1 |k|_1} - \frac{6^n\, c_1\, r_k}{2\,\mathtt{K}\,|k|_1} \tag{1.3.39}$$

taking $\mathtt{K} \geqslant \frac{32}{23} c_1^2 6^n$ one can obtain

$$\frac{3\alpha n}{\mathtt{C}\,\gamma |k|_1^2} \overset{(1.3.2)}{=} \frac{3}{2} \frac{\alpha}{\mathtt{C}\,\gamma_k} \leqslant \frac{1}{4} \frac{r_k}{c_1 |k|_1} = \frac{1}{4} \frac{\alpha}{L\, c_1\, |k|_1^2} \iff \frac{3\alpha n}{\mathtt{C}\,\gamma |k|_1^2} \leqslant \frac{1}{4} \frac{\alpha}{L\, c_1\, |k|_1^2} \tag{1.3.40}$$

that is satisfied taking $\mathtt{C} \geqslant \dfrac{12 c_1 n L}{\gamma}$. □

Before starting the main proposition of this section, we have to define a special class of diffeomorphisms that will be useful for our intent.

Definition 1.3.5. *Given a domain $\hat{\mathrm{D}} \subseteq \mathbb{R}^{n-1}$, we denote by $\mathfrak{G}_\dagger$ the abelian group of symplectic diffeomorphisms $\Psi_{\mathtt{g}}$ of $(\mathbb{R} \times \hat{\mathrm{D}}) \times \mathbb{R}^n$ given by*

$$(p, q) \in (\mathbb{R} \times \hat{\mathrm{D}}) \times \mathbb{R}^n \overset{\Psi_{\mathtt{g}}}{\mapsto} (P, Q) = (p_1 + \mathtt{g}(\hat{p}), \hat{p}, q_1, \hat{q} - q_1 \partial_{\hat{p}} \mathtt{g}(\hat{p})) \in (\mathbb{R} \times \hat{\mathrm{D}}) \times \mathbb{R}^n\,, \tag{1.3.41}$$

with $\mathtt{g} : \hat{\mathrm{D}} \to \mathbb{R}$ smooth.

Remark 1.3.9. The group properties of $\mathfrak{G}_\dagger$ are trivial:

$$\mathrm{id}_{\mathfrak{G}_\dagger} = \Psi_0\,, \qquad \Psi_{\mathtt{g}}^{-1} = \Psi_{-\mathtt{g}}\,, \qquad \Psi_{\mathtt{g}} \circ \Psi_{\mathtt{g}'} = \Psi_{\mathtt{g}+\mathtt{g}'}\,. \tag{1.3.42}$$

Notice that, unless $\partial_{\hat{p}} \mathtt{g} \in \mathbb{Z}^{n-1}$, maps $\Psi_{\mathtt{g}} \in \mathfrak{G}_\dagger$ *do not induce well defined maps*[5]

$$q \in \mathbb{T}^n \mapsto (q_1, \hat{q} - q_1 \partial_{\hat{p}} \mathtt{g}(\hat{p})) \in \mathbb{T}^n\,,$$

a fact that will create a problem in applying the theory of this and next part to the normalized Hamiltonians $\mathcal{H}_k$ of Theorem 1.2.1.

Proposition 1.3.1. *For all $k \in \mathcal{G}^n_{\mathtt{K}_0}$ let $\overline{\mathtt{H}}_k$ be the secular hamiltonian as in 1.2.61 and and, for $c = c(n) > 1$ large enough, set*

$$\rho_2 := \frac{\gamma_k^2\, \tilde{r}_k^3}{c\, |k|_1^2\, M^2}\,, \qquad \rho_1 := \frac{\gamma_k\, \tilde{r}_k^2}{c\, |k|_1\, M} \leqslant \boldsymbol{\varrho}\,.$$

Let $\mathcal{S}_k$ be the critical surface of h^k, and $\bar{y} \in \mathcal{S}_k$ a critical point. The following statements hold:

(i) *In the neighborhood of $\bar{\mathtt{y}}$ defined by $\mathtt{D}(\bar{\mathtt{y}}_1, \hat{\bar{\mathtt{y}}})$, $\overline{\mathtt{H}}_k$ is symplectically conjugated to a*

[5] In general, given $A \in \mathrm{SL}(n, \mathbb{Z})$ and a 2π–multi–periodic function $f : \mathbb{R}^n \to \mathbb{R}^n$, we identify the $\mathbb{R}^n$–map $x \in \mathbb{R}^n \to f(x) = Ax + g(x) \in \mathbb{R}^n$ with the $\mathbb{T}^n$–map given by $\theta \in \mathbb{T}^n \to F(\theta) = \pi_{\mathbb{T}^n}\big(Ax + f(x)\big) \in \mathbb{T}^n$ where $\theta = x + 2\pi\mathbb{Z}^n$ and $x \to \pi_{\mathbb{T}^n}(x) = x + 2\pi\mathbb{Z}^n$ is the projection of $\mathbb{R}^n$ onto $\mathbb{T}^n$.

suitable Hamiltonian in standard form $\mathtt{H}_k$ *(according to definition 1.3.4). In particular, for* $p \in \mathtt{D}(0, \hat{\bar{\mathtt{y}}})$*, there exists real analytic symplectic transformation*

$$\Phi_* : (p, q) \ \in \mathtt{D}(0, \bar{\mathtt{y}}) \times \mathbb{R}^n \to (\mathtt{y}, \mathtt{x}) = \Phi_*(p, q) \in \mathbb{R}^{2n}\,, \tag{1.3.43}$$

such that[6]*:* Φ_* *fixes* $\hat{p}$ *and* q_1*; for every* $\hat{p} \in \hat{\mathtt{D}}(\bar{\mathtt{y}}_1, \hat{\bar{\mathtt{y}}})$ *the map* $(p_1, q_1) \mapsto (\mathtt{y}_1, \mathtt{x}_1)$ *is symplectic; the* $(n+1)$*–dimensional map* $\check{\Phi}_*$ *depends only on the first* $n+1$ *coordinates* (p, q_1)*, is* 2π*–periodic in* $\mathtt{x}_1$ *and one has*

$$\begin{gathered}\check{\Phi}_* : (p, q_1) \in \mathtt{D}_{\rho_1+2\rho_2,\rho_2}(0, \hat{\bar{\mathtt{y}}}) \times \mathbb{T}^1_{\check{\mathtt{s}}_1} \to (\mathtt{y}, \mathtt{x}_1) \in \mathtt{D}_{\rho_1,\rho_2}(\bar{\mathtt{y}}_1, \hat{\bar{\mathtt{y}}}) \times \mathbb{T}^1_{\check{\mathtt{s}}_1}\\ \overline{\mathtt{H}}_k \circ \check{\Phi}_*(p, q_1) = h^k(0, \hat{p}) + \frac{1}{2}\partial^2_{p_1} h^k(0, \hat{p})\ \mathtt{H}_k(p, q_1).\end{gathered} \tag{1.3.44}$$

where, given B *a subset of* $\mathbb{R}^n$*, we denote by* $B_{r,r'} = \bigcup_{y \in B} \{z \in \mathbb{C}^n : |z_1 - y_1| \leqslant r, |\hat{z} - \hat{y}| \leqslant r'\}$*.*

(ii) $\mathtt{H}_k$ *has reference potential*

$$\bar{G}^k := \frac{2\varepsilon}{|k|_1^2} \pi_{\mathbb{Z}k} f \tag{1.3.45}$$

and analicity characteristic

$$\hat{\mathtt{D}} := \hat{\mathtt{D}}(\hat{\bar{\mathtt{y}}})\,, \qquad \mathtt{r} := \rho_2,\ \mathtt{R},\ \mathtt{s}_1,\ \upbeta,\ \epsilon,\ \mu\,, \tag{1.3.46}$$

with κ *given by*

$$\kappa = \kappa(n, s, \beta) := \max\{4\mathtt{c}_s, \mathtt{c}_s/\beta\}. \tag{1.3.47}$$

(iii) *The map* Φ_* *is obtained as composition of two symplectic maps:*

$$\Phi_* = \Phi_2 \circ \Phi_1\,, \tag{1.3.48}$$

where:

- $\Phi_1 := \Psi_{\mathtt{g}_1} \in \mathfrak{G}_\dagger$ *for a suitable real analytic function* $\mathtt{g}_1(\hat{y})$ *satisfying*

$$|\mathtt{g}_1|_{\rho_2} < \frac{2\varepsilon_k \chi_k}{\gamma\,\rho_2}\mu\,; \tag{1.3.49}$$

  where γ *is the convexity-constant of* h*.*

- $\Phi_2(\mathtt{p}, \mathtt{q}) = (\mathtt{p}_1 + \upeta_2, \hat{\mathtt{p}}, \mathtt{q}_1, \hat{\mathtt{q}} + \upchi_2)$ *for suitable real analytic functions* $\upeta_2 = \upeta_2(\hat{\mathtt{p}}, \mathtt{q}_1)$ *and* $\upchi_2 = \upchi_2(\hat{\mathtt{p}}, \mathtt{q}_1)$ *satisfying*

$$|\upeta_2|_{2\rho_2,\check{\mathtt{s}}_1} < \frac{2\varepsilon_k \chi_k}{\gamma\,\rho_2}\mu\,, \qquad |\upchi_2|_{2\rho_2,\check{\mathtt{s}}_1} < \frac{2\varepsilon_k \chi_k}{\gamma\,\rho_2^2}\mu\,. \tag{1.3.50}$$

[6]We are omitting the dependence upon vector k on the coordinates in this statement.

Remark 1.3.10. (i) The main point of the above theorem is item (ii), which shows that the 'simply–resonant Hamiltonians' $\mathtt{H}_k$ in 2.7.2 are in *uniform* Generic Standard Form. The word 'uniform' refers to the fact that the parameter κ (defined in 2.7.5 and satisfying (1.3.27)) – which rules the scaling properties of the normalized Hamiltonians $\mathtt{H}_k$ – *does not depend upon* k, allowing, e.g., for a uniform (in $k \in \mathcal{G}^n_{\mathtt{K}_\mathrm{o}}$) treatment of action–angle variables (compare next Section).

(ii) There is, however, a drawback in the construction of the above normal forms, namely, that the map Φ_1 appearing in the definition of $\Phi_\star$ (item (iii) in the above theorem), *do not induce well defined maps on* $\mathbb{T}^n$; compare Remark 1.3.9. Therefore, a non trivial homotopy issue will have to be faced in considering the global secondary nearly–integrable structure of the system near simple resonances. On the other hand, the map Φ_2 is well defined also on $\mathbb{T}^n$.

Proof. We want to use the Implicit Function Theorem to put all the critical points of h^k to zero, i.e. we want to find $\eta_k(\hat{y})$ such that $\partial_{\mathrm{y}_1} h^k(\eta_k(\hat{y}), \hat{y}) = 0$ in a proper neighborhood of the fixed critical point $\bar{\mathrm{y}} \in \mathcal{S}^k$.

So we use Theorem 3.5.1 in Appendix A with

$$\begin{aligned} &\mathcal{B}^n(y_0, r) = \mathcal{B}^1(\bar{\mathrm{y}}_1, \rho_1), \qquad \mathcal{B}^m(x_0, s) = \mathcal{B}^{n-1}(\hat{\bar{\mathrm{y}}}, \rho_2); \\ &F(\mathrm{y}_1, \hat{\mathrm{y}}) = \partial_{\mathrm{y}_1} h^k(\mathrm{y}_1, \hat{\mathrm{y}}), \quad T = \frac{1}{\partial^2_{\mathrm{y}_1} h^k(\bar{\mathrm{y}}_1, \hat{\bar{\mathrm{y}}})} \quad \text{such that} \quad ||T|| \leqslant \frac{1}{\gamma_k}. \end{aligned} \tag{1.3.51}$$

Now we check the hypotesis of theorem 3.5.1. For the second condition, using the assumption on h in (1.1.18), we know that for every $\mathrm{y} \in \mathscr{D}^k_{\tilde{r}_k}$

$$|\partial_{\mathrm{y}_1} h^k| \leqslant |\partial_{\mathrm{y}_1} h(y)| + \varepsilon |\partial_{\mathrm{y}_1} g^k_0(y)| \overset{(1.1.18)}{\leqslant} |k|_1 M + \varepsilon \mathcal{O}\left(\frac{1}{\mathtt{K}^b}\right) \leqslant 2|k|_1 M \tag{1.3.52}$$

taking $\mathtt{K}$ big enough. Furthermore, due to Cauchy estimates we have

$$|\partial^2_{\mathrm{y}_1} h^k(\mathrm{y})|_{\tilde{r}_k/2} \leqslant \frac{2}{\tilde{r}_k} |\partial_{\mathrm{y}_1} h^k(\mathrm{y})|_{\tilde{r}_k} \overset{(1.5.1)}{\leqslant} \frac{2}{\tilde{r}_k} 2|k|_1 M = \frac{4|k|_1 M}{\tilde{r}_k}\,. \tag{1.3.53}$$

Then, by using a one dimensional Lagrange theorem and by considering the fact that $\bar{\mathrm{y}} \in \mathcal{S}^k$, we obtain the following estimate:

$$\begin{aligned} \sup_{\mathcal{B}(\hat{\bar{\mathrm{y}}}, \rho_2)} |\partial_{\mathrm{y}_1} h^k(\bar{\mathrm{y}}_1, \hat{\mathrm{y}})| &\leqslant |\partial_{\mathrm{y}_1} h^k(\bar{\mathrm{y}})| + \sup_{\mathcal{B}(\hat{\bar{\mathrm{y}}}, \rho_2)} |\partial^2_{\mathrm{y}_1} h^k(\bar{\mathrm{y}}_1, \hat{\mathrm{y}})| |\hat{\mathrm{y}} - \hat{\bar{\mathrm{y}}}| \\ &\overset{(1.3.53)}{\leqslant} \frac{4|k|_1 M}{\tilde{r}_k} \sup_{\mathcal{B}(\hat{\bar{\mathrm{y}}}, \rho_2)} |\hat{\mathrm{y}} - \hat{\bar{\mathrm{y}}}| \leqslant \frac{4|k|_1 M\, \rho_2}{\tilde{r}_k}\,. \end{aligned} \tag{1.3.54}$$

Finally, using the fact that $\dfrac{\rho_1}{2||T||} \geqslant \dfrac{\rho_1 \gamma_k}{2}$ we can write

$$\sup_{\mathcal{B}(\hat{\bar{\mathrm{y}}},\rho_2)} |\partial_{\mathrm{y}_1} h^k(\bar{\mathrm{y}}_1, \hat{\mathrm{y}})| \leqslant \frac{\rho_1}{2||T||} \quad \Longleftrightarrow \quad \frac{8|k|_1 M}{\gamma_k \, \tilde{r}_k} \leqslant \frac{\rho_1}{\rho_2}\,. \tag{1.3.55}$$

Regarding the first condition in (3.5.2), we notice that

$$\begin{aligned}
\sup_{\mathcal{B}(\bar{\mathrm{y}}_1,\rho_1)\times\mathcal{B}(\hat{\bar{\mathrm{y}}},\rho_2)} \left| 1 - \frac{\partial^2_{\mathrm{y}_1} h^k(\mathrm{y}_1, \hat{\mathrm{y}})}{\partial^2_{\mathrm{y}_1} h^k(\bar{\mathrm{y}}_1, \hat{\bar{\mathrm{y}}})} \right| &= \sup_{\mathcal{B}(\bar{\mathrm{y}}_1,\rho_1)\times\mathcal{B}(\hat{\bar{\mathrm{y}}},\rho_2)} \left| \frac{\partial^2_{\mathrm{y}_1} h^k(\mathrm{y}_1, \hat{\mathrm{y}}) - \partial^2_{\mathrm{y}_1} h^k(\bar{\mathrm{y}}_1, \hat{\bar{\mathrm{y}}})}{\partial^2_{\mathrm{y}_1} h^k(\bar{\mathrm{y}}_1, \hat{\bar{\mathrm{y}}})} \right| \\
&\overset{(1.3.51)}{\leqslant} \frac{1}{\gamma_k} \sup_{\mathcal{B}(\bar{\mathrm{y}}_1,\rho_1)\times\mathcal{B}(\hat{\bar{\mathrm{y}}},\rho_2)} |\partial^2_{\mathrm{y}_1} h^k(\mathrm{y}_1, \hat{\mathrm{y}}) - \partial^2_{\mathrm{y}_1} h^k(\bar{\mathrm{y}}_1, \hat{\bar{\mathrm{y}}})| \\
&\leqslant \frac{1}{\gamma_k} \sup_{\mathcal{B}(\bar{\mathrm{y}}_1,\rho_1)\times\mathcal{B}(\hat{\bar{\mathrm{y}}},\rho_2)} |\partial^3_{\mathrm{y}_1} h^k| |\mathrm{y} - \bar{\mathrm{y}}|\,,
\end{aligned} \tag{1.3.56}$$

where, as we did before, in the last inequality we used the one-dimensional Lagrange theorem. Then Cauchy estimates ensure

$$|\partial^3_{\mathrm{y}_1} h^k(\mathrm{y})|_{\tilde{r}_k/2} \leqslant \frac{8}{\tilde{r}_k^2} |\partial_{\mathrm{y}_1} h^k(\mathrm{y})|_{\tilde{r}_k} \overset{(1.5.1)}{\leqslant} \frac{8}{\tilde{r}_k^2} 2|k|_1 M = \frac{16|k|_1 M}{\tilde{r}_k^2}\,. \tag{1.3.57}$$

Combining (1.3.56) and (1.3.57) together we have

$$\begin{aligned}
\sup_{\mathcal{B}(\bar{\mathrm{y}}_1,\rho_1)\times\mathcal{B}(\hat{\bar{\mathrm{y}}},\rho_2)} \left| 1 - \frac{\partial^2_{\mathrm{y}_1} h^k(\mathrm{y}_1, \hat{\mathrm{y}})}{\partial^2_{\mathrm{y}_1} h^k(\bar{\mathrm{y}}_1, \hat{\bar{\mathrm{y}}})} \right| &\overset{(1.3.56),(1.3.57)}{\leqslant} \frac{1}{\gamma_k} \frac{16|k|_1 M}{\tilde{r}_k^2} \sup_{\mathcal{B}(\bar{\mathrm{y}}_1,\rho_1)\times\mathcal{B}(\hat{\bar{\mathrm{y}}},\rho_2)} |\mathrm{y} - \bar{\mathrm{y}}| \\
&\overset{(\rho_1,\rho_2\geqslant 0)}{\leqslant} \frac{16|k|_1 M}{\gamma_k \, \tilde{r}_k^2} (\rho_1 + \rho_2)\,.
\end{aligned} \tag{1.3.58}$$

So in order to comply with (3.5.2) we impose

$$\sup_{\mathcal{B}(\bar{\mathrm{y}}_1,\rho_1)\times\mathcal{B}(\hat{\bar{\mathrm{y}}},\rho_2)} \left| 1 - \frac{\partial^2_{\mathrm{y}_1} h^k(\mathrm{y}_1, \hat{\mathrm{y}})}{\partial^2_{\mathrm{y}_1} h^k(\bar{\mathrm{y}}_1, \hat{\bar{\mathrm{y}}})} \right| \leqslant \frac{1}{2} \quad \Longleftrightarrow \quad \frac{4|k|_1 M}{\gamma_k \, \tilde{r}_k^2} (\rho_1 + \rho_2) \leqslant \frac{1}{8}\,. \tag{1.3.59}$$

Finally, as $\rho_2 = \rho_1 \frac{\gamma_k \tilde{r}_k}{|k|_1 M}$ by (1.3.55), we can say that a choice that satisfies both (1.3.59) and (1.3.55) simultaneously is

$$\rho_1 := \frac{\gamma_k \, \tilde{r}_k^2}{c\,|k|_1 M} \leqslant \frac{\tilde{r}_k}{8} =: \boldsymbol{\varrho}, \qquad \rho_2 = \frac{\gamma_k^2 \, \tilde{r}_k^3}{c\,|k|_1^2 M^2} \leqslant \boldsymbol{\varrho}\,, \tag{1.3.60}$$

for a constant $c > 1$ large enough.
Hence, by the Implicit Function Theorem there exists one function $\eta_k(\hat{y})$ holomorphic in $\hat{\mathtt{D}}_{\rho_2}(\bar{\mathtt{y}})$, such that $\bar{y}_1 = \eta_k(\hat{\bar{y}})$, and
$\partial_{\mathtt{y}_1} h^k(\eta_k(\hat{y}), \hat{y}) = \partial_{\mathtt{y}_1} h(\eta_k(\hat{y}), \hat{y}) + \varepsilon \partial_{\mathtt{y}_1} g_0^k(\eta_k(\hat{y}), \hat{y}) = 0$ for $|\eta_k(\hat{y}) - \bar{y}_1| \leqslant \rho_1, \ \forall\, |\hat{y} - \hat{\bar{y}}| < \rho_2$.
Then we define the symplectic transformation Φ_1 as

$$\begin{cases} y_1 = \eta_k(\hat{y}) + I_1 \\ \hat{y} = \hat{I} \end{cases} \qquad \begin{cases} \mathtt{x}_1 = \varphi_1 \\ \hat{\mathtt{x}} = \hat{\varphi} - x_1 \partial_{\hat{\mathtt{y}}} \eta_k(\hat{\mathtt{y}})\,, \end{cases} \tag{1.3.61}$$
$$\check{\Phi}_1 : (I, \varphi_1) \in \mathtt{D}_{\rho_1+\rho_2,\rho_2}(0, \hat{\bar{\mathtt{y}}}) \times \mathbb{T}^1_{\check{\mathtt{s}}_1} \to (\mathtt{y}, \mathtt{x}_1) \in \ \mathtt{D}_{\rho_1,\rho_2}(\bar{\mathtt{y}}_1, \hat{\bar{\mathtt{y}}}) \times \mathbb{T}^1_{\check{\mathtt{s}}_1}\,,$$

in which we have used the fact that, as in (1.3.35) we have the following estimates

$$|\eta_k|_{\rho_2} \leqslant \frac{2\varepsilon_k \chi_k}{\gamma\, \rho_2} \mu \leqslant \rho_2 \tag{1.3.62}$$

Furthermore, by Cauchy estimates, taking c large enough we have $|\partial^2_{I_1} h^k(I)| \geqslant \gamma_k/2, \ \forall\, I_k \in \mathtt{D}_{\rho_1+\rho_2,\rho_2}(0, \hat{\bar{\mathtt{y}}})$. With the same procedure as above, with Implicit Function Theorem we solve

$$\begin{gathered} \partial_{I_1} \overline{\mathtt{H}}_k(v_k(\hat{I}, \varphi_1), \hat{I}, \varphi_1) = \partial_{I_1} h^k(v_k(\hat{I}, \varphi_1), \hat{I}) + \varepsilon \partial_{I_1} \mathtt{g}^k(v_k(\hat{I}, \varphi_1), \hat{I}, \varphi_1) = 0 \\ \text{for } |v_k(\hat{I}, \varphi_1)| \leqslant \rho_2, \ \forall\, |\hat{I} - \hat{\bar{y}}| < \rho_2 \times \mathbb{T}^1_{\check{\mathtt{s}}_1} \end{gathered} \tag{1.3.63}$$

So for ε small enough, we found our symplectic transformation Φ_2

$$\begin{cases} I_1 = v_k(\hat{I}, \varphi_1) + p_1 \\ \hat{I} = \hat{p} \end{cases} \qquad \begin{cases} \varphi_1 = q_1 \\ \phi(\varphi_1, \hat{I}) = \int_0^{\varphi_1} v_k d\tilde{\varphi}_1 - \varphi_1 \langle v_k \rangle_{\varphi_1}) \\ \hat{\varphi} = \hat{q} - \partial_{\hat{I}} \phi(\varphi_1, \hat{I}) \end{cases} \tag{1.3.64}$$
$$\check{\Phi}_2 : \mathtt{D}_{\rho_1+2\rho_2,\rho_2}(0, \hat{\bar{\mathtt{y}}}) \times \mathbb{T}^1_{\check{\mathtt{s}}_1} \to \mathtt{D}_{\rho_1+\rho_2,\rho_2}(0, \hat{\bar{\mathtt{y}}}) \times \mathbb{T}^1_{\check{\mathtt{s}}_1}$$

where v_k is 2π-periodic in φ_1 and with the same idea as in (1.3.35) we have $|\langle v_k \rangle_{\varphi_1}|_{\rho_1+\rho_2} \leqslant |v_k|_{\rho_1+\rho_2,\check{\mathtt{s}}_1} \leqslant \frac{2\varepsilon_k\chi_k}{\gamma\,\rho_2}\mu$, moreover it is easy to see that $|\phi|_{\rho_1+2\rho_2,\check{\mathtt{s}}_1} \leqslant \frac{4\varepsilon_k\chi_k}{\gamma\,\rho_2}\mu$. Finally, applying the composition $\check{\Phi}_* = \check{\Phi}_2 \circ \check{\Phi}_1$, we have the last Hamiltonian holomorphic in
$\{|\operatorname{Re} p_1| < \mathtt{R} + \rho_1 + 2\rho_2;\ |\operatorname{Im} p_1| < \rho_1 + 2\rho_2\} \times \{|\operatorname{Re}(\hat{p} - \hat{\bar{\mathtt{y}}})| < \boldsymbol{\varrho} + \rho_2;\ |\operatorname{Im}(\hat{p} - \hat{\bar{\mathtt{y}}})| < \rho_2\} \times \mathbb{T}^1_{\check{\mathtt{s}}_1}$
given by

$$\begin{aligned} \overline{\mathtt{H}}_k(v_k + p_1, \hat{p}, q_1) &= \overline{\mathtt{H}}_k(v_k, \hat{p}, q_1) + p_1^2 \int_0^1 (1-t) \partial^2_{p_1} \overline{\mathtt{H}}_k(v_k + tp_1, \hat{p}, q_1) dt \\ &= E^k(\hat{p}) + E_0^k \mathtt{H}_k, \qquad \mathtt{H}_k := (1 + \nu_k(p, q_1))(p_1)^2 + G^k(\hat{p}, q_1), \end{aligned} \tag{1.3.65}$$

where

- $E^k(\hat{p}) := h^k(0, \hat{p})$;
- $E^k_0 = \frac{1}{2}\partial^2_{p_1} h^k(0, \hat{p})$;
- $\nu_k(p, q_1) = \frac{1}{E^k_0}\int_0^1 (1-t)[\partial^2_{p_1} h^k(u_k + v_k + tp_1, \hat{p}) - \partial^2_{p_1} h^k(\bar{\mathtt{y}}) +$
  $+ \varepsilon \partial^2_{p_1} \mathtt{g}^k(u_k + v_k + tp_1, \hat{p}, q_1)]dt$;
- $G^k(\hat{p}, q_1) := \frac{1}{E^k_0}\int_0^1 (1-t)\partial^2_{p_1} h^k(u_k + tv_k, \hat{p})(v_k)^2 dt + \varepsilon \mathtt{g}^k(u_k + v_k, \hat{p}, q_1)$;
- $\bar{G}^k(q_1) := \frac{2\varepsilon}{|k|_1^2}\pi_{\mathbb{Z}_k} f(q_1)$

Remark 1.3.11. Point (iii) follows calling $\eta_k =: \mathtt{g}_1$, and $v_k =: \upeta_2$, $\partial_{\hat{I}}\phi =: \chi_2$, omitting the depence on k for simplicity of notation.

Now we have to check the uniform behaviour on k of the analyticity characteristic of this "standard hamiltonian". We notice that

$$\sqrt{\varepsilon} \leqslant \gamma_k \tilde{r}_k^2 / c\, M, \qquad |v_k| \leqslant \rho_2$$
$$|u_k + v_k + tp_1 - \bar{y}_1| \leqslant 2\rho_2 \leqslant \tilde{r}_k/4 \;\; \forall\; 0 \leqslant t \leqslant 1, \quad |E^k_0| \geqslant \gamma_k/2, \quad \gamma_k \leqslant 2M'/\tilde{r}_k^2$$

and using Cauchy estimates and the fact that $\rho_2 \leqslant \boldsymbol{\varrho}$, we obtain

$$|\nu_k|_{\rho_2, s'_{k,1}} \leqslant \frac{2}{\gamma_k}\Big[\frac{\tilde{r}_k}{4} + 2\varepsilon\vartheta_{\mathrm{o}}\left(\frac{\tilde{r}_k}{32}\right)^{-2}\Big] \leqslant \frac{\tilde{r}_k}{2\gamma_k} + \frac{2\,050\vartheta_{\mathrm{o}} L^2 c_1^2 |k|^4}{\mathtt{K}^{2\nu}} \leqslant \frac{1}{\mathtt{K}^{6n+1}} := \mu \tag{1.3.66}$$

Regarding $\bar{G}^k$, for the case of $|k|_1 < \mathtt{N}$:

$$|\bar{G}^k|_{\mathtt{s}_1} \leqslant \frac{2\varepsilon}{|k|_1^2}|\pi_{\mathbb{Z}_k} f|_{s_1/2} = \frac{2\varepsilon}{|k|_1^2}\sum_{j\neq 0}|f_{jk}|e^{\frac{|j||k|_1 s_1}{s_1}} \leqslant \frac{8\varepsilon}{|k|_1^2}\frac{e^{-s_1/2}}{2(1-e^{-s_1/2})} < \frac{8\varepsilon}{|k|_1^2}\frac{1}{s_\flat} \tag{1.3.67}$$

while for $|k|_1 \geqslant \mathtt{N}$ one has

$$|\bar{G}^k|_1 = \frac{4\varepsilon}{|k|_1^2}|f_k||\cos(\theta + \theta_k) + F^k_\star(\theta)|_1 \leqslant \frac{8\varepsilon}{|k|_1^2}|f_k|. \tag{1.3.68}$$

So that $|\bar{G}_k|_{\mathtt{s}} \leqslant \epsilon$ as defined in (1.3.30). Moreover, for $|k|_1 < \mathtt{N}$ one has

$$|G^k - \bar{G}^k|_{\rho_2, s'_{k,1}} \leqslant \frac{\tilde{r}_k^3}{16^2} + \frac{2\varepsilon}{|k|_1^2}\vartheta_{\mathrm{o}} \leqslant \frac{\varepsilon}{|k|_1^2}\left[\frac{\sqrt{\varepsilon}K^{3\nu}}{c_1^3 L^3 |k|_1^6} + 2\vartheta_{\mathrm{o}}\right] \leqslant \epsilon\;\vartheta_{\mathrm{o}} := \epsilon\;\mu. \tag{1.3.69}$$

while for $|k| \geqslant \mathtt{N}$, reminding that $\mathtt{g}^k$ has the form in (2.6.15) and $\pi_{\mathbb{Z}k} f$ in (1.1.24), one has

$$|G^k - \bar{G}^k|_{\rho_2,1} \leqslant \frac{\tilde{r}_k^3}{16^2} + \frac{2\varepsilon}{|k|_1^2}|f_k||\mathtt{g}_\star^k|_{\rho_2,1} \leqslant \epsilon\, \vartheta \leqslant \epsilon\, \mu. \tag{1.3.70}$$

In this way we can look to the parameters defined in (1.3.30) and (1.3.60), so that analyticity characteristic of $\mathtt{H}_k$ are given in (2.7.4).
The smallness condition $\epsilon \leqslant \mathtt{r}^2/2^{16}$ is guaranteed taking $\mathtt{K}$ large enough, and finally by the definition in (1.3.30) one has

$$\frac{\epsilon}{\beta} = \begin{cases} \dfrac{4\mathtt{c}_s}{\beta} & \text{if } |k|_1 < \mathtt{N} \\ 4\mathtt{c}_s & \text{if } |k|_1 \geqslant \mathtt{N} \end{cases} \tag{1.3.71}$$

so that with the definition of κ in (2.7.5) the condition in (1.3.27) is easily verified due to the choice of parameters in (1.3.30). $\square$

So we have obtained the conjugation to an hamiltonian in Standard form in a neighborhood of each critical point $\bar{y}$ of a fixed critical surface. Moreover we have showed that we can cover $\check{\mathscr{D}}^k$ with a finite number of neighborhood in which one can conjugate the secular hamiltonian to a standard form hamiltonian, so that we have a standard form hamiltonian defined on the entire $\check{\mathscr{D}}^k$.

Now we apply Singular KAM theory made by Biasco and Chierchia in [4] with the standard form Hamiltonian made for the convex case.

1.4 Action-angles variables for 1D standard Hamiltonians

In this subsection we review the general theory of action–angle variables for Hamiltonian systems in standard form as developed in [3], where complete proofs may be found. This would be so useful for our intent because our Hamiltonian near simple resonance is conjugated with a standard form Hamiltonian near critical surfaces. If we transform this Hamiltonian with its action-angles coordinates we can try to apply classical KAM theory obtaining an estimate on Secondary tori.

Topology of the phase space of 1D Hamiltonians in standard form

We begin by describing the topological structure of the $\hat{p}$–dependent phase space of a givern Hamiltonian $(p_1, q_1) \mapsto \mathtt{H}_\flat(p_1, \hat{p}, q_1)$ in generic standard form according to

Definition 1.3.4. We will refer to $\mathtt{D}$ to a generic neighborhood of a fixed critical point $\bar{\mathtt{y}} \in \mathcal{S}_k$ defined in (1.3.30) in which we have done the standard form conjugation.
For a fixed $\hat{p} \in \hat{\mathtt{D}}$, we take as phase space of $\mathtt{H}_\flat$ the subset of $\mathbb{R} \times \mathbb{T}$ given by

$$\mathcal{M} = \mathcal{M}(\hat{p}) := \{(p_1, q_1) \in \mathbb{R} \times \mathbb{T} \,\big|\, \mathtt{H}_\flat(p_1, \hat{p}, q_1) < \mathtt{E}_\flat\}\,, \quad \mathtt{E}_\flat := \mathtt{R}^2 + \mathtt{R}\mathtt{r}\,, \tag{1.4.1}$$

where $\mathtt{R}$ and $\mathtt{r}$ are as in Definition 1.3.4. Although such sets depend on the parameter $\hat{p} \in \hat{D}$, for μ small enough, they are close to a box:

Lemma 1.4.1. *Let $\mathtt{H}_\flat$ be as in Definition 1.3.4 and $\mathcal{M}$ be as in (1.4.1), and assume that*[7]

$$\mu \leqslant 1/(4\kappa)^2\,. \tag{1.4.2}$$

Then, for all $\hat{p} \in \hat{\mathtt{D}}$, one has

$$\big(-\mathtt{R} - \tfrac{\mathtt{r}}{3}, \mathtt{R} + \tfrac{\mathtt{r}}{3}\big) \times \mathbb{T} \subseteq \mathcal{M}(\hat{p}) \subseteq \big(-\mathtt{R} - \tfrac{\mathtt{r}}{2}, \mathtt{R} + \tfrac{\mathtt{r}}{2}\big) \times \mathbb{T}\,. \tag{1.4.3}$$

The simple proof is given in Appendix of [4].
Since the reference potential $\bar{G}$ is a β–Morse function, it has $2N$ critical points, for some $N \in \mathbb{N}$, with different critical values. Let $\bar{\theta}_0 \in [0, 2\pi)$ be the unique point of absolute maximum of the reference potential $\bar{G}$ of $\mathtt{H}_\flat$. Then, the relative strict non–degenerate maximum and minimum points of $\bar{G}$, $\bar{\theta}_i \in [\bar{\theta}_0, \bar{\theta}_0 + 2\pi]$, $(0 \leqslant i \leqslant 2N)$ follow in alternating order, $\bar{\theta}_0 < \bar{\theta}_1 < \bar{\theta}_2 < \ldots < \bar{\theta}_{2N} := \bar{\theta}_0 + 2\pi$, in particular, $\bar{\theta}_i$ *are relative maxima/minima points for i even/odd*. The corresponding distinct critical energies will be denoted by

$$\bar{E}_i := \bar{G}(\bar{\theta}_i)\,, \quad \bar{E}_{2N} = \bar{E}_0 \;\text{ being the unique global maximum of } \bar{G}\,. \tag{1.4.4}$$

By the Implicit Function Theorem, for μ small enough with respect to κ, one can continue the $2N$ critical points $\bar{\theta}_i$ of $\bar{G}$ obtaining $2N$ critical points $\theta_i = \theta_i(\hat{p})$ of $G(\hat{p}, \cdot)$ for $\hat{p} \in \hat{\mathtt{D}}$. The corresponding distinct critical energies become

$$E_i = E_i(\hat{p}) := G(\hat{p}, \theta_i(\hat{p}))\,. \tag{1.4.5}$$

Furthermore, for μ small , the functions $\theta_i(\hat{p})$ and $E_i(\hat{p})$ preserve the same order of $\bar{\theta}_i$ and $\bar{E}_i$. Indeed, from Definition 1.1.9 and the Implicit Function Theorem, the following result proven in [3] holds[8]:

[7] Recall the definition of κ in (1.3.27).
[8] See Lemma 3.1 in [3].

Lemma 1.4.2. *Let* $\mathtt{H}_\flat$ *be as in Definition 1.3.4 and assume that*[9]

$$\mu \leqslant 1/(2\kappa)^6\,. \tag{1.4.6}$$

Then, the functions $\theta_i(\hat p)$ *and* $E_i(\hat p)$ *defined above are real analytic in* $\hat p \in \hat{\mathtt{D}}_{\mathtt{r}}$ *and*

$$\sup_{\hat p\in\hat{\mathtt{D}}_{\mathtt{r}}} |\theta_i(\hat p) - \bar\theta_i| \leqslant \tfrac{2\epsilon\mu}{\beta\mathtt{s}}\,, \qquad \sup_{\hat p\in\hat{\mathtt{D}}_{\mathtt{r}}} |E_i(\hat p) - \bar E_i| \leqslant 3\kappa^3\epsilon\mu\,. \tag{1.4.7}$$

Furthermore, the relative order of $\theta_i(\hat p)$ *and* $E_i(\hat p)$ *is, for every* $\hat p \in \hat{\mathtt{D}}_{\mathtt{r}}$*, the same as that of, respectively,* $\bar\theta_i$ *and* $\bar E_i$*.*

Therefore, under the assumption (1.4.6), we see that the phase space $\mathcal{M}$ is disconnected by the separatrices [10] into exactly $2N + 1$ open connected components $\mathcal{M}^i = \mathcal{M}^i(\hat p)$, for $0 \leqslant i \leqslant 2N$, which can be labelled so that:

- the *odd regions* $\mathcal{M}^{2j-1}$ (for $1 \leqslant j \leqslant N$) contain the elliptic points $(0, \theta_{2j-1})$ and have as boundary parts of separatrices; topologically, such regions are discs;
- the *outer even regions* $\mathcal{M}^0$ and $\mathcal{M}^{2N}$ are homotopically *non trivial* annuli bounded by the most external separatrices and one of the two curves $\mathtt{H}_\flat^{-1}(\mathtt{E}_\flat)$;
- when $N > 1$, the *inner even regions* $\mathcal{M}^{2j}$ (for $1 \leqslant j \leqslant N-1$) are homotopically trivial annuli[11] whose boundary is given by two pieces of separatrices (with different energies).

More formally, we can define the $2N+1$ regions $\mathcal{M}^i$ in terms of suitable energy intervals $(E_-^{(i)}, E_+^{(i)})$ as follows.

Let E_i be the critical energies defined in (1.4.5), and let $\mathtt{E}_\flat$ the reference energy defined in (1.4.1).

Definition 1.4.1. (i) (Outer regions) *For* $i = 0, 2N$*, let* $E_-^{(0)} = E_-^{(2N)} := E_0$*, and* $E_+^{(0)} = E_+^{(2N)} := \mathtt{E}_\flat$*. Then, the 'lower outer region'* $\mathcal{M}^{(0)}$ *is the connected component of* $\mathtt{H}_\flat^{-1}\big((E_-^{(0)}, E_+^{(0)})\big)$ *contained in* $\{p_1 < 0\}$*, while the 'upper outer region'* $\mathcal{M}^{(2N)}$ *is the connected component of* $\mathtt{H}_\flat^{-1}\big((E_-^{(2N)}, E_+^{(2N)})\big)$ *contained in* $\{p_1 > 0\}$*.*
(ii) (Inner region, $N = 1$) *When* $N = 1$*,* $\mathcal{M}^{(1)}$ *is just the region enclosed by the unique separatrix* $\mathtt{H}_\flat^{-1}(E_0)$*; the orbits in* $\mathcal{M}^{(1)}$ *have energies ranging in the critical interval*

[9] Notice that condition (1.4.6) is stronger than (1.4.2).
[10] I.e., the stable manifolds (curves) of the hyperbolic points $(0, \theta_{2j})$.
[11] I.e., annuli in the cylinder $\mathbb{R} \times \mathbb{T}$ which are contractible.

$[E_-^{(1)}, E_+^{(1)}) := [E_1, E_0)$.

(ii) (Inner regions, $N > 1$) *Define* $E_-^{(i)} := E_i$.

For i *odd, let* $E_+^{(i)} := \min\{E_{i-1}, E_{i+1}\}$ *and define* $\mathcal{M}^{(i)}$ *as the connected component of* $\mathtt{H}_\flat^{-1}\big([E_-^{(i)}, E_+^{(i)})\big)$ *containing the elliptic equilibrium* $(0, \theta_i)$.

Finally, for $0 < i = 2j < 2N$ *even, define*

$$j_- := \max\{\ell < j \big| \; E_{2\ell} > E_{2j}\}\,,\ j_+ := \min\{\ell > j \big| \; E_{2\ell} > E_{2j}\}\,,\ E_+^{(i)} := \min\{E_{2j_-}, E_{2j_+}\}\,;$$

and define $\mathcal{M}^{(i)}$ *as the connected component of* $\mathtt{H}_\flat^{-1}\big((E_-^{(i)}, E_+^{(i)})\big)$ *whose boundary contains the hyperbolic point* $(0, \theta_i)$.

Notice that the phase space $\mathcal{M}$ is the union of the regions $\mathcal{M}^{(i)}$ and the singular zero–measure set $S = S(\hat{p})$ formed by the N separatrices:

$$\mathcal{M} = \mathcal{M}(\hat{p}) = \bigcup_{i=0}^{2N} \mathcal{M}^i \ \cup \ S = \bigcup_{i=0}^{2N} \mathcal{M}^i(\hat{p}) \ \cup \ S(\hat{p})\,. \tag{1.4.8}$$

Below we shall also consider the following $(n+1)$–dimensional domains:

$$\begin{aligned} \check{\mathcal{M}} \ &:= \ \{(p, q_1) \ \text{ s.t. } \ \hat{p} \in \hat{\mathtt{D}}, \ (p_1, q_1) \in \mathcal{M}(\hat{p})\}\,, \\ \check{\mathcal{M}}^i \ &:= \ \{(p, q_1) \ \text{ s.t. } \ \hat{p} \in \hat{\mathtt{D}}, \ (p_1, q_1) \in \mathcal{M}^i(\hat{p})\}\,. \end{aligned} \tag{1.4.9}$$

Notice that $\bigcup_{0 \leqslant i \leqslant 2N} \check{\mathcal{M}}^i$ covers $\check{\mathcal{M}}$ up to a set of measure zero.

Arnol'd–Liouville's action/energy functions

Let $E \in [E_-^i(\hat{p}), E_+^i(\hat{p})]$ and let γ^i be the (possibly, piece–wise) smooth closed curve in the clusure of $\mathcal{M}^i(\hat{p})$ given by

$$\gamma^i = \gamma^i(E, \hat{p}) := \{(p_1, q_1) \in \overline{\mathcal{M}^i(\hat{p})} \ \text{ s.t. } \ \mathtt{H}_\flat(p_1, \hat{p}, q_1) = E\}\,,$$

oriented clockwise[12]; for $2 \leqslant j \leqslant N$ consider also the trivial curves $\gamma_j^i = \{(p_j, s) : s \in \mathbb{T}\}$. Then, the classical *Arnol'd–Liouville's action functions* are given by

$$\begin{aligned} I_1^{(i)}(E) &= I_1^{(i)}(E, \hat{p}) := \frac{1}{2\pi} \oint_{\gamma^i} p_1 dq_1\,, \\ I_j &= \frac{1}{2\pi} \oint_{\gamma_j^i} p_j dq_j = \frac{p_j}{2\pi} \int_{\mathbb{T}} dq_j = p_j\,, \qquad \forall\ 2 \leqslant j \leqslant N\,. \end{aligned}$$

[12]For the non contractible curves ($i = 0, 2N$) the orientation is 'to the right' on $\mathcal{M}^{2N}$, 'to the left' on $\mathcal{M}^0$.

The action function $E \to I_1^i(E, \hat{I})$ is strictly monotone and its inverse is, by definition, the *energy function* $I_1 \to \mathtt{E}^i(I_1, \hat{I})$. We also define $\bar{I}_1^i := I_1^i|_{\mu=0}$ and its inverse function[13] $\bar{\mathtt{E}}^i := \mathtt{E}^i|_{\mu=0}$.

We can now describe the fine analytic properties of the action/energy functions.

Critical holomorphic behaviour and action estimates

The first result describes the exact behaviour of the action functions as the energy approaches the critical energy of separatrices and contains estimates on the derivatives of the action functions that will play a central rôle in the discussion on the twist Hessian matrix in § 1.6. The following theorem has been proven in [3, Theorem 3.1].

Theorem 1.4.1. *Let $\mathtt{H}_\flat$ be a Hamiltonian in standard form as in Definition 1.3.4, let $\kappa \geqslant 4$ be such that (1.3.27) holds and let $2N$ be the number of critical points of the reference potential $\bar{G}$. Then, there exists a suitable constant $\mathbf{c} = \mathbf{c}(n, \kappa) \geqslant 2^8\kappa^3$ such that, if*[14]

$$\mu \leqslant 1/\mathbf{c}^2 \leqslant 1/(2^{16}\kappa^6)\,, \tag{1.4.10}$$

then, for all $0 \leqslant i \leqslant 2N$ and $\hat{I} \in \hat{\mathtt{D}}$, the action functions $E \in (E_-^i(\hat{I}), E_+^i(\hat{I})) \mapsto I_1^i(E, \hat{I})$ verify the following properties.

(i) **(Universal behaviour at critical energies)** *There exist functions $\phi_-^i(z, \hat{I})$, $\psi_-^i(z, \hat{I})$ for $0 \leqslant i \leqslant 2N$, and, functions $\phi_+^i(z, \hat{I})$, $\psi_+^i(z, \hat{I})$, for $0 < i < 2N$, which are real analytic in a complex neighborhood of the set $\{z = 0\} \times \hat{D}$ and satisfy*

$$I_1^i\big(E_\mp^i(\hat{I}) \pm \epsilon z,\, \hat{I}\big) = \phi_\mp^i(z, \hat{I}) + \psi_\mp^i(z, \hat{I})\; z \log z\;, \quad \forall\; 0 < z < 1/\mathbf{c}\,,\; \hat{I} \in \hat{\mathtt{D}}\,. \tag{1.4.11}$$

the functions $\phi_\pm^i(z, \hat{I})$, $\psi_\pm^i(z, \hat{I})$ are real analytic on $\{z \in \mathbb{C} : |z| < 1/\mathbf{c}\} \times \hat{\mathtt{D}}_{\mathtt{r}}$, where satisfy:

$$\begin{aligned}
&\sup_{|z|<1/\mathbf{c},\, \hat{I}\in\hat{D}_{\mathtt{r}}} \big(|\phi_\pm^i| + |\psi_\pm^i|\big) \leqslant \mathbf{c}\sqrt{\epsilon}\,,\\
&\sup_{|z|<1/\mathbf{c},\, \hat{I}\in\hat{D}_{\mathtt{r}/2}} \big(|\partial_{\hat{I}}\phi_\pm^i| + |\partial_{\hat{I}}\psi_\pm^i|\big) \leqslant \mathbf{c}\mu_{\mathrm{o}}\,, \qquad \mu_{\mathrm{o}} := \tfrac{\sqrt{\epsilon}}{\mathtt{r}}\mu \overset{(1.3.26)}{\leqslant} 2^{-8}\mu\,.
\end{aligned} \tag{1.4.12}$$

Moreover,

$$|\phi_\pm^i - \bar{\phi}_\pm^i|\,,\; |\psi_\pm^i - \bar{\psi}_\pm^i| \;\leqslant\; \mathbf{c}\sqrt{\epsilon}\mu\,, \tag{1.4.13}$$

[13] Note that when $\mu = 0$, $\mathtt{H}_\flat$ becomes simply $\bar{\mathtt{H}}_\flat = p_1^2 + \bar{G}(q_1)$.

[14] Note that (1.4.10) implies the hypothesis of Lemma 1.4.2. Thus, in particular also $\mathtt{H}_\flat$ has $2N$ critical points.

where $\bar\phi^i_\pm := \phi^i_\pm|_{\mu=0}$ *and* $\bar\psi^i_\pm := \psi^i_\pm|_{\mu=0}$.

(ii) **(Limiting critical values)** *The following bounds at the limiting critical energy values hold:*

$$\begin{array}{l} |\psi^i_+(0,\hat I)| \geqslant \sqrt{\epsilon}/\mathbf{c}\,, \qquad 0<i<2N\,, \qquad \forall\ \hat I \in \hat{\mathtt{D}}_{\mathtt{r}}\,, \\ |\psi^{2j}_-(0,\hat I)| \geqslant \sqrt{\epsilon}/\mathbf{c}\,, \qquad 0\leqslant j \leqslant N\,, \qquad \forall\ \hat I \in \hat{\mathtt{D}}_{\mathtt{r}}\,, \\ \psi^i_+(0,\hat I) > 0\,, \qquad 0<i<2N\,, \quad \forall\ \hat I \in \hat{\mathtt{D}}\,, \\ \psi^{2j}_-(0,\hat I) < 0\,, \qquad 0\leqslant j\leqslant N\,, \quad \forall\ \hat I\in\hat{\mathtt{D}}\,, \end{array} \tag{1.4.14}$$

while, in the case of relative minimal critical energies, one has, $\forall\ \hat I \in \hat{\mathtt{D}}$, $0<z<1/\mathbf{c}$,

$$\phi^{2j-1}_-(0,\hat I) = 0\,, \qquad \psi^{2j-1}_-(z,\hat I)=0\,, \qquad \forall\ 1\leqslant j\leqslant N\,. \tag{1.4.15}$$

(iii) **(Estimates on derivatives of actions on real domains)** *The derivatives of the actions with respect to energy verify, on real domains, the following estimates:*

$$\inf_{(E^i_-,E^i_+)} \partial_E I^i_1 \geqslant \frac{1}{\mathbf{c}\sqrt{\epsilon}}\,, \qquad \forall\ \hat I\in\hat{\mathtt{D}}\,,\ \forall\ 0<i<2N\,; \tag{1.4.16}$$

$$\min\left\{\partial_E I^{2N}_1\,,\ \partial_E I^0_1\right\} \geqslant \frac{1}{\mathbf{c}\sqrt{E+\epsilon}}\,, \quad \forall\, E>E_{2N}\,,\ \forall\ \hat I\in\hat{\mathtt{D}}\,.$$

(iv) **(Estimates on derivatives of actions on complex domains and perturbative bounds)** *For* $\lambda>0$ *satisfying*

$$\mathbf{c}\mu \leqslant \lambda \leqslant 1/\mathbf{c}\,, \tag{1.4.17}$$

define the following complex energy–domains:

$$\mathcal{E}^i_\lambda := \begin{cases} \{z\in\mathbb{C} : \bar E^i_- - \epsilon/\mathbf{c} \ <\ \mathrm{Re}\,z < \bar E^i_+ - \lambda\epsilon\,,\ |\mathrm{Im}\,z| < \epsilon/\mathbf{c}\}\,, & i \text{ odd}\,, \\ \{z\in\mathbb{C} : \bar E^i_- + \lambda\epsilon < \ \mathrm{Re}\,z < \bar E^i_+ - \lambda\epsilon\,,\ |\mathrm{Im}\,z| < \epsilon/\mathbf{c}\}\,, & i \text{ even}\,, i\neq 0,2N\,, \\ \{z\in\mathbb{C} : \bar E^i_- + \lambda\epsilon < \ \mathrm{Re}\,z < \bar E^i_+\,,\ |\mathrm{Im}\,z| < \epsilon/\mathbf{c}\}\,, & i = 0,2N\,. \end{cases} \tag{1.4.18}$$

Then, for $0\leqslant i\leqslant 2N$, *the functions* I^i_1 *and* $\bar I^i_1$ *are holomorphic on the domains* $\mathcal{E}^i_\lambda\times\hat{\mathtt{D}}_{\mathtt{r}}$, *and satisfy the following estimates:*

$$\sup_{\mathcal{E}^i_\lambda\times\hat{\mathtt{D}}_{\mathtt{r}/2}} |\partial_{\hat I} I^i_1| \leqslant \mathbf{c}^2\,\mu_{\mathrm{o}}\,,\quad \sup_{\mathcal{E}^i_\lambda}\left|\partial_E \bar I^i_1\right| \leqslant \mathbf{c}^2\,\frac{|\log\lambda|}{\sqrt{\epsilon}}\,,\quad \sup_{\mathcal{E}^i_\lambda\times\hat{\mathtt{D}}_{\mathtt{r}}}\left|\partial_E I^i_1 - \partial_E\bar I^i_1\right| \leqslant \frac{\mathbf{c}^2\mu}{\lambda\sqrt{\epsilon}}\,. \tag{1.4.19}$$

Remark 1.4.1. Eq. (1.4.15) confirms the known analyticity at minima of actions as function of energy.

We finally report a remarkable property of standard Hamiltonians $\mathtt{H}_\flat$, whose reference potential $\bar G$ is close enough to a cosine. In such a case, in fact, one has uniform concavity of the second derivative of the energy function:

Proposition 1.4.1. *Assume that, for some $\theta_0 \in \mathbb{R}$, $\bar G$ satisfies*

$$|\bar G(\theta) - \cos(\theta+\theta_0)|_1 := \sup_{\mathbb{T}_1} |\bar G(\theta) - \cos(\theta+\theta_0)| \leqslant 2^{-40}\,. \tag{1.4.20}$$

Then $N = 1$ and

$$\partial^2_{I_1} \bar{\mathtt{E}}^1(\bar I^1_1(E)) \leqslant -\frac{1}{27}\,, \qquad \forall E \in (\bar E_1, \bar E_2)\,.$$

Also this result is proven in [3]; compare Proposition 5.12 there.

Arnol'd–Liouville's action–angle variables in n d.o.f.

Let us now discuss the Arnol'd–Liouville's action–angle variables for the Hamiltonian $\mathtt{H}_\flat$ viewed as a n degrees of freedom Hamiltonian on the $2n$–dimensional phase space $\check{\mathcal{M}}^i \times \mathbb{T}^{n-1}$.

For every fixed $\hat p \equiv \hat I \in \hat{\mathtt{D}}$, the map $(p_1, q_1) \to I_1^{(i)}\big(\mathtt{H}_\flat(p_1, \hat I, q_1), \hat I\big)$ can be symplectically completed with the angular term[15] $(p_1, q_1) \to \varphi_1^{(i)}(p_1, q_1; \hat I) = \varphi_1^{(i)}(p_1, \hat I, q_1)$.

Defining the normal domains[16]

$$\mathcal{B}^i := \big\{ I = (I_1, \hat I) \mid \hat I \in \hat D,\ \ I_1^{(i)}(E^i_-(\hat I), \hat I) < I_1 < I_1^{(i)}(E^i_+(\hat I), \hat I) \big\}\,, \tag{1.4.21}$$

we see that, by construction, the map[17]

$$(p, q_1) \in \check{\mathcal{M}}^i \to (I, \varphi_1) = \big(I_1^{(i)}(\mathtt{H}_\flat(p, q_1), \hat I), \hat I, \varphi_1^{(i)}(p, q_1) \big) \in \mathcal{B}^i \times \mathbb{T}$$

is surjective and invertible; let us denote by

$$\check\Phi^i : (I, \varphi_1) \in \mathcal{B}^i \times \mathbb{T} \ \to\ (p, q_1) \in \check{\mathcal{M}}^i\,, \qquad (\hat p = \hat I)\,,$$

its inverse map. Note that such 'Arnol'd-Liouville suspended' transformation $\check\Phi^i$ integrates $\mathtt{H}_\flat$, i.e.,

$$\mathtt{H}_\flat \circ \check\Phi^i(I, \varphi_1) = \mathtt{E}^{(i)}(I)\,, \qquad dp_1 \wedge dq_1|_{\hat I = \mathrm{const}} = dI_1 \wedge d\varphi_1\,. \tag{1.4.22}$$

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[15]Such completion is unique if one fixes, e.g., $\varphi_1^{(i)}(p_1, 0; \hat I) = 0$.

[16]Recall Definition 1.4.1. For i odd, $I_1^{(i)}(E^i_-(\hat I), \hat I) = 0$, which is the action of the elliptic point.

[17]Recall the definition of $\check{\mathcal{M}}^i$ in (1.4.9).

By the standard Arnol'd–Liouville construction of the angle variables, one sees easily that the complete symplectic action–angle map $\Phi^i : (I, \varphi) \mapsto (p, q)$ has the form

$$\Phi^i(I,\varphi) = \begin{cases} (\eta^i, \hat I, \psi^i, \hat\varphi + \chi^i)\,, & \text{if } 0 < i < 2N\,, \\ (\eta^i, \hat I, \varphi_1 + \psi^i, \hat\varphi + \chi^i)\,, & \text{if } i = 0, 2N\,, \end{cases} \tag{1.4.23}$$

where η^i, χ^i, ψ^i are function of (I, φ_1) only and are 2π–periodic in φ_1, and, in the case $i = 0, 2N$, $\sup |\partial_{\varphi_1} \psi^i| < 1$.

By construction, $\Phi^i : \mathcal{B}^i \times \mathbb{T}^n \xrightarrow{\text{onto}} \check{\mathcal{M}}^i \times \mathbb{T}^{n-1}$ *is a global symplectomorphism*, and by (1.4.22), one has

$$(\mathtt{H}_\flat \circ \Phi^i)(I,\varphi) = (\mathtt{H}_\flat \circ \check\Phi^i)(I, \varphi_1) = \mathtt{E}^{(i)}(I)\,, \qquad \forall\ 0 \leqslant i \leqslant 2N\,. \tag{1.4.24}$$

Next, we introduce suitable decreasing subdomains $\mathcal{B}^i(\lambda)$ of $\mathcal{B}^i$ depending on a non negative parameter λ so that $\mathcal{B}^i(0) = \mathcal{B}^i$ and such that the map Φ^i has, for positive λ, a holomorphic extension on a suitable complex neighborhood of $\mathcal{B}^i(\lambda) \times \mathbb{T}^n$.

Define

$$\lambda_{\max} = \lambda_{\max}(\hat I) := \big(E_+(\hat I) - E_-(\hat I)\big)/\epsilon\,, \qquad \bar\lambda_{\max} := \big(\bar E_+ - \bar E_-\big)/\epsilon\,. \tag{1.4.25}$$

Notice that, by (1.3.27), Definitions 1.1.9, 1.3.4, and 1.3.26 one has

$$1/\kappa \leqslant \beta/\epsilon \leqslant \bar\lambda_{\max} \leqslant 2\,; \tag{1.4.26}$$

notice also that, by (1.4.7), we have[18]

$$|\lambda_{\max} - \bar\lambda_{\max}| \leqslant 6\kappa^3\mu\,, \qquad \lambda_{\max} \geqslant 1/2\kappa\,. \tag{1.4.27}$$

Then, for $0 \leqslant \lambda \leqslant \lambda_{\max}$ define[19]:

$$\begin{array}{rcl} a^i_\lambda(\hat I) & := & I^i_1(E^i_-(\hat I) + \lambda\epsilon, \hat I)\,, \qquad \forall\, 0 \leqslant i \leqslant 2N\,, \\ b^i_\lambda(\hat I) & := & \begin{cases} I^i_1(E^i_+(\hat I) - \lambda\epsilon, \hat I)\,, & \forall\ 0 < i < 2N \\ I^i_1\big(\mathtt{E}_\flat, \hat I\big)\,, & i = 0, 2N\,. \end{cases} \\ a^i(\hat I) & := & a^i_0(\hat I)\,,\ b^i(\hat I) := b^i_0(\hat I)\,,\ \ \forall\ 0 \leqslant i \leqslant 2N\,, \\ \mathcal{B}^i(\lambda) & := & \{I = (I_1, \hat I) :\ \hat I \in \hat{\mathtt{D}}\,,\ a^i_\lambda(\hat I) < I_1 < b^i_\lambda(\hat I)\}\,, \quad 0 \leqslant \lambda \leqslant \lambda_{\max}\,. \end{array} \tag{1.4.28}$$

[18] Recall that $\mu \leqslant 1/\mathbf{c}^2$ and $\mathbf{c} \geqslant 2^8\kappa^3$ (compare Theorem 1.4.1).

[19] Recall the definition of $\mathtt{E}_\flat$ in (1.4.1).

Remark 1.4.2. (i) By the above definitions one has that

$$a^{2j-1}(\hat{I}) := a_0^{2j-1}(\hat{I}) = I_1^{2j-1}(E_-^{2j-1}(\hat{I}), \hat{I}) \equiv 0\,, \tag{1.4.29}$$

reflecting the analyticity at the elliptic points; compare Remark 1.4.1–(i) above.
(ii) By (1.4.21) and (1.4.28) one sees that $\mathcal{B}^i = \mathcal{B}^i(0) = \bigcup_{0<\lambda<\lambda_{\max}} \mathcal{B}^i(\lambda)$.

The holomorphic properties of the Arnol'd–Liouville symplectic maps are described in following theorem, proven in [3, Theorem 4.1]. Recall the definition of the constant $\mathbf{c}$ in Theorem 1.4.1.

Theorem 1.4.2. *Under the hypotheses of Theorem 1.4.1 there exists a constant* $\hat{\mathbf{c}} = \hat{\mathbf{c}}(n,\kappa) \geqslant 4\mathbf{c}^2$ *depending only on* n *and* κ *such that, taking*

$$\mu \leqslant 1/\hat{\mathbf{c}}\,, \tag{1.4.30}$$

the symplectic transformation $\check{\Phi}^i$ *extends, for any* $0 \leqslant i \leqslant 2N$ *and* $0 < \lambda \leqslant 1/\hat{\mathbf{c}}$*, to a real analytic map*

$$\Phi^i : \big(\mathcal{B}^i(\lambda)\big)_{\rho_\lambda} \times \mathbb{T}^n_{\sigma_\lambda} \;\to\; \mathtt{D}_{\mathtt{r}}(0,\widehat{\widehat{\mathtt{y}}}) \times \mathbb{T}^n_{\mathtt{s}/4}\,, \qquad \forall\, 0<\lambda \leqslant 1/\hat{\mathbf{c}}\,, \tag{1.4.31}$$

where

$$\rho_\lambda := \tfrac{\sqrt{\epsilon}}{\hat{\mathbf{c}}}\,\lambda|\log\lambda|\,, \qquad \sigma_\lambda := \tfrac{1}{\hat{\mathbf{c}}|\log\lambda|}\,. \tag{1.4.32}$$

Now, let $0<\lambda\leqslant 1/\hat{\mathbf{c}}$*, then the function* $\mathtt{E}^i$ *admits a holomorphic extension on* $\big(\mathcal{B}^i(\lambda)\big)_{\rho_\lambda}$*, where, setting* $\hat{\lambda} := \lambda|\log\lambda|^3$*, one has*

$$\big|\partial_{I_1}\mathtt{E}^i\big| \leqslant \hat{\mathbf{c}}\sqrt{\epsilon + |\mathtt{E}^i|}\,, \quad \big|\partial^2_{I_1}\mathtt{E}^i\big| \leqslant \tfrac{\hat{\mathbf{c}}}{\hat{\lambda}}\,, \quad \big|\partial^2_{I_1\hat{I}}\mathtt{E}^i\big| \leqslant \hat{\mathbf{c}}\tfrac{\mu_{\mathrm{o}}}{\hat{\lambda}}\,, \quad \big|\partial^2_{\hat{I}}\mathtt{E}^i\big| \leqslant \hat{\mathbf{c}}\big(\tfrac{\sqrt{\epsilon}}{\mathtt{r}}I_1^i + \tfrac{\mu_{\mathrm{o}}}{\hat{\lambda}}\big)\mu_{\mathrm{o}}\,; \tag{1.4.33}$$

furthermore, defining

$$D^\flat := (-\mathtt{R}-\mathtt{r}/3, \mathtt{R}+\mathtt{r}/3) \times \widehat{\mathtt{D}}(\widehat{y})\,, \qquad \check{\mathcal{M}}^i(\lambda) := \check{\Phi}^i(\mathcal{B}^i(\lambda)\times\mathbb{T})\,, \tag{1.4.34}$$

one has

$$\mathrm{meas}\,\Big((D^\flat\times\mathbb{T}) \setminus \bigcup_{0\leqslant i\leqslant 2N} \check{\mathcal{M}}^i(\lambda)\Big) \leqslant \hat{\mathbf{c}}\,\sqrt{\epsilon}\,\mathrm{meas}(\widehat{\mathtt{D}}(\widehat{\widehat{\mathtt{y}}}))\;\lambda|\log\lambda|\,. \tag{1.4.35}$$

Remark 1.4.3. Observe that, by 1.3.30, (1.3.31), 1.2.22, (1.4.10), 1.2.23, 2.7.2 and 2.7.5, it is[20]

$$1/\kappa < \check{\mathtt{s}}/4\,, \qquad \tfrac{\varepsilon_k\chi_k}{\mathtt{r}}\,\mu < \mathtt{r}/6\,, \qquad \tfrac{4\varepsilon_k\chi_k}{\mathtt{r}^2}\,\mu < \tfrac{\check{\mathtt{s}}}{2^{20}\kappa^3} < \check{\mathtt{s}}/2^{20}\,. \tag{1.4.36}$$

Thus, since[21] $\lambda \leqslant 1/\hat{\mathbf{c}}$, by (1.4.10), σ_λ in (1.4.32) satisfies

$$\sigma_\lambda < \check{\mathtt{s}}/2^{20}\,. \tag{1.4.37}$$

[20]Recall that $\varepsilon < 1$; see 1.1.1.
[21]Recall the hypotheses of Theorem 1.4.2.

1.5 Secondary nearly–integrable structure at simple resonances

Now we go back to the original system in the simply–resonant zones governed by the Hamiltonians $\mathcal{H}_k(\mathtt{y}, \mathtt{x})$ in 1.2.60 and discuss their global nearly–integrable structure with exponential small perturbations (compare Theorem 1.5.1 below).

As mentioned above (see item (ii) in Remark 1.3.10), the problem here is that the symplectic transformations of Proposition 1.3.1, which put the simply–resonant Hamiltonians $\mathtt{H}_k$ in 2.7.2 in standard form, are, in general, *not well defined in the fast angles* $\hat{\mathtt{q}} = (\mathtt{q}_2, ..., \mathtt{q}_n)$, making the construction of *global* action–angle variables for the full Hamiltonians $\mathcal{H}_k(\mathtt{y}, \mathtt{x})$ in 1.2.60 not straightforward.

To overcome such homotopy problems, we shall exploit the particular group structure of the various symplectic transformations involved, and show that, introducing a special *ad hoc* conjugacy, one can indeed obtain globally well defined symplectic maps; see, in particular, (1.5.21) below.

Special sets of symplectic transformations

We shall introduce three special classes of symplectic transformations, which will be used in the proof of Theorem 1.5.1.

Definition 1.5.1. (a) *Given a domain* $\hat{\mathrm{D}} \subseteq \mathbb{R}^{n-1}$*,* $\mathfrak{G}$ *denotes the formal*[22] *group of symplectic transformations of the form*

$$(p, q) \in \mathrm{D} \times \mathbb{T}^n \overset{\Phi}{\mapsto} (P, Q) = (\eta, \hat{p}, q_1 + \psi, \hat{q} + \chi) \in \mathbb{R}^n \times \mathbb{T}^n\,,$$

where: $\mathrm{D} \subseteq \mathbb{R}^n$ *is a normal smooth domain*[23] *over* $\hat{\mathrm{D}}$*, the functions* η, ψ, χ *depend on* (p, q_1)*, are* 2π*–periodic in* q_1 *and the* $(n+1)$*–dimensional the map*

$$(p, q_1) \mapsto \check{\Phi}(p, q_1) = (\eta, \hat{p}, q_1 + \psi)$$

is injective.

(b) *Given a domain* $\hat{\mathrm{D}} \subseteq \mathbb{R}^{n-1}$*,* $\mathfrak{G}_0$ *denotes the set of smooth symplectic transformations of the form*

$$(p, q) \in \mathrm{D} \times \mathbb{T}^n \overset{\Phi}{\mapsto} (P, Q) = (\eta, \hat{p}, \psi, \hat{q} + \chi) \in \mathbb{R}^{n+1} \times \mathbb{T}^{n-1}\,,$$

where $\mathrm{D} \subseteq \mathbb{R}^n$ *is a normal smooth domain over* $\hat{\mathrm{D}}$*; the functions* η, ψ, χ *depend only on* (p, q_1) *and are* 2π*–periodic in* q_1*.*

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[22] See Remark 1.5.1–(iii) below.

[23] I.e., $D = \{(p_1, \hat{p}) : \ \alpha(\hat{p}) < p_1 < \hat{\beta}(\hat{p})\,,\ \hat{p} \in \hat{\mathrm{D}}\}$ where α and β are smooth function on $\hat{\mathrm{D}}$.

Let us collect a few observations and discuss the main properties of such classes, but, first of all, notice that all the above maps leave fixed the variable $\hat{p} \in \hat{\mathrm{D}} \subseteq \mathbb{R}^n$ and the set $\hat{\mathrm{D}}$. Thus, in the following discussion, the domain $\hat{\mathrm{D}}$ is fixed once and for all.

Remark 1.5.1. (i) The Arnol'd–Liouville map Φ^i in the outer cases (1.4.23) ($i = 0, 2N$) belongs to $\mathfrak{G}$ (since $\sup |\partial_{q_1}\psi| < 1$), while Φ^i in the inner case (1.4.23) ($0 < i < 2N$) belongs to $\mathfrak{G}_0$.
Notice also that Φ_2 in Theorem 1.3.1–(iii) is a near–to–the–identity symplectic map belonging to $\mathfrak{G}$.
(ii) In the definition of $\mathfrak{G}$ and $\mathfrak{G}_0$, the functions η and ψ are scalar functions, while χ has $(n-1)$ components. Notice that, since Φ is assumed to be symplectic, these maps are such that

$$
\begin{aligned}
d\eta \wedge dq_1 + d\eta \wedge d\psi + d\hat{p} \wedge d\chi &= dp_1 \wedge dq_1\,, && (\Phi \in \mathfrak{G})\,,\\
d\eta \wedge d\psi + d\hat{p} \wedge d\chi &= dp_1 \wedge dq_1\,, && (\Phi \in \mathfrak{G}_0).
\end{aligned}
$$

(iii) All maps in the group $\mathfrak{G}_\dagger$ in Definition 1.3.5 have a common domain of definition, i.e., $(\mathbb{R} \times \hat{\mathrm{D}}) \times \mathbb{R}^n$. On the other hand, every map $\Psi \in \mathfrak{G}$ has its own domain of definition D. Thus, the composition $\Psi_1 \circ \Psi_2$ of two maps in $\mathfrak{G}$

$$
\Psi_1 : \mathrm{D}_1 \times \mathbb{T}^n \to \mathbb{R}^n \times \mathbb{T}^n\,, \qquad \Psi_2 : \mathrm{D}_2 \times \mathbb{T}^n \to \mathbb{R}^n \times \mathbb{T}^n
$$

is well defined *only when the compatibility condition* $\Psi_2\big(\mathrm{D}_2 \times \mathbb{T}^n\big) \subseteq \mathrm{D}_1 \times \mathbb{T}^n$ *is satisfied*. This is the reason why the cautionary word 'formal' appears in the definition of $\mathfrak{G}$. However, as already noticed, all maps in $\mathfrak{G}$ verify $\pi_{\hat{p}}(\mathrm{D}) = \hat{\mathrm{D}}$, which is fixed *a priori*.
(iv) If $\Phi \in \mathfrak{G}$, by definition $\check{\Phi}$ is injective, so that also Φ itself is injective. Furthermore, for any fixed p, the map $q_1 \to Q_1 = q_1 + \psi$ is a continuous injective map on the circle $\mathbb{T}^1$, hence it is surjective, and, therefore, it is a smooth (orientation preserving) circle diffeomorphism. Thus, $q \to Q = (q_1 + \psi, \hat{q} + \chi)$ *is a global diffeomorphism of* $\mathbb{T}^n$, and $\Phi : \mathrm{D} \times \mathbb{T}^n \to \Phi(\mathrm{D} \times \mathbb{T}^n) \subseteq \mathbb{R}^n \times \mathbb{T}^n$ *is a global symplectomorphism*.
Notice also that *if* $\Phi, \Phi' \in \mathfrak{G}$ *and the composition* $\Phi \circ \Phi'$ *is well defined, then* $\Phi \circ \Phi' \in \mathfrak{G}$.
(v) The definition of the first $(n+1)$ component of any member of the above families depends only on the first $(n+1)$ variables (p, q_1). Therefore, any finite compositions of maps $\Psi_i \in \mathfrak{G}_\dagger \cup \mathfrak{G} \cup \mathfrak{G}_0$, $1 \leqslant i \leqslant m$, *whenever the composition is well defined*, satisfies

$$
\Psi_i \in \mathfrak{G}_\dagger \cup \mathfrak{G} \cup \mathfrak{G}_0 \quad \Longrightarrow \quad \big(\Psi_1 \circ \cdots \circ \Psi_m\big)\check{} = \big(\check{\Psi}_1 \circ \cdots \circ \check{\Psi}_m\big)\,. \tag{1.5.1}
$$

(vi) Finally, one readily verifies that the following property holds:

$$
\Phi \in \mathfrak{G}_0 \quad \text{and} \quad \Psi \in \mathfrak{G}_\dagger \cup \mathfrak{G} \qquad \Longrightarrow \qquad \Psi \circ \Phi \in \mathfrak{G}_0\,. \tag{1.5.2}
$$

Action–angles variables for the secular standard Hamiltonians $\mathtt{H}_k$ at simple resonances

For each $k \in \mathcal{G}^n_{\mathtt{K}_o}$, we may apply the theory of § 1.4 to the secular Hamiltonians described in Theorem 1.3.1 in standard form $\mathtt{H}_\flat = \mathtt{H}_k$; see 2.7.3 and 2.7.5.

By (1.4.24), we get that, for every $k \in \mathcal{G}^n_{\mathtt{K}_o}$ and $0 \leqslant i \leqslant 2N_k$, the Arnol'd–Liouville map

$$\Phi^i \ : \ \mathcal{B}^i_k \times \mathbb{T}^n \ \xrightarrow{\text{onto}} \ \check{\mathcal{M}}^i_k \times \mathbb{T}^{n-1} \tag{1.5.3}$$

integrates $\mathtt{H}_k$, i.e.:

$$(\mathtt{H}_k \circ \Phi^i)(I,\varphi) = (\mathtt{H}_k \circ \check{\Phi}^i)(I,\varphi_1) = \mathtt{E}^{(i)}_k(I)\,, \qquad \forall\ 0 \leqslant i \leqslant 2N_k\,, \tag{1.5.4}$$

where $\mathcal{B}^i_k$, $\check{\mathcal{M}}^i_k$ and $\mathtt{E}^{(i)}_k$ correspond to $\mathcal{B}^i$, $\check{\mathcal{M}}^i$ and $\mathtt{E}^{(i)}$ in § 1.4 in the case[24] $\mathtt{H}_\flat = \mathtt{H}_k$.

Beware that, even if sometimes, for ease of notation, we do not report the dependence upon the resonance label $k \in \mathcal{G}^n_{\mathtt{K}_o}$, we are treating different Hamiltonians in the neighbourhoods of simple resonances labelled by $k \in \mathcal{G}^n_{\mathtt{K}_o}$.

Finally, we shall use the following notations: Given a function $\mathtt{g} : \hat{\mathrm{D}} \to \mathbb{R}$, we shall denote by $\jmath_{\mathtt{g}}$ the translation

$$\jmath_{\mathtt{g}}(p) := (p_1 + \mathtt{g}(\hat{p}), \hat{p})\,. \tag{1.5.5}$$

Notice that, by the definition of $\Psi_{\mathtt{g}}$ in (1.3.41), one has

$$\check{\Psi}_{\mathtt{g}}(p, q_1) = (\jmath_{\mathtt{g}}(p), q_1)\,. \tag{1.5.6}$$

Global action–angle variables at simple resonances

We are now ready to state and prove the first step of the proof of Theorem 1.1.1, which consists in showing how to construct symplectic action–angle maps which put a generic nearly–integrable natural systems, near *critical surfaces* of simple resonances, *for all* $k \in \mathcal{G}^n_{\mathtt{K}_o}$, into uniform analytic nearly–integrable form *with exponentially small perturbations*:

Let $\mathcal{S}_k$ be a critical surface of h^k, and let $\mathtt{D}_{\mathtt{r}}(\bar{y})$ the neighborhood of a point of this surface in which one can do the standard form conjugation as be shown in proposition 1.3.1. Let assumptions 1.3.30, 1.2.23 and 1.2.22 hold; let $\mathbf{c}_{\mathtt{o}}$ be as in Theorem 1.2.2,

[24] Compare, in particular, (1.4.8) and (1.4.9) for the definitions of $\mathcal{M}^i_k$ and $\check{\mathcal{M}}^i_k$; (1.4.21) for the definition of $\mathcal{B}^i_k$; (1.4.28) and (1.4.25) for the definition of $\mathcal{B}^i_k(\lambda)$; the definition of $\mathcal{M}^i_k(\lambda)$ is given (1.4.34).

and $\hat{\mathbf{c}}$ as in Theorem 1.4.2 with κ as in (2.7.5). Let $\mathtt{g}_1$ be as in (ii) of Theorem 1.3.1, and define

$$B_k^i := \begin{cases} \mathcal{B}_k^i\,, & \text{if } \ 0 < i < 2N_k\,, \\ \jmath_{-\mathtt{g}_*}\big(\mathcal{B}_k^i\big)\,, & \text{if } \ i = 0, 2N_k\,, \end{cases} \qquad \mathtt{g}_* := -\mathtt{g}_1\,. \tag{1.5.7}$$

Then, the following result holds.

Theorem 1.5.1. (Secondary nearly–integrable structure at simple resonances) *There exists $\mathbf{c}_* = \mathbf{c}_*(n,s,\beta,\delta) \geqslant \max\{\mathbf{c}_o, \hat{\mathbf{c}}\}$ such that if $\mathtt{K}_o \geqslant \mathbf{c}_*$, then for any $k \in \mathcal{G}^n_{\mathtt{K}_o}$, $0 \leqslant i \leqslant 2N_k$, there exist real analytic symplectomorphisms*

$$\phi_k^i : B_k^i \times \mathbb{T}^n \to \mathrm{Re}\,(\mathtt{D}_\mathtt{r}(0, \hat{\mathtt{y}})) \times \mathbb{T}^n\,, \tag{1.5.8}$$

such that, if $\mathtt{E}_k^i = \mathtt{E}_k^i(I)$ is the integrable Hamiltonian $\mathtt{H}_k$ of Theorem 1.3.1 in its Arnol'd–Liouville action variables, $\widetilde{\mathtt{E}}_k^i := \mathtt{E}_k^i \circ \jmath_{\mathtt{g}_*}$, and $\hat{h}_k$ is as in Theorem 1.3.1, then

$$\begin{aligned} &\mathcal{H}_k^i := \mathtt{H} \circ \phi_k^i(I,\varphi) = h_k^i(I) + \varepsilon f_k^i(I,\varphi)\,, \quad \textit{with:} \\ &h_k^i := \mathtt{h}_k^i\,,\ \mathtt{h}_k^i := \begin{cases} \jmath^k(\widehat{I}) + \jmath_0^k(\widehat{I})\mathtt{E}_k^i\,, & \text{if } 0 < i < 2N_k\,, \\ \jmath^k(\widehat{I}) + \jmath_0^k(\widehat{I})\widetilde{\mathtt{E}}_k^i & \text{if } i = 0, 2N_k\,. \end{cases} \end{aligned} \tag{1.5.9}$$

Furthermore, for $0 < \lambda \leqslant 1/\mathbf{c}_*$ define:

$$\begin{aligned} &\rho_* := \frac{\sqrt{\epsilon}}{\mathbf{c}_* \mathtt{K}_o^n}\,\lambda|\log\lambda|\,, \qquad \sigma_* := \frac{1}{\mathbf{c}_* \mathtt{K}_o^n |\log\lambda|}\,, \\ &B_k^i(\lambda) := \begin{cases} \mathcal{B}_k^i(\lambda)\,, & \text{if } \ 0 < i < 2N_k\,, \\ \jmath_{-\mathtt{g}_*}\big(\mathcal{B}_k^i(\lambda)\big)\,, & \text{if } \ i = 0, 2N_k\,, \end{cases} \qquad \forall\ 0 \leqslant \lambda < 1/\mathbf{c}_*\,. \end{aligned} \tag{1.5.10}$$

Then, ϕ_k^i admits a holomorphic extension

$$\phi_k^i : (B_k^i(\lambda))_{\rho_*} \times \mathbb{T}^n_{\sigma_*} \to \mathtt{D}_\mathtt{r}(0, \hat{\mathtt{y}}) \times \mathbb{T}^n_{s_*} \tag{1.5.11}$$

and the perturbation f_k^i in (1.5.9) satisfies the exponential estimate

$$\sup_{(B_k^i(\lambda))_{\rho_*} \times \mathbb{T}^n_{\sigma_*}} |f_k^i| \leqslant e^{-\mathtt{K}s/3}\,. \tag{1.5.12}$$

Remark 1.5.2. (i) Notice that, since $\mu = 1/\mathtt{K}^{5n}$ (see (1.3.30)), and since[25]

$$\mathtt{K} > \mathtt{K}_o \geqslant \mathbf{c}_* > \mathbf{c}\,,$$

[25]The constant $\mathbf{c}$ is defined in Theorem 1.4.1.

condition (1.4.30) – which is stronger than condition (1.4.10) – is implied by the assumption $\mathtt{K}_o \geqslant \mathbf{c}_*$.
Observe also that from the definitions of the constants in Theorem 1.5.1, Theorem 1.4.1 and from (1.4.27) it follows that

$$\mathbf{c}_* \geqslant \mathbf{c} \geqslant 2^8\kappa^3 \geqslant 2^{14}\,, \qquad \lambda_{\max} \geqslant 2^{12}/\mathbf{c}_*\,. \tag{1.5.13}$$

Finally, we remark that, recalling the definitions of ρ_λ and σ_λ in (1.4.32), since $\mathbf{c}_* \geqslant \hat{\mathbf{c}}$, one has

$$\rho_* < \rho_\lambda\,, \qquad \sigma_* < \sigma_\lambda\,. \tag{1.5.14}$$

(ii) In the proof of the theorem the maps ϕ_k^i are explicitly given; compare (1.5.19) and (1.5.25) below.

The following simple lemma will be one of the key points of the proof of Theorem 1.5.1. Recall Definition 1.5.1.

Lemma 1.5.1. *Let* $\Phi : (p,q) \in \mathrm{D}\times\mathbb{T}^n \mapsto (\eta, \hat{p}, q_1 + \psi, \hat{q} + \chi) \in \mathbb{R}^n \times \mathbb{T}^n$ *be in* $\mathfrak{G}$, $\Psi_{\mathrm{g}} \in \mathfrak{G}_\dagger$, *and denote by* $\tau_{\mathrm{g}}\Phi$ *the map*

$$\tau_{\mathrm{g}}\Phi := \tau_{\mathrm{g}}\Phi(p,q) := \big(\eta_{\mathrm{g}} + \mathrm{g}, \hat{p}, q_1 + \psi_{\mathrm{g}}, \hat{q} + \chi_{\mathrm{g}} - \psi_{\mathrm{g}}\partial_{\hat{p}}\mathrm{g}\big)\,, \tag{1.5.15}$$

where for a function $u : D\times\mathbb{T} \to \mathbb{R}^m$, u_{g} *denotes the map*

$$u_{\mathrm{g}} := u \circ \check{\Psi}_{-\mathrm{g}} : j_{\mathrm{g}}(D)\times\mathbb{T} \to \mathbb{R}^m\,. \tag{1.5.16}$$

Then, $\tau_{\mathrm{g}}\Phi$ *belongs to* $\mathfrak{G}$ *and it is a symplectomorphism satisfying*

$$\tau_{\mathrm{g}}\Phi : j_{\mathrm{g}}(\mathrm{D})\times\mathbb{T}^n \xrightarrow{\text{onto}} \big(\check{\Psi}_{\mathrm{g}} \circ \Phi(\mathrm{D}\times\mathbb{T}^n)\big)\times\mathbb{T}^{n-1}\,, \tag{1.5.17}$$

and

$$(\tau_{\mathrm{g}}\Phi)\check{} = (\eta_{\mathrm{g}} + \mathrm{g}, \hat{p}, q_1 + \psi_{\mathrm{g}}) = \check{\Psi}_{\mathrm{g}} \circ \check{\Phi} \circ \check{\Psi}_{-\mathrm{g}}\,. \tag{1.5.18}$$

Proof. First observe that since $\eta_{\mathrm{g}}, \psi_{\mathrm{g}}, \chi_{\mathrm{g}}$ are 2π-periodic in q_1, the map

$$q \in \mathbb{T}^n \mapsto \pi_Q \tau_{\mathrm{g}}\Phi(p,q) = \big(q_1 + \psi_{\mathrm{g}}, \hat{q} + \chi_{\mathrm{g}} - \psi_{\mathrm{g}}\partial_{\hat{p}}\mathrm{g}\big) \in \mathbb{T}^n$$

is a well defined $\mathbb{T}^n$–map and (1.5.18) follows immediately by direct computation. Thus, $(\tau_{\mathrm{g}}\Phi)\check{}$ is injective being the composition of three injective maps, and, therefore, the whole map $\tau_{\mathrm{g}}\Phi$ is injective, and (1.5.17) follows. To check symplecticity, just note that, locally, on the universal cover $\mathbb{R}^{2n}$, $\tau_{\mathrm{g}}\Phi$ coincides (as it is immediate to check) with the composition $\Psi_{\mathrm{g}} \circ \Phi \circ \Psi_{-\mathrm{g}}$ of three symplectic maps. Hence $\tau_{\mathrm{g}}\Phi$ is symplectic and the claim follows. $\square$

Proof. **of Theorem 1.5.1** We start by defining the maps ϕ_k^i.
Consider, first, the inner case $0 < i < 2N_k$. Recall Definition 1.5.1. By Theorem 1.3.1–(iii), Φ_* is the composition of maps in $\mathfrak{G}_\dagger$ and $\mathfrak{G}$ while, for $0 < i < 2N_k$, $\Phi^i \in \mathfrak{G}_0$ (Remark 1.5.1–(i)). Hence, by (1.5.2), it follows that $\Phi_* \circ \Phi^i \in \mathfrak{G}_0$ and we may define[26]

$$\Phi_*^i := \Phi_* \circ \Phi^i\,, \quad \phi_k^i := \Psi^k \circ \Phi_*^i : B_k^i \times \mathbb{T}^n \to \mathbb{R}^n \times \mathbb{T}^n\,, \qquad (0 < i < 2N_k)\,, \quad (1.5.19)$$

provided the composition is well defined. To check that this is the case, we observe that by (1.5.1), (1.4.31), (1.3.32), (2.7.2), (1.5.14) for $0 < \lambda \leqslant 1/\hat{\mathtt{c}}$, we get

$$\check{\Phi}_*^i = (\Phi_* \circ \Phi^i)^\vee = \check{\Phi}_* \circ \check{\Phi}^i : (\mathcal{B}_k^i(\lambda))_{\rho_\lambda} \times \mathbb{T}_{\sigma_\lambda} \to \mathtt{D}_{\mathtt{r}}(0, \hat{\hat{\mathtt{y}}}) \times \mathbb{T}_{\check{\mathtt{s}}}\,, \qquad (0 < i < 2N_k)\,, \quad (1.5.20)$$

thus the composition is well defined and (1.5.19) is well posed.
Let us now consider the outer case $i = 0, 2N_k$. In this case $\Phi^i \in \mathfrak{G}$ (Remark 1.5.1–(i)). Recalling the definition in (1.5.15)–(1.5.16), by Lemma 1.5.1, we may define

$$\Phi_*^i := \Phi_2 \circ \tau_{\mathtt{g}_1} \Phi^i\,, \qquad (i = 0, 2N_k)\,. \qquad (1.5.21)$$

Recalling that $\Phi_2 \in \mathfrak{G}$, by Lemma 1.5.1 and Remark 1.5.1–(iv), $\Phi_*^i \in \mathfrak{G}$, provided the compositions are well defined. To check that this is the case, as above, it is enough to control the complex domains of the first $(n+1)$ components. By (1.5.18) (used twice), (1.3.42), (2.7.6), and (1.5.1), one finds[27]

$$\check{\Phi}_*^i = \check{\Phi}_* \circ \check{\Phi}^i \circ \check{\Psi}_{\mathtt{g}_*}\,, \qquad (i = 0, 2N_k)\,. \qquad (1.5.22)$$

Then, by (1.5.40), we get,

$$j_{\mathtt{g}_*}\big((B_k^i(\lambda))_{\rho'_\lambda}\big) \subseteq \big(j_{\mathtt{g}_*}(B_k^i(\lambda))\big)_{\rho_\lambda} \overset{(1.5.10)}{=} \big(\mathcal{B}_k^i(\lambda)\big)_{\rho_\lambda}\,, \quad \text{where } \rho'_\lambda := \tfrac{\rho_\lambda}{n+2}\,, \qquad (i = 0, 2N_k)\,. \quad (1.5.23)$$

Observing that $\check{\Psi}_{\mathtt{g}_*}(p, q_1) = \big(j_{\mathtt{g}_*}(p), q_1\big)$, by (1.5.22), (1.5.23), (1.4.31), (1.3.32) and (2.7.2), we get, for $0 < \lambda \leqslant 1/\hat{\mathtt{c}}$,

$$\check{\Phi}_*^i : (B_k^i(\lambda))_{\rho'_\lambda} \times \mathbb{T}_{\sigma_\lambda} \to \mathtt{D}_{\mathtt{r}}(0, \hat{\hat{\mathtt{y}}}) \times \mathbb{T}_{\check{\mathtt{s}}}\,, \qquad (i = 0, 2N_k)\,. \qquad (1.5.24)$$

Thus, the composition is well defined and (1.5.21) is well posed. So, we may define:

$$\phi_k^i := \Psi^k \circ \Phi_*^i : B_k^i \times \mathbb{T}^n \to \mathbb{R}^n \times \mathbb{T}^n\,, \qquad \Phi_*^i \text{ as in (1.5.21)}\,, \qquad (i = 0, 2N_k)\,. \quad (1.5.25)$$

[26] Ψ^k appears in Theorem 1.2.2. Recall that, when $0 < i < 2N_k$, $B_k^i := \mathcal{B}_k^i$.
[27] Recall that $\mathtt{g}_* = -\mathtt{g}_1$; compare (1.5.7).

We can, now, prove (1.5.9). Recall the definition of $\bar f^k$ in Theorem 1.2.2 and define

$$f_k^i := f \circ \phi_k^i \overset{(1.2.60)}{=} \bar f^k \circ \Phi_*^i\,, \qquad (0 \leqslant i \leqslant 2N_k)\,. \tag{1.5.26}$$

Then, by definition of ϕ_k^i in (1.5.19) and (1.5.25), we have, for $0 \leqslant i \leqslant 2N_k$,

$$\mathcal{H}_k^i := \mathtt{H} \circ \phi_k^i(I,\varphi) := \mathtt{H} \circ \Psi^k \circ \Phi_*^i \overset{(1.2.60),(1.5.26)}{=} \overline{\mathtt{H}}_k \circ \Phi_*^i + \varepsilon f_k^i\,. \tag{1.5.27}$$

Since $\overline{\mathtt{H}}_k$ in (1.2.61) depends only on the first $(n+1)$ variables, by (1.5.20) and (1.5.22), we find

$$\overline{\mathtt{H}}_k \circ \Phi_*^i = \overline{\mathtt{H}}_k \circ \check\Phi_*^i = \begin{cases} \overline{\mathtt{H}}_k \circ \check\Phi_* \circ \check\Phi^i\,, & \text{if } 0 < i < 2N_k \\ \overline{\mathtt{H}}_k \circ \check\Phi_* \circ \check\Phi^i \circ \check\Psi_{\mathtt{g}_*}\,, & \text{if } i = 0, 2N_k\,, \end{cases} \tag{1.5.28}$$

and, by (2.7.2) and (1.5.4),

$$\overline{\mathtt{H}}_k \circ \check\Phi_* \circ \check\Phi^i = j^k + j_0^k \mathtt{E}_k^{(i)}\,. \tag{1.5.29}$$

Thus, (1.5.9) follows from (1.5.27), (1.5.28), (1.5.29) and (1.5.6).
Next, we show that ϕ_k^i has, for $0 < \lambda \leqslant 1/\mathbf{c}_*$, a holomorphic extension satisfying (1.5.11). To do this we have to consider the last $n-1$ components of Φ_*^i, namely $\pi_{\hat\varphi}\Phi_*^i = \hat\Phi_*^i$. By definition of Φ_*^i in (1.5.19) and (1.5.21) it follows that

$$\hat\Phi_*^i(I,\varphi) = \begin{cases} \hat\varphi + \chi_*^i(I,\varphi_1)\,, & \text{if } 0 < i < 2N_k\,, \\ \hat\varphi + \chi_*^i\big(j_{\mathtt{g}_*}(I),\varphi_1\big)\,, & \text{if } i = 0, 2N_k\,, \end{cases} \tag{1.5.30}$$

with; $\mathtt{g}_*$ is defined in (1.5.7); χ_2 is as in Theorem 1.3.1–(iii).

$$\chi_*^i := \chi^i + \chi_2^\flat + \psi^i\,\partial_{\hat I}\mathtt{g}_*\,, \qquad \chi_2^\flat(I,\varphi_1) := \begin{cases} \chi_2(\hat I,\psi^i)\,, & \text{if } 0 < i < 2N_k\,, \\ \chi_2(\hat I,\varphi_1+\psi^i)\,, & \text{if } i = 0, 2N_k\,. \end{cases} \tag{1.5.31}$$

Now, we claim that

$$|\psi^i|_{\rho_\lambda,\sigma_\lambda} < \frac{3}{4}\check{\mathtt{s}}\,, \qquad \forall\ 0 \leqslant i \leqslant 2N_k\,. \tag{1.5.32}$$

Indeed, if $0 < i < 2N_k$, (1.5.32) follows directly from (1.4.31) and (1.3.32); in the case $i = 0, 2N_k$, (1.5.32) follows again from (1.4.31) and (1.3.32) observing that

$$|\psi^i|_{\rho_\lambda,\sigma_\lambda} = |(\varphi_1 + \psi^i) - \varphi_1|_{\rho_\lambda,\sigma_\lambda} \leqslant \frac{\mathtt{s}}{4} + \sigma_\lambda < \frac{3}{4}\check{\mathtt{s}}\,.$$

Next, since $\rho'_\lambda = \rho_\lambda/(n+2)$, by (1.5.23), (1.5.31), (1.4.31), (1.3.32), (2.7.8), (1.4.36), (1.4.37), (1.5.32), (1.5.40), we find, for every $0 \leqslant i \leqslant 2N_k$, and for every $2 \leqslant \ell \leqslant n$,

$$\begin{aligned} |\operatorname{Im}\hat{\Phi}^i_{*\ell}|_{\rho'_\lambda,\sigma_\lambda} &\leqslant |\operatorname{Im}(\varphi_\ell + \chi^i_{*\ell})|_{\rho_\lambda,\sigma_\lambda} \\ &\leqslant |\operatorname{Im}(\varphi_\ell + \chi^i_\ell)|_{\rho_\lambda,\sigma_\lambda} + |\chi^\flat_2|_{\rho_\lambda,\sigma_\lambda} + |\psi^i|_{\rho_\lambda,\sigma_\lambda}\,|\partial_{\hat{I}}\mathtt{g}_*|_{\rho_\lambda} \\ &\leqslant \frac{\check{\mathtt{s}}}{2} + \frac{\check{\mathtt{s}}}{2^{20}} + \frac{3}{4}(n+1)\check{\mathtt{s}} < 2n\check{\mathtt{s}}\,. \end{aligned} \tag{1.5.33}$$

Thus, by (1.5.20), (1.5.24) and (1.5.33), we get

$$\Phi^i_* : (B^i_k(\lambda))_{\rho'_\lambda} \times \mathbb{T}^n_{\sigma_\lambda} \to \mathtt{D}_\mathtt{r}(0,\hat{\hat{\mathtt{y}}}) \times \mathbb{T}^n_{2n\check{\mathtt{s}}}\,, \qquad (0 \leqslant i \leqslant 2N_k)\,.$$

We need, now, an elementary result on real analytic functions:

Lemma 1.5.2. *Let $g : D_r \times \mathbb{T}^n_s \to \mathbb{C}$ be a real analytic function satisfying $|\operatorname{Im} g| \leqslant \xi$. Then, for every $0 < \zeta \leqslant 1/2$, one has*

$$\sup_{D_{\zeta r}\times\mathbb{T}^n_{\zeta s}} |\operatorname{Im} g| \leqslant 8\zeta\xi\,.$$

Now, define

$$\zeta := \frac{1}{16\,c_1\,\mathtt{c}_s\,\mathtt{K}_\mathtt{o}^n}\,. \tag{1.5.34}$$

Then, since $|k| \leqslant \mathtt{K}_\mathtt{o}$, by (2.7.2), (1.3.30), we find

$$8\zeta(2n\check{\mathtt{s}}) < 16\,n\zeta\,\mathtt{K}_\mathtt{o}\max\{1,s\} \overset{(1.5.34)}{=} \frac{\max\{1,s\}}{c_1\,\mathtt{c}_s\,\mathtt{K}_\mathtt{o}^{n-1}} = \frac{s}{c_1\mathtt{K}_\mathtt{o}^{n-1}} \overset{(1.3.30)}{=} \tilde{s}_k\,.$$

Thus, by Lemma 1.5.2 (applied with $g = \hat{\Phi}^i_{*\ell}$ for $2 \leqslant \ell \leqslant n$, ζ as in (1.5.34) and $\xi = 2\check{\mathtt{s}}$), it follows that

$$\Phi^i_* : (B^i_k(\lambda))_{\rho_*} \times \mathbb{T}^n_{\sigma_*} \to \mathtt{D}_\mathtt{r}(0,\hat{\hat{\mathtt{y}}}) \times \mathbb{T}^n_{\tilde{s}_k}\,, \qquad (0 \leqslant i \leqslant 2N_k)\,,$$

with ρ_* and σ_* as in (1.5.10), provided

$$\mathbf{c}_* := \max\{\mathbf{c}_0\,,\,\hat{\mathbf{c}}\,c_1\,\mathtt{c}_s\,16\,(n+2)\}\,.$$

In conclusion, (1.5.11) follows by the definition of ϕ^i_k in (1.5.19), (1.5.25) and by (1.2.58).

Finally, estimate (1.5.12) follows at once from (1.5.26), (1.5.11) and (1.3.29). The proof is complete. $\square$

The following measure estimate will play a crucial rôle in the proof of Theorem 1.1.1.

Proposition 1.5.1. *For every* $0 \leqslant \lambda < 1/\mathbf{c}_*$, *the following measure estimate holds*[28]*:*

$$\mathrm{meas}\Big(\big(\mathtt{D}_{\mathtt{r}}(0,\hat{\hat{\mathtt{y}}}) \times \mathbb{T}^n\big) \setminus \bigcup_{0\leqslant i\leqslant 2N_k} \phi_k^i\big(B_k^i(\lambda)\times\mathbb{T}^n\big)\Big) \leqslant \mathbf{c}_*\, \mathrm{meas}\big(\widetilde{\mathfrak{R}}^{1,k}\times\mathbb{T}^n\big)\,\lambda|\log\lambda|\,. \tag{1.5.35}$$

Proof. Since $\check{\Phi}_*^i$ depends only on the first $(n+1)$ variables, by (1.5.30), (1.5.19), (1.5.22) and the definitions of $B_k^i(\lambda)$ in (1.5.10) and $\check{\mathcal{M}}_k^i(\lambda)$ in (1.5.3), one has

$$\begin{aligned}\Phi_*^i\big(B_k^i(\lambda)\times\mathbb{T}^n\big) &= \check{\Phi}_*^i\big(B_k^i(\lambda)\times\mathbb{T}\big)\times\mathbb{T}^{n-1} = \big(\check{\Phi}_*\circ\check{\Phi}^i(\mathcal{B}_k^i(\lambda)\times\mathbb{T})\big)\times\mathbb{T}^{n-1}\\ &\overset{(1.4.35)}{=} \big(\check{\Phi}_*\circ\check{\mathcal{M}}_k^i(\lambda)\big)\times\mathbb{T}^{n-1}\,.\end{aligned} \tag{1.5.36}$$

Analogously, one has

$$\Phi_*^{-1}(\mathtt{D}_{\mathtt{r}}(0,\hat{\hat{\mathtt{y}}})\times\mathbb{T}^n) = \check{\Phi}_*^{-1}(\mathtt{D}_{\mathtt{r}}(0,\hat{\hat{\mathtt{y}}})\times\mathbb{T})\times\mathbb{T}^{n-1}\,. \tag{1.5.37}$$

Observe also that, by (2.7.7), (2.7.8) and (the second estimate in) (1.4.36) it follows that[29]

$$\check{\Phi}_{{}_1}^{-1}\circ\check{\Phi}_{{}_2}^{-1}(\mathtt{D}(0,\hat{\hat{\mathtt{y}}})\times\mathbb{T}) \subseteq \big((-\mathtt{R}-\mathtt{r}/3,\mathtt{R}+\mathtt{r}/3)\times\hat{\mathtt{D}}_{\mathtt{r}}(\hat{\hat{\mathtt{y}}})\big)\times\mathbb{T} = (D^\flat\times\mathbb{T})\,. \tag{1.5.38}$$

Then[30], recalling Theorem 1.2.2, using the fact that $(\Psi^k)^{-1}$ and Φ_*^{-1} are diffeomorphysms preserving Liouville measure, we find

$$\begin{aligned}&\mathrm{meas}(\mathtt{D}_{\mathtt{r}}(0,\hat{\hat{\mathtt{y}}})\times\mathbb{T}^n\setminus\textstyle\bigcup\phi_k^i(B_k^i(\lambda)\times\mathbb{T}^n))\\
\overset{(1.5.19),(1.5.25)}{=}\;& \mathrm{meas}((\Psi^k)^{-1}(\mathtt{D}_{\mathtt{r}}(0,\hat{\hat{\mathtt{y}}})\times\mathbb{T}^n)\setminus\textstyle\bigcup\Phi_*^i(B_k^i(\lambda)\times\mathbb{T}^n))\\
\overset{(2.6.1)}{\leqslant}\;& \mathrm{meas}((\check{\mathscr{D}}^k\times\mathbb{T}^n)\setminus\textstyle\bigcup\Phi_*^i(B_k^i(\lambda)\times\mathbb{T}^n))\\
=\;& \mathrm{meas}(\Phi_*^{-1}(\check{\mathscr{D}}^k\times\mathbb{T}^n)\setminus\textstyle\bigcup\Phi_*^{-1}\Phi_*^i(B_k^i(\lambda)\times\mathbb{T}^n))\\
\overset{(1.5.36),(1.5.37)}{=}\;& (2\pi)^{n-1}\,\mathrm{meas}(\check{\Phi}_*^{-1}(\check{\mathscr{D}}^k\times\mathbb{T})\setminus\textstyle\bigcup\check{\mathcal{M}}_k^i(\lambda))\\
\overset{(2.7.6,??)}{=}\;& (2\pi)^{n-1}\,\mathrm{meas}(\check{\Phi}_{{}_1}^{-1}\circ\check{\Phi}_{{}_2}^{-1}(\mathtt{D}(0,\hat{\hat{\mathtt{y}}})\times\mathbb{T})\setminus\textstyle\bigcup\check{\mathcal{M}}_k^i(\lambda))\\
\overset{(1.5.38)}{\leqslant}\;& (2\pi)^{n-1}\,\mathrm{meas}(D^\flat\times\mathbb{T}\setminus\textstyle\bigcup\check{\mathcal{M}}_k^i(\lambda))\\
\overset{(1.4.35)}{\leqslant}\;& (2\pi)^{n-1}\hat{\mathbf{c}}\sqrt{\epsilon}\,\mathrm{meas}(\hat{\mathtt{D}}(\hat{\hat{\mathtt{y}}}))\,\lambda|\log\lambda|\\
\overset{(1.3.30)}{<}\;& (2\pi)^{n-1}\hat{\mathbf{c}}\;\mathtt{R}\;\mathrm{meas}(\hat{\mathtt{D}}(\hat{\hat{\mathtt{y}}}))\,\lambda|\log\lambda|\\
=\;& \frac{\hat{\mathbf{c}}}{2\pi}\,\mathrm{meas}(\widetilde{\mathfrak{R}}^{1,k}\times\mathbb{T}^n)\,\lambda|\log\lambda|\,,\end{aligned}$$

[28] The sets $\widetilde{\mathcal{R}}^{1,k}$ are defined in (1.2.6).

[29] Observe that $\check{\Phi}_{{}_1}^{-1}(p,q) = (p_1 - \mathtt{g}_{{}_1}(\hat{p}),\hat{p},q_1)$ and $\check{\Phi}_{{}_2}^{-1}(p,q) = (p_1 - \eta_{{}_2}(\hat{p},q_1),\hat{p},q_1)$. Recall the definition of $D^\flat$ in (1.4.34).

[30] The unions are over $0\leqslant i\leqslant 2N_k$.

which yields (1.5.35) since $\mathbf{c}_* \geqslant \hat{\mathbf{c}}$. $\square$

Remark 1.5.3. The measure estimate (1.5.35) holds in view of the covering property (2.6.1), which takes care of the deformations near the boundaries.
The logarithmic correction is unavoidable and is related to the Lyapunov exponents of the hyperbolic equilibria issuing the separatrices of the secondary integrable systems at simple resonances.

The final result of this section deals with the size of the domains B^i_k, which depends on k and actually *grows* with k. It is therefore important to control such a growth.

Proposition 1.5.2. *Assume that*[31] $\alpha < 1$. *Then, there exists a constant* $c_* = c_*(n) > 1$ *such that*

$$\operatorname{diam} B^i_k \leqslant c_* |k|^{n-1}\,, \qquad \operatorname{meas} B^i_k \leqslant c_*\,. \tag{1.5.39}$$

Proof. For the purpose of this proof, we denote by 'c' suitable (possibly different) constants greater than one and depending only on n.
Since $\alpha < 1$, by the definition of $\mathcal{B}^i_k$ in (1.4.21), by (1.3.27) and the definition of $\mathtt{R}$ in (1.3.30), we have, for every $0 \leqslant i \leqslant 2N_k$,

$$\operatorname{diam} \mathcal{B}^i_k \leqslant c\big(\mathtt{R} + \operatorname{diam} \hat{\mathtt{D}}\big) < c(1 + \operatorname{diam} \hat{\mathtt{D}})\,.$$

Since

$$|\widehat{A^T I}| = |A^T \tbinom{0}{\hat{I}}| \geqslant \tfrac{|\hat{I}|}{\|A^{-1}\|} \geqslant \tfrac{|\hat{I}|}{c|k|^{n-1}}\,,$$

by definition of $\hat{\mathtt{D}}$, it follows that $\operatorname{diam} \mathcal{B}^i_k \leqslant c|k|^{n-1}\rho_2 \leqslant c'|k|^{n-1}$, proving the first relation in (1.5.39) in the case $0 < i < 2N_k$.
In the case $i = 0, 2N_k$, we need to estimate the Lipschitz constant of[32] $\mathtt{g}_i$. By Cauchy estimates, one sees that

$$|\mathtt{g}_*|_{2\mathtt{r}} < \tfrac{4\varepsilon}{\gamma|k|^2}\tfrac{\mu}{\mathtt{r}} < \tfrac{c_2\mathbf{c_0}}{2}\,\tfrac{\sqrt{\varepsilon}}{\gamma\mathtt{K}^{14n+5}}\,, \qquad |\partial_{\hat{I}}\mathtt{g}_*|_{\mathtt{r}} \leqslant \tfrac{4\varepsilon\mu}{\gamma\mathtt{r}|k|^2\mathtt{r}^2} \leqslant c\,\tfrac{\mathbf{c_0}}{\mathtt{K}^{14n+3}} < \tfrac{1}{4}\,, \tag{1.5.40}$$

by taking $\mathtt{K}_{\rm o}$ big enough (recall that $\mathtt{K} \geqslant 6\mathtt{K}_{\rm o}$). Hence[33],

$$|\partial_{\hat{I}}\mathtt{g}_*|_{\mathtt{r}} \leqslant n+1\,, \qquad \operatorname{Lip}_{D_{\mathtt{r}}}(j_{\mathtt{g}_*}) \leqslant n+2\,, \tag{1.5.41}$$

[31] Notice that, since $\gamma = 2(\nu + n)$, the hypothesis $\alpha = \varepsilon\mathtt{K}^\nu < 1$ is implied by the second condition in (1.1.28).
[32] $\mathtt{g}_*$ is defined (1.5.7).
[33] $\operatorname{Lip}_B(g)$ denotes absolute value of the Lipschitz constant of a function g over a domain B.

choosing c_* suitably, the first relation in (1.5.39) follows also in this case.
Let us check the second relation in (1.5.39). Since ϕ_k^i in (1.5.8) is symplectic, we have

$$\begin{aligned}
\text{meas}\, B_k^i &= \frac{1}{(2\pi)^n}\text{meas}(B_k^i \times \mathbb{T}^n) = \frac{1}{(2\pi)^n}\text{meas}\left(\phi_k^i(B_k^i \times \mathbb{T}^n)\right) \\
&\overset{(1.5.8)}{\leqslant} \text{meas}\left(\text{Re}\,(\widetilde{\mathfrak{R}}^{1,k}_{r_k})\right).
\end{aligned}$$

Now, since $\widetilde{\mathfrak{R}}^{1,k} \subseteq \mathrm{B}$ and $r_k \leqslant \alpha < 1$, choosing c_* suitably, also the second relation in (1.5.39) follows, and claim (i) has been proved. □

1.6 Twist at Simple Resonance

In this section we discuss the main issue in singular KAM theory developed by Biasco and Chierchia, namely, the twist of the integrable (rescaled) secular Hamiltonians $\mathtt{h}_k^i$ in (1.5.9) near simple resonances and, in particular, in neighborhoods of secular separatrices, where the action become singular.

In general, it has to be expected that *there are points where the twist of the secular Hamiltonians* $\mathtt{h}_k^i$ *vanishes*; compare [4, Remark 4.1]. Furthermore, and more importantly, when approaching separatrices, the evaluation of the twist becomes a *singular perturbation problem*, where no standard tools can be applied and a new strategy is needed.

Our approach – which exploits in an essential way the fine analytic structure of the action functions described in Theorem 1.4.1 – roughly speaking, consists in constructing a suitable differential operator with non–constant coefficients, which does not vanish on (a suitable regularization of) the Kolmogorov's twist determinant. This will be enough to prove that *the Liouville measure of the set where the twist is smaller than a positive quantity η may be bounded, uniformly in k, by a power of η*. This is the content of the Twist Theorem 1.6.1 below.

Twist Theorem near simple resonances (statement)

To state the Twist Theorem we need to introduce two parameters ($\xi > 0$, $\mathtt{m} \geqslant 1$) which measure the non–degeneracy (in a suitable sense to be specified below) of the energy as function of actions in the inner regions $0 < i < 2N_k$. This requires some preparation.

Non–degenerate functions and theirs sub–levels

First, let us recall a standard quantitative definition of non–degenerate functions.

Definition 1.6.1. *Given $\xi > 0$, an open set $A \subseteq \mathbb{R}$ and $f \in C^m(A, \mathbb{R})$, we say that f is ξ–non–degenerate at order $m \geqslant 1$ on A (or, in short, (ξ, m)–non–degenerate), if*

$$\inf_{x \in A} \max_{1 \leqslant j \leqslant m} |f^{(j)}(x)| \geqslant \xi\,. \tag{1.6.1}$$

An important property of non–degenerate functions is that one can easily estimate the measure of their sub–levels:

Lemma 1.6.1. *Let f be a (ξ, m)–non–degenerate function on a bounded interval (a, b) and let*[34] *$M := \|f\|_{C^{m+1}(a,b)}$. Then, there exist a constants $c_m > 1$ depending only on m such that, for all $\eta > 0$, one has*

$$\text{meas}\{x \in (a, b) : |f(x)| \leqslant \eta\} \leqslant \frac{c_m}{\xi^{1/m}} \big(\tfrac{M}{\xi}(b-a) + 1\big)\, \eta^{1/m}\,.$$

The proof of this lemma can be found, e.g., in [48, Lemma B.1].

Non–degeneracy of the rescaled reference potentials for $|k|_1 \leqslant \mathtt{N}$

Consider a general Hamiltonian (1.3.4) in standard form, recall Definition 1.4.1, recall (1.4.28), and define also, for $0 \leqslant \lambda \leqslant \bar\lambda_{\text{max}}$ (defined in (1.4.25)),

$$\bar a^i := a^i|_{\mu=0}\,, \quad \bar b^i := b^i|_{\mu=0}\,, \quad \bar a^i_\lambda := a^i_\lambda|_{\mu=0}\,, \quad \bar b^i_\lambda := b^i_\lambda|_{\mu=0}\,, \qquad \forall\, 0 \leqslant i \leqslant 2N_k\,. \tag{1.6.2}$$

In the following, we shall explicitly indicate the dependence upon the reference potential $\bar G$ and write, e.g, $\bar I^i_{1,\bar G}$, $\bar{\mathtt{E}}^i_{\bar G}$, $\bar a^i_{\bar G}$, $\bar b^i_{\bar G}$ for $\bar I^i_1$, $\bar{\mathtt{E}}^i$, $\bar a^i$, $\bar b^i$, respectively.

Definition 1.6.2. *Given $\mathtt{H}_\flat$ in standard form with reference potential $\bar G$, we denote by*

$$\mathtt{F}^i_{\bar G}(x) := (\partial^2_{I_1} \bar{\mathtt{E}}^i_{\bar G})\big(\bar a^i_{\bar G} + (\bar b^i_{\bar G} - \bar a^i_{\bar G})x\big)\,, \quad \forall\ x \in (0,1)\,, \qquad (0 < i < 2N_k)\,, \tag{1.6.3}$$

the 'normalized second derivative of the energy function within separatrices'.

These functions satisfy a remarkable rescaling property:

Lemma 1.6.2. *If $\mathtt{F}^i_{\bar G}$ is as in Definition 1.6.2, then, for any $\lambda > 0$, one has $\mathtt{F}^i_{\bar G} = \mathtt{F}^i_{\lambda \bar G}$.*

Proof. Indeed, from the definition of actions, there follows easily that

$$\bar I^i_{1,\lambda\bar G}(E) = \sqrt{\lambda}\, \bar I^i_{1,\bar G}(E/\lambda)\,, \qquad \bar{\mathtt{E}}^i_{\lambda\bar G}(I_1) = \lambda \bar{\mathtt{E}}^i_{\bar G}(I_1/\sqrt{\lambda})\,, \qquad \forall \lambda > 0\,. \tag{1.6.4}$$

[34] $\|f\|_{C^{m+1}(a,b)} := \max_{0 \leqslant j \leqslant m+1} \sup_{(a,b)} |f^{(j)}|$.

Indeed, considering the case $i = 2N_k$ (the other cases being similar), one has

$$\bar{I}^{2N_k}_{1,\lambda\bar{G}}(E) = \frac{1}{2\pi}\int_0^{2\pi}\sqrt{E-\lambda\bar{G}(x)}dx = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi}\sqrt{\frac{E}{\lambda}-\bar{G}(x)}dx = \sqrt{\lambda}\bar{I}^i_{1,\bar{G}}(E/\lambda)\,,$$

which proves the first equality in (1.6.4), which, in turns, implies immediately the second inequality. From (1.6.4), then , follows that

$$\bar{a}^i_{\lambda\bar{G}} = \sqrt{\lambda}\bar{a}^i_{\bar{G}}\,,\qquad \bar{b}^i_{\lambda\bar{G}} = \sqrt{\lambda}\bar{b}^i_{\bar{G}}\,, \tag{1.6.5}$$

and the claim follows at once from (1.6.4) and (1.6.5). □

Let us go back to the Hamiltonians in standard form $\mathtt{H}_k$ of Theorem 1.5.1, and let us prove that the functions $\mathtt{F}^i_{\bar{G}}$ – and hence $\bar{\mathtt{E}}^i_{\lambda\bar{G}}$ – with $\bar{G}$ as in (**??**), are (ξ, m)–non–degenerate.

Lemma 1.6.3. *For every $0 < i < 2N_k$, the function $\mathtt{F}^i_{\bar{G}}$ defined in (1.6.3) is (ξ, m)–non–degenerate for some $\xi, m > 0$.*

Proof. We consider only the case i odd, the even case being similar. Deriving (**??**) we get, for $\mu = 0$,

$$\partial^3_{I_1}\bar{\mathtt{E}}^i(\bar{I}^i_1(E)) = -\frac{\partial^3_E\bar{I}^i_1(E)}{\big(\partial_E\bar{I}^i_1(E)\big)^4} + 3\frac{\big(\partial^2_E\bar{I}^i_1(E)\big)^2}{\big(\partial_E\bar{I}^i_1(E)\big)^5}\,. \tag{1.6.6}$$

By (1.4.11)–(1.4.16) (which hold also for $\bar{I}^i_1$, corresponding to $\mu = 0$), we have that the dominant term in (1.6.6) as $z := (\bar{E}^i_+ - E)/\epsilon \to 0^+$ has the form $-1/(\mathtt{c}^3 z^2 \log^4 z)$ with $\mathtt{c} := \psi^i_+(0)|_{\mu=0}$. Then,

$$\lim_{E\to(\bar{E}^i_+)^-}\big|\partial^3_{I_1}\bar{\mathtt{E}}^i(\bar{I}^i_1(E))\big| = \lim_{I_1\to(\bar{b}^i)^-}\big|\partial^3_{I_1}\bar{\mathtt{E}}^i(I_1)\big| = +\infty\,.$$

By (1.6.3) we obtain

$$\lim_{x\to1^-}|\partial_x\mathtt{F}^i_{\bar{G}}(x)| = +\infty\,. \tag{1.6.7}$$

Moreover $\partial_x\mathtt{F}^i_{\bar{G}}(x)$ is analytic in a neighborhood of $x = 0$ (recall in particular (1.4.15)). Assume now by contradiction that (2.7.9) does not hold, namely that there exists a sequence $x_m \in (0, 1)$ such that

$$|\partial^j_x\mathtt{F}^i_{\bar{G}}(x_m)| < 1/m\,,\qquad \forall\, 1 \leqslant j \leqslant m\,.$$

By (1.6.7), up to a subsequence, x_m converges to some $\bar{x} \in [0, 1)$ such that $\partial^j_x\mathtt{F}^i_{\bar{G}}(\bar{x}) = 0$ for every $j \geqslant 1$. By analyticity we would have that $\mathtt{F}^i_{\bar{G}}$ is constant on $[0, 1)$ leading to a contradiction with (1.6.7). □

This lemma allows us to introduce *uniform* non–degeneracy parameters $\xi > 0$ and $\mathtt{m} \geqslant 1$ for the function $\mathtt{F}^i_{\bar G}$ in (1.6.3) associated to the reference potentials $\bar G \overset{(??)}{=} \frac{2\varepsilon}{|k|^2}\pi_{\mathbb{Z}k} f$, for $k \in \mathcal{G}^n$, $|k|_1 < \mathtt{N}$ and $0 < i < 2N_k$. Indeed, by Lemma 1.6.2,

$$\mathtt{F}^i_{\bar G} = \mathtt{F}^i_{\frac{2\varepsilon}{|k|^2}\pi_{\mathbb{Z}k} f} = \mathtt{F}^i_{\pi_{\mathbb{Z}k} f}\,, \tag{1.6.8}$$

and, by (2.3.1), every potential $\pi_{\mathbb{Z}k} f$ is β–Morse. By the above Lemma 1.6.3, every function in (1.6.8) is (ξ, m)–non–degenerate for some $\xi, m > 0$. We therefore can define uniform ε–independent non–degeneracy parameters ξ, $\mathtt{m}$ by setting:

Definition 1.6.3. *Let $\mathtt{F}^i_{\pi_{\mathbb{Z}k} f}$ be as in Definition 1.6.2 with rescaled reference potential $\bar G = \pi_{\mathbb{Z}k} f$. We define $\xi > 0$ and $\mathtt{m} \geqslant 1$ to be, respectively, the largest and smallest number such that all the functions $\mathtt{F}^i_{\pi_{\mathbb{Z}k} f}$, for $0 < i < 2N_k$, $k \in \mathcal{G}^n$ with $|k|_1 \leqslant \mathtt{N}$, are $(\xi, \mathtt{m})$–non–degenerate (Definition 2.7.1).*

The Twist Theorem

Let 1.3.30, 1.2.23 e 1.2.22 hold, let κ be as in (2.7.5), let $\xi, \mathtt{m}$ be as in Definition 1.6.3, let B^i_k be as in (1.5.7), let $\mathtt{h}^i_k$ be as in (1.5.9), and define

$$\delta_{\mathrm{o}} := |k|^{-2n}\,. \tag{1.6.9}$$

Then, the following result holds.

Theorem 1.6.1. *There exists a constant $\mathtt{c}_0 = \mathtt{c}_0(n, \kappa, \xi, \mathtt{m}) > 1$ such that, for $\mathtt{K}_{\mathrm{o}} \geqslant \mathtt{c}_0$, $k \in \mathcal{G}^n_{\mathtt{K}_{\mathrm{o}}}$, $0 \leqslant i \leqslant 2N_k$, and $0 < \eta < \delta_{\mathrm{o}}/2^5$, one has:*

$$\operatorname{meas}\left(\left\{ I \in B^i_k : \ \left|\det \partial^2_I \mathtt{h}^i_k(I)\right| \leqslant \eta\right\}\right) \leqslant \mathtt{c}_0 (|k|^{2n}\eta)^{\mathtt{b}} \operatorname{meas} B^i_k\,, \qquad \mathtt{b} := \min\{\tfrac{1}{9n^4}, \tfrac{1}{\mathtt{m}}\}\,. \tag{1.6.10}$$

The proof of this theorem is particularly complicated and it will not be reported in this work, for complete details one can read [4].

The proof is based on checking non degenerate condition of the twist determinant, and there is a crucial division between two cases. Far from separatrices the strategy is essentially perturbative, and the twist comes from the non degeneracy condition satisfied by the twist determinant of the reference hamiltonian. Near separatrices, instead, the situation is dramatically difficult because in such regions perturbative arguments do not hold, and, in particular the energy function $\mathtt{E}^i$ is singular at the boundary (corresponding to separatrices) and its derivatives diverge as the boundary is approached. Furthermore, $\mathtt{E}^i$ and $\bar{\mathtt{E}}^i = \mathtt{E}^i|_{\mu=0}$ have singularities in different points. Exploiting the singularity structure described in Theorem 1.4.1, they have proved that a suitable regularization of the twist determinant is a non–degenerate function.

1.7 Maximal KAM tori and proof of the main results

In this final section we show that primary and secondary maximal KAM tori of H span the complementary of $\mathcal{R}^2 \times \mathbb{T}^n$ apart from an exponentially small (in $1/\mathtt{K}$) set and prove the main result of this chapter.

To construct such tori we shall use the following 'KAM theorem'.

Theorem 1.7.1 ([46]). *Fix $n \geqslant 2$ and let $\mathtt{D}$ be any non–empty, bounded subset of $\mathbb{R}^n$. Let*

$$\mathrm{H}(\mathrm{p},\mathrm{q}) := \mathrm{h}(\mathrm{p}) + \mathrm{f}(\mathrm{p},\mathrm{q})$$

be real analytic on $\mathtt{D}_{\mathfrak{r}} \times \mathbb{T}^n_{\mathfrak{s}}$, for some $\mathfrak{r} > 0$ and $0 < \mathfrak{s} \leqslant 1$, and having finite norms

$$\mathtt{M} := |\partial^2_{\mathrm{p}}\mathrm{h}|_{\mathfrak{r}}\,, \qquad\qquad |\mathrm{f}|_{\mathfrak{r},\mathfrak{s}}\,. \tag{1.7.1}$$

Assume that the frequency map $\mathrm{p} \in \mathtt{D} \to \omega = \partial_{\mathrm{p}}\mathrm{h}$ is a local diffeomorphism, namely, assume:

$$\mathrm{d} := \inf_{\mathtt{D}} |\det \partial^2_{\mathrm{p}}\mathrm{h}| > 0\,, \tag{1.7.2}$$

and let $\mathrm{d}_\star := \mathrm{d}/\mathtt{M}^n$ and $r_\star := \mathrm{d}^2_\star\mathfrak{r}$. Then, there exists $\mathrm{C}_* = \mathrm{C}_*(n) > 1$ such that, if

$$\epsilon := \frac{|\mathrm{f}|_{\mathfrak{r},\mathfrak{s}}}{\mathtt{M}\mathfrak{r}^2} \leqslant \frac{\mathrm{d}^8_\star\ \mathfrak{s}^{4(\mathrm{n}+1)}}{\mathrm{C}_*}\,, \tag{1.7.3}$$

there exists a set $\mathcal{T} \subseteq (\mathtt{D}_{\mathrm{r}_\star} \cap \mathbb{R}^{\mathrm{n}}) \times \mathbb{T}^{\mathrm{n}}$ formed by primary KAM tori such that[35]

$$\operatorname{meas}\big((\mathtt{D} \times \mathbb{T}^n) \setminus \mathcal{T}\big) \leqslant \mathrm{C}\sqrt{\epsilon}\,, \qquad \mathrm{C} := \big(\max\{\mathrm{d}^2_\star\mathfrak{r}\,,\,\operatorname{diam}\mathtt{D}\}\big)^{\mathrm{n}} \cdot \frac{\mathrm{C}_*}{\mathrm{d}^{n+5}_\star\ \mathfrak{s}^{3(\mathrm{n}+1)}}\,. \tag{1.7.4}$$

This statement is an immediate corollary of Theorem 1 in[36] [46].

Remark 1.7.1. (i) Note that in the formulation of Theorem 1.7.1 the action domain $\mathtt{D}$ is a completely arbitrary bounded set and that the smallness quantitative condition (1.7.3) depends on $\mathtt{D}$ only through its diameter, which in our application depends on k. For a similar statement, which takes into account the geometry of $\mathtt{D}$, see [47].
(ii) We point out that the smallness condition (1.7.3) can be rewritten as

$$|\mathrm{f}|_{\mathtt{D},\mathfrak{r},\mathfrak{s}} \leqslant \frac{\mathfrak{r}^2\mathrm{d}^8\ \mathfrak{s}^{4n+4}}{\mathrm{C}_*\,\mathtt{M}^{8n-1}}\,. \tag{1.7.5}$$

[35] Here 'meas' denotes the outer Lebesgue measure.
[36] In Theorem 1 of [46] take $\tau = n$ and substitute λ with its maximal value $2 \cdot n!\,\mathrm{d}_\star^{-1}$ (see (14) of [46]).

(iii) Finally, observe that, since[37] $\mathrm{d}_* \leqslant 1$, estimate (1.7.4) implies

$$\operatorname{meas}\big((\mathtt{D}\times\mathbb{T}^n)\setminus\mathcal{T}\big) \leqslant \big(\max\{\mathfrak{r}\,,\,\operatorname{diam}\mathtt{D}\}\big)^n \cdot \frac{\mathrm{C}_*\,\mathtt{M}^{n^2+5n-1/2}}{\mathrm{d}^{n+5}\;\mathfrak{s}^{3n+3}\mathfrak{r}}\sqrt{|\mathrm{f}|_{\mathtt{D},\mathfrak{r},\mathfrak{s}}}\,. \tag{1.7.6}$$

KAM tori in the non–resonant region

Proposition 1.7.1. *Let the assumptions of Theorem 1.2.2 hold. There exists a constant* $\mathtt{C}_{\rm o} = \mathtt{C}_{\rm o}(n,s) \geqslant \mathbf{c}_{\rm o}$ *such that, if* $\mathtt{K}_{\rm o} \geqslant \mathtt{C}_{\rm o}$*, then there exists a family of primary maximal KAM tori* $\mathcal{T}^0$ *invariant for the Hamiltonian* H *in* 1.1.1*, satisfying*

$$\operatorname{meas}\big((\mathfrak{R}^0\times\mathbb{T}^n)\setminus\mathcal{T}^0\big) \leqslant \mathtt{C}_{\rm o}\sqrt{\varepsilon}\,e^{-\mathtt{K}_{\rm o}s/6}\,. \tag{1.7.7}$$

Remark 1.7.2. The above result is essentially classical, and, in fact, no genericity assumptions on the potentials are needed. However, there is one delicate point related to the KAM tori near the boundary. Indeed, primary tori oscillates, in general, by a quantity of order $\sqrt{\varepsilon}$, and naive applications of classical KAM theorems would leave out regions near the boundary of the phase space of measure $\sim\sqrt{\varepsilon}$. Such a problem is overcome by using the second covering in 2.6.1; compare, also, Remark 1.2.5–(ii).

Proof. **of Theorem 1.7.1** We apply the KAM Theorem 1.7.1 to the nearly–integrable Hamiltonian $\mathtt{H}_{\rm o}$ in Theorem 1.2.2–(ii). More precisely, we let[38]

$$\mathrm{h}(\mathrm{p}) = h(p) + \varepsilon g^{\rm o}(\mathrm{p})\,,\quad \mathrm{f} = \varepsilon f^{\rm o}\,,\quad \mathtt{D} = \mathfrak{R}^0\,,\quad \mathfrak{r} = \frac{\mathfrak{r}_0'}{2} = \frac{\sqrt{\varepsilon}\mathtt{K}^{\frac{9}{2}n+2}}{16\mathtt{C}\mathtt{K}_{\rm o}}\,,\quad \mathfrak{s} = \min\{\tfrac{s}{2}\,,1\}\,.$$

By 2.5.19 and Cauchy estimates we get

$$\mathtt{M} \leqslant \frac{2M}{\mathtt{r}^2}\,,\qquad |\mathrm{f}|_{\mathfrak{r},\mathfrak{s}} \leqslant \varepsilon e^{-\mathtt{K}_{\rm o}s/3}\,.$$

If $\mathtt{K}_{\rm o}$ is taken large enough (larger than a constant despending on n and s) the KAM smallness condition (1.7.5) is satisfied, and the KAM Theorem 1.7.1 yields the existence of a set $\widetilde{\mathcal{T}}^0$ of invariant tori for the Hamiltonian $\mathtt{H}_{\rm o}$ in Theorem 1.2.2–(ii), which, by[39] (1.7.6), satisfy

$$\operatorname{meas}\big((\mathfrak{R}^0\times\mathbb{T}^n)\setminus\widetilde{\mathcal{T}}^0\big) \leqslant \mathtt{C}_{\rm o}\sqrt{\varepsilon}\,e^{-\mathtt{K}_{\rm o}s/6}\,, \tag{1.7.8}$$

[37] Indeed the absolute value of any eigenvalues of the symmetric matrix $\partial^2_{\mathrm{p}}\mathrm{h}$ is bounded by $\mathtt{M}$, which implies $\mathrm{d} \leqslant \sup_{\mathtt{D}}|\det\partial^2_{\mathrm{p}}\mathrm{h}| \leqslant \mathtt{M}^n$.

[38] Recall Theorem 1.2.2.

[39] Notice that the hypothesis $\mathtt{K} < \varepsilon^{-1/(9n+4)}$ implies that $\mathfrak{r} < 1$, so that $\max\{\mathrm{d}^2\mathtt{M}^{-2n}\mathfrak{r}\,,\,\operatorname{diam}\mathtt{D}\} = 2$.

for a suitable constant $\mathtt{C}_{\rm o} = \mathtt{C}_{\rm o}(n,s)$ large enough (so that also the condition on $\mathtt{K}_{\rm o}$ is met). Since the map $\psi_{\rm o}$ in (1.2.2) is symplectic, the family of tori $\mathcal{T}^0 := \psi_{\rm o}(\widetilde{\mathcal{T}}^0)$ is formed by KAM invariant for H in (1.1.1). Lemma (2.6.1) and the bound (1.7.8) imply (1.7.7). $\square$

KAM tori near simple resonances

Now, we turn to the construction, in all neighbourhoods of simple resonances, of families of primary tori for the nearly–integrable Hamiltonians $\mathcal{H}^i_k$ of Theorem 1.5.1, for all $k \in \mathcal{G}^n_{\mathtt{K}_{\rm o}}$ and $0 \leqslant i \leqslant 2N_k$. Note that such tori correspond, in the inner case $0 < i < 2N_k$, to *secondary tori for the Hamiltonian* $\mathtt{H}$.

Let us introduce zones $B^i_k(\lambda,\eta) \subseteq B^i_k$, which are λ–away in energy from separatrices and where the twist is bounded away from zero by a quantity $\eta > 0$, namely (recall (1.5.10), (1.5.7)), let us define:

$$B^i_k(\lambda,\eta) := \{I \in B^i_k(\lambda) \text{ s.t. } |\det \partial^2_I \mathtt{h}^i_k(I)| > \eta\} \subset B^i_k \,. \tag{1.7.9}$$

Proposition 1.7.2. (KAM tori for $\mathcal{H}^i_k$) near critical surface *Let the assumptions of Theorem 1.5.1 hold. There exist positive constants $\bar{\mathtt{C}}_1 = \bar{\mathtt{C}}_1(n,s,\beta) > 1$ and $\mathtt{C}_1 = \mathtt{C}_1(n,s,\beta,\delta) \geqslant \mathbf{c}_*$ such that the following holds. Let $k \in \mathcal{G}^n_{\mathtt{K}_{\rm o}}$, $0 \leqslant i \leqslant 2N_k$; $0 \leqslant j \leqslant P$ $0 < \lambda \leqslant 1/\mathbf{c}_*$ and $0 < \eta < 1/2$. Then, if*

$$\mathtt{K} \geqslant \mathtt{C}_1 \log \frac{1}{\lambda\eta} \,, \tag{1.7.10}$$

there exists a set $\mathcal{T}^{i,j}_k$ of maximal KAM tori for the Hamiltonian $\mathcal{H}^i_k$ in (1.5.9) such that

$$\text{meas}\left((B^i_k(\lambda,\eta) \times \mathbb{T}^n) \setminus \mathcal{T}^{i,j}_k \right) \leqslant \bar{\mathtt{C}}_1 e^{-\mathtt{K}s/7} \,. \tag{1.7.11}$$

Proof. We apply the KAM Theorem 1.7.1 to the Hamiltonian $\mathcal{H}^i_k$ of Theorem 1.5.1 with (recall (1.5.9) and (1.5.10)):

$$\begin{aligned}
&\mathrm{h} = h^i_k = \mathtt{h}^i_k \,, \qquad \mathrm{f} = \varepsilon f^i_k \,, \qquad \mathtt{D} = B^i_k(\lambda,\eta) \,, \\
&\mathfrak{r} = \rho_* = \frac{\sqrt{\epsilon}}{\mathbf{c}_* \mathtt{K}^n_{\rm o}} \lambda |\log\lambda| \,, \qquad \mathfrak{s} = \sigma_* = \frac{1}{\mathbf{c}_* \mathtt{K}^n_{\rm o} |\log\lambda|} \,.
\end{aligned} \tag{1.7.12}$$

Note that, by (1.5.13), $0 < \lambda \leqslant 1/\mathbf{c}_* \leqslant 1/8$, which implies easily $\mathfrak{r} \leqslant \mathtt{r}$ and $\mathfrak{s} \leqslant 1$. Also, since $\mathbf{c}_* \geqslant \hat{\mathbf{c}}$ (see Theorem 1.5.1) and $\mathtt{K}^n_{\rm o} \geqslant 2^n \geqslant n$, one has $\rho_* \leqslant \rho_\lambda/n$.

In the following arguments we denote by $c(\cdot)$ possibly different constants depending only on the quantities inside brackets.

We first have to estimate $\mathtt{M}$ in (1.7.1), namely, $\partial_I^2 \mathtt{h}_k^i$. By (1.4.33), (1.4.12) and (1.4.10) we get

$$\sup_{(\mathcal{B}_k^i(\lambda))_{\rho_\lambda}} \left|\partial_I^2 \mathtt{E}^i\right| \leqslant \frac{n\hat{\mathbf{c}}}{\lambda}\,. \tag{1.7.13}$$

In the case $0 < i < 2N_k$, by (1.5.10), we have $\mathcal{B}_k^i(\lambda) = B_k^i(\lambda)$. Therefore, recalling (1.5.9), we can bound $|\partial_I^2 \mathtt{h}_k^i|$ by $c(n,s,\beta)/\lambda$.
The estimate on $|\partial_I^2 \mathtt{h}_k^i|$ in the case $i = 0, 2N_k$ needs some extra attention. In particular fix $i = 2N_k$ (the case $i = 0$ being analogous). Recalling the definition of $j_{\mathtt{g}_*}$ in (1.5.5), (1.5.7) we have that $\partial_I^2 j_{\mathtt{g}_*}$ depends only on $\hat{I}$ and not on I_1. Moreover by (1.2.22), (1.2.51), (2.7.7), (1.3.30) and Cauchy estimates we get

$$\sup_{\hat{I}\in\hat{D}_{3\mathrm{r}}} |\partial_I j_{\mathtt{g}_*}| \leqslant c(n)\,, \qquad \sup_{\hat{I}\in\hat{D}_{3\mathrm{r}}} |\partial_I^2 j_{\mathtt{g}_*}| \leqslant \frac{c(n)|k|^2}{\sqrt{\varepsilon}\mathtt{K}^\nu}\,. \tag{1.7.14}$$

Recalling Definition 1.4.1, (1.4.33) and (1.3.30), we have that

$$\sup_{(\mathcal{B}_k^{2N_k}(\lambda))_{\rho_\lambda}} |\mathtt{E}^{2N_k}| \leqslant 4\mathtt{R}^2 = \frac{4\varepsilon\mathtt{K}^{2\nu}}{|k|^4}\,.$$

Then, by (1.4.33) and (1.3.30) we get

$$\sup_{(\mathcal{B}_k^{2N_k}(\lambda))_{\rho_\lambda}} |\partial_{I_1}\mathtt{E}^{2N_k}| \leqslant \hat{\mathbf{c}}\sqrt{8\mathtt{c}_s\varepsilon + 4\varepsilon\mathtt{K}^{2\nu}|k|^{-4}} \leqslant 4\hat{\mathbf{c}}\sqrt{\varepsilon}\mathtt{K}^\nu|k|^{-2}$$

(taking $\mathtt{K} \geqslant \mathtt{c}_s$ defined in (1.3.30)). Finally, recalling also (1.5.10), (1.5.23), (1.7.13), (1.7.14) , we get by the chain rule

$$\sup_{(B_k^{2N_k}(\lambda))_{\mathfrak{r}}} \left|\partial_I^2\big(\mathtt{E}^{2N_k}\circ I_*\big)\right| \leqslant \frac{c(n,s,\beta)}{\lambda}\,.$$

By (1.7.1), (1.7.12), (1.5.9), (and that $\mathfrak{r} \leqslant \mathtt{r}$), we finally get

$$\mathtt{M} \leqslant |k|^2\frac{c(n,s,\beta)}{\lambda}\,, \qquad \forall\ 0 \leqslant i \leqslant 2N_k\,. \tag{1.7.15}$$

Next, by (1.7.12) and (1.5.12),

$$|\mathrm{f}|_{\mathfrak{r},\mathfrak{s}} \leqslant \varepsilon\, e^{-\mathtt{K}s/3}\,. \tag{1.7.16}$$

By (1.7.2), (1.7.12) and (1.7.9), we get

$$\mathrm{d} \geqslant 2^{-n}|k|^{2n}\eta \qquad \text{and} \qquad \frac{\mathtt{M}^n}{\mathrm{d}} \leqslant \frac{c(n,s,\beta)}{\lambda^n\eta}\,. \tag{1.7.17}$$

By (1.3.30), (1.1.23) and using $\mathtt{K_o} \leqslant 6\mathtt{K}$, we have:

$$\frac{\varepsilon}{\epsilon} \leqslant \frac{\mathtt{K_o}^{n+2}}{8\mathtt{c}_s \delta}\, e^{\mathtt{K_o} s} \leqslant \frac{\mathtt{K}^{n+2}}{6^{n+3}\mathtt{c}_s \delta}\, e^{\mathtt{K}s/6}\,. \tag{1.7.18}$$

It is now easy to check, by (1.7.15), (1.7.16), (1.7.17) and (1.7.18), that the KAM smallness condition (1.7.5) is satisfied taking $\mathtt{K}$ as in (1.7.10) with $\mathtt{C_1}$ large enough. By the KAM Theorem 1.7.1 we, then, obtain a set $\mathcal{T}^{i,j}_k$ of invariant tori for the Hamiltonian in (1.5.9), which, in view of (1.7.6) and by (1.7.15), (1.7.16), (1.7.17) and (1.7.18), satisfies (1.7.11) with a suitable constant $\bar{\mathtt{C}}_1 = \bar{\mathtt{C}}_1(n,s,\beta)$; in particular, note that, by (1.5.39) and (1.7.12), the maximum in (1.7.6) is estimated by $c(n)\mathtt{K_o}^{n^2}$. □

This result holds near a single fixed point of a critical surfaces, now we have to taking into account also the secondary tori for H that comes from $\overline{\mathtt{H}}_k$ near simple resonance but far from critical points.

Putting together these KAM statements and the Twist Theorem 1.6.1, the proof of the main result follow easily.

Proof of Theorem 1.1.1 and its corollaries

Since $f \in \mathbb{G}^n_s$, there exist $\delta, \beta > 0$ such that (1.1.23) and (2.3.1) hold with $\mathtt{N}$ as in (1.1.10). Let

$$\mathtt{K_o} := \mathtt{K}/6\,,$$

with $\mathtt{K} \geqslant 12$ and let α be as in 1.2.22. Then and we may let the Definitions 1.3.30,1.2.23 hold. Let $\mathbf{c_o} = \mathbf{c_o}(n,s,\delta)$ be as in Theorem 1.3.1, and assume that[40]

$$\mathtt{K} \geqslant 6\mathbf{c_o}\,. \tag{1.7.19}$$

Then, Theorem 1.3.1 holds and we may define the parameters $\xi > 0$ and $\mathtt{m} \geqslant 1$ as in Definition 1.6.3 with respect to standard Hamiltonians $\mathtt{H}_k$ (with $|k|_1 \leqslant \mathtt{K_o}$) of Theorem 1.3.1–(ii).

We now let $\mathtt{b} < 1$ as in (1.6.10), $\mathtt{C_1} = \mathtt{C_1}(n,s,\beta,\delta)$ be as in Proposition 1.7.2, and define

$$\eta := e^{-\frac{\mathtt{K}}{\mathtt{C_1}(1+\mathtt{b})}}\,, \qquad \lambda := \eta^{\mathtt{b}}\,. \tag{1.7.20}$$

Notice that, with such definitions, it is

$$\mathtt{K} = \mathtt{C_1} \log \frac{1}{\lambda\eta}\,, \tag{1.7.21}$$

[40]Eq. 1.7.19 implies that $\mathtt{K} > 12$.

(compare (1.7.10)).
With these premises, let us turn to the proof of the claims of Theorem 1.1.1.
Claim (ii) has already been proven in 1.2.15 above.
Next, we define the set of maximal KAM tori $\mathcal{T}$ for $\mathtt{H}$ as it appears in item (iv) of the theorem.
Let $\mathtt{C}_{\rm o} = \mathtt{C}_{\rm o}(n, s)$ as in Proposition 1.7.1.There exists a constant

$$\hat{\mathfrak{c}} = \hat{\mathfrak{c}}(n, s, \beta, \delta, \mathtt{m}) \geqslant \max\{\mathtt{C}_{\rm o}, 2\mathtt{C}_1 \mathbf{c}_* / \mathtt{b}\} \, ,$$

such that, if $\mathtt{K} \geqslant \hat{\mathfrak{c}}$, then

$$\mathtt{K}^{2n} \eta \overset{(1.7.20)}{=} \mathtt{K}^{2n} e^{-\frac{\mathtt{K}}{\mathtt{C}_1 (1+\mathtt{b})}} \leqslant 1 \, .$$

Assume that

$$\mathtt{K} \geqslant \hat{\mathfrak{c}} \, . \tag{1.7.22}$$

Then, $\lambda = \eta^{\mathtt{b}}$ in (1.7.20) is smaller than $1/\mathbf{c}_*$ and (recall (1.6.9))

$$\eta \leqslant \frac{\delta_{\rm o}}{2^5} < \frac{1}{2} \, . \tag{1.7.23}$$

Thus, in view of (1.7.21), by (1.7.22) the assumptions of Propositions 1.7.1 and 1.7.2 are satisfied, and we can define the following families of tori[41]:

$$\begin{cases} \mathcal{T}^{1,k}_{i,j} := \phi^i_k(\mathcal{T}^{i,j}_k) \, , \quad \mathcal{T}^{1,k} := \displaystyle\bigcup_{j=1}^{J} \bigcup_{0 \leqslant i \leqslant 2N_k} \mathcal{T}^{1,k}_{i,j} \, , \quad \mathcal{T}^1 := \displaystyle\bigcup_{k \in \mathcal{G}^n_{\mathtt{K}_{\rm o}}} \mathcal{T}^{1,k} \, , \\ \mathcal{T} := \mathcal{T}^0 \cup \mathcal{T}^1 \, . \end{cases} \tag{1.7.24}$$

where, for a fixed k, $\bigcup_{0 \leqslant i \leqslant 2N_k} \mathcal{T}^{1,k}_{i,j}$ are the tori founded on a single neighborhood of a point $\bar{y}$ of a single surface $\mathcal{S}_k$, so $\bigcup_P$ represent the sum over all the neighborhoods that cover $\mathcal{S}_k$, so that $\bigcup_P \bigcup_{0 \leqslant i \leqslant 2N_k} \mathcal{T}^{1,k}_{i,j}$ represent all the tori near first resonance for a fixed k. Summing over k we can obtain the total union of tori for simple resonance near surfaces. Observe that $\mathcal{T}^i_k$ are invariant tori for $\mathcal{H}^i_k$ in (1.5.9), while $\mathcal{T}^{1,k}_i$, $\mathcal{T}^1$ and $\mathcal{T}^0$ are invariant for the original Hamiltonian $\mathtt{H}$.
Thus, $\mathcal{T}$ is a family of maximal KAM tori for $\mathtt{H}$ as in item (iv) of Theorem 1.1.1.
Claim (i) follows, now, immediately by (2.5.1), setting

$$\mathcal{A} := \big((\mathfrak{R}^0 \cup \mathfrak{R}^1) \times \mathbb{T}^n \big) \setminus \mathcal{T} \, . \tag{1.7.25}$$

[41] $\mathcal{T}^i_k$ is defined in Proposition 1.7.2, $\mathcal{T}^0$ in Proposition 1.7.1 and ϕ^i_k in (1.5.11).

It remains to prove claim (iii), namely, the exponential measure estimate on $\mathcal{A}$. Observe that by (1.7.25) and (1.7.24)

$$
\begin{aligned}
\mathcal{A} &\subseteq \big((\mathfrak{R}^0 \times \mathbb{T}^n)\setminus\mathcal{T}^0\big) \cup \big((\mathfrak{R}^1 \times \mathbb{T}^n)\setminus(\mathcal{T}^1)\big) \\
&\subseteq \big((\mathfrak{R}^0 \times \mathbb{T}^n)\setminus\mathcal{T}^0\big) \cup \bigcup_{k\in\mathcal{G}^n_{\mathrm{K}_{\mathrm{o}}}} \Phi_0(\check{\mathscr{D}}^k \times \mathbb{T}^n)\setminus\mathcal{T}^{1,k}\,.
\end{aligned}
\tag{1.7.26}
$$

We now need the some elementary results:

Lemma 1.7.1. *Let $F \in C^2(\mathbb{T},\mathbb{R})$, $\bar\theta$ and $0 < \mathtt{c} < \frac{1}{2}$ are such that*[42] $\|F-\cos(\theta+\bar\theta)\|_{C^2} \leqslant \mathtt{c}$. *Then, F has only two critical points and it is $(1-2\mathtt{c})$–Morse.*

Proof. By considering the translated function $\theta \to F(\theta-\bar\theta)$, one can reduce oneself to the case $\theta = 0$ (note that F is β–Morse, if and only if $\theta \to F(\theta-\bar\theta)$ is β–Morse). Thus, set $\bar\theta = 0$, and note that, by assumption $|F'| = |F' + \sin\theta - \sin\theta| \geqslant |\sin\theta| - \mathtt{c}$, and, analogously, $|F''| \geqslant |\cos\theta| - \mathtt{c}$. Hence, $|F'| + |F''| \geqslant |\sin\theta| + |\cos\theta| - 2\mathtt{c} \geqslant 1 - 2\mathtt{c}$. Next, let us show that F has a unique strict maximum $\theta_0 \in I := (-\pi/6, \pi/6) \pmod{2\pi}$. Writing $F = \cos\theta + g$, with $g := F - \cos\theta$, one has that $F'(-\pi/6) = 1/2 + g'(\pi/6) \geqslant 1/2 - \mathtt{c} > 0$, and, similarly $F'(\pi/6) \leqslant -1/2 + \mathtt{c}$, thus F has a critical point in I, and, since $-F'' = \cos\theta - g'' \geqslant \cos\theta - \mathtt{c} \geqslant \sqrt{3}/2 - \mathtt{c} > 0$, F is strictly concave in I, showing that such critical point is unique and it is a strict local minimum. In fact, similarly one shows that F has a second critical point $\theta_1 \in (\pi - \pi/6, \pi + \pi/6)$ where F is strictly convex, so that θ_1 is a strict local minimum; but, since in the complementary of these intervals F is strictly monotone (as it is easy to check), it follows that F has a unique global strict maximum and a unique global strict minimum. Finally, $F(\theta_0) - F(\theta_1) \geqslant \sqrt{3} - 2\mathtt{c} > 1 - 2\mathtt{c}$ and the claim follows. □

Lemma 1.7.2. *If G is β–Morse, then the number $2N$ of its critical points is bounded by $\pi\sqrt{2\max_{\mathbb{R}}|G''|/\beta}$.*

Proof. If θ_i and θ_j are different critical points of G, then, by Taylor expansion at order two and by 1.1.6 one has $\beta \leqslant |G(\theta_i) - G(\theta_j)| \leqslant \frac{1}{2}(\max_{\mathbb{R}}|G''|)|\theta_i - \theta_j|^2$, which implies that the minimal distance between two critical points is at least $\sqrt{2\beta/\max_{\mathbb{R}}|G''|}$, from which the claim follows. □

Thanks to this lemmata we can state the useful

Lemma 1.7.3. *If $f \in \mathbb{B}^n_s$ satisfies (2.3.1), then, for any $k \in \mathcal{G}^n$, the number $2N_k$ of critical points of $\pi_{\mathbb{Z}k} f$ is bounded by $\bar{\mathtt{c}} := \max\{4, \pi\sqrt{8/\beta}\}$.*

[42] $\|F\|_{C^2} := \max_{0\leqslant k\leqslant 2}\sup|F^{(k)}|$. Note that, by Cauchy estimates, $\|F\|_{C^2} \leqslant 2|F|_1$.

Proof. Consider first the case $|k|_1 \geqslant \mathtt{N}$. By previous lemmata, $F_*^k := \pi_{\mathbb{Z}k} f/2|f_k|$ satisfies

$$|F_*^k - \cos(\theta + \theta_k)|_1 \leqslant 2^{-40}\,.$$

Thus, by Cauchy estimates we get $\|F_*^k - \cos(\theta+\theta_k)\|_{C^2} \leqslant 2^{-39}$, so that by Lemma 2.2.1 it follows that $2N_k = 4$.

For the case $|k|_1 \leqslant \mathtt{N}$ by (2.3.1) we know that $\pi_{\mathbb{Z}k} f$ is β–Morse, and since $\|f\|_s \leqslant 1$ we have $\sup_{\mathbb{R}} |(\pi_{\mathbb{Z}k} f)''| \leqslant \sum_{j\neq 0} |f_{jk}| j^2 \leqslant \sum_{j\neq 0} e^{-|j|} j^2 < 4$. Then, by Lemma 1.7.2, the claim follows also in this case. □

Obviously, the hypothesis of this lemma are met by our fixed potential in $\mathbb{G}_s^n$, and the following measure estimate holds.

Lemma 1.7.4. *Let* λ *as above in* (1.7.20) *and* $\bar{\mathtt{c}}$ *as in Lemma 1.7.3. Then, for any* k *in* $\mathcal{G}^n_{\mathtt{K}_{\mathtt{o}}}$, *one has*

$$\begin{aligned}&\mathrm{meas}\left(\Phi_0(\check{\mathscr{D}}^k \times \mathbb{T}^n) \setminus \mathcal{T}^{1,k}\right) \leqslant \mathrm{meas}\left((\mathfrak{R}^{1,k} \times \mathbb{T}^n) \setminus \mathcal{T}^{1,k}\right) \leqslant \\ &\leqslant \mathtt{c}_* \,\mathrm{meas}\left(\tilde{\mathfrak{R}}^{1,k} \times \mathbb{T}^n\right)\lambda|\log\lambda| + \bar{\mathtt{c}} \max_{0\leqslant i\leqslant 2N_k} \mathrm{meas}\left(\left(B_k^i(\lambda) \times \mathbb{T}^n\right)\setminus\mathcal{T}_k^i\right).\end{aligned} \tag{1.7.27}$$

Proof. Since ϕ_k^i in Theorem 1.5.1 is a diffeomorphism, one has

$$\begin{aligned}&(\mathfrak{R}^{1,k} \times \mathbb{T}^n) \setminus \mathcal{T}^{1,k} \overset{(1.7.24)}{=} (\mathfrak{R}^{1,k} \times \mathbb{T}^n) \setminus \Big(\bigcup_{0\leqslant i\leqslant 2N_k} \phi_k^i(\mathcal{T}_k^i)\Big) \\ &\subseteq \left((\mathfrak{R}^{1,k} \times \mathbb{T}^n) \setminus \bigcup_{0\leqslant i\leqslant 2N_k} \phi_k^i\big(B_k^i(\lambda) \times \mathbb{T}^n\big)\right) \cup \bigcup_{0\leqslant i\leqslant 2N_k} \phi_k^i\Big(\big(B_k^i(\lambda) \times \mathbb{T}^n\big)\setminus\mathcal{T}_k^i\Big)\,,\end{aligned}$$

then, passing to measures, using (1.5.35), the fact that ϕ_k^i is symplectic and Lemma 1.7.3, we get (1.7.27). □

Now, assume that, together with (1.7.19) and (1.7.22), it is also $\mathtt{K} \geqslant \mathtt{c}_0$. Then, recalling (1.7.23), Theorem 1.6.1 holds. Thus, recalling (1.7.9), observing that

$$B_k^i(\lambda) \;=\; \left\{I \in B_k^i \text{ s.t. } |\det \partial_I^2 \mathtt{h}_k^i(I)| \leqslant \eta\right\} \cup B_k^i(\lambda,\eta)\,,$$

by (1.6.10) and (1.7.11) we get

$$\mathrm{meas}\left(\left(B_k^i(\lambda) \times \mathbb{T}^n\right)\setminus\mathcal{T}_k^i\right) \;\leqslant\; \mathtt{c}_0(|k|^{2n}\eta)^{\mathtt{b}}\,\mathrm{meas}\, B_k^i \;+\; \bar{\mathtt{C}}_1 e^{-\mathtt{K}s/7}\,. \tag{1.7.28}$$

Now, by (1.7.26), (1.7.7), (1.7.27), (1.7.28), (1.5.39), (1.7.20) and since $|k|_1 \leqslant \mathtt{K}_{\mathtt{o}} = \mathtt{K}/6$ we get, for a suitable constant[43] $\mathtt{c}_1 = \mathtt{c}_1(n, s, \delta, \beta, \xi, \mathtt{m})$,

$$\mathrm{meas}(\mathcal{A}) \leqslant \mathtt{c}_1\, \mathtt{K}^{2n} e^{-\mathtt{K}/\mathtt{c}_\star}\,, \qquad \mathtt{c}_\star := \max\left\{36/s, 2\mathtt{C}_1/\mathtt{b}\right\}. \tag{1.7.29}$$

[43]To get (1.7.29), use the following: $\varepsilon \leqslant \mathtt{K}^{-\gamma} < 1$ (compare (1.1.28)); $\mathrm{meas}\left(\tilde{\mathcal{R}}^{1,k} \times \mathbb{T}^n\right) \leqslant c(n)$, as $\tilde{\mathcal{R}}^{1,k} \subset \{y : |y| \leqslant 2\}$; $|\log\lambda| = \frac{\mathtt{b}}{\mathtt{C}_1(1+\mathtt{b})}\mathtt{K}$; $\mathrm{meas}\, B_k^i \leqslant c_{\circledast}$ by (1.5.39); $\#\mathcal{G}^n_{\mathtt{K}_{\mathtt{o}}} < (2\mathtt{K}_{\mathtt{o}}+1)^n$.

Finally, let

$$\mathfrak{c} = \mathfrak{c}(n, s, \delta, \beta, \xi, \mathtt{m}) \geqslant 1 + \mathfrak{c}_\star \tag{1.7.30}$$

be such that, if $\mathtt{K} \geqslant \mathfrak{c}$, then $\mathfrak{c}_1\, \mathtt{K}^{2n} e^{-\mathtt{K}/\mathfrak{c}_\star} \leqslant e^{-\mathtt{K}/(1+\mathfrak{c}_\star)}$. Then, if $\mathtt{K} \geqslant \mathfrak{c}$, claim (iii) follows, and the proof of Theorem 1.1.1 is complete. □

Remark 1.7.3. Notice that $\mathcal{T}^0$ is a family of maximal primary tori for $\mathtt{H}$, and so are the families $\mathcal{T}_i^{1,k}$ for all $k \in \mathcal{G}^n_{\mathtt{K}_o}$ and $i = 0, 2N_k$. On the other hand, $\mathcal{T}_i^{1,k}$ for all $k \in \mathcal{G}^n_{\mathtt{K}_o}$ and $0 < i < 2N_k$ are families of *maximal secondary tori* for $\mathtt{H}$. In particular these families do not bifurcate from integrable tori.

Chapter 2

Generalizing the angles analyticity domain

2.1 Set up and weighted norms

Let $n \geqslant 2$, we consider analytic Hamiltonian systems composed of the sum of an integrable part (in the sense of Arnold-Liouville) and a small perturbation. Namely, indicating the n-dimensional torus by $\mathbb{T}^n := \mathbb{R}^n \backslash (2\pi\mathbb{Z}^n)$, and making use of standard action-angle coordinates $(y, x) \in B \subset \mathbb{R}^n \times \mathbb{T}^n$ where B is a *compact* set in $\mathbb{R}^n$, associated to the symplectic two-form $\Omega := \sum_{i=1}^n dx_i \wedge dy_i$, we are interested in those systems described by

$$H(y, x) = H_\varepsilon(y, x) = h(y) + \varepsilon f(y, x) \tag{2.1.1}$$

where ε is a small parameter measuring the size of the analytic perturbation εf w.r.t. the analytic integrable part h.

In this part we want to generalize the result done in 1 to an Hamiltonian which has different width of analyticity stripes for each component of the n-dimensional action variable.

Precisely, Let $r > 0$ and $s \in \mathbb{R}^n_+$ a vector with positive components (i.e. $s_i > 0 \; \forall \; i = 1, ..., n$) and $|\cdot|$ be the standard Euclidean norm on vectors $u \in \mathbb{C}^n$ (and its subspaces); 'bar', as usual, denotes complex–conjugated, we define

$$B_r := \bigcup_{y \in B} \{z \in \mathbb{C}^n : |z - y| \leqslant r\},$$

$$\mathbb{T}^n_s := \{x \in \mathbb{C}^n : |\mathrm{Im}\; x_i| \leqslant s_i \; \forall \; i = 1, ..., n\} \backslash (2\pi\mathbb{Z}^n).$$

Assume that H_ε in 2.1.1 admits an holomorphic extension for some $r > 0$, $s \in \mathbb{R}^n_+$ on the complex domain $B_r \times \mathbb{T}^n_s \subset \mathbb{C}^{2n}$.

Let the definitions in 1 from 1.1.1 to 1.1.6 and 1.1.8 holds. We need to change some definitions with respect to the first chapter. For the rest of this chapter we will use the following notation: given $s \in \mathbb{R}^n_+$ we denote by

$$s_\flat := \min_{i=1,\ldots,n} s_i, \qquad s_\sharp := \max_{i=1,\ldots,n} s_i \tag{2.1.2}$$

Moreover we will often use the following two trivial estimates, for $s \in \mathbb{R}^n_+$, $k \in \mathbb{R}^n$

$$|k|_1\, s_\flat \leqslant \sum_{i=1}^n |k_i| s_i \leqslant |k|_1\, s_\sharp \leqslant |k|_1\, |s|_1. \tag{2.1.3}$$

Definition 2.1.1 (BANACH SPACES OF REAL ANALYTIC PERIODIC FUNCTIONS AND NORMS)**.** *For $s \in \mathbb{R}^n_+$ and $n \in \mathbb{N}$, consider the Banach space of zero-average real analytic periodic functions on $\mathbb{T}^n$ with finite norm*

$$\|f\|_s := \sup_{k\in\mathbb{Z}^n} |f_k| e^{|k_1|s_1+\ldots+|k_n|s_n}, \tag{2.1.4}$$

and denote by $\mathbb{B}^n_s$ its closed unit ball. Besides the norm $\|\cdot\|_s$, we shall also use the following two (non equivalent) norms

$$|f|_s := \sup_{\mathbb{T}^n_s} |f|, \qquad \| f \|\!|_s := \sum_{k\in\mathbb{Z}^n} |f_k| e^{\sum_{i=1}^n |k_i| s_i}. \tag{2.1.5}$$

Note that in general $\|f\|_s \leqslant |f|_s \leqslant \| f \|\!|_s$.
For functions (not necessarily holomorphic in y) $f : B_r \times \mathbb{T}^n_s \longmapsto \mathbb{C}$ we will also use the norms

$$\begin{gathered} |f|_{B,r,s} = |f|_{r,s} = \sup_{B_r\times\mathbb{T}^n_s} |f|, \qquad \| f \|\!|_{B,r,s} = \| f \|\!|_{r,s} = \sup_{y\in B_r} \sum_{k\in\mathbb{Z}^n} |f_k(y)| e^{\sum_{i=1}^n |k_i| s_i} \\ \|f\|_{B,r,s} = \|f\|_{r,s} := \sup_{B_r\times\mathbb{T}^n_s} |f_k(y)| e^{\sum_{i=1}^n |k_i| s_i}. \end{gathered} \tag{2.1.6}$$

For a function depending only on $y \in B_r$ we use $|f|_{B,r} = |f|_r = \sup_{B_r} |f|$.

Remark 2.1.1. The three norms are obviously not equivalent. Indeed, for any $\sigma =$

$(\sigma_1, ..., \sigma_n) \in \mathbb{R}^n_+$ with $\sigma_\flat := \min_{i=1,...,n} \sigma_i$, one has [1]

$$\|f\|_{r,s} \leqslant |f|_{r,s} \leqslant \| f \|_{r,s} \leqslant \left(\coth^n(\frac{\sigma_\flat}{2}) - 1 \right) \|f\|_{r,s+\sigma} \leqslant \left(\frac{2n}{\sigma_\flat} \right)^n \|f\|_{r,s+\sigma} \tag{2.1.7}$$

Remark 2.1.2. Given $f(y,x) = \sum_{k\in\mathbb{Z}^n} f_k(y)e^{\mathrm{i}k\cdot x}$ and a sublattice Λ of $\mathbb{Z}^n$, we denote by π_Λ the projection on the Fourier coefficients in Λ, namely

$$\pi_\Lambda f := \sum_{k\in\Lambda} f_k(y)e^{\mathrm{i}k\cdot x} \, . \tag{2.1.8}$$

and by $\pi_\Lambda^\perp$ its "orthogonal" operator (projection on the Fourier modes in $\mathbb{Z}^n \setminus \Lambda$):

$$\pi_\Lambda^\perp f := \sum_{k\notin\Lambda} f_k(y)e^{\mathrm{i}k\cdot x} \, .$$

Obviously

$$\| \pi_\Lambda f \|_{r,s} \, , \, \| \pi_\Lambda^\perp f \|_{r,s} \leqslant \| f \|_{r,s} \, . \tag{2.1.9}$$

Remark 2.1.3. Given $N > 1$, one can consider the standard following 'truncation' and 'high–mode' operators T_N and $T_N^\perp$:

$$T_N f(y,x) := \sum_{|k|_1 \leqslant N} f_k(y)e^{\mathrm{i}k\cdot x} \, , \qquad T_N^\perp f(y,x) := \sum_{|k|_1 > N} f_k(y)e^{\mathrm{i}k\cdot x} \, . \tag{2.1.10}$$

Note that π_Λ and T_N commute. Note, also, that

$$\| T_N f \|_{r,s} \, , \, \| T_N^\perp f \|_{r,s} \leqslant \| f \|_{r,s} \, , \tag{2.1.11}$$

and that if $s'_i \leqslant s_i \; \forall i = 1, ..., n$

$$\| T_N^\perp f \|_{r,s'} \leqslant e^{-(\sum_{i=1}^n (s_i - s'_i))N} \| f \|_{r,s} \, . \tag{2.1.12}$$

Lemma 2.1.1. *For all $y \in B_r$, $\| \phi(y,\cdot)\psi(y,\cdot) \|_s \leqslant \| \phi(y,\cdot) \|_s \| \psi(y,\cdot) \|_s$.*

[1]We have $\sum_{k\in\mathbb{Z}^n\setminus 0} e^{-\sum_i |k_i|\sigma_i} \leqslant \sum_{k\in\mathbb{Z}^n\setminus 0} e^{-|k|_1\sigma_\flat} = \coth^n(\sigma_\flat/2) - 1$. Moreover $\coth^n x - 1 \leqslant (n/x)^n$. Indeed for $0 < x \leqslant 1$ the estimates follows by $\coth x < 2/\sinh x < 2/x$. In the case $x > 1$ we have

$$\coth^n x - 1 \leqslant (1 + e^{1-2x})^n - 1 \leqslant n(1 + 1/e)^{n-1} e^{1-2x} \leqslant (n/x)^n \, ,$$

where in the second inequality we have used that $(1+y)^n \leqslant 1 + n(1+1/e)^{n-1}y$ for $0 \leqslant y \leqslant 1/e$, while in the last one we exploit $\max_{x\geqslant 1} x^n e^{-2x} = (n/2e)^n$.

Proof. The k-th Fourier coefficient of $\phi\psi$ is $(\phi\psi)_k = \sum_m \phi_{k-m}\psi_m$. Hence

$$\begin{aligned}\|\phi(y,\cdot)\psi(y,\cdot)\|_s &= \sum_k |(\phi\psi)_k(y)| e^{\sum_i |k_i| s_i} \\ &\leqslant \sum_k \sum_m |\phi_{k-m}(y)| e^{\sum_i |k_i - m_i| s_i} \cdot |\psi_m(y)| e^{\sum_i |m_i| s_i} \qquad (2.1.13) \\ &= \|\phi(y,\cdot)\|_s \|\psi(y,\cdot)\|_s.\end{aligned}$$

□

Remark 2.1.4. The space of functions $f : \mathbb{T}^n_s \to \mathbb{C}^m$ endowed with the sup–norm $|\cdot|_s$ or the ℓ^1–Fourier norm $\|\cdot\|_s$ is a Banach algebra, while $\{f : \mathbb{T}^n_s \to \mathbb{C}^m : \|f\|_s < \infty\}$ is just a Banach space (not a Banach algebra). However, the norm $\|\cdot\|_s$ is particularly suited to describe $\{f : \mathbb{T}^n_s \to \mathbb{C} : \|f\|_s < \infty\}$ as a probability space.

Remark 2.1.5. If $f(x) \in \mathbb{B}^n_s$ and f_k denotes its Fourier coefficients of mode k, one has that

$$|f_k| \leqslant \|f\|_s e^{-(|k_1|s_1 + \ldots + |k_n|s_n)} \qquad (2.1.14)$$

As we have done in the first chapter, in order to apply singular KAM Theory, when k is a generator of maximal $1d$-lattices, we want to control in a quantitative way the k-Fourier coefficient from below. For this reason, we introduce the following class of potential

Definition 2.1.2 (THE ANALYTIC CLASS $\mathbb{G}^n_s$)**.** *We denote by $\mathbb{G}^n_s$ the subset of functions $f \in \mathbb{B}^n_s$ such that the following two properties hold:*

$$\begin{aligned}&\varliminf_{\substack{|k|_1 \to +\infty \\ k \in \mathcal{G}^n}} |f_k| e^{(|k_1|s_1 + \ldots + |k_n|s_n)} |k|_1^n > 0\,, \qquad (2.1.15) \\ &\forall\ k \in \mathcal{G}^n\,,\ \pi_{\mathbb{Z}k} f \text{ is a Morse function with distinct critical values}.\end{aligned}$$

Remark 2.1.6. (i) A simple example of function in $\mathbb{G}^n_s$ is given by

$$f(x) := 2 \sum_{k \in \mathcal{G}^n} e^{-(|k_1|s_1 + \ldots + |k_n|s_n)} \cos(k \cdot x)\,.$$

Indeed, one checks immediately that

$$\|f\|_s = 1\,, \qquad \varliminf_{\substack{|k|_1 \to +\infty \\ k \in \mathcal{G}^n}} |f_k| e^{(|k_1|s_1 + \ldots + |k_n|s_n)} |k|_1^n = +\infty\,, \qquad \pi_{\mathbb{Z}k} f(\theta) = 2e^{-(|k_1|s_1 + \ldots + |k_n|s_n)} \cos\vartheta\,.$$

(ii) The critical points of an analytic Morse function on $\mathbb{T}$, by compactness, cannot accumulate, hence, there are a finite, even number of them, which are, alternately, a relative strict maximum and a relative strict minimum. In particular, if G is β–Morse, then the number of its critical points can be bounded by $\pi\sqrt{2\max|G''|/\beta}$. Indeed, if $\theta \neq \theta'$ are critical points of G, then, by definition one has

$$\beta \leqslant |G(\theta) - G(\theta')| \leqslant \tfrac{1}{2}(\max|G''|)\,\mathrm{dist}(\theta,\theta')^2\,,$$

which implies that the minimal distance between two critical points is $\sqrt{2\beta/\max|G''|}$ and the claim follows.

2.2 Uniform behaviour of large-mode Fourier projections

If a function $f \in \mathbb{B}^n_s$ satisfies (2.1.15), then, *apart from a finite number of Fourier modes, its Fourier projections* $\pi_{\mathbb{Z}k} f$ *are close to a shifted rescaled cosine*, a fact that allows, e.g., to have a uniform analytic theory of high order perturbation theory.

To discuss this matter, let us first point out that for any sequence of real numbers $\{a_k\}$ and for any function $N(\delta)$ such that $\lim_{\delta\downarrow 0} N(\delta) = +\infty$ one has

$$\underline{\lim}\, a_k > 0 \qquad \Longleftrightarrow \qquad \exists\ \delta > 0 \ \text{ s.t. } \ a_k \geqslant \delta\,,\ \forall\ k \geqslant N(\delta)\,. \tag{2.2.1}$$

We shall apply this remark to the minimum limit in (2.1.15) with a particular choice of the function $N(\delta)$, namely, calling $s_\flat = \min_{i=1,\dots,n} s_i$ we define $\mathtt{N}(\delta) = \mathtt{N}(\delta; n, s)$ as

$$\mathtt{N}(\delta) := 2\,\max\Big\{1\,,\ \frac{1}{s_\flat}\ \log\frac{c_{{}_n}}{s_\flat^n\,\delta}\Big\}\,, \qquad c_{{}_n} := 2^{44}\left(\frac{2n}{e}\right)^n . \tag{2.2.2}$$

For later use, we point out that[2]

$$\mathtt{N} \geqslant 2\mathtt{c}_s\,, \qquad \text{where} \quad \mathtt{c}_s := \max\big\{1, \frac{1}{s_\flat}\big\}\,. \tag{2.2.3}$$

From (2.2.1) it follows that if f satisfies (2.1.15), one can find $0 < \delta \leqslant 1$ such that

$$|f_k| \geqslant \delta |k|_{{}_1}^{-n}\, e^{-(\sum_{i=1}^n |k_i| s_i)}\,, \qquad \forall\ k \in \mathcal{G}^n\,,\ |k|_{{}_1} \geqslant \mathtt{N}\,. \tag{2.2.4}$$

The main feature of the above choice of $\mathtt{N}$ is that, for $|k|_{{}_1} \geqslant \mathtt{N}$, $\pi_{\mathbb{Z}k} f$ is very close to a shifted rescaled cosine function:

[2] In fact, if $s_\flat \geqslant 1$ then $\mathtt{N} \geqslant 2 \geqslant 2/s_\flat$, while if $s_\flat < 1$ then the logarithm in (2.2.2) is larger than one, so that $\mathtt{N} \geqslant 2/s_\flat$ also in this case.

Proposition 2.2.1. *Let $\delta > 0$, $f \in \mathbb{B}^n_s$ and assume (2.2.4). Then, for any $k \in \mathcal{G}^n$ with $|k|_1 \geqslant \mathtt{N}$, $\pi_{\mathbb{Z}k} f$ is 2^{-40}–cosine–like (Definition 1.1.8).*

Proof. We shall prove something slightly stronger, namely, that there exists $\theta_k \in [0, 2\pi)$ so that

$$\pi_{\mathbb{Z}k} f(\theta) = 2|f_k|\big(\cos(\theta + \theta_k) + F^k_\star(\theta)\big)\,, \qquad F^k_\star(\theta) := \frac{1}{2|f_k|} \sum_{|j|\geqslant 2} f_{jk} e^{\mathrm{i}j\theta}\,, \tag{2.2.5}$$

with $F^k_\star \in \mathbb{B}^1_1$ and (recall the definition of the norms in 2.1.1)

$$|F^k_\star|_1 \leqslant \|F^k_\star\|_1 \leqslant 2^{-40}\,. \tag{2.2.6}$$

Indeed, by definition of $\pi_{\mathbb{Z}k} f$,

$$\pi_{\mathbb{Z}k} f(\theta) := \sum_{j\in\mathbb{Z}\setminus\{0\}} f_{jk} e^{\mathrm{i}j\theta} = \sum_{|j|=1} f_{jk} e^{\mathrm{i}j\theta} + \sum_{|j|\geqslant 2} f_{jk} e^{\mathrm{i}j\theta}\,,$$

and, defining $\theta_k \in [0, 2\pi)$ so that $e^{\mathrm{i}\theta_k} = f_k/|f_k|$, one has

$$\frac{1}{2|f_k|} \sum_{|j|=1} f_{jk} e^{\mathrm{i}j\theta} = \mathrm{Re}\left(\frac{f_k}{|f_k|} e^{\mathrm{i}\theta}\right) = \mathrm{Re}\, e^{\mathrm{i}(\theta+\theta_k)} = \cos(\theta + \theta_k)\,,$$

which yields (2.2.5). Now, since $f \in \mathbb{B}^n_s$ it is $|f_k| \leqslant e^{-(\sum_{i=1}^n |k_i| s_i)}$ and, by (2.2.4), $|f_k| \geqslant \delta |k|_1^{-n}\, e^{-(\sum_{i=1}^n |k_i| s_i)}$. Therefore, for $|k|_1 \geqslant \mathtt{N}$, one has

$$\begin{aligned}
\|F^k_\star\|_1 &\overset{(2.2.5)}{=} \frac{1}{2|f_k|} \sum_{|j|\geqslant 2} |f_{jk}| e^{|j|} \leqslant \frac{|k|_1^n e^{-(\sum_{i=1}^n |k_i| s_i)}}{2\delta} \sum_{|j|\geqslant 2} |f_{jk}| e^{|j|} \\
&\leqslant \frac{|k|_1^n e^{(\sum_{i=1}^n |k_i| s_i)}}{2\delta} \sum_{|j|\geqslant 2} e^{-|j|((\sum_{i=1}^n |k_i| s_i)-1)} \\
&\leqslant \frac{2e^2 |k|_1^n}{\delta}\, e^{-(\sum_{i=1}^n |k_i| s_i)} = \frac{2^{n+1} e^2}{s_\flat^n \delta} e^{-\frac{1}{2}(\sum_{i=1}^n |k_i| s_i)} \left(\frac{|k|_1 s_\flat}{2}\right)^n e^{-\frac{1}{2}(|k|_1 s_\flat)} \\
&\leqslant \left(\frac{2n}{e s_\flat}\right)^n \frac{2e^2}{\delta}\, e^{-\frac{\mathtt{N} s_\flat}{2}} \leqslant 2^{-40}\,,
\end{aligned} \tag{2.2.7}$$

where the geometric series converges since $|k|_1 s_\flat \geqslant \mathtt{N} s_\flat \geqslant 2$ (by (2.2.3)) and last inequality follows by definition of $\mathtt{N}$ in (2.2.2). $\square$

Remark 2.2.1. In fact, the particular form of $\mathtt{N}$ is used *only* in the last inequality in (2.2.7).

Next, we need an elementary calculus lemma:

Lemma 2.2.1. *Assume that $F \in C^2(\mathbb{T}, \mathbb{R})$, $\bar\theta$ and $0 < \mathtt{c} < 1/2$ are such that*

$$\|F - \cos(\theta + \bar\theta)\|_{C^2} \leqslant \mathtt{c}\,,$$

where $\|F\|_{C^2} := \max_{0\leqslant k\leqslant 2} \sup |F^{(k)}|$. Then, F has only two critical points and it is $(1-2\mathtt{c})$–Morse (Definition 1.1.6).

Proof. By considering the translated function $\theta \to F(\theta - \bar\theta)$, one can reduce oneself to the case $\bar\theta = 0$ (F is β–Morse, if and only if $\theta \to F(\theta - \bar\theta)$ is β–Morse).
Thus, we set $\bar\theta = 0$, and note that, by assumption $|F'| = |F' + \sin\theta - \sin\theta| \geqslant |\sin\theta| - \mathtt{c}$, and, analogously, $|F''| \geqslant |\cos\theta| - \mathtt{c}$. Hence, $|F'| + |F''| \geqslant |\sin\theta| + |\cos\theta| - 2\mathtt{c} \geqslant 1 - 2\mathtt{c}$.
Next, let us show that F has a unique strict maximum $\theta_0 \in I := (-\pi/6, \pi/6) \pmod{2\pi}$. Writing $F = \cos\theta + g$, with $g := F - \cos\theta$, one has that $F'(-\pi/6) = 1/2 + g'(\pi/6) \geqslant 1/2 - \mathtt{c} > 0$, and, similarly $F'(\pi/6) \leqslant -1/2 + \mathtt{c}$, thus F has a critical point in I, and, since $-F'' = \cos\theta - g'' \geqslant \cos\theta - \mathtt{c} \geqslant \sqrt{3}/2 - \mathtt{c} > 0$, F is strictly concave in I, showing that such critical point is unique and it is a strict local minimum. In fact, similarly one shows that F has a second critical point $\theta_1 \in (\pi - \pi/6, \pi + \pi/6)$ where F is strictly convex, so that θ_1 is a strict local minimum; but, since in the complementary of these intervals F is strictly monotone (as it is easy to check), it follows that F has a unique global strict maximum and a unique global strict minimum. Finally, $F(\theta_0) - F(\theta_1) \geqslant \sqrt{3} - 2\mathtt{c} > 1 - 2\mathtt{c}$ and the claim follows. □

From Proposition 2.2.1 and Lemma 2.2.1 one gets immediately:

Proposition 2.2.2. *Let $\delta > 0$, $f \in \mathbb{B}^n_s$ and assume (2.2.4). Then, for every $k \in \mathcal{G}^n$ with $|k|_1 \geqslant \mathtt{N}$, the function $\pi_{\mathbb{Z}k} f$ is $|f_k|$–Morse.*

Proof. As in the proof of Proposition 2.2.1, we get

$$\left|\frac{\pi_{\mathbb{Z}k} f}{2 f_k} - \cos(\theta + \theta^k)\right|_1 \overset{(2.2.5)}{=} |F^k_\star|_1 \leqslant \|F^k_\star\|_1 \overset{(2.2.6)}{\leqslant} 2^{-40}\,, \tag{2.2.8}$$

which implies that the function $F := \pi_{\mathbb{Z}k} f/(2f_k)$ is C^2–close to a (shifted) cosine: Indeed, by Cauchy estimates $\|\cdot\|_{C^2} \leqslant 2|\cdot|_1$, so that

$$\|F - \cos(\theta + \theta^k)\|_{C^2} = \max_{0\leqslant j\leqslant 2} \max_{\mathbb{T}} |\partial_\theta^j (F - \cos(\theta + \theta^k))| \leqslant 2|F^k_\star|_1 \overset{(2.2.9)}{\leqslant} 2^{-39}\,.$$

By Lemma 2.2.1 we see that F is $(1 - 2^{-38})$–Morse, and the claim follows by rescaling. □

From Proposition 2.2.1 and Lemma 2.2.1 one gets immediately:

Proposition 2.2.3. *Let $\delta > 0$, $f \in \mathbb{B}^n_s$ and assume (2.2.4). Then, for every $k \in \mathcal{G}^n$ with $|k|_1 \geqslant \mathtt{N}$, the function $\pi_{\mathbb{Z}k} f$ is $|f_k|$–Morse.*

Proof. As in the proof of Proposition 2.2.1, we get

$$\left|\frac{\pi_{\mathbb{Z}k} f}{2f_k} - \cos(\theta + \theta^k)\right|_1 \overset{(2.2.5)}{=} |F^k_\star|_1 \leqslant \|F^k_\star\|_1 \overset{(2.2.6)}{\leqslant} 2^{-40}\,, \tag{2.2.9}$$

which implies that the function $F := \pi_{\mathbb{Z}k} f/(2f_k)$ is C^2–close to a (shifted) cosine: Indeed, by Cauchy estimates $\|\cdot\|_{C^2} \leqslant 2|\cdot|_1$, so that

$$\|F - \cos(\theta + \theta^k)\|_{C^2} = \max_{0 \leqslant j \leqslant 2} \max_{\mathbb{T}} |\partial^j_\theta(F - \cos(\theta + \theta^k))| \leqslant 2|F^k_\star|_1 \overset{(2.2.9)}{\leqslant} 2^{-39}\,.$$

By Lemma 2.2.1 we see that F is $(1 - 2^{-38})$–Morse, and the claim follows by rescaling. □

2.3 Genericity

In this section we prove that $\mathbb{G}^n_s$ is a generic set in $\mathbb{B}^n_s$.

Definition 2.3.1. *Given $n > 0, s \in \mathbb{R}^n_+$, $0 < \delta \leqslant 1$ and $\beta > 0$ and $\mathtt{N}$ as in (2.2.2) we call $\mathbb{G}^n_s(\delta, \beta)$ the set of functions in $\mathbb{B}^n_s$ which satisfy (2.2.4) together with:*

$$\pi_{\mathbb{Z}k} f \text{ is } \beta\text{–Morse}\,, \qquad \forall\ k \in \mathcal{G}^n\,,\ |k|_1 \leqslant \mathtt{N}\,. \tag{2.3.1}$$

Then, the following lemma holds:

Lemma 2.3.1. *Let $n > 0, s \in \mathbb{R}^n_+$. Then, $\mathbb{G}^n_s = \bigcup\limits_{\substack{\delta \in (0,1] \\ \beta > 0}} \mathbb{G}^n_s(\delta, \beta)$.*

Proof. Assume $f \in \mathbb{G}^n_s$ and let $0 < \delta_0 \leqslant 1$ be smaller than limit inferior in (2.1.15). Then, there exists N_0 such that $|f_k| > \delta_0 |k|_1^{-n} e^{\sum_i |k_i| s_i}$, for any $|k|_1 \geqslant N_0$, $k \in \mathcal{G}^n$. Since $\lim_{\delta \to 0} \mathtt{N} = +\infty$, there exists $0 < \delta < \delta_0$ such that $\mathtt{N} > N_0$. Hence, if $|k|_1 \geqslant \mathtt{N}$ and $k \in \mathcal{G}^n$, (2.2.4) holds.
Since $\pi_{\mathbb{Z}k} f$ is, for any $|k|_1 \leqslant \mathtt{N}$, a Morse function with distinct critical values one can, obviously, find a $\beta > 0$ for which (2.3.1) holds. Hence $f \in \mathbb{G}^n_s(\delta, \beta)$.
Now, let $f \in \bigcup \mathbb{G}^n_s(\delta, \beta)$. Then, there exist $\delta \in (0, 1]$ and $\beta > 0$ such that (2.2.4) and (2.3.1) hold. Then, (2.1.15) follows immediately from (2.2.4). By Proposition 2.2.1, for any $k \in \mathcal{G}^n$ with $|k|_1 > \mathtt{N}$, $\pi_{\mathbb{Z}k} f$ is 2^{-40}–cosine–like, showing (Lemma 2.2.1) that $\pi_{\mathbb{Z}k} f$ is Morse with distinct critical values also for $|k|_1 \geqslant \mathtt{N}$. The proof is complete. □

Proposition 2.3.1. $\mathbb{G}^n_s$ *contains an open and dense set in* $\mathbb{B}^n_s$.

To prove this result we need a preliminary elementary result on real analytic periodic functions:

Lemma 2.3.2. *Let* $F = \sum F_j e^{\mathrm{i}j\theta}$ *be a real analytic function on* $\mathbb{T}$*. There exists a compact set* $\Gamma \subseteq \mathbb{C}$ *(depending on* F_j *for* $|j| \geqslant 2$*) of zero Lebesgue measure such that if the Fourier coefficient* F_1 *does not belong to* Γ*, then* F *is a Morse function with distinct critical values.*

Proof. Without loss of generality we may assume that F has zero average. Then, letting $z := F_1 \in \mathbb{C}$, we write F as

$$F(\theta) = ze^{\mathrm{i}\theta} + \bar{z}e^{-\mathrm{i}\theta} + G(\theta) := ze^{\mathrm{i}\theta} + \bar{z}e^{-\mathrm{i}\theta} + \sum_{|j|\geqslant 2} F_j e^{\mathrm{i}j\theta}\,. \tag{2.3.2}$$

When $G \equiv 0$ the claim is true with $\Gamma = \{0\}$.
Assume that $G \not\equiv 0$. Observe that, since G is real–analytic, the equations $F'(\theta) = 0 = F''(\theta)$ are equivalent to the single equation $z = \frac{1}{2}e^{-\mathrm{i}\theta}\big(\mathrm{i}G'(\theta) + G''(\theta)\big)$, which, as $\theta \in \mathbb{T}$, describes a smooth closed 'critical' curve Γ_1 in $\mathbb{C}$.
Observe also that F has distinct critical points $\theta_1, \theta_2 \in \mathbb{T}$ with the same critical values if and only if the following three real equations are satisfied:

$$F'(\theta_1) = 0\,, \qquad F'(\theta_2) = 0\,, \qquad F(\theta_1) - F(\theta_2) = 0\,. \tag{2.3.3}$$

We claim that if z, θ_1, θ_2 satisfy (2.3.3) then

$$z = \zeta(\theta_1, \theta_2)\,, \qquad g(\theta_1, \theta_2) = 0\,, \tag{2.3.4}$$

with ζ and g real analytic on $\mathbb{T}^2$ given by

$$\zeta(\theta_1,\theta_2) := \begin{cases} \frac{\mathrm{i}}{2(e^{\mathrm{i}\theta_1} - e^{\mathrm{i}\theta_2})}\big(G'(\theta_1) - G'(\theta_2) + \mathrm{i}G(\theta_1) - \mathrm{i}G(\theta_2)\big)\,, & \text{for} \quad \theta_1 \neq \theta_2\,; \\ \frac{1}{2e^{\mathrm{i}\theta_1}}\big(G''(\theta_1) + \mathrm{i}G'(\theta_1)\big)\,, & \text{for} \quad \theta_1 = \theta_2\,, \end{cases}$$

$$g(\theta_1,\theta_2) := \big(1 - \cos(\theta_1 - \theta_2)\big)\big(G'(\theta_1) + G'(\theta_2)\big) - \sin(\theta_1 - \theta_2)\big(G(\theta_1) - G(\theta_2)\big)\,.$$

Indeed, summing up the the third equation in (2.3.3) with the difference of the first two equations multiplied by $-\mathrm{i}$, we get

$$2(e^{\mathrm{i}\theta_1} - e^{\mathrm{i}\theta_2})z - \mathrm{i}\big(G'(\theta_1) - G'(\theta_2) + \mathrm{i}G(\theta_1) - \mathrm{i}G(\theta_2)\big) = 0\,,$$

which is equivalent to $z = \zeta(\theta_1, \theta_2)$. Then, by definition $g(\theta_1, \theta_1) = 0$, while if $\theta_1 \neq \theta_2$, substituting $z = \zeta(\theta_1, \theta_2)$ in the first equation in (2.3.3) and multiplying by $1-\cos(\theta_1 - \theta_2)$ we get $g(\theta_1, \theta_2) = 0$ also for $\theta_1 \neq \theta_2$. Thus, (2.3.4) holds.
Next, we claim that the real analytic function $g(\theta_1, \theta_2)$ is not identically zero. Assume by contradiction that g is identically zero. Then $g(\theta_2 + t, \theta_2) \equiv 0$ for every θ_2 and t, and taking the fourth derivative with respect to t evaluated at $t = 0$, we see that $G'''(\theta_2) + G'(\theta_2) = 0$, for all θ_2. The general (real) solution of the such differential equation is given by $G(\theta_2) = ce^{\mathrm{i}\theta_2} + \bar{c}e^{-\mathrm{i}\theta_2} + c_0$, with $c \in \mathbb{C}$, $c_0 \in \mathbb{R}$, which contradicts the fact that, by definition, $G_j = 0$ for $|j| \leqslant 1$. Thus, $g(\theta_1, \theta_2)$ is not identically zero and, therefore, the set $\mathcal{Z} \subseteq \mathbb{T}^2$ of its zeros is compact and has zero Lebesgue measure[3]. Clearly, also the set $\Gamma_2 := \zeta(\mathcal{Z}) \subseteq \mathbb{C}$ is compact and has zero measure, and, therefore, if we define $\Gamma = \Gamma_1 \cup \Gamma_2$, we see that the lemma holds also in the case $G \not\equiv 0$. $\square$

Proof. **of Proposition 2.3.1** Let $\tilde{\mathbb{G}}^n_s(\delta, \beta)$ denote the subset of functions in $\mathbb{G}^n_s(\delta, \beta)$ satisfying the (stronger) condition[4]

$$|f_k| > \delta\, e^{-\sum_i |k_i| s_i}\,, \qquad \forall\ k \in \mathcal{G}^n\,,\ |k|_1 \geqslant \mathtt{N} = \mathtt{N}(\delta)\,, \tag{2.3.5}$$

and let $\tilde{\mathbb{G}}^n_s = \bigcup_{\substack{0<\delta\leqslant 1\\ \beta>0}} \tilde{\mathbb{G}}^n_s(\delta, \beta)$. We claim that $\tilde{\mathbb{G}}^n_s$ is an open subset of $\mathbb{B}^n_s$. Let $f \in \tilde{\mathbb{G}}^n_s(\delta, \beta)$ for some $0 < \delta \leqslant 1, \beta > 0$ and let us show that there exists $0 < \delta' \leqslant \delta/2$ such that if $g \in \mathbb{B}^n_s$ with $\|g - f\|_s < \delta' \leqslant \delta/2$, then $g \in \tilde{\mathbb{G}}^n_s(\delta', \beta')$ with $\beta' := \min\{\beta, \delta e^{-s_\flat \mathtt{N}(\delta/2)}\}/2$. Indeed

$$|\tilde{f}_k| e^{\sum_i |k_i| s_i} \geqslant |f_k| e^{\sum_i |k_i| s_i} - \|g - f\|_s > \delta - \delta' \geqslant \delta/2\,, \qquad \forall\ k \in \mathcal{G}^n\,,\ |k|_1 \geqslant \mathtt{N}(\delta)\,,$$

namely g satisfies (2.3.5) with $\delta/2$ instead of δ. We already know that $\pi_{\mathbb{Z}k} f$ is β–Morse $\forall\ k \in \mathcal{G}^n$, $|k|_1 < \mathtt{N}(\delta)$. Moreover, by Proposition 2.2.3, we know that $\pi_{\mathbb{Z}k} f$ is $|f_k|$–Morse for $k \in \mathcal{G}^n$ with $|k|_1 \geqslant \mathtt{N}(\delta)$. In conclusion, by (2.3.5), we get that $\pi_{\mathbb{Z}k} f$ is $2\beta'$–Morse $\forall\ k \in \mathcal{G}^n$, $|k|_1 < \mathtt{N}(\delta/2)$. Since the $\|\cdot\|_s$–norm is stronger than the C^2–one, taking δ' small enough we get that $\pi_{\mathbb{Z}k} g$ is β'–Morse $\forall\ k \in \mathcal{G}^n$, $|k|_1 < \mathtt{N}(\delta/2)$.
Let us now show that $\tilde{\mathbb{G}}^n_s$ is dense in $\mathbb{B}^n_s$. Fix f in $\mathbb{B}^n_s$ and $0 < \lambda < 1$. We have to find $g \in \tilde{\mathbb{G}}^n_s$ with $\|g - f\|_s \leqslant \lambda$. Let $\delta := \lambda/4$ and denote by f_k and g_k (to be defined) be the Fourier coefficients of, respectively, f and g. It is enough to define g_k only for $k \in \mathbb{Z}^n_\star$ since, for $k \in -\mathbb{Z}^n_\star$ we set $g_k := \bar{g}_{-k}$, since g must be real analytic. Set $g_k := f_k$ for $k \in \mathbb{Z}^n_\star \backslash \mathcal{G}^n$. For $k \in \mathcal{G}^n$, $|k|_1 \geqslant \mathtt{N}(\delta)$, we set $g_k := f_k$ if $|f_k| e^{\sum_i |k_i| s_i} > \delta$

[3] Compare, e.g., Corollary 10, p. 9 of [50].
[4] Here, we explicitly indicate the dependence on δ, while n and s are fixed. Recall that $\mathtt{N}(\delta)$ is decreasing.

and $g_k := 2\delta e^{-(\sum_i |k_i| s_i)}$ otherwise. Consider now $k \in \mathcal{G}^n$, $|k|_1 < \mathtt{N}(\delta)$. We make use of Lemma 2.3.2 with $F = \pi_{\mathbb{Z}k} g$, $z = F_1 = g_k$. Thus, by Lemma 2.3.2, there exists a compact set $\Gamma_k \subseteq \mathbb{C}$ (depending on F_k for $|k| \geqslant 2$) of zero measure such that if $g_k \notin \Gamma_k$ the function $\pi_{\mathbb{Z}k} g$ is a Morse function with distinct critical values. We conclude the proof of the density choosing $|g_k| < e^{-(\sum_i |k_i| s_i)}$, $|f_k - g_k| \leqslant \lambda e^{-(\sum_i |k_i| s_i)}$ with $g_k \notin \Gamma_k$. $\square$

2.3.1 Full measure

Here we show that $\mathbb{G}^n_s$ *is a set of probability 1 with respect to the standard product probability measure on* $\mathbb{B}^n_s$. More precisely, consider the space[5] $\mathrm{D}^{\mathbb{Z}^n_\star}$, where $\mathrm{D} := \{w \in \mathbb{C} : |w| \leqslant 1\}$, endowed with the product topology[6]. The product σ-algebra of the Borel sets of $\mathrm{D}^{\mathbb{Z}^n_\star}$ is the σ–algebra generated by the cylinders $\bigotimes_{k \in \mathbb{Z}^n_\star} A_k$, where A_k are Borel sets of D, which differs from D only for a finite number of k. The probability product measure $\mu_\otimes$ on $\mathrm{D}^{\mathbb{Z}^n_\star}$ is then defined by letting

$$\mu_\otimes\Big(\bigotimes_{k \in \mathbb{Z}^n_\star} A_k\Big) := \prod_{k \in \mathbb{Z}^n_\star} |A_k|\,,$$

where $|\cdot|$ denotes the normalized ($|\mathrm{D}| = 1$) Lebesgue measure on D. The (weighted) Fourier bijection[7]

$$\mathcal{F} : f(x) = \sum_{k \in \mathbb{Z}^n_\star} f_k e^{\mathrm{i}k\cdot x} + \bar{f}_k e^{-\mathrm{i}k\cdot x} \in \mathbb{B}^n_s \to \big\{f_k e^{\sum_i |k_i| s_i}\big\}_{k \in \mathbb{Z}^n_\star} \in \ell^\infty(\mathbb{Z}^n_\star) \tag{2.3.6}$$

induces a product topology on $\mathbb{B}^n_s$ and a *probability product measure* μ on the product σ-algebra $\mathcal{B}$ of the Borellians in $\mathbb{B}^n_s = \mathcal{F}^{-1}\big(\mathrm{D}^{\mathbb{Z}^n_\star}\big)$ (with respect to the induced product topology), i.e., given $B \in \mathcal{B}$, we set $\mu(B) := \mu_\otimes(\mathcal{F}(B))$. Then one has:

Proposition 2.3.2. $\mathbb{G}^n_s \in \mathcal{B}$ *and* $\mu(\mathbb{G}^n_s) = 1$.

Proof. First note that, for every $\delta, \beta > 0$ the set $\mathbb{G}^n_s(\delta, \beta)$ is closed with respect to the product topology. Indeed for every $k \in \mathcal{G}^n$ the set $\{f \in \mathbb{B}^n_s \text{ s.t. } |f_k| \geqslant \delta |k|_1^{-n} e^{-(\sum_i |k_i| s_i)}\}$ is a closed cylinder. Moreover also the set $\{f \in \mathbb{B}^n_s \text{ s.t. } \pi_{\mathbb{Z}k} f \text{ is } \beta\text{–Morse}\}$ is closed w.r.t the product topology. In fact we prove that the complementary $E := \{f \in \mathbb{B}^n_s \text{ s.t. } \pi_{\mathbb{Z}k} f \text{ is not } \beta\text{–Morse}\}$ is open w.r.t the product topology. Indeed if $f^* \in E$ there

[5] $\mathbb{Z}^n_\star$ was defined in 1.1.2.

[6] By Tychonoff's Theorem, $\mathrm{D}^{\mathbb{Z}^n_\star}$ with the product topology is a compact Hausdorff space.

[7] f is real analytic so that $f_{-k} = \bar{f}_k$.

exists a $r > 0$ small enough such that $E_r := \{f \in \mathbb{B}^n_s \text{ s.t. } \|\pi_{\mathbb{Z}k} f - \pi_{\mathbb{Z}k} f^*\|_{C^2} < r\} \subseteq E$. Define the open cylinder

$$E_{\rho,J} := \{f \in \mathbb{B}^n_s \text{ s.t. } |f_{jk} - f^*_{jk}| < \frac{\rho}{|j|^2} e^{-|j|(\sum_i |k_i| s_i)} \text{ for } j \in \mathbb{Z}\,,\ 0 < |j| \leqslant J\}\,.$$

We claim that $E_{\rho,J} \subseteq E_r$ for suitably small ρ and large J (depending on r and s). Indeed, when $f \in E_{\rho,J}$

$$\|\pi_{\mathbb{Z}k} f - \pi_{\mathbb{Z}k} f^*\|_{C^2} \leqslant 3 \sum_{j \neq 0} |j|^2 |f_{jk} - f^*_{jk}| \leqslant 3\rho \sum_{0<|j|\leqslant J} e^{-|j|(\sum_i |k_i| s_i)} + 6 \sum_{|j|>J} |j|^2 e^{-|j|(\sum_i |k_i| s_i)} < r$$

for suitably small ρ and large J. Therefore $E_{\rho,J} \subseteq E_r \subseteq E$ and E is open in the product topology. In conclusion, taking the intersection over $k \in \mathcal{G}^n$, we get that $\mathbb{G}^n_s(\delta, \beta)$ is closed with respect to the product topology.

Recalling Lemma 2.3.1, we note that $\mathbb{G}^n_s$ can be written as $\mathbb{G}^n_s = \bigcup_{h \in \mathbb{N}} \mathbb{G}^n_s(1/h, 1/h)$. Thus $\mathbb{G}^n_s$ is Borellian.

Let us now prove that $\mu(\mathbb{G}^n_s) = 1$. Fix $0 < \delta \leqslant 1$ and denote by $\mathbb{G}^n_s(\delta)$ be the subset of functions in $\mathbb{B}^n_s$ satisfying (2.2.4) and such that $\pi_{\mathbb{Z}k} f$ is a Morse function with distinct critical values for every $k \in \mathcal{G}^n$. Recall (2.3.6) and define

$$\mathbb{P}_\delta := \mathcal{F}(\mathbb{G}^n_s(\delta)) \subseteq \ell^\infty(\mathbb{Z}^n_\star)\,.$$

Fix $\hat{g} = (g_k)_{k \in \mathbb{Z}^n_\star \backslash \mathcal{G}^n} \in \ell^\infty(\mathbb{Z}^n_\star \backslash \mathcal{G}^n)$ with $|g_k| \leqslant 1$ for every $k \in \mathbb{Z}^n_\star \backslash \mathcal{G}^n$. Consider the section

$$\mathbb{P}^{\hat{g}}_\delta := \Big\{\check{g} = (g_k)_{k \in \mathcal{G}^n},\ |g_k| \leqslant 1 \text{ s.t } |g_k| \geqslant \delta |k|_1^{-n} \text{ if } |k|_1 \geqslant \mathtt{N}\,,\ g_k e^{-(\sum_i |k_i| s_i)} \notin \Gamma_k\,,\ \text{if } |k|_1 < \mathtt{N}\Big\},$$

where the sets Γ_k (depending on $\hat{g}$) were defined in the proof of Proposition 2.3.1 so that, for every $k \in \mathcal{G}^n$, $|k|_1 < \mathtt{N}$, if $g_k e^{-(\sum_i |k_i| s_i)} \notin \Gamma_k$ then the function[8]

$$g_k e^{-(\sum_i |k_i| s_i)} e^{\mathrm{i}\theta} + \bar{g}_k e^{-(\sum_i |k_i| s_i)} e^{-\mathrm{i}\theta} + \sum_{|j|\geqslant 2} \hat{g}_{jk} e^{-|j|(\sum_i |k_i| s_i)} e^{\mathrm{i}j\theta} = \pi_{\mathbb{Z}k} f\,,\ \text{ with } f := \mathcal{F}^{-1}(g)\,,\ g = (\check{g}, \hat{g})\,,$$

is a Morse function with distinct critical values. Then, since every Γ_k has zero measure

$$\mu_\otimes|_{\ell^\infty(\mathcal{G}^n)}(\mathbb{P}^{\hat{g}}_\delta) = \prod_{k \in \mathcal{G}^n, |k|_1 \geqslant \mathtt{N}} (1 - \delta^2 |k|_1^{-2n}) \geqslant 1 - c\delta^2\,,$$

[8]Recall (2.3.2).

for a suitable constant $c = c(n)$. Since the above estimate holds for every $\hat{g} \in \ell^\infty(\mathbb{Z}^n_\star \backslash \mathcal{G}^n)$, by Fubini's Theorem we get

$$\mu_\otimes|_{\ell^\infty(\mathcal{G}^n)}(\mathbb{P}^{\hat{g}}_\delta) = \mu_\otimes(\mathbb{P}_\delta) = \mu(\mathbb{G}^n_s(\delta)) \geqslant 1 - c\delta^2 \,.$$

Then,

$$\mu(\mathbb{G}^n_s) = \lim_{\delta \to 0^+} \mu(\mathbb{G}^n_s(\delta)) = 1 \,. \quad \blacksquare$$

Assumptions 2.3.1. *Also in this case, For the rest of the work we will assume two proprieties on our Hamiltonian in 2.1.1:*
*i) The integrable part $h(y)$ is a δ-**convex** function of the action variable (see Assumpion 1.1.1 in Chapter 1) and we will call L as the Lipschitz constant of h. This function is supposed to be real-analytic on B_r where B is a compact subset of $\mathbb{R}^n$ and $r > 0$.*
ii) The perturbation $f \in \mathbb{G}^n_s$.

2.4 A normal form lemma with "small" analyticity loss

In this section, as we have done in Chapter 1, we describe an analytic normal form lemma for nearly–integrable Hamiltonians $H_\varepsilon(y, x) = h(y) + \varepsilon f(y, x)$, which allows to average out non–resonant Fourier modes of the perturbation f on suitable non–resonant regions, and allows for "very small" analyticity loss in the angle variables, a fact, which will be crucial in our applications. In this case, we have to do something different because each analyticity stripe has different width. These differences are only technical.

We recall ([51], [25]) that, given an integrable Hamiltonian $h(y)$, positive numbers α, K and a lattice $\Lambda \subset \mathbb{Z}^n$, a (real or complex) domain U is (α, K) non–resonant modulo Λ (with respect to h) if

$$|h'(y) \cdot k| \geqslant \alpha \,, \quad \forall\, y \in U \,, \forall\, k \in \mathbb{Z}^n \setminus \Lambda \,, \; |k|_1 \leqslant K \,. \tag{2.4.1}$$

As in the first chapter, the main point of the following "Normal Form Lemma" is that the "new" averaged Hamiltonian is defined, in the fast variable (angle) domain, in a region "almost equal" to the original domain, "almost equal" meaning a complex strip with of width $s_i(1 - 1/K)$ for each component if s is the vector of width of the initial angle analyticity. More precisely, we have:

Proposition 2.4.1 (Normal form with "small" analyticity loss).
Let $r>0, s\in\mathbb{R}^n_+, \alpha>0,\ K\in\mathbb{N},\ K\geqslant 2,\ B\subseteq\mathbb{R}^n$, and let Λ be a lattice of $\mathbb{Z}^n$. Let

$$H(y,x)=h(y)+f(y,x) \tag{2.4.2}$$

be real–analytic on $B_r\times\mathbb{T}^n_s$ with $\|f\|_{r,s}<\infty$. Assume that B_r is (α,K) non–resonant modulo Λ and that

$$\vartheta_\star:=\frac{2^{11}K^2}{\alpha\, r s_\sharp}\|f\|_{r,s}<1\,. \tag{2.4.3}$$

where $s_\sharp:=\min_{1\leqslant i\leqslant n} s_i$. Then, there exists a real–analytic symplectic change of variables

$$\Psi:(y',x')\in B_{r_\star}\times\mathbb{T}^n_{s_\star}\ \mapsto\ (y,x)\in B_r\times\mathbb{T}^n_s\quad\text{with}\quad r_\star:=r/2\,,\quad s_\star:=s(1-1/K)^{9} \tag{2.4.4}$$

satisfying

$$|y-y'|_1\leqslant\frac{\vartheta_\star}{2^7K}r\,,\qquad \max_{1\leqslant i\leqslant n}|x_i-x_i'|\leqslant\frac{\vartheta_\star}{16K^2}s_\flat\,, \tag{2.4.5}$$

and such that

$$H\circ\Psi=h+f^\flat+f_\star\,,\qquad f^\flat:=\pi_\Lambda f+T_K^\perp\pi_\Lambda^\perp f \tag{2.4.6}$$

with

$$\|f_\star\|_{r_\star,s_\star}\leqslant\frac{1}{K}\vartheta_\star\|f\|_{r,s}\,,\qquad \|T_K\,\pi_\Lambda^\perp f_\star\|_{r_\star,s_\star}\leqslant(\vartheta_\star/8)^K\frac{8}{eK}\|f\|_{r,s}\,. \tag{2.4.7}$$

Moreover, re-writing (2.4.6) as

$$H\circ\Psi=h+g+f_{\star\star}\qquad\text{where}\quad \pi_\Lambda g=g\ ,\quad \pi_\Lambda f_{\star\star}=0\,, \tag{2.4.8}$$

one has

$$\|g-\pi_\Lambda f\|_{r_\star,s_\star}\leqslant\frac{1}{K}\vartheta_\star\|f\|_{r,s}\ ,\qquad \|f_{\star\star}\|_{r_\star,s/2}\leqslant 2e^{-(K-2)\bar s}\|f\|_{r,s}\,, \tag{2.4.9}$$

where

$$\bar s:=\min\left\{\frac{s_\flat}{2},\ \log\frac{8}{\vartheta_\star}\right\}\,. \tag{2.4.10}$$

Remark 2.4.1.

(i) As we have just said, the "novelty" of this lemma is that the bounds in (2.4.7) and the first one in (2.4.9) hold on the arbitrary large angle domain $\mathbb{T}^n_{s_\star}$ with $s_\star=s(1-1/K)$, that will be crucial for our work (as we have just seen in chapter one). The drawback of the gain in angle–analyticity strip is that the power of K in the smallness condition (2.4.3) is not optimal: for example in [25] the power of K is one (but $s_\star=s/6$, which would not work in our applications).

[9] Given a vector $\xi\in\mathbb{R}^n$, with the notation $a\xi$ with $a\in\mathbb{R}$ we intend the vector with component $(a\xi)_i=a\xi_i$ for all $i=1,\dots,n$.

(ii) Having information on non–resonant Fourier modes up to order K, the best one can do is to average out the non–resonant Fourier modes up to order K, namely, to "kill" the term $T_K\,\pi_\Lambda^\perp f$ of the Fourier expansion of the perturbation. This explains the "flat" term $f^\flat = \pi_\Lambda f + T_K^\perp\,\pi_\Lambda^\perp f$ surviving in (2.4.6) and which cannot be removed in general. Now, think of the remainder term f_* as

$$f_* = \pi_\Lambda f_* + \left(T_K^\perp\,\pi_\Lambda^\perp f_* + T_K\,\pi_\Lambda^\perp f_*\right)\ ;$$

then, $\pi_\Lambda f_*$ is a $(\vartheta_* \|f\|_{r,s}/K)$–perturbation of the part in normal form (i.e., with Fourier modes in Λ), while $T_K^\perp\,\pi_\Lambda^\perp f_*$ is, by (2.1.12), a term exponentially small with K (see also below) and $T_K\,\pi_\Lambda^\perp f_*$ is a very small remainder bounded by $8(\vartheta_*/8)^K \|f\|_{r,s}/eK$.

(iii) We note that (2.4.8) follows from (2.4.6). Indeed we take

$$g = \pi_\Lambda f + \pi_\Lambda f_*\,,\qquad f_{**} = T_K^\perp\,\pi_\Lambda^\perp f + \pi_\Lambda^\perp f_* = T_K\,\pi_\Lambda^\perp f_* + T_K^\perp\,\pi_\Lambda^\perp (f_* + f)\,.$$

Then the first estimate in (2.4.9) follows by the first bound in (2.4.7) and (2.1.9). Regarding the second estimate in (2.4.9), we first note by (2.4.7) and (2.1.12) (used with $f \rightsquigarrow f_*$ $N \rightsquigarrow K$, $r \rightsquigarrow r_*$, $s \rightsquigarrow s_*$, and $\sigma \rightsquigarrow \frac{1}{2}s - \frac{1}{K}s$ so that $s_* - \sigma = s/2$ and $e^{-(K+1)\sigma_\flat} \leqslant e^{-(K-2)s_\flat/2}$)

$$\|T_K^\perp f_*\|_{r_*,s/2} = \|T_K^\perp f_*\|_{r_*,s_*-\sigma} \leqslant e^{-(K+1)\sigma_\flat}\|f_*\|_{r_*,s_*} \leqslant e^{-(K-2)s_\flat/2}\frac{\vartheta_*}{K}\|f\|_{r,s}\,.$$

By (2.1.9), (2.4.7) and (2.1.12) we get

$$\begin{aligned}
\|f_{**}\|_{r_*,s/2} &\leqslant \|T_K\,\pi_\Lambda^\perp f_*\|_{r_*,s/2} + \|T_K^\perp f_*\|_{r_*,s/2} + \|T_K^\perp f\|_{r_*,s/2}\\
&\leqslant (\vartheta_*/8)^K\frac{8}{eK}\|f\|_{r,s} + e^{-(K-2)s_\flat/2}(\vartheta_*/K + e^{-3s_\flat/2})\|f\|_{r,s}\\
&\leqslant 2e^{-(K-2)\bar{s}}\|f\|_{r,s}\ .
\end{aligned}$$

(iv) For comparison with standard normal form theory like the one in [25] see Remark 4.1,*iv* in [1], the considerations are essentially the same.

In order to do this procedure we need some technical lemmata that are slightly different from the standard one (e.g. [25]).
Given a function ϕ we denote by X_ϕ^t the hamiltonian flow at time t generated by ϕ and by "ad" the linear operator $u \mapsto \mathrm{ad}_\phi u := \{u,\phi\}$ and ad^ℓ its iterates:

$$\mathrm{ad}_\phi^0 u := u\,,\qquad \mathrm{ad}_\phi^\ell u := \{\mathrm{ad}_\phi^{\ell-1}u, \phi\}\,,\qquad \ell \geqslant 1\,,$$

as standard, $\{\cdot,\cdot\}$ denotes Poisson bracket[10].

[10] Explicitly, $\{u,v\} = \sum_{i=1}^n (u_{x_i}v_{y_i} - u_{y_i}v_{x_i})$.

Recall the identity ("Lie series expansion")

$$u \circ X_\phi^1 = \sum_{\ell \geqslant 0} \frac{1}{\ell!} \mathrm{ad}_\phi^\ell u = \sum_{\ell=0}^{\infty} \frac{\partial_t^\ell (u \circ X_\phi^t)}{\ell!}\Big|_{t=0}, \tag{2.4.11}$$

valid for analytic functions and small ϕ. We recall the following technical lemma, a slightly different version of the lemma B.3 of [?].

Lemma 2.4.1. *For $r > 0, s \in \mathbb{R}_+^n$ and $\rho > 0, \sigma \in \mathbb{R}_+^n$ such that $\rho < r$, $\sigma_i < s_i \ \forall\ i = 1, ..., n$, for $B \subseteq \mathbb{R}^n$ and denoting by $\sigma_\flat := \min_i \sigma_i$ one has*

$$\sup_{y \in B_r} \sum_{1 \leqslant i \leqslant n} \| \partial_{x_i} \phi(y, \cdot) \|_{s-\sigma} \leqslant \frac{1}{e\sigma_\flat} \| \phi \|_{r,s}, \qquad \sup_{y \in B_{r-\rho}} \max_{1 \leqslant i \leqslant n} \| \partial_{y_i} \phi(y, \cdot) \|_s \leqslant \frac{1}{\rho} \| \phi \|_{r,s},$$

Proof. For the x-derivatives we have

$$\begin{aligned} \sup_{y \in B_r} \sum_{1 \leqslant i \leqslant n} \| \partial_{x_i} \phi(y, \cdot) \|_{s-\sigma} &= \sup_{y \in B_r} \sum_{1 \leqslant i \leqslant n} \sum_k |k_i| |\phi_k(y)| e^{\sum_i |k_i|(s_i - \sigma_i)} \\ &\leqslant \sup_{y \in B_r} \sum_{1 \leqslant i \leqslant n} \sum_k |k_i| |\phi_k(y)| e^{\sum_i |k_i|(s_i - \sigma_\flat)} \\ &\leqslant \sup_{t \geqslant 0} t e^{-t\sigma_\flat} \cdot \| \phi \|_{r,s} \\ &\leqslant \frac{1}{e\sigma_\flat} \| \phi \|_{r,s} \end{aligned} \tag{2.4.12}$$

uniformly for all $y \in B_r$. For the partial derivative with respect to y_i at a point y in $B_{r-\rho}$ we write

$$\partial_{y_i} \phi(y, x) = \frac{1}{2\pi i} \int_\gamma \frac{\phi(y + \xi, x)}{\xi} d\xi \tag{2.4.13}$$

with a circle γ in the y_i plane around the origin of radius ρ. We obtain

$$\| \partial_{y_i} \phi \|_s \leqslant \frac{1}{2\pi} \int_\gamma \frac{\| \phi(y + \xi, \cdot) \|_s}{|\xi|} d|\xi| \leqslant \frac{1}{\rho} \| \phi \|_{r,s} \tag{2.4.14}$$

uniformly in $y \in B_{r-\rho}$. □

Lemma 2.4.2. *For $r, r_0, \rho > 0$; $s, s_0, \sigma \in \mathbb{R}_+^n$ such that $0 < r - \rho < r_0$, $0 < s_i - \sigma_i < (s_0)_i$ for all $i = 1, ..., n$ one has*

$$\| \{u, \phi\} \|_{r-\rho, s-\sigma} \leqslant \frac{1}{e} \left(\frac{1}{(r_0 - r + \rho)\, \sigma_\flat} + \frac{1}{(s_0 - s + \sigma)_\flat\, \rho} \right) \| u \|_{r_0, s_0} \| \phi \|_{r,s}. \tag{2.4.15}$$

Proof. The proof is a simple application of the previous lemma noting that fixing $y \in B_{r-\rho}$ one has

$$\begin{aligned}
\|\partial_{y_i} u(y,\cdot)\cdot\partial_{x_i}\phi(y,\cdot)\|_{s-\sigma} &\leqslant \sum_{i=1,\dots,n}\|\partial_{y_i}u(y,\cdot)\|_{s-\sigma}\|\partial_{x_i}\phi(y,\cdot)\|_{s-\sigma}\\
&\leqslant \max_{1\leqslant i\leqslant n}\|\partial_{y_i}u(y,\cdot)\|_{s-\sigma}\sum_{1\leqslant i\leqslant n}\|\partial_{x_i}\phi(y,\cdot)\|_{s-\sigma} \qquad (2.4.16)\\
&\leqslant \frac{1}{(r_0-r+\rho)}\|u\|_{r_0,s_0}\cdot\frac{1}{e\sigma_\flat}\|\phi\|_{r,s}.
\end{aligned}$$

Likewise for $\|\partial_{x_i}u(y,\cdot)\cdot\partial_{y_i}\phi(y,\cdot)\|_{s-\sigma}$ □

Lemma 2.4.3. *Let $r, r_0, \rho > 0$; $s, s_0, \sigma \in \mathbb{R}^n_+$ such that $0 < \rho < r \leqslant r_0 - \rho$, and $0 < \sigma_i < s_i \leqslant (s_0-\sigma)_i$ for all $i = 1, \dots, n$. Then if $\|\phi\|_{r_0,s_0} < \frac{1}{2}\rho\sigma_\flat$, one has*

$$\|u\circ X^1_\phi\|_{r-\rho,s-\sigma} \leqslant \left(1-\frac{2}{\rho\sigma_\flat}\|\phi\|_{r_0,s_0}\right)^{-1}\|u\|_{r,s}. \qquad (2.4.17)$$

Proof. Consider the expression in 2.4.11. For $\ell \geqslant 1$, let $\tilde\rho = \frac{1}{\ell}\rho$, $\tilde\sigma = \frac{1}{\ell}\sigma$. Let $\|\cdot\|_i := \|\cdot\|_{r-i\tilde\rho,s-i\tilde\sigma}$ for $1 \leqslant i \leqslant \ell$. Using the above lemmata We then have

$$\begin{aligned}
\|\mathrm{ad}^i_\phi u\|_i &\leqslant \frac{1}{e}\left(\frac{1}{(s_0-s+i\tilde\sigma)_\flat\tilde\rho}+\frac{1}{(r_0-r+i\tilde\rho)\tilde\sigma_\flat}\right)\|\phi\|_{r_0,s_0}\|\mathrm{ad}^{i-1}_\phi u\|_{i-1}.\\
&\leqslant \frac{2}{e\tilde\rho\tilde\sigma_\flat}\frac{1}{\ell+i}\|\phi\|_{r_0,s_0}\|\mathrm{ad}^{i-1}_\phi u\|_{i-1}
\end{aligned} \qquad (2.4.18)$$

Hence,

$$\|\mathrm{ad}^\ell_\phi u\|_{r-\rho,s-\sigma} \leqslant \left(\frac{2}{e\tilde\rho\tilde\sigma_\flat}\right)^\ell\frac{\ell!}{(2\ell)!}\|\phi\|^\ell_{r_0,s_0}\|u\|_{r,s}. \qquad (2.4.19)$$

Now observing that

$$\left(\frac{2}{e\tilde\rho\tilde\sigma_\flat}\right)^\ell\frac{1}{(2\ell)!} \leqslant \left(\frac{2}{\rho\sigma_\flat}\right)^\ell \qquad (2.4.20)$$

we arrive at

$$\begin{aligned}
\|u\circ X^1_\phi\|_{r-\rho,s-\sigma} &\leqslant \sum_{\ell\geqslant 0}\frac{1}{\ell!}\|\mathrm{ad}^\ell_\phi u\|_{r-\rho,s-\sigma}\\
&\leqslant \sum_{\ell\geqslant 0}\left(\frac{2}{\rho\sigma_\flat}\right)^\ell\|\phi\|^\ell_{r_0,s_0}\|u\|_{r,s} \qquad (2.4.21)\\
&= \left(1-\frac{2}{\rho\sigma_\flat}\|\phi\|_{r_0,s_0}\right)^{-1}\|u\|_{r,s}.
\end{aligned}$$

$\square$

Lemma 2.4.4. *Let $r_0, \rho > 0$; $s_0, \sigma \in \mathbb{R}^n_+$ such that $0 < \rho < r_0$, and $0 < \sigma_i < (s_0)_i$ for all $i = 1, ..., n$. Assume that*

$$\hat{\vartheta} := \frac{4e\|\phi\|_{r_0,s_0}}{\rho\,\underline{\sigma}_\flat} \leqslant 1\,. \tag{2.4.22}$$

Then for every $r' > 0$; $s' \in \mathbb{R}^n_+$ such that $\rho < r' \leqslant r_0$, $\underline{\sigma}_i < s'_i \leqslant (s_0)_i$, the time-1-flow X^1_ϕ of vector field X_ϕ define a good canonical transformation

$$X^1_\phi : B_{r'-\rho} \times \mathbb{T}^n_{r'-\underline{\sigma}} \;\to\; B_{r'-\rho/2} \times \mathbb{T}^n_{s'-\underline{\sigma}/2} \tag{2.4.23}$$

satisfying

$$|y - y'|_1 \leqslant \hat{\vartheta}\frac{\rho}{4e}\,, \qquad \max_{1\leqslant i\leqslant n} |x_i - x'_i| \leqslant \hat{\vartheta}\frac{\sigma_\flat}{4} \tag{2.4.24}$$

Moreover let $r > 0$; $s \in \mathbb{R}^n_+$ such that $r > \rho, s_i > \sigma_i$ for all i, and set for $i = 1, ..., n$

$$\bar{r} := \min\{r_0, r\}\,, \qquad \bar{s}_i := \min\{(s_0)_i, s_i\}\,.$$

Then for any $j \geqslant 0$

$$\begin{aligned} \Big\|u \circ X^1_\phi - \sum_{\ell\leqslant j} \mathrm{ad}^\ell_\phi u\Big\|_{\bar{r}-\rho,\bar{s}-\sigma} &\leqslant \sum_{\ell>j} \frac{1}{\ell!}\|\mathrm{ad}^\ell_\phi u\|_{\bar{r}-\rho,\bar{s}-\sigma} \\ &\leqslant 2(\hat{\vartheta}/2)^j \|\{u,\phi\}\|_{\bar{r}-\rho/2,\bar{s}-\sigma/2} \end{aligned} \tag{2.4.25}$$

for every function u with $\|u\|_{r,s} < \infty$.
In particular when $r \leqslant r_0, s_i \leqslant (s_0)_i$ for all $i = 1, ..., n$

$$\|u \circ X^1_\phi - u\|_{r-\rho,s-\sigma} \leqslant \sum_{\ell\geqslant 1} \frac{1}{\ell!}\|\mathrm{ad}^\ell_\phi u\|_{r-\rho,s-\sigma} \leqslant 2\hat{\vartheta}\,\|u\|_{r,s}\,, \tag{2.4.26}$$

$$\|u \circ X^1_\phi - u - \{u,\phi\}\|_{r-\rho,s-\sigma} \leqslant \hat{\vartheta}^2\,\|u\|_{r,s}\,, \tag{2.4.27}$$

Proof. We first note that by Lemma 2.4.1 (applied with $r_0 \rightsquigarrow r$, $s_0 \rightsquigarrow s$) for every $(y,x) \in B_{r_0-\rho} \times \mathbb{T}^n_{s_0-\sigma}$ we have

$$|\partial_x\phi(y,x)|_1 \leqslant \frac{1}{e\sigma_\flat}\|\phi\|_{r_0,s_0} = \frac{\hat{\vartheta}\rho}{4e} \leqslant \frac{\rho}{4e}\,, \qquad \max_{1\leqslant i\leqslant n} |\partial_{y_i}\phi(y,x)| \leqslant \frac{1}{\rho}\|\phi\|_{r_0,s_0} = \frac{\hat{\vartheta}\sigma_\flat}{4} \leqslant \frac{\sigma_\flat}{4}\,.$$

Then (2.4.23) holds.
For $h \geqslant 1$, set for brevity

$$\|\cdot\|_i := \|\cdot\|_{\bar{r}-\frac{\rho}{2}-i\tilde{\rho},\bar{s}-\frac{\sigma}{2}-i\tilde{\sigma}}\,, \qquad 0 \leqslant i \leqslant h\,, \qquad \tilde{\rho} := \frac{\rho}{2h}\,, \quad \tilde{\sigma} := \frac{\sigma}{2h}\,.$$

We get

$$
\begin{aligned}
&\|\mathrm{ad}^i_\phi\{u,\phi\}\|_i \\
&\overset{(2.4.15)}{\leqslant} \frac{1}{e}\left(\frac{1}{\tilde\rho(s_0-\bar s+i\tilde\sigma+\sigma/2)_\flat}+\frac{1}{\tilde\sigma_\flat(r_0-\bar r+i\tilde\rho+\rho/2)}\right)\|\phi\|_{r_0,s_0}\|\mathrm{ad}^{i-1}_\phi\{u,\phi\}\|_{i-1} \\
&\leqslant \frac{8h^2}{e\rho\,\sigma_\flat}\frac{1}{h+i}\|\phi\|_{r_0,s_0}\|\mathrm{ad}^{i-1}_\phi\{u,\phi\}\|_{i-1}\,,
\end{aligned}
$$

and, iterating,

$$
\|\mathrm{ad}^h_\phi\{u,\phi\}\|_h \leqslant \frac{8h^2}{e\rho\sigma_\flat}\frac{h!}{(2h)!}\|\phi\|_{r_0,s_0}\|\{u,\phi\}\|_{r-\rho/2,s-\sigma/2} \leqslant h!(\hat\vartheta/2)^h\|\{u,\phi\}\|_{r-\rho/2,s-\sigma/2}
$$

by Stirling's formula. Then

$$
\sum_{h\geqslant j}\frac{1}{(h+1)!}\|\mathrm{ad}^{h+1}_\phi u\|_{\bar r-\rho,\bar s-\sigma} \leqslant \sum_{h\geqslant j}\frac{1}{h+1}(\hat\vartheta/2)^h\|\{u,\phi\}\|_{r-\rho/2,s-\sigma/2}
$$

proving (2.4.25) in view of (2.4.22).
Finally (2.4.26) and (2.4.27) follows by (2.4.25) and since $\|\{u,\phi\}\|_{\bar r-\rho/2,\bar s-\sigma/2} \leqslant 2e^{-1}\hat\vartheta\|u\|_{r,s}$ by (2.4.15). □

Given $K \geqslant 2$ and a lattice Λ, define

$$
f^\flat := \pi_\Lambda f + T_K^\perp\pi_\Lambda^\perp f; \qquad f^K := f - f^\flat = T_K\,\pi_\Lambda^\perp f\ ,
$$

so that we have the decomposition (valid for any f):

$$
f = f^\flat + f^K\,, \qquad f^\flat := P_\Lambda f + T_K^\perp\pi_\Lambda^\perp f\,, \qquad f^K := T_K\,\pi_\Lambda^\perp f\,. \tag{2.4.28}
$$

Lemma 2.4.5. *Let $r>0$; $s\in\mathbb{R}^n_+$ and $\rho>0$; $\sigma\in\mathbb{R}^n_+$ such that $\rho<r$, $\sigma_i<s_i$ $\forall\ i=1,\dots,n$. Consider a real–analytic Hamiltonian*

$$
H = H(y,x) = h(y) + f(y,x) \qquad \textit{analytic on } B_r\times\mathbb{T}^n_s\,. \tag{2.4.29}
$$

Suppose that B_r is (α,K) non–resonant modulo Λ for h (with $K\geqslant 2$). Assume that

$$
\check\vartheta := \frac{4e}{\alpha\,\rho\sigma_\flat}\|f^K\|_{r,s} \leqslant 1\,. \tag{2.4.30}
$$

Then there exists a real–analytic symplectic change of coordinates

$$
\Psi := X^1_\phi\ :\ B_{r_+}\times\mathbb{T}^n_{s_+}\ni(y',x')\ \to\ (y,x)\in B_r\times\mathbb{T}^n_s\,,\quad r_+ := r-\rho\,,\quad (s_+)_i := s_i-\sigma_i\,,
$$

generated by a function $\phi = \phi^K = T_K\, \pi_\Lambda^\perp \phi$ *with*

$$\|\phi\|_{r,s} \leqslant \frac{1}{\alpha} \|f^K\|_{r,s}\,, \tag{2.4.31}$$

satisfying

$$|y-y'|_1 \leqslant \check\vartheta \frac{\rho}{4e}\,, \qquad \max_{1\leqslant i\leqslant n} |x_i - x_i'| \leqslant \check\vartheta \frac{\sigma_\flat}{4}\,, \tag{2.4.32}$$

such that

$$H\circ\Psi = h(y') + f_+(y',x')\,, \qquad f_+ := f^\flat + f_\star \tag{2.4.33}$$

with

$$\|f_\star\|_{r_+,s_+} \leqslant\ 4\check\vartheta\, \|f\|_{r,s}\,. \tag{2.4.34}$$

Remark 2.4.2. Notice that, by (2.4.28) and (2.4.34), one has

$$f_+^K = f_\star^K\,, \quad \|f_+\|_{r_+,s_+} \leqslant \|f_\star\|_{r_+,s_+} + \|f\|_{r,s} \leqslant (1+4\check\vartheta)\|f\|_{r,s}\,. \tag{2.4.35}$$

Moreover notice also that

$$f_+^\flat - f^\flat \overset{(2.4.33)}{=} f_\star^\flat \quad \Longrightarrow \quad \|f_+^\flat - f^\flat\|_{r_+,s_+} \leqslant \|f_\star\|_{r_+,s_+} \overset{(2.4.34)}{\leqslant} 4\check\vartheta\|f\|_{r,s}\,. \tag{2.4.36}$$

Proof. Let us define

$$\phi = \phi(y,x) := \sum_{|m|\leqslant K, m\notin\Lambda} \frac{f_m(y)}{\mathrm{i}h'(y)\cdot m} e^{\mathrm{i}m\cdot x}\,,$$

and note that ϕ solves the homological equation

$$\{h,\phi\} + f^K = 0\,. \tag{2.4.37}$$

Since B_r is (α,K) non–resonant modulo Λ the estimate (2.4.31) holds. We now use Lemma 2.4.4 with parameters $r_0 \rightsquigarrow r$, $s_0 \rightsquigarrow s$. With these choices it is $\hat\vartheta = \check\vartheta$, and, by (2.4.30) $\check\vartheta \leqslant 1$. Thus, (2.4.22) holds and Lemma 2.4.4 applies. (2.4.32) follows by (2.4.24). We have

$$H\circ\Psi = h + f^\flat + f_\star$$

with

$$f_\star = (h\circ\Psi - h - \{h,\phi\}) + (f\circ\Psi - f)\,.$$

Since

$$h\circ\Psi - h - \{h,\phi\} = \sum_{\ell\geqslant 2} \frac{1}{\ell!}\mathrm{ad}_\phi^\ell h = \sum_{\ell\geqslant 1} \frac{1}{(\ell+1)!}\mathrm{ad}_\phi^\ell\{h,\phi\} \overset{(2.4.37)}{=} -\sum_{\ell\geqslant 1}\frac{1}{(\ell+1)!}\mathrm{ad}_\phi^\ell f^K\,,$$

we have

$$\|h\circ\Psi - h - \{h,\phi\}\|_{r_+,s_+} \leqslant \sum_{\ell\geqslant 1}\frac{1}{\ell!}\|\mathrm{ad}_\phi^\ell f^K\|_{r_+,s_+} \overset{(2.4.26)}{\leqslant} 2\check\vartheta\,\|f^K\|_{r,s} \leqslant 2\check\vartheta\,\|f\|_{r,s}\,.$$

Finally, applying again Lemma 2.4.4 with $u = f$, by (2.4.26), we get $\|f\circ\Psi - f\|_{r_+,s_+} \leqslant 2\check\vartheta\,\|f\|_{r,s}$, proving (2.4.34) and concluding the proof of Lemma 2.4.5. □

As a preliminary step we apply Lemma 2.4.5 to the Hamiltonian $H = h + f$ in (2.4.29) with $\rho = r/4$ and $\sigma = s/2K$. By (2.1.9), (2.1.11), (2.4.28) and (2.4.3) hypothesis (2.4.30) holds, namely

$$\vartheta_{-1} := \frac{2^5 eK}{\alpha\, r s_\flat}\|f^K\|_{r,s} \leqslant 1\,. \tag{2.4.38}$$

Then there exists a real–analytic symplectic change of coordinates

$$\Psi_{-1} : B_{r_0}\times\mathbb{T}^n_{s_0} \ni (y^{(0)},x^{(0)}) \;\to\; (y,x)\in B_r\times\mathbb{T}^n_s\,,\quad r_0 := \frac{3}{4}r\,,\quad (s_0)_i := \left(1-\frac{1}{2K}\right)s_i\,,$$

for all $i = 1,\dots,n$, satisfying

$$|y-y^{(0)}|_1 \leqslant \vartheta_{-1}\frac{r}{16e}\,,\qquad \max_{1\leqslant i\leqslant n}|x_i - x_i^{(0)}| \leqslant \vartheta_{-1}\frac{s_\flat}{8K}\,, \tag{2.4.39}$$

such that

$$H\circ\Psi_{-1} =: H_0 = h(y^{(0)}) + f_0(y^{(0)},x^{(0)})\,,\quad f_0 = f^\flat + f_*\,,\quad f^\flat := P_\Lambda f + T_K^\perp \pi_\Lambda^\perp f\,, \tag{2.4.40}$$

with

$$\|f_*\|_{r_0,s_0} \leqslant 4\vartheta_{-1}\|f\|_{r,s}\,. \tag{2.4.41}$$

Recalling (2.4.28) and (2.4.40) we get

$$f_0^K = f_*^K$$

and, by (2.4.41) and (2.4.38),

$$\|f_0^K\|_{r_0,s_0} \leqslant 4\vartheta_{-1}\|f\|_{r,s} \leqslant \frac{2^7 eK}{\alpha\, r s_\flat}\|f\|_{r,s}^2\,. \tag{2.4.42}$$

Then, setting

$$\vartheta_0 := \gamma\,\|f_0^K\|_{r_0,s_0} \qquad\text{with}\qquad \gamma := \frac{2^5 e\, K^3}{\alpha\, r s_\flat}\,, \tag{2.4.43}$$

we have

$$\vartheta_0 \leqslant \left(\frac{2^6 e K^2}{\alpha\, r s_\flat} \| f \|_{r,s}\right)^2 \overset{(2.4.3)}{\leqslant} (\vartheta_\star/8)^2 \leqslant \frac{1}{2^6}\ . \tag{2.4.44}$$

Finally, since $f_0^\flat - f^\flat = f_\star^\flat$ by (2.4.36) we get

$$\| f_0^\flat - f^\flat \|_{r_0,s_0} \leqslant\ 4\vartheta_{-1} \| f \|_{r,s} \overset{(2.4.38)}{\leqslant} \frac{2^7 e K}{\alpha\, r s_\flat} \| f \|_{r,s}^2 \overset{(2.4.3)}{\leqslant} \frac{1}{4K} \vartheta_\star \| f \|_{r,s}\ . \tag{2.4.45}$$

The idea is to construct Ψ by applying K times Lemma 2.4.5.
Let

$$\rho := \frac{r}{4K}\,, \qquad \sigma_i := \frac{s_i}{2K^2}\,,$$

$$r^{(j)} := \frac{3}{4} r - j\rho\,, \qquad s^{(j)} := \left(1 - \frac{1}{2K}\right) s - j\sigma\,, \qquad \| \cdot \|_j := \| \cdot \|_{r^{(j)},s^{(j)}}\,. \tag{2.4.46}$$

Fix $1 \leqslant h \leqslant K$ and make the following **inductive assumptions**:
Assume that there exist, for $1 \leqslant j \leqslant h$, real–analytic symplectic transformations

$$\Psi_{j-1} := X^1_{\phi_{j-1}} \ :\ B_{r^{(j)}} \times \mathbb{T}^n_{s^{(j)}} \ni (y^{(j)}, x^{(j)}) \ \to\ (y^{(j-1)}, x^{(j-1)}) \in B_{r^{(j-1)}} \times \mathbb{T}^n_{s^{(j-1)}}\,,$$

generated by a function $\phi_{j-1} = \phi^K_{j-1}$ with

$$\| \phi_{j-1} \|_{j-1} \leqslant \frac{1}{\alpha} \| f^K_{j-1} \|_{j-1}\,, \tag{2.4.47}$$

satisfying

$$|y^{(j-1)} - y^{(j)}|_1 \leqslant \vartheta_{j-1} \frac{r}{16eK}\,, \qquad \max_{1 \leqslant \ell \leqslant n} |x^{(j-1)}_\ell - x^{(j)}_\ell| \leqslant \vartheta_{j-1} \frac{s_\flat}{8K^2}\,, \tag{2.4.48}$$

such that

$$H_i := H_{j-1} \circ \Psi_{j-1} =: h + f_j = h + f^K_j + f^\flat_j \tag{2.4.49}$$

satisfies, for $1 \leqslant j \leqslant h$, the estimates

$$\vartheta_j \leqslant \left(\frac{2^8 K^2 \| f \|_{r,s}}{\alpha r s_\flat}\right)^{j+1} \overset{(2.4.3)}{=} \left(\frac{\vartheta_\star}{8}\right)^{j+1}\,, \qquad \| f^\flat_j - f^\flat_{j-1} \|_j \leqslant \frac{1}{\gamma} \left(\frac{\vartheta_\star}{8}\right)^{j+1}\,, \tag{2.4.50}$$

where

$$\vartheta_j := \gamma |f^K_j|_j\ . \tag{2.4.51}$$

Let us first show that the inductive hypothesis is true for $h = 1$ (which implies $j = 1$). Indeed by (2.4.44) we see that we can apply Lemma 2.4.5 with $f \rightsquigarrow f_0^K$ and $\check{\vartheta} \rightsquigarrow \vartheta_0 = \gamma \| f_0^K \|$. Thus, we obtain the existence of $\Psi_0 = X^1_{\phi_0}$, generated by a function $\phi_0 = \phi_0^K$ with

$$\| \phi_0 \|_{r_0, s_0} \leqslant \frac{1}{\alpha} \| f_0^K \|_{r_0, s_0} \overset{(2.4.42)}{\leqslant} \frac{2^7 eK}{\alpha^2 r s_\flat} \| f \|^2_{r,s} , \tag{2.4.52}$$

satisfying (2.4.47) and[11] (2.4.48), so that $(h + f_0^K) \circ \Psi_0 =: h + \tilde{f}_1$ and, by (2.4.33) and (2.4.34),

$$\| \tilde{f}_1 \|_1 \leqslant 4\vartheta_0 \| f_0^K \|_0 \overset{(2.4.44)}{\leqslant} \frac{1}{4} \| f_0^K \|_0 \overset{(2.4.42)}{\leqslant} \frac{2^5 eK}{\alpha r s_\flat} \| f \|^2_{r,s} . \tag{2.4.53}$$

We have that $f_1 = \tilde{f}_1 + f_0^\flat \circ \Psi_0$. Then[12]

$$f_1^K = \tilde{f}_1^K + (f_0^\flat \circ \Psi_0 - f_0^\flat)^K , \qquad f_1^\flat - f_0^\flat = \tilde{f}_1^\flat + (f_0^\flat \circ \Psi_0 - f_0^\flat)^\flat . \tag{2.4.54}$$

Write

$$f_0^\flat \circ \Psi_0 - f_0^\flat = (f_0^\flat - f^\flat) \circ \Psi_0 - (f_0^\flat - f^\flat) + (f^\flat \circ \Psi_0 - f^\flat - \{f^\flat, \phi_0\}) + \{f^\flat, \phi_0\} .$$

By (2.4.26) (with $u \rightsquigarrow f_0^\flat - f^\flat$, $r \rightsquigarrow r_0$, $s \rightsquigarrow s_0$) we have

$$\| (f_0^\flat - f^\flat) \circ \Psi_0 - (f_0^\flat - f^\flat) \|_1 \leqslant 2\vartheta_0 \| f_0^\flat - f^\flat \|_0 \leqslant \frac{2^4 eK}{\alpha r s_\flat} \| f \|^2_{r,s}$$

by (2.4.44) and (2.4.45). By (2.4.25) with $u \rightsquigarrow f^\flat$, $\phi \rightsquigarrow \phi_0$, $j \rightsquigarrow 1$, $\bar{r} \rightsquigarrow r_0$, $\bar{s} \rightsquigarrow s_0$,

$$\| f^\flat \circ \Psi_0 - f^\flat - \{f^\flat, \phi_0\} \|_1 \leqslant 2\vartheta_0 \| \{f^\flat, \phi_0\} \|_{r_0 - \rho/2, s_0 - \sigma/2} \leqslant \frac{2^9 K^3}{\alpha^2 r^2 s_\flat^2} \| f \|^3_{r,s} \overset{(2.4.3)}{\leqslant} \frac{K}{4\alpha r s_\flat} \| f \|^2_{r,s} ,$$

by (2.4.44), (2.4.52) and (2.4.15) (with $f \rightsquigarrow \phi_0$, $g \rightsquigarrow f^\flat$). Analaogously by (2.4.15) we get

$$\| \{f^\flat, \phi_0\} \|_1 \leqslant \frac{2^4 K^2}{e r s_\flat} \| \phi_0 \|_0 \| f \|_{r,s} \overset{(2.4.52)}{\leqslant} \frac{2^{11} K^3}{\alpha^2 r^2 s_\flat^2} \| f \|^3_{r,s} \overset{(2.4.3)}{\leqslant} \frac{K}{\alpha r s_\flat} \| f \|^2_{r,s} .$$

Summarizing:

$$\| f_0^\flat \circ \Psi_0 - f_0^\flat \|_1 \leqslant \frac{2^6 K}{\alpha r s_\flat} \| f \|^2_{r,s} .$$

[11] Note also that $(f_0^K)^\flat = 0$

[12] Note that $(f_0^\flat)^K = 0$ and $(f_0^\flat)^\flat = f_0^\flat$.

Then, by (2.4.53) and (2.4.54) we get

$$\|f_1^K\|_1\,,\ \|f_1^\flat - f_0^\flat\|_1 \ \leqslant\ \frac{2^7 K}{\alpha r s_\flat}\|f\|_{r,s}^2 \tag{2.4.55}$$

checking (2.4.50) in the case $h = j = 1$.

Now take $2 \leqslant h \leqslant K$ and assume that the inductive hypothesis holds true for $1 \leqslant j \leqslant h$ and let us prove that it holds also for $j = h+1$. By (2.4.50) and (2.4.3) we can apply Lemma 2.4.5 with $f \rightsquigarrow f_h^K$ and $\check{\vartheta} \rightsquigarrow \vartheta_h$. Thus, we obtain the existence of $\Psi_h = X^1_{\phi_h}$, generated by a function $\phi_h = \phi_h^K$ with

$$\|\phi_h\|_h \overset{(2.4.47)}{\leqslant} \frac{1}{\alpha}\|f_h^K\|_h \overset{(2.4.51)}{=} \frac{\vartheta_h}{\alpha\gamma}\,, \tag{2.4.56}$$

so that $(h + f_h^K)\circ\Psi_h =: h + \tilde f_{h+1}$ and, by (2.4.33) and (2.4.34),

$$\|\tilde f_{h+1}\|_{h+1} \leqslant 4\vartheta_h\|f_h^K\|_h \overset{(2.4.51)}{=} \frac{4}{\gamma}\vartheta_h^2 \overset{(2.4.50)}{\leqslant} \frac{4}{\gamma}(\vartheta_\star/8)^{2h+2} \overset{(2.4.3)}{\leqslant} \frac{1}{2^{3h-2}\gamma}(\vartheta_\star/8)^{h+2} \leqslant \frac{1}{2^4\gamma}(\vartheta_\star/8)^{h+2}\,, \tag{2.4.57}$$

since $h \geqslant 2$. We have that $f_{h+1} = \tilde f_{h+1} + f_h^\flat \circ \Psi_h$. Then[13]

$$f_{h+1}^K = \tilde f_{h+1}^K + (f_h^\flat\circ\Psi_h - f_h^\flat)^K\,, \qquad f_{h+1}^\flat - f_h^\flat = \tilde f_{h+1}^\flat + (f_h^\flat\circ\Psi_h - f_h^\flat)^\flat\,. \tag{2.4.58}$$

Writing

$$f_h^\flat = f^\flat + (f_0^\flat - f^\flat) + \sum_{m=1}^{h} f_m^\flat - f_{m-1}^\flat$$

we have

$$\begin{aligned} f_h^\flat\circ\Psi_h - f_h^\flat \ &=\ \{f^\flat,\phi_h\} \\ &\quad + f^\flat\circ\Psi_h - f^\flat - \{f^\flat,\phi_h\} \\ &\quad + (f_0^\flat - f^\flat)\circ\Psi_h - (f_0^\flat - f^\flat) \\ &\quad + \sum_{j=1}^{h}\Big((f_j^\flat - f_{j-1}^\flat)\circ\Psi_h - (f_j^\flat - f_{j-1}^\flat)\Big) \end{aligned} \tag{2.4.59}$$

where $\Psi_h = X^1_{\phi_h}$. By (2.4.15) with $f \rightsquigarrow \phi_h$, $g \rightsquigarrow f^\flat$, $r_0 \rightsquigarrow r^{(j)}$, $s_0 \rightsquigarrow s^{(j)}$, we get, by (2.4.47) and (2.4.51),

$$\|\{f^\flat,\phi_h\}\|_{h+1} \leqslant \frac{2^4K^2}{ers_\flat}\|\phi_h\|_h\|f\|_{r,s} \leqslant \frac{2^4K^2\vartheta_h}{e\alpha r s_\flat\gamma}\|f\|_{r,s} \overset{(2.4.3)}{=} \frac{1}{e2^4\gamma}(\vartheta_\star/8)\vartheta_h \overset{(2.4.50)}{\leqslant} \frac{1}{e2^4\gamma}(\vartheta_\star/8)^{h+2}\,.$$

[13] Note that $(f_h^\flat)^K = 0$ and $(f_h^\flat)^\flat = f_h^\flat$.

By (2.4.25) with $u \rightsquigarrow f^\flat$, $\phi \rightsquigarrow \phi_h$, $h \rightsquigarrow 1$, $\bar r \rightsquigarrow r^{(h)}$, $\bar s \rightsquigarrow s^{(h)}$, reasoning as above we get

$$\|f^\flat \circ \Psi_j - f^\flat - \{f^\flat, \phi_h\}\|_{h+1} \leqslant \vartheta_h \|\{f^\flat, \phi_h\}\|_{r^{(h)}-\rho/2, s^{(h)}-\sigma/2} \leqslant \frac{\vartheta_h}{4e\gamma}(\vartheta_\star/8)^{h+2} \leqslant \frac{1}{2^6 e\gamma}(\vartheta_\star/8)^{h+2}$$

by (2.4.50) and (2.4.3). By (2.4.26) (with $u \rightsquigarrow f_0^\flat - f^\flat$, $r \rightsquigarrow r^{(h)}$, $s \rightsquigarrow r^{(h)}$) we have

$$\|(f_0^\flat - f^\flat) \circ \Psi_h - (f_0^\flat - f^\flat)\|_{h+1} \leqslant 2\vartheta_h \|f_0^\flat - f^\flat\|_h \leqslant \frac{2^8 eK}{\alpha r s_\flat}\|f\|^2_{r,s}\vartheta_h \leqslant \frac{1}{4\gamma}(\vartheta_\star/8)^{h+2}$$

by (2.4.45), (2.4.50), (2.4.43) and (2.4.3). Analogously, for $1 \leqslant j \leqslant h$, by (2.4.26) (now with $u \rightsquigarrow f_j^\flat - f_{j-1}^\flat$)

$$\|(f_j^\flat - f_{j-1}^\flat) \circ \Psi_h - (f_j^\flat - f_{j-1}^\flat)\|_{h+1} \leqslant 2\vartheta_h \|f_j^\flat - f_{j-1}^\flat\|_h \leqslant \frac{2}{\gamma}(\vartheta_\star/8)^{h+j+2}$$

by (2.4.50). Then by (2.4.3)

$$\Big\|\sum_{j=1}^{h}\Big((f_j^\flat - f_{j-1}^\flat) \circ \Psi_h - (f_j^\flat - f_{j-1}^\flat)\Big)\Big\|_{h+1} \leqslant \frac{2}{7\gamma}(\vartheta_\star/8)^{h+2}\,.$$

Whence:

$$\|f_h^\flat \circ \Psi_h - f_h^\flat\|_{h+1} \leqslant \frac{4}{7\gamma}(\vartheta_\star/8)^{h+2}\,.$$

Then by (2.4.57) we get

$$\|\tilde f_{h+1}\|_{h+1} + \|f_h^\flat \circ \Psi_h - f_h^\flat\|_{h+1} \leqslant \frac{1}{\gamma}(\vartheta_\star/8)^{h+2}\,.$$

By (2.4.58) we get (2.4.50) with $j = h+1$. This completes the proof of the induction. Now, we can conclude the proof of Proposition 2.4.1. Set

$$\Psi := \Psi_{-1} \circ \Psi_0 \circ \cdots \circ \Psi_{K-1}\ .$$

Notice that, by (2.4.46), $r^{(K)} = r/2 = r_\star$ and $s^{(K)} = s(1-1/K) = s_\star$. [14]

By the induction, it is

$$H \circ \Psi = H_{K-1} \circ \Psi_{K-1} \overset{(2.4.49)}{=} h + f_K =: h + f^\flat + f_\star\,, \tag{2.4.60}$$

[14]Given a vector $\xi \in \mathbb{R}^n$, with the notation $a\xi$ with $a \in \mathbb{R}$ we intend the vector with component $(a\xi)_i = a\xi_i$ for all $i = 1, ..., n$.

with $f^\flat = \pi_\Lambda f + T_K^\perp \pi_\Lambda^\perp f$ (recall (2.4.6)). Note that by (2.4.50) and (2.4.44)

$$\sum_{j=1}^{K} \vartheta_{j-1} \leqslant \sum_{j=1}^{K} (\vartheta_*/8)^j \leqslant \vartheta_*/7 \,. \tag{2.4.61}$$

Since $(y', x') = (y^{(K)}, x^{(K)})$ by (2.4.39), (2.4.48) and triangular inequality we get

$$\begin{aligned} |y' - y|_1 &\leqslant |y - y^{(0)}|_1 + \sum_{j=1}^{K} |y^{(j)} - y^{(j-1)}|_1 \leqslant \frac{r\vartheta_{-1}}{16e} + \frac{r}{16eK} \sum_{j=1}^{K} \vartheta_{j-1} \\ &\overset{(2.4.61)}{\leqslant} \frac{r}{16e}\left(\vartheta_{-1} + \frac{\vartheta_*}{7K}\right) \overset{(2.4.38)}{\leqslant} \frac{r}{16e}\left(\frac{\vartheta_*}{8K} + \frac{\vartheta_*}{7K}\right) \,, \end{aligned}$$

then (2.4.5) follows (the estimate on the angle being analogous).
Since $T_K P_\Lambda^\perp f^\flat = (f^\flat)^K = 0$ (for any f, recall (2.4.28)) we have

$$\| T_K P_\Lambda^\perp f_* \|_{r_*, s_*} = \| f_K^K \|_K \overset{(2.4.51)}{=} \gamma^{-1} \vartheta_K \overset{(2.4.50)}{\leqslant} \gamma^{-1} (\vartheta_*/8)^{K+1} = (\vartheta_*/8)^K \frac{8}{eK} \| f \|_{r,s} \tag{2.4.62}$$

proving the second estimates in (2.4.7).
Finally, (using that $K \geqslant 2$ and that $\vartheta_* \leqslant 1$)

$$\begin{aligned} \| f_* \|_{r_*, s_*} &\overset{(2.4.60)}{=} \| f_K - f^\flat \|_K \overset{(2.4.28)}{=} \| f_K^K + f_K^\flat - f^\flat \|_K \\ &\leqslant \| f_K^K \|_K + \| f_0^\flat - f^\flat \|_0 + \sum_{j=1}^{K} \| f_j^\flat - f_{j-1}^\flat \|_j \\ &\overset{(2.4.45),(2.4.50)}{\leqslant} \| f_K^K \|_K + \frac{1}{4K} \vartheta_* \| f \|_{r,s} + \frac{1}{\gamma} \sum_{j=1}^{K} (\vartheta_*/8)^{j+1} \\ &\overset{(2.4.62),(2.4.61)}{\leqslant} (\vartheta_*/8)^K \frac{8}{eK} \| f \|_{r,s} + \frac{1}{4K} \vartheta_* \| f \|_{r,s} + \frac{\vartheta_*^2}{56\gamma} \leqslant \frac{1}{K} \vartheta_* \| f \|_{r,s} \,, \end{aligned} \tag{2.4.63}$$

which proves also the first estimate in (2.4.7). □

2.5 Averaging

Regarding geometry of resonances there is no difference between this case and the prevoius case in chapter 1. In fact, the analiticity stripes of the action-angle variables do not influence in any way the geometry of resonances.

So we recall the first covering lemma proved in section 5 of [1] and written in the first chapter of this work. In order to face the notation and the definition of resonances we are going to write down once again only the first covering lemma.

Lemma 2.5.1 (First covering lemma)**.** *Let h be KAM non-degenerate and let ω denote its gradient. Fix $\mathtt{K} \geqslant 6\mathtt{K}_0 \geqslant 12$ and $\alpha > 0$. Then, the domain B can be covered by three sets $\mathcal{R}^i \subseteq B$,*

$$B = \mathcal{R}^0 \cup \mathcal{R}^1 \cup \mathcal{R}^2 \tag{2.5.1}$$

so that the following holds.
a) $\mathcal{R}^0$ is $(\frac{\alpha}{2\mathtt{C}}, \mathtt{K}_0)$ completely non-resonant (i.e. non-resonant modulus $\{0\}$), namely,

$$y \in \mathcal{R}^0 \Longrightarrow |\omega(y) \cdot k| \geqslant \frac{\alpha}{2\mathtt{C}},\ \ \forall\ 0 < |k|_1 \leqslant \mathtt{K}_0. \tag{2.5.2}$$

where $\mathtt{C} = \mathtt{C}(n, L, \gamma) = \frac{12c_1nL}{\gamma} \geqslant 1$ is a constant.
b) $\mathcal{R}^1 = \bigcup_{k\in\mathcal{G}^n_{1,\mathtt{K}_0}} \mathcal{R}^{1,k}$, where, for each $k \in \mathcal{G}^n_{1,\mathtt{K}_0}$, $\mathcal{R}^{1,k}$ is a closed neighbourhood of a simple resonance $\{y \in B : \omega(y) \cdot k = 0\}$, which is $(3\alpha\mathtt{K}^{(n+3)}/|k|, \mathtt{K})$ non-resonant modulo $\mathbb{Z}k$, namely

$$y \in \mathcal{R}^{1,k} \rightarrow |\omega(y) \cdot k| \leqslant \frac{\alpha}{\mathtt{C}};\ \ |\omega(y) \cdot \ell| \geqslant \frac{3\alpha\mathtt{K}^{(n+3)}}{|k|},\ \ \forall\ \ell \in \mathbb{Z}^n,\ \ \ell \notin \mathbb{Z}k,\ \ |\ell|_1 \leqslant \mathtt{K}. \tag{2.5.3}$$

c) $\mathcal{R}^2$ contains all the resonance of order two or more and has Lebesgue measure small with α^2: more precisely, there exists a costant $c > 0$ depending only on n such that

$$meas(\mathcal{R}^2) \leqslant c(n)\alpha^2\mathtt{K}^{2n+2} \tag{2.5.4}$$

This lemma is equivalently written in the first part of this thesis.

Remark 2.5.1. *If a set $B \subseteq \mathbb{R}^n$ is (α, K) non–resonant modulo Λ for h, then the complex domain B_r is $(\alpha - L|r|K, K)$ non–resonant modulo Λ,* [15] *provided $L|r|K < \alpha$, where L is the Lipschitz constant of ω on the complex domain B_r.*
Indeed, if $y \in B_r$ there exists $y_0 \in B$ such that $|y_i - (y_0)_i| < r_i\ \forall i = 1, ..., n \Rightarrow |y - y_0| < |r|$ and $|\omega(y_0) \cdot k| \geqslant \alpha$ for all $k \in \mathbb{Z}^n \setminus \Lambda$, $|k|_1 \leqslant K$. Thus, for such k's, one has

$$|\omega(y) \cdot k| = |\omega(y_0) \cdot k - (\omega(y_0) - \omega(y)) \cdot k| \geqslant |\omega(y_0) \cdot k| - L|r|K \geqslant \alpha - L|r|K\ .$$

[15] $|r| := \left(\sum_{i=1}^n r_i^2\right)^{1/2}$ if r is a vector, and it is the standard modulus if $r > 0$ is a number.

In our case, i.e. $r > 0$ is a number, obviously B_r is $(\alpha - LrK, K)$ non–resonant modulo Λ. So $\mathcal{R}^{1,k}_{\tilde{r}_k}$ is

$$\left(\frac{3\alpha\mathtt{K}^{(n+3)}}{|k|} - L\alpha\tilde{r}_k\right) \text{ non-resonant mod } \mathbb{Z}k \Rightarrow \left(\frac{\alpha\mathtt{K}^{(n+3)}}{|k|}\right) \text{ non-resonant mod } \mathbb{Z}k \tag{2.5.5}$$

As in the previous chapter, we want to perform averaging in which the actions remains more stable, so we need to introduce a few parameters (Fourier cut–offs, a small divisor threshold, radii of analyticity) and some notation. Let $\mathtt{K}, \mathtt{K}_0, \nu$ and α such that

$$\mathtt{K} \geqslant 6\frac{s_\sharp}{s_\flat}\mathtt{K}_0 \geqslant 6\mathtt{K}_0 \geqslant 12, \quad \nu \geqslant \frac{9}{2}n + 2 \quad \alpha := \sqrt{\varepsilon}\mathtt{K}^\nu. \tag{2.5.6}$$

For a generic $k \in \mathcal{G}^n_{1,\mathtt{K}_0}$ we define

$$\begin{aligned}
&r_0 := \frac{\alpha}{4L\mathtt{C}\mathtt{K}_0} = \sqrt{\varepsilon}\frac{\mathtt{K}^\nu}{4L\mathtt{C}\mathtt{K}_0}; \quad r_0' := \frac{r_0}{2}; \qquad r_k := \frac{\alpha}{L|k|} = \sqrt{\varepsilon}\frac{\mathtt{K}^\nu}{L|k|}; \quad r_k' := \frac{r_k}{2}\\
&s_0 := s(1 - \frac{1}{\mathtt{K}_0}) \in \mathbb{R}^n_+; \quad s_0' := s_0(1 - \frac{1}{\mathtt{K}_0}); \qquad s_* := s(1 - \frac{1}{\mathtt{K}}) \in \mathbb{R}^n_+; \quad s_*' := s_*(1 - \frac{1}{\mathtt{K}})\\
&\bar{\vartheta} := 2^{14}n^{2n}\frac{L}{s_\flat^{2n+1}\mathtt{K}^{2\nu-2n-3}}; \quad \vartheta := 2^{2(n-2)k}\bar{\vartheta}; \quad s_k' := |k|_1 s_*'
\end{aligned} \tag{2.5.7}$$

Remark 2.5.2. Observe that $r_0 \leqslant r_k \leqslant \sqrt{\varepsilon}\mathtt{K}^\nu/L$, and we have to impose the condition $r_k \leqslant r$ (where r is the analyticity radius of the unperturbed Hamiltonian, which here is a free vector of parameters, and r indicates the smallest component). So we have to verify the smallness condition

$$\varepsilon \leqslant \frac{r^2L^2}{\mathtt{K}^{2\nu}}$$

but one can take $r = \frac{\mathtt{K}^\nu}{L}$ so that the condition becomes simply $\varepsilon \leqslant 1$.

Remark 2.5.3. With the choice of parameters in 2.5.7 one can notice that choose a "vector" of different analiticity stripes for the actions is pointless. In fact the value of r_0 and r_k depends only on α and $\mathtt{K}$ that are numbers that do not depend on the original analiticity stripe r. If one wants to consider an initial "vector" of different analiticity stripes $r = (r_1, ..., r_n)$ for the actions the only consideration that changes is that one has to ask the condition $r_0 \leqslant r_k \leqslant r_\flat$ instead of $r_0 \leqslant r_k \leqslant r$.

Theorem 2.5.1 (Averaging theorem). *Let H_ε as in 2.1.1 with $\|f\|_{B,r,s} = 1$ and let 2.5.6,2.5.7 holds. There exists a costant $b_0 = b_0(n,s) > 1$ such that if $\mathtt{K}_0 \geqslant b_0$, the following holds.*
a) There exists a symplectic change of variables

$$\psi_0 : \mathcal{R}^0_{r_0'} \times \mathbb{T}^n_{s_0'} \longmapsto \mathcal{R}^0_{r_0} \times \mathbb{T}^n_{s_0} \tag{2.5.8}$$

such that

$$H_\varepsilon \circ \psi_0 := h(y) + \varepsilon g^0(y) + \varepsilon f^0(y,x), \qquad \langle f^0 \rangle = 0 \tag{2.5.9}$$

with g_0 and f_0 real analytic on $\mathcal{R}^0_{r_0'} \times \mathbb{T}^n_{s_0'}$ and satisfies

$$|g^0 - \langle f \rangle|_{\mathcal{R}^0, r_0'} \leqslant \bar{\vartheta}, \qquad \| f^0 \|_{\mathcal{R}^0, r_0', s_0'} \leqslant 2\left(\frac{2n\mathtt{K}_0}{s_\flat}\right)^n e^{-(\mathtt{K}_0 - 3)s_\flat/2}. \tag{2.5.10}$$

b) for any $k \in \mathcal{G}^n_{1,\mathtt{K}_0}$ there exists a symplectic change of variables

$$\psi_k : \mathcal{R}^{1,k}_{r_k'} \times \mathbb{T}^n_{s_*'} \longmapsto \mathcal{R}^{1,k}_{r_k} \times \mathbb{T}^n_{s_*} \tag{2.5.11}$$

such that

$$\begin{aligned} H_k = H_\varepsilon \circ \psi_k &= h^k(y) + \varepsilon g^k(y, k\cdot x) + \varepsilon f^k(y,x) \\ &= h(y) + \varepsilon g_0^k(y) + \varepsilon g^k(y, k\cdot x) + \varepsilon f^k(y,x), \quad p_{\mathbb{Z}k} f^k = 0 \end{aligned} \tag{2.5.12}$$

where g_0^k is real-analytic on $\mathcal{R}^{1,k}_{r_k'}$, $g^k(y,\cdot) \in \mathbb{B}^1_{s_k'}$ for every $y \in \mathcal{R}^{1,k}_{r_k'}$, f^k is real-analytic on $\mathcal{R}^{1,k}_{r_k'} \times \mathbb{T}^n_{s_*'}$, and

$$\begin{aligned} &|g_0^k|_{\mathcal{R}^{1,k}, r_k'} \leqslant \vartheta, \qquad \| g^k(y, k\cdot x) - p_{\mathbb{Z}k} f(y,x) \|_{\mathcal{R}^{1,k}, r_k', s_k'} \leqslant \vartheta. \\ &\| f^k \|_{\mathcal{R}^{1,k}, r_k', \frac{s_*}{2}} < 2\left(\frac{2n\mathtt{K}}{s_\flat}\right)^n e^{-(\mathtt{K}-3)\frac{s_\flat}{2}}. \end{aligned} \tag{2.5.13}$$

c) Finally,

$$\| \pi_y \psi_0 - y \|_{r_0', s_0'} \leqslant \frac{r_0}{2^7 \mathtt{K}_0}, \qquad \| \pi_y \psi_k - y \|_{r_k', s_*'} \leqslant \frac{r_k}{2^7 \mathtt{K}^{n+1}} \tag{2.5.14}$$

and, for every fixed $y \in B$, $\pi_x \psi_0(y,\cdot)$ and $\pi_x \psi_k(y,\cdot)$ are diffeomorphisms on $\mathbb{T}^n$.

Remark 2.5.4. i) In order to apply lemma 2.4.1, we want to check condition 2.4.3 with our parameters in 2.5.7. But since $\| f \|_{r, s(1-1/\mathtt{K})} \leqslant (\frac{2n\mathtt{K}}{s_\flat})^n ||f||_{B,r,s}$, with simple replacing of parameters and calculation (remember that, for part b we use that $|k| \leqslant \mathtt{K}$), the condition 2.4.3 becomes

$$\mathtt{K}^{2\nu-n-4} \geqslant 2^{13+n} n^n \frac{L\, e^{s_\flat/2}}{s_\flat^{n+1}}. \tag{2.5.15}$$

Our choice of ν and b_0 ensures that $\mathtt{K}^{2\nu-n-4} \geqslant \mathtt{K}^{8n} \geqslant b_0^{8n}$, so that by taking b_0 large enough 2.5.15 holds. To be more specific for the non resonant part, taking b_0 large enough one has

$$\begin{aligned} \vartheta_* \rightsquigarrow \vartheta_*^0 \quad &:= \quad 2^{15} \frac{L\mathtt{K}_o^3 \| f \|_{r_0, s(1-1/\mathtt{K}_o)}}{\mathtt{K}^{2\nu} s_\flat (1-1/\mathtt{K}_o)} \\ &\overset{(2.5.6)}{<} \quad 2^{16} \frac{L\mathtt{K}_o^3}{s_\flat \mathtt{K}^{2\nu}} \left(\frac{2n\mathtt{K}_o}{s_\flat} \right)^n \\ &\overset{(2.5.6)}{\leqslant} \quad 2^{13} n^n \frac{L}{s_\flat^{n+1}} \frac{1}{\mathtt{K}^{2\nu-n-3}} \overset{2.5.15}{\leqslant} e^{-s_\flat/2} \leqslant 1\ , \end{aligned} \tag{2.5.16}$$

while for the resonant part one has

$$\begin{aligned} \vartheta_* \rightsquigarrow \vartheta_*^k \quad &:= \quad 2^{11} \frac{L\mathtt{K}^2 |k|^2 \varepsilon \| f \|_{r_k, s(1-1/\mathtt{K})}}{\alpha^2 s_\flat (1-1/\mathtt{K})} \\ &\overset{(2.5.6)}{\leqslant} \quad 2^{n+10} n^n \frac{L}{s_\flat^{n+1}} \frac{1}{\mathtt{K}^{2\nu-n-4}} \overset{(2.5.15)}{\leqslant} e^{-s_\flat/2} \leqslant 1\ , \end{aligned} \tag{2.5.17}$$

ii) if we define

$$\vartheta_0 := \frac{1}{\mathtt{K}^{6n+1}} \geqslant \vartheta \geqslant \bar{\vartheta} \tag{2.5.18}$$

and by taking b_0 large enough, the smallness condition 2.5.10 becomes

$$|g^0|_{r_0'} \leqslant \vartheta_0, \qquad \| f^0 \|_{r_0', s_0'} \leqslant e^{-\mathtt{K}_0 s_\flat/3} \tag{2.5.19}$$

while the condition 2.5.13 becomes

$$|g_0^k|_{r_k'} \leqslant \vartheta_0; \qquad \| g^k - \pi_{\mathbb{Z}k} f \|_{r_k', s_k'} \leqslant \vartheta_0; \qquad \| f^k \|_{r_k', s_*/2} \leqslant e^{-\mathtt{K} s_\flat/3}. \tag{2.5.20}$$

For the rest of the work we will refer to these simplier smallness conditions.

Proof. a) By the choice of r_0, the domain $\mathcal{R}^0_{r_0}$ is $(\alpha/(4\mathtt{C}), \mathtt{K}_0)$ completely non-resonant because $\mathcal{R}^0$ is $(\alpha/(2\mathtt{C}), \mathtt{K}_0)$ completely non-resonant, so taking b_0 big enough in such a way that 2.4.3 holds, we can apply normal form lemma 2.4.1 to H_ε in 2.1.1 with

$f, B, r, \Lambda, \alpha, K, s$ replaced respectively by $\varepsilon f, \mathcal{R}^0, r_0, \{0\}, \frac{\alpha}{4\mathtt{C}}, \mathtt{K}_{\rm o}, s_0$. The estimates on 2.5.10 comes from 2.4.9 in this way

$$\sup_{\mathcal{R}^0_{r'_0}} |g^{\rm o} - \langle f \rangle| \leqslant \vartheta^0_\star \left(\frac{2n\mathtt{K}_{\rm o}}{s_\flat}\right)^n \overset{(2.5.6)}{\leqslant} \left(\frac{n\mathtt{K}}{s_\flat}\right)^n \vartheta^0_\star \overset{(2.5.16),(2.5.7)}{\leqslant} \bar\vartheta\,,$$

$$\|f^0\|_{\mathcal{R}^0, r'_0, s'_0} \leqslant 2e^{-(\mathtt{K}_{\rm o}-2)s_\flat(1-1/\mathtt{K}_{\rm o})/2}\left(\frac{2n\mathtt{K}_{\rm o}}{s_\flat}\right)^n \leqslant 2\left(\frac{2n\mathtt{K}_{\rm o}}{s_\flat}\right)^n e^{-(\mathtt{K}_{\rm o}-3)s_\flat/2}\,.$$

b) By the definition of r_k, the domain $\mathcal{R}^{1,k}_{r_k}$ is $(\alpha\mathtt{K}^{(n+3)}/|k|, \mathtt{K})$ non-resonant modulo $\mathbb{Z}k$. So taking b_0 big enough we can use again lemma 2.4.1 with $f, B, r, \Lambda, \alpha, K, s$ replaced by $\varepsilon f, \mathcal{R}^{1,k}, r_k, \mathbb{Z}k, \frac{\alpha\mathtt{K}^{n+3}}{|k|}, \mathtt{K}, s_\star$.
For this part, using 2.4.5

$$\|g^k - \pi_{k\mathbb{Z}} f\|_{\mathcal{R}^{1,k}, r'_k, s'_k} \leqslant \frac{1}{\mathtt{K}}\left(\frac{2n\mathtt{K}}{s_\flat}\right)^n \vartheta^k_\star \overset{(2.5.17),(2.5.7)}{\leqslant} \vartheta\,,$$

$$\|f^k\|_{\mathcal{R}^{1,k}, r'_k, s_\star/2} \leqslant 2e^{-(\mathtt{K}-2)s_\flat(1-1/\mathtt{K})/2}\left(\frac{2n\mathtt{K}}{s_\flat}\right)^n \leqslant 2\left(\frac{2n\mathtt{K}}{s_\flat}\right)^n e^{-(\mathtt{K}-3)s_\flat/2}\,,$$

□

Lemma 2.5.2 (Cosine–like Normal forms)**.** *Let* $\mathtt{H}$ *be as in* (2.1.1) *with* $f \in \mathbb{B}^n_s$ *satisfying* (2.2.4) *and let* (2.5.7) *hold. There exists a constant* $\mathbf{c}_{\rm o} = \mathbf{c}_{\rm o}(n, s, \delta) \geqslant \max\{\mathtt{N}\,, b_0\}$ *such that if* $\mathtt{K}_{\rm o} \geqslant \mathbf{c}_{\rm o}$ *then the following holds. For any* $k \in \mathcal{G}^n_{\mathtt{K}_{\rm o}}$ *such that* $|k|_1 \geqslant \mathtt{N}$*, then the Hamiltonian* $\mathtt{H}_k$ *in* (2.5.12) *takes the form:*

$$\mathtt{H}_k = h^k(y) + \varepsilon g^k_{\rm o}(y) + 2|f_k|\varepsilon\left[\cos(k\cdot x + \theta_k) + F^k_\star(k\cdot x) + g^k_\star(y, k\cdot x) + f^k_\star(y, x)\right], \quad (2.5.21)$$

where θ_k *and* $F^k_\star$ *are as in Proposition* 2.2.1 *and:*

$$g^k_\star := \frac{1}{2|f_k|}\left(g^k - \pi_{\mathbb{Z}k} f\right), \qquad f^k_\star := \frac{1}{2|f_k|} f^k\,. \qquad (2.5.22)$$

Furthermore, $g^k_\star(y, \cdot) \in \mathbb{B}^1_1$ *(for every* $y \in \mathcal{R}^{1,k}_{r'_k}$*),* $\pi_{\mathbb{Z}k} f^k_\star = 0$*, and one has:*

$$\|g^k_\star\|_{r'_k, 1} \leqslant \vartheta := \frac{1}{\mathtt{K}^{5n}}\,, \qquad \|f^k_\star\|_{r'_k, \frac{s_\star}{2}} \leqslant e^{-\mathtt{K}s_\flat/7}\,. \qquad (2.5.23)$$

Observe that, under the assumptions of Lemma 2.5.2, by (2.5.6) and (2.2.3) it is

$$\mathtt{K} \geqslant 6\mathtt{K}_{\rm o} \geqslant 6\mathtt{N} \geqslant 12\mathtt{c}_s \geqslant 12\,. \qquad (2.5.24)$$

Proof. First of all observe that the hypotheses of Lemma 2.5.2 imply those of Lemma 2.5.1 so that the results of Lemma 2.5.1 hold.
From (2.5.22) it follows that $g^k(y,\theta) = \pi_{\mathbb{Z}k} f(\theta) + 2|f_k| g^k_\star(y,\theta)$, which together with (2.2.5) and (2.5.12) of Lemma 2.5.1, implies immediately the relations (2.5.21). To prove the first estimate in (2.5.23), we observe that, since $|k|_1 \geqslant \mathtt{N}$, recalling (2.5.6) and (2.5.24) one has

$$s'_{k,\flat} = |k|_1 s_\flat \left(1 - \tfrac{1}{\mathtt{K}}\right)^2 > \mathtt{N} s_\flat \tfrac{4}{5} > 1\,. \tag{2.5.25}$$

Thus, $g^k_\star(y,\cdot)$ is bounded on a 'large' angle–domain of size larger than 1 and has zero average (since $g^k_\star(y,\cdot) \in \mathbb{B}^1_{|k|_1 s'_\star}$). Now, recall the smoothing property (2.1.12) (with $N = 1$), recall that $\mathtt{K}_{\rm o} \leqslant \mathtt{K}/6$, and take $\mathbf{c_o}$ large enough. Then,

$$\begin{aligned}
\|g^k_\star\|_{r'_k,1} &\overset{(2.5.22)}{:=} \frac{1}{2|f_k|}\|g^k - \pi_{\mathbb{Z}k} f\|_{r'_k,1} \overset{(2.2.4)}{\leqslant} \frac{|k|_1^n e^{\sum_i |k_i| s_i}}{2\delta}\|g^k - \pi_{\mathbb{Z}k} f\|_{r'_k,1} \\
&\overset{(2.1.12),(2.5.25)}{\leqslant} \frac{|k|_1^n e^{\sum_i |k_i| s_i}}{2\delta}\|g^k - \pi_{\mathbb{Z}k} f\|_{r'_k,s'_k} \cdot e^{-(|s'_k|_1 - n)} \overset{(2.5.20)}{\leqslant} \frac{|k|_1^n e^n}{2\delta}\,\vartheta_{\rm o}\; e^{\sum_i |k_i|(s_i - s'_{\star i})} \\
&\overset{(2.5.7)}{=} \frac{|k|_1^n e^n}{2\delta}\,\vartheta_{\rm o}\; e^{\sum_i \frac{|k_i|}{\mathtt{K}} s_i \left(2 - \frac{1}{\mathtt{K}}\right)} \overset{(2.5.19)}{\leqslant} \frac{\mathtt{K}_{\rm o}^n e^n}{2\delta}\,\frac{1}{\mathtt{K}^{6n+1}}\; e^{2|s|_1 \frac{\mathtt{K}_{\rm o}}{\mathtt{K}}} \leqslant \frac{1}{\mathtt{K}^{5n}} \overset{(2.5.23)}{=} \vartheta\,.
\end{aligned}$$

Furthermore, possibly increasing $\mathbf{c_o}$, one also has

$$\begin{aligned}
\|f^k_\star\|_{r'_k,\frac{s_\star}{2}} &\overset{(2.5.22)}{=} \frac{1}{2|f_k|}\|f^k\|_{r'_k,\frac{s_\star}{2}} \overset{(2.2.4)}{\leqslant} \frac{|k|_1^n e^{\sum_i |k_i| s_i}}{2\delta}\|f^k\|_{r'_k,\frac{s_\star}{2}} \overset{(2.5.20)}{\leqslant} \frac{|k|_1^n e^{\sum_i |k_i| s_i}}{2\delta}\, e^{-\frac{\mathtt{K}s_\flat}{3}} \\
&\leqslant \frac{\mathtt{K}_{\rm o}^n}{2\delta}\, e^{-\mathtt{K}s_\flat \frac{1}{3} + \mathtt{K}s_\sharp \frac{\mathtt{K}_{\rm o}}{\mathtt{K}}} \leqslant \frac{\mathtt{K}^n}{2\delta \cdot 6^n}\, e^{-\mathtt{K}s_\flat/6} \leqslant e^{-\mathtt{K}s_\flat/7}\,.
\end{aligned}$$

where we have used the fact that $\frac{\mathtt{K}_0}{\mathtt{K}} \leqslant \frac{s_\flat}{6 s_\sharp}$. ▮ □

2.6 Normal form theorem

As in the first chapter, we need a second covering lemma

Definition 2.6.1.

$$\tilde{\mathcal{R}}^0 = Re\,(\mathcal{R}^0_{r'_0/2}) \qquad \tilde{\mathcal{R}}^{1,k} = Re\,(\mathcal{R}^{1,k}_{\check{r}_k}), \qquad \check{r}_k = \frac{r_k}{2\mathtt{K}^{n+1}}, \qquad k \in \mathcal{G}^n_{\mathtt{K}_0} \tag{2.6.1}$$

Lemma 2.6.1 (Second covering lemma)**.**

$$\begin{aligned}&\mathcal{R}^0 \times \mathbb{T}^n \subseteq \psi_0(\tilde{\mathcal{R}}^0 \times \mathbb{T}^n)\\ &\mathcal{R}^{1,k} \times \mathbb{T}^n \subseteq \psi_k(\tilde{\mathcal{R}}^{1,k} \times \mathbb{T}^n), \qquad \forall\ k \in \mathcal{G}^n_{\mathtt{K}_0}\\ &\mathcal{R}^2 := D\backslash(\mathcal{R}^0 \cup \mathcal{R}^1) \subseteq \bigcup_{k\in\mathcal{G}^n_{\mathtt{K}_0}} \bigcup_{\substack{\ell\in\mathcal{G}^n_{\mathtt{K}}\\ \ell\notin\mathbb{Z}k}} \mathcal{R}^2_{k,\ell}\end{aligned} \tag{2.6.2}$$

where $\mathcal{R}^2_{k,\ell}$ is the pull back of the following set in frequency space

$$\Omega^2_{k,\ell} := \{|\omega\cdot k| < \alpha\} \cap \{|p_k^\perp \omega| < M\} \cap \{|p_k^\perp \omega\cdot \ell| \leqslant 3\alpha\mathtt{K}/|k|\}. \tag{2.6.3}$$

for this proof one can see [2]

Lemma 2.6.2. *Let the hypotheses of Lemma 2.5.2 hold.*
(i) *For any $k \in \mathcal{G}^n_{\mathtt{K}_0}$ there exists a matrix $\hat{\mathrm{A}} \in \mathbb{Z}^{(n-1)\times n}$ such that*[16]

$$\begin{aligned}&\mathrm{A} := \begin{pmatrix} k \\ \hat{\mathrm{A}}\end{pmatrix} = \begin{pmatrix} k_1 \cdots k_n \\ \hat{\mathrm{A}}\end{pmatrix} \in \quad \mathrm{SL}(n,\mathbb{Z})\,,\\ &|\hat{\mathrm{A}}|_\infty \leqslant |k|_\infty\,, \quad |\mathrm{A}|_\infty = |k|_\infty\,, \quad |\mathrm{A}^{-1}|_\infty \leqslant (n-1)^{\frac{n-1}{2}}|k|_\infty^{n-1}\,.\end{aligned} \tag{2.6.4}$$

(ii) *Let Φ_0 be the linear, symplectic map on $\mathbb{R}^n \times \mathbb{T}^n$ onto itself defined by*

$$\Phi_0 : (\mathbf{y},\mathbf{x}) \mapsto (y,x) = (\mathrm{A}^T\mathbf{y}, \mathrm{A}^{-1}\mathbf{x})\,. \tag{2.6.5}$$

Then,

$$\mathbf{x}_1 = k\cdot x\,, \qquad y = \mathbf{y}_1 k + \hat{\mathrm{A}}^T\hat{\mathbf{y}}\,, \qquad \big[\hat{\mathbf{y}} := (\mathbf{y}_2, ..., \mathbf{y}_n)\big]\,. \tag{2.6.6}$$

Furthermore, letting[17]

$$\mathscr{D}^k := \mathrm{A}^{-T}\widetilde{\mathcal{R}}^{1,k}\,, \qquad \begin{cases} \tilde{r}_k := \frac{r_k}{c_1|k|} \\ (\tilde{s}_k)_i := \frac{1}{c_1|k|^{n-1}} s_i \end{cases}, \qquad c_1 := 5n(n-1)^{\frac{n-1}{2}}\,, \tag{2.6.7}$$

with A *as in* (i)*, we find*

$$\Phi_0 : \mathscr{D}^k_{\tilde{r}_k} \times \mathbb{T}^n_{\tilde{s}_k} \to \widetilde{\mathcal{R}}^{1,k}_{r_k'/2} \times \mathbb{T}^n_{s_*/2}\,, \qquad \Phi_0(\mathscr{D}^k \times \mathbb{T}^n) = \widetilde{\mathcal{R}}^{1,k} \times \mathbb{T}^n\,. \tag{2.6.8}$$

[16] $\mathrm{SL}(n,\mathbb{Z})$ denotes the group of $n\times n$ matrices with entries in $\mathbb{Z}$ and determinant 1; $|M|_\infty$, with M matrix (or vector), denotes the maximum norm $\max_{ij}|M_{ij}|$ (or $\max_i |M_i|$).
[17] $\widetilde{\mathcal{R}}^{1,k}$ is defined in (2.6.1); recall, also, (2.5.7).

(iii) $\mathtt{H}_k$ *in* (2.5.12), *in the symplectic variables* $(\mathbf{y},\mathrm{x}) = \big((\mathbf{y}_1,\hat{\mathbf{y}}),\mathrm{x}\big)$, *takes the form:*

$$\mathcal{H}_k(\mathbf{y},\mathrm{x}) := \mathtt{H}_k \circ \Phi_0(\mathbf{y},\mathrm{x}) = \bar{\mathtt{H}}_k(\mathbf{y},\mathrm{x}_1) + \varepsilon \bar{f}^k(\mathbf{y},\mathrm{x})\,, \quad (\mathbf{y},\mathrm{x}) \in \mathscr{D}^k_{\tilde{r}_k} \times \mathbb{T}^n_{\tilde{s}_k}\,, \tag{2.6.9}$$

where the 'secular Hamiltonian'

$$\bar{\mathtt{H}}_k(\mathbf{y},\mathrm{x}_1) := h^k(\mathbf{y}) + \varepsilon g^k_0(\mathrm{A}^T\mathbf{y}) + \varepsilon g^k(\mathrm{A}^T\mathbf{y},\mathrm{x}_1)\,, \quad \bar{f}^k(\mathbf{y},\mathrm{x}) := f^k(\mathrm{A}^T\mathbf{y},\mathrm{A}^{-1}\mathrm{x}) \tag{2.6.10}$$

is a real analytic function for $\mathbf{y} \in \mathscr{D}^k_{\tilde{r}_k}$ *and*[18] $\mathrm{x}_1 \in \mathbb{T}_{(s'_k)_1}$.

proof **of Lemma 2.6.2** (i) From Bézout's lemma it follows that[19]:
given $k \in \mathbb{Z}^n$, $k \neq 0$ *there exists a matrix* $\mathrm{A} = (\mathrm{A}_{ij})_{1\leqslant i,j\leqslant n}$ *with integer entries such that* $A_{nj} = k_j\ \forall\ 1 \leqslant j \leqslant n$, $\det \mathrm{A} = \gcd(k_1,...,k_1)$, *and* $|\mathrm{A}|_\infty = |k|_\infty$.
Hence, since $k \in \mathcal{G}^n$, $\gcd(k_1,...,k_1) = 1$, and (2.6.4) follows[20].
(ii) Φ_0 is symplectic since it is generated by the generating function $\mathbf{y}\cdot Ax$.
The relations in (2.6.6) follow at once from the definition of Φ_0.
Let us prove (2.6.8): $\mathbf{y} \in \mathscr{D}^k_{\tilde{r}_k}$ if and only if $\mathbf{y} = \mathbf{y}_0 + z$ with $\mathbf{y}_0 \in \mathscr{D}^k$ and $|z| < \tilde{r}_k$. Thus,

$$|\mathrm{A}^T z| \overset{(2.6.4)}{\leqslant} n|k||z| < n|k|\tilde{r}_k \overset{(2.6.7)}{<} \frac{r_k}{4} \overset{(2.5.7)}{=} \frac{r'_k}{2}\,.$$

Since, by definition of $\mathscr{D}^k$, $\mathrm{A}^T\mathbf{y}_0 \in \tilde{\mathcal{R}}^{1,k}$, we have that $\mathrm{A}^T\mathbf{y} \in \tilde{\mathcal{R}}^{1,k}_{r'_k/2}$.
Let, now, x belong to $\mathbb{T}^n_{\tilde{s}_k}$.Then, for any $1 \leqslant j \leqslant n$, recalling the definitions of $s_\star$ and $s'_\star$ in (2.5.7), we find

$$\Big|\mathrm{Im}\,(\mathrm{A}^{-1}\mathrm{x})_j\Big| = \Big|\sum_{i=1}^n (\mathrm{A}^{-1})_{ij}\,\mathrm{Im}\,\mathrm{x}_j\Big| \overset{(2.6.4)}{<} n(n-1)^{\frac{n-1}{2}}|k|^{n-1}(\tilde{s}_k)_j \overset{(2.6.7)}{\leqslant} \frac{(s_\star)_j}{2} < (s'_\star)_j\,.$$

Thus, $\mathrm{A}^{-1}\mathrm{x}$ belong to $\mathbb{T}^n_{s'_\star}$, and (2.6.8) follows.
(iii) Eq.'s (2.6.9)–(2.6.10) follow immediately from the definition of the symplectic map Φ_0 in (2.6.5) and (2.6.6). The statement on the angle–analyticity domain of $\bar{\mathtt{H}}_k$ follows from part (b) of Lemma 2.5.1. $\square$

Finally we are ready to state the following

[18] Recall (2.5.7).
[19] See Appendix A of [1, p. 3564] for a detailed proof.
[20] Notice that the bound on $|\mathrm{A}^{-1}|_\infty$ follows from D'Alembert expansion of determinants, observing that for any $m \times m$ matrix M, one has $|\det \mathrm{M}| \leqslant m^{m/2}|\mathrm{M}|^m_\infty$

Theorem 2.6.1 (Normal Form Theorem). *Let* $\mathtt{H}$ *be as in* (2.1.1) *with* $f \in \mathbb{B}^n_s$ *satisfying* (2.2.4) *with* $\mathtt{N}$ *as in* (2.2.2), *and let* (2.5.7) *hold. There exists a constant*[21] $\mathbf{c_o} = \mathbf{c_o}(n, s, \delta) \geqslant \max\{\mathtt{N}, b_0\}$ *such that, if* $\mathtt{K_o} \geqslant \mathbf{c_o}$, $k \in \mathcal{G}^n_{\mathtt{K_o}}$, *and* $\mathscr{D}^k$, $\tilde{r}_k$, $\tilde{s}_k$ *are as in* (2.6.7), *then there exist real analytic symplectic maps*

$$\Psi_{\rm o} : \mathcal{R}^0_{r'_{\rm o}} \times \mathbb{T}^n_{s'_{\rm o}} \to \mathcal{R}^0_{r_{\rm o}} \times \mathbb{T}^n_{s_{\rm o}}\,, \qquad \Psi^k : \mathscr{D}^k_{\tilde{r}_k} \times \mathbb{T}^n_{\tilde{s}_k} \to \mathcal{R}^{1,k}_{r_k} \times \mathbb{T}^n_{s_*} \tag{2.6.11}$$

having the following properties.

(i) $\mathtt{H_o}(y,x) := \big(\mathtt{H} \circ \Psi_{\rm o}\big)(y,x) = h(y) + \varepsilon g^0(y) + \varepsilon f^0(y,x)$,, *with* g^0 *and* f^0 *satisfying* (2.5.19) *and* $\langle f^0 \rangle = 0$.

(ii)

$$\mathcal{H}_k(\mathbf{y},\mathbf{x}) := \mathtt{H} \circ \Psi^k(\mathbf{y},\mathbf{x}) = \bar{\mathtt{H}}_k(\mathbf{y},\mathbf{x}_1) + \varepsilon \bar{f}^k(\mathbf{y},\mathbf{x})\,, \quad (\mathbf{y},\mathbf{x}) \in \mathscr{D}^k_{\tilde{r}_k} \times \mathbb{T}^n_{\tilde{s}_k}\,, \tag{2.6.12}$$

where

$$\bar{\mathtt{H}}_k(A^T\mathbf{y},\mathbf{x}_1) := h^k(A^T\mathbf{y}) + \varepsilon \mathtt{g}^k(\mathbf{y},\mathbf{x}_1), \qquad h^k(A^T\mathbf{y}) := h(A^T\mathbf{y}) + \varepsilon \mathtt{g}^k_0(\mathbf{y}); \tag{2.6.13}$$

is a real analytic function for $\mathbf{y} \in \mathscr{D}^k_{\tilde{r}_k}$ *and* $\mathbf{x}_1 \in \mathbb{T}_{(s'_k)_1}$. *In particular* $\mathtt{g}^k(\mathbf{y},\cdot) \in \mathbb{B}^1_{s'_k}$ *for every* $\mathbf{y} \in \mathscr{D}^k_{\tilde{r}_k}$. *Furthermore, the following estimates hold:*

$$|\mathtt{g}^k_{\rm o}|_{\tilde{r}_k} \leqslant \vartheta_{\rm o} = \frac{1}{\mathtt{K}^{6n+1}}\,, \qquad \|\mathtt{g}^k - \pi_{\mathbb{Z}k} f\|_{\tilde{r}_k, s'_k} \leqslant \vartheta_{\rm o}\,, \qquad \|\bar{f}^k\|_{\tilde{r}_k,\tilde{s}_k} \leqslant e^{-\mathtt{K}s_\flat/3}\,. \tag{2.6.14}$$

(iii) *If* $|k|_1 \geqslant \mathtt{N}$, *there exists* $\theta_k \in [0, 2\pi)$ *such that*

$$\mathcal{H}_k = h(A^T\mathbf{y}) + \varepsilon \mathtt{g}^k_0(\mathbf{y}) + 2|f_k|\varepsilon[\cos(\mathbf{x}_1 + \theta_k) + F^k_*(\mathbf{x}_1) + \mathtt{g}^k_*(\mathbf{y},\mathbf{x}_1) + \mathtt{f}^k_*(\mathbf{y}, A^{-1}\mathbf{x})] \tag{2.6.15}$$

where F^k_* *is as in Proposition* 2.2.1 *and satisfies* $F^k_* \in \mathbb{B}^1_1$ *and* $|F^k_*|_1 \leqslant 2^{-40}$. *Moreover,* $\mathtt{g}^k_*(\mathbf{y},\cdot) \in \mathbb{B}^1_1$ *(for every* $\mathbf{y} \in \mathscr{D}^k_{\tilde{r}_k}$*),* $\mathtt{f}^k_* = 0$, *and one has*

$$\|\mathtt{g}^k_*\|_{\tilde{r}_k,1} \leqslant \vartheta = \tfrac{1}{\mathtt{K}^{5n}}\,, \qquad \|\mathtt{f}^k_*\|_{\tilde{r}_k,\tilde{s}_k} \leqslant e^{-\mathtt{K}s_\flat/7}\,. \tag{2.6.16}$$

Proof. The first relation in (2.6.11) is (2.5.8). Define

$$\Psi^k := \psi_k \circ \Phi_0\,. \tag{2.6.17}$$

Then, since $(s_*)_i/2 < (s'_*)_i$ for all $i = 1, \ldots, n$ (compare (2.5.7)), by (2.6.8) we get the second relation in (2.6.11).

---

[21] b_0 is defined in Lemma 2.5.1.

(i) follows from point (a) of Lemma 2.5.1.

(ii) (2.6.12), (2.6.13) and (2.6.14) follow from, respectively, (2.6.9), (2.6.10), (2.5.20) and point (ii) of Lemma 2.6.2 setting

$$\mathtt{g}_{\mathrm{o}}^{k}(\mathbf{y}) := g_{\mathrm{o}}^{k}(\mathrm{A}^T\mathbf{y}), \qquad \mathtt{g}^{k}(\mathbf{y},\mathtt{x}_1) := g^{k}(\mathrm{A}^T\mathbf{y},\mathtt{x}_1). \tag{2.6.18}$$

(iii) follows by Proposition 2.2.1 and Lemma 2.5.2. In particular (2.6.15) follows from (2.5.21). Furthermore,

$$\mathtt{g}_{*}^{k} := \frac{1}{2|f_k|}\big(\mathtt{g}^{k} - \pi_{\mathbb{Z}k} f\big), \qquad \mathtt{f}_{*}^{k} := \frac{1}{2|f_k|}\bar{f}^{k} \tag{2.6.19}$$

and noting that $\mathtt{g}_{*}^{k}(\mathbf{y},\mathtt{x}_1) = g_{*}^{k}(\mathrm{A}^T\mathbf{y},\mathtt{x}_1)$ and that, by (2.6.10), $\mathtt{f}_{*}^{k}(\mathbf{y},\mathtt{x}) = f_{*}^{k}(\mathrm{A}^T\mathbf{y},\mathrm{A}^{-1}\mathtt{x})$, we see that (2.6.16) follows from (2.5.23) and (2.6.8). $\square$

Now, in order to apply Singular KAM Theory there are three more steps to do. This steps are analogous to the ones in chapter 1, so we will briefly report the main strategies.

2.7 An outline of conclusion

2.7.1 Step 2: Secondary nearly–integrable structures

After averaging on each neighbourhood $\mathcal{R}^{1,k} \times \mathbb{T}^n$, the strategy is straightforward: put the 1 degree–of–freedom systems into Arnol'd–Liouville action–angle variables of the averaged system getting Hamiltonians of the form $h_k(I) + \varepsilon \tilde{f}^k(I,\varphi)$ with a perturbing function $\tilde{f}^k \approx e^{(-cK)}$, check Kolmogorov's non–degeneracy of $h_k(I)$ (i.e., that its Hessian is uniformly invertible), apply a classical KAM Theorem showing that that the KAM tori cover the region outside the non–perturbative set up to an exponentially small term.

However, there arise major technical problems to overcome. In short: in order to gain significantly on the measure of KAM tori (passing, from the classical density of $1 - O(\sqrt{\varepsilon})$ to $1 - O(\varepsilon|\log\varepsilon|^b)$, or better in the $n = 2$ case), one hast to analyze the dynamics *close* to the singularities of the action–angle variables, namely, close to the 'secular separatrices' appearing in the averaged systems. Here 'close', means, essentially, exponentially close (i.e., $O(e^{-\mathtt{K}})$). But then, near separatrices, no perturbative approach is possible, and checking Kolmogorov's non–degeneracy become a singular perturbation problem.

To carry out the above strategy, it essential to have full control of the analytic properties of the action–angle variables of the averaged systems near their singularities. In [3], such a theory has been worked out for so–called 'Standard Form Hamiltonians', namely, one–degrees–of–freedom real–analytic Hamiltonians

$$\mathtt{H}_\flat(p,q_1) = (1+\nu(p,q_1))p_1^2 + G(\hat p, q_1)\,,$$

where: $(p_1,q_1)\in\mathbb{R}\times\mathbb{T}$ are symplectic variables, $p=(p_1,p_2,\dots,p_n)$, $\hat p := (p_2,\dots,p_n)$ are external parameters ('dumb actions'), ν is small, and G is close to a reference Morse potential $\bar G(q_1)$; for precise quantitative definitions, see Definition 1.3.4 below[22].
In Singular KAM Theory an important step (carried out in [2]) is to show that the averaged Hamiltonian can be put in Standard Form for every Fourier mode k with[23] $|k|_1 \leqslant \mathtt{K}$ having a *uniform* analytic control of the standard form $\mathtt{H}_\flat$ (which depend on k).
The proposition is the following

Proposition 2.7.1. *For all $k\in\mathcal{G}^n_{\mathtt{K}_0}$ let $\overline{\mathtt{H}}_k$ be the secular hamiltonian as in 1.2.61 and and, for $c=c(n)>1$ large enough, set*

$$\rho_2 := \frac{\gamma_k^2\,\tilde r_k^3}{c\,|k|_1^2\,M^2}\,,\qquad \rho_1 := \frac{\gamma_k\,\tilde r_k^2}{c\,|k|_1\,M}\leqslant \boldsymbol{\varrho}\,.$$

Let $\mathcal{S}_k$ be the critical surface of h^k, and $\bar y\in\mathcal{S}_k$ a critical point. The following statements hold:
(i) *In the neighborhood of $\bar{\mathtt{y}}$ defined by $\mathtt{D}(\bar{\mathtt{y}}_1,\hat{\bar{\mathtt{y}}})$, $\overline{\mathtt{H}}_k$ is symplectically conjugated to a suitable Hamiltonian in standard form $\mathtt{H}_k$ (according to definition 1.3.4). In particular, for $p\in\mathtt{D}(0,\hat{\bar{\mathtt{y}}})$, there exists real analytic symplectic transformation*

$$\Phi_* : (p,q)\ \in \mathtt{D}(0,\bar{\mathtt{y}})\times\mathbb{R}^n \to (\mathtt{y},\mathtt{x}) = \Phi_*(p,q)\in\mathbb{R}^{2n}\,, \tag{2.7.1}$$

such that[24]: Φ_* fixes $\hat p$ and q_1; for every $\hat p\in\hat{\mathtt{D}}(\bar{\mathtt{y}}_1,\hat{\bar{\mathtt{y}}})$ the map $(p_1,q_1)\mapsto(\mathtt{y}_1,\mathtt{x}_1)$ is symplectic; the $(n+1)$–dimensional map $\check\Phi_*$ depends only on the first $n+1$ coordinates (p,q_1), is 2π–periodic in $\mathtt{x}_1$ and one has

$$\begin{aligned}\check\Phi_* : (p,q_1)\in\mathtt{D}_{\rho_1+2\rho_2,\rho_2}(0,\hat{\bar{\mathtt{y}}})\times\mathbb{T}^1_{\check{\mathtt{s}}_1} &\to (\mathtt{y},\mathtt{x}_1)\in\mathtt{D}_{\rho_1,\rho_2}(\bar{\mathtt{y}}_1,\hat{\bar{\mathtt{y}}})\times\mathbb{T}^1_{\check{\mathtt{s}}_1}\\ \overline{\mathtt{H}}_k\circ\check\Phi_*(p,q_1) &= h^k(0,\hat p)+\frac12\partial^2_{p_1}h^k(0,\hat p)\ \mathtt{H}_k(p,q_1).\end{aligned} \tag{2.7.2}$$

[22]For example, a crucial property of action–angle variables for standard form Hamiltonians is that, near critical energies E_c (corresponding to hyperbolic equilibria and separatrices), the action I_1 as function of energy is given by $I_1(E_c\pm\epsilon z) = a(z)+b(z)\,z\,\log(z)$ for suitable analytic functions a,b where ϵ is a suitable reference energy (and everything depends on the external $(n-1)$ dumb actions).
[23]To be more precise, for every k with $|k|_1\leqslant\mathtt{K}$, having components with no common divisors, and positive first non–null component.
[24]We are omitting the dependence upon vector k on the coordinates in this statement.

where, given B a subset of $\mathbb{R}^n$, we denote by $B_{r,r'} = \bigcup_{y\in B}\{z \in \mathbb{C}^n : |z_1 - y_1| \leqslant r, |\hat{z} - \hat{y}| \leqslant r'\}$.
(ii) $\mathtt{H}_k$ *has reference potential*

$$\bar{G}^k := \frac{2\varepsilon}{|k|_1^2}\pi_{\mathbb{Z}k} f \tag{2.7.3}$$

and analicity characteristic

$$\hat{\mathtt{D}} := \hat{\mathtt{D}}(\hat{\bar{\mathtt{y}}}) , \qquad \mathtt{r} := \rho_2,\ \mathtt{R},\ \mathtt{s}_1,\ \beta,\ \epsilon,\ \mu , \tag{2.7.4}$$

with κ given by

$$\kappa = \kappa(n, s, \beta) := \max\{4\mathtt{c}_s, \mathtt{c}_s/\beta\}. \tag{2.7.5}$$

(iii) *The map $\Phi_\star$ is obtained as composition of two symplectic maps:*

$$\Phi_\star = \Phi_2 \circ \Phi_1 , \tag{2.7.6}$$

where:

- $\Phi_1 := \Psi_{\mathtt{g}_1} \in \mathfrak{G}_\dagger$ *for a suitable real analytic function $\mathtt{g}_1(\hat{y})$ satisfying*

  $$|\mathtt{g}_1|_{\rho_2} < \frac{2\varepsilon_k \chi_k}{\gamma\, \rho_2}\mu ; \tag{2.7.7}$$

  where γ is the convexity-constant of h.

- $\Phi_2(\mathtt{p}, \mathtt{q}) = (\mathtt{p}_1 + \eta_2, \hat{\mathtt{p}}, \mathtt{q}_1, \hat{\mathtt{q}} + \chi_2)$ *for suitable real analytic functions $\eta_2 = \eta_2(\hat{\mathtt{p}}, \mathtt{q}_1)$ and $\chi_2 = \chi_2(\hat{\mathtt{p}}, \mathtt{q}_1)$ satisfying*

  $$|\eta_2|_{2\rho_2,\check{\mathtt{s}}_1} < \frac{2\varepsilon_k \chi_k}{\gamma\, \rho_2}\mu , \qquad |\chi_2|_{2\rho_2,\check{\mathtt{s}}_1} < \frac{2\varepsilon_k \chi_k}{\gamma\, \rho_2^2}\mu . \tag{2.7.8}$$

The proof is essentially the same of the previous chapter, with all the same local discussion. There are some minor differences due to the fact that now the analyticity strip is a vector, by these difficulties are very easy to overcome. For this reason, we will omit the proof.

2.7.2 Step 3: Twist

Once all the secular systems (as k varies) are piut into action–angle variables, in order to construct KAM tori, one has to check Kolmogorov's non–degeneracy (or some other weaker non–degeneracy assumption). This is not a perturbative problem as one approaches critical energies, where lie the primary and secondary tori which contribute essentially in the change of order of measure of persistent tori. This is the main problem solved in [4], where the following result has been proven:

Theorem 2.7.1 (The twist theorem)**.** *The Liouville measure of the phase set where the twist determinant of the secular Hamiltonians* $\overline{\mathtt{H}}_k(\mathtt{y}, \mathtt{x}_1) + \varepsilon f^k(\mathtt{y}, \mathtt{x})$ *in* (**??**) *(with respect to their action variable) is smaller than a positive quantity* η *may be bounded, uniformly in* k*, by a power of* η*.*

The approach is based on the following

Definition 2.7.1. *Given* $\xi > 0$*, an open set* $A \subseteq \mathbb{R}$ *and* $f \in C^m(A, \mathbb{R})$*, we say that* f *is* ξ*–non–degenerate at order* $m \geqslant 1$ *on* A *(or, in short,* (ξ, m)*–non–degenerate), if*

$$\inf_{x\in A} \max_{1\leqslant j\leqslant m} |f^{(j)}(x)| \geqslant \xi \,. \tag{2.7.9}$$

indeed one can show that these non–degenerate functions satisfies

Lemma 2.7.1. *Let* f *be a* (ξ, m)*–non–degenerate function on a bounded interval* (a, b) *and let*[25] $M := \|f\|_{C^{m+1}(a,b)}$*. Then, there exist a constants* $c_m > 1$ *depending only on* m *such that, for all* $\eta > 0$

$$\text{meas}\{x \in (a, b) : |f(x)| \leqslant \eta\} \leqslant \frac{c_m}{\xi^{1/m}} \big(\tfrac{M}{\xi}(b - a) + 1\big)\, \eta^{1/m} \,.$$

so roughly speaking, the proof consists in constructing a suitable differential operator with non–constant coefficients, which does not vanish on (a suitable regularization of) the Kolmogorov's twist determinant. In this way using the above lemma the theorem follows.

Far from separatrices the strategy is essentially perturbative, and the twist comes from the non degeneracy condition satisfied by the twist determinant of the reference hamiltonian. Near separatrices, instead, perturbative arguments do not hold, and, in particular the energy function $\mathtt{E}^i$ is singular at the boundary (corresponding to separatrices) and its derivatives diverge as the boundary is approached. Furthermore, $\mathtt{E}^i$ and $\bar{\mathtt{E}}^i = \mathtt{E}^i|_{\mu=0}$ have singularities in different points.

In order to deal with these problems, in [4] one defines

Definition 2.7.2. *We denote by* $\mathcal{F}$ *the set of functions of the form*

$$f(z, \hat{I}) = z^h \sum_{j=0}^{\ell} u_j(z, \hat{I}) \log^j z \,, \tag{2.7.10}$$

[25] $\|f\|_{C^{m+1}(a,b)} := \max_{0\leqslant j\leqslant m+1} \sup_{(a,b)} |f^{(j)}|$.

where $h, \ell \in \mathbb{Z}$ with $\ell \geqslant 0$ and the u_j are real analytic functions. [26] *We say that $f(z, \hat{I}) = \mathcal{O}_\varrho(h, \ell)$ if $f \in \mathcal{F}$ as in (2.7.10) and there exists $\varrho > 0$ such that*

$$\|\!|f|\!\|_\varrho := \sup_{0 \leqslant j \leqslant \ell} \sup_{\substack{\{z \in \mathbb{C}: |z| < \varrho\} \\ \hat{I} \in \hat{D}}} |u_j| < +\infty \,.$$

in such a way that there exists a suitable differential operator with non–constant coefficients

$$\mathcal{L} = \sum_{j=1}^{\bar{\mathrm{m}}} a_j(z) \partial_z^j; \quad \bar{\mathrm{m}} = 3n^2 - 2n - 1$$

such that

$$\mathcal{L}[\bar{\delta}] = \bar{n}!^{3\bar{n}+1} (3\bar{n})! \, \gamma^{3\bar{n}} + \mathcal{O}_\varrho(1, 3\bar{n} + 1) \,,$$

where $\bar{\delta}$ is a suitable *regularization of the twist determinant*, $\bar{n} = n - 1$ and $\gamma = \gamma(\widehat{I})$ is a smooth function of the last "dumb" action. In this sense the twist determinant is *non–degenerate* and applying lemma 2.7.1 one can conclude the proof of the twist theorem.

2.7.3 Step 4: Primary and secondary maximal KAM tori

At this point, choosing carefully the various free parameters of the game, a suitable KAM Theorem (like the one in [46]) yields the existence of maximal primary and secondary KAM tori, which fill the complementary phase set of $\mathcal{R}^2 \times \mathbb{T}^n$ up to a very small set.

Concerning the non–resonant set, the hamiltonian is conjugated to the sum of an integrable system and an exponentially small term, so that by classical KAM theory, it follows that this set is filled by primary[27] KAM tori up to a set of measure of order $O(e^{-\mathtt{K}_\mathrm{o} s_\flat})$.

For the case in $\mathscr{D}_\star^k$ there is much more work to do. In particular we have to use carefully the estimates of the *Twist theorem*, but the result is always an exponentially small measure of the non–torus set in each neighborhood of a single critical point in which we have done the *Standard form conjugation*. Then, having a control of the finite number of neighborhoods that cover the entire $\mathscr{D}_\star^k$, one can find the measure of the non–torus set in $\mathcal{R}^1$ that is always exponentially small.

[26] Given a vector $I \in \mathbb{R}^n$, we will denote by $\hat{I} = (I_2, ..., I_n)$ the vector of the last $n - 1$ coordinates such that $I = (I_1, \hat{I})$.

[27] Primary tori are smooth deformation of the flat Lagrangian integrable ($\varepsilon = 0$) tori.

In this way, calling $\mathcal{A} = ((\mathcal{R}^1 \cup \mathcal{R}^0) \times \mathbb{T}^n) \backslash \mathcal{T}$, where $\mathcal{T}$ is union of the maximal primary and secondary tori for H, one has proved that $\mathrm{meas}(\mathcal{A}) \leqslant c\, e^{-\mathtt{K}/c'}$ for some constants c, c' that summed to the double resonances ($\mathrm{meas}(\mathcal{R}^2) \leqslant c''\varepsilon\ \mathtt{K}^b$ for some constants c'', b) and choosing $\mathtt{K} \approx |\log \varepsilon|$ gives us the desired result.

Theorem 1.1.2 is due to the fact that the two degrees of freedom is special: in this case the only double resonance is the origin and one can take as $\mathcal{R}^2$ a disk of measure ε^a with any $0 < a < 1$ getting a set of KAM tori of exponential density in the complementary of $\mathcal{R}^2 \times \mathbb{T}^2$.

Chapter 3

Application to celestial mechanics

The intent of this chapter is to apply Singular KAM Theory to some physical relevant models in order to obtain some interesting new estimates on the total measure of the invariant tori for these nearly-integrable systems. The system that we consider is the *Restricted, planar, circular three body problem* which from now on will be called RCPTBP.

Roughly speaking this model describe the bounded planar motion of a *"zero mass"* body subject to the gravitational field generated by two primary bodies revolving on circular Keplerian orbits (which are assumed to be not influenced by the small body).

When the mass ratio of the two primary bodies is small the RCPTBP is described by a nearly-integrable Hamiltonian system with two degrees of freedom; in a region of phase space corresponding to neaerly elliptical motions with non small eccentricities, the system is well described by Delaunay variables.

Before diving up into the application of singular KAM theory, we are going to study briefly the considered model from a physical and a mathematical point of view. This first part is based on [21],[20],[52].

3.1 The restricted three-doby problem

First of all we want introduce our model starting for the *restricted three-body problem*; then we will add the other semplifications. The restricted 3BP is simply a "zero mass" body subject to the gravitationale attraction by as assigned two-body system. To describe mathematically such system, let P_0, P_1, P_2 be three bodies (viewed as "point masses" because for the gravitational field we can consider all the masses concentrated

in the center of mass) with masses m_1, m_2 and m_3 interacting only through the gravitational attraction.
If $u^{(i)} \in \mathbb{R}^3$, $i = 0, 1, 2$ denote the position of the bodies in some (intertial) reference frame (and assuming, without loss of generality, that the gravitational constante g is one), the Newton equations for this system have the form

$$\begin{aligned} \frac{d^2u^{(0)}}{dt^2} &= -\frac{m_1(u^{(0)} - u^{(1)})}{|u^{(1)} - u^{(0)}|^3} - \frac{m_2(u^{(0)} - u^{(2)})}{|u^{(2)} - u^{(0)}|^3}; \\ \frac{d^2u^{(1)}}{dt^2} &= -\frac{m_0(u^{(1)} - u^{(0)})}{|u^{(1)} - u^{(0)}|^3} - \frac{m_2(u^{(1)} - u^{(2)})}{|u^{(2)} - u^{(1)}|^3}; \\ \frac{d^2u^{(2)}}{dt^2} &= -\frac{m_0(u^{(2)} - u^{(0)})}{|u^{(2)} - u^{(0)}|^3} - \frac{m_1(u^{(2)} - u^{(1)})}{|u^{(2)} - u^{(1)}|^3}. \end{aligned} \tag{3.1.1}$$

The *restricted three-body problem* with "primary bodies" P_0 and P_1 is, by definition, the problem of studying the bounded motions of the system 3.1.1 after having set $m_2 = 0$, i.e. of the system

$$\begin{aligned} \frac{d^2u^{(0)}}{dt^2} &= -\frac{m_1(u^{(0)} - u^{(1)})}{|u^{(1)} - u^{(0)}|^3}; \\ \frac{d^2u^{(1)}}{dt^2} &= -\frac{m_0(u^{(1)} - u^{(0)})}{|u^{(1)} - u^{(0)}|^3}; \\ \frac{d^2u^{(2)}}{dt^2} &= -\frac{m_0(u^{(2)} - u^{(0)})}{|u^{(2)} - u^{(0)}|^3} - \frac{m_1(u^{(2)} - u^{(1)})}{|u^{(2)} - u^{(1)}|^3}. \end{aligned} \tag{3.1.2}$$

Notice that the equations for the two primaries P_0 and P_1 *decouple* and describe an *unperturbed two-body system*, which can be solved and the solution can be plugged into the equation for $u^{(2)}$, which becomes a second-order, *periodically forced* equation in $\mathbb{R}^3$.

3.2 Delaunay action-angle variables for the two-body problem

In this section we take the two-primaries-body problem and we review the classical Delaunay action-angle variables construction developed in [Delaunay 1860].
The equation of motion of the two bodies P_0 and P_1 of masses m_0 and m_1, interacting through gravtiational (setting $g = 1$) are given by

$$\frac{d^2u^{(0)}}{dt^2} = -\frac{m_1(u^{(0)} - u^{(1)})}{|u^{(1)} - u^{(0)}|^3}; \quad \frac{d^2u^{(1)}}{dt^2} = -\frac{m_0(u^{(1)} - u^{(0)})}{|u^{(1)} - u^{(0)}|^3}; \quad u^{(i)} \in \mathbb{R}^3 \tag{3.2.1}$$

As it is well known, the total energy, momentum and angular momentum are preserved. We shall therefore fix an inertial frame $\{k_1, k_2, k_3\}$, with origin in the center of the mass and with k_3-axis parallel to the total angular momentum. In such frame we have

$$u_3^{(0)} \equiv 0 \equiv u_3^{(1)}, \quad m_0 u^{(0)} + m_1 u^{(1)} = 0. \tag{3.2.2}$$

Now we pass to a *heliocentric frame* by letting

$$(x, 0) := u^{(1)} - u^{(0)}, \quad x \in \mathbb{R}^2. \tag{3.2.3}$$

In view of (3.2.1) and (3.2.2), the equations for x become

$$\ddot{x} = -M \frac{x}{|x|^3}, \quad M := m_0 + m_1. \tag{3.2.4}$$

This equation is obviously Hamiltonian: let $\mu > 0$ and set

$$H_{Kep}(x, X) := \frac{|X|^2}{2\mu} - \frac{\mu M}{|x|}, \quad X := \mu \dot{x}, \tag{3.2.5}$$

then 3.2.4 is equivalent to the Hamiltonian equation associated to H_{Kep} with respect to the standard symplectic form $dx \wedge dX$, the phase space being $\mathbb{R}^2 \backslash \{0\} \times \mathbb{R}^2$; the (free) parameter μ is traditionally chosen as the "reduced mass" $\frac{m_0 m_1}{M}$.
The motion in the u-coordinates is recovered (via 3.2.3 and 3.2.2) by the relation

$$u^{(0)} = \Big(-\frac{m_1}{M} x, 0 \Big), \quad u^{(1)} = \Big(-\frac{m_0}{M} x, 0 \Big) \tag{3.2.6}$$

The dependence of the Hamiltonian on x through the absolute value suggests to introduce polar coordinates in the x-plane and, in order to get a symplectit transformation, one is led to the symplectic map $\phi_{pc} : ((r, \varphi), (R, \Phi)) \to (x, X)$ given by

$$\phi_{pc} : \begin{cases} x = r(\cos\varphi, \sin\varphi), \\ X = (R\cos\varphi - \frac{\Phi}{r}\sin\varphi, R\sin\varphi + \frac{\Phi}{r}\cos\varphi), \\ dx_1 \wedge dX_1 + dx_2 \wedge dX_2 = dr \wedge dR + d\varphi \wedge d\Phi. \end{cases} \tag{3.2.7}$$

The variables r and φ are commonly called, in celestial mechanics, the *orbital radius* and the *longitude of the planet* P_1. In the new symplectic variables the Hamiltonian takes the form

$$H_{pc}(r, \varphi, R, \Phi) := H_{Kep} \circ \phi_{pc}(r, \varphi, R, \Phi) = \frac{1}{2\mu}\Big(R^2 + \frac{\Phi^2}{r^2} \Big) - \frac{\mu M}{r}. \tag{3.2.8}$$

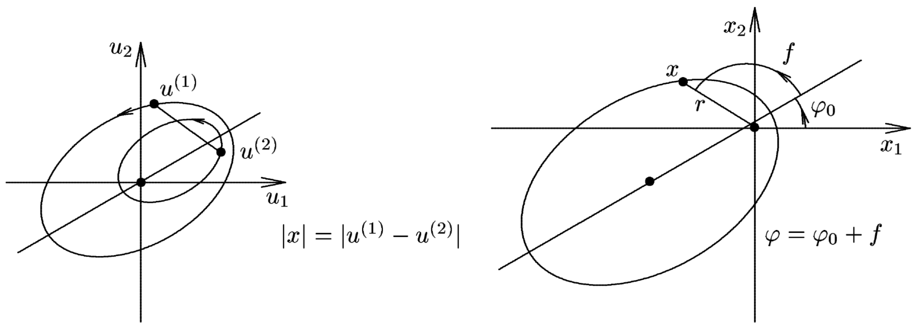

Figure 3.1: The geometry of the Kepler two-body problem

The variable φ is *cyclic*, i.e. $\partial H_{pc}/\partial\varphi = 0$ so that $\Phi \equiv$ const. This fact implies that H_{pc} is actually a one-degree-of-freedon Hamiltonian system and is therefore integrable. The momentumn variable Φ conjugated to φ is obviously an integral of motions and

$$\dot{\varphi} = \frac{\partial H_{pc}}{\partial\varphi} = \frac{\Phi}{\mu r^2} \quad \Rightarrow \quad \Phi = \mu r^2\dot{\varphi} \equiv \text{const.} \tag{3.2.9}$$

Remark 3.2.1. The total angular momentum C in the inertial frame referred to the center of mass denoting the standard "vector product" with $\times$, is given by

$$C = m_0 u^{(0)} \times \dot{u}^{(0)} + m_1 u^{(1)} \times \dot{u}^{(1)}. \tag{3.2.10}$$

With as easy calculation this expression is equivalent to

$$C = \frac{m_0 m_1}{\mu M} x \times X \tag{3.2.11}$$

and using polar coordinates it becomes

$$C = k_3 \frac{m_0 m_1}{M} r^2\dot{\varphi} = k_3 \frac{m_0 m_1}{\mu M}\Phi; \tag{3.2.12}$$

thus if μ is chosen to be the reduced mass $\frac{m_0 m_1}{M}$, then Φ is exactly the absolute value of the total angular momentum.

The analysis of the (r, R) motion is quite standard and well known in the literature: introducing the "effective potential"

$$V_{eff}(r) := V_{eff}(r; \Phi) := \frac{\Phi^2}{2\mu r^2} - \frac{\mu M}{r}, \tag{3.2.13}$$

one is led to the "effective Hamiltonian" (parameterized by Φ)

$$H_{eff} = \frac{R^2}{2mu} + V_{eff}(r), \quad R = \mu\dot{r}. \tag{3.2.14}$$

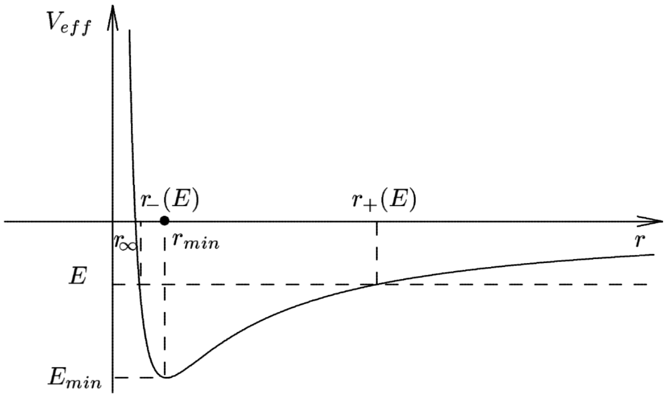

Figure 3.2: The effective potential of the two-body problem

The motion on the energy level $H_{eff}^{-1}(E)$ is bounded (and periodic) if and only if

$$E \in [E_{min}, 0), \quad E_{min} := V_{eff}(r_{min}) = -\frac{\mu^3 M^2}{2\Phi^2}, \quad r_{min} := \frac{\Phi^2}{\mu^2 M}. \tag{3.2.15}$$

For $E \in (E_{min}, 0)$, the period $T(E)$ is given by

$$T(E) = 2\int_{r_-(E)}^{r_+(E)} \frac{dr}{\sqrt{\frac{2}{\mu}(E - V_{eff}(r))}} \tag{3.2.16}$$

where $r_\pm(E) = r_\pm(E;\Phi)$ are the two real roots of $E - V_{eff}(r) = 0$, i.e.

$$\begin{aligned} E - V_{eff}(r) &=: -\frac{E}{r^2}(r_+ - r)(r - r_-), \\ r_\pm(E;\Phi) &= \frac{\mu M \pm \sqrt{(\mu M)^2 + \frac{2E\Phi^2}{\mu}}}{-2E}. \end{aligned} \tag{3.2.17}$$

The integral in 3.2.16 is readily computer yielding Kepler's second law

$$T(E) = 2\pi M\left(\frac{\mu}{-2E}\right)^{3/2}. \tag{3.2.18}$$

Let us now integrate the motion in the (r, φ) coordinates. The equations of motion in such coordinates are given by

$$\dot{\varphi} = \frac{\Phi}{\mu r^2}, \quad \dot{r}^2 = \frac{2}{\mu}(E - V_{eff}(r)). \tag{3.2.19}$$

By symmetry arguments, it is enough to consider the motion for $0 \leqslant t \leqslant \frac{T(E)}{2}$; furthermore, we shall choose the initial time so that $r(0) = r_-$ (i.e., at the initial time the system is at the "*perihelion*"): the corresponding angle will be a certain φ_0 and we shall make the (trivial) change of variables

$$\varphi = \varphi_0 + f, \quad \text{so that} \quad r(0) = r_-(E), \ f(0) = 0. \tag{3.2.20}$$

The angle f is commonly called the *true anomaly*; the angle φ_0 (i.e., the constant angle between the perihelion line, joining the foci of the ellipse and the x_1 axis) is called the *argument of the perihelion* (compare figure 3.1)

Equations 3.2.19 become

$$\begin{cases} \dot{f} = \frac{\Phi}{\mu r^2}, & f(0) = 0 \\ \dot{r}^2 = \frac{2}{\mu}(E - V_{eff}(r)), & r(0) = r_-(E) \end{cases} \tag{3.2.21}$$

Eliminating time (for $t \in (0, T(E))$, $\dot{r} > 0$) we find

$$f = \Phi \int_{r_-(E)}^{r} \frac{d\rho/\rho^2}{\sqrt{2\mu(E - V_{eff}(\rho))}} = \arccos\left(\frac{\frac{r_{min}}{r} - 1}{\sqrt{1 - \frac{E}{E_{min}}}} \right). \tag{3.2.22}$$

Setting

$$e = \sqrt{1 - \frac{E}{E_{min}}}, \qquad p := r_{min}, \tag{3.2.23}$$

we know from the literstuve that the motion of P_0 and P_1 describe two ellipses of eccentricity $e \in (0, 1)$ with common focus in the center of mass (known as the first Kepler law)

Let now recall some basic features of the geometry of an ellipse. Let $a \geqslant b > 0$ denote the semi-axis; the cartesian equation of an ellipse with respect to a reference plane $(x_1, x_2) \in \mathbb{R}^2$ with orighin chosen as the middle point of the segment joining the two foci is given by

$$\left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 = 1, \tag{3.2.24}$$

where $2a$ is the (constant) sum of distances between a point of the ellipse x and the foci, and

$$\Big(\pm a\sqrt{1-(\frac{b}{a})^2}, 0 \Big) \tag{3.2.25}$$

are the coordinates of the foci. As we just know, the number

$$e = \sqrt{1-(\frac{b}{a})^2} \tag{3.2.26}$$

is called the eccentricity of the ellipse that, in our case, is expressed in 3.2.23. As it follows from 3.2.25, the distance c between one focus and the center of the elliopse is given by

$$c = e\ a \tag{3.2.27}$$

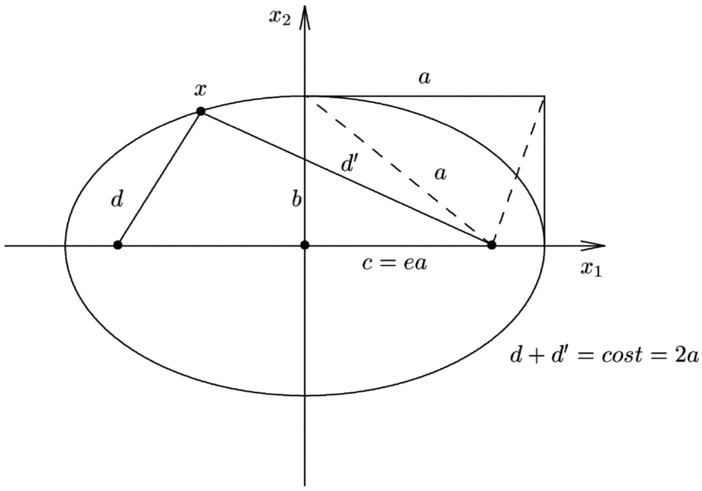

Figure 3.3: Ellipse of eccentricity $e = 0.78$

Now, if one inctrocues polar coordinates (f, r) in the above x-plane taking as pole the focus $O = (c, 0)$, as f the angle between the x_1-axis and the axis joining O with the point x on the ellipse and $r = r(f)$ as the distance $|x - O|$, one can find from simple geometrical consideration the following *focal equation*

$$r = r(f) := \frac{p}{1 + e\cos f}, \tag{3.2.28}$$

where p is called the parameter of the ellipse and is given by

$$p = a(1 - e^2) = \frac{b^2}{a}. \tag{3.2.29}$$

The angle f is called the *true anomaly*.

Moreover, one can describe the above ellipse by the following parametric equations

$$x_1 = a \cos u, \quad x_2 = b \sin u \tag{3.2.30}$$

where u is the angle between the origin of the plane and the planet during his elliptical motion. This angle u is called the *eccentric anomaly*.

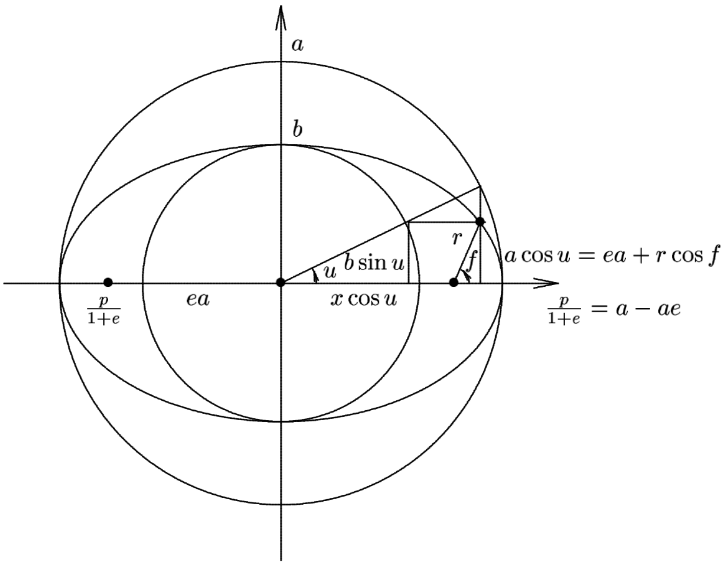

Figure 3.4: Ellipse parameters

Thus a point x on the ellipse has the double representation

$$x = (a \cos u, b \sin u) = (ea + r \cos f, r \sin f) \tag{3.2.31}$$

which relates the true and the eccentric anomalies. In particular, one finds:

$$
\begin{aligned}
r\cos f &= a(\cos u - e),\\
r\sin f &= b\sin u = a\sqrt{1-e^2}\sin u,\\
r &= a(1-e\cos u),\\
\tan\frac{f}{2} &= \sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2},\\
\mathrm{Area}(\mathcal{E}(f)) &= \frac{ab}{2}(u - e\sin u),
\end{aligned}
\tag{3.2.32}
$$

where

$$
\mathcal{E}(f) := \{x = x(r', f') : 0 \leqslant r' \leqslant r(f),\ 0 \leqslant f' \leqslant f\}. \tag{3.2.33}
$$

So joining the focal equation in 3.2.28 with the change of variable in 3.2.20, for our case we get

$$
r = \frac{p}{1 + e\cos(\varphi - \varphi_0)}, \tag{3.2.34}
$$

and it follows immediately from the geometrical considerations made above (with our notation) that

$$
r_\pm = \frac{p}{1\mp e}, \quad r_+ + r_- = 2a, \quad p = a(1-e^2), \quad r_\pm = a(1\pm e). \tag{3.2.35}
$$

From the definition of E_{min} in 3.2.15, the expression for $E - V_{eff}$ in 3.2.17, and the relations 3.2.23, 3.2.35 one finds

$$
E_{min} = -\frac{\mu M}{2p}, \qquad E = -\frac{\mu M}{2a}, \qquad E - V_{eff} = \frac{\mu M}{2a}\left(\frac{e\sin u}{1 - e\cos u}\right)^2. \tag{3.2.36}
$$

Remark 3.2.2. Since we will later assume the two-body-Keplerian motion to be circular, we notice trhat this circular motion is obrained for the minimal value of the energy $E = E_{min} = -\frac{\mu^3 M^2}{2\Phi^2}$. In such a case

$$
e = 0, \qquad r \equiv p = r_{min} = \frac{\Phi^2}{\mu^2 M}; \tag{3.2.37}
$$

the constant angular velocity and the period are respectively given by

$$
\omega_{circ} = \frac{\mu^3 M^2}{\Phi^3}, \qquad T_{circ} = 2\pi\frac{\Phi^3}{\mu^3 M^2}. \tag{3.2.38}
$$

Eliminating Φ in 3.2.37 and 3.2.38, one gets

$$\omega_{circ} = \sqrt{\frac{M}{r^3}}, \qquad T_{circ} = 2\pi\sqrt{\frac{r^3}{M}}. \tag{3.2.39}$$

Thus the motion in the x-variables is given by

$$x(t) = r(\cos(\varphi_0 + \omega_{circ}t), \sin(\varphi_0 + \omega_{circ}t)). \tag{3.2.40}$$

We now turn to the construction of the *action angle variables*. For $E \in (E_{min}, 0)$, we denote the energy level at a fixed value of Φ by

$$S_E := \{(r, R) : H_{eff}(r, R) = E\}. \tag{3.2.41}$$

The area $A(E)$ encircled by such a curve in the (r, R)-plane is given by

$$A(E) = 2\int_{r_-(E)}^{r_+(E)} \sqrt{2\mu(E - V_{eff}(r))}dr = -2\pi\mu M\sqrt{\frac{\mu}{-2E}}\Phi. \tag{3.2.42}$$

Thus, by the theorem of Liouville-Arnold the action variable is given by

$$I(E) = \frac{A(E)}{2\pi} = -\mu M\Phi\sqrt{\frac{\mu}{-2E}}, \tag{3.2.43}$$

which, inverted, gives the form of the effective Hamiltonian in the action-angle variables (θ, I) (and parametrized by Φ):

$$h(I) := h(I; \Phi) := -\frac{\mu^3 M^2}{2(I + \Phi)^2}. \tag{3.2.44}$$

Furthermore (again using Liouville-Arnold theorem) the symplectic transformation between (r, R) (in a neighborhood of a point with $R > 0$) and the action-angle variables (θ, I), for the Hamiltonian H_{eff} is *generated* by the generating function

$$S_0(I, r; \Phi) := \int_{(r_-(h(I)),0)}^{(r,R_+(r;I))} Rdr; \qquad R_+(r; I) := \sqrt{2\mu(h(I) - V_{eff}(r))} \tag{3.2.45}$$

where the integration is performed over the curve $S_{h(I)}$ oriented *clockwise* : the orientation of $S_{h(I)}$ and the choice of the base point as $(r_-(h(I)), 0)$ is done so that an integration over the closed curve gives $+A(E)$ and so that $\theta = 0$ corresponds to the perihelion position.

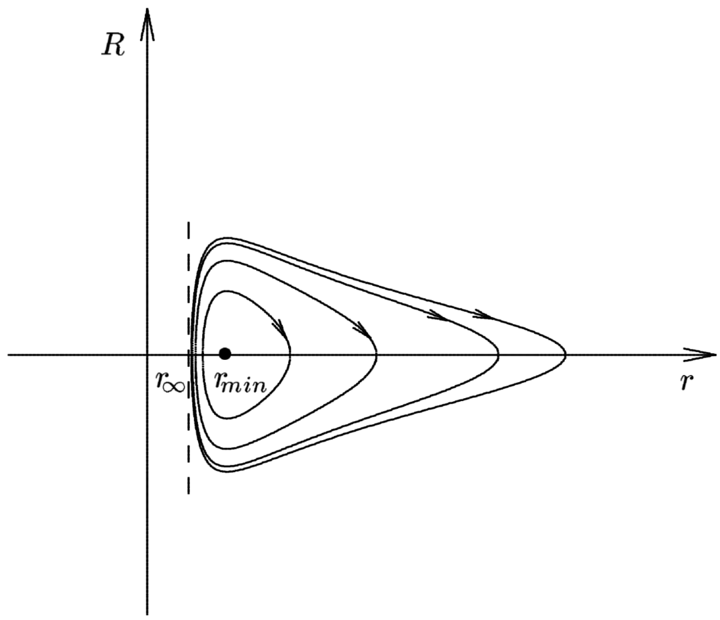

Figure 3.5: Level curves of the effective Hamiltonian

The full symplectic transformation (in the four dimensional phase space of H_{pc})

$$\phi_{aa} : \begin{cases} (\theta, \psi, I, J) \to (r, \varphi, R, \Phi) \\ d\theta \wedge dI + d\psi \wedge dJ = dr \wedge dR + d\varphi \wedge d\Phi \end{cases} \tag{3.2.46}$$

will be generated by the generating function

$$S_1(I, J, r, \varphi) := S_0(I, r; J) + J\varphi, \qquad (J = \Phi). \tag{3.2.47}$$

The form of $h(I)$ suggests to introduce one more (linear,symplectic) change of variables given by

$$\phi_{lin}^{-1} : \begin{cases} \Lambda = I + J, \qquad \Gamma = J, \\ \lambda = \theta, \qquad \gamma = \psi - \theta \end{cases} \tag{3.2.48}$$

The variables $(\lambda, \gamma, \Lambda, \Gamma)$ are the celebrated *Delaunay variables for the two-body-problem.* If we set

$$\phi_D := \phi_{pc} \circ \phi_{aa} \circ \phi_{lin} \tag{3.2.49}$$

be the above analysis we get

$$h_{Kep} \circ \phi_D(\lambda, \gamma, \Lambda, \Gamma) = h_{Kep}(\Lambda) := -\frac{\mu^3 M^2}{2\Lambda^2}. \tag{3.2.50}$$

The symplectic trasformation $\phi_{aa} \circ \phi_{lin}$ is generated by $(\Gamma = J = \Phi)$

$$\begin{aligned} S_2(\Lambda, \Gamma, r, \varphi) :&= S_0(\Lambda - \Gamma, r; \Gamma) + \Gamma\varphi \\ &= \int_{r_-(h_{Kep}(\Lambda))}^{r} \sqrt{-\frac{\mu^4 M^2}{\Lambda^2} + \frac{2\mu^2 M}{\rho} - \frac{\Gamma^2}{\rho^2}} d\rho + \Gamma\varphi \\ &= \sqrt{2\mu} \int_{r_-(h_{Kep}(\Lambda))}^{r} \sqrt{h_{Kep}(\Lambda) - V_{eff}(\rho;\Gamma)} d\rho + \Gamma\varphi. \end{aligned} \tag{3.2.51}$$

Replacing E by $h_{Kep}(\Lambda)$ and Φ with Γ in the expression for the eccentricity e in 3.2.23 (recall the definition of E_{min} in 3.2.36) one finds

$$e = e(\Lambda, \Gamma) = \sqrt{1 - \left(\frac{\Gamma}{\Lambda}\right)^2}. \tag{3.2.52}$$

Recalling the relation between the parameter p (in the focal equation for an ellipse), the eccentricity and the major semi-axis (see 3.2.35), from 3.2.52 it follows that

$$a = \frac{\Lambda^2}{\mu^2 M}, \qquad \Lambda = \mu\sqrt{Ma}. \tag{3.2.53}$$

Remark 3.2.3. Recall that

$$\Gamma = \Phi = \frac{\mu M}{m_0 m_1}|C|, \qquad C := \text{total angular momentum}, \tag{3.2.54}$$

so that

$$\Gamma > 0. \tag{3.2.55}$$

Recall also that

$$E_{min} = -\frac{\mu^3 M^2}{2\Gamma^2}, \tag{3.2.56}$$

so that $E > E_{min}$ means, by 3.2.50

$$\Gamma < \Lambda. \tag{3.2.57}$$

The momentum space $\{L, G\}$ is therefore the *positive cone* $\{0 < \Gamma < \Lambda\}$.

The angle λ is computed from the generating function S_2:

$$
\begin{aligned}
\lambda = \frac{\partial S_2}{\partial \Lambda} &= \sqrt{\frac{\mu}{2}}\frac{\mu^3 M^2}{\Gamma^3}\int_{r_-}^{r} \frac{d\rho}{\sqrt{h_{Kep}(\Lambda) - V_{eff}(\rho;\Gamma)}} \\
&= \sqrt{\frac{\mu M}{2a}}\frac{1}{a}\int_{r_-}^{r} \frac{d\rho}{\sqrt{h_{Kep}(\Lambda) - V_{eff}(\rho;\Gamma)}} \\
&\overset{3.2.36}{=} \frac{1}{a}\int_{r_-}^{r} \frac{1 - e\cos u}{e\sin u} \\
&\overset{3.2.32}{=} \int_0^u (1 - e\cos u)du \\
&= u - e\sin u \\
&\overset{3.2.32}{=} 2\pi\frac{\mathrm{Area}(\mathcal{E}(f))}{\mathrm{Area}(\mathcal{E}(2\pi))},
\end{aligned}
\tag{3.2.58}
$$

where we have used the fact that ρ as a function of $u \in [0, \pi]$ is a strictly increasing function and that $\rho(0) = r_-$.

In view of 3.2.58, λ is called the *mean anomaly*. Analogously, the angle γ is recognized to be argument of the perihelion φ_0 introduced above

$$
\begin{aligned}
\gamma = \frac{\partial S_2}{\partial \Gamma} &= \varphi - \Gamma\int_{r_-}^{r} \frac{1}{\sqrt{2\mu(h_{Kep}(\Lambda) - V_{eff}(\rho))}}\frac{d\rho}{\rho^2} \\
&\overset{3.2.22}{=} \varphi - f \\
&\overset{3.2.20}{=} \varphi_0.
\end{aligned}
\tag{3.2.59}
$$

In order to conclude this classical section, we review some important analytical features about the eccentric anomaly u and the true anomaly f in terms of the Delaunay variables.

The Kepler equation

$$\lambda = u - e\sin u,$$

can be inverted for $|e| < 1$ for the implicit function theorem. In particular we use this

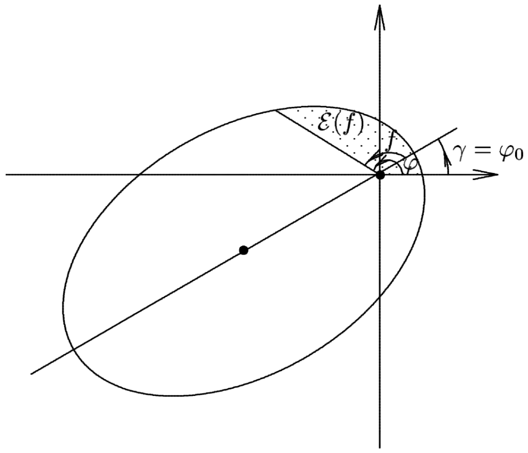

Figure 3.6: The Delaunay angles.

theorem to show that

$$
\begin{aligned}
\frac{du}{d\lambda} &= \frac{1}{1 - e\cos u} \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{d\lambda}{1 - e\cos u} + \sum_{n=1}^{\infty} \frac{\cos(n\lambda)}{\pi} \int_0^{2\pi} \frac{\cos(n\lambda) d\lambda}{1 - e\cos u} \\
&= \frac{1}{2\pi}\int_0^{2\pi} du + \sum_{n=1}^{\infty} \frac{\cos(n\lambda)}{\pi} \int_0^{2\pi} \cos\{n(u - e\sin u)\} du \\
&= 1 + 2\sum_{n=1}^{\infty} J_n(ne)\cos(n\lambda),
\end{aligned}
\tag{3.2.60}
$$

so that

$$u(\lambda, e) = \lambda + 2\sum_{n=1}^{\infty} \frac{J_n(ne)}{n} \sin(n\lambda) \tag{3.2.61}$$

where $J_n(x)$ is the *Bessel function* of the first kind, i.e. if $\Gamma(z)$ represents the Gamma function

$$J_n(x) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!\ \Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m+n}. \tag{3.2.62}$$

Since in our specific case n is an integer, the expression below becomes

$$J_n(x) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!\ (m+n)!} \left(\frac{x}{2}\right)^{2m+n}, \text{ s.t. } J_{-n} = (-1)^n J_n. \tag{3.2.63}$$

and it is well known in the literature that

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x) \Rightarrow J_n'(x) = J_{n-1}(x) - \frac{n}{x} J_n(x). \tag{3.2.64}$$

The most important part of this result is the expansion of u in power of e that helps us to know its behaviour in for small eccentricity:

$$\begin{aligned} u &:= \lambda + e\widetilde{u}(\lambda, e) \\ &= \lambda + e\sin(\lambda) + \frac{e^2}{2}\sin(2\lambda) + \frac{e^3}{8}(-\sin(\lambda) + 3\sin(3\lambda)) + O(e^4) \end{aligned} \tag{3.2.65}$$

where $\tilde{u}$ is analytics in $\lambda \in \mathbb{T}$ and $|e| < 1$.

As we have seen in 3.2.35, we have a precise relation between the true anomaly f and the eccentric anomaly u, so that we can express f for $|e| < 1$ in terms of Bessel function, i.e. we can know its expansion in e powers

$$\begin{aligned} f = f(\lambda, e) &:= \lambda + e\widetilde{f}(\lambda, e) \\ &= \lambda + 2\sum_{k=1}^{\infty} \frac{1}{k}\left[\sum_{n=-\infty}^{+\infty} J_n(-ke)\ (\frac{e}{2})^{|k+n|}\right]\sin(k\lambda) \\ &= \lambda + (2e - \frac{1}{4}e^3)\sin(\lambda) + \frac{5}{4}e^2\sin(2\lambda) + \frac{13}{12}e^3\sin(3\lambda) + O(e^4). \end{aligned} \tag{3.2.66}$$

The longitude φ is simply $\varphi = \gamma + f$ and can, therefore, be expressed as a function of $\lambda, \gamma, \Lambda, \Gamma$.

From 3.2.35 we know $r = a(1 - e\cos u)$, such that we find

$$\begin{aligned} \frac{r}{a} &= \frac{r_0(\lambda, e)}{a} \\ &= 1 - e\cos\lambda + \frac{e^2}{2}(1 - \cos 2\lambda) + \frac{3}{8}e^3(\cos\lambda - \cos 3\lambda) + ... \end{aligned} \tag{3.2.67}$$

where $e = e(\Lambda, \Gamma)$ and $a = a(\Lambda) := \frac{\Lambda^2}{\mu^2 M}$.

3.3 The restricted, circular, planar three-body problem viewed as nearly-integrable Hamiltonian system

Let us go back to the system in 3.1.2. Since we shall study the *planar three-body problem*, we will *assume that the motion takes place on the plane hosting the Keplerian motion* *of* P_0 *and* P_1, i.e. the two primaries bodies. This amounts to require

$$u_3^{(i)} \equiv 0 \equiv \dot{u}_3^{(i)}, \qquad i = 0, 1, 2. \tag{3.3.1}$$

Notice that, since we are considering the *restricted* problem (i.e. we have set in 3.1.1 $m_2 = 0$), the "conservation laws" are those of the two-body system $P_0 - P_1$: in particular the total angular momentum is parallel to the u_3-axis (consistently with 3.3.1) and the center of mass (and hence the origin of the u-frame) is simply

$$m_0 u^{(0)} + m_1 u^{(1)} = 0. \tag{3.3.2}$$

Next, we pass, to *heliocentric coordinates*:

$$(x^{(1)}, 0) := u^{(1)} - u^{(0)}, \qquad (x^{(2)}, 0) := u^{(2)} - u^{(0)}, \qquad x^{(1)}, x^{(2)} \in \mathbb{R}^2 \tag{3.3.3}$$

which transform 3.1.2 into

$$\begin{aligned} \ddot{x}^{(1)} &:= -M_0 \frac{x^{(1)}}{|x^{(1)}|^3}, \qquad M_0 := m_0 + m_1, \\ \ddot{x}^{(2)} &:= -m_0 \frac{x^{(2)}}{|x^{(2)}|^3} - m_1 \frac{x^{(1)}}{|x^{(1)}|^3} - m_1 \frac{x^{(2)} - x^{(1)}}{|x^{(2)} - x^{(1)}|^3}. \end{aligned} \tag{3.3.4}$$

In view of 3.3.2 the motion in the original u-coordinates is related to the motion in the heliocentric coordinates by

$$u^{(0)} = \left(-\frac{m_1}{M_0} x^{(1)}, 0\right), \qquad u^{(1)} = \left(\frac{m_0}{M_0} x^{(1)}, 0\right), \qquad u^{(2)} = \left(x^{(2)} - \frac{m_1}{M_0} x^{(1)}, 0\right). \tag{3.3.5}$$

The equation in 3.3.4 describes the decoupled two-body system $P_0 - P_1$, which has been discussed in the previous section.

In the RCPTBP such motion is assumed to be *circular*.

It is convenient to fix the measure units for lenghts and masses so that the (fixed) distances between the two primary bodies is one and the sum of their masses is one:

$$\mathrm{dist}(P_0, P_1) = 1, \qquad M_0 := m_0 + m_1 = 1. \tag{3.3.6}$$

Notice that the period of revolution of P_0 and P_1 around their center of mass (the "year") is, in such units, 2π; the $x^{(1)}$-motion is simply (compare 3.2.40)

$$\widehat{x}^{(1)}_{circ}(t) = \widehat{x}^{(1)}_{circ}(t_0 + t) := \Big(\cos(t_0 + t), \sin(t_0 + t)\Big). \tag{3.3.7}$$

Even though the ststem of equations 3.3.4 is not a Hamiltonian system, the equations in 3.3.4 *taken separately* are Hamiltonian: we have already seen that the first of the two equations (the one for $x^{(1)}$) represent a two-body system; the second one represent a $2\frac{1}{2}$-degree-of-freedom Hamiltonian system with Hamiltonian

$$\widetilde{H}_1(x^{(2)}, X^{(2)}, t) := \frac{|X^{(2)}|^2}{2\mu} - \mu m_0 \frac{1}{|x^{(2)}|} + \mu m_1 \Big(x^{(2)} \cdot \widehat{x}^{(1)}_{circ}(t) \Big) - \mu m_1 \frac{1}{|x^{(2)} - \widehat{x}^{(1)}_{circ}(t)|}, \\ (x^{(2)}, X^{(2)}) \in \mathbb{R}^2 \backslash \{0\} \times \mathbb{R}^2, \quad t \in \mathbb{T}, \tag{3.3.8}$$

with respect to the standard symplectic form $dx^{(2)} \wedge dX^{(2)}$; here, $\mu > 0$ is a free parameter. To make the system 3.3.8 autonomous, we introduce a linear symplectic variable T conjugated to time $\tau = t$:

$$\widetilde{H}_1(x^{(2)}, X^{(2)}, \tau, T) := \frac{|X^{(2)}|^2}{2\mu} - \mu m_0 \frac{1}{|x^{(2)}|} + T + \mu m_1 \Big(x^{(2)} \cdot \widehat{x}^{(1)}_{circ}(\tau) \Big) - \mu m_1 \frac{1}{|x^{(2)} - \widehat{x}^{(1)}_{circ}(\tau)|}, \\ (x^{(2)}, X^{(2)}) \in \mathbb{R}^2 \backslash \{0\} \times \mathbb{R}^2, \quad (\tau, T) \in \mathbb{T}, \tag{3.3.9}$$

Remark 3.3.1. In the limiting case of a primary body with mass $m_1 = 0$, the Hamiltonian $\widetilde{H}_1$ describes a two-body system with total mass $M = m_0$ reflecting the fact that the asteroid mass has been set equal to zero.

If the mass m_1 does not vanish but it is small compared to the mass of m_0, the system 3.3.9 may be viewed as a *nearly-integrable system*. This is more transparent if we use, for the integrable part, the Delaunay variables introduced in prevoius section. Recall in particular that the symplectic transformation ϕ_D, mapping the Delaunay variables to the original Cartesian variables, depends parametrically also on μ and M and that M is now m_0. Next, we choose the free parameter μ so as to make the Keplerian part equal to $\frac{-1}{2\Lambda^2}$ (see 3.3.12 below) and we introduce also a perturbation parameter ε closely related to the mass m_1 of the primary body:

$$\mu := \frac{1}{m_0^{2/3}}, \qquad \varepsilon := \frac{m_1}{m_0^{2/3}} = \frac{m_1}{(1 - m_1)^{2/3}}. \tag{3.3.10}$$

Now, letting

$$(\lambda, \gamma, \Lambda, \Gamma) = \phi_D^{-1}(x^{(2)}, X^{(2)}), \\ \widehat{\phi}_D\Big((\lambda, \gamma, \Lambda, \Gamma), (\tau, T) \Big) := \Big(\phi_D(\lambda, \gamma, \Lambda, \Gamma), (\tau, T) \Big), \tag{3.3.11}$$

we find that

$$\widetilde{H}_2 := \widetilde{H}_1 \circ \widehat{\phi}_D = -\frac{1}{2\Lambda^2} + T + \varepsilon\left(x^{(2)} \cdot x^{(1)}_{circ}(\tau) - \frac{1}{|x^{(2)} - x^{(1)}_{circ}(\tau)|}\right), \tag{3.3.12}$$

where, of course, $x^{(2)}$ is now a function of the new symplectic variables.

Let us now analysize more in detail the perturbing function in 3.3.12. Recalling the definition of φ in 3.2.7, one sees that the angle between the rays $(0, x^{(2)})$ and $(0, x^{(1)}_{circ})$ is $\varphi - \tau$.

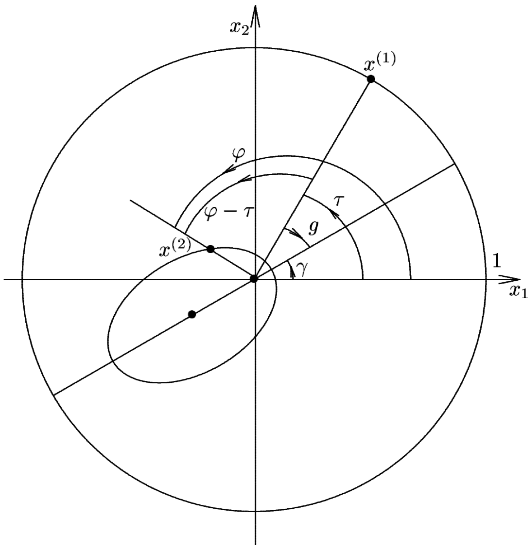

Figure 3.7: Angle variables for the RCPTBP

Therefore, if we let

$$r_2 := |x^{(2)}|, \tag{3.3.13}$$

we get

$$\begin{aligned} \widetilde{H}_2 &= -\frac{1}{2\Lambda^2} + T + \varepsilon\left(r_2 \cos(\varphi - \tau) - \frac{1}{\sqrt{1 + r_2^2 - 2r_2\cos(\varphi - \tau)}}\right) \\ &:= -\frac{1}{2\Lambda^2} + T + \varepsilon R(r_2, \varphi, \tau) \end{aligned} \tag{3.3.14}$$

Recall 3.2.59 that $\varphi = \gamma + f$ and that $f := \lambda + e\widetilde{f}(\lambda, e)$. Thus

$$\varphi - \tau = f + \gamma - \tau = \lambda + \gamma - \tau + e\widetilde{f}(\lambda, e). \tag{3.3.15}$$

Such relations suggests to make a new linear symplectic change of variables, by setting

$$\widehat{\phi}_{lin}^{-1} : \begin{cases} L = \Lambda, & G = \Gamma, \qquad \widehat{T} = T + \Gamma, \\ \ell = \lambda, & g = \gamma - \tau, \qquad \widehat{\tau} = \tau. \end{cases} \tag{3.3.16}$$

Now, recalling 3.2.67, 3.3.6 and 3.3.10 we see that

$$a = \frac{L^2}{\mu^2 M} = m_0^{1/3} L^2, \tag{3.3.17}$$

so that, in the new symplectic variables, it is:

$$\begin{aligned} &\varphi - \tau = f + g = f(\ell, e) + g = \ell + g + e\widetilde{f}(\ell, e), \\ &r_2 = r_0(\ell, e) = m_0^{1/3} L^2 (1 - e\cos u(\ell, e)). \end{aligned} \tag{3.3.18}$$

where, as above, $e = e(L, G) = \sqrt{1 - (G/L)^2}$.

Notice that the positions 3.3.10 and 3.3.6 define implicitly m_0 and hence $m_0^{1/3}$ as a (analytic) function of ε:

$$\begin{aligned} m_0(\varepsilon) &= 1 - \varepsilon + \frac{2}{3}\varepsilon^2 - \frac{1}{3}\varepsilon^3 + ..., \\ m_0(\varepsilon)^{1/3} &= 1 - \frac{\varepsilon}{3} + \frac{1}{9}e^2 - \frac{2}{81}e^3 + ... \end{aligned} \tag{3.3.19}$$

Thus, introducing the functions

$$\begin{aligned} &a_\varepsilon := a_\varepsilon(L) := m_0(\varepsilon)^{1/3} L^2 := \left(1 - \frac{\varepsilon}{3} + \frac{\varepsilon^2}{9} - \frac{2\varepsilon^3}{81} + ...\right) L^2 \\ &\rho_\varepsilon := \rho_\varepsilon(\ell, e(L, G)) := a_\varepsilon(L)\Big(1 - e\cos(u(\ell, e))\Big) \\ &\sigma := \sigma(\ell, e(L, G)) := e(L, G)\widetilde{f}(\ell, e(L, G)) = \left[2\arctan\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2}\right) - \ell\right] \end{aligned} \tag{3.3.20}$$

we get

$$\widetilde{H}_3 := \widetilde{H}_2 \circ \widehat{\phi}_{lin} = -\frac{1}{2L^2} + \widehat{T} - G + \varepsilon F_\varepsilon(\ell, g, L, G) \tag{3.3.21}$$

where

$$F_\varepsilon(\ell, g, e(L, G), a_\varepsilon(L)) := \rho_\varepsilon \cos(\ell + g + \sigma) - \frac{1}{\sqrt{1 + \rho_\varepsilon^2 - 2\rho_\varepsilon \cos(\ell + g + \sigma)}}. \tag{3.3.22}$$

The variable $\widehat{\tau}$ si cyclic (this is the reason for having introduced $\widehat{\phi}_{lin}$) and the linear constant of motion $\widehat{T}$ can be dropped from $\widetilde{H}_3$. The final form of the *Hamiltonian for the restricted, circular, planar, three-body-problem* is:

$$\begin{aligned} H_{rpc}(\ell, g, L, G) &:= -\frac{1}{2L^2} - G + \varepsilon F_\varepsilon(\ell, g, L, G) \\ &:= H_0(L, G) + \varepsilon F_\varepsilon(\ell, g, L, G); \end{aligned} \tag{3.3.23}$$

the phase space is the two dimensional torus $\mathbb{T}^2$ times the positive cone $\{0 < G < L\}$. The symplectic form is the standard $d\ell \wedge dL + dg \wedge dG$.

In order to apply in some way singular KAM theory, we firstly have to check non-degeneracy propriety of the unperturbed hamiltonian. Given a function $f : \mathbb{R}^2 \to \mathbb{R}$, for the rest of this work we are going to call Q_f the hessian matrix of f. In this way, with a simple calculation one obtain

$$Q_{H_0} = \begin{pmatrix} \partial_L^2 H_0(L, G) & \partial_L \partial_G H_0(L, G) \\ \partial_G \partial_L H_0(L, G) & \partial_G^2 H_0(L, G) \end{pmatrix} = \begin{pmatrix} -\frac{3}{L^4} & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.3.24}$$

So this is a degenarate matrix, and we cannot use it.

Our idea is to consider the square of the initial hamiltonian, knowing that from a dynamical point of view there is no difference.

Thanks to this result we can consider

$$\begin{aligned} H^2(\ell, g, L, G) = H_{rpc}^2(\ell, g, L, G) &:= \Big(H_0(L, G) + \varepsilon F_\varepsilon(\ell, g, L, G) \Big)^2 \\ &:= H_0^2 + \varepsilon \Big(2 H_0 F_\varepsilon + \varepsilon F_\varepsilon^2 \Big) \end{aligned} \tag{3.3.25}$$

and notice that the hessian of the unpertubed problem now is

$$Q_{H_0^2} = \begin{pmatrix} \frac{5}{L^6} + \frac{6G}{L^4} & -\frac{2}{L^3} \\ -\frac{2}{L^3} & 2 \end{pmatrix} \Longrightarrow \begin{cases} \mathrm{Tr}(Q) = 2 + \frac{5}{L^6} + \frac{6G}{L^4} \\ \det(Q) = 6(\frac{1}{L^6} + \frac{2G}{L^4}). \end{cases} \tag{3.3.26}$$

So one can calculates its eigenvalues

$$\lambda_{1,2} = \frac{1}{2}\Big[2 + \frac{5}{L^6} + \frac{6G}{L^4} \pm \sqrt{(2 + \frac{5}{L^6} + \frac{6G}{L^4})^2 - 24(\frac{1}{L^6} + \frac{2G}{L^4})} \Big] \tag{3.3.27}$$

since the term $2 + \frac{5}{L^6} + \frac{6G}{L^4} > 2$ and thanks to the positivity of L, G is really easy to check that the eigenvalues are both positive. So this integrable part is non degenerate and H_0 is also a strictly convex function of actions.

Now we have to concentrate on the perturbation, checking if this function belongs to the special class of analytic functions that we have studied in the first part of this work, i.e. if the potential belongs to $\mathbb{G}^n$. In order to do this we want to know the analytic form of this potential and its Fourier series.

The potential as written in 3.3.22 is quite untractable. So the first approximation that we are going to do is to neglect every term that is $O(\varepsilon)$ (motivated by the ε that multiplies the potential), so that we obtain

$$\begin{aligned} F_0(\ell, g, e(L,G), a(L)) &:= \rho\cos(\ell+g+\sigma) - \frac{1}{\sqrt{1+\rho^2-2\rho\cos(\ell+g+\sigma)}} \\ &:= \mathcal{A}_0(\ell, g, e(L,G), a(L)) + \mathcal{B}_0(\ell, g, e(L,G), a(L)). \end{aligned} \tag{3.3.28}$$

where

$$a(L) := L^2; \quad \rho(\ell, e(L,G), a(L)) := a(1-e\cos(u)) \tag{3.3.29}$$

As we have seen above, we can consider F_0 as a function of ℓ, g, a, e.

Now for some reasons that will become clear later, is useful to do the following symplectic change of variables that maps $(L, G, \ell, g) \to (\mathtt{L}, \mathtt{G}, \ell, \mathtt{g})$ as follows

$$\widetilde{\phi}^{-1} : \begin{cases} \ell \to \ell, & L \to \mathtt{L} := L - G \\ g \to \mathtt{g} := \ell + g, & G \to \mathtt{G} := G \end{cases} \tag{3.3.30}$$

In this way

$$\begin{aligned} F_0 \circ \widetilde{\phi}(\ell, \mathtt{g}, a(\mathtt{L},\mathtt{G}), e(\mathtt{L},\mathtt{G})) &= \rho\cos(\mathtt{g}+\sigma) - \frac{1}{\sqrt{1+\rho^2-2\rho\cos(\mathtt{g}+\sigma)}} \\ H_0 \circ \widetilde{\phi}(\mathtt{L},\mathtt{G}) &= -\frac{1}{2(\mathtt{L}+\mathtt{G})^2} - \mathtt{G} \end{aligned} \tag{3.3.31}$$

where

$$a(\mathtt{L},\mathtt{G}) := (\mathtt{L}+\mathtt{G})^2; \quad \rho(\ell, a(\mathtt{L},\mathtt{G}), e(\mathtt{L},\mathtt{G})) := a(1-e\cos(u)); \quad \sigma = \sigma(\ell, e(\mathtt{L},\mathtt{G})). \tag{3.3.32}$$

Due to the simplicity of this change of variable, for the rest of the work I will omit the presence of $\widetilde{\phi}$ calling $F_0 \circ \widetilde{\phi} := F_0$ and $H_0 \circ \widetilde{\phi} := H_0$.

3.3.1 Analytic continuation

In this section we find the analyticity radii of the perturbing function, i.e. we will find the size of the complex neighborhood in which this function is analytic. Since our model depends in a crucial way by the solution to the Kepler equation, fistly we will anaylize analyticity of the eccentric anomaly, and then we will consider the complete potential.

Proposition 3.3.1. *The solution to the Kepler Equation, i.e. the Eccentric Anomaly* $u : (\ell, e) \in \mathbb{T} \times (0,1) \to \mathbb{T}$ *admits an analytic complex continuation*

$$u : \mathbb{T}_s \times \{e \in \mathbb{C} : 0 < |e| < e_0\} \to \mathbb{T}_\sigma \tag{3.3.33}$$

such that

$$\begin{aligned} \sigma &= \operatorname{arccosh}\left(\sqrt{\frac{1}{2}\left(1 + \frac{1}{|e|^2}(1 + \sqrt{(|e|^2+1)^2 - 4e^2})\right)}\right) \\ s &= \sigma - \sqrt{(\operatorname{Im} e)^2 \cosh^2(\sigma) + e^2 \sinh^2(\sigma)} \end{aligned} \tag{3.3.34}$$

and $|e|^2 = (\operatorname{Re} e)^2 + (\operatorname{Im} e)^2$.

Proof. The map

$$\mathcal{W} : \mathbb{T} \to \mathbb{T},\ u \to u - e \sin u \tag{3.3.35}$$

is diffeomorphic if and only if $|e| < 1$ (it is an easy check studying the derivative), that is our case. Let us now consider its analytic continuation. We will call the complex anomaly $\widetilde{u} = u + iu'$ and complex eccentrictiy $\widetilde{e} = e + ie'$

$$\widetilde{\mathcal{W}} : \mathbb{T}_{\mathbb{C}} \to \mathbb{T}_{\mathbb{C}},\ \widetilde{u} \to \widetilde{\ell} = \widetilde{u} - \widetilde{e} \sin \widetilde{u} \tag{3.3.36}$$

We are interested in determining if $\widetilde{\mathcal{W}}$ is an analytic diffeomorphism (at least locally), in every point of a set $\mathbb{T} \times (-u'_{max}, u'_{max})$. Hence, we want to determine the singular points of $\widetilde{\mathcal{W}}$. Consider the function $(\operatorname{Re} \widetilde{\mathcal{W}}, \operatorname{Im} \widetilde{\mathcal{W}}) : (\operatorname{Re} \widetilde{u}, \operatorname{Im} \widetilde{u}) \to (\operatorname{Re} \widetilde{\ell}, \operatorname{Im} \widetilde{\ell})$, it has the same singular points as $\widetilde{\mathcal{W}}$. This function is defined by the formulas

$$\begin{cases} \operatorname{Re} \widetilde{\mathcal{W}} = \ell = u - e \sin(u) \cosh(u') + e' \cos(u) \sinh(u') \\ \operatorname{Im} \widetilde{\mathcal{W}} = \ell' = u' - e' \sin(u) \cosh(u') - e \cos(u) \sinh(u') \end{cases} \tag{3.3.37}$$

The derivative is $\widetilde{\mathcal{W}}'(\widetilde{u}) = 1 - \widetilde{e} \cos(\widetilde{u})$ so that there is a singular point if and only if

$$\begin{cases} \operatorname{Re} (\widetilde{e} \cos(\widetilde{u})) = 1 \\ \operatorname{Im} (\widetilde{e} \cos(\widetilde{u})) = 0. \end{cases} \tag{3.3.38}$$

In the real variables it gives

$$\begin{cases} e \sin u \sinh u' = e' \cos u \cosh u' \\ e \cos u \cosh u' + e' \sin u \sinh u' = 1. \end{cases} \tag{3.3.39}$$

The first case that one has to take into account is when the eccentricity is real, i.e. $e' = 0$. In this case, if $0 < |e| < 1$ there exists two singular points $(u, u') = (0, \pm \operatorname{arccosh}(1/e))$ for $e > 0$ and respectively $(\pi, \pm \operatorname{arccosh}(1/e))$ for $e < 0$. Now, let us concentrate to the complex case, i.e. $e' \neq 0$. For the sake of simplicity, we consider that $e \geqslant 0$. Assume $u \in [0, 2\pi], e' > 0, u' > 0$ (by symmetry of the equations, the study of this case is enough to compute the other case) the first equation gives

$$e^2(1 - \cos^2(u)) \sinh^2(u') = (e')^2 \cos^2(u) \cosh^2(u'). \tag{3.3.40}$$

such that

$$\begin{cases} \cos(u) = \frac{e \sinh(u')}{\sqrt{e'^2 \cosh^2(u') + e^2 \sinh^2(u')}} \\ \sin(u) = \frac{e' \cosh(u')}{\sqrt{e'^2 \cosh^2(u') + e^2 \sinh^2(u')}}. \end{cases} \tag{3.3.41}$$

Notice that there exists two solutions (obtained by adding π to the first solution) of the first equation in $\mathbb{T}$. The second equation can now be written as

$$e^2 \cosh(u') \sinh(u') + e'^2 \cosh(u') \sinh(u') = \sqrt{e'^2 \cosh^2(u') + e^2 \sinh^2(u')} \tag{3.3.42}$$

Let $|e|^2 = e^2 + e'^2 > 0$, squaring this equation and using hyperbolic trigonometry identities, we obtain a unique possible values of u' that is

$$\sigma =: u'_{max} = \sigma = \operatorname{arccosh}\left(\sqrt{\frac{1}{2}\left(1 + \frac{1}{|e|^2}(1 + \sqrt{(|e|^2 + 1)^2 - 4e^2})\right)}\right). \tag{3.3.43}$$

Now, for $t \leqslant \sigma$ consider

$$\begin{aligned} \widetilde{\mathcal{W}} : U_t =: \mathbb{T} \times (-t, t) &\to g(U) \\ (u, u') &\to (u - e \sin(u) \cosh(u') + e' \cos(u) \sinh(u'), \\ &\qquad u' - e' \sin(u) \cosh(u') - e \cos(u) \sinh(u')) \end{aligned} \tag{3.3.44}$$

we want to find $s =: \ell'_{max}$ such that g^{-1} is a analytic diffeomorphism on the set $\mathbb{T} \times (-s, s)$ and its image is contained in the set U_σ. It is therefore enough to concentrate on the minimum of ℓ' when u varies. Besides, we show that this value is maximum for $\sigma =: u'_{max}$, which means that the maximum value $s =: \ell'_{max}$ on which there exists a diffeomorphism is as well a singular point of Kepler's equation. Indeed, we have

$$\ell' = u' - e' \sin(u) \cosh(u') - e \cos(u) \sinh(u'), \tag{3.3.45}$$

and for $u' = u'_{max} =: \sigma$, the location of a minimum of this function is a point for which u and u' goes to zero at the same time. Therefore, the width $\ell'_{max} =: s$ corresponds to the minimal value of ℓ' such that the inverse map has a singular point.

Now, equations in 3.3.41 was giving:

$$\begin{cases} \cos(u) = \pm\frac{e\sinh(u')}{\sqrt{e'^2\cosh^2(u')+e^2\sinh^2(u')}} \\ \sin(u) = \pm\frac{e'\cosh(u')}{\sqrt{e'^2\cosh^2(u')+e^2\sinh^2(u')}}. \end{cases} \tag{3.3.46}$$

injecting in the equation of ℓ', it gives

$$\ell' = u' \mp \sqrt{e'^2\cosh^2(u') + e^2\sinh^2(u')}. \tag{3.3.47}$$

For $\sqrt{e'^2\cosh^2(u') + e^2\sinh^2(u')} \geqslant 0$ the minimum of the right term is $\ell' = u' - \sqrt{e'^2\cosh^2(u') + e^2\sinh^2(u')}$. As said before, this value is maximal in the set $u' \in [-\sigma, \sigma]$ when $u' = u'_{max} =: \sigma$.

The width s is hence given by the formula

$$s := \ell'_{max} = \sigma - \sqrt{e'^2\cosh^2(\sigma) + e^2\sinh^2(\sigma)}. \tag{3.3.48}$$

Notice that for small values of e, e' (that is our case) this value is positive. $\square$

Proposition 3.3.2. *Consider the perturbing function in 3.3.31*

$$F_0(\ell, \mathrm{g}, a, e) = \rho\cos(\mathrm{g} + \sigma) - \frac{1}{\sqrt{1 + \rho^2 - 2\rho\cos(\mathrm{g} + \sigma)}}. \tag{3.3.49}$$

This function admits a complex continuation that is analytical in $\{\ell \in \mathbb{T}_s\} \times \{\mathrm{g} \in \mathbb{T}_\beta\} \times \{e \in \mathbb{C} : 0 < |e| < e_0\} \times \{a \in \mathbb{C} : 0 < |a| < a_0\}$ *where*

$$\begin{aligned} \beta &= \operatorname{arccosh}\left(\frac{1 + a^2}{2a}\right) \\ s &= \sigma - \sqrt{(\operatorname{Im} e)^2\cosh^2(\sigma) + e^2\sinh^2(\sigma)} \end{aligned} \tag{3.3.50}$$

and

$$\sigma = \operatorname{arccosh}\left(\sqrt{\frac{1}{2}\left(1 + \frac{1}{|e|^2}(1 + \sqrt{(|e|^2 + 1)^2 - 4e^2})\right)}\right). \tag{3.3.51}$$

Proof. Studying the perturbing function for $\ell = 0$, we have singularities for

$$\cos\tilde{\mathrm{g}} = \frac{1 + a^2}{2a} \tag{3.3.52}$$

solving for $\widetilde{\mathrm{g}} = \mathrm{g} + i\mathrm{g}'$ we finds

$$\begin{cases} \mathrm{Re}\,(\cos\widetilde{\mathrm{g}}) = \frac{1+a^2}{2a} \Rightarrow \cos\mathrm{g}\cosh\mathrm{g}' = \frac{1+a^2}{2a} \\ \mathrm{Im}\,(\cos\widetilde{\mathrm{g}}) = 0 \qquad \Rightarrow \sin\mathrm{g}\sinh\mathrm{g}' = 0. \end{cases} \tag{3.3.53}$$

In this way, as we have done in the above proposition, we can find the value

$$\beta := \mathrm{g}'_{max} = \operatorname{arccosh}\left(\frac{1+a^2}{2a}\right) \tag{3.3.54}$$

For the width s, our perturbing function depends on ℓ via the eccentric anomaly u, so it is a analytic composition of that function, and that is why the analyticity width is the same as the eccentric anomaly. $\square$

3.3.2 Expansion of the perturbing function

[1]Since our conditions on potential regard mainly its Fourier coefficients, in this section we want to obtain the Fourier expansion of the perturbing function. There is a standard and simpler way of doing that, and it is based on the well-known *Laguerre polynomials* expressed in terms of the *Hansen coefficients*.

In order to do this, we have start considering the hamiltonian in 3.3.14 before using the Delaunay variables, with perturbing function

$$R(r_2, \varphi, \tau) = r_2\cos(\varphi - \tau) - \frac{1}{\sqrt{1 + r_2^2 - 2r_2\cos(\varphi - \tau)}} \tag{3.3.55}$$

where the expression for r_2 is shown in 3.2.67, while in Delaunay variables $\varphi - \tau = g + \ell + \sigma(\ell, e)$.

For this kind of function that arises from gravitational force in 1782 Legendre finds an useful expansion on some polynomials (then extended by Laplace) such that they can be expressed by the generating formula

$$\frac{1}{\sqrt{1 + z^2 - 2z\xi}} = \sum_{n=0}^{\infty} z^n P_n(\xi); \quad (\text{in our case } \xi = \cos(\varphi - \tau), z = r_2) \tag{3.3.56}$$

or equivalently by the differential equation (which we are most often used to)

$$-\frac{d}{d\xi}[(1 - \xi^2)\frac{d}{d\xi}P_n(\xi)](\xi) = n(n+1)P_n(\xi). \tag{3.3.57}$$

[1]I want to thank Giacomo Longaroni for the useful discussions about this expansion.

These Legendre polynomials that we have called P_n can be expressed as

$$P_n(\xi) = \sum_{k=0}^{[n/2]} p_{n,k} \xi^{n-2k} \tag{3.3.58}$$

where the notation $[n/2]$ represent the largest integer less than or equal to $n/2$ (namely, the floor of $n/2$) and

$$p_{n,k} = \frac{(-1)^k}{2^n} \frac{(2n-2k)!}{k!(n-k)!(n-2k)!}. \tag{3.3.59}$$

For instance, the explicit expression of the first few Legendre polynomials when the argument ξ is equal to $\cos(\varphi - \tau)$ (as the form of the RCPTBP perturbing function suggests) are

$$\begin{aligned}
P_0(\cos(\varphi-\tau)) &= 1; \qquad P_1(\cos(\varphi-\tau)) = \cos(\varphi-\tau) \\
P_2(\cos(\varphi-\tau)) &= \frac{1}{4} + \frac{3}{4}\cos 2(\varphi-\tau) \\
P_3(\cos(\varphi-\tau)) &= \frac{3}{8}\cos(\varphi-\tau) + \frac{5}{8}\cos 3(\varphi-\tau) \\
P_4(\cos(\varphi-\tau)) &= \frac{9}{64} + \frac{5}{16}\cos 2(\varphi-\tau) + \frac{35}{64}\cos 4(\varphi-\tau) \\
P_5(\cos(\varphi-\tau)) &= \frac{54}{64}\cos(\varphi-\tau) + \frac{35}{128}\cos 3(\varphi-\tau) + \frac{63}{128}\cos 5(\varphi-\tau).
\end{aligned} \tag{3.3.60}$$

In this way one can obtain an expasion

$$\begin{aligned}
R(r_2,\varphi,\tau) &= r_2\cos(\varphi-\tau) - P_0(\cos(\varphi-\tau)) - r_2 P_1(\cos(\varphi-\tau)) - \sum_{j=2}^{\infty} P_j(\cos(\varphi-\tau)) r_2^j \\
&= -(1 + \sum_{j=2}^{\infty} P_j(\cos(\varphi-\tau)) r_2^j)
\end{aligned} \tag{3.3.61}$$

In order to obtain some better expansion, one can do a classical computation (see [75],[73]) that is for all $r_2 \in [0,1)$

$$\begin{aligned}
(1 + r_2^2 - 2r_2\cos(\varphi-\tau))^{-1/2} &= \big(1 - r_2 \exp\left(-i(\varphi-\tau)\right)\big)^{-1/2} \big(1 - r_2\exp(-i(\varphi-\tau))\big)^{-1/2} \\
&= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(2p)!(2q)!}{2^{2p+2q}(p!)^2(q!)^2} \exp\big(i(p-q)(\varphi-\tau)\big) r_2^{p+q}
\end{aligned} \tag{3.3.62}$$

and, with $n = p + q$, and after changing q to $n - q$,

$$= \sum_{n=0}^{\infty} \left(\sum_{q=0}^{n} \frac{(2q)!(2n-2q)!}{2^{2n}(q!)^2((n-q)!)^2} \exp\big(i(2q-n)(\varphi-\tau)\big) \right) r_2^n. \tag{3.3.63}$$

Thus, comparing two expression above with 3.3.56, for $0 \leqslant q \leqslant n$ we have

$$P_n(\cos(\varphi - t)) := \sum_{q=0}^{n} \widetilde{\mathcal{F}}_{q,n} \exp\big(i(2q-n)(\varphi-\tau)\big) := \sum_{q=0}^{n} \frac{(2q)!(2n-2q)!}{2^{2n}(q!)^2((n-q)!)^2} \exp\big(i(2q-n)(\varphi-\tau)\big) \tag{3.3.64}$$

such that the perturbing function becomes

$$\begin{aligned} R &= -1 - \sum_{n=2}^{\infty} \left(\frac{r_2}{a}\right)^n \left(\sum_{q=0}^{n} \widetilde{\mathcal{F}}_{q,n} \exp\big(i(2q-n)(\varphi-\tau)\big) \right) a^n \\ &= -1 - \sum_{n=2}^{\infty} \sum_{q=0}^{n} \mathcal{F}_{q,n} \exp\big(i(2q-n)(\varphi-\tau)\big)\, a^n; \qquad \mathcal{F}_{q,n} := \widetilde{\mathcal{F}}_{q,n} \left(\frac{r_2}{a}\right)^n, \end{aligned} \tag{3.3.65}$$

using Delaunay variables in 3.3.15 we write $\varphi - \tau = f + g$ and the above expression becomes

$$F_0 = -1 - \sum_{n=2}^{\infty} \sum_{q=0}^{n} \mathcal{F}_{q,n} \exp\big(i(2q-n)f\big) \exp\big(i(2q-n)g\big)\, a^n. \tag{3.3.66}$$

where F_0 is the name given to the perturbing function R when is composited with $\widehat{\phi}_{lin}^{-1}$ as shown in 3.3.16. Now the crucial point is that the coefficients $\mathcal{F}_{q,n}$ can be expressed in terms of Delaunay variables $(\ell, g, a(L), e(L,G))$ via the Hansen coefficients $X_k^{n,m}(e)$ defined for $n, m \in \mathbb{Z}$ such that

$$\left(\frac{r_2}{a}\right)^n \exp\big(imf\big) = \sum_{k=-\infty}^{+\infty} X_k^{n,m}(e) \exp\big(ik\ell\big). \tag{3.3.67}$$

In the next section we will review some key features and proprieties of these coefficients. Thus we have

$$\begin{aligned} F &= -1 - \sum_{n=2}^{\infty} \sum_{q=0}^{n} \sum_{k=-\infty}^{+\infty} \widetilde{\mathcal{F}}_{q,n} X_k^{n,2q-n}(e) \exp\big(i[(2q-n)g + k\ell]\big)\, a^n. \\ &= -1 - \sum_{n=2}^{\infty} {\sum_{r=-n}^{n}}' \sum_{k=-\infty}^{+\infty} \widetilde{\mathcal{F}}_{\frac{r+n}{2},n} X_k^{n,r}(e) \exp\big(i(rg + k\ell)\big)\, a^n \\ &= -1 - \sum_{n=2}^{\infty} {\sum_{r=-n}^{n}}' \sum_{k=-\infty}^{+\infty} \frac{(r+n)!(n-r)!}{2^{2n}((\frac{r+n}{2})!)^2((\frac{n-r}{2})!)^2} X_k^{n,r}(e) \exp\big(i(rg + k\ell)\big)\, a^n \end{aligned} \tag{3.3.68}$$

where $\sum'$ indicates that the sum is over every term between $-n$ and n separated by 2, i.e. $r = 2j$ with $j = -\frac{n}{2}, ..., \frac{n}{2}$.
Thanks to the parity of the perturbing function, we just know that

$$\begin{aligned} F_0(a,e,\ell,g) &= -1 - \sum_{n=2}^{\infty} \sum_{r=-n}^{n}{}' \sum_{k=-\infty}^{+\infty} \frac{(r+n)!(n-r)!}{2^{2n}((\frac{r+n}{2})!)^2((\frac{n-r}{2})!)^2} X_k^{n,r}(e) a^n \cos(rg+k\ell) \\ &= -1 - \sum_{n=2}^{\infty} \sum_{p=-n/2}^{n/2} \sum_{k=-\infty}^{+\infty} \frac{(2p+n)!(n-2p)!}{2^{2n}((p+\frac{n}{2})!)^2((\frac{n}{2}-p)!)^2} X_k^{n,2p}(e) a^n \cos(2pg+k\ell) \end{aligned} \tag{3.3.69}$$

and applying the change of variables $j' = n + r$; $m' = n - r$ such that $j', m' = 0, 2, 4, 6..., +\infty$ (they count only the even number) and $j' + m' = 2n \geqslant 4$, one can obtain (keeping in mind that $r = \frac{j'-m'}{2}$ and $n = \frac{j'+m'}{2}$)

$$F_0(a,e,\ell,g) = -1 - \sum_{\substack{j',m'=0 \\ j'+m'\geqslant 4}}^{\infty}{}' \sum_{k=-\infty}^{+\infty} \frac{j'!m'!}{2^{j'+m'}((\frac{j'}{2})!)^2((\frac{m'}{2})!)^2} X_k^{\frac{j'+m'}{2},\frac{j'-m'}{2}}(e) a^{\frac{j'+m'}{2}} \cos\left(\frac{j'-m'}{2}g + k\ell\right) \tag{3.3.70}$$

that with a rescalation $j' = 2j, m' = 2m$ becomes

$$F_0(a,e,\ell,g) = -1 - \sum_{\substack{j,m=0 \\ j+m\geqslant 2}}^{\infty} \sum_{k=-\infty}^{+\infty} \frac{(2j)!(2m)!}{4^{j+m}(j!)^2(m!)^2} X_k^{j+m,j-m}(e) a^{j+m} \cos\Big((j-m)g + k\ell\Big) \tag{3.3.71}$$

A crucial propriety that we will understand in the section dedicated to Hansen coefficients is that $X_k^{n,m} = X_{-k}^{n,-m}$.
So in general we have reached our expansion in Fourier coefficients such that

$$\begin{aligned} F_0(a,e,\ell,g) &= -1 - \sum_{\substack{j,m=0 \\ j+m\geqslant 2}}^{\infty} \sum_{k=-\infty}^{+\infty} f_{j,m,k}(a,e) \cos(k\ell + (j-m)g); \\ f_{j,m,k}(a,e) &:= \widetilde{f}_{j,m} X_k^{j+m,j-m}(e) a^{j+m} \\ &:= \frac{(2j)!(2m)!}{4^{j+m}(j!)^2(m!)^2} X_k^{j+m,j-m}(e) a^{j+m} \end{aligned} \tag{3.3.72}$$

where F_0 is the name given to the perturbing function R when is composited with $\widehat{\phi}_{lin}^{-1}$ as shown in 3.3.16.

From 3.3.69, fixing (r, k) in $\mathbb{N} \times \mathbb{Z}$ one can obtain the expression for the *Fourier coefficient of the perturbing function*

$$f_{r,k} = \sum_{n=r}^{\infty}{}' \frac{2(r+n)!(n-r)!}{2^{2n}((\frac{r+n}{2})!)^2((\frac{n-r}{2})!)^2} X_k^{n,r}(e) a^n \tag{3.3.73}$$

Remark 3.3.2. Thanks to analytical proprieties of this function, one can compute the above coefficients also with its taylor coefficients for $|e|, |a| < 1$ near $(e, a) = (0, 0)$, i.e.

$$F_0(a, e, \ell, g) = \sum_{n,m \geqslant 0} f_{n,m}(a, e) \cos(n\ell + mg) = \sum_{j,k \geqslant 0} F^0_{j,k}(\ell, g) e^j a^k \tag{3.3.74}$$

where $F^0_{j,k}$ represent the coefficients of the Taylor series of F_0.

Thanks to [54] we can use the following expansion for Hansen coefficients (for $k = m+s$) if $s = k - m \geqslant 0$

$$\begin{aligned} X^{n,m}_{m+s}(e) = & (-1)^s \left(\frac{e}{2}\right)^s \sum_{t=0}^{\infty} \left\{ \sum_{j=0}^{t} \sum_{p=0}^{j} \binom{n+m+1}{j-p} \frac{(m+s)^p}{p!} \sum_{q=0}^{s+j} \binom{n-m+1}{s+j-q} \frac{(m+s)^q}{q!} (-1)^q \right. \\ & \left. \left[2\binom{2t-n+s-p-q-2}{t-j} - \binom{2t-n+s-p-q-1}{t-j} \right] \right\} \left(\frac{e}{2}\right)^{2t} \end{aligned} \tag{3.3.75}$$

where binomial coefficient $\binom{-\mu}{p}$ where $\mu > 0$ must be computed as being equal to $(-1)^p \binom{\mu+p-1}{p}$, p being always positive.

If $s = k - m < 0$, one can use the fact that $X^{n,m}_k = X^{n,-m}_{-k}$ to calculate $X^{n,m}_{m+s}(e) = X^{n,-m}_{-m-s}(e)$ using the formula in 3.3.75. In this way it is easy to see that, for $s = k-m \geqslant 0$, the leading term is for $t = 0$, i.e. it is e^{k-m}, and the same holds for $-s = m - k \geqslant 0$ with the right change of sign, namely that the leading term is e^{m-k}.

So thanks to this formula, it is clear that $X^{n,m}_k(e) = o(e^{|k-m|})$, i.e. using 3.3.73, that

$$f_{r,k}(a, e) = t_{r,k}\, e^{|r-k|}\, a^r \left[1 + \mathcal{O}(e^2; a)\right] \tag{3.3.76}$$

and so for e, a small enough the form of the coefficient $t_{r,k}$ is crucial. From expansions 3.3.73 we also know that

$$t_{r,k} = \frac{2(2r)!}{2^{2r}(r!)^2} [X^{r,r}_k] \tag{3.3.77}$$

where $[X^{r,r}_k]$ indicates the coefficient that multiplies the term $e^{|k-r|}$ in the e-power expansion of the Hansen coefficient.

As we can see from 3.3.75, in the series expansion of the Hansen coefficient, the eccentricity has a power expressed by $s+2t$, so in order to evaluate the coefficient of the leading term ($e^{|k-r|}$), we have to look at the term with

$$s+2t=|k-r|$$

so it is clear that we have to distinguish between two cases:
Case $\mathbf{r}-\mathbf{k}<\mathbf{0}$: for our intent, w.r.t. 3.3.75 we set $n=m=r$ and $s=k-r$, and if we want the term $t_{r,k}$ we need $s+2t=|k-r|$ that means, for $r<k$, $k-r+2t=k-r \Rightarrow t=0$.

$$[X_k^{r,r}]=\frac{(-1)^{k-r}}{2^{k-r}}\binom{2r+1}{0}\sum_{q=0}^{k-r}\binom{1}{k-r-q}\frac{k^q}{q!}(-1)^q\left[2\binom{k-2r-q-2}{0}-\binom{k-2r-q-1}{0}\right]. \tag{3.3.78}$$

so that the first binomial coefficient that involves q is different from zero only if $q=k-r$ or $q=k-r-1$ and such that

$$\begin{aligned}[X_k^{r,r}]&=\frac{(-1)^{k-r}}{2^{k-r}}\Big[\frac{k^{k-r}}{(k-r)!}(-1)^{k-r}-\frac{k^{k-r}}{(k-r)!}\frac{k-r}{k}(-1)^{k-r}\Big]=\frac{k^{k-r}}{2^{k-r}\,(k-r)!}\Big[1-\frac{k-r}{k}\Big]\\&=-\frac{1}{2^{k-r}}\frac{r\,k^{k-r}}{k\,(k-r)!}.\end{aligned} \tag{3.3.79}$$

Finally for this case ($k>r$) we have obtained

$$t^-_{r,k}=-\frac{(2r)!}{2^{r+k-1}(r!)^2}\frac{r\,k^{k-r}}{k\,(k-r)!}. \tag{3.3.80}$$

Case $\mathbf{r}-\mathbf{k}>\mathbf{0}$: Now, setting always $s=k-r$, in order to control only the dominant term, we have to ask that $s+2t=|k-r|=r-k$ namely that $t=r-k$. So from 3.3.75 we have

$$\begin{aligned}[X_k^{r,r}]=&(-1)^{k-r}\left(\frac{1}{2}\right)^{r-k}\Bigg\{\sum_{j=0}^{r-k}\sum_{p=0}^{j}\binom{2r+1}{j-p}\frac{k^p}{p!}\sum_{q=0}^{k-r+j}\binom{1}{k-r+j-q}\frac{k^q}{q!}(-1)^q\\&\left[2\binom{-k-p-q-2}{r-k-j}-\binom{-k-p-q-1}{r-k-j}\right]\Bigg\}.\end{aligned} \tag{3.3.81}$$

We start analyzing the sum in q; in order to make the binomial coefficient different from zero, one finds the condition $k-r+j\geqslant 0$, i.e. $j\geqslant r-k$, but since the index j

goes from 0 to $r-k$, the only possible contribution is from $j = r-k$, and so $q = 0$. In this way the expression becomes

$$\begin{aligned}[X_k^{r,r}] &= (-1)^{k-r+1}\left(\frac{1}{2}\right)^{r-k}\left\{\sum_{p=0}^{r-k}\binom{2r+1}{r-k-p}\frac{k^p}{p!}\left[2\binom{-k-p-2}{0}-\binom{-k-p-1}{0}\right]\right\}\\ &= \frac{(-1)^{k-r+1}}{2^{r-k}}\left\{\sum_{p=0}^{r-k}\binom{2r+1}{r-k-p}\frac{k^p}{p!}\right\}.\end{aligned} \tag{3.3.82}$$

Notice that this sum has contributes different from zero until $2r+1 \geqslant r-k-p$, i.e. $p \geqslant -(r+k+1)$. So if $r+k+1 < 0$ the contributions different from zero start from $p = -(r+k+1)$.

So finally we have found the following

$$t^+_{r,k} = (-1)^{k-r+1}\frac{(2r)!}{2^{3r-k-1}(r!)^2}\sum_{p=0}^{r-k}\binom{2r+1}{r-k-p}\frac{k^p}{p!}. \tag{3.3.83}$$

So finally

$$f_{r,k}(a,e) = t_{r,k}\, e^{|r-k|}\, a^r \left[1+\mathcal{O}(e^2;a)\right] \tag{3.3.84}$$

with

$$t_{r,k} = \begin{cases} -\frac{(2r)!}{2^{r+k-1}(r!)^2}\frac{r\,k^{k-r}}{k\,(k-r)!} & \text{if} \quad k \geqslant r \\ (-1)^{k-r+1}\frac{(2r)!}{2^{3r-k-1}(r!)^2}\sum\limits_{p=0}^{r-k}\binom{2r+1}{r-k-p}\frac{k^p}{p!} & \text{if} \quad r > k. \end{cases} \tag{3.3.85}$$

Where $k = r$ the expressions are completely equivalent, and obviously when $k = 0$ all terms of the sum are null except the one with $p = 0$ when we use the convention $0^0 = 1$.

3.4 On the Hansen Coefficients

In this section we are going to review the standard theory about Hansen coefficients used in the expansion of the RCPTBP perturbing function.

Hansen coefficient(Cefola [59]) is an important class of functions which frequently occur in many branches of Celestial Mechanics such as planetary theory (Newcomb [66]) and artificial satellite motion (Allan[53]; Hughes[62]). Moreover , there are extensive forms of Hansen like expansions (Klioner et. al. [64] ; Sharaf [69], [70]) which play important roles in the expansion theories of elliptic motion.

As we have seen in the expansion of the RCPTBP perturbing function, Giacalia ([60]) noted that Hansen's coefficients appears in satellite theory in expression of the disturbing function due to the primary and due to the presence of a third body and they are usually called Eccentricity Functions. He derived recurrence relation for these functions and their derivatives, as they appear in the evaluation of geopotential and third body perturbations of an artificial satellite.

Also in [61], he proved Hansen's coefficients for Fourier series in terms of the mean anomaly correspond to a rotation of the orbital plane proportional to the eccentricity of the orbit. They are given in terms of Bessel functions and generalized associated Legendre functions which arise through the transformation of spherical harmonics under rotation. In [63], Hughes computed tables of analytical expressions for the Hansen coefficients $\mathrm{X}_o^{n,\pm m}(e)$ and $\mathrm{X}_o^{-(n+1),\pm m}(e)$ when $1 \leqslant n \leqslant 30$ and $0 \leqslant m \leqslant n$.

In [56], Branham derived a recursive calculation of Hansen coefficients which are used in expansions of elliptic motion by three methods: Tisserand's method, Von Zeipel-Andoyer method with explicit representation of the polynomials required to compute the Hansen coefficients and von Zeipel-Andoyer method with the value of the polynomials calculated recursively. Vakhidov ([72]) studied in detail efficient approximations of Hansen coefficients using polynomials in terms of the eccentricity.

He and Zhang ([65]) used Hansen coefficients to compute general perturbations of the asteroids of Flora group due to Jupiter. Breiter et.al ([57]) show that most of the theory of Hansen coefficients remains valid for $\mathrm{X}_k^{\gamma j}$, when γ is a real number, also, the generalized coefficients can be applied in a variety of perturbed problems that involve some drag effects.

Sadov ([68]) deals analytically with the properties of Hansen's coefficients in the theory of elliptic motion considered as functions of the parameter $\eta = \sqrt{1-e^2}$ where e is the eccentricity.

We are going to use this coefficients only to study some behaviours of the Fourier coefficients of the disturbing functions, but in order to do this, we firstly recollect some crucial proprieties of these coefficients.

3.4.1 Computation of $X_0^{n,m}$ and $X_0^{-(n+1),m}$

As we have seen before in 3.3.67 the general Hansen coefficient $X_k^{n,m}(e)$ is a function of the orbital eccentricity and is defined bu the generating function

$$\left(\frac{r}{a}\right)^n \mathrm{e}^{imf} = \sum_{k=-\infty}^{+\infty} X_k^{n,m}(e)\ \mathrm{e}^{ik\ell}. \tag{3.4.1}$$

where n, m and k are integers which may be positive or negative, r the radius vector, a the semi-major axis, e the orbital eccentricity, f the true anomaly and ℓ the mean anomaly. The individual coefficients being given by the integral

$$X_k^{n,m}(e) = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^n \cos(mf - k\ell)d\ell. \tag{3.4.2}$$

that shows easily that $X_k^{n,m} = X_{-k}^{n,-m}$.

A number of authors have given extensive table of these coefficients, the most important are by Cayley (1861) and Newcomb (1895) and Cherniack (1972) but they are quite tedious and time consuming. We prefer following the expansion by Hughes [63] starting by compute $X_0^{n,m}$ and $X_0^{-(n+1),m}$ for $1 \leqslant n \leqslant 30$ and $0 \leqslant m \leqslant n$.

If we put $k = 0$, then the integrals 3.4.2 for $X_0^{n,m}$ and $X_0^{-(n+1),m}$ become

$$X_0^{n,m} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^n \cos(mf)d\ell, \qquad X_0^{-(n+1),m} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{a}{r}\right)^{n+1} \cos(mf)d\ell. \tag{3.4.3}$$

On putting $m = -m$ into 3.4.3 it is obvious that $X_0^{n,m} = X_0^{n,-m}$ and $X_0^{-(n+1),m} = X_0^{-(n+1),-m}$, therefore it is only necessary to obtain relations for positive m. If the integrals 3.4.3 are evalueted (see for example [Kozai,1973])

$$\begin{aligned} X_0^{n,m} &= \left(-\frac{e}{2}\right)^m \binom{n+m+1}{m} F\left(\frac{m-n-1}{2}, \frac{m-n}{2}, m+1; e^2\right), \\ X_0^{-(n+1),m} &= \left(-\frac{e}{2}\right)^m \frac{1}{(1-e)^{(2n-1)/2}} \sum_{j=0}^{[(n-m-1)/2]} \frac{1}{2^j}\binom{n-1}{2j+m}\binom{2j+m}{j} e^2 j, \end{aligned} \tag{3.4.4}$$

where $F(\ \ )$ is the standard hypergeometric function and $[\ \ ]$ denotes the nearest lowest integer. From these equations, replacing the hypergeometric functional expressions, one can obtain the recursive formulae (see [Cefola and Broucke,1975]) for $X_0^{n,m}$

$$\begin{aligned} X_0^{n+1,m} &= \frac{(2n+3)}{(n+2)} X_0^{n,m} - \frac{(n+1-m)(n+1+m)}{(n+1)(n+2)}(1-e^2) X_0^{n-1,m} \\ eX_0^{n,m+1} &= \frac{1}{(n-m+1)}(e(n+m+1)X_0^{n,m-1} + 2mX_0^{n,m}), \end{aligned} \tag{3.4.5}$$

the corresponding recoursive relations for $X_0^{-(n+1),m}$ are

$$\begin{aligned} (n-m+1)(n-m-1)X_0^{-(n+3),m} &= \frac{(n+1)}{(1-e^2)}[(2n+1)X_0^{-(n+2),m} - nX_0^{-(n+1),m}] \\ X_0^{-(n+1),m} &= \frac{1}{(n-m-1)}[2(m+1)\sqrt{1-e^2}X_0^{-(n+1),(m+1)} + \\ &\quad + (n+m+1)e^2(1-e^2)^{3/2}X_0^{-(n+1),(m+2)}]. \end{aligned} \tag{3.4.6}$$

The tables of some of the Hansen coefficients $X_0^{n,m}$ for $1 \leqslant n \leqslant 30$ and $0 \leqslant m \leqslant n$ are now given (taken by [73])

n	m	$X_0^{n,m}$	n	m	$X_0^{n,m}$
0	0	1	7	0	$+1+14e^2+105/4\,e^4+35/4\,e^6+35/128\,e^8$
1	0	$+1+1/2\,e^2$	7	1	$-9/2\,e-189/8\,e^3-315/16\,e^5-315/128\,e^7$
1	1	$-3/2\,e$	7	2	$+45/4\,e^2+225/8\,e^4+675/64\,e^6+45/128\,e^8$
2	0	$+1+3/2\,e^2$	7	3	$-165/8\,e^3-825/32\,e^5-495/128\,e^7$
2	1	$-2e-1/2\,e^3$	7	4	$+495/16\,e^4+297/16\,e^6+99/128\,e^8$
2	2	$+5/2\,e^2$	7	5	$-1287/32\,e^5-1287/128\,e^7$
3	0	$+1+3e^2+3/8\,e^4$	7	6	$+3003/64\,e^6+429/128\,e^8$
3	1	$-5/2\,e-15/8\,e^3$	7	7	$-6435/128\,e^7$
3	2	$+15/4\,e^2+5/8\,e^4$	8	0	$+1+18e^2+189/4\,e^4+105/4\,e^6+315/128\,e^8$
3	3	$-35/8\,e^3$	8	1	$-5e-35e^3-175/4\,e^5-175/16\,e^7-35/128\,e^9$
4	0	$+1+5e^2+15/8\,e^4$	8	2	$+55/4\,e^2+385/8\,e^4+1925/64\,e^6+385/128\,e^8$
4	1	$-3e-9/2\,e^3-3/8\,e^5$	8	3	$-55/2\,e^3-825/16\,e^5-495/32\,e^7-55/128\,e^9$
4	2	$+21/4\,e^2+21/8\,e^4$	8	4	$+715/16\,e^4+715/16\,e^6+715/128\,e^8$
4	3	$-7e^3-7/8\,e^5$	8	5	$-1001/16\,e^5-1001/32\,e^7-143/128\,e^9$
4	4	$+63/8\,e^4$	8	6	$+5005/64\,e^6+2145/128\,e^8$
5	0	$+1+15/2\,e^2+45/8\,e^4+5/16\,e^6$	8	7	$-715/8\,e^7-715/128\,e^9$
5	1	$-7/2\,e-35/4\,e^3-35/16\,e^5$	8	8	$+12155/128\,e^8$
5	2	$+7e^2+7e^4+7/16\,e^6$	9	0	$+1+45/2\,e^2+315/4\,e^4+525/8\,e^6+1575/128\,e^8+63/256\,e^{10}$
5	3	$-21/2\,e^3-63/16\,e^5$	9	1	$-11/2\,e-99/2\,e^3-693/8\,e^5-1155/32\,e^7-693/256\,e^9$
5	4	$+105/8\,e^4+21/16\,e^6$	9	2	$+33/2\,e^2+77e^4+1155/16\,e^6+231/16\,e^8+77/256\,e^{10}$
5	5	$-231/16\,e^5$	9	3	$-143/4\,e^3-3003/32\,e^5-3003/64\,e^7-1001/256\,e^9$
6	0	$+1+21/2\,e^2+105/8\,e^4+35/16\,e^6$	9	4	$+1001/16\,e^4+3003/32\,e^6+3003/128\,e^8+143/256\,e^{10}$
6	1	$-4e-15e^3-15/2\,e^5-5/16\,e^7$	9	5	$-3003/32\,e^5-5005/64\,e^7-2145/256\,e^9$
6	2	$+9e^2+15e^4+45/16\,e^6$	9	6	$+1001/8\,e^6+429/8\,e^8+429/256\,e^{10}$
6	3	$-15e^3-45/4\,e^5-9/16\,e^7$	9	7	$-2431/16\,e^7-7293/256\,e^9$
6	4	$+165/8\,e^4+99/16\,e^6$	9	8	$+21879/128\,e^8+2431/256\,e^{10}$
6	5	$-99/4\,e^5-33/16\,e^7$	9	9	$-46189/256\,e^9$
6	6	$+429/16\,e^6$	10	0	$+1+55/2\,e^2+495/4\,e^4+1155/8\,e^6+5775/128\,e^8+693/256\,e^{10}$
			10	1	$-6e-135/2\,e^3-315/2\,e^5-1575/16\,e^7-945/64\,e^9-63/256\,e^{11}$
			10	2	$+39/2\,e^2+117e^4+2457/16\,e^6+819/16\,e^8+819/256\,e^{10}$
			10	3	$-91/2\,e^3-637/4\,e^5-1911/16\,e^7-637/32\,e^9-91/256\,e^{11}$
			10	4	$+1365/16\,e^4+5733/32\,e^6+9555/128\,e^8+1365/256\,e^{10}$
			10	5	$-273/2\,e^5-1365/8\,e^7-585/16\,e^9-195/256\,e^{11}$
			10	6	$+1547/8\,e^6+1105/8\,e^8+3315/256\,e^{10}$
			10	7	$-1989/8\,e^7-5967/64\,e^9-663/256\,e^{11}$
			10	8	$+37791/128\,e^8+12597/256\,e^{10}$
			10	9	$-20995/64\,e^9-4199/256\,e^{11}$
			10	10	$+88179/256\,e^{10}$

Figure 3.8: Explicit formulae for some of the Hansen coefficients

3.4.2 Computation of $X_k^{n,m}$ and $X_k^{-(n+1),m}$ when $k \neq 0$.

If $k \neq 0$ then the computation of $X_k^{n,m}$ and $X_k^{-(n+1),m}$ presents some difficulty in that the analytical expressions for such coefficients do not terminate, consequently the series have to be truncated at some particular order in the eccentricity.

Since most planets and satellites both natural and artificial have small or moderate eccentricities ($0 \leqslant e \leqslant 0.1$), a series expansion in the eccentricity is usually fine.

$$X_k^{n,m} = \sum_q \widehat{X}_{k,q}^{n,m} e^q \tag{3.4.7}$$

The coefficients $\widehat{X}_{k,q}^{n,m}$ with shifted indices are known as Newcomb's operators defined as follows

$$X_{\rho,\sigma}^{n,m} = \widehat{X}_{m+\rho-\sigma,\rho+\sigma}^{n,m} \tag{3.4.8}$$

in such a way that the expansion in 3.4.7 becomes the well-known

$$X_k^{n,m} = \sum_{\rho-\sigma=k+m} X_{\rho,\sigma}^{n,m} e^{\rho+\sigma}. \tag{3.4.9}$$

For $\sigma = 0$, knowing that $X_{0,0}^{n,m} = 1$ and $X_{1,0}^{n,m} = \left(m - \frac{n}{2}\right)$, the recursive relations are easily founded

$$4\rho X_{\rho,0}^{n,m} = 2(2m-n)X_{\rho-1,0}^{n,m+1} + (m-n)X_{\rho-2,0}^{n,m+2}, \tag{3.4.10}$$

while for $\sigma \neq 0$ the relation

$$\begin{aligned} 4\sigma X_{\rho,\sigma}^{n,m} =& -2(2m+n)X_{\rho,\sigma-1}^{n,m-1} - (m+n)X_{\rho,\sigma-2}^{n,m-2} - (\rho - 5\sigma + 4 + 4m + n)X_{\rho-1,\sigma-1}^{n,m} + \\ &+ 2(\rho - \sigma + m)\sum_{j \geqslant 2}(-1)^j \binom{3/2}{j} X_{\rho-j,\sigma-j}^{n,m} \end{aligned} \tag{3.4.11}$$

is used. From the above relations we notice that $X_{\rho,\sigma}^{n,m} = 0$ whenever ρ or σ is negative, and that

$$X_{\rho,\sigma}^{n,m} = X_{\sigma,\rho}^{n,-m} \quad \text{if } \sigma > \rho. \tag{3.4.12}$$

As we can see, when $k \neq 0$ the recursive relations are more more complicate than the ones for $k = 0$. Indeed in a computational way, the more rapide mode to calculate these Hansen coefficients with $k \neq 0$ is using the Bessel function as is shown in [74]. In particular we will use the following expansion, known as *Wnuk's method*:

$$\begin{aligned} X_k^{n,m} =& (1+\beta^2)^{-(n+1)} \sum_{t=-\infty}^{\infty} E_{k-t}^{n,m} J_t(ke), \\ 2(1-e^2)\frac{dX_k^{n,m}}{de} =& -\frac{2m}{e} X_k^{n,m} - (n+m)eX_k^{n,m} + \frac{2k(1-e^2)^{3/2}}{e} X_k^{n,m} - \\ & (2n+4m)X_k^{n,m-1} - (n+m)eX_k^{n,m-2}, \end{aligned} \tag{3.4.13}$$

where

$$\begin{aligned} \beta &= \frac{e}{1+\sqrt{1-e^2}}, \\ E_{k-t}^{n,m} &= \begin{cases} (-\beta)^{k-t-m} \sum_{s=0}^{\infty} \binom{n-m+1}{k-t-m+s}\binom{n+m+1}{s}\beta^{2s}, & (k-t-m \geqslant 0), \\ (-\beta)^{t-k+m} \sum_{s=0}^{\infty} \binom{n+m+1}{t-k+m+s}\binom{n-m+1}{s}\beta^{2s}, & (k-t-m < 0), \end{cases} \end{aligned} \tag{3.4.14}$$

and $J_t(ke)$ is the Bessel function of ke, for which the following relation holds (for $t < 0$ or $ke < 0$)

$$J_{-t}(ke) = J_t(-ke) = (-1)^t J_t(ke).$$

3.5 Checking the Singular KAM condition for the perturbing function

[2] The new expansion of the Fourier coefficient of the perturbing function of RCP3BP developed on the previous section allow us to check the Singular KAM condition, i.e. to check thar for high Fourier modes, exists a δ such that

$$|f_{r,k}(a,e)| \geqslant \delta\ e^{-|r|s-|k|\beta}, \tag{3.5.1}$$

where s and β are the analyticity strips of the disturbing function w.r.t. the two angles ℓ, g.

To be more precise, we are neglecting the polynomial part on that condition because it is present only to adjust the measure of the set $\mathbb{G}_s^n$ (to be 1). In fact, the polynomial term, for high Fourier modes is negligible w.r.t. the exponential term.

[2] Regarding this section, I would like to warmly thank Prof. Corrado Falcolini for his invaluable help.

If there is presence of zeros on $f_{r,k}(a,e)$ for high Fourier modes, the condition (3.5.1) will be certainly false. So our request implies (as a necessary but not sufficient condition) the absence of zeros on the analytic function $f_{r,k}(a,e)$.

In order to study the zeros of this coefficient, is useful to do some numerical work so that we can computer explicitly the Coefficients. If one uses the expansion in [21] with the Legendre polynomials, this numerical work will be not very precise and very long to do. The expansion in Hansen coefficient makes it more reliable and so much faster (moving from computations that take hours to a few minutes), since they are obtained from iterative and quite simple relations.

To make this clear, we show some Fourier coefficients computed using Mathematica, keeping in mind the following notation used in the previous section:

$$F(\ell,g,a,e)=\sum_{r\in\mathbb{N}}\sum_{k\in\mathbb{Z}} f_{k,r}(a,e)\cos(rg+k\ell);$$

$$f_{r,k}(a,e)=t_{r,k}\,e^{|r-k|}\,a^r\left[1+\mathcal{O}(e^2;a)\right]:=t_{r,k}\,e^{|r-k|}\,a^r\left[1+\mathscr{F}_{r,k}(a,e)\right].$$

Since our expansion allows us to compute coefficients at very high modes with arbitrary precision, we will not show coefficients related to small modes (which are really easy to compute), but only those with really high Fourier modes that do not appear in the literature.

We want to stress that every coefficient shown in the following pages is computed in not more than 10 seconds. In the following figures one can see a list that represents

$$\{r,k,t_{r,k},r,|r-k|,\mathscr{F}_{r,k}(a,e)\}.$$

The truncation has been chosen so as not to make the figures too large but can be increased without changing the computation time too much.

$$
\begin{aligned}
&\Big\{85,\ 90,\ -\frac{132\,810\,966\,781\,307\,063\,652\,233\,264\,539\,385\,287\,822\,753\,847\,148\,118\,003\,125}{748\,288\,838\,313\,422\,294\,120\,286\,634\,350\,736\,906\,063\,837\,462\,003\,712},\ 85,\ 5,\\
&\frac{97\,033\,a^2}{219\,300}+\frac{120\,677\,928\,613\,a^4}{412\,108\,560\,000}+\frac{249\,142\,441\,387\,a^6}{1\,160\,497\,704\,960}+\frac{7\,562\,252\,685\,883\,a^8}{45\,904\,131\,440\,640}+\frac{7\,138\,398\,409\,719\,689\,a^{10}}{55\,084\,957\,728\,768\,000}+\frac{6\,338\,823\,049\,407\,149\,a^{12}}{61\,270\,969\,024\,512\,000}+\\
&\frac{2\,457\,554\,829\,307\,328\,363\,a^{14}}{29\,485\,475\,555\,180\,544\,000}-\frac{3955\,e^2}{4}-\frac{151\,128\,907\,a^2\,e^2}{350\,880}-\frac{92\,458\,886\,529\,611\,a^4\,e^2}{329\,686\,848\,000}-\frac{117\,288\,635\,499\,917\,a^6\,e^2}{580\,248\,852\,480}-\\
&\frac{20\,983\,817\,376\,368\,063\,a^8\,e^2}{137\,712\,394\,321\,920}-\frac{576\,076\,819\,272\,505\,117\,a^{10}\,e^2}{4\,896\,440\,687\,001\,600}-\frac{127\,929\,940\,252\,729\,709\,689\,a^{12}\,e^2}{1\,388\,808\,631\,222\,272\,000}-\frac{643\,194\,898\,236\,103\,221\,757\,a^{14}\,e^2}{8\,845\,642\,666\,554\,163\,200}+\\
&\frac{47\,125\,355\,e^4}{112}+\frac{110\,806\,446\,389\,a^2\,e^4}{614\,040}+\frac{1\,067\,109\,948\,621\,529\,567\,a^4\,e^4}{9\,231\,231\,744\,000}+\frac{101\,394\,924\,348\,690\,431\,a^6\,e^4}{1\,237\,864\,218\,624}+\frac{11\,137\,308\,842\,060\,768\,927\,a^8\,e^4}{183\,616\,525\,762\,560}+\\
&\frac{900\,300\,126\,109\,777\,213\,499\,a^{10}\,e^4}{19\,585\,762\,748\,006\,400}+\frac{117\,633\,234\,355\,144\,407\,605\,861\,a^{12}\,e^4}{3\,333\,140\,714\,933\,452\,800}+\frac{18\,025\,928\,270\,771\,018\,468\,942\,119\,a^{14}\,e^4}{660\,474\,652\,436\,044\,185\,600}\Big\}
\end{aligned}
$$

Figure 3.9: $r = 85, k = 90$.

$$
\begin{aligned}
&\Big\{117,\ 112,\ \frac{924\,034\,633\,556\,965\,623\,388\,565\,762\,413\,145\,329\,573\,510\,622\,626\,566\,042\,026\,921\,153\,744\,487\,007\,942\,815}{6\,901\,746\,346\,790\,563\,787\,434\,755\,862\,277\,025\,452\,451\,108\,972\,170\,386\,555\,162\,524\,223\,799\,296},\ 117,\ 5,\\
&\frac{149\,157\,622\,503\,515\,a^2}{291\,081\,330\,369\,648}+\frac{1\,732\,241\,348\,462\,855\,a^4}{4\,398\,562\,325\,585\,792}+\frac{11\,832\,245\,303\,261\,095\,a^6}{35\,188\,498\,604\,686\,336}+\frac{14\,215\,921\,845\,929\,184\,445\,a^8}{47\,163\,415\,360\,619\,593\,728}+\frac{5\,278\,873\,579\,624\,661\,967\,657\,a^{10}}{18\,997\,124\,257\,001\,314\,451\,456}+\frac{147\,781\,919\,474\,360\,609\,450\,985\,a^{12}}{566\,459\,705\,117\,857\,376\,370\,688}-\\
&\frac{1\,827\,423\,531\,200\,e^2}{1\,129\,483\,029}-\frac{2\,861\,735\,956\,924\,818\,895\,a^2\,e^2}{3\,492\,975\,964\,435\,776}-\frac{16\,415\,451\,280\,879\,477\,405\,a^4\,e^2}{26\,391\,373\,953\,514\,752}-\frac{94\,906\,760\,229\,171\,409\,985\,a^6\,e^2}{180\,969\,421\,395\,529\,728}-\frac{50\,209\,443\,788\,759\,256\,565\,735\,a^8\,e^2}{108\,198\,423\,474\,362\,597\,376}-\\
&\frac{32\,077\,198\,720\,940\,080\,558\,203\,675\,a^{10}\,e^2}{75\,988\,497\,028\,005\,257\,805\,824}-\frac{442\,866\,086\,169\,507\,604\,228\,771\,755\,a^{12}\,e^2}{1\,132\,919\,410\,235\,714\,752\,741\,376}+\frac{2\,543\,503\,731\,618\,229\,e^4}{2\,258\,966\,058}+\frac{984\,041\,388\,100\,057\,492\,405\,a^2\,e^4}{1\,746\,487\,982\,217\,888}+\\
&\frac{8\,006\,724\,481\,990\,730\,039\,395\,a^4\,e^4}{18\,947\,653\,094\,831\,104}+\frac{99\,043\,307\,114\,645\,461\,072\,815\,a^6\,e^4}{281\,507\,988\,837\,490\,688}+\frac{2\,261\,018\,239\,701\,258\,276\,518\,073\,625\,a^8\,e^4}{7\,357\,492\,796\,256\,656\,621\,568}+\frac{1\,232\,968\,891\,259\,563\,229\,508\,504\,597\,a^{10}\,e^4}{4\,469\,911\,589\,882\,662\,223\,872}+\\
&\frac{1\,141\,703\,171\,598\,367\,090\,297\,096\,942\,185\,a^{12}\,e^4}{4\,531\,677\,640\,942\,859\,010\,965\,504}-\frac{1\,554\,288\,003\,897\,536\,105\,e^6}{3\,388\,449\,087}-\frac{9\,508\,015\,736\,093\,083\,016\,769\,925\,a^2\,e^6}{41\,915\,711\,573\,229\,312}-\frac{53\,223\,913\,079\,916\,059\,248\,281\,125\,a^4\,e^6}{316\,696\,487\,442\,177\,024}-\\
&\frac{1\,867\,089\,514\,910\,976\,904\,361\,485\,855\,a^6\,e^6}{13\,512\,383\,464\,199\,553\,024}-\frac{21\,036\,754\,609\,356\,569\,857\,701\,653\,801\,785\,a^8\,e^6}{176\,579\,827\,110\,159\,758\,917\,632}-\frac{128\,284\,104\,729\,367\,422\,737\,772\,895\,053\,855\,a^{10}\,e^6}{1\,215\,815\,952\,448\,084\,124\,893\,184}-\\
&\frac{1\,722\,941\,199\,516\,943\,413\,732\,752\,566\,413\,525\,a^{12}\,e^6}{18\,126\,710\,563\,771\,436\,043\,862\,016}+\frac{243\,975\,516\,604\,441\,439\,504\,345\,e^8}{1\,951\,746\,674\,112}+\frac{23\,045\,621\,461\,714\,422\,026\,764\,639\,105\,a^2\,e^8}{377\,241\,404\,159\,063\,808}+\frac{509\,762\,127\,294\,215\,321\,482\,689\,602\,395\,a^4\,e^8}{11\,401\,073\,547\,918\,372\,864}+\\
&\frac{2\,838\,018\,869\,288\,017\,137\,680\,666\,846\,435\,a^6\,e^8}{78\,178\,790\,042\,868\,842\,496}+\frac{392\,794\,772\,125\,298\,139\,015\,962\,761\,436\,754\,065\,a^8\,e^8}{12\,713\,747\,551\,931\,502\,642\,069\,504}+\frac{131\,301\,097\,282\,967\,824\,830\,789\,633\,590\,556\,225\,a^{10}\,e^8}{4\,863\,263\,809\,792\,336\,499\,572\,736}+\\
&\frac{204\,616\,523\,813\,331\,430\,000\,032\,687\,634\,096\,835\,a^{12}\,e^8}{8\,530\,216\,735\,892\,440\,491\,229\,184}-\frac{18\,760\,950\,457\,106\,410\,788\,277\,049\,e^{10}}{762\,401\,044\,575}-\frac{179\,335\,629\,655\,518\,425\,913\,639\,768\,190\,187\,a^2\,e^{10}}{15\,089\,656\,166\,362\,552\,320}-\\
&\frac{326\,564\,976\,498\,930\,933\,891\,860\,615\,880\,719\,a^4\,e^{10}}{38\,003\,578\,493\,061\,242\,880}-\frac{1\,795\,361\,005\,867\,803\,466\,862\,632\,758\,180\,283\,a^6\,e^{10}}{260\,595\,966\,809\,562\,808\,320}-\frac{1\,642\,489\,688\,635\,172\,366\,208\,844\,667\,809\,463\,539\,a^8\,e^{10}}{283\,789\,007\,855\,613\,898\,260\,480}-\\
&\frac{12\,978\,202\,844\,302\,726\,244\,947\,483\,598\,630\,068\,167\,a^{10}\,e^{10}}{2\,600\,675\,834\,113\,548\,930\,252\,800}-\frac{6\,341\,007\,874\,420\,668\,447\,180\,710\,686\,112\,691\,630\,117\,a^{12}\,e^{10}}{1\,450\,136\,845\,101\,714\,883\,508\,961\,280}+\frac{4\,742\,262\,877\,200\,022\,116\,308\,078\,909\,539\,e^{12}}{1\,288\,152\,804\,913\,920}+\\
&\frac{31\,249\,593\,664\,877\,272\,131\,365\,196\,021\,929\,293\,a^2\,e^{12}}{17\,784\,237\,624\,641\,579\,520}+\frac{18\,888\,355\,202\,309\,356\,741\,172\,608\,369\,630\,760\,579\,a^4\,e^{12}}{15\,049\,417\,083\,252\,252\,180\,480}+\frac{34\,181\,292\,342\,844\,647\,526\,755\,804\,394\,928\,166\,841\,a^6\,e^{12}}{34\,398\,667\,618\,862\,290\,698\,240}+\\
&\frac{493\,868\,890\,023\,818\,007\,299\,107\,561\,734\,276\,474\,307\,907\,a^8\,e^{12}}{599\,362\,384\,591\,056\,553\,126\,133\,760}+\frac{374\,996\,912\,609\,089\,602\,091\,945\,918\,051\,135\,849\,352\,779\,a^{10}\,e^{12}}{534\,959\,019\,077\,157\,014\,953\,000\,960}+\frac{38\,650\,650\,933\,442\,439\,529\,802\,292\,102\,747\,920\,666\,937\,047\,a^{12}\,e^{12}}{63\,806\,021\,184\,475\,454\,874\,394\,296\,320}\Big\}
\end{aligned}
$$

Figure 3.10: $r = 117, k = 112$.

$$\{207,\ 215,\ -(361\,530\,446\,917\,515\,616\,638\,513\,418\,032\,057\,643\,857\,534\,922\,233\,063\,652\,049\,608\,305\,335\,568\,589\,816\,849\,066\,084\,379\,996\,944\,082\,099\,483\,454\,456\,581\,181\,761\,580\,090\,769\,709\,682\,890\,625\ /$$
$$10\,830\,740\,992\,659\,433\,045\,228\,180\,406\,808\,920\,716\,548\,582\,325\,686\,783\,496\,759\,685\,861\,775\,864\,483\,615\,725\,089\,999\,900\,023\,844\,295\,226\,942\,934\,417\,817\,982\,702\,456\,930\,304),$$
$$207,\ 8,\ \frac{735\,774\,997\,a^2}{1\,592\,210\,880}+\frac{875\,296\,805\,768\,147\,a^4}{2\,734\,646\,954\,124\,800}+\frac{134\,032\,789\,958\,896\,218\,989\,a^6}{544\,531\,315\,803\,650\,560\,000}+\frac{1\,311\,967\,299\,219\,845\,034\,742\,803\,083\,a^8}{6\,606\,197\,914\,394\,548\,789\,862\,400\,000}+\frac{102\,505\,329\,551\,240\,206\,529\,655\,606\,993\,a^{10}}{622\,450\,647\,934\,064\,152\,644\,812\,800\,000}+$$
$$\frac{893\,338\,111\,230\,649\,769\,365\,517\,976\,021\,a^{12}}{6\,428\,217\,600\,482\,698\,885\,495\,521\,280\,000}+\frac{10\,048\,488\,744\,825\,553\,575\,169\,713\,163\,861\,a^{14}}{84\,654\,681\,015\,587\,542\,245\,910\,249\,472\,000}+\frac{11\,913\,195\,648\,826\,698\,849\,285\,381\,834\,569\,883\,a^{16}}{116\,484\,841\,077\,448\,458\,130\,372\,503\,273\,472\,000}+$$
$$\frac{4\,959\,112\,100\,177\,343\,516\,338\,059\,438\,808\,227\,729\,a^{18}}{55\,912\,723\,717\,175\,259\,902\,578\,801\,571\,266\,560\,000}+\frac{33\,216\,341\,950\,700\,235\,483\,758\,411\,416\,908\,326\,311\,a^{20}}{429\,700\,145\,281\,429\,542\,901\,077\,813\,944\,647\,680\,000}-\frac{11\,389\,e^2}{3}-\frac{3\,843\,038\,819\,741\,a^2\,e^2}{2\,204\,599\,680}-\frac{7\,381\,200\,952\,768\,886\,941\,a^4\,e^2}{6\,152\,955\,646\,780\,800}-$$
$$\frac{350\,329\,960\,514\,025\,477\,485\,872\,109\,a^6\,e^2}{382\,260\,983\,694\,162\,693\,120\,000}-\frac{1\,898\,703\,357\,056\,207\,061\,249\,661\,549\,381\,a^8\,e^2}{2\,585\,033\,966\,502\,214\,743\,859\,200\,000}-\frac{150\,614\,072\,838\,706\,365\,211\,789\,931\,770\,909\,a^{10}\,e^2}{248\,980\,259\,173\,625\,661\,057\,925\,120\,000}-\frac{93\,793\,944\,469\,918\,454\,123\,434\,153\,181\,a^{12}\,e^2}{185\,013\,074\,442\,582\,042\,869\,760\,000}-$$
$$\frac{655\,172\,260\,578\,387\,041\,603\,097\,077\,713\,900\,679\,a^{14}\,e^2}{1\,523\,784\,258\,280\,575\,760\,426\,384\,490\,496\,000}-\frac{996\,376\,000\,169\,844\,014\,280\,380\,823\,696\,110\,131\,a^{16}\,e^2}{2\,708\,949\,792\,498\,801\,351\,869\,127\,983\,104\,000}-\frac{318\,688\,140\,457\,590\,941\,695\,904\,937\,573\,383\,801\,684\,813\,a^{18}\,e^2}{1\,006\,429\,026\,909\,154\,678\,246\,418\,428\,282\,798\,080\,000}-$$
$$\frac{2\,213\,947\,025\,518\,199\,148\,861\,270\,515\,265\,342\,320\,029\,a^{20}\,e^2}{8\,082\,055\,466\,914\,387\,691\,674\,569\,039\,622\,963\,200}+\frac{3\,742\,674\,499\,e^4}{576}+\frac{2\,718\,952\,986\,463\,399\,583\,a^2\,e^4}{917\,113\,466\,880}+\frac{354\,771\,160\,978\,898\,401\,121\,817\,a^4\,e^4}{175\,017\,405\,063\,987\,200}+$$
$$\frac{38\,962\,343\,349\,312\,796\,715\,387\,259\,683\,a^6\,e^4}{25\,325\,779\,457\,998\,356\,480\,000}+\frac{4\,660\,374\,953\,064\,486\,242\,192\,096\,490\,019\,881\,169\,a^8\,e^4}{3\,805\,169\,998\,691\,260\,102\,960\,742\,400\,000}+\frac{725\,635\,602\,485\,896\,140\,158\,489\,080\,766\,160\,621\,a^{10}\,e^4}{724\,306\,208\,505\,092\,832\,168\,509\,440\,000}+$$
$$\frac{343\,025\,324\,455\,604\,252\,854\,436\,056\,787\,616\,317\,103\,a^{12}\,e^4}{411\,405\,926\,430\,892\,728\,671\,713\,361\,920\,000}+\frac{171\,195\,695\,985\,367\,666\,235\,850\,814\,490\,727\,710\,240\,659\,a^{14}\,e^4}{243\,805\,481\,324\,892\,121\,668\,221\,518\,479\,360\,000}+\frac{4\,495\,440\,326\,497\,408\,830\,862\,161\,275\,861\,125\,650\,237\,a^{16}\,e^4}{7\,537\,947\,248\,692\,316\,805\,201\,051\,779\,072\,000}+$$
$$\frac{76\,348\,039\,914\,002\,168\,179\,026\,888\,084\,232\,269\,804\,626\,501\,a^{18}\,e^4}{149\,794\,087\,726\,013\,719\,552\,955\,300\,953\,718\,784\,000}+\frac{1\,655\,793\,137\,205\,487\,608\,915\,715\,010\,190\,773\,484\,025\,493\,445\,673\,a^{20}\,e^4}{3\,783\,325\,621\,997\,866\,512\,582\,746\,832\,719\,503\,687\,680\,000}\}$$

Figure 3.11: $r = 207, k = 215$.

$$\{503,\ 507,$$
$$-(1\,472\,107\,163\,563\,234\,668\,774\,608\,875\,914\,225\,229\,222\,898\,387\,431\,903\,936\,064\,467\,856\,792\,540\,526\,078\,131\,436\,230\,389\,314\,985\,834\,541\,885\,757\,357\,401\,020\,686\,535\,023\,072\,769\,869\,125\,890\,230\,063\,447\,\backslash$$
$$099\,676\,973\,992\,133\,851\,905\,455\,853\,539\,968\,295\,137\,760\,951\,355\,625\,605\,502\,809\,861\,756\,647\,265\,201\,667\,932\,935\,699\,983\,106\,121\,610\,382\,968\,027\,796\,102\,173\,237\,579\,853\,654\,166\,942\,428\,747\,236\,605\,\backslash$$
$$113\,904\,956\,064\,825\ /$$
$$171\,441\,377\,149\,802\,771\,351\,748\,007\,849\,600\,289\,689\,824\,769\,872\,885\,377\,191\,000\,062\,139\,256\,168\,179\,989\,779\,598\,911\,740\,610\,511\,337\,300\,415\,147\,666\,808\,503\,492\,029\,943\,245\,710\,770\,246\,975\,753\,241\,\backslash$$
$$195\,177\,196\,862\,953\,084\,397\,187\,695\,766\,737\,193\,680\,997\,938\,270\,047\,266\,914\,448\,743\,599\,737\,311\,060\,278\,380\,280\,946\,648\,703\,137\,233\,006\,633\,139\,143\,642\,984\,674\,682\,566\,877\,306\,441\,990\,189\,395\,290\,\backslash$$
$$689\,110\,016),\ 503,\ 4,\ \frac{42\,716\,095\,127\,a^2}{86\,886\,674\,784}+\frac{3\,072\,305\,296\,963\,583\,a^4}{8\,474\,678\,010\,789\,120}+\frac{7\,132\,407\,348\,985\,807\,085\,a^6}{24\,013\,847\,611\,372\,050\,432}+\frac{789\,984\,854\,470\,842\,143\,237\,a^8}{3\,092\,068\,759\,102\,382\,112\,768}+\frac{20\,582\,377\,732\,127\,060\,752\,063\,a^{10}}{91\,059\,184\,326\,029\,571\,784\,704}+$$
$$\frac{12\,146\,144\,622\,029\,747\,890\,518\,923\,a^{12}}{59\,638\,034\,735\,934\,444\,580\,896\,768}+\frac{193\,286\,698\,404\,124\,312\,260\,185\,459\,a^{14}}{1\,039\,842\,656\,934\,241\,597\,820\,764\,160}-\frac{192\,148\,e^2}{5}-\frac{16\,372\,178\,113\,216\,717\,a^2\,e^2}{868\,866\,747\,840}-\frac{152\,238\,630\,007\,210\,269\,847\,a^4\,e^2}{10\,985\,693\,717\,689\,600}-$$
$$\frac{5\,384\,364\,766\,083\,517\,875\,961\,a^6\,e^2}{475\,521\,734\,878\,654\,464}-\frac{444\,311\,358\,139\,316\,897\,491\,883\,a^8\,e^2}{45\,740\,662\,116\,899\,143\,680}-\frac{59\,832\,229\,410\,757\,342\,315\,575\,383\,613\,a^{10}\,e^2}{6\,981\,204\,131\,662\,267\,170\,160\,640}-\frac{2\,296\,419\,616\,250\,016\,410\,268\,669\,746\,549\,a^{12}\,e^2}{298\,190\,173\,679\,672\,222\,904\,483\,840}-$$
$$\frac{218\,659\,181\,246\,345\,789\,062\,528\,289\,801\,201\,a^{14}\,e^2}{31\,195\,279\,708\,027\,247\,934\,622\,924\,800}+\frac{197\,080\,285\,357\,e^4}{320}+\frac{8\,374\,006\,515\,109\,042\,187\,291\,a^2\,e^4}{27\,803\,735\,930\,880}+\frac{199\,695\,326\,860\,070\,091\,915\,390\,713\,a^4\,e^4}{903\,965\,654\,484\,172\,800}+$$
$$\frac{1\,383\,338\,046\,338\,413\,826\,518\,748\,141\,137\,a^6\,e^4}{7\,684\,431\,235\,639\,056\,138\,240}+\frac{60\,115\,245\,428\,789\,180\,120\,556\,355\,399\,a^8\,e^4}{390\,320\,316\,730\,872\,692\,736}+\frac{179\,141\,007\,908\,740\,047\,040\,084\,299\,503\,439\,a^{10}\,e^4}{1\,321\,884\,805\,995\,222\,186\,065\,920}+$$
$$\frac{2\,317\,596\,477\,353\,177\,274\,356\,742\,024\,963\,513\,719\,a^{12}\,e^4}{19\,084\,171\,115\,499\,022\,265\,886\,965\,760}+\frac{110\,033\,840\,738\,495\,227\,047\,935\,955\,217\,156\,521\,661\,a^{14}\,e^4}{998\,248\,950\,656\,871\,933\,907\,933\,593\,600}-\frac{12\,644\,469\,406\,423\,163\,e^6}{2240}-\frac{1\,071\,693\,735\,519\,891\,171\,756\,946\,373\,a^2\,e^6}{389\,252\,303\,032\,320}-$$
$$\frac{6\,372\,171\,763\,057\,548\,472\,049\,560\,543\,993\,a^4\,e^6}{3\,163\,879\,790\,694\,604\,800}-\frac{176\,093\,037\,267\,128\,610\,429\,802\,425\,845\,084\,077\,a^6\,e^6}{107\,582\,037\,298\,946\,785\,935\,360}-\frac{19\,079\,406\,947\,191\,616\,816\,602\,064\,865\,346\,639\,a^8\,e^6}{13\,661\,211\,085\,580\,544\,245\,760}-$$
$$\frac{9\,720\,237\,104\,092\,568\,191\,449\,372\,577\,869\,972\,793\,a^{10}\,e^6}{7\,931\,308\,835\,971\,333\,116\,395\,520}-\frac{7\,729\,868\,041\,667\,095\,756\,671\,243\,111\,552\,512\,227\,a^{12}\,e^6}{7\,057\,755\,590\,051\,413\,559\,869\,440}-\frac{13\,854\,711\,945\,137\,929\,184\,978\,938\,685\,159\,494\,946\,245\,669\,a^{14}\,e^6}{13\,975\,485\,309\,196\,207\,074\,711\,070\,310\,400}\}$$

Figure 3.12: $r = 503, k = 507$.

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{810, 813,
 -(2 576 756 560 683 605 614 678 799 949 429 972 756 318 014 825 838 717 076 874 482 991 686 147 777 170 731 945 336 844 801 923 529 980 776 873 127 725 288 756 726 501 420 155 717 667 550 499 847 722 \
    117 887 265 709 132 155 376 769 709 986 818 618 443 282 950 009 027 159 812 501 653 177 220 742 373 128 909 349 848 585 310 471 971 895 493 075 718 061 668 632 024 308 530 105 525 383 640 043 285 \
    110 091 163 272 067 378 664 681 182 414 090 313 562 030 495 153 355 181 236 079 319 196 428 272 581 939 823 670 555 411 018 006 487 175 042 050 533 938 152 040 445 032 033 750 885 501 394 224 664 \
    924 376 126 400 076 758 377 868 260 357 591 745 240 801 838 373 155 /
   5 827 777 852 485 670 616 434 843 853 195 491 975 603 058 705 974 930 875 274 785 337 344 482 747 320 782 322 268 213 421 824 274 519 488 718 899 922 999 624 826 238 672 103 706 981 736 957 216 909 \
    113 794 247 991 307 151 424 935 162 902 047 057 293 487 248 380 517 516 741 847 787 407 331 242 287 486 490 857 004 733 010 922 453 597 110 539 692 179 043 164 360 524 059 328 182 670 049 644 324 \
    964 544 941 875 588 173 922 555 043 691 477 397 529 185 339 599 224 554 629 138 444 620 961 706 180 503 492 983 609 073 953 121 050 920 879 385 516 492 671 135 687 892 796 495 164 050 655 741 467 \
    590 235 029 053 051 892 001 314 722 365 618 866 798 723 072), 810, 3,
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$$\frac{35\,893\,918\,091\,a^2}{72\,366\,190\,965}+\frac{11\,014\,253\,943\,731\,a^4}{29\,847\,033\,429\,120}+\frac{62\,178\,962\,876\,148\,625\,a^6}{203\,831\,360\,694\,146\,304}+\frac{6\,085\,636\,370\,925\,892\,625\,a^8}{22\,984\,412\,482\,082\,783\,232}+\frac{60\,130\,951\,530\,165\,407\,775\,a^{10}}{254\,389\,207\,286\,261\,915\,648}+\frac{2\,336\,178\,099\,680\,143\,890\,261\,545\,a^{12}}{10\,869\,727\,059\,259\,970\,958\,655\,488}-$$
$$\frac{1\,979\,643\,e^2}{16}-\frac{1\,970\,561\,058\,385\,289\,a^2\,e^2}{32\,162\,751\,540}-\frac{152\,127\,013\,089\,751\,488\,443\,a^4\,e^2}{3\,342\,867\,744\,061\,440}-\frac{150\,655\,449\,047\,535\,696\,875\,a^6\,e^2}{4\,011\,441\,292\,873\,728}-\frac{43\,881\,862\,750\,093\,324\,421\,586\,875\,a^8\,e^2}{1\,348\,418\,865\,615\,523\,282\,944}-$$
$$\frac{236\,106\,748\,397\,489\,810\,631\,786\,775\,a^{10}\,e^2}{8\,140\,454\,633\,160\,381\,300\,736}-\frac{29\,927\,092\,680\,265\,815\,649\,446\,475\,735\,a^{12}\,e^2}{1\,136\,703\,483\,321\,304\,152\,539\,136}+\frac{3\,921\,106\,831\,989\,e^4}{640}+\frac{1\,731\,864\,905\,968\,336\,363\,931\,a^2\,e^4}{571\,782\,249\,600}+$$
$$\frac{33\,369\,701\,806\,326\,525\,724\,333\,633\,a^4\,e^4}{14\,857\,189\,973\,606\,400}+\frac{59\,385\,745\,468\,803\,469\,712\,915\,425\,a^6\,e^4}{32\,091\,530\,342\,989\,824}+\frac{21\,240\,647\,538\,784\,602\,234\,128\,064\,125\,a^8\,e^4}{13\,268\,574\,323\,399\,983\,104}+$$
$$\frac{4\,216\,303\,100\,535\,937\,698\,355\,496\,145\,705\,a^{10}\,e^4}{2\,960\,165\,321\,149\,229\,563\,904}+\frac{199\,539\,733\,584\,222\,701\,553\,032\,276\,639\,028\,881\,a^{12}\,e^4}{154\,591\,673\,731\,697\,364\,745\,322\,496}-\frac{863\,419\,512\,305\,115\,363\,e^6}{5120}-$$
$$\frac{142\,772\,145\,709\,249\,843\,658\,814\,011\,a^2\,e^6}{1\,715\,346\,748\,800}-\frac{2\,441\,244\,734\,633\,629\,744\,970\,879\,992\,521\,a^4\,e^6}{39\,619\,173\,262\,950\,400}-\frac{4\,337\,307\,901\,827\,405\,699\,321\,512\,014\,825\,a^6\,e^6}{85\,577\,414\,247\,972\,864}-$$
$$\frac{41\,816\,299\,420\,157\,971\,874\,832\,923\,618\,949\,275\,a^8\,e^6}{955\,337\,351\,284\,798\,783\,488}-\frac{74\,746\,965\,577\,691\,770\,702\,243\,973\,983\,001\,165\,a^{10}\,e^6}{1\,922\,468\,990\,857\,064\,218\,624}-\frac{130\,505\,549\,373\,551\,312\,107\,482\,638\,792\,027\,719\,759\,033\,a^{12}\,e^6}{3\,710\,200\,169\,560\,736\,753\,887\,739\,904}\}$$

Figure 3.13: $r = 810, k = 813$.

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{999, 937,
 -(265 618 634 092 485 079 064 455 393 938 243 998 833 990 785 803 828 085 659 812 658 167 142 124 575 891 209 042 361 498 687 790 960 843 371 148 480 118 228 942 892 786 279 233 093 870 945 354 336 210 336 563 268 219 871 966 367 995 107 633 910 361 078 460 144 952 973 696 \
    736 781 509 697 480 831 429 775 330 390 656 269 732 898 744 927 827 785 501 212 727 030 556 556 773 043 041 951 836 077 882 984 667 016 638 441 777 996 406 641 940 550 638 042 671 997 209 574 793 940 091 016 868 703 891 685 730 777 099 200 831 742 356 281 343 105 115 621 \
    101 223 220 930 125 740 588 505 785 042 159 252 074 806 321 520 109 631 295 711 333 707 228 918 679 425 572 287 972 624 055 562 253 262 034 470 198 606 968 522 202 559 943 525 061 695 933 839 096 053 757 019 421 476 687 049 827 923 806 339 679 649 693 764 885 620 388 253 \
    243 318 410 838 892 027 574 006 875 992 290 241 626 988 874 104 231 849 299 549 052 840 647 667 856 192 494 907 002 924 296 211 013 081 834 308 072 739 837 688 052 151 869 233 701 712 855 662 541 109 080 122 501 172 017 310 136 445 407 /
   16 712 432 544 432 247 422 822 687 483 897 483 448 322 154 682 234 841 461 603 604 421 016 217 747 737 387 830 887 911 873 737 032 931 082 041 521 795 931 892 448 585 501 104 744 663 483 843 404 476 841 166 178 185 915 198 399 769 062 723 587 803 491 080 493 171 597 774 \
    975 241 914 314 013 708 940 605 072 132 929 500 523 425 455 469 130 643 178 259 678 784 942 977 060 235 750 559 903 782 038 777 462 287 134 224 806 153 946 364 715 511 934 734 273 241 065 927 710 790 254 468 008 220 266 210 841 463 174 359 656 683 828 303 419 206 838 \
    578 485 321 279 648 903 827 475 912 812 858 960 325 951 958 872 598 748 601 854 505 151 072 274 993 392 254 827 649 011 047 259 661 635 512 517 811 763 927 892 265 437 406 543 060 953 163 639 603 358 260 681 853 505 363 223 893 831 570 038 902 129 098 145 850 171 041 \
    026 520 942 888 523 043 066 679 175 367 502 965 199 417 166 051 156 762 236 593 111 040 000 000 000), 999, 62,
 (134 763 976 887 634 613 674 466 569 482 464 321 501 996 760 301 468 309 669 219 970 950 732 842 006 646 922 661 875 986 031 876 250 912 432 331 332 000 498 543 753 159 895 136 819 033 112 156 791 326 817 357 310 488 102 269 237 183 441 047 325 026 345 654 673 131 924 278 \
     503 414 984 184 737 a^2) /
  258 474 077 536 527 201 841 145 566 480 660 544 624 961 127 136 872 969 296 632 391 922 273 221 601 256 934 283 096 884 528 867 031 536 663 786 663 342 301 967 137 367 799 945 228 276 069 926 469 047 493 861 174 608 436 878 446 860 282 201 702 971 813 094 168 514 780 052 \
   176 008 406 308 000 +
 (93 775 888 608 417 697 236 809 341 961 914 566 857 130 286 150 237 726 527 929 460 571 779 241 226 432 199 548 966 795 169 540 163 853 126 030 441 054 132 180 153 739 568 284 556 789 061 428 523 837 785 425 861 359 118 021 929 790 708 676 536 619 322 047 715 980 331 609 \
     446 770 633 224 841 481 a^4) /
  229 984 490 323 612 203 593 765 966 264 125 515 706 298 745 123 564 304 236 381 354 945 951 550 953 651 725 526 559 983 478 574 132 060 622 622 622 227 239 350 315 115 704 662 376 448 307 552 351 570 258 982 254 029 373 613 622 495 237 763 470 822 031 028 677 940 706 517 \
   536 163 924 190 496 000 -
 (89 560 580 346 286 042 567 265 639 880 538 943 511 257 205 525 393 282 798 563 524 579 787 449 818 466 580 929 088 429 486 479 482 985 763 768 124 463 939 647 920 115 511 507 821 438 400 371 375 896 179 712 674 140 922 334 125 040 280 397 616 436 833 096 222 343 570 367 \
     597 873 820 122 169 e^2) /
  8 150 083 525 926 533 391 387 472 816 957 764 920 606 882 387 198 697 230 073 994 339 891 497 978 418 011 441 358 910 773 432 744 237 642 551 831 727 009 521 486 313 399 097 372 062 758 961 645 420 416 473 100 100 266 027 698 774 873 763 116 760 372 484 951 259 475 046 \
   690 234 499 298 -
 (46 584 518 897 110 520 332 079 159 064 427 388 581 411 294 212 940 123 379 645 567 287 829 091 586 066 096 169 362 046 473 155 751 976 323 587 446 379 538 769 861 799 474 817 536 149 186 896 972 355 828 665 097 661 063 855 155 962 190 885 887 578 703 473 700 520 836 927 \
     406 265 358 200 246 145 823 a^2 e^2) /
  8 141 933 442 400 606 857 996 085 344 140 807 155 686 275 504 811 498 532 843 920 345 551 606 480 439 593 429 917 551 862 659 311 493 404 909 279 895 282 511 964 827 085 698 274 690 696 202 683 774 996 056 627 000 165 761 671 076 098 889 353 643 612 112 466 308 215 571 \
   643 544 264 798 702 000 -
 (129 484 954 495 540 349 463 136 877 633 457 197 586 432 374 558 163 289 045 622 933 978 153 820 287 402 761 702 997 053 762 556 655 457 152 638 356 646 873 468 103 233 852 589 430 529 725 946 572 268 280 812 798 528 406 506 611 050 693 425 581 437 701 281 822 084 709 340 \
     760 902 479 618 201 086 491 301 a^4 e^2) /
  28 978 045 780 775 137 652 814 511 749 279 814 978 993 641 885 569 102 333 784 050 723 189 895 420 160 117 416 346 557 918 300 340 639 638 450 450 400 632 158 139 704 578 787 459 432 486 751 596 297 852 631 764 007 701 075 316 434 399 958 197 323 575 909 613 420 529 021 \
   209 556 654 448 002 496 000 +
 (496 236 539 822 061 228 966 771 010 196 157 750 183 792 791 319 516 713 859 431 487 162 810 860 278 541 351 911 221 529 961 329 720 675 443 864 292 867 335 112 731 390 948 887 937 586 099 898 835 208 451 408 642 349 830 890 118 241 322 165 623 341 600 109 899 765 581 374 \
     984 365 957 740 667 403 113 e^4) /
  8 345 685 530 548 770 192 780 772 164 564 751 278 701 447 564 491 465 963 595 770 204 048 893 929 900 043 715 951 524 631 995 130 099 345 973 075 688 457 750 001 984 920 675 708 992 265 176 724 910 506 468 454 502 672 412 363 545 470 733 431 562 621 424 590 089 702 447 \
   810 800 127 281 152 +
 (147 291 585 134 136 883 624 372 081 548 824 025 840 906 676 036 667 626 084 469 724 272 128 102 828 001 173 397 942 234 595 640 477 513 699 250 534 040 376 675 193 337 711 609 561 951 634 141 815 317 555 558 323 208 210 889 198 610 016 495 897 108 941 416 163 121 889 982 \
     200 564 086 729 166 396 924 891 481 a^2 e^4) /
  4 764 194 197 153 269 384 335 995 081 371 535 158 527 283 495 386 842 570 075 528 247 911 340 020 554 367 812 706 041 775 636 077 125 283 786 915 778 725 309 858 275 963 288 590 447 584 520 884 677 483 406 849 170 382 708 543 532 528 721 541 789 176 458 951 714 064 425 \
   921 708 186 945 069 056 000 +
 (2 409 004 905 945 001 704 267 174 115 213 308 766 436 911 881 805 675 142 249 771 205 565 417 251 864 696 082 111 266 411 081 851 909 170 740 911 162 466 698 604 709 104 771 616 946 557 759 670 122 632 748 799 307 805 752 161 887 651 766 637 366 444 553 802 841 674 342 943 \
     158 896 933 219 013 418 550 925 263 a^4 e^4) /
  99 910 837 978 160 743 961 219 057 344 318 284 641 378 751 820 952 056 531 295 851 651 671 558 620 350 034 459 053 452 216 631 477 491 548 058 118 553 021 312 912 651 477 031 509 962 513 244 560 973 067 659 684 659 548 488 633 093 688 744 761 142 564 752 337 180 544 504 \
   102 983 212 642 271 232 000}
```

Figure 3.14: $r = 999, k = 937$.

Now that we have this power of computation, we can directly check wheter the considered coefficients have zeros, i.e. we want to find whether there exists $(a, e) \in (0, 1) \times (0, 1)$

such that

$$\mathscr{F}_{r,k}(a,e) = -1.$$

In order to do this, we have represented this zeros as implicit function $e(a)$ such that $\mathscr{F}_{r,k}(a, e(a)) - 1 = 0$. The complete result is in the following figure.

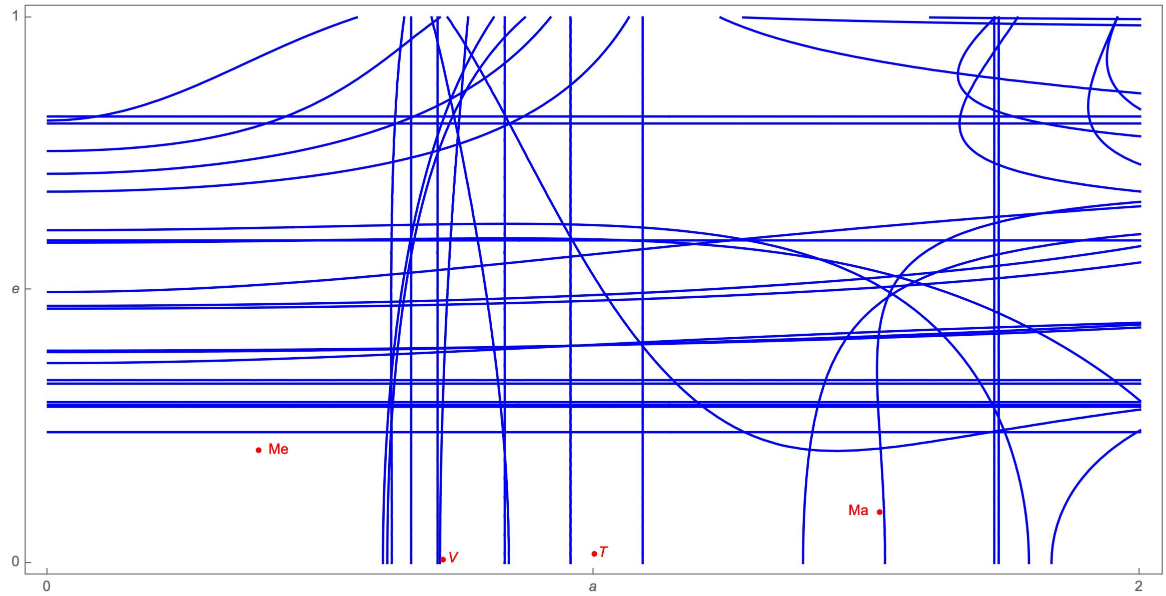

Figure 3.15: Zeros of $f_{k,r}(a,e)$ for some r, k truncated to 8^{th} order. In red some planets value of (e, a).

The figure gives us rather negative results. There are a lot of Zeros, and if one increases the value of the modes one finds that the zeros accumulate more and more towards the origin. This tells us that it is not possible to find a region of space (a, e) in which one is uniformly far from zeros of the Fourier coefficients.

So, one must change the approach and try something different. The idea suggested by S. Barbieri in his [15] is that it is normal that a smooth function in general has zeros, but it is more complicated that the sum of several smooth functions has zeros (because there must be coincident zeros). For this reason we check if there are zeros in $f_{r,k}(a,e) + f_{2r,2k}(a,e)$, or in $f_{r,k}(a,e) + f_{3r,3k}(a,e)$. This condition, for S. Barbieri, should be sufficient to apply Singular KAM Theory. In the following figures we show our result.

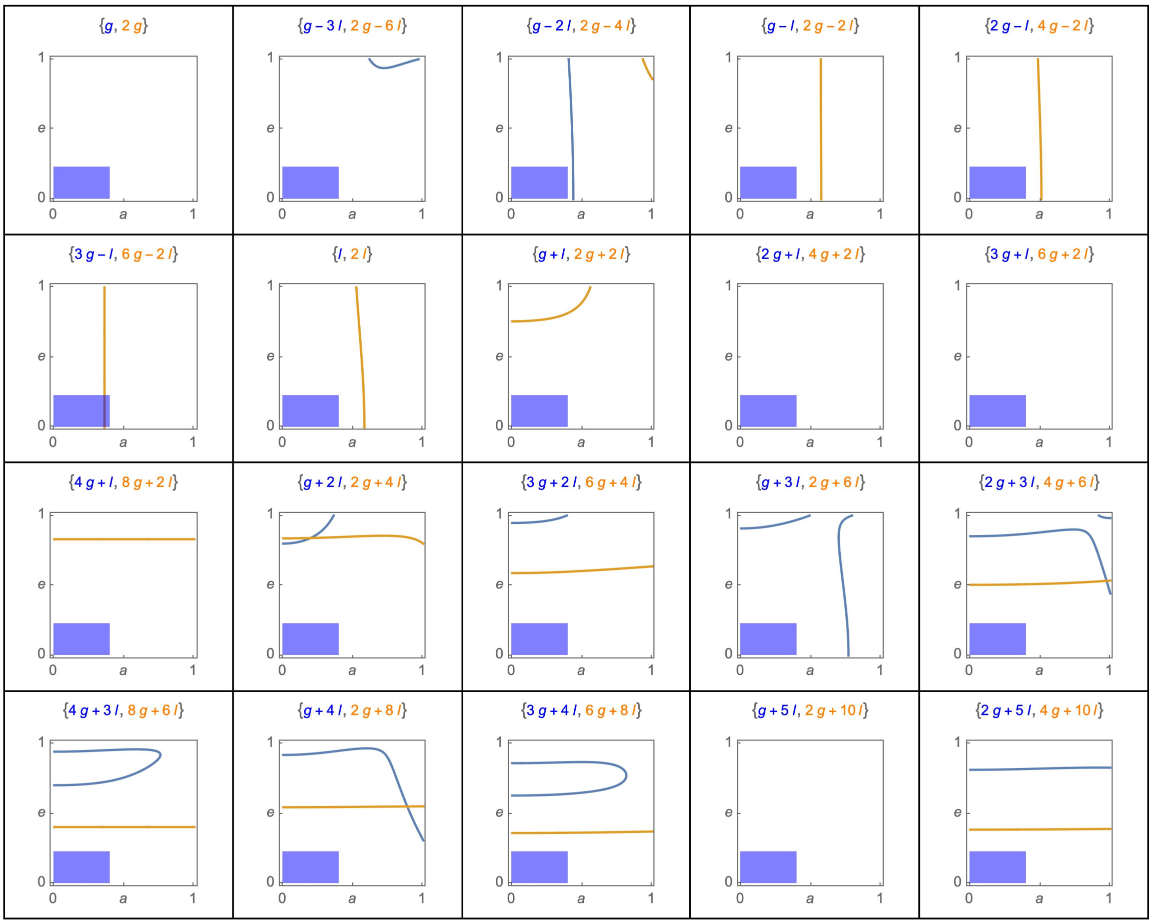

Figure 3.16: Zeros of $f_{k,r}(a,e) + f_{2k,2r}(a,e)$ truncated to 10^{th} order. .

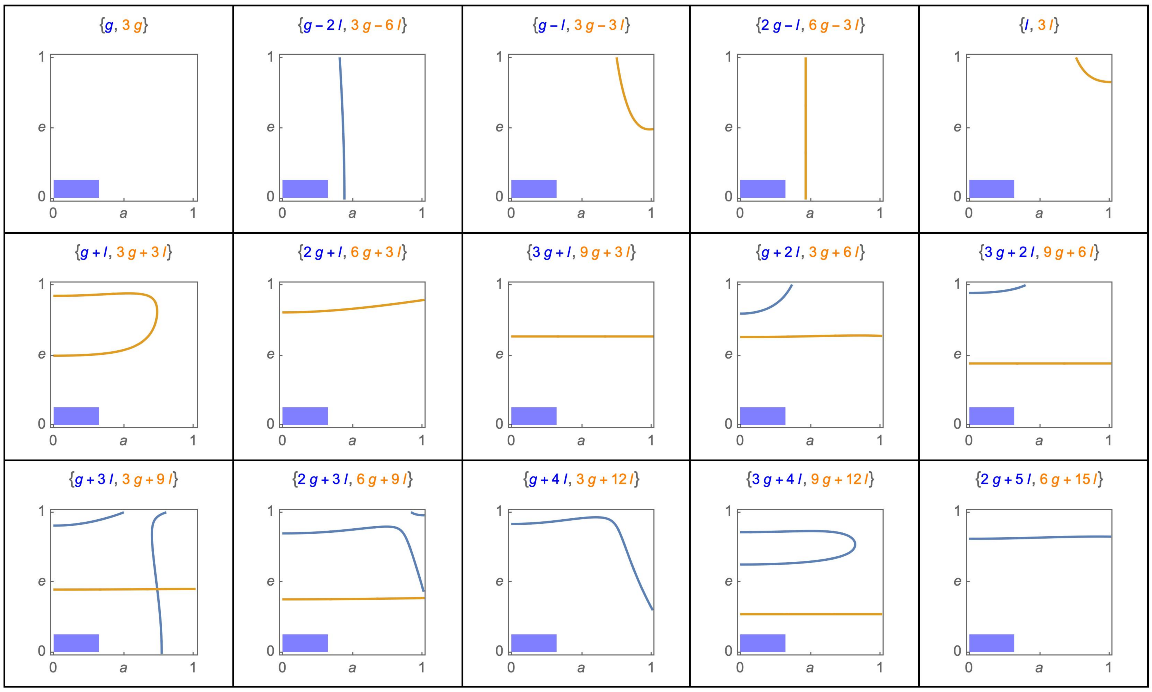

Figure 3.17: Zeros of $f_{k,r}(a,e) + f_{3k,3r}(a,e)$ truncated to 10^{th} order.

A question that appear naturally looking at this figures is if these behaviours change by modifying the order of truncations. So we have done the same drawings changing the order, but the result is that from a certain order onwards, the figures remain stable.

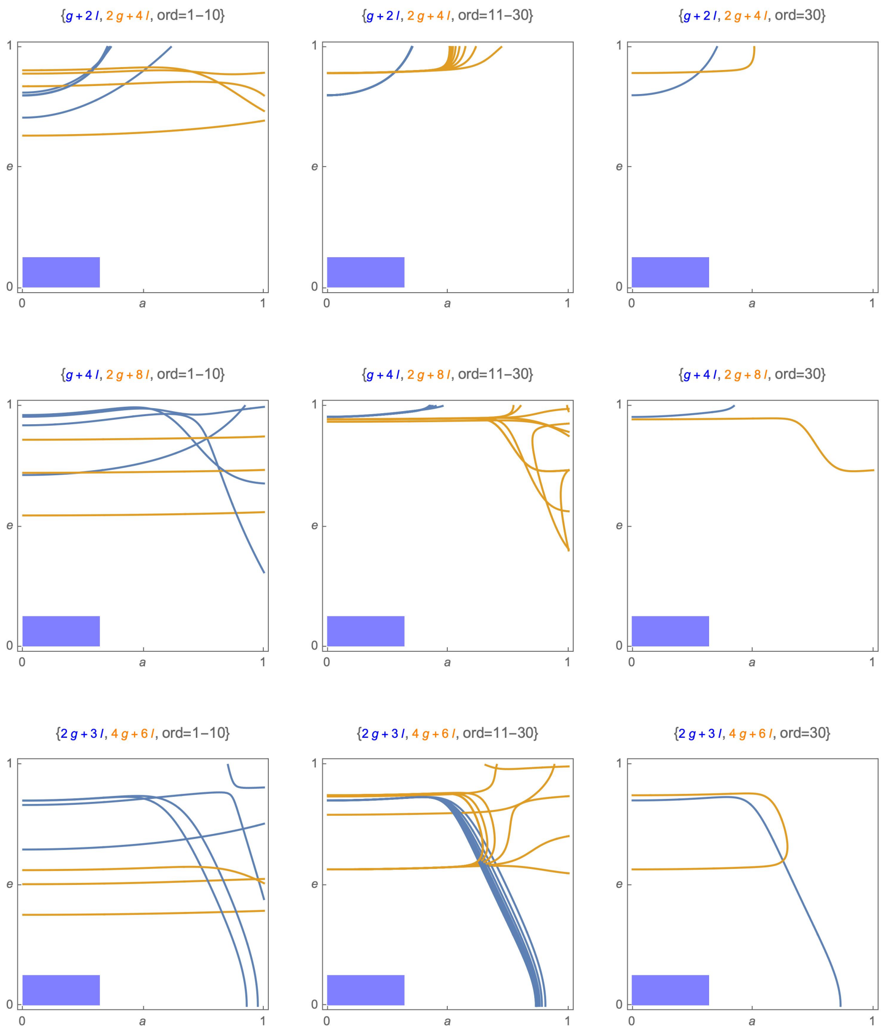

Figure 3.18: How the "double zeros" changes under the modification of truncation order

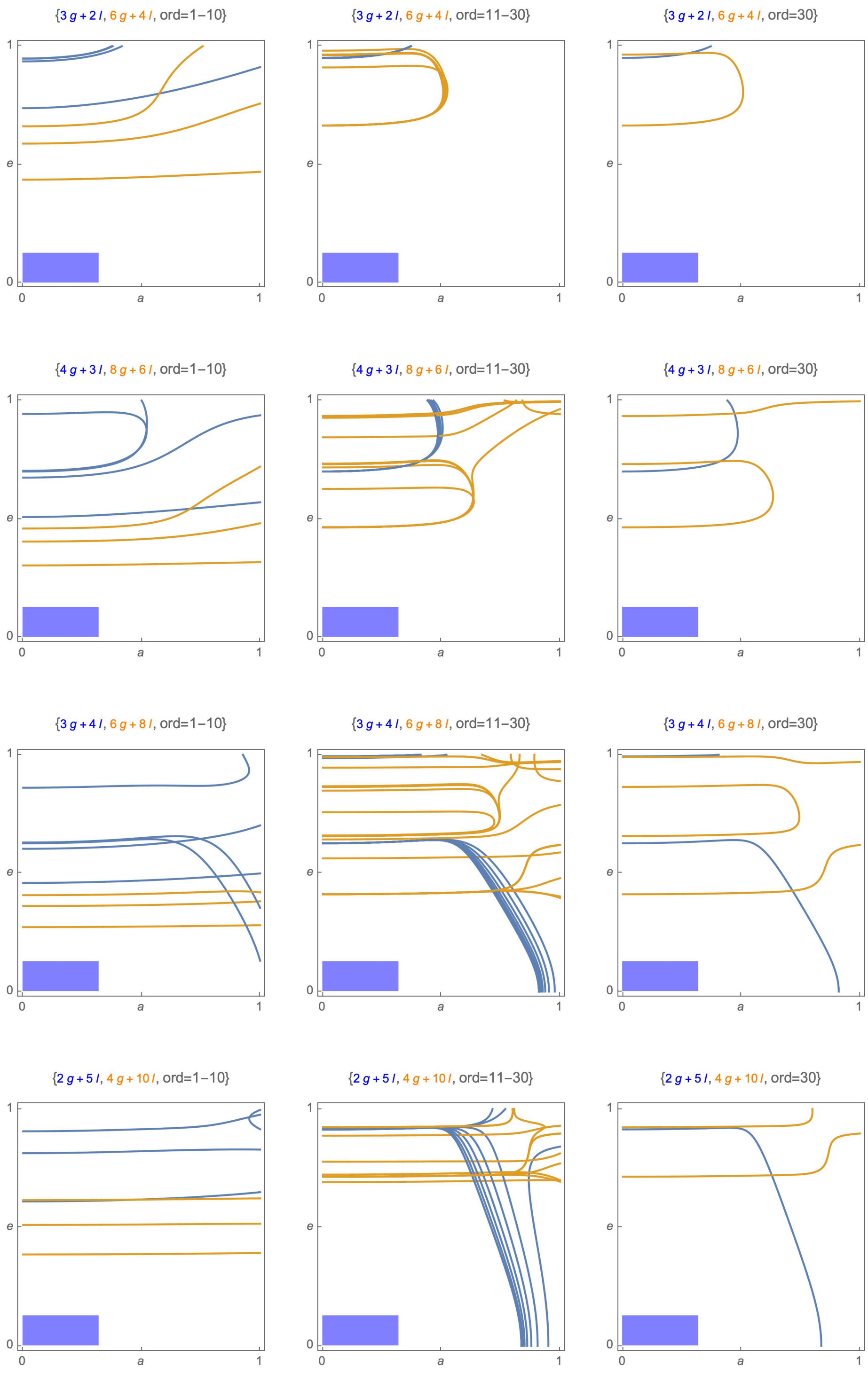

Figure 3.19: How the "double zeros" changes under the modification of truncation order

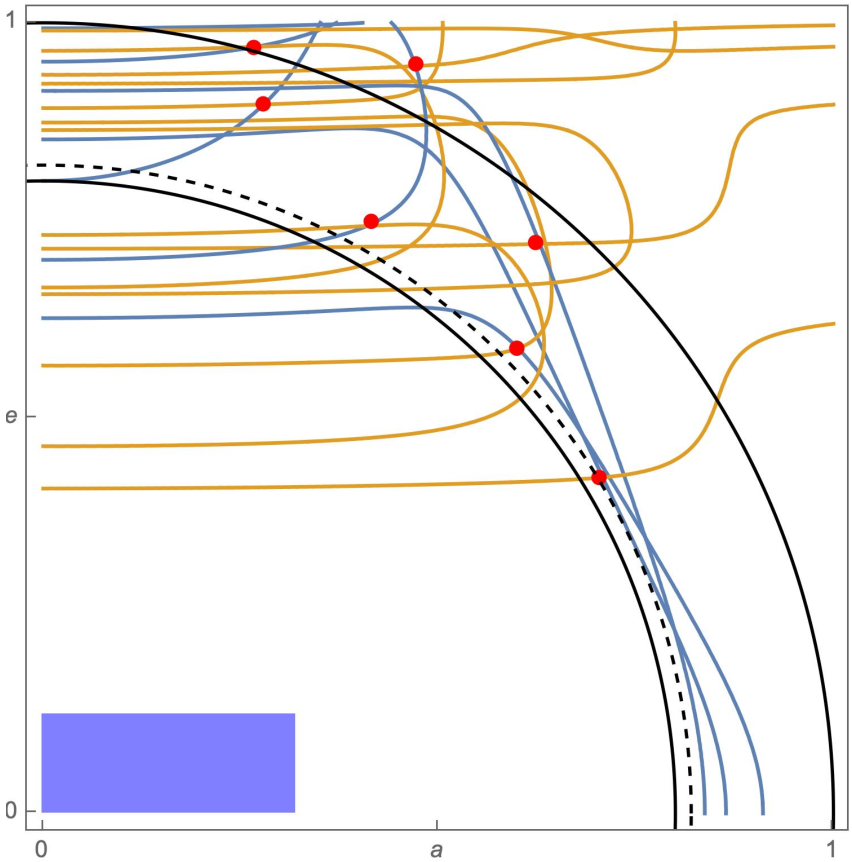

Figure 3.20: Some "double" intersection point at order 30.

Fortunately we have enough numerical power to be able to consider coefficients with valid truncation order, i.e. to represent the real behaviour of the functions. After noting the presence of a great number of double zeros, we try to draw "triple zeros".

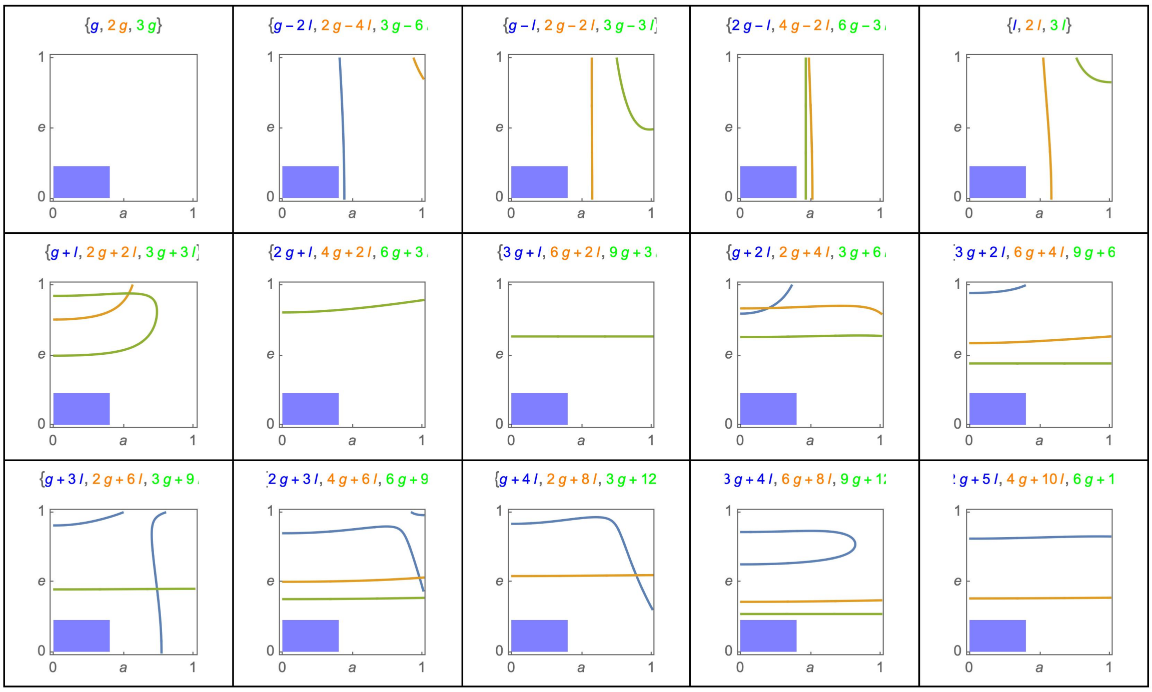

Figure 3.21: Zeros of $f_{k,r}(a,e) + f_{2k,2r}(a,e) + f_{3k,3r}(a,e)$ truncated to 10^{th} order.

Now this truncation is not enough because with $f_{3k,3r}$ the modes become really huge and if one wants to have some real behaviour of that function, one has to deeply increase the truncation order. Finally, we can say that these numerical results are very negative. The condition on the potential necessary to use Singular KAM theory seems inapplicable to RCP3BP because of the continued presence of zeros in the Fourier coefficient. In particular, these zeros seem to accumulate toward the origin by increasing the value of the Fourier modes, so there does not seem to be a rectangle in the (a,e) space in which one can apply this theory. The only hope left concerns the summation of three multiple Fourier modes, but that seems a pretty complicated plan.

A quantitative version of the Implicit Function Theorem

Theorem 3.5.1 (Implicit Function Theorem)**.** *Setting* $\mathcal{B}^n(y_0, r') := \{y \in \mathbb{C}^n : |y - y_0| \leqslant r'\}$*, let*

$$F : (y, x) \in \mathcal{B}^n(y_0, r) \times \mathcal{B}^m(x_0, s) \subset \mathbb{C}^{n+m} \longmapsto F(y, x) \in \mathbb{C}^n$$

be a continuous function with continuous Jacobian matrix $\partial_y F$*; assume that* $\partial_y F(y_0, x_0)$ *is invertible and denote by* T *its inverse. Assume also that*

$$\begin{aligned} &\sup_{\mathcal{B}(y_0,r)\times\mathcal{B}(x_0,s)} ||\mathbb{I}_n - T\partial_y F(y, x)|| \leqslant \frac{1}{2} \\ &\sup_{\mathcal{B}(x_0,s)} |F(y_0, x)| \leqslant \frac{r}{2||T||} \end{aligned} \tag{3.5.2}$$

Then, all solutions $(y, x) \in \mathcal{B}(y_0, r) \times \mathcal{B}(x_0, s)$ *of* $F(y, x) = 0$ *are given by the graph of a unique continuous function* $g : \mathcal{B}(x_0, s) \to \mathcal{B}(y_0, r)$ *satisfying, in particular*

$$\sup_{\mathcal{B}(x_0,s)} |g| \leqslant 2||T|| \sup_{\mathcal{B}(x_0,s)} |F(y_0, x)|. \tag{3.5.3}$$

Moreover, if F *is real-analytic, then so is* g *on its domain.*

This version of Implicit Function Theorem is taken from Appendix A of [22] with some different notations.

Proof. Let $X = \mathcal{C}(\mathcal{B}^m(x_0, s), \mathcal{B}^n(y_0, r))$ be the closed ball of continuous function from $\mathcal{B}^m(x_0, s)$ to $\mathcal{B}^n(y_0, r)$ w.r.t. the sup-norm $|| \cdot ||_\infty$ (X is a nonempty metric space with distance $d(u, v) := ||u - v||$) and denote $\Phi(y; x) := y - TF(y, x)$.
Then, $u \to \Phi(u) := \Phi(u, \cdot)$ maps $\mathcal{C}(\mathcal{B}^m(x_0, s))$ into $\mathcal{C}(\mathbb{C}^m)$ and, since

$$\partial_y \Phi = \mathbb{I}_n - TF_y(y, x),$$

from the first relation in (3.5.2), it follows that is a contraction.
Furthermore, for any $u \in \mathcal{C}(\mathcal{B}^m(x_0, s), \mathcal{B}^n(y_0, r))$,

$$|\Phi(u) - y_0| \leqslant |\Phi(u) - \Phi(y_0)| + |\Phi(y_0) - y_0| \leqslant \frac{1}{2}||u - y_0||_\infty + ||T||\ ||F(y_0, x)||_\infty$$
$$\leqslant \frac{1}{2}r + ||T||\frac{r}{2||T||} = r$$

showing that $\Phi : X \to X$. Thus, by Contraction lemma [3], there is a unique $g \in X$ such that $\Phi(g) = g$, with is equivalent to $F(g, x) = 0\ \forall\, x$.
If $F(y_1, x_1) = 0$ for some $(y_1, x_1) \in \mathcal{B}(y_0, r) \times \mathcal{B}(x_0, s)$, it follows that

$$|y_1 - g(x_1)| = |\Phi(y_1; x_1) - \Phi(g(x_1), x_1) \leqslant \alpha|y_1 - g(x_1)|$$

which implies that $y_1 = g(x_1)$ and that all solutions of $F = 0$ in $\mathcal{B}(y_0, r) \times \mathcal{B}(x_0, s)$ coincide with the graph of g.
Finally, (3.5.3) follows by observing that

$$||g-y_0||_\infty = ||\Phi(g)-y_0||_\infty \leqslant ||\Phi(g)-\Phi(y_0)||_\infty+||\Phi(y_0)-y_0||_\infty \leqslant \frac{1}{2}||g-y_0||_\infty+||T||\ ||F(y_0, \cdot)||_\infty,$$

finishing the proof. □

Remark 3.5.1. (i) If F is periodic in x or/and real on reals, then (by uniqueness) so is g.
(ii) If F is analytic, then so is g (Weierstrass theorem, since g is attained as uniform limit of analytic functions).

[3]Given a non-empty complete metric space (X, d) with a contraction mapping $T : X \to X$ (i.e. there exists $k \in [0, 1)$ such that $d(T(x), T(y)) \leqslant kd(x, y)$), then Contration lemma states that T admits a unique fixed-point $x^* \in X$.

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