VI: Bounded Operators

I was at the mathematical school, where the master taught his pupils after a method scarce imaginable to us in Europe. The proposition and demonstration were fairly written on a thin wafer, with ink composed of a cephalic tincture. This the student was to swallow upon a fasting stomach, and for three days following eat nothing but bread and water. As the wafer digested the tincture mounted to the brain, bearing the proposition along with it.

*Jonathan Swift in **Gulliver's Travels***

VI.1 Topologies on bounded operators

We have already introduced $\mathscr{L}(X, Y)$, the Banach space of operators from one Banach space to another. In this chapter we will study $\mathscr{L}(X, Y)$ more closely. We emphasize the case which will arise most frequently later, namely, $\mathscr{L}(\mathscr{H}, \mathscr{H}) \equiv \mathscr{L}(\mathscr{H})$ where $\mathscr{H}$ is a separable Hilbert space. Theorem III.2 shows that $\mathscr{L}(X, Y)$ is a Banach space with the norm

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X}$$

The induced topology on $\mathscr{L}(X, Y)$ is called the **uniform operator topology** (or **norm** topology). In this topology the map $\langle A, B\rangle \to BA$ of $\mathscr{L}(X, Y) \times \mathscr{L}(Y, Z) \to \mathscr{L}(X, Z)$ is jointly continuous.

We now introduce two new topologies on $\mathscr{L}(X, Y)$, the weak and strong operator topologies. There are other interesting and useful topologies on $\mathscr{L}(X, Y)$, but we delay their introduction until we need them in a later volume (see however the discussion at the end of Section 6 and the Notes).

The **strong operator topology** is the weakest topology on $\mathscr{L}(X, Y)$ such that the maps

$$E_x : \mathscr{L}(X, Y) \to Y$$

given by $E_x(T) = Tx$ are continuous for all $x \in X$. A neighborhood basis at the origin is given by sets of the form

$$\{S \mid S \in \mathscr{L}(X, Y), \quad \|Sx_i\|_Y < \varepsilon, \quad i = 1, \ldots, n\}$$

where $\{x_i\}_{i=1}^n$ is a finite collection of elements of X and ε is positive. In this topology a net $\{T_\alpha\}$ of operators converges to an operator T (written $T_\alpha \xrightarrow{s} T$) if and only if $\|T_\alpha x - Tx\| \to 0$ for all $x \in X$. The map $\langle A, B\rangle \to AB$ is separately but not jointly continuous if X, Y, and Z are infinite dimensional (see Problem 6a, b). We sometimes denote strong limits by the symbol s-lim.

The **weak operator topology** on $\mathscr{L}(X, Y)$ is the weakest topology such that the maps

$$E_{x,\ell}\colon \mathscr{L}(X, Y) \to \mathbb{C}$$

given by $E_{x,\ell}(T) = \ell(Tx)$ are all continuous for all $x \in X$, $\ell \in Y^*$. A basis at the origin is given by sets of the form

$$\{S \mid S \in \mathscr{L}(X, Y), \quad |\ell_i(Tx_j)| < \varepsilon, \quad i = 1, \ldots, n, \quad j = 1, \ldots, m\}$$

where $\{x_i\}_{i=1}^n$ and $\{\ell_j\}_{j=1}^m$ are finite families of elements of X and Y^* respectively. A net of operators $\{T_\alpha\}$ converges to an operator T in the weak operator topology (written $T_\alpha \xrightarrow{w} T$) if and only if $|\ell(T_\alpha x) - \ell(Tx)| \to 0$ for each $\ell \in Y^*$ and $x \in X$. Notice that in the case $\mathscr{L}(\mathscr{H})$, $T_\gamma \to T$ weakly just means that the "matrix elements" $(y, T_\gamma x)$ converge to (y, Tx). In the weak topology the map $\langle A, B\rangle \to AB$ is separately, but not jointly continuous if X, Y, and Z are infinite dimensional (see Problem 6c).

Remark The reader should not confuse the weak operator topology on $\mathscr{L}(X, Y)$ with the weak (Banach space) topology on $\mathscr{L}(X, Y)$. The former is the weakest topology such that the bounded linear functionals on $\mathscr{L}(X, Y)$ of the form $\ell(\cdot x)$ are continuous for all $x \in X$ and $\ell \in Y^*$. The latter is the weakest topology such that *all* bounded linear functionals on $\mathscr{L}(X, Y)$ are continuous (see Section VI.6).

Notice that the weak operator topology is weaker than the strong operator topology which is weaker than the uniform operator topology. In general, the weak and strong operator topologies on $\mathscr{L}(X, Y)$ will not be first countable so that questions of compactness, net convergence, and sequential convergence are complicated. The following simple example illustrates the different topologies on $\mathscr{L}(\ell_2)$.

Example Consider the bounded operators on ℓ_2.

(i) Let T_n be defined by

$$T_n(\xi_1, \xi_2, \ldots) = \left(\frac{1}{n}\xi_1, \frac{1}{n}\xi_2, \ldots\right)$$

Then $T_n \to 0$ uniformly.

(ii) Let S_n be defined by

$$S_n(\xi_1, \xi_2, \ldots) = (\underbrace{0, 0, \ldots, 0}_{n \text{ places}}, \xi_{n+1}, \xi_{n+2}, \ldots)$$

Then $S_n \to 0$ strongly but not uniformly.

(iii) Let W_n be defined by

$$W_n(\xi_1, \xi_2, \ldots) = (\underbrace{0, 0, \ldots, 0}_{n \text{ places}}, \xi_1, \xi_2, \ldots)$$

Then $W_n \to 0$ in the weak operator topology but not in the strong or uniform topologies.

The following result in the Hilbert space case is sometimes useful and provides a nice application of the uniform boundedness theorem.

Theorem VI.1 Let $\mathscr{L}(\mathscr{H})$ denote the bounded operators on a Hilbert space $\mathscr{H}$. Let T_n be a *sequence* of bounded operators and suppose that $(T_n x, y)$ converges as $n \to \infty$ for each $x, y \in \mathscr{H}$. Then there exists $T \in \mathscr{L}(\mathscr{H})$ such that $T_n \xrightarrow{w} T$.

Proof We begin by showing that for each x, $\sup_n \|T_n x\| < \infty$. Since for any $x \in \mathscr{H}$, $(x, T_n y)$ converges we have

$$\sup_n |(T_n x, y)| < \infty$$

For each n, $T_n x \in \mathscr{L}(\mathscr{H}, \mathbb{C})$, and since $\sup_n |(T_n x)(y)|_{\mathbb{C}} < \infty$, the uniform boundedness theorem implies that the operator norms of the $T_n x$ in $\mathscr{L}(\mathscr{H}, \mathbb{C})$ are uniformly bounded. But the norm of $T_n x$ as an operator in $\mathscr{L}(\mathscr{H}, \mathbb{C})$ is the same as its norm in $\mathscr{H}$; thus $\|T_n x\|_{\mathscr{H}}$ is uniformly bounded.

Now, we use the uniform boundedness theorem again. Since

$$\sup_n \|T_n x\|_{\mathscr{H}} < \infty,$$

we conclude

$$\sup_n \|T_n\|_{\mathscr{L}(\mathscr{H})} < \infty$$

Define $B(x, y) = \lim_n (T_n x, y)$. Then it is easily verified that $B(x, y)$ is sesquilinear and

$$|B(x, y)| \le \overline{\lim_n} |(T_n x, y)| \le \|x\| \|y\| (\sup_n \|T_n\|)$$

Thus $B(x, y)$ is a bounded sesquilinear form on $\mathscr{H}$ and so, by the corollary to the Riesz lemma, there is a bounded operator $T \in \mathscr{L}(\mathscr{H})$ such that $B(x, y) = (Tx, y)$. Clearly $T_n \xrightarrow{w} T$. ∎

If a *sequence* of operators T_n on a Hilbert space has the property that $T_n x$ converges for each $x \in \mathscr{H}$, then there exists $T \in \mathscr{L}(\mathscr{H})$ such that $T_n \xrightarrow{s} T$. The reader is asked to prove this theorem and various generalizations in Problem 3.

Let $T \in \mathscr{L}(X, Y)$. The set of vectors $x \in X$ so that $Tx = 0$ is called the **kernel** of T, written Ker T. The set of vectors $y \in Y$ so that $y = Tx$ for some $x \in X$ is called the **range** of T, written Ran T. Notice that both Ker T and Ran T are subspaces. Ker T is necessarily closed, but Ran T may not be closed (Problem 7).

VI.2 Adjoints

In this section we define adjoints of bounded operators on Banach and Hilbert spaces. The reader should be cautioned at the outset that the Hilbert space adjoint of an operator $T \in \mathscr{L}(\mathscr{H})$ is not equal to the Banach space adjoint although it is closely related to it.

Definition Let X and Y be Banach spaces, T a bounded linear operator from X to Y. The Banach space **adjoint** of T, denoted by T', is the bounded linear operator from Y^* to X^* defined by

$$(T'\ell)(x) = \ell(Tx)$$

for all $\ell \in Y^*$, $x \in X$.

Example Let $X = \ell_1 = Y$ and let T be the right shift operator

$$T(\xi_1, \xi_2, \ldots) = (0, \xi_1, \xi_2, \ldots)$$

Then $T' : \ell_\infty \to \ell_\infty$ is the operator

$$T'(\xi_1, \xi_2, \ldots) = (\xi_2, \xi_3, \ldots)$$

In this example, $\|T\| = 1 = \|T'\|$. In fact the norms of T and T' are always equal:

Theorem VI.2 Let X and Y be Banach spaces. The map $T \to T'$ is an isometric isomorphism of $\mathscr{L}(X, Y)$ *into* $\mathscr{L}(Y^*, X^*)$.

Proof The map $T \to T'$ is linear. The fact that T' is bounded and that the map is an isometry follows from the computation

$$\begin{aligned}
\|T\|_{\mathscr{L}(X,Y)} &= \sup_{\|x\| \le 1} \|Tx\|_Y \\
&= \sup_{\|x\| \le 1} \Big(\sup_{\|\ell\| \le 1} |\ell(Tx)| \Big) \qquad \ell \in Y^* \\
&= \sup_{\|\ell\| \le 1} \Big(\sup_{\|x\| \le 1} |(T'\ell)(x)| \Big) \\
&= \sup_{\|\ell\| \le 1} \|T'\ell\| \\
&= \|T'\|_{\mathscr{L}(Y^*, X^*)}
\end{aligned}$$

The second equality uses a corollary of the Hahn–Banach theorem. ∎

We are mostly interested in the case where T is a bounded linear transformation of a Hilbert space $\mathscr{H}$ to itself. The Banach space adjoint of T is then a mapping of $\mathscr{H}^*$ to $\mathscr{H}^*$. Let $C: \mathscr{H} \to \mathscr{H}^*$ be the map which assigns to each $y \in \mathscr{H}$, the bounded linear functional $(y, \cdot)$ in $\mathscr{H}^*$. C is a *conjugate* linear isometry which is surjective by the Riesz lemma. Now define a map T^*: $\mathscr{H} \to \mathscr{H}$ by

$$T^* = C^{-1}T'C$$

Then T^* satisfies

$$(x, Ty) = (Cx)(Ty) = (T'Cx)(y) = (C^{-1}T'Cx, y) = (T^*x, y)$$

T^* is called the **Hilbert space adjoint** of T, but usually we will just call it the adjoint and let the * distinguish it from T'. Notice that the map $T \to T^*$ is *conjugate* linear, that is, $\alpha T \to \bar{\alpha} T^*$. This is because C is conjugate linear. We summarize the properties of the map $T \to T^*$:

Theorem VI.3 (a) $T \to T^*$ is a conjugate linear isometric isomorphism of $\mathscr{L}(\mathscr{H})$ onto $\mathscr{L}(\mathscr{H})$.

(b) $(TS)^* = S^*T^*$.

(c) $(T^*)^* = T$.

(d) If T has a bounded inverse, T^{-1}, then T^* has a bounded inverse and $(T^*)^{-1} = (T^{-1})^*$.

(e) The map $T \to T^*$ is always continuous in the weak and uniform operator topologies but is only continuous in the strong operator topology if $\mathscr{H}$ is finite dimensional.

(f) $\|T^*T\| = \|T\|^2$.

Proof (a) follows from Theorem VI.2 and the fact that C is an isometry. (b) and (c) are easily checked. Since $T^{-1}T = I = TT^{-1}$ we have from (b)

$$T^*(T^{-1})^* = I^* = I = I^* = (T^{-1})^*T^*$$

which proves (d).

Continuity of $T \to T^*$ in the weak and uniform operator topologies is trivial. In the case $\mathscr{H} = \ell_2$, here is a counter example which shows that $T \to T^*$ is not continuous in the strong operator topology. The general infinite dimensional case is similar. Let W_n be right shift on ℓ_2 by n places. Then W_n converges weakly but not strongly to zero. However, $W_n^* = V_n$ converges strongly to zero. Thus $V_n \xrightarrow{s} 0$, but $V_n^* = W_n$ does not converge strongly to zero.

(f) Note that $\|T^*T\| \le \|T\| \; \|T^*\| = \|T\|^2$ and

$$\|T^*T\| \ge \sup_{\|x\|=1} (x, T^*Tx) = \sup_{\|x\|=1} \|Tx\|^2 = \|T\|^2 \quad ∎$$

Definition A bounded operator T on a Hilbert space is called **self-adjoint** if $T = T^*$.

Self-adjoint operators play a major role in functional analysis and mathematical physics and much of our time is devoted to studying them. Chapter VII is devoted to proving a structure theorem for bounded self-adjoint operators. In Chapter VIII we introduce unbounded self-adjoint operators and continue their study in Chapter X. We remind the reader that on $\mathbb{C}^n$, a linear transformation is self-adjoint if and only if its matrix in any orthonormal basis is invariant under the operation of reflection across the diagonal followed by complex conjugation.

An important class of operators on Hilbert spaces is that of the projections.

Definition If $P \in \mathscr{L}(\mathscr{H})$ and $P^2 = P$, then P is called a **projection**. If in addition $P = P^*$, then P is called an **orthogonal projection**.

Notice that the range of a projection is always a closed subspace on which P acts like the identity. If in addition P is orthogonal, then P acts like the zero operator on $(\text{Ran } P)^\perp$. If $x = y + z$, with $y \in \text{Ran } P$ and $z \in (\text{Ran } P)^\perp$, is the decomposition guaranteed by the projection theorem, then $Px = y$. P is called the orthogonal projection onto Ran P. Thus, the projection theorem sets up a one to one correspondence between orthogonal projections and closed subspaces. Since orthogonal projections arise more frequently than non-orthogonal ones, we normally use the word projection to mean orthogonal projection.

VI.3 The spectrum

If T is a linear transformation on $\mathbb{C}^n$, then the eigenvalues of T are the complex numbers λ such that the determinant of $\lambda I - T$ is equal to zero. The set of such λ is called the spectrum of T. It can consist of at most n points since $\det(\lambda I - T)$ is a polynomial of degree n. If λ is not an eigenvalue, then $\lambda I - T$ has an inverse since $\det(\lambda I - T) \neq 0$.

The spectral theory of operators on infinite-dimensional spaces is more complicated, more interesting, and very important for an understanding of the operators themselves.

Definition Let $T \in \mathscr{L}(X)$. A complex number λ is said to be in the **resolvent set** $\rho(T)$ of T if $\lambda I - T$ is a bijection with a bounded inverse. $R_\lambda(T) = (\lambda I - T)^{-1}$ is called the **resolvent** of T at λ. If $\lambda \notin \rho(T)$, then λ is said to be in the **spectrum** $\sigma(T)$ of T.

We note that by the inverse mapping theorem, $\lambda I - T$ automatically has a bounded inverse if it is bijective. We distinguish two subsets of the spectrum.

Definition Let $T \in \mathscr{L}(X)$.

(a) An $x \neq 0$ which satisfies $Tx = \lambda x$ for some $\lambda \in \mathbb{C}$ is called an **eigenvector** of T; λ is called the corresponding **eigenvalue**. If λ is an eigenvalue, then $\lambda I - T$ is not injective so λ is in the spectrum of T. The set of all eigenvalues is called the **point spectrum** of T.

(b) If λ is not an eigenvalue and if $\mathrm{Ran}(\lambda I - T)$ is not dense, then λ is said to be in the **residual spectrum.**

At the end of this section we present an example which illustrates these kinds of spectra. The reason that we single out the residual spectrum is that it does not occur for a large class of operators, for example, for self-adjoint operators (see Theorem VI.8).

The spectral analysis of operators is very important for mathematical physics. For example, in quantum mechanics the Hamiltonian is an unbounded self-adjoint operator on a Hilbert space. The point spectrum of the Hamiltonian corresponds to the energy levels of bound states of the system. The rest of the spectrum plays an important role in the scattering theory of the system (see Chapter XII).

We will shortly prove that the resolvent set $\rho(T)$ is open and that $R_\lambda(T)$ is an analytic operator-valued function on $\rho(T)$. This fact allows one to use complex analysis to study $R_\lambda(T)$ and thus to obtain information about T. We begin with a brief aside about vector-valued analytic functions.

Let X be a Banach space and let D be a region in the complex plane, i.e., a connected open subset of $\mathbb{C}$. A function, $x(\cdot)$, defined on D with values in X, is said to be **strongly analytic** at $z_0 \in D$ if the limit of $(x(z_0 + h) - x(z_0))/h$ exists in X as h goes to zero in $\mathbb{C}$. Starting from this point one can develop a theory of vector-valued analytic functions which is almost exactly parallel to the usual theory; in particular, a strongly analytic function has a norm-convergent Taylor series. We do not repeat this development here; see the notes for references. We do want to discuss one important point. There is another natural way to define Banach-valued analytic functions. Namely: a function $x(\cdot)$ on D with values in X is said to be **weakly analytic** if $\ell\,(x(\cdot))$ is a complex valued analytic function on D for each $\ell \in X^*$. Although this second definition of analytic is a priori weaker than the first, the two definitions are equivalent, a fact we will prove in a moment. This is very important, since weak analyticity is often much easier to check.

Lemma Let X be a Banach space. Then a sequence $\{x_n\}$ is Cauchy if and only if $\{\ell(x_n)\}$ is Cauchy, uniformly for $\ell \in X^*$, $\|\ell\| \le 1$.

Proof If $\{x_n\}$ is Cauchy, then $|\ell(x_n) - \ell(x_m)| \le \|x_n - x_m\|$ for all ℓ with $\|\ell\| \le 1$, so $\{\ell(x_n)\}$ is Cauchy uniformly. Conversely,

$$\|x_n - x_m\| = \sup_{\|\ell\| \le 1} |\ell(x_n - x_m)|$$

Thus, if $\{\ell(x_n)\}$ is Cauchy, uniformly for $\|\ell\| \le 1$, then $\{x_n\}$ is norm-Cauchy. ∎

Theorem VI.4 Every weakly analytic function is strongly analytic.

Proof Let $x(\cdot)$ be a weakly analytic function on D with values in X. Let

$z_0 \in D$ and suppose that Γ is a circle in D containing z_0 whose interior is contained in D. If $\ell \in X^*$ then $\ell(x(z))$ is analytic and

$$\ell\left(\frac{x(z_0+h)-x(z_0)}{h}\right) - \frac{d}{dz}\ell(x(z_0))$$
$$= \frac{1}{2\pi i}\oint_\Gamma \left[\frac{1}{h}\left(\frac{1}{z-(z_0+h)} - \frac{1}{z-z_0}\right) - \frac{1}{(z-z_0)^2}\right] \ell(x(z))\, dz$$

Since $\ell(x(z))$ is continuous on Γ and Γ is compact, $|\ell(x(z))| \le C_\ell$ for all $z \in \Gamma$. Regarding $x(z)$ as a family of mappings $x(z)\colon X^* \to \mathbb{C}$ we see that $x(z)$ is pointwise bounded at each ℓ so by the uniform boundedness theorem $\sup_{z\in\Gamma} \|x(z)\| \le C < \infty$. Thus

$$\left|\ell\left(\frac{x(z_0+h)-x(z_0)}{h}\right) - \frac{d}{dz}\ell(x(z_0))\right|$$
$$\le (2\pi)^{-1}\|\ell\| \left(\sup_{z\in\Gamma}\|x(z)\|\right) \oint_\Gamma \left|\frac{1}{(z-(z_0+h))(z-z_0)} - \frac{1}{(z-z_0)^2}\right| dz$$

This estimate shows that $[x(z_0+h)-x(z_0)]/h$ is uniformly Cauchy for $\|\ell\| \le 1$. By the lemma, $[x(z_0+h)-x(z_0)]/h$ converges in X, proving that $x(\cdot)$ is strongly analytic. ∎

We now prove the theorem we promised about the resolvent.

Theorem VI.5 Let X be a Banach space and suppose $T \in \mathscr{L}(X)$. Then $\rho(T)$ is an open subset of $\mathbb{C}$ and $R_\lambda(T)$ is an analytic $\mathscr{L}(X)$-valued function on each component (maximal connected subset) of D. For any two points λ, $\mu \in \rho(T)$, $R_\lambda(T)$ and $R_\mu(T)$ commute and

$$R_\lambda(T) - R_\mu(T) = (\mu - \lambda) R_\mu(T) R_\lambda(T) \tag{VI.1}$$

Proof We begin with the following formal computation, temporarily ignoring questions of convergence. Let $\lambda_0 \in \rho(T)$.

$$\frac{1}{\lambda - T} = \frac{1}{\lambda - \lambda_0 + (\lambda_0 - T)} = \left(\frac{1}{\lambda_0 - T}\right) \frac{1}{1 - \left(\dfrac{\lambda_0 - \lambda}{\lambda_0 - T}\right)}$$
$$= \left(\frac{1}{\lambda_0 - T}\right)\left[1 + \sum_{n=1}^{\infty}\left(\frac{\lambda_0-\lambda}{\lambda_0 - T}\right)^n\right]$$

This suggests that we define

$$\tilde{R}_\lambda(T) = R_{\lambda_0}(T)\left\{I + \sum_{n=1}^{\infty}(\lambda_0 - \lambda)^n [R_{\lambda_0}(T)]^n\right\}$$

Since

$$\|[R_{\lambda_0}(T)]^n\| \leq \|R_{\lambda_0}(T)\|^n$$

the series on the right converges in the uniform operator topology if

$$|\lambda - \lambda_0| < \|R_{\lambda_0}(T)\|^{-1}$$

For such λ, $\tilde{R}_\lambda(T)$ is well defined, and it is easily checked that

$$(\lambda I - T)\tilde{R}_\lambda(T) = I = \tilde{R}_\lambda(T)(\lambda I - T)$$

This proves that $\lambda \in \rho(T)$ if $|\lambda - \lambda_0| < \|R_{\lambda_0}(T)\|^{-1}$ and that $\tilde{R}_\lambda(T) = R_\lambda(T)$. Thus $\rho(T)$ is open. Since $R_\lambda(T)$ has a power series expansion, it is analytic.

The expression

$$R_\lambda(T) - R_\mu(T) = R_\lambda(T)(\mu I - T)R_\mu(T) - R_\lambda(T)(\lambda I - T)R_\mu(T)$$

proves (VI.1). Interchanging μ and λ shows that $R_\lambda(T)$ and $R_\mu(T)$ commute. ∎

Equation (VI.1) is called the **first resolvent formula**. A nice example of the use of complex analytic methods is given by the proof of the following corollary.

Corollary Let X be a Banach space, $T \in \mathscr{L}(X)$. Then the spectrum of T is not empty.

Proof Formally,

$$\frac{1}{\lambda - T} = \left(\frac{1}{\lambda}\right)\frac{1}{1 - T/\lambda} = \frac{1}{\lambda}\left(1 + \sum_{n=1}^{\infty}\left(\frac{T}{\lambda}\right)^n\right)$$

which suggests that for large values of $|\lambda|$,

$$R_\lambda(T) = \frac{1}{\lambda}\left\{I + \sum_{n=1}^{\infty}\left(\frac{T}{\lambda}\right)^n\right\} \tag{VI.2}$$

If $|\lambda| > \|T\|$, then the series on the right converges in norm and it is easily checked that for such λ, its limit is indeed the inverse of $(\lambda I - T)$. Thus, as $|\lambda| \to \infty$, $\|R_\lambda(T)\| \to 0$. If $\sigma(T)$ were empty, $R_\lambda(T)$ would be an entire bounded analytic function. By Liouville's theorem, $R_\lambda(T)$ would be zero which is a contradiction. Thus, $\sigma(T)$ is not empty. ∎

The series (VI.2) is called the **Neumann series** for $R_\lambda(T)$. The proof of the corollary shows that $\sigma(T)$ is contained in the closed disc of radius $\|T\|$. Actually, we can say more about $\sigma(T)$.

Definition Let $$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$$

$r(T)$ is called the **spectral radius** of T.

Theorem VI.6 Let X be a Banach space, $T \in \mathscr{L}(X)$. Then $\lim_{n\to\infty} \|T^n\|^{1/n}$ exists and is equal to $r(T)$. If X is a Hilbert space and A is self-adjoint, then $r(A) = \|A\|$.

Proof The reader can check that $\lim_{n\to\infty} \|T^n\|^{1/n}$ exists by following the clever subadditivity argument outlined in Problem 11. The crux of the proof of the theorem is to establish that the radius of convergence of the Laurent series of $R_\lambda(T)$ about ∞ is just $r(T)^{-1}$. First notice that the radius of convergence cannot be smaller than $r(T)^{-1}$ since we have proven that $R_\lambda(T)$ is analytic on $\rho(T)$ and $\{\lambda \mid |\lambda| > r(T)\} \subset \rho(T)$. On the other hand, (VI.2) is just the Laurent series about ∞ and we have seen that where it converges absolutely, $R_\lambda(T)$ exists. Since a Laurent series converges absolutely inside the circle of convergence, we conclude that the radius of convergence cannot be larger than $r(T)^{-1}$. That $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$ follows from the vector-valued version of Hadamard's theorem which says that the radius of convergence of (VI.2) is just the inverse of

$$\overline{\lim_n}\, \|T^n\|^{1/n} = \lim_{n\to\infty} \|T^n\|^{1/n}$$

Finally, if X is a Hilbert space and A is self-adjoint, then $\|A\|^2 = \|A^2\|$ by part (f) of Theorem VI.3. This implies that $\|A^{2^n}\| = \|A\|^{2^n}$ so

$$r(A) = \lim_{k\to\infty} \|A^k\|^{1/k} = \lim_{n\to\infty} \|A^{2^n}\|^{2^{-n}} = \|A\| \quad \blacksquare$$

The following theorem is sometimes useful in determining spectra.

Theorem VI.7 (Phillips) Let X be a Banach space, $T \in \mathscr{L}(X)$. Then $\sigma(T) = \sigma(T')$ and $R_\lambda(T') = R_\lambda(T)'$. If $\mathscr{H}$ is a Hilbert space, then $\sigma(T^*) = \{\bar{\lambda} \mid \lambda \in \sigma(T)\}$ and $R_{\bar\lambda}(T^*) = R_\lambda(T)^*$.

We note that the Hilbert space case follows from (d) of Theorem VI.3.

We now work out in some detail an example which illustrates the various kinds of spectra.

Example Let T be the operator on ℓ_1 which acts by

$$T(\xi_1, \xi_2, \ldots) = (\xi_2, \xi_3, \ldots)$$

The adjoint of T, T', acts on ℓ_∞ by

$$T'(\xi_1, \xi_2, \ldots) = (0, \xi_1, \xi_2, \ldots)$$

We first observe that $\|T\| = \|T'\| = 1$, so that all λ with $|\lambda| > 1$ are in $\rho(T)$ and $\rho(T')$. Suppose $|\lambda| < 1$. Then the vector $x_\lambda = (1, \lambda, \lambda^2, \ldots)$ is in ℓ_1 and satisfies $(\lambda I - T)x_\lambda = 0$. Thus all such λ are in the point spectrum of T. Since the spectrum is closed, $\sigma(T) = \{\lambda \mid |\lambda| \le 1\}$. By Theorem VI.7 this set is also the spectrum of T'.

We want to show that T' has no point spectrum. Suppose that $\{\xi_n\}_{n=1}^{\infty} \in \ell_\infty$ and $(\lambda I - T')\{\xi_n\} = 0$. Then

$$\begin{aligned} \lambda\xi_0 &= 0 \\ \lambda\xi_1 - \xi_0 &= 0 \\ &\vdots \end{aligned}$$

These equations together imply that $\{\xi_n\}_{n=1}^{\infty} = 0$ so $\lambda I - T'$ is one to one and T' has no point spectrum. Next, suppose $|\lambda| < 1$. Then for all $L \in \ell_\infty$

$$[(\lambda I - T')L](x_\lambda) = L((\lambda I - T)x_\lambda) = 0.$$

where $x_\lambda \in \ell_1$ is the eigenvector with eigenvalue λ. By the Hahn–Banach theorem we know that there is a linear functional in ℓ_∞ which does not vanish on x_λ so the range of $\lambda I - T'$ is not dense. Thus $\{\lambda;\ |\lambda| < 1\}$ is in the residual spectrum of T'.

It remains to consider the boundary $|\lambda| = 1$. Suppose that $|\lambda| = 1$ and $(\lambda I - T)\{\xi_n\} = 0$ for some $\{\xi_n\}$ in ℓ_1. Then

$$\begin{aligned} \xi_1 &= \lambda\xi_0 \\ \xi_2 &= \lambda\xi_1 \\ &\vdots \end{aligned}$$

so $\{\xi_n\}_{n=1}^{\infty} = \xi_0(1, \lambda, \lambda^2, \ldots)$ which is not in ℓ_1. Thus λ is not point spectrum. If the range of $\lambda I - T$ were not dense there would be a nonzero $L \in \ell_\infty$ such that $L[(\lambda I - T)x] = 0$ for all $x \in \ell_1$. But then $[(\lambda I - T')L](x) = 0$ which would imply that λ is in the point spectrum of T' which we have proven cannot occur. Thus, $\{\lambda \mid |\lambda| = 1\}$ is neither in the point spectrum of T nor in the residual spectrum of T.

Finally, we prove that $\{\lambda \mid |\lambda| = 1\}$ is in the residual spectrum of T' by explicitly finding an open ball disjoint from $\mathrm{Ran}(\lambda - T')$. If $a = \{a_n\}$ and $b = \{b_n\}$ are in ℓ_∞ and obey $a = (\lambda I - T')b$, then

$$\begin{aligned} a_0 &= \lambda b_0 \\ &\vdots \\ a_n &= \lambda b_n - b_{n-1} \end{aligned}$$

so $b_n = (\bar{\lambda})^{n+1} \sum_{m=0}^{n} \lambda^m a_m$. Let $c = \{c_n\}$ with $c_n = \bar{\lambda}^n$ and suppose that $d \in \ell_\infty$ and $\|d - c\|_\infty \le \frac{1}{2}$. Then

$$\operatorname{Re}\{\lambda^n d_n\} \ge \operatorname{Re}\{\lambda^n c_n\} - \|d - c\|_\infty \ge \tfrac{1}{2}$$

Thus, if $(\lambda - T')e = d$ for some $e \in \ell_\infty$, then since

$$e_n = (\bar{\lambda})^{n+1} \sum_{m=0}^{n} \lambda^m d_m$$

$|e_n| \ge n/2$ which is impossible. Therefore, $\operatorname{Ran}(\lambda - T')$ does not intersect the ball of radius $\frac{1}{2}$ about c so λ is in the residual spectrum.

Operator	Spectrum	Point spectrum	Residual spectrum
T	$\lvert\lambda\rvert \le 1$	$\lvert\lambda\rvert < 1$	$\varnothing$
T'	$\lvert\lambda\rvert \le 1$	$\varnothing$	$\lvert\lambda\rvert \le 1$

As in the above example, one can prove in general

Proposition Let X be a Banach space and $T \in \mathscr{L}(X)$. Then,

(a) If λ is in the residual spectrum of T, then λ is in the point spectrum of T'.
(b) If λ is in the point spectrum of T, then λ is in either the point or the residual spectrum of T'.

Finally, we note:

Theorem VI.8 Let T be a self-adjoint operator on a Hilbert space $\mathscr{H}$. Then,

(a) T has no residual spectrum.
(b) $\sigma(T)$ is a subset of $\mathbb{R}$.
(c) Eigenvectors corresponding to distinct eigenvalues of T are orthogonal.

Proof If λ and μ are real, we compute

$$\|[T - (\lambda + i\mu)]x\|^2 = \|(T - \lambda)x\|^2 + \mu^2\|x\|^2$$

Thus $\|[T - (\lambda + i\mu)]x\|^2 \ge \mu^2\|x\|^2$, so if $\mu \ne 0$, then $T - (\lambda + i\mu)$ is one to one and has a bounded inverse on its range, which is closed. If $\operatorname{Ran}(T - (\lambda + i\mu)) \ne \mathscr{H}$, then, by the above proposition, $\lambda - i\mu$ would be in the point spectrum of T, which is impossible by the inequality. Thus if $\mu \ne 0$, $\lambda + i\mu$ is in $\rho(T)$. This proves (b). If a real λ were in the residual spectrum of T, then $\bar{\lambda} = \lambda$ would be in the point spectrum of $T^* = T$, which is impossible since the point and residual spectrum are disjoint by definition. This proves (a). The easy proof of (c) is left as an exercise (Problem 8). ∎

VI.4 Positive operators and the polar decomposition

We want to prove the existence of a special decomposition for operators on a *Hilbert space* which is analogous to the decomposition $z = |z|e^{i\arg z}$ for complex numbers. First we must describe a suitable analogue of the positive numbers.

Definition Let $\mathscr{H}$ be a Hilbert space. An operator $B \in \mathscr{L}(\mathscr{H})$ is called **positive** if $(Bx, x) \geq 0$ for all $x \in \mathscr{H}$. We write $B \geq 0$ if B is positive and $B \leq A$ if $A - B \geq 0$.

Every (bounded) positive operator on a complex Hilbert space is self-adjoint. To see this, notice that $(x, Ax) = \overline{(x, Ax)} = (Ax, x)$ if (Ax, x) takes only real values. By the polarization identity (Chapter II, Problem 4), $(Ax, y) = (x, Ay)$ if $(Ax, x) = (x, Ax)$ for all x. Thus, if A is positive, it is self-adjoint. This is false on real Hilbert spaces because it is not possible to recover (x, Ay) by knowing (x, Ax) for all x.

For any $A \in \mathscr{L}(\mathscr{H})$, notice that $A^*A \geq 0$ since $(A^*Ax, x) = \|Ax\|^2 \geq 0$. Just as $|z| = \sqrt{\bar{z}z}$ we would like to define $|A| = \sqrt{A^*A}$. To do this we must show that we can take square roots of positive operators. We begin with a lemma.

Lemma The power series for $\sqrt{1-z}$ about zero converges absolutely for all complex numbers z satisfying $|z| \leq 1$.

Proof Let $\sqrt{1-z} = 1 + c_1 z + c_2 z^2 + \cdots$ be the power series of $\sqrt{1-z}$ about the origin. Since $\sqrt{1-z}$ is analytic for $|z| < 1$, the series converges absolutely there. The derivatives of $\sqrt{1-z}$ at the origin are all negative, so the c_i are negative if $i \geq 1$. Thus

$$\begin{aligned}
\sum_{n=0}^{N} |c_n| &= 2 - \sum_{n=0}^{N} c_n \\
&= 2 - \lim_{x \to 1-} \sum_{n=0}^{N} c_n x^n \\
&\leq 2 - \lim_{x \to 1-} \sqrt{1-x} \\
&= 2
\end{aligned}$$

where $\lim_{x\to 1^-}$ means the limit as x approaches one from below. Since this is true for all N, $\sum_{n=0}^{\infty} |c_n| \le 2$, which implies that the series converges absolutely for $|z| = 1$. ∎

Theorem VI.9 (square root lemma) Let $A \in \mathscr{L}(\mathscr{H})$ and $A \ge 0$. Then there is a unique $B \in \mathscr{L}(\mathscr{H})$ with $B \ge 0$ and $B^2 = A$. Furthermore, B commutes with every bounded operator which commutes with A.

Proof It is sufficient to consider the case where $\|A\| \le 1$. Since

$$\|I - A\| = \sup_{\|\varphi\|=1} |((I-A)\varphi, \varphi)| \le 1$$

the above lemma implies that the series $1 + c_1(I - A) + c_2(I - A)^2 + \cdots$ converges in norm to an operator B. Since the convergence is absolute we can square the series and rearrange terms which proves that $B^2 = A$. Furthermore, since $0 \le I - A \le I$ we have $0 \le (\varphi, (I - A)^n\varphi) \le 1$ for all $\varphi \in \mathscr{H}$ with $\|\varphi\| = 1$. Thus

$$\begin{aligned}(\varphi, B\varphi) &= 1 + \sum_{n=1}^{\infty} c_n(\varphi, (I-A)^n\varphi)\\ &\ge 1 + \sum_{n=1}^{\infty} c_n = 0\end{aligned}$$

where we have used the fact that $c_n < 0$ and the estimate in the lemma. Thus, $B \ge 0$. Since the series for B converges absolutely, it commutes with any operator that commutes with A.

Suppose there is a B', with $B' \ge 0$ and $(B')^2 = A$. Then since

$$B'A = (B')^3 = AB'$$

B' commutes with A and thus with B. Therefore

$$(B - B')B(B - B') + (B - B')B'(B - B') = (B^2 - B'^2)(B - B') = 0 \qquad \text{(VI.3)}$$

Since both terms in (VI.3) are positive, they must both be zero, so their difference $(B - B')^3 = 0$. Since $B - B'$ is self-adjoint, $\|B - B'\|^4 = \|(B - B')^4\| = 0$, so $B - B' = 0$. ∎

We are now ready to define $|A|$.

Definition Let $A \in \mathscr{L}(\mathscr{H})$. Then $|A| = \sqrt{A^*A}$.

The reader should be wary of the emotional connotations of the symbol $|\cdot|$. While it is true that $|\lambda A| = |\lambda|\,|A|$ for $\lambda \in \mathbb{C}$, it is in general *false* that

$|AB| = |A|\ |B|$ or that $|A| = |A^*|$. Furthermore it is not true in general that $|A + B| \leq |A| + |B|$ (Problem 16). In fact, while it is known that $|\cdot|$ is norm continuous (see Problem 15), it is not known whether it is Lipschitz, that is, whether $\|\, |A| - |B|\, \| \leq c\, \|A - B\|$ for some constant c (however, see Problem 17).

The analogue of the complex numbers of modulus one is a little more complicated. At first one might expect that the unitary operators would be sufficient, but the following example shows that this is not the case.

Example Let A be the right shift operator on ℓ_2. Then $|A| = \sqrt{A^*A} = I$ so if we write $A = U|A|$ we must have $U = A$. However, A is not unitary since $(1, 0, 0, \ldots)$ is not in its range.

Definition An operator $U \in \mathscr{L}(\mathscr{H})$ is called an **isometry** if $\|Ux\| = \|x\|$ for all $x \in \mathscr{H}$. U is called a **partial isometry** if U is an isometry when restricted to the closed subspace $(\text{Ker } U)^\perp$.

Thus, if U is a partial isometry, $\mathscr{H}$ can be written as $\mathscr{H} = \text{Ker } U \oplus (\text{Ker } U)^\perp$ and $\mathscr{H} = \text{Ran } U \oplus (\text{Ran } U)^\perp$ and U is a unitary operator between $(\text{Ker } U)^\perp$, the **initial subspace** of U, and $\text{Ran } U$, the **final subspace** of U. It is not hard to see that U^* is a partial isometry from $\text{Ran } U$ to $(\text{Ker } U)^\perp$ which acts as the inverse of the map U: $(\text{Ker } U)^\perp \to \text{Ran } U$.

Proposition Let U be a partial isometry. Then $P_\text{i} = U^*U$ and $P_\text{f} = UU^*$ are respectively the projections onto the initial and final subspaces of U. Conversely, if $U \in \mathscr{L}(\mathscr{H})$ with U^*U and UU^* projections, then U is a partial isometry.

The proof of the proposition is left to Problem 18. We are now ready to prove the analogue of the decomposition $z = |z|\, e^{i \arg z}$.

Theorem VI.10 (Polar decomposition) Let A be a bounded linear operator on a Hilbert space $\mathscr{H}$. Then there is a partial isometry U such that $A = U\,|A|$. U is uniquely determined by the condition that $\text{Ker } U = \text{Ker } A$. Moreover, $\text{Ran } U = \overline{\text{Ran } A}$.

Proof Define U: $\text{Ran } |A| \to \text{Ran } A$ by $U(|A|\ \psi) = A\psi$. Since

$$\|\, |A|\psi\|^2 = (\psi, |A|^2\, \psi) = (\psi, A^*A\psi) = \|A\psi\|^2$$

U is well-defined, that is, if $|A|\psi = |A|\phi$ then $A\psi = A\phi$. U is isometric and so extends to an isometry of $\overline{\text{Ran}\,|A|}$ to $\overline{\text{Ran}\,A}$. Extend U to all of $\mathscr{H}$ by defining it to be zero on $(\text{Ran}\,|A|)^\perp$. Since $|A|$ is self-adjoint, $(\text{Ran}\,|A|)^\perp = \text{Ker}\,|A|$. Furthermore, $|A|\psi = 0$ if and only if $A\psi = 0$ so that $\text{Ker}\,|A| = \text{Ker}\,A$. Thus $\text{Ker}\,U = \text{Ker}\,A$. Uniqueness is left to the reader. ∎

In Problem 20 of Chapter VII, the reader will prove that U is a strong limit of polynomials in A and A^* so that U is in the "von Neumann algebra" generated by A.

VI.5 Compact operators†

Many problems in classical mathematical physics can be handled by reformulating them in terms of integral equations. A famous example is the Dirichlet problem discussed at the end of this section. Consider the simple operator K, defined in $C[0, 1]$ by

$$(K\varphi)(x) = \int_0^1 K(x, y)\varphi(y)\,dy \tag{VI.4}$$

where the function $K(x, y)$ is continuous on the square $0 \le x, y \le 1$. $K(x, y)$ is called the **kernel** of the **integral operator** K. Since

$$|(K\varphi)(x)| \le \left(\sup_{0 \le x, y \le 1} |K(x, y)|\right)\left(\sup_{0 \le y \le 1} |\varphi(y)|\right)$$

we see that

$$\|K\varphi\|_\infty \le \left(\sup_{0 \le x, y < 1} |K(x, y)|\right)\|\varphi\|_\infty$$

so K is a bounded operator on $C[0, 1]$. K has another property which is very important. Let B_M denote the functions φ in $C[0, 1]$ such that $\|\varphi\|_\infty \le M$. Since $K(x, y)$ is continuous on the square $0 \le x, y \le 1$ and since the square is compact, $K(x, y)$ is uniformly continuous. Thus, given an $\varepsilon > 0$, we can find $\delta > 0$ such that $|x - x'| < \delta$ implies $|K(x, y) - K(x', y)| < \varepsilon$ for all $y \in [0, 1]$. Thus, if $\varphi \in B_M$

$$\begin{aligned}|(K\varphi)(x) - (K\varphi)(x')| &\le \left(\sup_{y \in [0, 1]} |K(x, y) - K(x', y)|\right)\|\varphi\|_\infty \\ &\le \varepsilon M\end{aligned}$$

† A supplement to this section begins on p. 368.

Therefore the functions $K[B_M]$ are equicontinuous. Since they are also uniformly bounded by $\|K\|M$, we can use the Ascoli theorem (Theorem I.28) to conclude that for every sequence $\varphi_n \in B_M$, the sequence $K\varphi_n$ has a convergent subsequence (the limit may not be in $K[B_M]$). Another way of saying this is that the set $K[B_M]$ is **precompact**; that is, its closure is compact in $C[0, 1]$. It is clear that the choice of M was not important so what we have shown is that K takes bounded sets into precompact sets. It is this property which makes the so called "Fredholm alternative" hold for nice integral equations like (VI.4). This section is devoted to studying such operators.

Definition Let X and Y be Banach spaces. An operator $T \in \mathscr{L}(X, Y)$ is called **compact** (or completely continuous) if T takes bounded sets in X into precompact sets in Y. Equivalently, T is compact if and only if for every bounded sequence $\{x_n\} \subset X$, $\{Tx_n\}$ has a subsequence convergent in Y.

The integral operator (VI.4) is one example of a compact operator. Another class of examples is:

Example (finite rank operators) Suppose that the range of T is finite dimensional. That is, every vector in the range of T can be written $Tx = \sum_{i=1}^{N} \alpha_i y_i$, for some fixed family $\{y_i\}_{i=1}^{N}$ in Y. If x_n is any bounded sequence in X, the corresponding α_i^n are bounded since T is bounded. The usual subsequence trick allows one to extract a convergent subsequence from $\{Tx_n\}$ which proves that T is compact.

An important property of compact operators is given by (compare Problem 34):

Theorem VI.11 A compact operator maps weakly convergent *sequences* into norm convergent sequences.

Proof Suppose $x_n \xrightarrow{w} x$. By the uniform boundedness theorem, the $\|x_n\|$ are bounded. Let $y_n = Tx_n$. Then $\ell(y_n) - \ell(y) = (T'\ell)(x_n - x)$ for any $\ell \in Y^*$. Thus, y_n converges weakly to $y = Tx$ in Y. Suppose that y_n does not converge to y in norm. Then, there is an $\varepsilon > 0$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ so that $\|y_{n_k} - y\| \geq \varepsilon$. Since the sequence $\{x_{n_k}\}$ is bounded and T is compact $\{y_{n_k}\}$ has a subsequence which converges to a $\tilde{y} \neq y$. This subsequence must then also converge weakly to $\tilde{y}$, but this is impossible since y_n converges weakly to y. Thus y_n converges to y in norm. ∎

We note that if X is reflexive then the converse of Theorem VI.11 holds (Problem 20). The following theorem is important since one can use it to prove that an operator is compact by exhibiting it as a norm limit of compact operators or as an adjoint of a compact operator.

Theorem VI.12 Let X and Y be Banach spaces, $T \in \mathscr{L}(X, Y)$.

(a) If $\{T_n\}$ are compact and $T_n \to T$ in the norm topology, then T is compact.

(b) T is compact if and only if T' is compact.

(c) If $S \in \mathscr{L}(Y, Z)$ with Z a Banach space and if T *or* S is compact, then ST is compact.

Proof (a) Let $\{x_m\}$ be a sequence in the unit ball of X. Since T_n is compact for each n, we can use the diagonalization trick of I.5 to find a subsequence of $\{x_m\}$, call it $\{x_{m_k}\}$, so that $T_n x_{m_k} \to y_n$ for each n as $k \to \infty$. Since $\|x_{m_k}\| \le 1$ and $\|T_n - T\| \to 0$, an $\varepsilon/3$-argument shows that the sequence $\{y_n\}$ is Cauchy, so $y_n \to y$. It is not difficult to show using an $\varepsilon/3$ argument that $Tx_{m_k} \to y$. Thus T is compact.

(b) See the Notes and Problem 36.

(c) The proof is elementary (Problem 37). ∎

We are mostly interested in the case where T is a compact operator from a separable Hilbert space to itself, so we will not pursue the general case any further (however, see the discussion in the Notes). We denote the Banach space of compact operators on a separable Hilbert space by $\mathrm{Com}(\mathscr{H})$. By the first example and Theorem VI.12 the norm limit of a sequence of finite rank operators is compact. The converse is also true in the Hilbert space case.

Theorem VI.13 Let $\mathscr{H}$ be a separable Hilbert space. Then every compact operator on $\mathscr{H}$ is the norm limit of a sequence of operators of finite rank.

Proof Let $\{\varphi_j\}_{j=1}^{\infty}$ be an orthonormal set in $\mathscr{H}$. Define

$$\lambda_n = \sup_{\substack{\psi \in [\varphi_1, \ldots, \varphi_n]^\perp \\ \|\psi\| = 1}} \|T\psi\|$$

Clearly, $\{\lambda_n\}$ is monotone decreasing so it converges to a limit $\lambda \ge 0$. We first show that $\lambda = 0$. Choose a sequence $\psi_n \in [\varphi_1, \ldots, \varphi_n]^\perp$, $\|\psi_n\| = 1$, with

$\|T\psi_n\| \geq \lambda/2$. Since $\psi_n \xrightarrow{w} 0$, $T\psi_n \to 0$ by Theorem VI.11. Thus, $\lambda = 0$. As a result

$$\sum_{j=1}^{n} (\varphi_j, \cdot)T\varphi_j \to T$$

in norm since λ_n is just the norm of the difference. ∎

We have discussed a wide variety of properties of compact operators but we have not yet described any property which explains our special interest in them. The basic principle which makes compact operators important is the Fredholm alternative: If A is compact, then either $A\psi = \psi$ has a solution or $(I - A)^{-1}$ exists. This is not a property shared by all bounded linear transformations. For example, if A is the operator $(A\varphi)(x) = x\varphi(x)$ on $L^2[0, 2]$, then $A\varphi = \varphi$ has no solutions but $(I - A)^{-1}$ does not exist (as a bounded operator). In terms of "solving equations" the Fredholm alternative is especially nice: It tells us that if for any φ there is at most one ψ with $\psi = \varphi + A\psi$, then there is always exactly one. That is, compactness and uniqueness together imply existence; for an example, see the discussion of the Dirichlet problem at the end of the section.

As one might expect, since the Fredholm alternative holds for finite-dimensional matrices, it is possible to prove the Fredholm alternative for compact operators (in the Hilbert space case) by using the fact that any compact operator A can be written as $A = F + R$ where F has finite rank and R has small norm. Compactness combines very nicely with analyticity so we first prove an elegant result which is of great use in itself (see Sections XI.6, XI.7, XIII.4, and XIII.5).

Theorem VI.14 (analytic Fredholm theorem) Let D be an open connected subset of $\mathbb{C}$. Let f: $D \to \mathscr{L}(\mathscr{H})$ be an analytic operator-valued function such that $f(z)$ is compact for each $z \in D$. Then, either

(a) $(I - f(z))^{-1}$ exists for no $z \in D$.

or

(b) $(I - f(z))^{-1}$ exists for all $z \in D\backslash S$ where S is a discrete subset of D (i.e. a set which has no limit points in D). In this case, $(I - f(z))^{-1}$ is meromorphic in D, analytic in $D\backslash S$, the residues at the poles are finite rank operators, and if $z \in S$ then $f(z)\psi = \psi$ has a nonzero solution in $\mathscr{H}$.

Proof We will prove that near any z_0 either (a) or (b) holds. A simple connectedness argument allows one to convert this into a statement about all of D

(Problem 21). Given $z_0 \in D$, choose an r so that $|z - z_0| < r$ implies $\|f(z) - f(z_0)\| < \frac{1}{2}$ and pick F, an operator with finite rank so that

$$\|f(z_0) - F\| < \tfrac{1}{2}$$

Then, for $z \in D_r$, the disc of radius r about z_0, $\|f(z) - F\| < 1$. By expanding in a geometric series we see that $(I - f(z) + F)^{-1}$ exists and is analytic.

Since F has finite rank, there are independent vectors $\psi_1, \ldots, \psi_N$ so that $F(\varphi) = \sum_{i=1}^{N} \alpha_i(\varphi)\psi_i$. The $\alpha_i(\cdot)$ are bounded linear functionals on $\mathscr{H}$ so by the Riesz lemma there are vectors $\phi_1, \ldots, \phi_N$ so that $F(\varphi) = \sum_{n=1}^{N} (\phi_i, \varphi)\psi_i$ for all $\varphi \in \mathscr{H}$. Let $\phi_n(z) = ((I - f(z) + F)^{-1})^* \phi_n$ and

$$g(z) = F(I - f(z) + F)^{-1} = \sum_{n=1}^{N} (\phi_n(z), \cdot)\psi_n$$

By writing

$$(I - f(z)) = (I - g(z))(I - f(z) + F)$$

we see that $I - f(z)$ is invertible for $z \in D_r$ if and only if $I - g(z)$ is invertible and that $\psi = f(z)\psi$ has a nonzero solution if and only if $\varphi = g(z)\varphi$ has a solution.

If $g(z)\varphi = \varphi$, then $\varphi = \sum_{n=1}^{N} \beta_n \psi_n$ and the β_n satisfy

$$\beta_n = \sum_{m=1}^{N} (\phi_n(z), \psi_m)\beta_m \tag{VI.5a}$$

Conversely, if (VI.5a) has a solution $\langle \beta_1, \ldots, \beta_N \rangle$, then $\varphi = \sum_{n=1}^{N} \beta_n \psi_n$ is a solution of $g(z)\varphi = \varphi$. Thus $g(z)\varphi = \varphi$ has a solution if and only if the determinant

$$d(z) = \det\{\delta_{nm} - (\phi_n(z), \psi_m)\} = 0$$

Since $(\phi_n(z), \psi_m)$ is analytic in D_r so is $d(z)$ which means that either $S_r = \{z \mid z \in D_r, d(z) = 0\}$ is a discrete set in D_r or $S_r = D_r$. Now, suppose $d(z) \neq 0$. Then, given ψ, we can solve $(I - g(z))\varphi = \psi$ by setting $\varphi = \psi + \sum_{n=1}^{N} \beta_n \psi_n$ if we can find β_n satisfying

$$\beta_n = (\phi_n(z), \psi) + \sum_{m=1}^{N} (\phi_n(z), \psi_m)\beta_m \tag{VI.5b}$$

But, since $d(z) \neq 0$, this equation has a solution. Thus $(I - g(z))^{-1}$ exists if and only if $z \notin S_r$.

The meromorphic nature of $(I - f(z))^{-1}$ and the finite rank residues follow from the fact that there is an explicit formula for the β_n in (VI.5b) in terms of cofactor matrices. ▌

This theorem has four important consequences:

Corollary (the Fredholm alternative) If A is a compact operator on $\mathscr{H}$, then either $(I - A)^{-1}$ exists or $A\psi = \psi$ has a solution.

Proof Take $f(z) = zA$ and apply the last theorem at $z = 1$. ∎

Theorem VI.15 (Riesz–Schauder theorem) Let A be a compact operator on $\mathscr{H}$, then $\sigma(A)$ is a discrete set having no limit points except perhaps $\lambda = 0$. Further, any nonzero $\lambda \in \sigma(A)$ is an eigenvalue of finite multiplicity (i.e. the corresponding space of eigenvectors is finite dimensional).

Proof Let $f(z) = zA$. Then $f(z)$ is an analytic compact operator-valued function on the entire plane. Thus $\{z \mid zA\psi = \psi \text{ has a solution } \psi \neq 0\}$ is a discrete set (it is not the entire plane since it does not contain $z = 0$) and if $1/\lambda$ is not in this discrete set then

$$(\lambda - A)^{-1} = \frac{1}{\lambda}\left(I - \frac{1}{\lambda}A\right)^{-1}$$

exists. The fact that the nonzero eigenvalues have finite multiplicity follows immediately from the compactness of A. ∎

Theorem VI.16 (the Hilbert–Schmidt theorem) Let A be a self-adjoint compact operator on $\mathscr{H}$. Then, there is a complete orthonormal basis, $\{\phi_n\}$, for $\mathscr{H}$ so that $A\phi_n = \lambda_n \phi_n$ and $\lambda_n \to 0$ as $n \to \infty$.

Proof For each eigenvalue of A choose an orthonormal basis for the set of eigenvectors corresponding to the eigenvalue. The collection of all these vectors, $\{\phi_n\}$, is an orthonormal set since eigenvectors corresponding to distinct eigenvalues are orthogonal. Let $\mathscr{M}$ be the closure of the span of $\{\phi_n\}$. Since A is self-adjoint and $A: \mathscr{M} \to \mathscr{M}$, $A: \mathscr{M}^\perp \to \mathscr{M}^\perp$. Let $\tilde{A}$ be the restriction of A to $\mathscr{M}^\perp$. Then $\tilde{A}$ is self-adjoint and compact since A is. By the Riesz–Schauder theorem, if any $\lambda \neq 0$ is in $\sigma(\tilde{A})$, it is an eigenvalue of $\tilde{A}$ and thus of A. Therefore the spectral radius of $\tilde{A}$ is zero since the eigenvectors of A are in $\mathscr{M}$. Because $\tilde{A}$ is self-adjoint, it is the zero operator on $\mathscr{M}^\perp$ by Theorem VI.6. Thus, $\mathscr{M}^\perp = \{0\}$ since if $\varphi \in \mathscr{M}^\perp$, then $A\varphi = 0$ which implies that $\varphi \in \mathscr{M}$. Therefore, $\mathscr{M} = \mathscr{H}$.

The fact that $\lambda_n \to 0$ is a consequence of the first part of the Riesz–Schauder theorem which says that each nonzero eigenvalue has finite multiplicity and the only possible limit point of the λ_n is zero. ∎

Theorem VI.17 (canonical form for compact operators) Let A be a compact operator on $\mathscr{H}$. Then there exist (not necessarily complete) orthonormal sets $\{\psi_n\}_{n=1}^N$ and $\{\phi_n\}_{n=1}^N$ and positive real numbers $\{\lambda_n\}_{n=1}^N$ with $\lambda_n \to 0$ so that

$$A = \sum_{n=1}^{N} \lambda_n(\psi_n, \cdot)\phi_n \tag{VI.6}$$

The sum in (VI.6), which may be finite or infinite, converges in norm. The numbers, $\{\lambda_n\}$, are called the **singular values** of A.

Proof Since A is compact, so is A^*A (Theorem VI.12). Thus A^*A is compact and self-adjoint. By the Hilbert–Schmidt theorem, there is an orthonormal set $\{\psi_n\}_{n=1}^N$ so that $A^*A\psi_n = \mu_n\psi_n$ with $\mu_n \neq 0$ and so that A^*A is the zero operator on the subspace orthogonal to $\{\psi_n\}_{n=1}^N$. Since A^*A is positive, each $\mu_n > 0$. Let λ_n be the positive square root of μ_n and set $\phi_n = A\psi_n/\lambda_n$. A short calculation shows that the ϕ_n are orthonormal and that

$$A\psi = \sum_{n=1}^{N} \lambda_n(\psi_n, \psi)\phi_n \quad \blacksquare$$

The proof shows that the singular values of A are precisely the eigenvalues of $|A|$.

We conclude with a classical example.

Example (Dirichlet problem) The main impetus for the study of compact operators arose from the use of integral equations in attempting to solve the classical boundary value problems of mathematical physics. We briefly describe this method. Let D be an open bounded region in $\mathbb{R}^3$ with a smooth boundary surface ∂D. The Dirichlet problem for Laplace's equation is: given a continuous function f on ∂D, find a function u, twice differentiable in D and continuous on $\bar{D}$, which satisfies

$$\begin{aligned} \Delta u(x) &= 0 & x &\in D \\ u(x) &= f(x) & x &\in \partial D \end{aligned}$$

Let $K(x, y) = (x - y, n_y)/2\pi|x - y|^3$ where n_y is the outer normal to ∂D at the point $y \in \partial D$. Then, as a function of x, $K(x, y)$ satisfies $\Delta_x K(x, y) = 0$ in the interior which suggests that we try to write u as a superposition

$$u(x) = \int_{\partial D} K(x, y)\varphi(y)\, dS(y) \tag{VI.6a}$$

where $\varphi(y)$ is some continuous function on ∂D and dS is the usual surface measure. Indeed, for $x \in D$, the integral makes perfectly good sense and

$\Delta u(x) = 0$ in D. Furthermore, if x_0 is any point in ∂D and $x \to x_0$ from *inside* D, it can be proven that

$$u(x) \to -\varphi(x_0) + \int_{\partial D} K(x_0, y)\varphi(y)\, dS(y) \tag{VI.6b}$$

If $x \to x_0$ from outside D, the minus is replaced by a plus. Also,

$$\int_{\partial D} K(x_0, y)\varphi(y)\, dS(y)$$

exists and is a continuous function on ∂D if φ is a continuous function on ∂D. The proof depends on the fact that the boundary of D is smooth which implies that for $x, y \in \partial D$, $(x - y, n_y) \approx c|x - y|^2$ as $x \to y$.

Since we wish $u(x) = f(x)$ on ∂D, the whole question reduces to whether we can find φ so that

$$f(x) = -\varphi(x) + \int_{\partial D} K(x, y)\varphi(y)\, dS(y), \qquad x \in \partial D$$

Let T: $C(\partial D) \to C(\partial D)$ be defined by

$$T\varphi = \int_{\partial D} K(x, y)\varphi(y)\, dS(y)$$

Not only is T bounded but (as we will shortly see) T is also compact. Thus, by the Fredholm alternative, either $\lambda = 1$ is in the point spectrum of T in which case there is a $\psi \in C(\partial D)$ such that $(I - T)\psi = 0$, or $-f = (I - T)\varphi$ has a unique solution for each $f \in C(\partial D)$. If u is defined by (VI.6a) with ψ replacing φ, then $u \equiv 0$ in D by the maximum principle. Further, $\partial u/\partial n$ is continuous across ∂D and therefore equals zero on ∂D. By an integration by parts this implies that $u \equiv 0$ outside ∂D. Therefore, by (VI.6b), $2\psi(x) \equiv 0$ on ∂D, so the first alternative does not hold.

The idea of the compactness proof is the following. Let

$$K_\delta(x, z) = \frac{(x - z, n_z)}{|x - z|^3 + \delta}$$

If $\delta > 0$, the kernel K_δ is continuous, so, by the discussion at the beginning of this section, the corresponding integral operators T_δ, are compact. To prove that T is compact, we need only show that $\|T - T_\delta\| \to 0$ as $\delta \to 0$. By the estimate

$$|(T_\delta f)(x) - (Tf)(x)| \le \|f\|_\infty \int_{\partial D} |K(x, z) - K_\delta(x, z)|\, dS(z)$$

we must only show that the integral converges to zero uniformly in x as $\delta \to 0$. To prove this, divide the integration region into the set where $|x-z| \geq \varepsilon$ and its complement. For fixed ε, the kernels converge uniformly on the first region. By using the fact that K is integrable, the contribution from the second region can be made arbitrarily small for ε sufficiently small.

VI.6 The trace class and Hilbert–Schmidt ideals

In the last section we saw that compact operators have many nice properties and are useful for applications. It is therefore important to have effective criteria for determining when a given operator is compact or, better yet, general statements about whole classes of operators. In this section we will prove that the integral operator

$$(Tf)(x) = \int_M K(x, y) f(y)\, d\mu(y)$$

on $L^2(M, d\mu)$ is compact if $K(\cdot, \cdot) \in L^2(M \times M, d\mu \otimes d\mu)$. First we will develop the trace, a tool which is of great interest in itself. Theorem VI.12 shows that $\mathrm{Com}(\mathscr{H})$, the compact operators on a separable Hilbert space $\mathscr{H}$, form a Banach space. At the conclusion of the section, we will compute the dual and double dual of $\mathrm{Com}(\mathscr{H})$. These calculations illustrate the difference between the weak Banach space topology on $\mathscr{L}(\mathscr{H})$ and the weak operator topology and give a foretaste of the structure of abstract von Neumann algebras which we will study later.

The trace is a generalization of the usual notion of the sum of the diagonal elements of a matrix, but because infinite sums are involved, not all operators will have a trace. The construction of the trace is analogous to the construction of the Lebesgue integral where one first defines $\int f\, d\mu$ for $f \geq 0$; it has values in $[0, \infty]$, including ∞. Then $\mathscr{L}^1$ is defined as those f so that $\int |f|\, d\mu < \infty$. $\mathscr{L}^1$ is a vector space and $f \mapsto \int f\, d\mu$ a linear functional. Similarly we first define the trace, $\mathrm{tr}(\cdot)$, on the positive operators; $A \to \mathrm{tr}\, A$ has values in $[0, \infty]$. We then define the **trace class**, $\mathscr{I}_1$, to be all $A \in \mathscr{L}(\mathscr{H})$ such that $\mathrm{tr}\, |A| < \infty$. We will then show that $\mathrm{tr}(\cdot)$ is a linear functional on $\mathscr{I}_1$ with the right properties.

Theorem VI.18 Let $\mathscr{H}$ be a separable Hilbert space, $\{\varphi_n\}_{n=1}^{\infty}$ an orthonormal basis. Then for any positive operator $A \in \mathscr{L}(\mathscr{H})$ we define $\mathrm{tr}\, A = \sum_{n=1}^{\infty} (\varphi_n, A\varphi_n)$. The number $\mathrm{tr}\, A$ is called the **trace of** A and is independent of the orthonormal basis chosen. The trace has the following properties:

(a) $\operatorname{tr}(A + B) = \operatorname{tr} A + \operatorname{tr} B$.
(b) $\operatorname{tr}(\lambda A) = \lambda \operatorname{tr} A$ for all $\lambda \geq 0$.
(c) $\operatorname{tr}(UAU^{-1}) = \operatorname{tr} A$ for any unitary operator U.
(d) If $0 \leq A \leq B$, then $\operatorname{tr} A \leq \operatorname{tr} B$.

Proof Given an orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$, define $\operatorname{tr}_\varphi(A) = \sum_{n=1}^{\infty} (\varphi_n, A\varphi_n)$. If $\{\psi_m\}_{m=1}^{\infty}$ is another orthonormal basis then

$$\begin{aligned}
\operatorname{tr}_\varphi(A) &= \sum_{n=1}^{\infty} (\varphi_n, A\varphi_n) = \sum_{n=1}^{\infty} \|A^{1/2}\varphi_n\|^2 \\
&= \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} |(\psi_m, A^{1/2}\varphi_n)|^2 \right) \\
&= \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} |(A^{1/2}\psi_m, \varphi_n)|^2 \right) \\
&= \sum_{m=1}^{\infty} \|A^{1/2}\psi_m\|^2 \\
&= \sum_{m=1}^{\infty} (\psi_m, A\psi_m) \\
&= \operatorname{tr}_\psi(A)
\end{aligned}$$

Since all the terms are positive, interchanging the sums is allowed.

Properties (a), (b), and (d) are obvious. To prove (c) we note that if $\{\varphi_n\}$ is an orthonormal basis, then so is $\{U\varphi_n\}$. Thus,

$$\operatorname{tr}(UAU^{-1}) = \operatorname{tr}_{(U\varphi)}(UAU^{-1}) = \operatorname{tr}_\varphi(A) = \operatorname{tr}(A). \quad \blacksquare$$

Definition An operator $A \in \mathscr{L}(\mathscr{H})$ is called **trace class** if and only if $\operatorname{tr}|A| < \infty$. The family of all trace class operators is denoted by $\mathscr{I}_1$.

The basic properties of $\mathscr{I}_1$ are given in the following:

Theorem VI.19 $\mathscr{I}_1$ is a $*$-ideal in $\mathscr{L}(\mathscr{H})$, that is,

(a) $\mathscr{I}_1$ is a vector space.
(b) If $A \in \mathscr{I}_1$ and $B \in \mathscr{L}(\mathscr{H})$, then $AB \in \mathscr{I}_1$ and $BA \in \mathscr{I}_1$.
(c) If $A \in \mathscr{I}_1$, then $A^* \in \mathscr{I}_1$.

Proof (a) Since $|\lambda A| = |\lambda|\,|A|$ for $\lambda \in \mathbb{C}$, $\mathscr{I}_1$ is closed under scalar multiplication. Now, suppose that A and B are in $\mathscr{I}_1$, we wish to prove that

$A + B \in \mathscr{I}_1$. Let U, V, and W be the partial isometries arising from the polar decompositions

$$\begin{aligned} A + B &= U|A + B| \\ A &= V|A| \\ B &= W|B| \end{aligned}$$

Then

$$\begin{aligned} \sum_{n=1}^{N} (\varphi_n, |A + B|\varphi_n) &= \sum_{n=1}^{N} (\varphi_n, U^*(A + B)\varphi_n) \\ &\leq \sum_{n=1}^{N} |(\varphi_n, U^*V|A|\varphi_n)| + \sum_{n=1}^{N} |(\varphi_n, U^*W|B|\varphi_n)| \end{aligned}$$

However,

$$\begin{aligned} \sum_{n=1}^{N} |(\varphi_n, U^*V|A|\varphi_n)| &\leq \sum_{n=1}^{N} \| \, |A|^{1/2}V^*U\varphi_n\| \; \| \, |A|^{1/2}\varphi_n\| \\ &\leq \left(\sum_{n=1}^{N} \| \, |A|^{1/2}V^*U\varphi_n\|^2\right)^{1/2} \left(\sum_{n=1}^{N} \| \, |A|^{1/2}\varphi_n\|^2\right)^{1/2} \end{aligned}$$

Thus, if we can show

$$\sum_{n=1}^{N} \| \, |A|^{1/2}V^*U\varphi_n\|^2 \leq \operatorname{tr} |A| \tag{VI.7}$$

we can conclude that

$$\sum_{n=1}^{N} (\varphi_n, |A + B|\varphi_n) \leq \operatorname{tr} |A| + \operatorname{tr} |B| < \infty$$

and thus $A + B \in \mathscr{I}_1$. To show (VI.7), we need only prove that

$$\operatorname{tr}(U^*V|A|V^*U) \leq \operatorname{tr}|A|$$

Picking an orthonormal basis, $\{\varphi_n\}$ with each φ_n in Ker U or $(\text{Ker } U)^\perp$ we see that $\operatorname{tr}(U^*(V|A|V^*)U) \leq \operatorname{tr}(V|A|V^*)$. Similarly, picking an orthonormal basis, $\{\psi_m\}$, with each ψ_m in Ker V^* or $(\text{Ker } V^*)^\perp$ we find $\operatorname{tr}(V|A|V^*) \leq \operatorname{tr} |A|$.

(b) By the lemma proven below, each $B \in \mathscr{L}(\mathscr{H})$ can be written as a linear combination of four unitary operators so by (a) we need only show that $A \in \mathscr{I}_1$ implies $UA \in \mathscr{I}_1$ and $AU \in \mathscr{I}_1$ if U is unitary. But $|UA| = |A|$ and $|AU| = U^{-1}|A|U$, so by part (c) of Theorem VI.18, AU and UA are in $\mathscr{I}_1$.

(c) Let $A = U|A|$ and $A^* = V|A^*|$ be the polar decompositions of A and A^*. Then $|A^*| = V^*|A|U^*$. If $A \in \mathscr{I}_1$, then $|A| \in \mathscr{I}_1$, so by part (b) $|A^*| \in \mathscr{I}_1$ and $A^* = V|A^*| \in \mathscr{I}_1$. ∎

To complete the proof of part (b) above we need the following lemma which we will use in other contexts later.

Lemma Every $B \in \mathscr{L}(\mathscr{H})$ can be written as a linear combination of four unitary operators.

Proof Since $B = \frac{1}{2}(B + B^*) - \frac{i}{2}[i(B - B^*)]$, B can be written as a linear combination of two self-adjoint operators. So, suppose A is self-adjoint and without loss of generality assume $\|A\| \leq 1$. Then $A \pm i\sqrt{I - A^2}$ are unitary and $A = \frac{1}{2}(A + i\sqrt{I - A^2}) + \frac{1}{2}(A - i\sqrt{I - A^2})$. ∎

The proof of the following theorem is left to the reader (Problem 23).

Theorem VI.20 Let $\|\cdot\|_1$ be defined in $\mathscr{I}_1$ by $\|A\|_1 = \operatorname{tr} |A|$. Then $\mathscr{I}_1$ is a Banach space with norm $\|\cdot\|_1$ and $\|A\| \leq \|A\|_1$.

We note that $\mathscr{I}_1$ is *not* closed under the operator norm $\|\cdot\|$. The connection between the trace class operators and the compact operators is simple:

Theorem VI.21 Every $A \in \mathscr{I}_1$ is compact. A compact operator A is in $\mathscr{I}_1$ if and only if $\sum_{n=1}^\infty \lambda_n < \infty$ where $\{\lambda_n\}_{n=1}^\infty$ are the singular values of A.

Proof Since $A \in \mathscr{I}_1$, $|A|^2 \in \mathscr{I}_1$, so $\operatorname{tr}(|A|^2) = \sum_{n=1}^\infty \|A\varphi_n\|^2 < \infty$ for any orthonormal basis $\{\varphi_n\}_{n=1}^\infty$. Suppose $\psi \in [\varphi_1, \ldots, \varphi_N]^\perp$ and $\|\psi\| = 1$, then we have

$$\|A\psi\|^2 \leq \operatorname{tr}(|A|^2) - \sum_{n=1}^N \|A\varphi_n\|^2$$

since $\{\varphi_1, \varphi_2, \ldots, \varphi_N, \psi\}$ can always be completed to an orthonormal basis. Thus

$$\sup\{\|A\psi\| \mid \psi \in [\varphi_1, \ldots, \varphi_N]^\perp, \|\psi\| = 1\} \to 0 \qquad \text{as} \quad N \to \infty$$

Therefore $\sum_{n=1}^N (\varphi_n, \cdot)A\varphi_n$ is norm convergent to A. Thus A is compact. The second part of the theorem follows easily from the canonical form derived in Theorem VI.17 (Problem 24). ∎

Corollary The finite rank operators are $\|\cdot\|_1$-dense in $\mathscr{I}_1$.

The second class of operators which we will discuss are the Hilbert–Schmidt operators, the analogue of $\mathscr{L}^2$.

Definition An operator $T \in \mathscr{L}(\mathscr{H})$ is called **Hilbert–Schmidt** if and only if $\operatorname{tr} T^*T < \infty$. The family of all Hilbert–Schmidt operators is denoted by $\mathscr{I}_2$.

By arguments analogous to those we used for $\mathscr{I}_1$, one can prove.

Theorem VI.22 (a) $\mathscr{I}_2$ is a $*$-ideal.

(b) If $A, B \in \mathscr{I}_2$, then for any orthonormal basis $\{\varphi_n\}$,

$$\sum_{n=1}^{\infty} (\varphi_n, A^*B\varphi_n)$$

is absolutely summable, and its limit, denoted by $(A, B)_2$, is independent of the orthonormal basis chosen.

(c) $\mathscr{I}_2$ with inner product $(\cdot, \cdot)_2$ is a Hilbert space.

(d) If $\|A\|_2 = \sqrt{(A, A)_2} = (\operatorname{tr}(A^*A))^{1/2}$, then

$$\|A\| \le \|A\|_2 \le \|A\|_1 \qquad \text{and} \qquad \|A\|_2 = \|A^*\|_2$$

(e) Every $A \in \mathscr{I}_2$ is compact and a compact operator, A, is in $\mathscr{I}_2$ if and only if $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$ where λ_n are the singular values of A.

(f) The finite rank operators are $\|\cdot\|_2$-dense in $\mathscr{I}_2$.

(g) $A \in \mathscr{I}_2$ if and only if $\{\|A\varphi_n\|\} \in \ell_2$ for some orthonormal basis $\{\varphi_n\}$.

(h) $A \in \mathscr{I}_1$ if and only if $A = BC$ with B, C in $\mathscr{I}_2$.

We note that $\mathscr{I}_2$ is not $\|\cdot\|$-closed. The important fact about $\mathscr{I}_2$ is that when $\mathscr{H} = L^2(M, d\mu)$, $\mathscr{I}_2$ has a concrete realization.

Theorem VI.23 Let $\langle M, \mu \rangle$ be a measure space and $\mathscr{H} = L^2(M, d\mu)$. Then $A \in \mathscr{L}(\mathscr{H})$ is Hilbert–Schmidt if and only if there is a function

$$K \in L^2(M \times M, d\mu \otimes d\mu)$$

with

$$(Af)(x) = \int K(x, y) f(y)\, d\mu(y)$$

Moreover,

$$\|A\|_2^2 = \int |K(x, y)|^2\, d\mu(x)\, d\mu(y)$$

Proof Let $K \in L^2(M \times M, d\mu \otimes d\mu)$ and let A_K be the associated integral operator. It is easy to see (Problem 25) that A_K is a well-defined operator on $\mathscr{H}$ and that

$$\|A_K\| \le \|K\|_{L^2} \tag{VI.8}$$

Let $\{\varphi_n\}_{n=1}^\infty$ be an orthonormal basis for $L^2(M, d\mu)$. Then $\{\varphi_n(x)\overline{\varphi_m(y)}\}_{n,m=1}^\infty$ is an orthonormal base for $L^2(M \times M, d\mu \otimes d\mu)$ so

$$K = \sum_{n,m=1}^{\infty} \alpha_{nm}\,\varphi_n(x)\overline{\varphi_m(y)}$$

Let

$$K_N = \sum_{n,m=1}^{N} \alpha_{n,m}\,\varphi_n(x)\overline{\varphi_m(y)}$$

Then each K_N is the integral kernel of a finite rank operator. In fact, $A_{K_N} = \sum_{n,m=1}^N \alpha_{nm}(\varphi_m, \cdot)\varphi_n$. Since $\|K_N - K\|_{L^2} \to 0$ we have $\|A_K - A_{K_N}\| \to 0$ as $N \to \infty$ by (VI.8). Thus A_K is compact and in fact

$$\operatorname{tr}(A_K^* A_K) = \sum_{n=1}^{\infty} \|A_K \varphi_n\|^2 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\alpha_{nm}|^2 = \|K\|_{L^2}$$

Thus $A_K \in \mathscr{I}_2$ and $\|A_K\|_2 = \|K\|_{L^2}$.

We have shown that the map $K \to A_K$ is an isometry of $L^2(M \times M, d\mu \otimes d\mu)$ into $\mathscr{I}_2$, so its range is closed. But the finite rank operators clearly come from kernels and since they are dense in $\mathscr{I}_2$ the range of $K \mapsto A_K$ is all of $\mathscr{I}_2$. ∎

This theorem provides a simple sufficient condition for an operator to be compact and is therefore very useful. Notice that the condition is not necessary. Also, we have a sufficient condition for an operator on $\mathscr{H} = L^2(M, d\mu)$ to be an integral operator. This condition is also not necessary. Now, we return to defining the trace on $\mathscr{I}_1$.

Theorem VI.24 If $A \in \mathscr{I}_1$ and $\{\varphi_n\}_{n=1}^\infty$ is any orthonormal basis, then $\sum_{n=1}^\infty (\varphi_n, A\varphi_n)$ converges absolutely and the limit is independent of the choice of basis.

Proof We write $A = U|A|^{1/2}|A|^{1/2}$. Then

$$|(\varphi_n, A\varphi_n)| \le \|\,|A|^{1/2}U^*\varphi_n\|\,\|\,|A|^{1/2}\varphi_n\|$$

Thus

$$\sum_{n=1}^{\infty} |(\varphi_n, A\varphi_n)| \le \left(\sum_{n=1}^{\infty} \|\,|A|^{1/2}U^*\varphi_n\|^2\right)^{1/2}\left(\sum_{n=1}^{\infty} \|\,|A|^{1/2}\varphi_n\|^2\right)^{1/2}$$

so since $|A|^{1/2}U^*$ and $|A|^{1/2}$ are in $\mathscr{I}_2$, the sum converges. The proof of the independence of basis is identical to that for $\operatorname{tr} A$ when $A \ge 0$. ∎

Definition The map $\operatorname{tr}: \mathscr{I}_1 \to \mathbb{C}$ given by $\operatorname{tr} A = \sum_{n=1}^\infty (\varphi_n, A\varphi_n)$ where $\{\varphi_n\}$ is any orthonormal basis is called the **trace**.

We remark that it is not true that $\sum_{n=1}^{\infty} |(\varphi_n, A\varphi_n)| < \infty$ for some orthonormal basis implies $A \in \mathscr{I}_1$. For A to be in $\mathscr{I}_1$ the sum must be finite for all orthonormal bases. The spectral theorem which we will prove in the next chapter will tell us that any self-adjoint A can be written $A = A_+ - A_-$ where both A_+ and A_- are positive and $A_+A_- = 0$. Not surprisingly, $A \in \mathscr{I}_1$ if and only if $\operatorname{tr}(A_+) < \infty$, $\operatorname{tr}(A_-) < \infty$ and in this case $\operatorname{tr} A = \operatorname{tr} A_+ - \operatorname{tr} A_-$. We collect the properties of the trace.

Theorem VI.25 (a) $\operatorname{tr}(\cdot)$ is linear.
(b) $\operatorname{tr} A^* = \overline{\operatorname{tr} A}$.
(c) $\operatorname{tr} AB = \operatorname{tr} BA$ if $A \in \mathscr{I}_1$ and $B \in \mathscr{L}(\mathscr{H})$.

Proof (a) and (b) are obvious. To prove (c) it is sufficient to consider the case where B is unitary since any bounded operator is the sum of four unitaries. In that case

$$\begin{aligned}\operatorname{tr} AB &= \sum_{n=1}^{\infty} (\varphi_n, AB\varphi_n) \\ &= \sum_{n=1}^{\infty} (B^*\psi_n, A\psi_n) \\ &= \sum_{n=1}^{\infty} (\psi_n, BA\psi_n) \\ &= \operatorname{tr} BA\end{aligned}$$

where $\psi_n = B\varphi_n$ for all n. ∎

If $A \in \mathscr{I}_1$, the map $B \mapsto \operatorname{tr} AB$ is a linear functional on $\mathscr{L}(\mathscr{H})$. These are not all the continuous linear functionals on $\mathscr{L}(\mathscr{H})$ but such functionals do yield the entire dual of $\operatorname{Com}(\mathscr{H})$, the compact operators. We can also hold $B \in \mathscr{L}(\mathscr{H})$ fixed and obtain a linear functional on $\mathscr{I}_1$ given by the map $A \mapsto \operatorname{tr} BA$. The set of these functionals is just the dual of $\mathscr{I}_1$ (with the operator norm topology). We state this as a theorem; the interested reader can follow the outline of the proof given in Problem 30.

Theorem VI.26 (a) $\mathscr{I}_1 = [\operatorname{Com}(\mathscr{H})]^*$. That is, the map $A \mapsto \operatorname{tr}(A\,\cdot)$ is an isometric isomorphism of $\mathscr{I}_1$ onto $[\operatorname{Com}(\mathscr{H})]^*$.
(b) $\mathscr{L}(\mathscr{H}) = \mathscr{I}_1^*$. That is, the map $B \mapsto \operatorname{tr}(B\,\cdot)$ is an isometric isomorphism of $\mathscr{L}(\mathscr{H})$ onto $\mathscr{I}_1^*$.

We now return to the distinction between the weak operator topology on $\mathscr{L}(\mathscr{H})$ (see Section VI.1) and the weak Banach space topology, i.e. the $\sigma(\mathscr{L}(\mathscr{H}), \mathscr{L}(\mathscr{H})^*)$ topology. If $\mathscr{F}$ is the family of finite rank operators, then $\mathscr{F} \subset \mathscr{I}_1$ and each $F \in \mathscr{F}$ can be realized as linear functional on $\mathscr{L}(\mathscr{H})$ via the dual action of $\mathscr{I}_1$ on $\mathscr{L}(\mathscr{H})$. The topology on $\mathscr{L}(\mathscr{H})$ generated by these functionals, that is, $\sigma(\mathscr{L}(\mathscr{H}), \mathscr{F})$ is just the weak operator topology. The set $\mathscr{F}$ is not closed in the $\mathscr{L}(\mathscr{H})^*$-norm. As a matter of fact, the $\mathscr{L}(\mathscr{H})^*$ norm on $\mathscr{F}$ is just $\|\cdot\|_1$ so the closure of $\mathscr{F}$ in this norm is just $\mathscr{I}_1$. The weak topology on $\mathscr{L}(\mathscr{H})$ generated by the functionals in $\mathscr{I}_1$, that is, $\sigma(\mathscr{L}(\mathscr{H}), \mathscr{I}_1)$, is called the **ultraweak** topology on $\mathscr{L}(\mathscr{H})$. Notice that it is stronger than the weak operator topology, since more functionals are required to be continuous, but weaker than the weak Banach space topology on $\mathscr{L}(\mathscr{H})$, since $\mathscr{I}_1$ is not the entire dual of $\mathscr{L}(\mathscr{H})$. In fact, since $\mathscr{L}(\mathscr{H}) = \mathscr{I}_1^*$, the ultraweak topology on $\mathscr{L}(\mathscr{H})$ is just the weak-$*$ topology. This realization of $\mathscr{L}(\mathscr{H})$ as the dual of the Banach space of linear functionals continuous in the $\sigma(\mathscr{L}(\mathscr{H}), \bar{\mathscr{F}})$ topology is valid for a larger class of algebras than just $\mathscr{L}(\mathscr{H})$. Problem 31 gives another example: the multiplication algebra L_∞ on L_2. We will study such algebras in detail in Chapter XVIII. We study the $\mathscr{I}_p$ spaces for $p \neq 1, 2, \infty$ in Sections IX.4 and XIII.17.

NOTES

Section VI.1 The reader may be bewildered by the many topologies we have introduced on $\mathscr{L}(\mathscr{H})$: the weak, strong, and uniform operator topologies, the weak Banach space topology, the ultraweak topology (Section VI.6). Later on we will even encounter the ultrastrong topology. Why is it necessary to introduce all these topologies? The answer is that many of the operators we are interested in are given as some sort of limit of simpler operators. It is important to know exactly what one means by "some sort" and to know what properties of the limiting operator follow from properties in the sequence, for example, the uniform limit of compact operators is compact. Furthermore, when one begins a problem one doesn't always know in what sense limits will exist, so it is useful to have a wide range of topologies at hand. In general it is the weak, strong, and uniform operator topologies which are important in Volumes I and II. The ultraweak and ultrastrong topologies will play a role when we deal with von Neumann algebras. The weak, strong, and ultrastrong operator topologies were introduced in J. von Neumann, "Zur Algebra der Functionaloperationen und Theorie der Normalen Operatoren," *Math. Ann.* **102** (1929–1930), 370–427.

Section VI.2 The spectral theorem for self-adjoint operators on finite dimensional vector spaces is nicely described in P. R. Halmos, *Finite Dimensional Vector Spaces*, Van Nostrand–Reinhold, Princeton, New Jersey, 1958.

Section VI.3 The definitions of various kinds of spectra will also be used for unbounded operators. Theorem VI.5 holds as long as we require that T be closed. If T is bounded it is, of course, automatically closed.

The theory of Banach space-valued analytic functions is described in great detail in *Functional Analysis and Semi-groups*, Amer. Math. Soc., Providence, Rhode Island, 1957, by E. Hille and R. S. Phillips. They also discuss the more difficult notion of analytic functions from one Banach space to another. A proof of Theorem VI.7 can be found in *Functional Analysis*, Academic Press, New York, 1965, by K. Yosida.

Some authors (for example: Yosida or Hille, Phillips) use the term "continuous spectrum" to denote any $\lambda \in \sigma(T)$ which is neither in the point spectrum, nor in the residual spectrum. Other authors (such as Kato or Riesz, Nagy) use the definition that we give in Section VII.2. One important distinction is that with our definition the continuous spectrum and the point spectrum need not be disjoint.

Section VI.4 The polar decomposition has a simple geometric meaning for linear transformations on $\mathbb{R}^n$. Any linear transformation A on $\mathbb{R}^n$ can be written as $A = OS$ where O is orthogonal and S is self-adjoint. By the spectral theorem, S can be thought of as a dilation, contraction, or annihilation in certain preferred orthogonal directions.

The notion of positivity has a natural generalization to operator algebras and will play an important role in our investigations in Volume III.

The statement that the triangle inequality fails for $|\cdot|$, that is, $|A + B|$ may *not* be less than or equal to $|A| + |B|$ (see Problem 16) is a statement that $f(x) = |x|$ is not a convex *operator-valued* function, that is, for $0 \le t \le 1$, $f(tA + (1-t)B) \le tf(A) + (1-t)f(B)$ can be false for general operators A and B despite the fact that $f(tx - (1-t)y) \le tf(x) + (1-t)f(y)$ is true for x and y real and $0 \le t \le 1$. Exactly which matrix and operator valued functions are convex has been studied in: F. Krauss, "Über konvexe Matrixfunktionen," *Math. Z.* **41** (1936) 18–42, and J. Bendat and S. Sherman, "Monotone and Convex Operator Functions," *Trans. Amer. Math. Soc.* **79** (1955), 58–71.

Section VI.5 The proof of the second part of Theorem VI.12 can be found in Yosida's book; it is a nice application of the Ascoli–Arzela and Alaoglu theorems (see also Problem 36).

In a very real sense, the theory of compact operators goes back to Fredholm's great paper on integral operators, "Sur une class d'équations fonctionnelles," *Acta Math.* **27** (1903), 365–390. Fredholm considered solving equations of the form

$$f(x) = g(x) + \lambda \int_a^b K(x, y) f(y)\, dy$$

where g and K are given continuous functions and $-\infty < a < b < \infty$. Fredholm showed that there exists an explicit entire function $d(\lambda)$, not identically zero, and an explicit function $D_\lambda(x, y)$, entire in λ and continuous in x and y, so that if $d(\lambda) \neq 0$, then $f(x) = g(x) + d(\lambda)^{-1} \int_a^b D_\lambda(x, y) g(y)\, dy$ solves the equation. Moreover, he showed that when $d(\lambda) = 0$, then

$$f(x) = \lambda \int_a^b K(x, y) f(y)\, dy$$

has a solution $f \neq 0$. Fredholm thus had Theorem VI.15 and the preceding corollary in this special case. Readable expositions of the Fredholm theory may be found in W. Lovitt: *Linear Integral Equations*, Dover, New York (reprinted 1950; original edition, McGraw-Hill, New York, 1926), and F. Smithies, *Integral Equations*, Cambridge Univ. Press, London and New York, 1958.

Fredholm's work produced considerable interest among Hilbert and his school and led to the abstraction of many notions we now associate with Hilbert space theory. Hilbert first defined completely continuous operators in a manner whose modern form would be the criterion of Theorem VI.11: D. Hilbert, "Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen, I–VI," *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl.* 49–91 (1904), 213–259, 307–388 (1905); 157–222, 439–480 (1906); 355–417 (1910); esp. IV. The extension of the notion of compact operator to arbitrary Banach spaces by the precompactness criteria is due to F. Riesz "Über lineare Functionalgleichungen," *Acta. Math.* **41** (1918), 71–98.

Theorem VI.12b is due to J. Schauder: "Über lineare, vollstetige Functionaloperationen," *Studia Math.* **2** (1930), 183–196.

The idea of using Theorem IV.13 to develop the general theory is due to E. Schmidt, "Auflösung der allgemeinen linearen Integralgleichung," *Math. Ann.* **64** (1907), 161–174. While it is true that compact operators in most explicit Banach spaces are norm limits of finite rank operators, there are Banach spaces where this is false. The earliest examples were constructed by P. Enflo. For extensive discussion, see M. M. Day, *Normed Linear Spaces*, Springer, Berlin, 1973, and J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, Springer Lecture Notes in Math **388**, Springer-Verlag, 1973.

Theorem VI.14, its corollary, and Theorem VI.15 hold in an arbitrary Banach space. For their proof in that case, see N. Dunford and J. Schwartz, *Linear Operators*, Vol. I. Wiley (Interscience), 1958. Our technique of proof for Theorem VI.14 is taken from a technical appendix in W. Hunziker, "On the Spectra of Schrödinger Multiparticle Hamiltonians," *Helv. Phys. Acta.* **39** (1966), 451–462. A similar approach can be found in an appendix of G. Tiktopolous, "Analytic Continuation in Complex Angular Momentum and Integral Equations," *Phys. Rev.* **133B** (1964), 1231–1238. One part of Theorem VI.14 is not proven in the general case in Dunford–Schwartz; a discussion of this extra point can be found in S. Steinberg, "Meromorphic Families of Compact Operators," *Arch. Rat. Mech. Anal.* **31** (1968), 372–379. For extensions to locally convex spaces, see J. Leray, "Valeurs propres et vecteurs propres d'un endomorphisme conplètement continu d'un espace vectoriel à voisinages convexes," *Acta Sci. Math. Szcg.* **12**, Part B, (1950), 177–186. Theorem VI.15 was first proven by Riesz and Schauder in the above cited works (Schauder filled in some details for the general case) and Theorem VI.16 is due to Hilbert and Schmidt in the aforementioned papers.

For a discussion of the use of integral equations in the solution of Dirichlet problem, see *Boundary Value Problems of Mathematical Physics*, Vol. 2, (especially sections 6.4 and 6.5), Macmillan, New York, 1968, by Ivor Stakgold and Volume II of R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Wiley (Interscience).

Section VI.6 For a discussion on $\mathscr{I}_1$, $\mathscr{I}_2$, and the $\mathscr{I}_p$ analogues, see R. Schatten, *Norm Ideals of Completely Continuous Operators*, Springer–Verlag, Berlin and New York, 1960. $\mathscr{I}_p$ is defined as those A with $\mathrm{Tr}(|A|^p) < \infty$ and is equivalently those compact operators with $\sum |\lambda_n|^p < \infty$. For further discussion, see Sections IX.4 and XIII.17.

These norm ideals have been extended to other situations with traces (von Neumann algebras) and more general settings in a manner emphasizing the analogy with L^p by I. Segal: "A Non-Commutative Extension of Abstract Integration," *Ann. Math.* **57** (1953), 401–457; **58** (1953), 595–596, and R. A. Kunze, "L_p Fourier Transforms on Locally Compact Unimodular Groups," *Trans. Amer. Math. Soc.* **89** (1958), 519.

PROBLEMS

†1. Prove that the weak operator topology is weaker than the strong operator topology which is weaker than the uniform operator topology.

†2. Prove the statements in the example in Section VI.1.

3. (a) Let X and Y be Banach spaces. Prove that if $T_n \in \mathscr{L}(X, Y)$ and $\{T_n x\}$ is a Cauchy sequence for each $x \in X$, then there exists a $T \in \mathscr{L}(X, Y)$ so that $T_n \to T$ strongly.
*(b) Is the theorem in (a) true if T_n is replaced by a net T_α ?

4. (a) Let X and Y be Banach spaces. Prove that a theorem for $\mathscr{L}(X, Y)$ analogous to Theorem VI.1 holds if Y is weakly sequentially complete (which means that every weakly Cauchy sequence has a weak limit.)
(b) Prove that if a Banach space is reflexive, then it is weakly sequentially complete.

5. (a) Let T_t be the operator $T_t: \varphi(x) \to \varphi(x+t)$ on $L^2(\mathbb{R})$. What is the norm of T_t? To what operator does T_t converge as $t \to \infty$ and in what topology?
(b) Answer the same question for T_t if the Hilbert space is $L^2(\mathbb{R}, e^{-x^2}\,dx)$.

6. (a) Let $\mathscr{H}$ be an infinite dimensional Hilbert space. Suppose $\psi_1, \ldots, \psi_n$ orthonormal are given and that ε, ψ are given. Show there are A and B with $\|A\psi_i\| < \varepsilon$, $\|B\psi_i\| = \varepsilon$; $i = 1, \ldots, n$, but that $\|AB\psi\| > 1$.
(b) Prove that multiplication from $\mathscr{L}(\mathscr{H}) \times \mathscr{L}(\mathscr{H}) \to \mathscr{L}(\mathscr{H})$ is not jointly continuous when $\mathscr{L}(\mathscr{H})$ is given the strong topology.
(c) Suppose $\{A_\alpha\}_{\alpha \in I}$ and $\{B_\alpha\}_{\alpha \in I}$ are *nets*. Let $A_\alpha^* \xrightarrow{s} A^*$, $B_\alpha \xrightarrow{s} B$. Prove that $A_\alpha B_\alpha \xrightarrow{w} AB$.
(d) Let A_n, B_n be *sequences* so that $A_n \xrightarrow{s} A$, $B_n \xrightarrow{s} B$. Prove that $A_n B_n \xrightarrow{s} AB$.
(e) Let A_n, B_n be *sequences* so that $A_n \xrightarrow{w} A$, $B_n \xrightarrow{w} B$. Give an example where $A_n B_n \xrightarrow{w} AB$ is *false*.

7. Give an example to show that the range of a bounded operator need not be closed. Prove that if T is bounded, everywhere defined, and an isometry, then Ran T is closed.

†8. (a) Let A be a self-adjoint bounded operator on a Hilbert space. Prove that the eigenvalues of A are real and that the eigenvectors corresponding to distinct eigenvalues are orthogonal.
(b) From the proof of Theorem VI.8 derive a universal (but λ-dependent) bound for the norm of the resolvent of a self-adjoint operator at a nonreal $\lambda \in \mathbb{C}$.

9. (a) Let A be a self-adjoint operator on a Hilbert space, $\mathscr{H}$. Prove that

$$\|A\| = \sup_{\|x\|=1} |(Ax, x)|$$

Hint: First note that

$$\mathrm{Re}(\psi, A\phi) = \tfrac{1}{4}[(\psi+\phi, A(\psi+\phi)) - (\psi-\phi, A(\psi-\phi))]$$

Then using

$$|(\eta, A\eta)| \le \|\eta\|^2 \sup_{\|\eta\|=1} |(\eta, A\eta)|$$

and the parallelogram law, prove that

$$|(\psi, A\phi)| \le \sup_{\|\eta\|=1} |(\eta, A\eta)|$$

if $\|\varphi\| = \|\psi\| = 1$.

(b) Find an example which shows that the conclusion of (a) need not be true if A is not self-adjoint.

10. Show that the spectral radius of the Volterra integral operator

$$(Tf)(x) = \int_0^x f(y)\,dy$$

as a map on $C[0, 1]$ is equal to zero. What is the norm of T?

†*11.* Let $T \in \mathscr{L}(X)$. Prove that $\lim_{n\to\infty} \|T^n\|^{1/n}$ exists and is equal to $\inf_n \|T^n\|^{1/n}$ as follows:
(a) Set $a_n = \log \|T^n\|$ and prove that $a_{m+n} \le a_m + a_n$.
(b) For a fixed positive integer m set $n = mq + r$ where q and r are positive integers and $0 \le r \le m - 1$. Using (a) conclude that

$$\overline{\lim_n}\, \frac{a_n}{n} \le \frac{a_m}{m}$$

(c) Prove that $\lim_{n\to\infty} a_n/n = \inf_n a_n/n$ and thus the desired equality.

†*12.* Prove the proposition at the end of Section VI.3.

13. (a) Give an example which shows that a linear transformation on $\mathbb{C}^n$ can be positive without all the entries in a given matrix representation being positive.
*(b) Derive a necessary and sufficient condition for a $n \times n$ matrix to be positive.

14. (a) Prove that if $A_n \ge 0$, $A_n \to A$ in norm, then $\sqrt{A_n} \to \sqrt{A}$ norm.
(b) Suppose $A_n \to A$ strongly for a sequence $\{A_n\}$. Prove that $\sqrt{A_n} \to \sqrt{A}$ strongly.

15. (a) Let $A_n \to A$ in norm. Prove that $|A_n| \to |A|$ in norm.
(b) Suppose $A_n \to A$ and $A_n^* \to A^*$ strongly where A_n is a sequence. Prove that $|A_n| \to |A|$ strongly.
(c) Find an example which shows that $|\cdot|$ is not weakly continuous on $\mathscr{L}(\mathscr{H})$.

16. Let $\sigma_3 = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, $\sigma_1 = \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$. Prove that it is false that

$$|(\sigma_3 + 1) + (\sigma_1 - 1)| \le |(\sigma_3 + 1)| + |(\sigma_1 - 1)|$$

Remark: This example is due to E. Nelson.

17. Show that it is not necessarily true that

$$\|\,|A| - |B|\,\| \le \|A - B\|$$

(*Hint*: See Problem 16.)

†*18.* (a) Prove the proposition preceding Theorem VI.10.
(b) Prove the uniqueness in Theorem VI.10.

19. Write the matrix $\left(\begin{smallmatrix}-1 & -2\\ \frac{1}{2} & 1\end{smallmatrix}\right)$ as the product of a rotation and a positive symmetric matrix.

**20.* Suppose that X is a reflexive Banach space and that T: $X \to X$ a bounded linear operator. Prove that if T takes weakly convergent sequences into norm convergent sequences, then T is compact.

†*21.* Complete the proof of Theorem VI.14 by extending the result to all of D.

22. Using the Stone–Weierstrass theorem prove that every Fredholm integral operator on $C[a, b]$

$$(Tf)(x) = \int_a^b K(x, y) f(y)\, dy$$

where K is continuous, is a norm limit of operators of finite rank.

†23. (a) Prove that $\|A\| \le \|A\|_1$.
(b) Suppose $\{A_n\}$ is an $\|\cdot\|_1$-Cauchy sequence. Show that $\{A_n\}$ has a $\|\cdot\|$-limit A and that $\operatorname{tr}|A| < \infty$. Then conclude the proof of Theorem VI.20 by showing that A is the $\|\cdot\|_1$-limit of $\{A_n\}$.

†24. (a) Use the canonical form given by Theorem VI.17 to prove the second statement in Theorem VI.21.
(b) Prove the corollary to Theorem VI.21.

†25. Let $K \in L^2(M \times M, d\mu \otimes d\mu)$ and let A_K be the integral operator

$$(A_K \varphi)(x) = \int_M K(x, y) \varphi(y)\, d\mu(y)$$

Prove that A_K is well defined and $\|A_K\| \le \|K\|_{L^2}$.

26. (a) Prove that if $\sum_{n=1}^{\infty} |(A\varphi_n, \varphi_n)| < \infty$ for all orthonormal bases, then $A \in \mathscr{I}_1$.
(b) Find an $A \notin \mathscr{I}_1$ so that $\sum_{n=1}^{\infty} |(A\varphi_n, \varphi_n)| < \infty$ for some fixed orthonormal basis.

27. Prove that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ if $A, B \in \mathscr{I}_2$.

28. Prove that (a) $\|AB\|_1 \le \|A\|\ \|B\|_1$
(b) $\|AB\|_2 \le \|A\|\ \|B\|_2$
(c) $\|AB\|_1 \le \|A\|_2 \|B\|_2$

·29. Prove that $A \in \mathscr{I}_1$ if and only if $A = BC$ with B and C in $\mathscr{I}_2$.

·30. The goal of this problem is to prove Theorem VI.26.
(a) Let f be a bounded linear functional on $\operatorname{Com}(\mathscr{H})$. Let $(\psi, \cdot)\phi$ be the operator on $\mathscr{H}$ which takes η into $(\psi, \eta)\phi$. Show that there is a unique bounded linear operator, B, with

$$(\psi, B\phi) = f[(\psi, \cdot)\phi]$$

(b) Using the fact that

$$\sum_{n=1}^{N} (\phi_n, |B|\phi_n) = f\left[\sum_{n=1}^{N} (U\phi_n, \cdot)\phi_n\right]$$

prove that $B \in \mathscr{I}_1$ and $\|B\|_1 \le \|f\|_{\operatorname{Com}(\mathscr{H})^*}$.
(c) Prove that $A \mapsto \operatorname{tr}(BA)$ is a bounded linear functional on $\operatorname{Com}(\mathscr{H})$ which is in fact equal to $f(\cdot)$.
(d) Prove that $\|B\|_1 = \|f\|_{\operatorname{Com}(\mathscr{H})^*}$.
(e) Let g be a bounded linear functional on $\mathscr{I}_1$. Show that there is a unique bounded linear operator, B, with

$$(\psi, B\phi) = g[(\psi, \cdot)\phi]$$

(f) Prove that $A \mapsto \operatorname{tr}(BA)$ is a bounded linear functional on $\mathscr{I}_1$ which agrees with g and that $\|g\|_{\mathscr{I}_1^*} = \|B\|$.

31. Let $\langle M, \mu \rangle$ be a measure space and let $L^\infty(M, d\mu)$ act on $\mathscr{H} = L^2(M, d\mu)$ by

$$(T_f\varphi)(x) = f(x)\varphi(x)$$

Prove that the topology on L^∞ induced by the weak operator topology on $\mathscr{L}(\mathscr{H})$ is identical to the weak-$*$ topology induced on L^∞ by L^1.

32. Let $C[0, 1]$ act on $L^2(0, 1)$ as in Problem 31. Find a sequence in $C[0, 1]$ convergent in the weak operator topology on $C[0, 1]$ to $f \in C[0, 1]$ which is not convergent in the weak Banach space topology on $C[0, 1]$.

33. Consider $\mathscr{I}_2$ as a Hilbert space with inner product $(A, B)_2 = \operatorname{tr}(A^*B)$. Let $A \mapsto L_A$ and $A \mapsto R_A$ be the maps of $\mathscr{L}(\mathscr{H})$ into $\mathscr{L}(\mathscr{I}_2)$ given by

$$L_A(B) = AB, \qquad R_A(B) = BA^*$$

(a) Prove that $A \mapsto L_A$ is a homomorphism of $\mathscr{L}(\mathscr{H})$ into $\mathscr{L}(\mathscr{I}_2)$.
(b) Prove that $A \mapsto R_A$ is a conjugate linear homomorphism of $\mathscr{L}(\mathscr{H})$ into $\mathscr{L}(\mathscr{I}_2)$.
(c) Suppose that $C \in \mathscr{L}(\mathscr{I}_2)$ and obeys $CL_A = L_A C$ for all $A \in \mathscr{L}(\mathscr{H})$. Prove that $C = R_B$ for some $B \in \mathscr{L}(\mathscr{H})$.

*34. Show that in a Hilbert space, a map $T: \mathscr{H} \to \mathscr{H}$ is continuous if the domain is given the weak topology and the range the norm topology (that is, $x_\alpha \xrightarrow{w} x$ implies $Tx_\alpha \xrightarrow{\|\cdot\|} Tx$ for arbitrary *nets*) if and only if T has finite rank! (Compare with Theorem VI.11.)

35. (a) Suppose T is an operator in $\mathscr{L}(\mathscr{H})$ so that $x_n \xrightarrow{\|\cdot\|} x$ implies $Tx_n \xrightarrow{w} Tx$. Prove T is bounded (so $Tx_n \xrightarrow{\|\cdot\|} Tx$).
(b) Identify the continuous linear maps of $\mathscr{L}(\mathscr{H})$ into itself if both the domain and range are given the weak topology.

36. Use (c) of Theorem VI.12 and the polar decomposition to prove (b) of Theorem VI.12 when $X = Y$ is a Hilbert space.

†37. Prove part (c) of Theorem VI.12.

38. Let P and Q be orthogonal projections onto subspaces $\mathscr{M}$ and $\mathscr{N}$ in a Hilbert space $\mathscr{H}$. Suppose that $PQ = QP$.
(a) Prove $1 - P, 1 - Q, PQ, P + Q - PQ$ and $P + Q - 2PQ$ are orthogonal projections.
(b) How are the ranges of the projections in (a) related to $\mathscr{M}$ and $\mathscr{N}$.

*39. Let P and Q be orthogonal projections onto subspaces $\mathscr{M}$ and $\mathscr{N}$ in a Hilbert space $\mathscr{H}$. Prove that $\text{s-lim}_{n\to\infty} (PQ)^n$ exists and is the orthogonal projection onto $\mathscr{M} \cap \mathscr{N}$.

*40. Let $\mathscr{I}$ be a norm closed ideal in $\mathscr{L}(\mathscr{H})$, $\mathscr{I} \neq 0$. Prove $\operatorname{Com}(\mathscr{H}) \subset \mathscr{I}$ by proving that any finite rank operator is in $\mathscr{I}$.

Remark: We will see (Chapter VII, Problem 31) that the only norm closed ideals when $\mathscr{H}$ is separable are $\{0\}$, $\operatorname{Com}(\mathscr{H})$, $\mathscr{L}(\mathscr{H})$.

41. Find a projection on $\mathbb{R}^2$ which is not an orthogonal projection.

42. Let $A \in \mathscr{L}(X)$. Prove that the set of λ such that λ is in $\sigma(A)$ but not an eigenvalue and $\operatorname{Ran}(\lambda I - A)$ is closed but not all of X is a open subset of $\mathbb{C}$.

43. Let M and N be subspaces of a Banach space X such that $M + N = X$ and $M \cap N = \{0\}$. Let P be the projection of X onto M. Prove that P is bounded if and only if *both* M and N are closed.

44. (a) Define the numerical range, $N(T)$, of a bounded operator, T, on a Hilbert space, $\mathscr{H}$, by $N(T) = \{(\psi, T\psi) \mid \psi \in \mathscr{H},\ \|\psi\| = 1\}$. Prove that $\sigma(T) \subset \overline{N(T)}$. (Hint: First show that if λ is an eigenvalue of T or T^*, then $\lambda \in N(T)$; then show that if $\lambda \in \sigma(T)$ and λ is not an eigenvalue of T or T^*, we can find $\psi_n \in \mathscr{H}$ so that $\|(T - \lambda)\psi_n\| \to 0$.)
(b) Find an example where $N(T)$ is not closed and $\sigma(T) \not\subset N(T)$.
(c) Find an example where $\sigma(T) \neq N(T) = \overline{N(T)}$.

Remark: There is a deep result of Hausdorff that $N(T)$ is convex.

45. (a) Let $\{\phi_n\}_{n=1}^{\infty}$ be an orthonormal basis for a Hilbert space $\mathscr{H}$. Let A be an operator with

$$\sup_{\substack{\psi \in [\phi_1, \ldots, \phi_n]^\perp \\ \|\psi\| = 1}} \|A\psi\| \to 0 \qquad \text{as} \quad n \to \infty.$$

Prove that A is compact.
(b) Let $\{\phi_n\}_{n=1}^{\infty}$ be any orthonormal basis for a Hilbert space $\mathscr{H}$ and let A be compact. Prove that

$$\sup_{\psi \in [\phi_1, \ldots, \phi_n]^\perp} \|A\psi\| \to 0 \qquad \text{as} \quad n \to \infty.$$

46. (a) Let $A \geq 0$ with A compact. Prove that $A^{1/2}$ is compact. (Hint: Use Problem 45.)
(b) Let $0 \leq A \leq B$. Prove that A is compact if B is compact. (Hint: Prove that $A^{1/2}$ is compact using Problem 45 and part (a).)

47. Let $\mathscr{H}$ and $\mathscr{H}'$ be two Hilbert spaces. If T is a bounded linear map from $\mathscr{H}$ to $\mathscr{H}'$ we define $T^* : \mathscr{H}' \to \mathscr{H}$ by $(T^*\psi, \phi)_{\mathscr{H}} = (\psi, T\phi)$. T is called *Hilbert–Schmidt* if and only if $T^*T : \mathscr{H} \to \mathscr{H}$ is trace class. Let T be Hilbert–Schmidt. Prove that there are real numbers, $\lambda_n > 0$, and orthonormal sets $\{\phi_n\}_{n=1}^{N} \subset \mathscr{H}$, $\{\psi_n\}_{n=1}^{N} \in \mathscr{H}'$ so that

$$T\phi = \sum_{n=1}^{N} \lambda_n(\phi_n, \phi)\psi_n$$

48. Let $\mathscr{H}$ and $\mathscr{H}'$ be the two Hilbert spaces and let $\mathscr{I}_2(\mathscr{H}, \mathscr{H}')$ denote the Hilbert–Schmidt operators from $\mathscr{H}$ to $\mathscr{H}'$.
(a) Prove that $\mathscr{I}_2(\mathscr{H}, \mathscr{H}')$ with the inner product

$$(S, T) = \mathrm{Tr}_{\mathscr{H}}(S^*T)$$

is a Hilbert space.
(b) Given $\psi \in \mathscr{H}$, $\phi \in \mathscr{H}'$ define $I(\psi, \phi) \in \mathscr{I}_2(\mathscr{H}^*, \mathscr{H}')$ by $I(\psi, \phi)\ell = \ell(\psi)\phi$ for any $\ell \in \mathscr{H}^*$. Prove that the map J, taking $\psi \otimes \phi$ into $I(\psi, \phi)$, is well defined and extends to an isometry of $\mathscr{H} \otimes \mathscr{H}'$ and $\mathscr{I}_2(\mathscr{H}^*, \mathscr{H}')$.
(c) Given $\eta \in \mathscr{H} \otimes \mathscr{H}'$ show that there exist reals, $\lambda_n > 0$, and orthonormal sets $\{\phi_n\}_{n=1}^{N} \subset \mathscr{H}$, $\{\psi_n\}_{n=1}^{N} \subset \mathscr{H}'$ with N finite or infinite, so that

$$\sum_{n=1}^{N} |\lambda_n|^2 = \|\eta\|^2 \qquad \text{and} \qquad \sum_{n=1}^{N} \lambda_n \phi_n \otimes \psi_n = \eta.$$