CHAPTER

NINE

FOURIER TRANSFORMS

Formal Properties

9.1 Definitions In this chapter we shall depart from the previous notation and use the letter m not for Lebesgue measure on R^1 but for Lebesgue measure divided by $\sqrt{2\pi}$. This convention simplifies the appearance of results such as the inversion theorem and the Plancherel theorem. Accordingly, we shall use the notation

$$\int_{-\infty}^{\infty} f(x)\, dm(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, dx, \tag{1}$$

where dx refers to ordinary Lebesgue measure, and we define

$$\|f\|_p = \left\{\int_{-\infty}^{\infty} |f(x)|^p\, dm(x)\right\}^{1/p} \qquad (1 \le p < \infty), \tag{2}$$

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - y)g(y)\, dm(y) \qquad (x \in R^1), \tag{3}$$

and

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{-ixt}\, dm(x) \qquad (t \in R^1). \tag{4}$$

Throughout this chapter, we shall write L^p in place of $L^p(R^1)$, and C_0 will denote the space of all continuous functions on R^1 which vanish at infinity.

If $f \in L^1$, the integral (4) is well defined for every real t. The function $\hat{f}$ is called the *Fourier transform* of f. Note that the term "Fourier transform" is also applied to the *mapping* which takes f to $\hat{f}$.

The formal properties which are listed in Theorem 9.2 depend intimately on the translation-invariance of m and on the fact that for each real α the mapping $x \to e^{i\alpha x}$ is a *character* of the additive group R^1. By definition, a function φ is a character of R^1 if $|\varphi(t)| = 1$ and if

$$\varphi(s + t) = \varphi(s)\varphi(t) \tag{5}$$

for all real s and t. In other words, φ is to be a homomorphism of the additive group R^1 into the multiplicative group of the complex numbers of absolute value 1. We shall see later (in the proof of Theorem 9.23) that every continuous character of R^1 is given by an exponential.

9.2 Theorem *Suppose $f \in L^1$, and α and λ are real numbers.*

(*a*) *If $g(x) = f(x)e^{i\alpha x}$, then $\hat{g}(t) = \hat{f}(t - \alpha)$.*
(*b*) *If $g(x) = f(x - \alpha)$, then $\hat{g}(t) = \hat{f}(t)e^{-i\alpha t}$.*
(*c*) *If $g \in L^1$ and $h = f * g$, then $\hat{h}(t) = \hat{f}(t)\hat{g}(t)$.*

Thus the Fourier transform converts multiplication by a character into translation, and vice versa, and it converts convolutions to pointwise products.

(*d*) *If $g(x) = \overline{f(-x)}$, then $\hat{g}(t) = \overline{\hat{f}(t)}$.*
(*e*) *If $g(x) = f(x/\lambda)$ and $\lambda > 0$, then $\hat{g}(t) = \lambda\hat{f}(\lambda t)$.*
(*f*) *If $g(x) = -ixf(x)$ and $g \in L^1$, then $\hat{f}$ is differentiable and $\hat{f}'(t) = \hat{g}(t)$.*

PROOF (*a*), (*b*), (*d*), and (*e*) are proved by direct substitution into formula 9.1(4). The proof of (*c*) is an application of Fubini's theorem (see Theorem 8.14 for the required measurability proof):

$$\begin{aligned}\hat{h}(t) &= \int_{-\infty}^{\infty} e^{-itx}\, dm(x) \int_{-\infty}^{\infty} f(x - y)g(y)\, dm(y) \\ &= \int_{-\infty}^{\infty} g(y)e^{-ity}\, dm(y) \int_{-\infty}^{\infty} f(x - y)e^{-it(x-y)}\, dm(x) \\ &= \int_{-\infty}^{\infty} g(y)e^{-ity}\, dm(y) \int_{-\infty}^{\infty} f(x)e^{-itx}\, dm(x) \\ &= \hat{g}(t)\hat{f}(t).\end{aligned}$$

Note how the translation-invariance of m was used.

To prove (*f*), note that

$$\frac{\hat{f}(s) - \hat{f}(t)}{s - t} = \int_{-\infty}^{\infty} f(x)e^{-ixt}\varphi(x, s - t)\, dm(x) \qquad (s \neq t), \tag{1}$$

where $\varphi(x, u) = (e^{-ixu} - 1)/u$. Since $|\varphi(x, u)| \le |x|$ for all real $u \neq 0$ and since $\varphi(x, u) \to -ix$ as $u \to 0$, the dominated convergence theorem applies to (1), if s tends to t, and we conclude that

$$\hat{f}'(t) = -i \int_{-\infty}^{\infty} xf(x)e^{-ixt}\, dm(x). \tag{2}$$

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9.3 Remarks

(*a*) In the preceding proof, the appeal to the dominated convergence theorem may seem to be illegitimate since the dominated convergence theorem deals only with *countable* sequences of functions. However, it does enable us to conclude that

$$\lim_{n\to\infty} \frac{\hat{f}(s_n) - \hat{f}(t)}{s_n - t} = -i \int_{-\infty}^{\infty} xf(x)e^{-ixt}\, dm(t)$$

for every sequence $\{s_n\}$ which converges to t, and this says exactly that

$$\lim_{s\to t} \frac{\hat{f}(s) - \hat{f}(t)}{s - t} = -i \int_{-\infty}^{\infty} xf(x)e^{-ixt}\, dm(t).$$

We shall encounter similar situations again, and shall apply convergence theorems to them without further comment.

(*b*) Theorem 9.2(*b*) shows that the Fourier transform of

$$[f(x + \alpha) - f(x)]/\alpha$$

is

$$\hat{f}(t)\, \frac{e^{i\alpha t} - 1}{\alpha}.$$

This suggests that an analogue of Theorem 9.2(*f*) should be true under certain conditions, namely, that the Fourier transform of f' is $it\hat{f}(t)$. If $f \in L^1$, $f' \in L^1$, and if f is the indefinite integral of f', the result is easily established by an integration by parts. We leave this, and some related results, as exercises. The fact that the Fourier transform converts differentiation to multiplication by ti makes the Fourier transform a useful tool in the study of differential equations.

The Inversion Theorem

9.4 We have just seen that certain operations on functions correspond nicely to operations on their Fourier transforms. The usefulness and interest of this correspondence will of course be enhanced if there is a way of returning from the transforms to the functions, that is to say, if there is an inversion formula.

Let us see what such a formula might look like, by analogy with Fourier series. If

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx}\, dx \qquad (n \in Z), \tag{1}$$

then the inversion formula is

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}. \tag{2}$$

We know that (2) holds, in the sense of L^2-convergence, if $f \in L^2(T)$. We also know that (2) does not necessarily hold in the sense of pointwise convergence, even if f is continuous. Suppose now that $f \in L^1(T)$, that $\{c_n\}$ is given by (1), and that

$$\sum_{-\infty}^{\infty} |c_n| < \infty. \tag{3}$$

Put

$$g(x) = \sum_{-\infty}^{\infty} c_n e^{inx}. \tag{4}$$

By (3), the series in (4) converges uniformly (hence g is continuous), and the Fourier coefficients of g are easily computed:

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)e^{-ikx}\, dx &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \sum_{n=-\infty}^{\infty} c_n e^{inx} \right\} e^{-ikx}\, dx \\ &= \sum_{n=-\infty}^{\infty} c_n \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n-k)x}\, dx \\ &= c_k. \end{aligned} \tag{5}$$

Thus f and g have the same Fourier coefficients. This implies $f = g$ a.e., so the Fourier series of f converges to $f(x)$ a.e.

The analogous assumptions in the context of Fourier transforms are that $f \in L^1$ and $\hat{f} \in L^1$, and we might then expect that a formula like

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(t)e^{itx}\, dm(t) \tag{6}$$

is valid. Certainly, if $\hat{f} \in L^1$, the right side of (6) is well defined; call it $g(x)$; but if we want to argue as in (5), we run into the integral

$$\int_{-\infty}^{\infty} e^{i(t-s)x}\, dx, \tag{7}$$

which is meaningless as it stands. Thus even under the strong assumption that $\hat{f} \in L^1$, a proof of (6) (which *is* true) has to proceed over a more devious route.

[It should be mentioned that (6) may hold even if $\hat{f} \notin L^1$, if the integral over $(-\infty, \infty)$ is interpreted as the limit, as $A \to \infty$, of integrals over $(-A, A)$. (Analogue: a series may converge without converging absolutely.) We shall not go into this.]

9.5 Theorem *For any function f on R^1 and every $y \in R^1$, let f_y be the translate of f defined by*

$$f_y(x) = f(x-y) \qquad (x \in R^1). \tag{1}$$

If $1 \le p < \infty$ and if $f \in L^p$, the mapping

$$y \to f_y \tag{2}$$

is a uniformly continuous mapping of R^1 into $L^p(R^1)$.

PROOF Fix $\epsilon > 0$. Since $f \in L^p$, there exists a continuous function g whose support lies in a bounded interval $[-A, A]$, such that

$$\|f - g\|_p < \epsilon$$

(Theorem 3.14). The uniform continuity of g shows that there exists a $\delta \in (0, A)$ such that $|s - t| < \delta$ implies

$$|g(s) - g(t)| < (3A)^{-1/p}\epsilon.$$

If $|s - t| < \delta$, it follows that

$$\int_{-\infty}^{\infty} |g(x-s) - g(x-t)|^p \, dx < (3A)^{-1}\epsilon^p(2A + \delta) < \epsilon^p,$$

so that $\|g_s - g_t\|_p < \epsilon$.

Note that L^p-norms (relative to Lebesgue measure) are translation-invariant: $\|f\|_p = \|f_s\|_p$. Thus

$$\begin{aligned} \|f_s - f_t\|_p &\le \|f_s - g_s\|_p + \|g_s - g_t\|_p + \|g_t - f_t\|_p \\ &= \|(f-g)_s\|_p + \|g_s - g_t\|_p + \|(g-f)_t\|_p < 3\epsilon \end{aligned}$$

whenever $|s - t| < \delta$. This completes the proof. ////

9.6 Theorem *If $f \in L^1$, then $\hat{f} \in C_0$ and*

$$\|\hat{f}\|_\infty \le \|f\|_1. \tag{1}$$

PROOF The inequality (1) is obvious from 9.1(4). If $t_n \to t$, then

$$|\hat{f}(t_n) - \hat{f}(t)| \le \int_{-\infty}^{\infty} |f(x)|\,|e^{-it_nx} - e^{-itx}|\, dm(x). \tag{2}$$

The integrand is bounded by $2|f(x)|$ and tends to 0 for every x, as $n \to \infty$. Hence $\hat{f}(t_n) \to \hat{f}(t)$, by the dominated convergence theorem. Thus $\hat{f}$ is continuous.

Since $e^{\pi i} = -1$, 9.1(4) gives

$$\hat{f}(t) = -\int_{-\infty}^{\infty} f(x)e^{-it(x+\pi/t)}\, dm(x) = -\int_{-\infty}^{\infty} f(x-\pi/t)e^{-itx}\, dm(x). \tag{3}$$

Hence

$$2\hat{f}(t) = \int_{-\infty}^{\infty} \left\{ f(x) - f\left(x - \frac{\pi}{t}\right)\right\} e^{-itx}\, dm(x), \tag{4}$$

so that

$$2|\hat{f}(t)| \le \|f - f_{\pi/t}\|_1, \tag{5}$$

which tends to 0 as $t \to \pm\infty$, by Theorem 9.5. ////

9.7 A Pair of Auxiliary Functions In the proof of the inversion theorem it will be convenient to know a positive function H which has a positive Fourier transform whose integral is easily calculated. Among the many possibilities we choose one which is of interest in connection with harmonic functions in a half plane. (See Exercise 25, Chap. 11.)

Put

$$H(t) = e^{-|t|} \tag{1}$$

and define

$$h_\lambda(x) = \int_{-\infty}^{\infty} H(\lambda t)e^{itx}\, dm(t) \qquad (\lambda > 0). \tag{2}$$

A simple computation gives

$$h_\lambda(x) = \sqrt{\frac{2}{\pi}}\,\frac{\lambda}{\lambda^2 + x^2} \tag{3}$$

and hence

$$\int_{-\infty}^{\infty} h_\lambda(x)\, dm(x) = 1. \tag{4}$$

Note also that $0 < H(t) \le 1$ and that $H(\lambda t) \to 1$ as $\lambda \to 0$.

9.8 Proposition *If $f \in L^1$, then*

$$(f * h_\lambda)(x) = \int_{-\infty}^{\infty} H(\lambda t)\hat{f}(t)e^{ixt}\, dm(t).$$

PROOF This is a simple application of Fubini's theorem.

$$
\begin{aligned}
(f * h_\lambda)(x) &= \int_{-\infty}^{\infty} f(x-y)\, dm(y) \int_{-\infty}^{\infty} H(\lambda t) e^{ity}\, dm(t) \\
&= \int_{-\infty}^{\infty} H(\lambda t)\, dm(t) \int_{-\infty}^{\infty} f(x-y) e^{ity}\, dm(y) \\
&= \int_{-\infty}^{\infty} H(\lambda t)\, dm(t) \int_{-\infty}^{\infty} f(y) e^{it(x-y)}\, dm(y) \\
&= \int_{-\infty}^{\infty} H(\lambda t) \hat{f}(t) e^{itx}\, dm(t).
\end{aligned}
$$ ////

9.9 Theorem *If $g \in L^\infty$ and g is continuous at a point x, then*

$$\lim_{\lambda \to 0} (g * h_\lambda)(x) = g(x). \tag{1}$$

PROOF On account of 9.7(4), we have

$$
\begin{aligned}
(g * h_\lambda)(x) - g(x) &= \int_{-\infty}^{\infty} [g(x-y) - g(x)] h_\lambda(y)\, dm(y) \\
&= \int_{-\infty}^{\infty} [g(x-y) - g(x)] \lambda^{-1} h_1\left(\frac{y}{\lambda}\right) dm(y) \\
&= \int_{-\infty}^{\infty} [g(x-\lambda s) - g(x)] h_1(s)\, dm(s).
\end{aligned}
$$

The last integrand is dominated by $2\|g\|_\infty h_1(s)$ and converges to 0 pointwise for every s, as $\lambda \to 0$. Hence (1) follows from the dominated convergence theorem. ////

9.10 Theorem *If $1 \le p < \infty$ and $f \in L^p$, then*

$$\lim_{\lambda \to 0} \|f * h_\lambda - f\|_p = 0. \tag{1}$$

The cases $p = 1$ and $p = 2$ will be the ones of interest to us, but the general case is no harder to prove.

PROOF Since $h_\lambda \in L^q$, where q is the exponent conjugate to p, $(f * h_\lambda)(x)$ is defined for every x. (In fact, $f * h_\lambda$ is continuous; see Exercise 8.) Because of 9.7(4) we have

$$(f * h_\lambda)(x) - f(x) = \int_{-\infty}^{\infty} [f(x-y) - f(x)] h_\lambda(y)\, dm(y) \tag{2}$$

and Theorem 3.3 gives

$$|(f * h_\lambda)(x) - f(x)|^p \le \int_{-\infty}^{\infty} |f(x-y) - f(x)|^p h_\lambda(y)\, dm(y). \tag{3}$$

Integrate (3) with respect to x and apply Fubini's theorem:

$$\|f * h_\lambda - f\|_p^p \le \int_{-\infty}^{\infty} \|f_y - f\|_p^p h_\lambda(y)\, dm(y). \tag{4}$$

If $g(y) = \|f_y - f\|_p^p$, then g is bounded and continuous, by Theorem 9.5, and $g(0) = 0$. Hence the right side of (4) tends to 0 as $\lambda \to 0$, by Theorem 9.9. ////

9.11 The Inversion Theorem *If $f \in L^1$ and $\hat{f} \in L^1$, and if*

$$g(x) = \int_{-\infty}^{\infty} \hat{f}(t) e^{ixt}\, dm(t) \qquad (x \in R^1), \tag{1}$$

then $g \in C_0$ and $f(x) = g(x)$ a.e.

PROOF By Proposition 9.8,

$$(f * h_\lambda)(x) = \int_{-\infty}^{\infty} H(\lambda t) \hat{f}(t) e^{ixt}\, dm(t). \tag{2}$$

The integrands on the right side of (2) are bounded by $|\hat{f}(t)|$, and since $H(\lambda t) \to 1$ as $\lambda \to 0$, the right side of (2) converges to $g(x)$, for every $x \in R^1$, by the dominated convergence theorem.

If we combine Theorems 9.10 and 3.12, we see that there is a sequence $\{\lambda_n\}$ such that $\lambda_n \to 0$ and

$$\lim_{n \to \infty} (f * h_{\lambda_n})(x) = f(x) \text{ a.e.} \tag{3}$$

Hence $f(x) = g(x)$ a.e. That $g \in C_0$ follows from Theorem 9.6. ////

9.12 The Uniqueness Theorem *If $f \in L^1$ and $\hat{f}(t) = 0$ for all $t \in R^1$, then $f(x) = 0$ a.e.*

PROOF Since $\hat{f} = 0$ we have $\hat{f} \in L^1$, and the result follows from the inversion theorem. ////

The Plancherel Theorem

Since the Lebesgue measure of R^1 is infinite, L^2 is not a subset of L^1, and the definition of the Fourier transform by formula 9.1(4) is therefore not directly applicable to every $f \in L^2$. The definition does apply, however, if $f \in L^1 \cap L^2$, and it turns out that then $\hat{f} \in L^2$. In fact, $\|\hat{f}\|_2 = \|f\|_2$! This isometry of $L^1 \cap L^2$ into L^2 extends to an isometry of L^2 onto L^2, and this extension defines the Fourier transform (sometimes called the *Plancherel transform*) of every $f \in L^2$. The resulting L^2-theory has in fact a great deal more symmetry than is the case in L^1. In L^2, f and $\hat{f}$ play exactly the same role.

9.13 Theorem *One can associate to each $f \in L^2$ a function $\hat{f} \in L^2$ so that the following properties hold:*

(*a*) *If $f \in L^1 \cap L^2$, then $\hat{f}$ is the previously defined Fourier transform of f.*
(*b*) *For every $f \in L^2$, $\|\hat{f}\|_2 = \|f\|_2$.*
(*c*) *The mapping $f \to \hat{f}$ is a Hilbert space isomorphism of L^2 onto L^2.*
(*d*) *The following symmetric relation exists between f and $\hat{f}$: If*

$$\varphi_A(t) = \int_{-A}^{A} f(x)e^{-ixt}\, dm(x) \quad \text{and} \quad \psi_A(x) = \int_{-A}^{A} \hat{f}(t)e^{ixt}\, dm(t),$$

then $\|\varphi_A - \hat{f}\|_2 \to 0$ and $\|\psi_A - f\|_2 \to 0$ as $A \to \infty$.

Note: Since $L^1 \cap L^2$ is dense in L^2, properties (*a*) and (*b*) determine the mapping $f \to \hat{f}$ uniquely. Property (*d*) may be called the *L^2 inversion theorem.*

PROOF Our first objective is the relation

$$\|\hat{f}\|_2 = \|f\|_2 \qquad (f \in L^1 \cap L^2). \tag{1}$$

We fix $f \in L^1 \cap L^2$, put $\tilde{f}(x) = \overline{f(-x)}$, and define $g = f * \tilde{f}$. Then

$$g(x) = \int_{-\infty}^{\infty} f(x-y)\overline{f(-y)}\, dm(y) = \int_{-\infty}^{\infty} f(x+y)\overline{f(y)}\, dm(y), \tag{2}$$

or

$$g(x) = (f_{-x}, f), \tag{3}$$

where the inner product is taken in the Hilbert space L^2 and f_{-x} denotes a translate of f, as in Theorem 9.5. By that theorem, $x \to f_{-x}$ is a continuous mapping of R^1 into L^2, and the continuity of the inner product (Theorem 4.6) therefore implies that g is a continuous function. The Schwarz inequality shows that

$$|g(x)| \le \|f_{-x}\|_2 \|f\|_2 = \|f\|_2^2 \tag{4}$$

so that g is bounded. Also, $g \in L^1$ since $f \in L^1$ and $\tilde{f} \in L^1$.

Since $g \in L^1$, we may apply Proposition 9.8:

$$(g * h_\lambda)(0) = \int_{-\infty}^{\infty} H(\lambda t)\hat{g}(t)\, dm(t). \tag{5}$$

Since g is continuous and bounded, Theorem 9.9 shows that

$$\lim_{\lambda \to 0} (g * h_\lambda)(0) = g(0) = \|f\|_2^2. \tag{6}$$

Theorem 9.2(*d*) shows that $\hat{g} = |\hat{f}|^2 \geq 0$, and since $H(\lambda t)$ increases to 1 as $\lambda \to 0$, the monotone convergence theorem gives

$$\lim_{\lambda \to 0} \int_{-\infty}^{\infty} H(\lambda t)\hat{g}(t)\, dm(t) = \int_{-\infty}^{\infty} |\hat{f}(t)|^2\, dm(t). \tag{7}$$

Now (5), (6), and (7) shows that $\hat{f} \in L^2$ and that (1) holds.

This was the crux of the proof.

Let Y be the space of all Fourier transforms $\hat{f}$ of functions $f \in L^1 \cap L^2$. By (1), $Y \subset L^2$. We claim that Y is dense in L^2, i.e., that $Y^{\perp} = \{0\}$.

The functions $x \to e^{i\alpha x}H(\lambda x)$ are in $L^1 \cap L^2$, for all real α and all $\lambda > 0$. Their Fourier transforms

$$h_\lambda(\alpha - t) = \int_{-\infty}^{\infty} e^{i\alpha x}H(\lambda x)e^{-ixt}\, dm(x) \tag{8}$$

are therefore in Y. If $w \in L^2$, $w \in Y^{\perp}$, it follows that

$$(h_\lambda * \bar{w})(\alpha) = \int_{-\infty}^{\infty} h_\lambda(\alpha - t)\bar{w}(t)\, dm(t) = 0 \tag{9}$$

for all α. Hence $w = 0$, by Theorem 9.10, and therefore Y is dense in L^2.

Let us introduce the temporary notation Φf for $\hat{f}$. From what has been proved so far, we see that Φ is an L^2-isometry from one dense subspace of L^2, namely $L^1 \cap L^2$, onto another, namely Y. Elementary Cauchy sequence arguments (compare with Lemma 4.16) imply therefore that Φ extends to an isometry $\tilde{\Phi}$ of L^2 onto L^2. If we write $\hat{f}$ for $\tilde{\Phi}f$, we obtain properties (*a*) and (*b*).

Property (*c*) follows from (*b*), as in the proof of Theorem 4.18. The *Parseval formula*

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dm(x) = \int_{-\infty}^{\infty} \hat{f}(t)\overline{\hat{g}(t)}\, dm(t) \tag{10}$$

holds therefore for all $f \in L^2$ and $g \in L^2$.

To prove (*d*), let k_A be the characteristic function of $[-A, A]$. Then $k_A f \in L^1 \cap L^2$ if $f \in L^2$, and

$$\varphi_A = (k_A f)\hat{}. \tag{11}$$

Since $\|f - k_A f\|_2 \to 0$ as $A \to \infty$, it follows from (*b*) that

$$\|\hat{f} - \varphi_A\|_2 = \|(f - k_A f)\hat{}\|_2 \to 0 \tag{12}$$

as $A \to \infty$.

The other half of (*d*) is proved the same way. ////

9.14 Theorem *If $f \in L^2$ and $\hat{f} \in L^1$, then*

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(t)e^{ixt}\, dm(t) \qquad a.e.$$

PROOF This is corollary of Theorem 9.13(d). ////

9.15 Remark If $f \in L^1$, formula 9.1(4) defines $\hat{f}(t)$ unambiguously for every t. If $f \in L^2$, the Plancherel theorem defines $\hat{f}$ uniquely as an element of the Hilbert space L^2, but as a point function $\hat{f}(t)$ is only determined almost everywhere. This is an important difference between the theory of Fourier transforms in L^1 and in L^2. The indeterminacy of $\hat{f}(t)$ as a point function will cause some difficulties in the problem to which we now turn.

9.16 Translation-Invariant Subspaces of L^2 A subspace M of L^2 is said to be *translation-invariant* if $f \in M$ implies that $f_\alpha \in M$ for every real α, where $f_\alpha(x) = f(x - \alpha)$. Translations have already played an important part in our study of Fourier transforms. We now pose a problem whose solution will afford an illustration of how the Plancherel theorem can be used. (Other applications will occur in Chap. 19.) The problem is:

Describe the closed translation-invariant subspaces of L^2.

Let M be a closed translation-invariant subspace of L^2, and let $\hat{M}$ be the image of M under the Fourier transform. Then $\hat{M}$ is closed (since the Fourier transform is an L^2-isometry). If f_α is a translate of f, the Fourier transform of f_α is $\hat{f}e_\alpha$, where $e_\alpha(t) = e^{-i\alpha t}$; we proved this for $f \in L^1$ in Theorem 9.2; the result extends to L^2, as can be seen from Theorem 9.13(d). *It follows that $\hat{M}$ is invariant under multiplication by e_α, for all $\alpha \in R^1$.*

Let E be any measurable set in R^1. If $\hat{M}$ is the set of all $\varphi \in L^2$ which vanish a.e. on E, then $\hat{M}$ certainly is a subspace of L^2, which is invariant under multiplication by all e_α (note that $|e_\alpha| = 1$, so $\varphi e_\alpha \in L^2$ if $\varphi \in L^2$), and $\hat{M}$ is also closed. *Proof*: $\varphi \in \hat{M}$ if and only if φ is orthogonal to every $\psi \in L^2$ which vanishes a.e. on the complement of E.

If M is the inverse image of this $\hat{M}$, under the Fourier transform, then M is a space with the desired properties.

One may now conjecture that every one of our spaces M is obtained in this manner, from a set $E \subset R^1$. To prove this, we have to show that to every closed translation-invariant $M \subset L^2$ there corresponds a set $E \subset R^1$ such that $f \in M$ if and only if $\hat{f}(t) = 0$ a.e. on E. The obvious way of constructing E from M is to associate with each $f \in M$ the set E_f consisting of all points at which $\hat{f}(t) = 0$, and to define E as the intersection of these sets E_f. But this obvious attack runs into a serious difficulty: Each E_f is defined only up to sets of measure 0. If $\{A_i\}$ is a countable collection of sets, each determined up to sets of measure 0, then $\bigcap A_i$ is also determined up to sets of measure 0. But there are uncountably many $f \in M$, so we lose all control over $\bigcap E_f$.

This difficulty disappears entirely if we think of our functions as elements of the Hilbert space L^2, and not primarily as point functions.

We shall now prove the conjecture. Let $\hat{M}$ be the image of a closed translation-invariant subspace $M \subset L^2$, under the Fourier transform. Let P be the orthogonal projection of L^2 onto $\hat{M}$ (Theorem 4.11): To each $f \in L^2$ there corresponds a unique $Pf \in \hat{M}$ such that $f - Pf$ is orthogonal to $\hat{M}$. Hence

$$f - Pf \perp Pg \qquad (f \text{ and } g \in L^2) \tag{1}$$

and since $\hat{M}$ is invariant under multiplication by e_α, we also have

$$f - Pf \perp (Pg)e_\alpha \qquad (f \text{ and } g \in L^2, \alpha \in R^1). \tag{2}$$

If we recall how the inner product is defined in L^2, we see that (2) is equivalent to

$$\int_{-\infty}^{\infty} (f - Pf) \cdot \overline{Pg} \cdot e_{-\alpha}\, dm = 0 \qquad (f \text{ and } g \in L^2, \alpha \in R^1) \tag{3}$$

and this says that the Fourier transform of

$$(f - Pf) \cdot \overline{Pg} \tag{4}$$

is 0. The function (4) is the product of two L^2-functions, hence is in L^1, and the uniqueness theorem for Fourier transforms shows now that the function (4) is 0 a.e. This remains true if $\overline{Pg}$ is replaced by Pg. Hence

$$f \cdot Pg = (Pf) \cdot (Pg) \qquad (f \text{ and } g \in L^2). \tag{5}$$

Interchanging the roles of f and g leads from (5) to

$$f \cdot Pg = g \cdot Pf \qquad (f \text{ and } g \in L^2). \tag{6}$$

Now let g be a fixed positive function in L^2; for instance, put $g(t) = e^{-|t|}$. Define

$$\varphi(t) = \frac{(Pg)(t)}{g(t)}. \tag{7}$$

$(Pg)(t)$ may only be defined a.e.; choose any one determination in (7). Now (6) becomes

$$Pf = \varphi \cdot f \qquad (f \in L^2). \tag{8}$$

If $f \in \hat{M}$, then $Pf = f$. This says that $P^2 = P$, and it follows that $\varphi^2 = \varphi$, because

$$\varphi^2 \cdot g = \varphi \cdot Pg = P^2 g = Pg = \varphi \cdot g. \tag{9}$$

Since $\varphi^2 = \varphi$, we have $\varphi = 0$ or 1 a.e., and if we let E be the set of all t where $\varphi(t) = 0$, then $\hat{M}$ consists precisely of those $f \in L^2$ which are 0 a.e. on E, since $f \in \hat{M}$ if and only if $f = Pf = \varphi \cdot f$.

We therefore obtain the following solution to our problem.

9.17 Theorem *Associate to each measurable set $E \subset R^1$ the space M_E of all $f \in L^2$ such that $\hat{f} = 0$ a.e. on E. Then M_E is a closed translation-invariant subspace of L^2. Every closed translation-invariant subspace of L^2 is M_E for some E, and $M_A = M_B$ if and only if*

$$m((A - B) \cup (B - A)) = 0.$$

The uniqueness statement is easily proved; we leave the details to the reader.

The above problem can of course be posed in other function spaces. It has been studied in great detail in L^1. The known results show that the situation is infinitely more complicated there than in L^2.

The Banach Algebra L^1

9.18 Definition A Banach space A is said to be a *Banach algebra* if there is a multiplication defined in A which satisfies the inequality

$$\|xy\| \le \|x\| \, \|y\| \qquad (x \text{ and } y \in A), \tag{1}$$

the associative law $x(yz) = (xy)z$, the distributive laws

$$x(y + z) = xy + xz, \quad (y + z)x = yx + zx \quad (x, y, \text{ and } z \in A), \tag{2}$$

and the relation

$$(\alpha x)y = x(\alpha y) = \alpha(xy) \tag{3}$$

where α is any scalar.

9.19 Examples

(*a*) Let $A = C(X)$, where X is a compact Hausdorff space, with the supremum norm and the usual pointwise multiplication of functions: $(fg)(x) = f(x)g(x)$. This is a commutative Banach algebra ($fg = gf$) with unit (the constant function 1).

(*b*) $C_0(R^1)$ is a commutative Banach algebra without unit, i.e., without an element u such that $uf = f$ for all $f \in C_0(R^1)$.

(*c*) The set of all linear operators on R^k (or on any Banach space), with the operator norm as in Definition 5.3, and with addition and multiplication defined by

$$(A + B)(x) = Ax + Bx, \quad (AB)x = A(Bx),$$

is a Banach algebra with unit which is not commutative when $k > 1$.

(*d*) L^1 is a Banach algebra if we define multiplication by convolution; since

$$\|f * g\|_1 \le \|f\|_1 \|g\|_1,$$

the norm inequality is satisfied. The associative law could be verified directly (an application of Fubini's theorem), but we can proceed as follows: We know that the Fourier transform of $f * g$ is $\hat{f} \cdot \hat{g}$, and we know that the mapping $f \to \hat{f}$ is one-to-one. For every $t \in R^1$,

$$\hat{f}(t)[\hat{g}(t)\hat{h}(t)] = [\hat{f}(t)\hat{g}(t)]\hat{h}(t),$$

by the associative law for complex numbers. It follows that

$$f * (g * h) = (f * g) * h.$$

In the same way we see immediately that $f * g = g * f$. The remaining requirements of Definition 9.18 are also easily seen to hold in L^1.

Thus L^1 is a commutative Banach algebra. The Fourier transform is an algebra isomorphism of L^1 into C_0. Hence there is no $f \in L^1$ with $\hat{f} \equiv 1$, and therefore L^1 has no unit.

9.20 Complex Homomorphisms The most important complex functions on a Banach algebra A are the homomorphisms of A into the complex field. These are precisely the linear functionals which also preserve multiplication, i.e., the functions φ such that

$$\varphi(\alpha x + \beta y) = \alpha\varphi(x) + \beta\varphi(y), \qquad \varphi(xy) = \varphi(x)\varphi(y)$$

for all x and $y \in A$ and all scalars α and β. Note that no boundedness assumption is made in this definition. It is a very interesting fact that this would be redundant:

9.21 Theorem *If φ is a complex homomorphism on a Banach algebra A, then the norm of φ, as a linear functional, is at most* 1.

PROOF Assume, to get a contradiction, that $|\varphi(x_0)| > \|x_0\|$ for some $x_0 \in A$. Put $\lambda = \varphi(x_0)$, and put $x = x_0/\lambda$. Then $\|x\| < 1$ and $\varphi(x) = 1$.

Since $\|x^n\| \le \|x\|^n$ and $\|x\| < 1$, the elements

$$s_n = -x - x^2 - \cdots - x^n \tag{1}$$

form a Cauchy sequence in A. Since A is complete, being a Banach space, there exists a $y \in A$ such that $\|y - s_n\| \to 0$, and it is easily seen that $x + s_n = xs_{n-1}$, so that

$$x + y = xy. \tag{2}$$

Hence $\varphi(x) + \varphi(y) = \varphi(x)\varphi(y)$, which is impossible since $\varphi(x) = 1$. ////

9.22 The Complex Homomorphisms of L^1 Suppose φ is a complex homomorphism of L^1, i.e., a linear functional (of norm at most 1, by Theorem 9.21) which also satisfies the relation

$$\varphi(f * g) = \varphi(f)\varphi(g) \qquad (f \text{ and } g \in L^1). \tag{1}$$

By Theorem 6.16, there exists a $\beta \in L^\infty$ such that

$$\varphi(f) = \int_{-\infty}^{\infty} f(x)\beta(x)\, dm(x) \qquad (f \in L^1). \tag{2}$$

We now exploit the relation (1) to see what else we can say about β. On the one hand,

$$\begin{aligned}\varphi(f * g) &= \int_{-\infty}^{\infty} (f * g)(x)\beta(x)\, dm(x) \\ &= \int_{-\infty}^{\infty} \beta(x)\, dm(x) \int_{-\infty}^{\infty} f(x-y)g(y)\, dm(y) \\ &= \int_{-\infty}^{\infty} g(y)\, dm(y) \int_{-\infty}^{\infty} f_y(x)\beta(x)\, dm(x) \\ &= \int_{-\infty}^{\infty} g(y)\varphi(f_y)\, dm(y). \end{aligned} \tag{3}$$

On the other hand,

$$\varphi(f)\varphi(g) = \varphi(f) \int_{-\infty}^{\infty} g(y)\beta(y)\, dm(y). \tag{4}$$

Let us now assume that φ is not identically 0. Fix $f \in L^1$ so that $\varphi(f) \neq 0$. Since the last integral in (3) is equal to the right side of (4) for every $g \in L^1$, the uniqueness assertion of Theorem 6.16 shows that

$$\varphi(f)\beta(y) = \varphi(f_y) \tag{5}$$

for almost all y. But $y \to f_y$ is a continuous mapping of R^1 into L^1 (Theorem 9.5) and φ is continuous on L^1. Hence the right side of (5) is a continuous function of y, and we may assume [by changing $\beta(y)$ on a set of measure 0 if necessary, which does not affect (2)] that β is continuous. If we replace y by $x + y$ and then f by f_x in (5), we obtain

$$\varphi(f)\beta(x+y) = \varphi(f_{x+y}) = \varphi((f_x)_y) = \varphi(f_x)\beta(y) = \varphi(f)\beta(x)\beta(y),$$

so that

$$\beta(x+y) = \beta(x)\beta(y) \qquad (x \text{ and } y \in R^1). \tag{6}$$

Since β is not identically 0, (6) implies that $\beta(0) = 1$, and the continuity of β shows that there is a $\delta > 0$ such that

$$\int_0^\delta \beta(y)\, dy = c \neq 0. \tag{7}$$

Then

$$c\beta(x) = \int_0^\delta \beta(y)\beta(x)\,dy = \int_0^\delta \beta(y+x)\,dy = \int_x^{x+\delta} \beta(y)\,dy. \tag{8}$$

Since β is continuous, the last integral is a differentiable function of x; hence (8) shows that β is differentiable. Differentiate (6) with respect to y, then put $y = 0$; the result is

$$\beta'(x) = A\beta(x), \qquad A = \beta'(0). \tag{9}$$

Hence the derivative of $\beta(x)e^{-Ax}$ is 0, and since $\beta(0) = 1$, we obtain

$$\beta(x) = e^{Ax}. \tag{10}$$

But β is bounded on R^1. Therefore A must be pure imaginary, and we conclude: There exists a $t \in R^1$ such that

$$\beta(x) = e^{-itx}. \tag{11}$$

We have thus arrived at the Fourier transform.

9.23 Theorem *To every complex homomorphism φ on L^1 (except to $\varphi = 0$) there corresponds a unique $t \in R^1$ such that $\varphi(f) = \hat{f}(t)$.*

The existence of t was proved above. The uniqueness follows from the observation that if $t \neq s$ then there exists an $f \in L^1$ such that $\hat{f}(t) \neq \hat{f}(s)$; take for $f(x)$ a suitable translate of $e^{-|x|}$.

Exercises

1 Suppose $f \in L^1, f > 0$. Prove that $|\hat{f}(y)| < \hat{f}(0)$ for every $y \neq 0$.

2 Compute the Fourier transform of the characteristic function of an interval. For $n = 1, 2, 3, \ldots$, let g_n be the characteristic function of $[-n, n]$, let h be the characteristic function of $[-1, 1]$, and compute $g_n * h$ explicitly. (The graph is piecewise linear.) Show that $g_n * h$ is the Fourier transform of a function $f_n \in L^1$; except for a multiplicative constant,

$$f_n(x) = \frac{\sin x \sin nx}{x^2}$$

Show that $\|f_n\|_1 \to \infty$ and conclude that the mapping $f \to \hat{f}$ maps L^1 into a *proper* subset of C_0.

Show, however, that the range of this mapping is dense in C_0.

3 Find

$$\lim_{A\to\infty} \int_{-A}^{A} \frac{\sin \lambda t}{t} e^{itx}\,dt \qquad (-\infty < x < \infty)$$

where λ is a positive constant.

4 Give examples of $f \in L^2$ such that $f \notin L^1$ but $\hat{f} \in L^1$. Under what circumstances can this happen?

5 If $f \in L^1$ and $\int |t\hat{f}(t)|\, dm(t) < \infty$, prove that f coincides a.e. with a differentiable function whose derivative is

$$i \int_{-\infty}^{\infty} t\hat{f}(t)e^{ixt}\, dm(t).$$

6 Suppose $f \in L^1$, f is differentiable almost everywhere, and $f' \in L^1$. Does it follow that the Fourier transform of f' is $ti\hat{f}(t)$?

7 Let S be the class of all functions f on R^1 which have the following property: f is infinitely differentiable, and there are numbers $A_{mn}(f) < \infty$, for m and $n = 0, 1, 2, \ldots$, such that

$$|x^n D^m f(x)| \le A_{mn}(f) \qquad (x \in R^1).$$

Here D is the ordinary differentiation operator.

Prove that the Fourier transform maps S onto S. Find examples of members of S.

8 If p and q are conjugate exponents, $f \in L^p$, $g \in L^q$, and $h = f * g$, prove that h is uniformly continuous. If also $1 < p < \infty$, then $h \in C_0$; show that this fails for some $f \in L^1$, $g \in L^\infty$.

9 Suppose $1 \le p < \infty$, $f \in L^p$, and

$$g(x) = \int_x^{x+1} f(t)\, dt.$$

Prove that $g \in C_0$. What can you say about g if $f \in L^\infty$?

10 Let C^∞ be the class of all infinitely differentiable complex functions on R^1, and let C_c^∞ consist of all $g \in C^\infty$ whose supports are compact. Show that C_c^∞ does not consist of 0 alone.

Let L^1_{loc} be the class of all f which belong to L^1 locally; that is, $f \in L^1_{\text{loc}}$ provided that f is measurable and $\int_I |f| < \infty$ for every bounded interval I.

If $f \in L^1_{\text{loc}}$ and $g \in C_c^\infty$, prove that $f * g \in C^\infty$.

Prove that there are sequences $\{g_n\}$ in C_c^∞ such that

$$\|f * g_n - f\|_1 \to 0$$

as $n \to \infty$, for every $f \in L^1$. (Compare Theorem 9.10.) Prove that $\{g_n\}$ can also be so chosen that $(f * g_n)(x) \to f(x)$ a.e., for every $f \in L^1_{\text{loc}}$; in fact, for suitable $\{g_n\}$ the convergence occurs at every point x at which f is the derivative of its indefinite integral.

Prove that $(f * h_\lambda)(x) \to f(x)$ a.e. if $f \in L^1$, as $\lambda \to 0$, and that $f * h_\lambda \in C^\infty$, although h_λ does not have compact support. (h_λ is defined in Sec. 9.7.)

11 Find conditions on f and/or $\hat{f}$ which ensure the correctness of the following formal argument: If

$$\varphi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)e^{-itx}\, dx$$

and

$$F(x) = \sum_{k=-\infty}^{\infty} f(x + 2k\pi)$$

then F is periodic, with period 2π, the nth Fourier coefficient of F is $\varphi(n)$, hence $F(x) = \sum \varphi(n)e^{inx}$. In particular,

$$\sum_{k=-\infty}^{\infty} f(2k\pi) = \sum_{n=-\infty}^{\infty} \varphi(n).$$

More generally,

$$\sum_{k=-\infty}^{\infty} f(k\beta) = \alpha \sum_{n=-\infty}^{\infty} \varphi(n\alpha) \qquad \text{if } \alpha > 0,\ \beta > 0,\ \alpha\beta = 2\pi. \tag{*}$$

What does (*) say about the limit, as $\alpha \to 0$, of the right-hand side (for "nice" functions, of course)? Is this in agreement with the inversion theorem?

[(*) is known as the Poisson summation formula.]

12 Take $f(x) = e^{-|x|}$ in Exercise 11 and derive the identity

$$\frac{e^{2\pi\alpha}+1}{e^{2\pi\alpha}-1} = \frac{1}{\pi}\sum_{n=-\infty}^{\infty} \frac{\alpha}{\alpha^2+n^2}.$$

13 If $0 < c < \infty$, define $f_c(x) = \exp(-cx^2)$.

(a) Compute $\hat{f}_c$. *Hint*: If $\varphi = \hat{f}_c$, an integration by parts gives $2c\varphi'(t) + t\varphi(t) = 0$.

(*b*) Show that there is one (and only one) c for which $\hat{f}_c = f_c$.

(*c*) Show that $f_a * f_b = \gamma f_c$; find γ and c explicitly in terms of a and b.

(*d*) Take $f = f_c$ in Exercise 11. What is the resulting identity?

14 The Fourier transform can be defined for $f \in L^1(R^k)$ by

$$\hat{f}(y) = \int_{R^k} f(x)e^{-ix\cdot y}\, dm_k(x) \qquad (y \in R^k),$$

where $x \cdot y = \sum \xi_i \eta_i$ if $x = (\xi_1, \ldots, \xi_k)$, $y = (\eta_1, \ldots, \eta_k)$, and m_k is Lebesgue measure on R^k, divided by $(2\pi)^{k/2}$ for convenience. Prove the inversion theorem and the Plancherel theorem in this context, as well as the analogue of Theorem 9.23.

15 If $f \in L^1(R^k)$, A is a linear operator on R^k, and $g(x) = f(Ax)$, how is $\hat{g}$ related to $\hat{f}$? If f is invariant under rotations, i.e., if $f(x)$ depends only on the euclidean distance of x from the origin, prove that the same is true of $\hat{f}$.

16 The *Laplacian* of a function f on R^k is

$$\Delta f = \sum_{j=1}^{k} \frac{\partial^2 f}{\partial x_j^2},$$

provided the partial derivatives exist. What is the relation between $\hat{f}$ and $\hat{g}$ if $g = \Delta f$ and all necessary integrability conditions are satisfied? It is clear that the Laplacian commutes with translations. Prove that it also commutes with rotations, i.e., that

$$\Delta(f \circ A) = (\Delta f) \circ A$$

whenever f has continuous second derivatives and A is a rotation of R^k. (Show that it is enough to do this under the additional assumption that f has compact support.)

17 Show that every Lebesgue measurable character of R^1 is continuous. Do the same for R^k. (Adapt part of the proof of Theorem 9.23.) Compare with Exercise 18.

18 Show (with the aid of the Hausdorff maximality theorem) that there exist real *discontinuous* functions f on R^1 such that

$$f(x + y) = f(x) + f(y) \tag{1}$$

for all x and $y \in R^1$.

Show that if (1) holds and f is Lebesgue measurable, then f is continuous.

Show that if (1) holds and the graph of f is not dense in the plane, then f is continuous.

Find all continuous functions which satisfy (1).

19 Suppose A and B are measurable subsets of R^1, having finite positive measure. Show that the convolution $\chi_A * \chi_B$ is continuous and not identically 0. Use this to prove that $A + B$ contains a segment.

(A different proof was suggested in Exercise 5, Chap. 7.)