III: Banach Spaces

Reductio ad absurdum is one of a mathematician's finest weapons. It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game.

G. H. Hardy

III.1 Definition and examples †

We defined normed linear spaces in Section I.2. Since normed linear spaces are metric spaces, they may have the property of being complete.

Definition A complete normed linear space is called a **Banach space**.

Banach spaces have many of the properties of $\mathbb{R}^n$: they are vector spaces, they have a notion of distance provided by the norm, and every Cauchy sequence has a limit. In general the norm does not arise from an inner product (see Problem 4 of Chapter II), so Banach spaces are not necessarily Hilbert spaces and will not have all of the same nice geometrical properties. In order to acquaint the reader with the types of Banach spaces he is likely to encounter, we discuss several examples in detail.

Example 1 ($L^\infty(\mathbb{R})$ and its subspaces) Let $L^\infty(\mathbb{R})$ be the set of (equivalence classes of) complex-valued measurable functions on $\mathbb{R}$ such that $|f(x)| \leq M$ a.e. with respect to Lebesgue measure for some $M < \infty$ ($f \sim g$ means $f(x) = g(x)$ a.e.). Let $\|f\|_\infty$ be the smallest such M. It is an easy exercise (Problem 1) to

† A supplement to this section begins on p. 348.

show that $L^\infty(\mathbb{R})$ is a Banach space with norm $\|\cdot\|_\infty$. The bounded continuous functions $C(\mathbb{R})$ is a subspace of $L^\infty(\mathbb{R})$ and restricted to $C(\mathbb{R})$ the $\|\cdot\|_\infty$-norm is just the usual supremum norm under which $C(\mathbb{R})$ is complete (since the uniform limit of continuous functions is continuous). Thus, $C(\mathbb{R})$ is a closed subspace of $L^\infty(\mathbb{R})$.

Consider the set $\kappa(\mathbb{R})$ of continuous functions with compact support, that is, the continuous functions that vanish outside of some closed interval. $\kappa(\mathbb{R})$ is a normed linear space under $\|\cdot\|_\infty$ but is not complete. The completion of $\kappa(\mathbb{R})$ is not all of $C(\mathbb{R})$; for example, if f is the function which is identically equal to one, then f cannot be approximated by a function in $\kappa(\mathbb{R})$ since $\|f-g\|_\infty \geq 1$ for all $g \in \kappa(\mathbb{R})$. The completion of $\kappa(\mathbb{R})$ is just $C_\infty(\mathbb{R})$, the continuous functions which approach zero at $\pm\infty$ (Problem 5). Some of the most powerful theorems in functional analysis (Riesz–Markov, Stone–Weierstrass) are generalizations of properties of $C(\mathbb{R})$ (see Sections IV.3 and IV.4).

Example 2 (L^p spaces) Let $\langle X, \mu\rangle$ be a measure space and $p \geq 1$. We denote by $L^p(X, d\mu)$ the set of equivalence classes of measurable functions which satisfy:

$$\|f\|_p \equiv \left(\int_X |f(x)|^p \, d\mu(x)\right)^{1/p} < \infty$$

Two functions are equivalent if they differ only on a set of measure zero. The following theorem collects many of the standard facts about L^p spaces.

Theorem III.1 Let $1 \leq p < \infty$, then

(a) (the Minkowski inequality) If $f, g \in L^p(X, d\mu)$, then

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

(b) (Riesz–Fisher) $L^p(X, d\mu)$ is complete.

(c) (the Hölder inequality) Let p, q, and r be positive numbers satisfying $p, q, r \geq 1$ and $p^{-1} + q^{-1} = r^{-1}$. Suppose $f \in L^p(X, d\mu)$, $g \in L^q(X, d\mu)$. Then $fg \in L^r(X, d\mu)$ and

$$\|fg\|_r \leq \|f\|_p \|g\|_q$$

Proofs of many of the basic facts about L^p spaces, including these inequalities, can be found in the second supplemental section. The Minkowski inequality shows that $L^p(X, d\mu)$ is a vector space and that $\|\cdot\|_p$ satisfies the triangle inequality. Combined with (b) this shows that $L^p(X, d\mu)$ is a Banach space. We have given the proof of (b) for the case where $p = 1$, $X = \mathbb{R}$ and $\mu =$ Lebesgue measure; the proof for the general case is similar.

Example 3 (sequence spaces) There is a nice class of spaces which is easy to describe and which we will often use to illustrate various concepts. In the following definitions,

$$a = \{a_n\}_{n=1}^{\infty}$$

always denotes a sequence of complex numbers.

$$\ell_\infty = \left\{a \,\middle|\, \|a\|_\infty \equiv \sup_n |a_n| < \infty\right\}$$

$$c_0 = \left\{a \,\middle|\, \lim_{n\to\infty} a_n = 0\right\}$$

$$\ell_p = \left\{a \,\middle|\, |a|_p \equiv \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p} < \infty\right\}$$

$$s = \left\{a \,\middle|\, \lim_{n\to\infty} n^p a_n = 0 \quad \text{for all positive integers } p\right\}$$

$$f = \left\{a \,\middle|\, a_n = 0 \quad \text{for all but a finite number of } n\right\}$$

It is clear that as sets $f \subset s \subset \ell_p \subset c_0 \subset \ell_\infty$.

The spaces ℓ_∞ and c_0 are Banach spaces with the $\|\cdot\|_\infty$ norm; ℓ_p is a Banach space with the $\|\cdot\|_p$ norm (note that this follows from Example 2 since $\ell_p = L^p(\mathbb{R}, d\mu)$ where μ is the measure with mass one at each positive integer and zero everywhere else). It will turn out that s is a Fréchet space (Section V.2). One of the reasons that these spaces are easy to handle is that f is dense in ℓ_p (in $\|\cdot\|_p$; $p < \infty$) and is dense in c_0 (in the $\|\cdot\|_\infty$ norm). Actually, the set of elements of f with only rational entries is also dense in ℓ_p and c_0. Since this set is countable, ℓ_p and c_0 are separable. ℓ_∞ is not separable (Problem 2).

Example 4 (the bounded operators) In Section I.3 we defined the concept of a bounded linear transformation or bounded operator from one normed linear space, X, to another Y; we will denote the set of all bounded linear operators from X to Y by $\mathscr{L}(X, Y)$. We can introduce a norm on $\mathscr{L}(X, Y)$ by defining

$$\|A\| = \sup_{x \in X,\, x \neq 0} \frac{\|Ax\|_Y}{\|x\|_X}$$

This norm is often called the **operator norm**.

Theorem III.2 If Y is complete, $\mathscr{L}(X, Y)$ is a Banach space.

Proof Since any finite linear combination of bounded operators is again a bounded operator, $\mathscr{L}(X, Y)$ is a vector space. It is easy to see that $\|\cdot\|$ is a norm; for example, the triangle inequality is proven by the computation

$$\begin{aligned}\|A + B\| &= \sup_{x \neq 0} \frac{\|(A + B)x\|}{\|x\|} \leq \sup_{x \neq 0} \frac{\|Ax\| + \|Bx\|}{\|x\|} \\ &\leq \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} + \sup_{x \neq 0} \frac{\|Bx\|}{\|x\|} \\ &= \|A\| + \|B\|\end{aligned}$$

To show that $\mathscr{L}(X, Y)$ is complete, we must prove that if $\{A_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the operator norm, then there is a bounded linear operator A so that $\|A_n - A\| \to 0$. Let $\{A_n\}_{n=1}^{\infty}$ be Cauchy in the operator norm; we construct A as follows. For each $x \in X$, $\{A_n x\}_{n=1}^{\infty}$ is a Cauchy sequence in Y. Since Y is complete, $A_n x$ converges to an element $y \in Y$. Define $Ax = y$. It is easy to check that A is a linear operator. From the triangle inequality it follows that

$$\big|\, \|A_n\| - \|A_m\| \,\big| \leq \|A_n - A_m\|$$

so $\{\|A_n\|\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers converging to some real number C. Thus,

$$\begin{aligned}\|Ax\|_Y &= \lim_{n \to \infty} \|A_n x\|_Y \leq \lim_{n \to \infty} \|A_n\| \; \|x\|_X \\ &= C\|x\|_X\end{aligned}$$

so A is a bounded linear operator. We must still show that $A_n \to A$ in the operator norm. Since $\|(A - A_n)x\| = \lim_{m \to \infty} \|(A_m - A_n)x\|$, we have

$$\frac{\|(A - A_n)x\|}{\|x\|} \leq \lim_{n \to \infty} \|A_m - A_n\|$$

which implies

$$\|A - A_n\| = \sup_{x \neq 0} \frac{\|(A - A_n)x\|}{\|x\|} \leq \lim_{m \to \infty} \|A_m - A_n\|$$

which is arbitrarily small for n large enough. The triangle inequality shows that the norm of A is actually equal to C. ∎

It is important to have criteria to determine whether normed linear spaces are complete. Such a criterion is given by the following theorem (whose proof is left to Problem 3). A sequence of elements $\{x_n\}_{n=1}^{\infty}$ in a normed linear space X is called **absolutely summable** if $\sum_{n=1}^{\infty} \|x_n\| < \infty$. It is called **summable** if $\sum_{n=1}^{N} x_n$ converges as $N \to \infty$ to an $x \in X$.

Theorem III.3 A normed linear space is complete if and only if every absolutely summable sequence is summable.

For a typical application of this theorem, see the construction of quotient spaces in Section III.4. We conclude this introductory section with some definitions.

Definition A bounded linear operator from a normed linear space X to a normed linear space Y is called an **isomorphism** if it is a bijection which is continuous and which has a continuous inverse. If it is norm preserving, it is called an **isometric isomorphism** (any norm preserving map is called an isometry).

For example, we proved in Section II.3 that all separable, infinite-dimensional Hilbert spaces are isometric to ℓ_2. Two Banach spaces which are isometric can be regarded as the same as far as their Banach space properties are concerned.

We will often encounter a situation in which we have two different norms on a normed linear space.

Definition Two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$, on a normed linear space X are called **equivalent** if there are positive constants C and C' such that, for all $x \in X$,

$$C\|x\|_1 \le \|x\|_2 \le C'\|x\|_1$$

For example, the following three norms on $\mathbb{R}^2$ are all equivalent:

$$\|\langle x, y\rangle\|_2 = \sqrt{|x|^2 + |y|^2}$$
$$\|\langle x, y\rangle\|_1 = |x| + |y|$$
$$\|\langle x, y\rangle\|_\infty = \max\{|x|, |y|\}$$

In fact, all norms on $\mathbb{R}^2$ are equivalent; see Problem 4. The usual situation we will encounter is an incomplete normed linear space with two norms. The completions of the space in the two norms will be isomorphic if and only if the norms are equivalent. An example is provided by the sequence spaces of Example 3. The completion of f in the $\|\cdot\|_\infty$ norm is c_0 while the completion in the $\|\cdot\|_p$ norm is ℓ_p. Two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$, on a normed linear space X are equivalent if and only if the identity map is an isomorphism from $\langle X, \|\cdot\|_1\rangle$ to $\langle X, \|\cdot\|_2\rangle$.

III.2 Duals and double duals

In the last section we proved that the set of bounded linear transformations from one Banach space X to another Y was itself a Banach space. In the case where Y is the complex numbers, this space $\mathscr{L}(X, \mathbb{C})$ is denoted by X^* and called the **dual space** of X. The elements of X^* are called bounded linear functionals on X. In this chapter when we talk about convergence in X^* we always mean convergence in the norm given in Theorem III.2. If $\lambda \in X^*$, then

$$\|\lambda\| = \sup_{x \in X, \|x\| \leq 1} |\lambda(x)|$$

In Section IV.5, we discuss another notion of convergence for X^*.

Dual spaces play an important role in mathematical physics. In many models of physical systems, whether in quantum mechanics, statistical mechanics, or quantum field theory, the possible states of the system in question can be associated with linear functionals on appropriate Banach spaces. Furthermore, linear functionals are important in the modern theory of partial differential equations. For these reasons, and because they are interesting in their own right, dual spaces have been studied extensively. There are two directions in which such study can proceed: either determining the dual spaces of particular Banach spaces or proving general theorems relating properties of Banach spaces to properties of their duals. In this section we study several examples of special interest and prove one general theorem. For an example of another general theorem see Theorem III.7.

Example 1 (L^p spaces) Suppose that $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$. If $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ then, according to the Hölder inequality (Theorem III.1), fg is in $L^1(\mathbb{R})$. Thus,

$$\int_{-\infty}^{\infty} \overline{g(x)} f(x)\, dx$$

makes sense. Let $g \in L^q(\mathbb{R})$ be fixed and define

$$G(f) = \int_{-\infty}^{\infty} \bar{g} f\, dx$$

for each $f \in L^p(\mathbb{R})$. The Hölder inequality shows that $G(\cdot)$ is a bounded linear functional on $L^p(\mathbb{R})$ with norm less than or equal to $\|g\|_q$; actually the norm is equal to $\|g\|_q$. The converse of this statement is also true. That is, every bounded linear functional on L^p is of the form $G(\cdot)$ for some $g \in L^q$. Furthermore, different functions in L^q give rise to different functionals on L^p. Thus, the mapping that assigns to each $g \in L^q$ the corresponding linear functional, $G(\cdot)$, on $L^p(\mathbb{R})$ is a (conjugate linear) isometric isomorphism of L^q onto $(L^p)^*$. In this sense, L^q is the dual of L^p. Since the roles of p and q in the expression $p^{-1} + q^{-1} = 1$ are symmetric, it is clear that $L^p = (L^q)^* = ((L^p)^*)^*$. That is, the dual of the dual of L^p is again L^p.

The case where $p = 1$ is different. The dual of $L^1(\mathbb{R})$ is $L^\infty(\mathbb{R})$ with the elements of $L^\infty(\mathbb{R})$ acting on functions in $L^1(\mathbb{R})$ in the natural way given by the above integral. However, the dual of $L^\infty(\mathbb{R})$ is not $L^1(\mathbb{R})$ but a much larger space (see Problems 7 and 8). As a matter of fact, we will prove later (Chapter XVI) that $L^1(\mathbb{R})$ is not the dual of any Banach space. The duality statements in this example hold for $L^p(X, d\mu)$ where $\langle X, \mu \rangle$ is a general measure space except that $L^1(X)$ may be the dual of $L^\infty(X)$ if $\langle X, \mu \rangle$ is trivially small.

Example 2 (Hilbert spaces) If we let $p = 2$ in Example 1, then $q = 2$ and we obtain the result that $L^2(\mathbb{R}) = L^2(\mathbb{R})^*$, that is, $L^2(\mathbb{R})$ is its own dual space. In fact, we have already shown (the Riesz lemma) in Section II.2 that this is true for all Hilbert spaces. The reader is cautioned again that the map which identifies $\mathscr{H}$ with its dual $\mathscr{H}^*$ is conjugate linear. If $g \in \mathscr{H}$, then the linear functional G corresponding to g is $G(f) = (g, f)$.

Example 3 ($\ell_\infty = \ell_1^*, \ell_1 = c_0^*$) Suppose that $\{\lambda_k\}_{k=1}^\infty \in \ell_1$. Then for each $\{a_k\}_{k=1}^\infty \in c_0$

$$\Lambda(\{a_k\}_{k=1}^\infty) = \sum_{k=1}^\infty \lambda_k a_k$$

converges and $\Lambda(\cdot)$ is a continuous linear functional on c_0 with norm equal to $\sum_{k=1}^\infty |\lambda_k|$. To see that all continuous linear functionals on c_0 arise in this way, we proceed as follows. Suppose $\lambda \in c_0^*$ and let e^k be the sequence in c_0 which has all its terms equal to zero except for a one in the kth place. Define $\lambda_k = \lambda(e^k)$ and let $f^\ell = \sum_{k=1}^\ell (|\lambda_k|/\lambda_k)e^k$. If some λ_k is zero, we simply omit that term from the sum. Then for each ℓ, $f^\ell \in c_0$ and $\|f^\ell\|_{c_0} = 1$. Since,

$$\lambda(f^\ell) = \sum_{k=1}^\ell |\lambda_k| \qquad \text{and} \qquad |\lambda(f^\ell)| \le \|f^\ell\|_{c_0}\|\lambda\|_{c_0^*}$$

we have

$$\sum_{k=1}^\ell |\lambda_k| \le \|\lambda\|_{c_0^*}.$$

Since this is true for all ℓ, $\sum_{k=1}^\infty |\lambda_k| < \infty$ and

$$\Lambda(\{a_k\}_{k=1}^\infty) = \sum_{k=1}^\infty \lambda_k a_k$$

is a well-defined linear functional on c_0. However, $\lambda(\cdot)$ and $\Lambda(\cdot)$ agree on finite linear combinations of the e_k. Because such finite linear combinations are dense in c_0, we conclude that $\lambda = \Lambda$. Thus every functional in c_0^* arises from a sequence in ℓ_1, and the reader can check for himself that the norms in ℓ_1 and c_0^* coincide. Thus $\ell_1 = c_0^*$. A similar proof shows that $\ell_\infty = \ell_1^*$.

Since the dual X^* of a Banach space is itself a Banach space (Theorem III.2), it also has a dual space, denoted by X^{**}. X^{**} is called the **second dual**, the **bidual**, or the **double dual** of the space X. In Example 3, ℓ_1 is the first dual of c_0 and ℓ_∞ is the second dual. It is not a priori evident that X^* is always nonzero and if $X^* = \{0\}$ then $X^{**} = \{0\}$ too. However, this situation does not occur; dual spaces always have plenty of linear functionals in them. We prove this fact in the next section. Using a corollary also proven there we will prove that X can be regarded in a natural way as a subset of X^{**}.

Theorem III.4 Let X be a Banach space. For each $x \in X$, let $\tilde{x}(\cdot)$ be the linear functional on X^* which assigns to each $\lambda \in X^*$ the number $\lambda(x)$. Then the map $J: x \to \tilde{x}$ is an isometric isomorphism of X onto a (possibly proper) subspace of X^{**}.

Proof Since

$$|\tilde{x}(\lambda)| = |\lambda(x)| \leq \|\lambda\|_{X^*}\|x\|_X$$

$\tilde{x}$ is a bounded linear functional on X^* with norm $\|\tilde{x}\|_{X^{**}} \leq \|x\|_X$. It follows from Theorems III.5 and III.6 that, given x, we can find a $\lambda \in X^*$ so that

$$\|\lambda\|_{X^*} = 1 \qquad \text{and} \qquad \lambda(x) = \|x\|_X$$

This shows that

$$\|\tilde{x}\|_{X^{**}} = \sup_{\lambda \in X^*,\ \|\lambda\| \leq 1} |\tilde{x}(\lambda)| \geq \|x\|_X$$

which implies that

$$\|\tilde{x}\|_{X^{**}} = \|x\|_X$$

Thus, J is an isometry of X into X^{**}. ∎

Definition If the map J, defined in Theorem III.4, is surjective, then X is said to be **reflexive**.

The $L^p(\mathbb{R})$ spaces are reflexive for $1 < p < \infty$ since $(L^p)^{**} = (L^q)^* = L^p$, but $L^1(\mathbb{R})$ is not reflexive. All Hilbert spaces are reflexive. c_0 is not reflexive, since its double dual is ℓ_∞. The theory of reflexive spaces is developed further in Problems 22 and 26 of this chapter and Problem 15 of Chapter V.

III.3 The Hahn–Banach theorem

In dealing with Banach spaces, one often needs to construct linear functionals with certain properties. This is usually done in two steps: first one defines the linear functional on a subspace of the Banach space where it is easy to verify the desired properties; second, one appeals to (or proves) a general theorem which says that any such functional can be extended to the whole space while retaining the desired properties. One of the basic tools of the second step is the following theorem, whose variants will reappear in Section V.1 and Chapter XIV.

Theorem III.5 (Hahn–Banach theorem) Let X be a real vector space, p a real-valued function defined on X satisfying $p(\alpha x + (1-\alpha)y) \le \alpha p(x) + (1-\alpha)p(y)$ for all x and y in X and all $\alpha \in [0, 1]$. Suppose that λ is a linear functional defined on a subspace Y of X which satisfies $\lambda(x) \le p(x)$ for all $x \in Y$. Then, there is a linear functional Λ, defined on X, satisfying $\Lambda(x) \le p(x)$ for all $x \in X$, such that $\Lambda(x) = \lambda(x)$ for all $x \in Y$.

Proof The idea of the proof is the following. First we will show that if $z \in X$ but $z \notin Y$, then we can extend λ to a functional having the right properties on the space spanned by z and Y. We then use a Zorn's lemma argument to show that this process can be continued to extend λ to the whole space X.

Let $\tilde{Y}$ denote the subspace spanned by Y and z. The extension of λ to Y, call it $\tilde{\lambda}$, is specified as soon as we define $\tilde{\lambda}(z)$ since

$$\tilde{\lambda}(az + y) = a\tilde{\lambda}(z) + \lambda(y)$$

Suppose that $y_1, y_2 \in Y$, $\alpha, \beta > 0$. Then

$$\begin{aligned}\beta\lambda(y_1) + \alpha\lambda(y_2) &= \lambda(\beta y_1 + \alpha y_2) = (\alpha+\beta)\lambda\left(\frac{\beta}{\alpha+\beta}\, y_1 + \frac{\alpha}{\alpha+\beta}\, y_2\right) \\ &\le (\alpha+\beta)p\left(\frac{\beta}{\alpha+\beta}(y_1 - \alpha z) + \frac{\alpha}{\alpha+\beta}(y_2 + \beta z)\right) \\ &\le \beta p(y_1 - \alpha z) + \alpha p(y_2 + \beta z)\end{aligned}$$

Thus, for all $\alpha, \beta > 0$ and $y_1, y_2 \in Y$,

$$\frac{1}{\alpha}[-p(y_1 - \alpha z) + \lambda(y_1)] \le \frac{1}{\beta}[p(y_2 + \beta z) - \lambda(y_2)]$$

We can therefore find a real number a such that

$$\sup_{\substack{y \in Y \\ \alpha > 0}} \left[\frac{1}{\alpha}(-p(y - \alpha z) + \lambda(y))\right] \le a \le \inf_{\substack{y \in Y \\ \alpha > 0}} \left[\frac{1}{\alpha}(p(y + \alpha z) - \lambda(y))\right]$$

We now define $\tilde{\lambda}(z) = a$. It may be easily verified that the resulting extension satisfies $\tilde{\lambda}(x) \le p(x)$ for all $x \in \tilde{Y}$. This shows that λ can be extended one dimension at a time.

We now proceed with the Zorn's lemma argument. Let $\mathscr{E}$ be the collection of extensions e of λ which satisfy $e(x) \le p(x)$ on the subspace where they are defined. We partially order $\mathscr{E}$ by setting $e_1 \prec e_2$ if e_2 is defined on a larger set than e_1 and $e_2(x) = e_1(x)$ where they are both defined. Let $\{e_\alpha\}_{\alpha \in A}$ be a linearly ordered subset of $\mathscr{E}$; let X_α be the subspace on which e_α is defined. Define e on $\bigcup_{\alpha \in A} X_\alpha$ by setting $e(x) = e_\alpha(x)$ if $x \in X_\alpha$. Clearly $e_\alpha \prec e$ so each linearly ordered subset of $\mathscr{E}$ has an upper bound. By Zorn's lemma, $\mathscr{E}$ has a maximal element Λ, defined on some set X', satisfying $\Lambda(x) \le p(x)$ for $x \in X'$. But, X' must be all of X, since otherwise we could extend Λ to a $\tilde{\Lambda}$ on a larger space by adding one dimension as above. Since this contradicts the maximality of Λ, we must have $X = X'$. Thus, the extension Λ is everywhere defined. ∎

In the theorem we have just proven, X is a real vector space. We now extend the theorem to the case where X is complex.

Theorem III.6 (complex Hahn–Banach theorem) Let X be a complex vector space, p a real-valued function defined on X satisfying $p(\alpha x + \beta y) \le |\alpha| p(x) + |\beta| p(y)$ for all $x, y \in X$, and $\alpha, \beta \in \mathbb{C}$ with $|\alpha| + |\beta| = 1$. Let λ be a complex linear functional defined on a subspace Y of X satisfying $|\lambda(x)| \le p(x)$ for all $x \in Y$. Then, there exists a complex linear functional Λ, defined on X, satisfying $|\Lambda(x)| \le p(x)$ for all $x \in X$ and $\Lambda(x) = \lambda(x)$ for all $x \in Y$.

Proof Let $\ell(x) = \mathrm{Re}\{\lambda(x)\}$. ℓ is a real linear functional on Y and since

$$\ell(ix) = \mathrm{Re}\{\lambda(ix)\} = \mathrm{Re}\{i\lambda(x)\} = -\mathrm{Im}\{\lambda(x)\}$$

we see that $\lambda(x) = \ell(x) - i\ell(ix)$. Since ℓ is real linear and $p(\alpha x + (1 - \alpha)y) \le \alpha p(x) + (1 - \alpha)p(y)$ for $\alpha \in [0, 1]$, ℓ has a real linear extension L to all of X obeying $L(x) \le p(x)$ (by Theorem III.5). Define $\Lambda(x) = L(x) - iL(ix)$. Λ clearly extends λ and is real linear. Moreover, $\Lambda(ix) = L(ix) - iL(-x) = i\Lambda(x)$, so Λ is also complex linear. To complete the proof, we need only show that $|\Lambda(x)| \leq p(x)$. First, note that $p(\alpha x) = p(x)$ if $|\alpha| = 1$. If we let $\theta = \operatorname{Arg}\{\Lambda(x)\}$ and use the fact that $\operatorname{Re}\Lambda = L$, we see that

$$\begin{aligned} |\Lambda(x)| &= e^{-i\theta}\Lambda(x) = \Lambda(e^{-i\theta}x) = L(e^{-i\theta}x) \\ &\leq p(e^{-i\theta}x) = p(x) \quad \blacksquare \end{aligned}$$

Corollary 1 Let X be a normed linear space, Y a subspace of X, and λ an element of Y^*. Then there exists a $\Lambda \in X^*$ extending λ and satisfying $\|\Lambda\|_{X^*} = \|\lambda\|_{Y^*}$.

Proof Choose $p(x) = \|\lambda\|_{Y^*}\|x\|$ and apply the above theorems. ∎

Corollary 2 Let y be an element of a normed linear space X. Then there is a nonzero $\Lambda \in X^*$ such that $\Lambda(y) = \|\Lambda\|_{X^*}\|y\|$.

Proof Let Y be the subspace consisting of all scalar multiples of y and define $\lambda(ay) = a\|y\|$. By using Corollary 1, we can construct Λ with $\|\Lambda\| = \|\lambda\|$ extending λ to all of X. But, since $\Lambda(y) = \|y\|$, $\|\Lambda\| = 1$ and therefore

$$\Lambda(y) = \|\Lambda\|_{X^*}\|y\| \quad \blacksquare$$

Corollary 3 Let Z be a subspace of a normed linear space X and suppose that y is an element of X whose distance from Z is d. Then there exists a $\Lambda \in X^*$ so that $\|\Lambda\| \leq 1$, $\Lambda(y) = d$, and $\Lambda(z) = 0$ for all z in Z.

The proof of the third corollary is left to the reader (Problem 10). To show how useful these corollaries are we prove the following general theorem.

Theorem III.7 Let X be a Banach space. If X^* is separable, then X is separable.

Proof Let $\{\lambda_n\}$ be a dense set in X^*. Choose $x_n \in X$, $\|x_n\| = 1$, so that

$$|\lambda_n(x_n)| \geq \|\lambda_n\|/2$$

Let $\mathscr{D}$ be the set of all finite linear combinations of the $\{x_n\}$ with rational coefficients. Since $\mathscr{D}$ is countable, it is sufficient to show that $\mathscr{D}$ is dense in X. If $\mathscr{D}$ is not dense in X, then there is a $y \in X\backslash\mathscr{D}$ and a linear functional $\lambda \in X^*$ so that $\lambda(y) \neq 0$, but $\lambda(x) = 0$ for all $x \in \mathscr{D}$ (Corollary 3). Let $\{\lambda_{n_k}\}$ be a subsequence of $\{\lambda_n\}$ which converges to λ. Then

$$\begin{aligned} \|\lambda - \lambda_{n_k}\|_{X^*} &\geq |(\lambda - \lambda_{n_k})(x_{n_k})| \\ &= |\lambda_{n_k}(x_{n_k})| \geq \|\lambda_{n_k}\|/2 \end{aligned}$$

which implies $\|\lambda_{n_k}\| \to 0$ as $k \to \infty$. Thus $\lambda = 0$ which is a contradiction. Therefore $\mathscr{D}$ is dense and X is separable. ▮

The example of ℓ_1 and ℓ_∞ shows that the converse of this theorem does not hold. Incidently, Theorem III.7 provides a proof that ℓ_1 is not the dual of ℓ_∞, since ℓ_1 is separable and ℓ_∞ is not.

III.4 Operations on Banach spaces

We have already seen several ways in which new Banach spaces can arise from old ones. The successive duals of a Banach space are Banach spaces and the bounded operators from one Banach space to another form a Banach space. Also, any closed linear subspace of a Banach space is a Banach space. There are two other ways of constructing new Banach spaces which we will need: direct sums and quotient spaces.

Let A be an index set (not necessarily countable), and suppose that for each $\alpha \in A$, X_α is a Banach space. Let

$$X = \{\{x_\alpha\}_{\alpha \in A} \mid x_\alpha \in X_\alpha, \ \sum_{\alpha \in A} \|x_\alpha\|_{X_\alpha} < \infty\}$$

Then X with the norm

$$\|\{x_\alpha\}\| = \sum_{\alpha \in A} \|x_\alpha\|_{X_\alpha}$$

is a Banach space. It is called the **direct sum** of the spaces X_α and is often written $X = \bigoplus_{\alpha \in A} X_\alpha$. We remark that the Hilbert space direct sum and the Banach space direct sum are not necessarily the same. For example, if we take a countable number of copies of $\mathbb{C}$, the Banach space direct sum is ℓ_1, while the Hilbert space direct sum is ℓ_2. However, if one has a *finite* number of Hilbert spaces, their Hilbert space direct sum and their Banach space direct sum are isomorphic in the sense of Section III.1.

Let M be a closed linear subspace of a Banach space X. If X were a Hilbert space, we could write $X = M \oplus M^\perp$. The Banach space that we now define can sometimes take the place of $M^\perp$ in the Banach space case where there is no orthogonality. If x and y are elements of X, we will write $x \sim y$ if $x - y \in M$. The relation $\sim$ is an equivalence relation; we denote the set of equivalence classes by X/M. As usual we denote the equivalence class containing x by $[x]$. We define addition and scalar multiplication of equivalence classes by

$$\alpha[x] + \beta[y] = [\alpha x + \beta y]$$

which makes sense since the equivalence class on the right only depends on the equivalence classes from which x and y are chosen, not on the elements themselves. With these operations X/M, becomes a complex vector space (the class M is the zero element). Now define

$$\|[x]\|_1 = \inf_{m \in M} \|x - m\|_X$$

It is not hard to show that $\|\cdot\|_1$ is a norm on X/M. $\|[x]\| = 0$ implies $[x] = 0$ because M is closed. We will show that X/M with this norm is complete by using Theorem III.3. Let $\{[x_n]\}_{n=1}^{\infty}$, be an absolutely summable sequence in X/M. That is,

$$\sum_{n=1}^{\infty} \inf_{m \in M} \|x_n - m\| < \infty$$

For each n, choose $m_n \in M$ so that

$$\|x_n - m_n\| \leq 2 \inf_{m \in M} \|x_n - m\|$$

Then $\{x_n - m_n\}$ is absolutely summable in X. Since X is complete, $\{x_n - m_n\}$ is summable. Let

$$y = \lim_{N \to \infty} \sum_{n=1}^{N} (x_n - m_n)$$

Then

$$\left\| \sum_{n=1}^{N} [x_n] - [y] \right\|_1 \leq \left\| \sum_{n=1}^{N} x_n - y - \sum_{n=1}^{N} m_n \right\| \to 0 \quad \text{as } N \to \infty$$

This proves that $\{[x_n]\}$ is summable. Using Theorem III.3 again we conclude that X/M is complete. X/M is called the **quotient space** of X by M. The reader should work out the easy details of the following example.

Example Let $X = C[0, 1]$ and let $M = \{f \mid f(0) = 0\}$. Then $X/M = \mathbb{C}$.

III.5 The Baire category theorem and its consequences

Many questions in Banach space theory involve proving that sets have nonempty interiors. For example:

Proposition Let X and Y be normed linear spaces. Then a linear map $T: X \to Y$ is bounded if and only if

$$T^{-1}[\{y \mid \|y\|_Y \le 1\}]$$

has a nonempty interior.

Proof Suppose that T is given and the set in question contains the ball

$$\{x \mid \|x - x_0\|_X < \varepsilon\}$$

Then $\|x\| < \varepsilon$ implies

$$\|Tx\| \le \|T(x + x_0)\| + \|Tx_0\| \le 1 + \|T(x_0)\|$$

since $x + x_0$ is in the ball of radius ε about x_0. Thus for all $x \in X$,

$$\|Tx\| \le \varepsilon^{-1}(\|Tx_0\| + 1)\|x\|$$

so T is bounded. The converse is easy. ▌

It is thus of great interest to know when sets must have nonempty interiors. There is an extraordinary theorem about complete metric spaces. Before stating it, we make the following definition.

Definition A set S in a metric space M is called **nowhere dense** if $\bar{S}$ has an empty interior.

Theorem III.8 (Baire category theorem) A complete metric space is never the union of a countable number of nowhere dense sets.

Proof The idea of the proof is simple. Suppose that M is the complete metric space and $M = \bigcup_{n=1}^{\infty} A_n$ with each A_n nowhere dense. We will construct a Cauchy sequence $\{x_m\}$ which stays away from each A_n so that its limit point x (which is in M by completeness) is in no A_n, thereby contradicting the statement $M = \bigcup_{n=1}^{\infty} A_n$.

Since A_1 is nowhere dense, we can find $x_1 \notin \bar{A}_1$. Pick an open ball B_1 about x_1 so that $B_1 \cap A_1 = \emptyset$ and so that the radius of B_1 is smaller than one. Since A_2 is nowhere dense, we can find $x_2 \in B_1 \backslash \bar{A}_2$. Let B_2 be an open ball about x_2 so that $\bar{B}_2 \subset B_1$, $B_2 \cap A_2 = \emptyset$, and with radius smaller than $\frac{1}{2}$. Proceeding inductively, we pick $x_n \in B_{n-1} \backslash \bar{A}_n$ and choose an open ball B_n about x_n satisfying $\bar{B}_n \subset B_{n-1}$, $B_n \cap A_n = \emptyset$, and having a radius smaller than 2^{1-n}. Now $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence since $n, m \ge N$ implies that x_n, $x_m \in B_N$ so

$$\rho(x_n, x_m) \le 2^{1-N} + 2^{1-N} = 2^{2-N} \to 0$$

as $N \to \infty$. Let $x = \lim_{n\to\infty} x_n$. Since $x_n \in B_N$ for $n \geq N$, we have

$$x \in \bar{B}_N \subset B_{N-1}$$

Thus $x \notin A_{N-1}$ for any N which contradicts $M = \bigcup_{n=1}^{\infty} A_n$. ∎

The Baire category theorem tells us that if $M = \bigcup_{n=1}^{\infty} A_n$, then some of the sets $\bar{A}_n$ must have nonempty interior. In practice, one rarely uses the Baire category theorem directly but rather one of the following consequences. The first is known as the Banach–Steinhaus theorem or the principle of uniform boundedness.

Theorem III.9 (principle of uniform boundedness) Let X be a Banach space. Let $\mathscr{F}$ be a family of bounded linear transformations from X to some normed linear space Y. Suppose that for each $x \in X$, $\{\|Tx\|_Y \mid T \in \mathscr{F}\}$ is bounded. Then $\{\|T\| \mid T \in \mathscr{F}\}$ is bounded.

Proof Let $B_n = \{x \mid \|Tx\| \leq n$ for all $T \in \mathscr{F}\}$. By the hypothesis each x is in some B_n, that is, $X = \bigcup_{n=1}^{\infty} B_n$. Moreover each B_n is closed (since each T is continuous). By the Baire category theorem, some B_n has a nonempty interior. By mimicking the argument in the proposition at the beginning of this section, we conclude that the $\|T\|$'s are uniformly bounded. ∎

As a typical application of this theorem we have (see also Problem 13):

Corollary Let X and Y be Banach spaces and let $B(\cdot, \cdot)$ be a separately continuous bilinear mapping from $X \times Y$ to $\mathbb{C}$, that is, for each fixed x, $B(x, \cdot)$ is a bounded linear transformation, and for each fixed y, $B(\cdot, y)$ is a bounded linear transformation. Then $B(\cdot, \cdot)$ is jointly continuous, that is, if $x_n \to 0$ and $y_n \to 0$ then $B(x_n, y_n) \to 0$.

Proof Let $T_n(y) = B(x_n, y)$. Since $B(x_n, \cdot)$ is continuous, each T_n is bounded. Since $x_n \to 0$ and $B(\cdot, y)$ is bounded, $\{\|T_n(y)\|\}$ is bounded for each fixed y. Therefore, there exists C so that

$$\|T_n(y)\| \leq C\|y\|$$

for all n. Thus

$$\|B(x_n, y_n)\| = \|T_n(y_n)\| \leq C\|y_n\| \to 0$$

as $n \to \infty$. ∎

We remark that even on $\mathbb{R}^2$, for nonlinear functions separate continuity does not imply joint continuity. The standard example is

$$f(x, y) = \frac{xy}{x^2 + y^2} \qquad \text{if} \quad \langle x, y \rangle \neq \langle 0, 0 \rangle$$

$$f(0, 0) = 0$$

The second application of the Baire category theorem is to the following series of results.

Theorem III.10 (open mapping theorem) Let $T: X \to Y$ be a bounded linear transformation of one Banach space *onto* another Banach space Y. Then if M is an open set in X, $T[M]$ is open in Y.

Proof We make a series of remarks which will simplify the proof. We need only show that, for every neighborhood N of x, $T[N]$ is a neighborhood of $T(x)$. Since $T[x + N] = T(x) + T[N]$ we need only show this for $x = 0$. Since neighborhoods contain balls it is sufficient to show that $T[B_r^X] \supset B_{r'}^Y$ for some r' where

$$B_r^X = \{x \in X \mid \|x\| < r\}$$

However, since $T[B_r^X] = rT[B_1^X]$, we need only show that $T[B_r^X]$ is a neighborhood of zero for some r. Finally, by the "translation argument" of the proposition, it is sufficient to show that $T[B_r^X]$ has a nonempty interior for some r.

Since T is onto,

$$Y = \bigcup_{n=1}^{\infty} T[B_n]$$

so some $\overline{T(B_n)}$ has a nonempty interior. Now the hard work begins, since we want $T(B_n)$ to have a nonempty interior. By scaling and translating we can suppose that B_ε is contained in $\overline{T[B_1]}$; we will show that $\overline{T[B_1]} \subset T[B_2]$ which will complete the proof.

Let $y \in \overline{T[B_1]}$. Pick $x_1 \in B_1$ so $y - Tx_1 \in B_{\varepsilon/2} \subset \overline{T[B_{1/2}]}$. Now pick $x_2 \in B_{1/2}$ so that

$$y - Tx_1 - Tx_2 \in B_{\varepsilon/4}$$

By induction, we choose $x_n \in B_{2^{1-n}}$ so that

$$y - \sum_{j=1}^{n} Tx_j \in B_{\varepsilon 2^{1-n}}$$

Then $x = \sum_{i=1}^{\infty} x_i$ exists and is in B_2 and

$$y = \sum_{j=1}^{\infty} Tx_i = Tx$$

Thus $y \in T[B_2]$. ∎

Theorem III.11 (inverse mapping theorem) A continuous bijection of one Banach space onto another has a continuous inverse.

Proof T is open so T^{-1} is continuous. ∎

For an application of this result see Problem 19.

Definition Let T be a mapping of a normed linear space X into a normed linear space Y. The **graph** of T, denoted by $\Gamma(T)$, is defined as

$$\Gamma(T) = \{\langle x, y\rangle \mid \langle x, y\rangle \in X \times Y, \quad y = Tx\}$$

Theorem III.12 (closed graph theorem) Let X and Y be Banach spaces and T a linear map of X into Y. Then T is bounded if and only if the graph of T is closed.

Proof Suppose that $\Gamma(T)$ is closed. Then, since T is linear, $\Gamma(T)$ is a subspace of the Banach space $X \oplus Y$. By assumption $\Gamma(T)$ is closed and thus is a Banach space in the norm

$$\|\langle x, Tx\rangle\| = \|x\| + \|Tx\|$$

Consider the continuous maps Π_1, Π_2,

$$\Pi_1 : \langle x, Tx\rangle \to x, \qquad \Pi_2 : \langle x, Tx\rangle \to Tx$$

Π_1 is a bijection so by the inverse mapping theorem Π_1^{-1} is continuous. But $T = \Pi_2 \circ \Pi_1^{-1}$, so T is continuous. The converse is trivial. ∎

To avoid future confusion, we emphasize that the T in this theorem is implicitly assumed to be defined on all of X. We will later deal with transformations defined on *algebraic* subspaces of X (not all of X) with closed graphs which are not continuous. To appreciate what the closed graph theorem really does, consider the three statements:

(a) x_n converges to some element x.
(b) Tx_n converges to some element y.
(c) $Tx = y$.

A priori, to prove that T is continuous one must show that (a) implies (b) and (c). What the closed graph theorem says is that it is sufficient to prove that (a) and (b) imply (c).

The following corollary of the closed graph theorem has important consequences in mathematical physics.

Corollary (the Hellinger–Toeplitz theorem) Let A be an everywhere-defined linear operator on a Hilbert space $\mathscr{H}$ with $(x, Ay) = (Ax, y)$ for all x and y in $\mathscr{H}$. Then A is bounded.

Proof We will prove that $\Gamma(A)$ is closed. Suppose that $\langle x_n, Ax_n \rangle \to \langle x, y \rangle$. We need only prove that $\langle x, y \rangle \in \Gamma(A)$, that is, that $y = Ax$. But, for any $z \in \mathscr{H}$,

$$(z, y) = \lim_{n\to\infty} (z, Ax_n) = \lim_{n\to\infty} (Az, x_n)$$
$$= (Az, x) = (z, Ax)$$

Thus $y = Ax$ and $\Gamma(A)$ is closed. ∎

As we shall see, this theorem is the cause of much technical pain because in quantum mechanics there are operators (like the energy) which are unbounded but which we want to obey

$$(x, Ay) = (Ax, y)$$

in some sense. The Hellinger–Toeplitz theorem tells us that such operators cannot be everywhere defined. Thus such operators are defined on subspaces $D(A)$ of $\mathscr{H}$ and defining what one means by $A + B$ or AB may be difficult. For example, $A + B$ is a priori only defined on $D(A) \cap D(B)$ which may equal $\{0\}$ even in the case where both $D(A)$ and $D(B)$ are dense. We return to these questions in Chapters VIII and X.

NOTES

Section III.1 The name Banach space honors the important work of S. Banach on normed linear spaces during the 1920's culminating in his book, *Théorie des Opérations Linéaires*, Monografie Math., I, Warsaw, 1932. A good elementary reference for the material in this chapter is the book *Foundations of Modern Analysis* by A. Friedman, Holt, New York, 1970. In the second supplement we prove Hölder's inequality only in the case $r = 1$. To prove the general case where $p^{-1} + q^{-1} = r^{-1}$ observe that

$$|fg|^r = |f|^r |g|^r$$

and use the Hölder inequality for the special case where

$$\frac{1}{(p/r)} + \frac{1}{(q/r)} = 1$$

obtaining

$$\int |fg|^r \le \left(\int |f|^{rp/r}\right)^{r/p} \left(\int |g|^{rq/r}\right)^{r/q}$$

or

$$\left(\int |fg|^r\right)^{1/r} \le \left(\int |f|^p\right)^{1/p} \left(\int |g|^q\right)^{1/q}$$

Suppose X is a Banach space. One of the ways of studying the Banach space of operators from X to itself, $\mathscr{L}(X, X)$, is to use the fact that it is also an algebra. Thus, one can use algebraic notions like ideals and commutators to investigate the structure of $\mathscr{L}(X, X)$. In Section VI.6 certain important ideals of $\mathscr{L}(\mathscr{H}, \mathscr{H})$, where $\mathscr{H}$ is a separable Hilbert space, are studied. The general theory of operator algebras is studied in later volumes.

Section III.2 The proof that $(L^p)^* = L^q$ may be found in Royden's book (see the Notes for Chapter I) or may be proven using the notion of uniformly convex space (see Problems 25 and 26 of Chapter III and Problem 15 of Chapter V). In Section VI.6 we discuss the duals of several subalgebras of $\mathscr{L}(\mathscr{H}, \mathscr{H})$.

Section III.3 The Hahn–Banach theorem dates back to the work of Helly in " Über Lineare Funktional Operationen," *Sitzgsber, Akad. Wiss. Wien Math-Nat. Kl.* **121 IIa**, (1912), 265–297, and " Über Systeme linearer Gleichungen mit Unendlich Vielen Unbekannten," *Monatsh. Math. Phys.* 31. (1921), 60–91. The modern version is due to H. Hahn, "Über lineare Gleichungssystem in linearen Räumen," *J. Reine Angew. Math.* **157**, (1926), 214–229, and S. Banach, "Sur les fonctionelles linéaires, I, II," *Studia Math.* **1** (1929), 211–216, 223–239. A nice example of the concrete applications of the Hahn–Banach theorem may be found in the book by Friedman mentioned above. There it is shown how to use the Hahn–Banach theorem to prove the existence of a Green's function for the Dirichlet problem in two dimensions.

Section III.5 The Baire theorem was proven in R. Baire, "Sur les fonctions de variables réelles," *Annali di Mat. Ser. 3* 3 (1899), 1–123. The general case is in C. Kuratowski, "La propriété de Baire dans les espaces métriques," *Fund. Math.* **16** (1930), 390–394, and S. Banach, "Théorèmes sur les ensembles de premières catégorie," *Fund. Math.* **16**, (1930), 395–398. The Banach–Steinhaus theorem was proven by S. Banach and H. Steinhaus in "Sur le principe de la condensation de singularités," *Fund. Math.* **9**, (1927), 50–61. There is a discussion of the Baire theorem and its consequences in Lorch's book (see the Notes for Section II.5). The term category comes from the following: A countable union of nowhere dense sets is called a **first category** set. All other sets are called **second category**. The Baire theorem says that any complete metric space is second category.

Complements of first category sets are often called residuals. A residual set is thus a set containing a countable intersection of dense open sets. The Baire theorem implies that any residual in a complete metric space is dense (Problem 21).

In metric spaces, one sometimes says something is "true almost everywhere" if it is true on a residual; thereby, first category sets play the role of "sets of measure zero". There are some amusing results on this notion of a.e. in G. Choquet's book, *Lectures in Analysis*, Vol. I, pp. 120–126, Benjamin, New York, 1969. Warning: There exist sets $X \subset [0, 1]$ which are first category with measure 1! Thus the two notions of a.e., Lebesgue and Baire, are quite different.

Other topological spaces besides complete metric spaces have the property that residuals are dense; such spaces are called Baire spaces. For example, every locally compact space is Baire. For additional discussion, see Choquet's book, pp. 105–120.

PROBLEMS

†*1.* Prove that $L^\infty(\mathbb{R})$ is a Banach space.

†*2.* (a) Prove that ℓ_p and c_0 are separable but ℓ_∞ is not.
(b) Prove that $s \subset \ell_p$ for all p.

†*3.* Prove that a normed linear space is complete if and only if every absolutely summable sequence is summable. (Hint for the "if" part: To show that a Cauchy sequence converges it is only necessary to show that a subsequence converges.)

•*4.* Prove that all norms on $\mathbb{R}^n$ are equivalent. (Hint: Use the fact that the unit sphere is compact in the Euclidean topology.)

†*5.* Prove that $C_\infty(\mathbb{R})$ is the completion of $\kappa(\mathbb{R})$.

6. Prove that if $\{\lambda_k\}_{k=1}^\infty \in \ell_1$ then the linear functional on c_0 given by

$$\Lambda(\{a_k\}_{k=1}^\infty) = \sum \lambda_k a_k$$

has norm $\sum_{k=1}^\infty |\lambda_k|$.

7. Prove that $\ell_\infty = \ell_1^*$ but that $\ell_\infty^* \neq \ell_1$ by using the Hahn–Banach theorem.

8. (a) Prove that there is a nonzero bounded linear functional on $L^\infty(\mathbb{R})$ which vanishes on $C(\mathbb{R})$.
(b) Prove that there is a bounded linear functional λ on $L^\infty(\mathbb{R})$ such that $\lambda(f) = f(0)$ for each $f \in C(\mathbb{R})$.

9. Suppose that $\mathscr{H}$ is a Hilbert space and that λ is a bounded linear functional on $\mathscr{M}$, a not necessarily closed subspace. Describe the continuous extensions of λ.

†*10.* Prove the third corollary to the Hahn–Banach theorem.

11. Prove that there is a linear functional λ on $\ell_\infty(\mathbb{R})$ so that

$$\underline{\lim}_{n\to\infty} a_n \leq \lambda(\{a_n\}_{n=1}^\infty) \leq \overline{\lim}_{n\to\infty} a_n$$

$\ell_\infty(\mathbb{R}) = \{a \mid a \in \ell_\infty,\quad a_n \in \mathbb{R}\quad \text{for all } n\}$.

†*12.* Prove the statement in the example at the end of Section 4.

13. Use the uniform boundedness principle to provide an alternative proof of the Hellinger–Toeplitz theorem.

**14.* Let X be a Banach space. Give an example of an everywhere-defined but discontinuous linear functional λ. Show directly that λ is not closed.

15. Let $\mathscr{H}$ be a separable Hilbert space with an orthonormal basis $\{x_n\}_{n=1}^{\infty}$. Let $\{y_n\}$ be a sequence of elements of $\mathscr{H}$ and prove that the following two statements are equivalent.

(a) $(x, y_n) \underset{n\to\infty}{\to} 0$, $\forall x \in \mathscr{H}$.

(b) $(x_m, y_n) \underset{n\to\infty}{\to} 0$, for each $m = 1, 2, \ldots$, and $\{\|y_n\|\}_{n=1}^{\infty}$ is bounded.

16. A subset S of a Banach space is called **weakly bounded** if and only if for all $\lambda \in X^*$, $\sup_{x\in S} |\lambda(x)| < \infty$. S is called **strongly bounded** if and only if $\sup_{x\in S} \|x\| < \infty$. Prove that a set is strongly bounded if and only if it is weakly bounded (see Section V.7).

17. Prove that a separately continuous multilinear functional on a Banach space is jointly continuous.

18. Extend the Hellinger–Toeplitz theorem to include pairs of operators A, B satisfying: $(Ax, y) = (x, By)$.

19. Let X be a Banach space in either of the norms $\|\cdot\|_1$ or $\|\cdot\|_2$. Suppose that $\|\cdot\|_1 \le C\|\cdot\|_2$ for some C. Prove that there is a D with $\|\cdot\|_2 \le D\|\cdot\|_1$.

20. Why doesn't a one-point space violate the Baire theorem?

**21.* Prove that any countable intersection of dense open sets in a complete metric space is dense.

22. (a) Prove that a Banach space X is reflexive if and only if X^* is reflexive. (Hint: If $X \ne X^{**}$ find a bounded linear functional on X^{**} which vanishes on X).
(b) Prove that whenever X is a nonreflexive Banach space, $(\cdots(X^*)^*\cdots)^*$ is not reflexive.

23. Let X be a Hilbert space and let $\mathscr{M}$ be a closed subspace. Show that the restriction of the natural map $\pi : X \to X/\mathscr{M}$ is an isomorphism of $\mathscr{M}^\perp$ and $X/\mathscr{M}$.

24. Let ℓ be a linear functional on a real Banach space X. Prove that $X/\ker \ell$ is isomorphic to $\mathbb{R}$ with the usual norm and that the natural projection $\pi : X \to X/\ker \ell = \mathbb{R}$ is related to ℓ by $\ell = \pm \|\ell\|\pi$.

25. A Banach space is called **uniformly convex** if for each $\varepsilon > 0$, there is a $\delta > 0$, so that $\|x\| = \|y\| = 1$ and $\|\frac{1}{2}(x+y)\| > 1 - \delta$ imply $\|x - y\| < \varepsilon$; thus the unit ball is uniformly round. We will see in Problem 15 of Chapter V, that every uniformly convex space is reflexive.

(a) Prove directly that $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$ are not uniformly convex.
(b) Prove that any Hilbert space is uniformly convex.
*(c) Prove that $L^p(X, d\mu)$ is uniformly convex for $p \geq 2$. Hint: Prove that for $\alpha, \beta \in \mathbb{C}$, one has $|\alpha+\beta|^p + |\alpha-\beta|^p \leq 2^{p-1}(|\alpha|^p + |\beta|^p)$ by first proving

$$(|\alpha+\beta|^p + |\alpha-\beta|^p)^{1/p} \leq \sqrt{2}(|\alpha|^2 + |\beta|^2)^{1/2}$$

Notes: 1. L^p is actually uniformly convex for all $1 < p < \infty$, but the proof for $1 < p < 2$ is harder; c.f. G. Köthe: *Topological Vector Spaces, I*, Springer (1969), 358–359.
2. Uniformly convex spaces were introduced by J. Clarkson, "Uniformly convex spaces," *Trans. A.M.S.* **40** (1936), 396–414.
3. M. Day has given examples of reflexive Banach spaces which are not uniformly convex in "Reflexive Banach spaces not isomorphic to uniformly convex spaces," *Bull. A.M.S.* **47** (1941), 313–317; see also Köthe, pp. 360–363.

26. (a) A pair of Banach spaces, X and Y, are said to be in **strict duality** if there is a map $f: X \to Y^*$ which is isometric, so that the induced map $f^*: Y \to X^*$ is also isometric. Prove that if X and Y are in strict duality and X is reflexive, then $Y = X^*$ and $X = Y^*$. (Hint: Use the Hahn–Banach theorem.)
(b) Prove that $L^p(X, d\mu)$ and $L^q(X, d\mu)$ are in strict duality if $p^{-1} + q^{-1} = 1$.
(c) Prove that $L^p(X, d\mu)^* = L^q(X, d\mu)$ if $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$. (Hint: Use Problem 25 and Problem 15 of Chapter V).

**27.* Prove the Banach–Schauder theorem: Let T be a continuous linear map, $T: E \to F$, where E and F are Banach spaces. Then either $T[A]$ is open in $T[E]$ for each open $A \subset E$, or $T[E]$ is of first category in $\overline{T[E]}$ (see the notes to Section 5 for the definition of first category).

28. (a) Prove that every quotient of ℓ_2 by a closed subspace is isometrically isomorphic to either ℓ_2 or $\mathbb{C}^N$ for some N.
(b) Prove that ℓ_1 is not topologically isomorphic to any quotient space of ℓ_2.

29. Let X be a separable Banach space. Let $\{x_1, \ldots, x_n, \ldots\}$ be a dense subset of the unit ball in X. Map $\ell_1 \to X$ by

$$A: \langle \alpha_1, \ldots, \alpha_n, \ldots \rangle \to \sum_{n=1}^{\infty} \alpha_n x_n$$

(a) Prove that A is well defined and continuous.
(b) Prove that Ker A is closed and that A "lifts" to a continuous map $\hat{A}: \ell_1/\text{Ker } A \to X$.
(c) Prove Ran $\hat{A}$ = Ran A is all of X. Hint: Given x with $\|x\| = 1$, choose $x_{n(i)}$ recursively by requiring

$$\left\| x - \sum_{i=1}^{k} 2^{-i+1} x_{n(i)} \right\| \leq 2^{-k}$$

(d) Conclude that any separable Banach space is topologically isomorphic to some quotient space of ℓ_1.
(e) By using (c) with 2 replaced by 3, 4, ..., show that $\hat{A}$ is actually an isometry.

30. Let X be a Banach space and let Y be a closed subspace of X. Let Y° in X^* be defined by

$$Y^\circ = \{\ell \in X^* \mid \ell \restriction Y = 0\}$$

Given a bounded linear functional f on X/Y, define $\pi^*(f) \in X^*$ by $[\pi^*(f)](x) = f([x])$. Prove that π^* is an isometric isomorphism of $(X/Y)^*$ onto Y°.

31. (a) Let E be a Banach space with separable dual and $\langle M, \mu\rangle$ a measure space with $L^p(M, d\mu)$ separable for all $1 < p < \infty$. Develop the theory of $L^p(M, d\mu; E)$ analogous to the theory of $L^2(M, d\mu; \mathscr{H})$ discussed in Sections II.1 and II.4.

(b) Prove $L^p(M \times N, d\mu \otimes d\nu)$ and $L^p(M, d\mu; L^p(N, d\nu))$ are naturally isomorphic.

*(c) Let E^{**} be a separable Banach space and let $1 \le p < \infty$. Prove that $L^p(M, d\mu, E)^*$ is naturally isometrically isomorphic to $L^q(M, d\mu, E^*)$ (Hint: First show that it is enough to prove that every bounded linear transformation T of E into $L^q(M, d\mu)$ is of the form $[T(x)](m) = [f(m)](x)$ for some $f \in L^q(M, d\mu; E^*)$. Prove this in the special case where $E = \ell_1$. Finally use Problems 29 and 30 to treat the general separable Banach space, E.)

**32.* Let S be a closed linear subspace of $L^1[0, 1]$. Suppose that $f \in S$ implies that $f \in L^p[0, 1]$ for some $p > 1$. Prove that $S \subset L^p[0, 1]$ for some $p > 1$.