Therefore $y - y'$ is orthogonal to all the x_α in S. Since S is a complete orthonormal system we must have $y - y' = 0$. Thus

$$y = \lim_{n\to\infty} \sum_{j=1}^{n} (x_{\alpha_j}, y)x_{\alpha_j}$$

and (II.1) holds. Furthermore,

$$\begin{aligned} 0 &= \lim_{n\to\infty} \left\| y - \sum_{j=1}^{n} (x_{\alpha_j}, y)x_{\alpha_j} \right\|^2 \\ &= \lim_{n\to\infty} \left(\|y\|^2 - \sum_{j=1}^{n} |(x_{\alpha_j}, y)|^2 \right) \\ &= \|y\|^2 - \sum_{\alpha\in A} |(x_\alpha, y)|^2 \end{aligned}$$

so that (II.2) holds also. We omit the easy proof of the converse statement. ▮

We note that (II.2) is called Parseval's relation. The coefficients (x_α, y) are often called the **Fourier coefficients** of y with respect to the basis $\{x_\alpha\}$. The reason for this terminology will become apparent shortly.

We now describe a useful procedure, called Gram–Schmidt orthogonalization, for constructing an orthonormal set from an arbitrary sequence of independent vectors. Suppose the independent vectors $u_1, u_2, \ldots$ are given and define

$$\begin{aligned} w_1 &= u_1, & v_1 &= w_1/\|w_1\| \\ w_2 &= u_2 - (v_1, u_2)v_1, & v_2 &= w_2/\|w_2\| \\ &\vdots & &\vdots \\ w_n &= u_n - \sum_{k=1}^{n-1} (v_k, u_n)v_k, & v_n &= w_n/\|w_n\| \\ &\vdots & &\vdots \end{aligned}$$

The family $\{v_j\}$ is an orthonormal set and has the property that for each m, $\{u_j\}_{j=1}^m$ and $\{v_j\}_{j=1}^m$ span the same vector space. In particular, the set of finite linear combinations of all the v's is the same as the finite linear combinations of the u's (see Figure II.2).

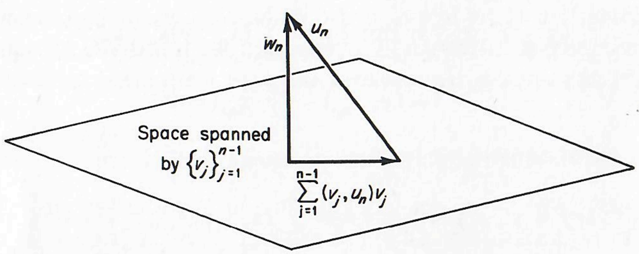

FIGURE II.2 Gram–Schmidt orthogonalization.

We remark that the Legendre polynomials (up to constant multiples) are obtained by applying the Gram–Schmidt process to the functions 1, x, x^2, $x^3, \ldots$, on the interval $[-1, 1]$ with the usual L^2 inner product.

Definition A metric space which has a countable dense subset is said to be **separable.**

Most Hilbert spaces that arise in practice are separable. The following theorem characterizes them up to isomorphism.

Theorem II.7 A Hilbert space $\mathscr{H}$ is separable if and only if it has a countable orthonormal basis S. If there are $N < \infty$ elements in S, then $\mathscr{H}$ is isomorphic to $\mathbb{C}^N$. If there are countably many elements in S, then $\mathscr{H}$ is isomorphic to ℓ_2 (Example 3, Section II.1).

Proof Suppose $\mathscr{H}$ is separable and let $\{x_n\}$ be a countable dense set. By throwing out some of the x_n's we can get a subcollection of independent vectors whose span (finite linear combinations) is the same as the $\{x_n\}$ and is thus dense. Applying the Gram–Schmidt procedure to this subcollection we obtain a countable complete orthonormal system. Conversely, if $\{y_n\}$ is a complete orthonormal system for a Hilbert space $\mathscr{H}$ then it follows from Theorem II.6 that the set of finite linear combinations of the y_n with rational coefficients is dense in $\mathscr{H}$. Since this set is countable, $\mathscr{H}$ is separable.

Suppose $\mathscr{H}$ is separable and $\{y_n\}_{n=1}^\infty$ is a complete orthonormal system. We define a map $\mathscr{U}: \mathscr{H} \to \ell_2$ by

$$\mathscr{U}: x \to \{(y_n, x)\}_{n=1}^\infty$$

Theorem II.6 shows that this map is well defined and onto. It is easy to show it is unitary. The proof that $\mathscr{H}$ is isomorphic to $\mathbb{C}^N$ if S has N elements is similar. ▮

Notice that in the separable case, the Gram–Schmidt process allows us to construct an orthonormal basis without using Zorn's lemma.

We conclude this section with an example that shows how Hilbert spaces arose naturally from problems in classical analysis. If $f(x)$ is an integrable function on $[0, 2\pi]$ we can define the numbers

$$c_n = \frac{1}{(2\pi)^{1/2}} \int_0^{2\pi} e^{-inx} f(x)\, dx$$

The formal series $\sum_{n=-\infty}^{\infty} c_n (2\pi)^{-1/2} e^{inx}$ is called the Fourier series of f. The classical problem is: for which f and in what sense does the Fourier series of